Semi-classical mass of quantum k-component topological kinks

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Abstract

We use the generalized zeta function regularization method to compute the one-loop quantum correction to the masses of the TK1 and TK2 kinks in a deformation of the O(N) linear sigma model on the real line.

1 Introduction

In this paper we shall apply the generalized zeta function regularization method to unveil the semi-classical behaviour of the quantum topological kinks arising as BPS states in multi-component scalar (1+1)-dimensional field theory. The interest of this investigation lies in the fact that these systems live at the heart of the low energy regime of string/M theory. Effective theories such as $N = 1$ SUSY QCD and/or the Wess-Zumino model [1] encompass N-component scalar fields and have a discrete set of vacuum states. From a one-dimensional perspective one can foresee the existence of k-component topological kinks, $k \leq N$, which are the seeds of the BPS domain walls. These extended states play such a prominent rôle in the three-dimensional world that analysis of the quantum behaviour of k-component topological kinks becomes an important issue.

Here, we shall focus on a one-parametric family of deformations of the $O(N)$ linear sigma model, in (1+1)-dimensional space-time. The model to be addressed forms the bosonic sector of a super-symmetric theory with N real super-fields and is of the general type of the Wess-Zumino model, like those studied in [1] and [2]. An important feature common to all these models is the non-existence of continuous symmetries in internal space and they can therefore be included in the class of systems considered in Reference [6]. The authors of these works use continuous phase-shift methods to calculate the one-loop correction to the kink energy in full super-symmetric theories.

Our approach differs in two ways: 1) we restrict ourselves to the bosonic sector and leave the fermionic fluctuations for future research; 2) the generalized zeta function regularization procedure is applied to deal with the infinite quantities arising in the quantization prescription. This method has been used previously in the description of quantum corrections to kink masses for theories with a single real scalar field, see [7], but here we shall follow the more elaborated procedure developed in [8] and [9] for one-component systems. Nevertheless, our method is particularly suited to models with several real fields because in general, in theories of this type, the spectrum of the second order fluctuation operator around kink solutions is only partially (asymptotically) known. Since this operator is a Schrodinger operator acting on functions belonging to the Hilbert space $L^2(R) \quad C^N$, one can write an associated heat equation. From the asymptotic expansion to the heat function one obtains enough information about the generalized zeta function of the second fluctuation operator [10] to provide a good approximation to the one-loop correction to the kink mass. Asymptotic methods to compute the mass of quantum solitons were first used in Reference [13].

Within a given range of the deformation parameter, the modified $O(2)$ linear sigma model that we shall consider is the celebrated Montonen-Sarker-Trullinger-Bishop (MSTB) model [3]. Over the years, several kinds of kink solitary waves with very noticeable properties have been discovered in this system, [4]. The kink moduli space of the $N \quad 3$ generalizations of the MSTB model has been described in [5], whereas more recently the stability of the different kinds of kink was established in [18]. Therefore, computation of the semi-classical mass of the stable topological kinks with one non-null component, TK1, or two non-null components, TK2, is compelling. To achieve this task, which is the main goal of this paper, the zeta function regularization method is specially appropriate, because of the impossibility of solving the spectral problem of the second order fluctuation operator.

The organization of the paper is as follows: In Section x.2 the general semi-classical formula for the mass of quantum solitons, the zeta function regularization procedure, and the zero point energy and mass renormalization prescriptions are explained. In Section x.3 we compute the one-

loop quantum corrections to the masses of topological kinks with one and two non-null components in the MSTB model. In Section x.4 similar formulas are given in the deformation of the linear O(N) sigma model, with N 3, that generalizes the MSTB model. Section x.5 offers some comments on possible extensions of our results. Finally, in Appendix A the first coefficients of the matrix heat kernel expansion are written, whereas, in Appendix B, it is shown that only the stable two-component topological kinks saturate the topological bound.

2 Semi-classical mass formula for k-component quantum topological kinks

N-component scalar fields are maps from the $R^{1;1}$ Minkowski space-time to the R^N "internal" space:

$$\tilde{}(y\ ) = \sum_{a=1}^{N} {}_a(y\ )e_a : R^{1;1} \ ! \ R^N :$$

Here, $y\ , \ = 0;1$, are coordinates in $R^{1;1}$; $@\ = \frac{@}{@y}$ is a basis in $TR^{1;1}$, and $e_a;a = 1;2; \ ;N$, are ortho-normal vectors in R^N, $e_a \ e = {}_{ab}$. We shall consider (1+1)-dimensional field theories whose classical dynamics is governed by the action

$$S = \int d^2y \ \frac{1}{2}@ \tilde{}@ \tilde{} \ U(\tilde{}) \ :$$

We choose the metric tensor in $T^2(R^{1;1})$ as $g = \mathrm{diag}(1; \ 1)$ and the Einstein convention will be used throughout the paper only for the indices in $R^{1;1}$. The system of units -identical to that chosen in [8]- is such that only the speed of light is set to $c = 1$.

The classical configuration space C is formed by the static configurations $\tilde{}(y)$ -we denote the spatial coordinate as $y^1 = y$ - for which the energy functional

$$E(\tilde{}) = \int dy \ \left(\frac{1}{2} \sum_{a=1}^{N} \frac{d \ _a}{dy}\frac{d \ _a}{dy} + U(\tilde{}) \right.$$

is finite: $C = f \tilde{}(y) = E(\tilde{}) < +1 \ g$. In the Schrodinger picture, the quantum evolution is ruled by the Schrodinger functional equation

$$i\tilde{}\frac{@}{@t} \ [\tilde{}(y);t = y^0] = H \ [\tilde{}(y);t = y^0]:$$

If $\tilde{}_V$ is a constant minimum of U, the masses of the fundamental quanta are: $v_a^2 \ {}_{ab} = \frac{{}^2U}{{}_a \ {}_b}j_{\tilde{}_V}$. The quantum Hamiltonian operator

$$H = \int dy \ \left(\frac{\tilde{}^2}{2} \sum_{a=1}^{N} v_a^2 \frac{}{{}_a(y)}\frac{}{{}_a(y)} \right. + E[\tilde{}(y)]$$

acts on wave functionals $[\tilde{}(y);t]$ that belong to $L^2(C)$.

A straightforward generalization of the arguments and definitions of the Section x.2 of [8] – following the classical papers [11], [12] – shows that the kink ground state energy at the semi-classical limit is:

$$E_0^K = E[\tilde{}_K] + \frac{\tilde{}}{2} \mathrm{Tr}(PK)^{\frac{1}{2}} + o(\tilde{}^2) = E[\tilde{}_K] + \frac{\tilde{}}{2}\ {}_{PK}(\ \tfrac{1}{2}) + o(\tilde{}^2)\quad : \tag{1}$$

Here K is the second variation operator, with spectral equation $K\ \tilde{}_n(x) = !_n^2\ \tilde{}_n(x)$, and

$${}_{PK}(s) = \sum_{!_n^2 > 0} \frac{1}{(!_n^2)^s}$$

is the associated generalized zeta function; P is the projector to the strictly positive part of Spec K. In this case, however, K is a $N \quad N$ "matrix" differential operator with "matrix elements"

$$K_{ab} = \quad \frac{@^2}{@y^2}\ {}_{ab} + \frac{{}^2U}{{}_a\ {}_b\ \tilde{}_K} \quad ; \quad a;b = 1;2; \quad ;N :$$

2.1 Generalized zeta function regularization method

We shall regularize ${}_{PK}(\ \frac{1}{2})$ by defining the analogous quantity ${}_{PK}(s)$ at some point in the complex s-plane where ${}_{PK}(s)$ does not have a pole. ${}_{PK}(s)$ is a meromorphic function of s such that its residues and poles can be derived through heat kernel methods, see [10]. If $K_K(y;z;\ )$ is the kernel of the $N \quad N$ "matrix" heat equation associated to K,

$$\frac{@}{@}1_N + K \quad K_K(y;z;\ ) = 0 \quad ; \quad K_K(y;z;0) = 1_N \quad (y \quad z); \tag{2}$$

the Mellin transformation tells us that

$${}_{PK}(s) = \frac{1}{(s)} \int_0^{\infty} d\ \ {}^{s\ 1}\mathrm{Tr}\ e^{\ PK}\quad ;$$

where

$$h_{PK}[\ ] = \mathrm{Tr}\ e^{\ PK} = \mathrm{Tr}\ e^{\ K}\quad j = \quad j + \int_1^{\infty} dy K_K(y;y;\ )$$

is the heat function $h_{PK}[\ ]$, if K is positive semi-definite and $\dim \mathrm{Ker} K = j$. The "regularized" kink energy is at the semi-classical limit:

$$E_0^K(s) = E[\ _K] + \frac{\tilde{}}{2}\ {}^{2s+1}\ {}_{PK}(s) + o(\tilde{}^2) \tag{3}$$

where is a unit of length ^{1} introduced to render the terms in (3) homogeneous from a dimensional point of view. The infiniteness of the bare quantum energy shows itself here in the pole that the zeta function develops for $s = \ \frac{1}{2}$.

The renormalization of $E_0^K(s)$ will be performed in the same three steps as in [8]:

A. The quantum fluctuations around the vacuum are governed by the Schrodinger operator:

$$V_{ab} = \quad \frac{d^2}{dy^2}\ {}_{ab} + \frac{{}^2U}{{}_a\ {}_b} j_{\ v}$$

where $\tilde{}_V$ is a constant minimum of U and $U_{ab}(\tilde{}_V) = \frac{^2U}{_a\ _b}j_{\tilde{}_V} = v_a^2\ _{ab}$ is the matrix of second variational derivatives of U at $\tilde{}_V$. The kernel of the heat equation

$$\frac{@}{@} + V \quad K_V(y;z;) = 0 \quad ; \quad K_V(y;z;0) = 1_N \quad (y \quad z)$$

provides the heat function $h_V( )$,

$$h_V( ) = \mathrm{Tr}e^{\ V} = \sum_{a=1}^{N} \int_{\ 1}^{\ 1} dy K_V(y;y;)$$

and, through the Mellin transformation, we obtain

$$_V(s) = \frac{1}{(s)} \int_0^{\ 1} d \quad ^{s\ 1}\mathrm{Tr}e^{\ V}:$$

The regularized kink energy measured with respect to the regularized vacuum energy is thus

$$\begin{aligned} E^K(s) &= E[\tilde{}_K] + \quad _1{}^{"K}(s) + o(\tilde{}^2) \\ &= E[\tilde{}_K] + \frac{\tilde{}}{2}^{\ 2s+1}[\ _{PK}(s) \quad _V(s)] + o(\tilde{}^2): \end{aligned}$$

B. If we now pass to the physical limit $E^K = \lim_{s!\ \frac{1}{2}} E^K(s)$, we still obtain an infinite result. The reason for this is that the physical parameters of the theory have not been renormalized. It is well known that in (1+1)-dimensional scalar field theory normal ordering takes care of all renormalizations in the system: the only ultraviolet divergences that occur in perturbation theory come from graphs that contain a closed loop consisting of a single internal line, [14]. From Wick's theorem, adapted to contractions of two fields at the same point in space-time, we see that normal ordering adds to the Hamiltonian the mass renormalization counter-terms

$$H(m^2) = \frac{\tilde{}}{2} \int dy \sum_{a=1}^{N} m_{aa}^2 :\frac{^2U}{_a\ _a}: + o(\tilde{}^2)$$

up to one-loop order. To regularize

$$m_{aa}^2 = \int \frac{dk}{4} \frac{1}{k^2 + U_{aa}(\tilde{}_V)} \quad ;$$

we first put the system in a 1D box of length L so that $m_{aa}^2 = \frac{1}{2L}\ _{V_{aa}}(\frac{1}{2})$, if the constant eigen-function of V_{aa} is not included in $_{V_{aa}}$. Then, we again use the zeta function regularization method and define: $m_{aa}^2(s) = \frac{1}{L}\frac{(s+1)}{(s)}^{\ 2s+1}\ _{V_{aa}}(s+1)$. Note that $m_{aa}^2 = \lim_{s!\ \frac{1}{2}} m_{aa}^2(s)$. The criterion behind this regularization prescription is the vanishing tadpole condition, which is shown in Appendix B of Reference [9] to be equivalent to the heat kernel subtraction scheme.

The one-loop correction to the kink energy due to $H(m^2(s))$ is thus

$$\begin{aligned} _2{}^{"K}(s) &= \left\langle \tilde{}_K \right| H(m^2(s)) \left| \tilde{}_K \right\rangle \quad \left\langle \tilde{}_V \right| H(m^2(s)) \left| \tilde{}_V \right\rangle = \\ &= \lim_{L!\ 1} \frac{\tilde{}}{2L}^{\ 2s+1} \frac{(s+1)}{(s)} \int_{\frac{L}{2}}^{\frac{L}{2}} dy \sum_{a=1}^{N} \ _{V_{aa}}(s+1) \left[\frac{^2U}{_a\ _a}_{\tilde{}_K} \quad \frac{^2U}{_a\ _a}_{\tilde{}_V} \right] \\ &= \lim_{L!\ 1} \frac{\tilde{}}{2L}^{\ 2s+1} \frac{(s+1)}{(s)} \sum_{a=1}^{N} \ _{V_{aa}}(s+1) \int_{\ 1}^{\ 1} dy V_{aa}(y) \end{aligned} \tag{4}$$

because the expectation values of normal ordered operators in coherent states are the corresponding c-number-valued functions.

C. Finally, the renormalized kink energy is:

$$E_R^K = E[\tilde{}_K] + \lim_{s \to \frac{1}{2}} \quad {}_1\zeta^K(s) + \quad {}_2\zeta^K(s) + o(\tilde{}^2) \quad : \tag{5}$$

2.2 Asymptotic approximation to semi-classical kink masses

In general it is very difficult to compute $h_{PK}[\ ]$ exactly. In such a case, we shall make use of the asymptotic expansion of $h_{PK}[\ ]$, which is well defined if $0 < \quad < 1$. In order to use the asymptotic expansion of the generalized zeta function of the K operator to compute the semi-classical expansion of the corresponding quantum kink mass, it is convenient to use non-dimensional variables. We define non-dimensional space-time coordinates $x = m_d y$ and field amplitudes $\tilde{}(x) = c_d \tilde{}(y)$, where m_d and c_d are constants with dimensions $[m_d] = L^{-1}$ and $[c_d] = M^{\frac{1}{2}} L^{-\frac{1}{2}}$. Also, we write $U(\tilde{}) = \frac{c_d^2}{m_d^2} U(\tilde{})$.

The action and the energy can now be written in terms of their non-dimensional counterparts:

$$S[\tilde{}] = \frac{1}{c_d^2} \int d^2x \left[\frac{1}{2} \frac{\partial \tilde{}}{\partial x} \frac{\partial \tilde{}}{\partial x} \quad U(\tilde{}) \right] = \frac{1}{c_d^2} S[\tilde{}]$$

$$E[\tilde{}] = \frac{m_d}{c_d^2} \int dx \left[\frac{1}{2} \frac{d\tilde{}}{dx} \frac{d\tilde{}}{dx} + U(\tilde{}) \right] = \frac{m_d}{c_d^2} E[\tilde{}]:$$

The important point is that the Hessians at the vacuum and kink configurations can always be written as the $N \quad N$ matrix differential operators $V = \frac{1}{m_d^2} V$ and $K = \frac{1}{m_d^2} K$, given by

$$V_{ab} = \quad \frac{d^2}{dx^2} \ {}_{ab} + v_a^2 \ {}_{ab}$$

$$K_{ab} = \quad \frac{d^2}{dx^2} \ {}_{ab} + v_a^2 \ {}_{ab} \quad V_{ab}(x)$$

where $\frac{\partial^2 U}{\partial \ {}_a^2} \Big|_{v} = v_a^2$ and $\frac{\partial^2 U}{\partial \ {}_a \ \partial \ {}_b} \Big|_{K} = v_a^2 \ {}_{ab} \quad V_{ab}(x)$. Therefore,

$${}_V(s) = \frac{1}{m_d^{2s}} \ {}_V(s) \quad ; \quad {}_K(s) = \frac{1}{m_d^{2s}} \ {}_K(s):$$

The asymptotic expansion of $N \quad N$ matrix heat kernels is given in Reference [15]. Nevertheless, we shall sketch this procedure in order to adapt it to our computational needs. We thus write the heat kernel in the form

$$K_K(x;x^0; \ ) = A(x;x^0; \ ) K_V(x;x^0; \ )$$

where

$$[K_V]_{ab}(x;x^0; \ ) = \frac{1}{\sqrt{4}} \exp \quad \frac{(x \quad x^0)^2}{4} \quad \exp(\quad v_a^2 \ )_{ab}$$

is the solution of the heat kernel equation for the V operator with the initial condition $[K_V]_{ab}(x;x^0;0) = {}_{ab}(x\ x^0)$. The matrix elements of the $N\ \ N$ matrix-valued function $A(x;x^0;\ )$ satisfy the system of N^2 coupled PDE:

$$\sum_{c=1}^{N}\sum_{d=1}^{N}\ \frac{@}{@}\ \ \frac{@^2}{@x^2}+v_a^2\ \ {}_{ac}\ \ V_{ac}(x)\ \ [A]_{cd}(x;x^0)[K_V]_{db}(x;x^0;\ )=0\ \ ; \qquad (6)$$

whereas $A(x;x^0;0)=1_N$ is the $N\ \ N$ identity matrix.

For <1, we solve (6) by means of an asymptotic (high-temperature) expansion: $A(x;x^0;\ )=\sum_{n=0}^{1} a_n(x;x^0)\ ^n$. In this regime the heat function is given by:

$$\begin{aligned}\mathrm{Tr}\,e^{\ K} &= \sum_{a=1}^{N}\int_{1}^{1} dx\,[K_K]_{aa}(x;x;\ ) = \sum_{a=1}^{N}\sum_{n=0}^{1}\frac{e^{\ v_a^2}}{\sqrt{4}}\int_{1}^{1} dx\,[a_n]_{aa}(x;x)\ ^n \\ &= \sum_{a=1}^{N}\sum_{n=0}^{1}\frac{e^{\ v_a^2}}{\sqrt{4}}[a_n]_{aa}(K)\ ^n:\end{aligned}$$

The coefficients $[a_n]_{ab}(x;x^0)$ can be found by means of an iterative procedure that starts from $[a_0]_{ab}(x;x^0)={}_{ab}$ and becomes more and more involved with larger and larger N. In Appendix A, this procedure is explained and the explicit expressions for the lower orders are shown.

The use of the power expansion of $h_{PK}[\ ]=\mathrm{Tr}\,e^{\ PK}$ in the formula for the quantum kink mass is developed in three steps:

1. First, we write the generalized zeta function of V in the form:

$$_V(s)=\frac{1}{(s)}\frac{m_d L}{\sqrt{4}}\sum_{a=1}^{N}\int_0^{\ 1} d\ \ ^{\,s\ \frac{3}{2}}e^{\ \ v_a^2\ }+B_V(s)\ \ ;$$

with

$$B_V(s)=\frac{m_d L}{\sqrt{4}}\sum_{a=1}^{N}\frac{[s\ \ \frac{1}{2};v_a^2]}{v_a^{2s\ 1}\ [s]}\ \ ;\quad {}_V(s)=\frac{m_d L}{\sqrt{4}}\sum_{a=1}^{N}\frac{[s\ \ \frac{1}{2};v_a^2]}{v_a^{2s\ 1}\ (s)}+B_V(s);$$

$[s;v^2]$ and $[s\ \ \frac{1}{2};v^2]$ being respectively the upper and lower incomplete gamma functions, see [16]. It follows that ${}_V(s)$ is a meromorphic function of s with poles at the poles of $[s\ \ \frac{1}{2};v_a^2]$, which occur when $s\ \ \frac{1}{2}$ is a negative integer or zero. $B_V(s)$, however, is an entire function of s.

2. Second, from the asymptotic expansion of $h_{PK}[\ ]$ we estimate the generalized zeta function of K:

$$\begin{aligned}{}_{PK}(s) &= \frac{1}{(s)}\left[\ j\int_0^{\ 1} d\ \ ^{\,s\ 1}+\frac{1}{\sqrt{4}}\sum_{a=1}^{N}\sum_{n<n_0}[a_n]_{aa}(K)\int_0^{\ 1} d\ \ ^{\,s+n\ \frac{3}{2}}e^{\ \ v_a^2\ }+h_{n_0,K}(s)\right]+B_{PK}(s)= \\ &= \frac{j}{s\ (s)}+\frac{1}{(s)\sqrt{4}}\sum_{a=1}^{N}\sum_{n<n_0}[a_n]_{aa}(K)\frac{[s+n\ \ \frac{1}{2};v_a^2]}{v_a^{2(s+n\ \frac{1}{2})}}+\frac{1}{(s)}h_{n_0,K}(s)+B_{PK}(s);\end{aligned}$$

where

$$h_{n_0,K}(s)=\frac{1}{\sqrt{4}}\sum_{a=1}^{N}\sum_{n\ \ n_0}^{1}[a_n]_{aa}(K)\frac{[s+n\ \ \frac{1}{2};v_a^2]}{v_a^{2(s+n\ \frac{1}{2})}}$$

is holomorphic for $Re\, s > -n_0 + \frac{1}{2}$, whereas

$$B_{PK}(s) = \frac{1}{(s)} \sum_{a=1}^{N} \int_0^{\infty} d\ Tr[e^{-\ PK}]_{aa}\ ^{s-1}$$

is an entire function of s. The values of s where $s + n - \frac{1}{2}$ is a negative integer or zero are the poles of $_{PK}(s)$ because the poles of $[s + n - \frac{1}{2}; v_a^2]$ lie at these points in the complex s-plane.

Renormalization of the zero point energy requires the subtraction of $_V(s)$ from $_{PK}(s)$. We find:

$$_{PK}(s) - _V(s) \quad \frac{1}{(s)} \left[\frac{j}{s} + \sum_{a=1}^{N} \sum_{n=1}^{n_X - 1} \frac{[a_n]_{aa}(K)}{\sqrt{4}} \frac{[s + n - \frac{1}{2}; v_a^2]}{v_a^{2(s+n-\frac{1}{2})}} \right] ;$$

and the error in this approximation with respect to the exact result to $_1"^K$ is:

$$error_1 = \frac{\tilde{m}_d}{2} [\ \frac{1}{2\sqrt{}} b_{n_0,K}(-\frac{1}{2}) + B_{PK}(-\frac{1}{2}) - B_V(-\frac{1}{2})].$$

Note that the subtraction of $_V(s)$ exactly cancels the contribution of $a_0(K)$ and hence the divergence arising at $s = -\frac{1}{2}, n = 0$.

3. Third, $_1"^K$ now reads:

$$_1"^K = \frac{\tilde{m}_d}{(-\frac{1}{2})} j + \frac{\tilde{}}{2} \lim_{s \to -\frac{1}{2}} \left(\frac{}{m_d^2} \right)^{s} \sum_{a=1}^{N} \frac{[a_1]_{aa}(K)}{\sqrt{4}} \frac{[s + \frac{1}{2}; v_a^2]}{(s)\, v_a^{2s+1}} +$$
$$+ \frac{\tilde{m}_d}{2} \sum_{a=1}^{N} \sum_{n=2}^{n_X - 1} \frac{[a_n]_{aa}(K)}{\sqrt{4}} \frac{[n - 1; v_a^2]}{(-\frac{1}{2})\, v_a^{2n-2}} .$$

The surplus in energy due to the mass renormalization counter-term is:

$$_2"^K = \lim_{L \to \infty} \sum_{a=1}^{N} \frac{\tilde{} [a_1]_{aa}(K)}{2L} \lim_{s \to -\frac{1}{2}} \left(\frac{}{m_d} \right)^{2s+1} \frac{(s+1)}{(s)}\ _{V_{aa}}(s+1) + o(\tilde{}^2)$$
$$= \frac{\tilde{m}_d}{2\sqrt{4}} \sum_{a=1}^{N} [a_1]_{aa}(K) \lim_{s \to -\frac{1}{2}} \left(\frac{}{m_d} \right)^{2s+1} \frac{[s + \frac{1}{2}; v_a^2]}{v_a^{2s+1}\ (s)} + o(\tilde{}^2);$$

and the deviation from the exact result is:

$$error_2 = \lim_{L \to \infty} \frac{\tilde{}}{4L} \sum_{a=1}^{N} [a_1]_{aa}(K) B_{V_{aa}}(\tfrac{1}{2}).$$

Therefore,

$$E_R^K = E[\ _K] + \ M\ _K = E[\ _K] - \frac{\tilde{m}_d}{2\sqrt{}} j + \frac{1}{4\sqrt{}} \sum_{a=1}^{N} \sum_{n=2}^{n_X - 1} [a_n]_{aa}(K) \frac{[n - 1; v_a^2]}{v_a^{2n-2}} +$$
$$+ \frac{\tilde{m}_d}{2\sqrt{4}} \sum_{a=1}^{N} [a_1]_{aa}(K) \lim_{s \to -\frac{1}{2}} \left(\frac{}{m_d} \right)^{2s+1} \frac{[s + \frac{1}{2}; v_a^2]}{v_a^{2s+1}\ (s)} - \frac{[s + \frac{1}{2}; v_a^2]}{v_a^{2s+1}\ (s)} + o(\tilde{}^2) .$$

The contributions proportional to $[a_1]_{aa}(K)$ of the poles at $s = \frac{1}{2}$ in ${}_1\zeta^K(s)$ and ${}_2\zeta^K(s)$ exactly cancel.

We thus obtain the very compact formula:

$$M_K = \tilde{m}_d[\ {}_0 + D_{n_0}] \qquad \begin{cases} {}_0 = \dfrac{\mathrm{j}}{2\sqrt{\ }} \\ D_{n_0} = \displaystyle\sum_{a=1}^{N}\sum_{n=2}^{n_0} \frac{[a_n]_{aa}(K)}{8}\frac{[n\ \ 1;v_a^2]}{v_a^{2n\ \ 2}} \end{cases} : \tag{7}$$

In summ: there are only two contributions to semi-classical kink masses obtained by means of the asymptotic method: 1) $\tilde{m}_d\ {}_0$ is due to the subtraction of the zero modes; 2) $\tilde{m}_d D_{n_0}$ comes from the partial sum of the asymptotic series up to the n_0 order. We stress that the merit of the asymptotic method lies in the fact that there is no need to solve the spectral problem of K: all the information is encoded in the potential $V(x)$.

3 The MSTB model: a deformed O(2) linear sigma system

Let us consider now a $N = 2$ model determined by the potential energy density, [3]:

$$U[\tilde{}] = \frac{}{4}(\tilde{}\ \tilde{}\ \ \frac{m^2}{\ }\ )^2 + \frac{^2}{4}\ {}_2^2 : \tag{8}$$

The system is a generalization of the $(\ {}^4)_2$ model to a $N = 2$ scalar field, the O(2)-linear sigma model, although it has been deformed by a quadratic term in ${}_2$ in order to avoid the Goldstone boson. The deformation parameter $$ has dimensions of inverse length, $[\ ] = L^{\ 1}$, and the choice of non-dimensional variables in the so called MSTB model comes from the choice of $c_d = \frac{\sqrt{\ }}{m}$, $m_d = \frac{m}{\sqrt{2}}$ and $^2 = m^{2\ 2}$:

$$U(\ {}_1;\ {}_2) = \frac{1}{2}(\ {}_1^2 + \ {}_2^2\ \ 1)^2 + \frac{^2}{2}\ {}_2^2 :$$

^{2} is the non-dimensional parameter that measures the deformation and we shall also use the related parameter $^2 = 1\ \ ^2$. $U(\tilde{})$ is not invariant under the O(2)-matrix group if ^{2} is not zero. The "internal" symmetry group is the $Z_2\ \ Z_2$ group of discrete transformations:

$$G_I = \ e_a \to \ e_a\ ;\ e_a \to \ (\ 1)^{1a}e_a\ \ :$$

The vacuum classical configurations are:

$$\tilde{}_V(x;t) = \ e_1\ \ ;\ \ \tilde{}_V(y;y^0) = \ \frac{m}{\sqrt{\ }}e_1 :$$

Therefore, if we denote by

$$H_I^{(b)} = \ e_a \to e_a\ ;\ e_a \to \ (\ 1)^{ba}e_a$$

the Z_2- subgroup that leaves e_b invariant, the vacuum orbit and the vacuum moduli space are respectively: $M\ = \frac{G_I}{H_I^{(1)}} = Z_2, \hat{M} = \frac{M}{G_I} =$ point.

There are two kinds of topological kinks which are thus loop kinks and candidates for being stable solitary waves of the system:

Topological kinks with one non-null component:

$$\tilde{}_{TK1}(x;t) = (\tanh x)\hat{e}_1 \quad ; \quad \tilde{}_{TK1}(y;y^0) = \frac{m}{\sqrt{}}(\tanh\frac{m}{\sqrt{}}\frac{y}{2})\hat{e}_1$$

Topological kinks with two non-null components:

$$\tilde{}_{TK2}(x;t) = [(\tanh \ x)\hat{e}_1 \quad (\operatorname{sech} \ x)\hat{e}_2];$$

$$\tilde{}_{TK2}(y;y^0) = \frac{1}{\sqrt{}}[m \ (\tanh\frac{y}{\sqrt{}}\frac{y}{2})\hat{e}_1 \quad \sqrt{(m^2 \quad ^2)}(\operatorname{sech}\frac{y}{\sqrt{}}\frac{y}{2})\hat{e}_2]:$$

Note that there are two two-component topological kinks, which only exist if $0 < \ ^2 < 1$.

The kink and vacuum solutions have classical energies: $E[\tilde{}_{TK1}] = \frac{4}{3}\frac{m^3}{\sqrt{}}$, $E[\tilde{}_{TK2}] = 2 \ (1 \ \frac{^2}{3})\frac{m^3}{\sqrt{}}$ and $E[\tilde{}_V] = 0$. Thus, $E[\tilde{}_{TK1}] > E[\tilde{}_{TK2}]$ and we shall see that the TK1 kink, as a quantum state, is unstable if $^2 < 1$. The lower bound in energy in the topological sector of the configuration space is attained, however, by the TK2 kink. This last statement is proved in Appendix B.

The Hessian operator for the vacuum solution is:

$$V = \begin{pmatrix} \frac{d^2}{dy^2} + 2m^2 & 0 \\ 0 & \frac{d^2}{dy^2} + \frac{^2}{2} \end{pmatrix} = \frac{m^2}{2}\begin{pmatrix} \frac{d^2}{dx^2} + 4 & 0 \\ 0 & \frac{d^2}{dx^2} + \ ^2 \end{pmatrix} = \frac{m^2}{2}V \quad ;$$

and hence the masses of the fundamental quanta are $v_1^2 = 2m^2 = \frac{m^2}{2}v_1^2$ and $v_2^2 = \ ^2 = \frac{m^2}{2}v_2^2$, in such a way that the $H_I^{(2)}$ symmetry is spontaneously broken.

The Hessian operators for the topological kinks read:

TK1,

$$K = \begin{pmatrix} \frac{d^2}{dx^2} + 4 \ \frac{6}{\cosh^2 x} & 0 \\ 0 & \frac{d^2}{dx^2} + \ ^2 \ \frac{^2}{\cosh^2 x} \end{pmatrix}$$

and $K = \frac{m^2}{2}K$. Note that K has a negative eigenvalue if $^2 < 1$, as it should be for a unstable solution.

TK2,

$$H = \begin{pmatrix} \frac{d^2}{dx^2} + 4 \ \frac{2(2+ \ ^2)}{\cosh^2 x} & 4 \ \frac{\tanh x}{\cosh x} \\ 4 \ \frac{\tanh x}{\cosh x} & \frac{d^2}{dx^2} + \ ^2 + \frac{2(2 \ 3 \ ^2)}{\cosh^2 x} \end{pmatrix}$$

and $H = \frac{m^2}{2}H$.

The corresponding generalized zeta functions satisfy

$$_V(s) = \left(\frac{2}{m^2}\right)^s \ _V(s) \quad ; \quad _{PK}(s) = \left(\frac{2}{m^2}\right)^s \ _{PK}(s) \quad ; \quad _{PH}(s) = \left(\frac{2}{m^2}\right)^s \ _{PH}(s) \quad :$$

3.1 The quantum TK1 kink: exact computation of the semi-classical mass

Generalized zeta function of V:

Acting on the $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ Hilbert space, we have that

$$\mathrm{Spec}V = \{k_1^2 + 4\} \cup \{k_2^2 + m^2\} = \mathrm{Spec}V_{11} \cup \mathrm{Spec}V_{22};$$

$k_1, k_2 \in \mathbb{R}$, whereas the spectral density over a large interval of length L is:

$$\rho_V(k_1,k_2) = \begin{cases} \dfrac{mL}{\sqrt{2\pi}} & \text{for } k_2^2 \geq 4m^2 \\ \dfrac{mL}{2\sqrt{2\pi}} & \text{for } k_2^2 < 4m^2 \end{cases}.$$

From these data, the heat and generalized zeta functions can be readily computed:

$$\mathrm{Tr}e^{-tV} = \mathrm{Tr}e^{-tV_{11}} + \mathrm{Tr}e^{-tV_{22}} = \frac{mL}{\sqrt{8\pi}}\left(e^{-4t} + e^{-m^2t}\right) \quad \text{quad};$$

$$\zeta_V(s) = \zeta_{V_{11}}(s) + \zeta_{V_{22}}(s) = \frac{mL}{\sqrt{8\pi}}\left(\frac{1}{4^{s-\frac{1}{2}}} + \frac{1}{(m^2)^{s-\frac{1}{2}}}\right)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}.$$

Generalized zeta function of K

In the case of the one-component topological kink, computation of the heat and generalized zeta functions is easy because K is diagonal. Moreover, K_{11} and K_{22} are respectively the Hessian operators for the $(\phi)^4$ kink and the sine-Gordon soliton (in the second case shifted by $m^2 - 1$). Therefore, we shall take advantage from the work performed in [8]. The spectrum of the Hessian operator for the TK1 kink is:

$$\mathrm{Spec}K = \{0, 3, m^2 - 1\} \cup \{k_1^2 + 4\}_{k_1 \in \mathbb{R}} \cup \{k_2^2 + m^2\}_{k_2 \in \mathbb{R}},$$

and the spectral density and the phase shifts for the continuous spectrum read:

$$\rho_K(k_1,k_2) = \rho_V(k_1,k_2) + \frac{1}{2\pi}\left(\frac{d\,\delta_1(k_1)}{dk_1} + \frac{d\,\delta_2(k_2)}{dk_2}\right)$$

$$\delta_1(k_1) = -2\arctan\frac{3k_1}{2(4-k_1^2)}, \quad \delta_2(k_2) = 2\arctan\frac{1}{k_2}$$

respectively. Besides the continuous spectrum there are three bound states with eigenvalues 0, 3 and $m^2 - 1$. The eigenfunction of 0 eigenvalue is the translational mode. There is a second bound state in the e_1 direction, but the third bound state points along the e_2 axis. Note that to develop a non-zero ϕ_2 component is energetically favorable if $m^2 < 1$ and this process costs energy when $m^2 > 1$ and the TK1 kink is stable. The second zero mode that occurs at $m^2 = 1$ is the signal of this phase transition.

The heat function

$$\begin{aligned}
&\mathrm{Tr}\, e^{-\beta K} = \mathrm{Tr}\, e^{-\beta K_{11}} + \mathrm{Tr}\, e^{-\beta K_{22}} = \\
&= e^{-3\beta} + e^{-4\beta} \int_{-\infty}^{\infty} dk_1 \left[\frac{mL}{\sqrt{8\pi}} + \frac{1}{2} \frac{d\,\delta(k_1)}{dk_1} \right] e^{-\beta k_1^2} + \\
&+ e^{-(\beta_2 - 1)\beta} + e^{-\beta_2 \beta} \int_{-\infty}^{\infty} dk_2 \left[\frac{mL}{\sqrt{8\pi}} + \frac{1}{2} \frac{d\,\delta(k_2)}{dk_2} \right] e^{-\beta k_2^2}
\end{aligned}$$

is therefore equal to :

$$\mathrm{Tr}\, e^{-\beta K} = \mathrm{Tr}\, e^{-\beta V} + e^{-3\beta} + e^{-(\beta_2 - 1)\beta} \left[\mathrm{Erf}[\sqrt{\beta}] - \mathrm{Erfc}[2\sqrt{\beta}] \right] ;$$

where Erf and Erfc, [16], are respectively the error and complementary error functions. The Mellin transform provides the corresponding zeta function:

$$\zeta_K(s) = \zeta_V(s) + \frac{1}{\sqrt{\pi}} \left[\frac{2}{3^{s+\frac{1}{2}}} {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{3}] - \frac{1}{4^s s} + \frac{2}{(\beta_2 - 1)^{s+\frac{1}{2}}} {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{\beta_2 - 1}] \right] \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)} ;$$

in terms of Gauss hypergeometric functions of the form ${}_2F_1[a;b;c;d]$, [16].

Applying these results we obtain :

$$\begin{aligned}
\Delta_1 \mathcal{F}^K &= \frac{\tilde{\mu}}{2} \lim_{s \to -\frac{1}{2}} \frac{1}{\sqrt{\pi}} \left(\frac{2\mu^2}{m^2} \right)^s \\
&\left[\frac{2}{3^{s+\frac{1}{2}}} {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{3}] - \frac{1}{4^s s} + \frac{2}{(\beta_2 - 1)^{s+\frac{1}{2}}} {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{\beta_2 - 1}] \right] \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)} ;
\end{aligned}$$

which is still a divergent quantity. The mass renormalization counter-terms add another divergent quantity :

$$\Delta_2 \mathcal{F}^K = \lim_{s \to -\frac{1}{2}} \lim_{L \to \infty} \frac{2\tilde{\mu}}{L} \left(\frac{2\mu^2}{m^2} \right)^{s+\frac{1}{2}} \frac{\Gamma(s+1)}{\Gamma(s)} [3\,\zeta_{V_{11}}(s+1) + \zeta_{V_{22}}(s+1)] \quad ; \qquad (9)$$

and we obtain:

$$\begin{aligned}
\Delta_1 \mathcal{F}^K + \Delta_2 \mathcal{F}^K &= \tilde{\mu} \lim_{\varepsilon \to 0} \left[\sqrt{\frac{1}{2} m} \left(\frac{2\mu^2}{m^2} \right)^{\varepsilon} \left[\frac{3}{4^{\varepsilon}} + \frac{1}{(\beta_2 - 1)^{\varepsilon}} \right] \frac{\Gamma(\varepsilon)}{\Gamma(\frac{1}{2} + \varepsilon)} + \right. \\
&\left. + \frac{\tilde{\mu}}{2} \lim_{\varepsilon \to 0} \frac{m\sqrt{2}}{\sqrt{\pi}} \left(\frac{2\mu^2}{m^2} \right)^{\varepsilon} \left[\frac{1}{3^{\varepsilon}} {}_2F_1[\tfrac{1}{2}; \varepsilon; \tfrac{3}{2}; -\tfrac{1}{3}] - \frac{1}{4^{\varepsilon}(\frac{1}{2} + \varepsilon)} + \frac{1}{(\beta_2 - 1)^{\varepsilon}} {}_2F_1[\tfrac{1}{2}; \varepsilon; \tfrac{3}{2}; \tfrac{1}{\beta_2 - 1}] \right] \frac{\Gamma(\varepsilon)}{\Gamma(\frac{1}{2} + \varepsilon)} :
\end{aligned}$$

It is appropriate to consider the contributions from K_{11} and K_{22} separately before taking this limit:

From K_{11} we obtain :

$$\begin{aligned}
{}^{(1)}\Delta_1 \mathcal{F}^K &= \lim_{\varepsilon \to 0} \frac{3\tilde{\mu} m}{2\sqrt{\pi}} + \\
&+ \frac{\tilde{\mu} m}{2\sqrt{\pi}} \left[-4 + 3\gamma_E + \log(3^4 2) - 3\log\frac{2\mu^2}{m^2} + 3\,\psi(\tfrac{3}{2}) - {}_2F_1^{(0,0,1,0)}[\tfrac{1}{2}; 0; \tfrac{3}{2}; -\tfrac{1}{3}] + o(\varepsilon) \right] ;
\end{aligned}$$

and

$$ {}^{(1)}_{2}\mathbb{M}^{K} = \lim_{\varepsilon \to 0} \frac{3\tilde{m}}{2\sqrt{2}} + \frac{\tilde{m}}{2\sqrt{2}} \qquad 3\,\gamma_E + 3\log\frac{2\,\mu^2}{m^2} \quad 6\log 2 \quad 3\,\psi(\tfrac{3}{2}) \;+ o(\varepsilon) \;: $$

Here, γ_E is the Euler gamma constant; ψ is the Psi(Digamma) function, and ${}_2F_1^{\,0}$ is the derivative of the Gauss hypergeometric function with respect to the second argument, [16].

From K_{22} we obtain :

$$ {}^{(2)}_{1}\mathbb{M}^{K} = \lim_{\varepsilon \to 0} \frac{\tilde{m}}{2\sqrt{2}} + $$

$$ + \frac{\tilde{m}}{2\sqrt{2}} \quad \gamma_E \quad \log\frac{2\,\mu^2}{m^2} + \log(\mu^2 \quad 1) + \psi(\tfrac{3}{2}) \quad {}_2F_1^{\,0}[\tfrac{1}{2};0;\tfrac{3}{2};\tfrac{1}{1-\mu^2}] + o(\varepsilon); $$

and

$$ {}^{(2)}_{2}\mathbb{M}^{K} = \lim_{\varepsilon \to 0} \frac{\tilde{m}}{2\sqrt{2}} + \frac{\tilde{m}}{2\sqrt{2}} \qquad \gamma_E + \log\frac{2\,\mu^2}{m^2} \quad \log\,\mu^2 \qquad \psi(\tfrac{3}{2}) \;+ o(\varepsilon) \;: $$

Gathering all this together, we finally find:

$$ \begin{aligned} {}_1\mathbb{M}^{K} + {}_2\mathbb{M}^{K} &= {}^{(1)}_{1}\mathbb{M}^{K} + {}^{(1)}_{2}\mathbb{M}^{K} + {}^{(2)}_{1}\mathbb{M}^{K} + {}^{(2)}_{2}\mathbb{M}^{K} \\ &= \frac{\tilde{m}}{2\sqrt{2}} \;4 + \ln\frac{4}{3} + {}_2F_1^{\,0}[\tfrac{1}{2};0;\tfrac{3}{2};\tfrac{1}{3}] + \ln(\frac{\mu^2}{\mu^2 \quad 1}) + {}_2F_1^{\,0}[\tfrac{1}{2};0;\tfrac{3}{2};\tfrac{1}{\mu^2 \quad 1}] \\ &= \frac{\tilde{m}}{2\sqrt{2}} \;(3 \quad \sqrt{12}) + (1 \quad \frac{\sqrt{\mu^2 \quad 1}}{\mu}\arcsin\frac{1}{\mu}) \quad : \end{aligned} \qquad (10) $$

Note that $M_{TK1}(\mu^2) = {}_1\mathbb{M}^{K} + {}_2\mathbb{M}^{K}$ acquires an imaginary part if $\mu^2 < 1$ and in this latter regime the quantum TK1 kink becomes a resonance. Formula (10) shows one interesting pattern in the one-loop correction to the kink masses: First, the arguments of the logarithms are always quotients of the eigenvalue where the continuous spectrum starts by the energy of the bound eigen-state. Second, the fourth argument of the derivatives of the Gauss hypergeometric functions is always minus the inverse of the bound-state eigenvalues.

3.2 The quantum topological kinks: asymptotic expansions

The difficulty in computing the quantum correction to the mass of the TK2 kink lies in the fact that the Hessian operator at this configuration is a Schrodinger operator for which the 2×2 matrix-valued potential $V(x)$ is non-diagonal. Therefore, the spectral problem of H is very difficult to solve and we are led to use asymptotic methods to calculate the quantum TK2 kink mass.

In the MSTB model there are two regimes:

1. $\mu^2 < 1$: The TK2 kink exists and is stable. In this case we have that

$$ V_{12}(x) = V_{21}(x) = 4\,\frac{\tanh\,x}{\cosh\,x} $$

is an odd function of x such that $[a_1]_{12}(H) = [a_1]_{21}(H) = 0$. Bearing in mind that $V_{11}(x) = \frac{2(2+\mu^2)}{\cosh^2 x}$ and $V_{22}(x) = \frac{2(2\,3\,\mu^2)}{\cosh^2 x}$, we present the coefficients $[a_n]_{aa}(H)$ up to $n_0 = 11$ with the help of Mathematica in the Tables 1 and 2 for several values of μ :

	= 0:5		= 0:7		= 0:8	
n	$[a_n\ h_1(H)$	$[a_n\ b_2(H)$	$[a_n\ h_1(H)$	$[a_n\ b_2(H)$	$[a_n\ h_1(H)$	$[a_n\ b_2(H)$
1	17.9984	-9.99909	14.2285	-3.02857	13.2000	-0.4000
2	34.9975	16.3313	27.5051	4.95579	25.6320	2.42133
3	47.9636	-16.8309	38.5365	-3.88117	36.3858	-1.46193
4	49.4219	13.2491	40.9039	2.58248	39.2877	0.984184
5	40.6415	-8.41779	34.5847	-1.43041	33.7529	-0.53935
6	27.7911	4.4983	24.2279	0.696572	23.9672	0.267282
7	16.2632	-2.08006	14.4768	-0.30375	14.4819	-0.117869
8	8.31759	-0.849979	7.53962	0.119982	7.61204	0.0464771
9	3.77771	-0.295903	3.47957	-0.0425253	3.53989	-0.0162638
10	1.51992	0.0884623	1.44002	0.0132872	1.47558	0.00497705

Table 1: Heat function coefficients for several values of the deformation parameter such that < 0.9.

	= 0:9		= 0:95			1	
n	$[a_n\ h_1(H)$	$[a_n\ b_2(H)$	$[a_n\ h_1(H)$	$[a_n\ b_2(H)$	n	$[a_n\ h_1(K)$	$[a_n\ b_2(K)$
1	12.4889	1.91111	12.2211	2.97895	1	12.0000	4.0000
2	24.5218	1.67378	24.1951	1.952444	2	24.0000	2.66667
3	35.3368	-0.241656	35.16.16	0.310882	3	35.2000	1.066667
4	38.8216	0.341514	38.9553	0.230323	4	39.3143	0.304762
5	33.8715	-0.16798	34.2227	-0.053986	5	34.7429	0.0677249
6	24.3610	0.0863412	24.7475	0.0372813	6	25.2306	0.0123136
7	14.8742	-0.0370868	15.1758	-0.0145382	7	15.5208	0.0018944
8	7.88516	0.0140749	8.07351	0.00547229	8	8.27702	0.000252587
9	3.69263	-0.0046464	3.79191	-0.00173035	9	3.89498	$2.97134\ 10^{5}$
10	1.54847	0.001313	1.594006	0.0003535	10	1.63998	$3.12591\ 10^{6}$

Table 2: Heat function coefficients for several values such that 0.9. If 1 the coefficients coming from K are shown.

We also need to know that $j = 1$: there is only one translational zero mode in the spectrum of H, and that $v_1^2 = 4$, $v_2^2 = {}^2$. Plugging this information in the asymptotic formula (7) we obtain the values shown in Table 3 for M_{TK2}.

2. > 1: The only solitary wave is the TK1 kink, which is stable. To determine the quantum TK1 kink mass by using the asymptotic method, the general formulas must be applied to K. Now, $V_{12}(x) = V_{21}(x) = 0$ and trivially $[a_1]_{12}(K) = [a_n]_{21}(K) = 0$. The $[a_n]_{aa}(K)$ must be read from the formulas in the Appendix applied to $V_{11}(x) = \frac{6}{\cosh^2 x}$ and $V_{22}(x) = \frac{2}{\cosh^2 x}$, and are shown in Table 2 up to $n_0 = 11$.

	M_{TK2}
0:4	1:103270~m
0:5	0:852622~m
0:6	0:689001~m
0:7	0:583835~m
0:8	0:524363~m
0:9	0:505708~m
0:95	0:511638~m

	M_{TK1}
1:0	0:528311~m
1:2	0:518426~m
1:4	0:509645~m
1:6	0:502291~m
1:8	0:496369~m
2:0	0:49172~m
2:5	0:484183~m
3:0	0:480101~m
4:0	0:476181~m

Table 3: One-loop corrections to the mass of topological kinks

Note that in this case the coefficients of the asymptotic expansion of the generalized zeta function of K are independent of . Nevertheless, the quantum correction to the TK1 kink mass depends on through the factors $\frac{[n\ 1,v_2^2]}{v_2^{2n\ 2}}$. Again $j = 1$: there is only one translational zero mode in the spectrum of K, and $v_1^2 = 4$, $v_2^2 = {}^2$. Application of formula (7) provides the values of M_{TK1}, also shown in Table 3 for several values of .

We summarize the results obtained in this Section in the next Figure, where M_K is depicted as a function of . It is understood that for > 1 we plot M_{TK1} and for $0 < < 1$ M_{TK2} is represented. The continuous line shows the exact value of M_{TK1} as a function of , whereas the dots correspond to the answer provided by the asymptotic method to M_{TK1}, (whites), and M_{TK2}, (blacks), for several values of .

We observe the following facts:

In the 1:2 range, the approximation established by the asymptotic method is extremely good : the discrepancy with the exact answer is compatible with zero.

In the $1 < < 1:2$ interval the error is larger and can be estimated exactly in terms of $B_K(\frac{1}{2})$, $b_{n_0,K}(\frac{1}{2})$, $B_V(\frac{1}{2})$ and $B_{V_a}(\frac{1}{2})$. The closer the value of to 1, the smaller the first eigenvalue in the spectrum of K and the larger $B_K(\frac{1}{2})$. The asymptotic method is better when fluctuations in the $_2$ direction of the TK1 kink cost more energy.

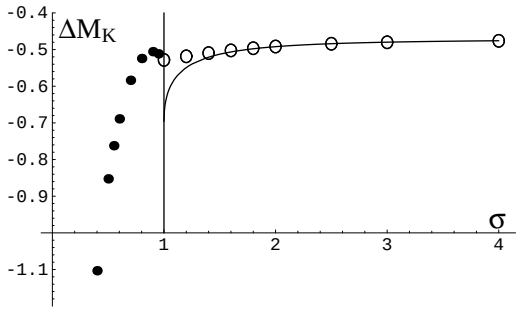

Figure 1: Quantum correction to the mass in the MSTB model: TK2 ($0 < {}^2 < 1$, black dots) and TK1 ($^2 \quad 1$, white dots).

The point $= 1$, where the error is maximum, is singular. There is a second zero eigenvalue which is the signal for an instability-type phase transition on the TK1 kink. The co-existing unstable TK1 and stable TK2 kinks for < 1 coalesce to a single TK1 kink at $= 1$, which becomes stable for > 1.

In the $0 < < 1$ regime there is no way of estimating the error in the approximation to M_{TK2} obtained by means of asymptotic methods because the exact value is not known. We expect, however, that the result will be better for smaller values of because of the same qualitative argument as above: the cost in energy for climbing from the TK2 to the TK1 kink is larger with smaller .

The value $^2 = 4$ is a very special one. In this case the system is $N = 2$ pre-supersymmetric in the sense that there exists a super-potential:

$$\tilde{W}(\tilde{}) = W^1(_1; _2)e_1 + W^2(_1; _2)e_2 \quad ;$$

where

$$W^1(_1; _2) = \quad _1 \quad \frac{^3_1}{3} + \quad _1 \, ^2_2 \quad ; \quad W^2(_1; _2) = \quad _2 \quad ^2_1 \, _2 + \frac{^3_2}{3}$$

and

$$\frac{1}{2}(\tilde{}\tilde{} \quad 1)^2 + 2 \, ^2_2 = \frac{1}{2}\frac{@W^1}{@ _1}\frac{@W^1}{@ _1} + \frac{1}{2}\frac{@W^1}{@ _2}\frac{@W^1}{@ _2} = \frac{1}{2}\frac{@W^2}{@ _1}\frac{@W^2}{@ _1} + \frac{1}{2}\frac{@W^2}{@ _2}\frac{@W^2}{@ _2} \quad :$$

This is no more than the super-potential of the holomorphic Wess-Zumino model,

$$\frac{@W^1}{@ _1} = \frac{@W^2}{@ _2} \quad ; \quad \frac{@W^1}{@ _2} = \quad \frac{@W^2}{@ _1} \quad ;$$

see [17], and we have thus calculated the quantum correction to the kink in this important system.

4 Quantum kinks in a deformed O(N) linear sigma model

Let us finally consider an $O(N)$, $N \quad 3$ model determined by the potential energy density:

$$U[\tilde{}] = \frac{}{4}(\tilde{}\tilde{} \quad \frac{m^2}{}\,)^2 + \sum_{l=2}^{N} \frac{^2_1}{4} \, ^2_1 quad; \qquad (11)$$

which generalizes the system discussed in Sect. 3. Without loss of generality, the deformation parameters are chosen in such a way that $_2 < _3 < ::: < _N$. Passing to non-dimensional variables, the potential is:

$$U(\tilde{}) = \frac{1}{2}(\tilde{}\tilde{} \quad 1)^2 + \sum_{l=2}^{N} \frac{^2_l}{2} \; ^2_l \quad ;$$

with $_2 < _3 < ::: < _N$. The "internal" symmetry group is $(Z_2)^N$; the vacuum classical configurations are:

$$\tilde{}_V(x;t) = \quad e_1 \qquad ; \qquad \tilde{}_V(y;y^0) = \quad \frac{m}{p=} e_1 ;$$

and the vacuum orbit and the vacuum moduli space are respectively: $M = \frac{G_I}{H_I^{(1)}} = Z_2, \hat{M} = \frac{M}{G_I} =$ point.

There is a very rich manifold of kinks in this model, see [5]. We shall only study the one-component and two-component topological kinks, which are loop kinks and candidate stable solitary waves in this system:

Topological kinks and antikinks with one non-null component:

$$\tilde{}_{TK1}(x;t) = \quad (\tanh x) e_1 \qquad ; \qquad \tilde{}_{TK1}(y;y^0) = \quad \frac{m}{p=} (\tanh \frac{m\,y}{p=2}) e_1 \quad :$$

Topological kinks and anti-kinks with two non-null components. For each n from 2 to N such that $_n < 1$, we have:

$$\tilde{}_{TK2_n}(x;t) = \quad [(\tanh \; _n x) e_1 \quad _n (\operatorname{sech} \; _n x) e_n];$$

$$\tilde{}_{TK2_n}(y;y^0) = \frac{1}{p=} [m \; (\tanh \frac{_n y}{p=2}) e_1 \quad \sqrt{(m^2 \quad ^2_n)} (\operatorname{sech} \frac{_n y}{p=2}) e_n]$$

where $_n = \sqrt{1 \quad ^2_n}$.

The kink and vacuum solutions have classical energies: $E[\tilde{}_{TK1}] = \frac{4}{3} \frac{m^3}{p=2}$, $E[\tilde{}_{TK2_n}] = 2 \; _n (1 \; \frac{^2_n}{3}) \frac{m^3}{p=2}$, $E[\tilde{}_V] = 0$. Thus, $E[\tilde{}_{TK1}] > E[\tilde{}_{TK2_N}] > ::: > E[\tilde{}_{TK2_2}]$. The lower bound in energy in the topological sector of the configuration space is attained by the $TK2_2$ kink. If $_2 > 1$, only the TK1 kink exists and is stable. If $_j < 1 < _{j+1}$, the kinks $TK1, TK2_2, :::, TK2_j$ exist, but only the $TK2_2$ is stable, see [18].

The Hessian operators in non-dimensional variables are as follows:

Vacuum:

$$V = \operatorname{diag}(V_{ll})$$

$$V_{11} = \quad \frac{d^2}{dx^2} + 4; \qquad V_{ll} = \quad \frac{d^2}{dx^2} + \; ^2_l; \; l \quad 2 \quad :$$

TK1:

$$K = \operatorname{diag}(K_{ll})$$

$$K_{11} = \quad \frac{d^2}{dx^2} + 4 \quad \frac{6}{\cosh^2 x}; \qquad K_{ll} = \quad \frac{d^2}{dx^2} + \; ^2_l \quad \frac{2}{\cosh^2 x}; \; l \quad 2 \quad :$$

TK2$_n$:

$$H^{(n)} = \begin{pmatrix} H^{(n)}_{ND} & 0 \\ 0 & H^{(n)}_{D} \end{pmatrix}$$

$$H^{(n)}_{ND} = \begin{pmatrix} \frac{d^2}{dx^2} + 4\,\frac{2(2+\,{}^2_n)}{\cosh^2\,{}_n x} & 4\,{}_n\frac{\tanh\,{}_n x}{\cosh\,{}_n x} \\ 4\,{}_n\frac{\tanh\,{}_n x}{\cosh\,{}_n x} & \frac{d^2}{dx^2} + {}^2_n + \frac{2(2\;3\,{}^2_n)}{\cosh^2\,{}_n x} \end{pmatrix}$$

$$H^{(n)}_{D} = \mathrm{diag}\Big(\ \frac{d^2}{dx^2} + {}^2_l\ \ \frac{2\,{}^2_n}{\cosh^2\,{}_n x}\Big);\quad l = \hat{1};2;3;:::;\hat{n};:::;N\ ;$$

where the hat over an index means that this index is absent and rows and columns have been rearranged in such a way that $H^{(n)}_{ND}$ acts on the small deformations of $_1$; $_n$ and $H^{(n)}_{D}$ on the remaining ones.

These operators are direct sums of the Hessian operators used in Sect.3. We can therefore take advantage of the calculations already made to give the quantum corrections to the kinks of the present model.

4.1 The quantum TK1 kink: exact computation of the semi-classical mass

The relevant quantities are:

Generalized zeta function of V:

$$_V(s) = \ _{V_{11}}(s) + \sum_{l=2}^{N}\ _{V_{ll}}(s) = \frac{m\,L}{\sqrt{\pi}}\ \ \frac{1}{4^{s\ \frac{1}{2}}} + \sum_{l=2}^{N}\frac{1}{(\ {}^2_l)^{s\ \frac{1}{2}}}\ \ \frac{(s\ \ \frac{1}{2})}{(s)}\ \ :$$

Generalized zeta function of PK :

$$_{PK}(s) = \ _V(s) + \frac{1}{\sqrt{\pi}}\ \ \frac{2}{3^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2};s+\ \frac{1}{2};\tfrac{3}{2};\ \ \tfrac{1}{3}]\ \ \frac{1}{4^s s}\ \ \frac{(s+\ \frac{1}{2})}{(s)}$$
$$+\ \ \frac{1}{\sqrt{\pi}}\sum_{l=2}^{N}\ \ \frac{2}{(\ {}^2_l)^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2};s+\ \frac{1}{2};\tfrac{3}{2};\tfrac{1}{{}^2_l}]\ \ \frac{(s+\ \frac{1}{2})}{(s)}:$$

Applying these results, we obtain :

$$_1{}^{"K} = \frac{\sim}{2}\lim_{s!\ \frac{1}{2}}\frac{1}{\sqrt{\pi}}\ \ \frac{2\ {}^2}{m^{\,2}}^{s}\ \ \frac{2}{3^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2};s+\ \frac{1}{2};\tfrac{3}{2};\ \ \tfrac{1}{3}]\ \ \frac{1}{4^s s}\ \ \frac{(s+\ \frac{1}{2})}{(s)}$$
$$+\ \ \frac{\sim}{2}\lim_{s!\ \frac{1}{2}}\frac{1}{\sqrt{\pi}}\ \ \frac{2\ {}^2}{m^{\,2}}^{s}\sum_{l=2}^{N}\ \ \frac{2}{(\ {}^2_l)^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2};s+\ \frac{1}{2};\tfrac{3}{2};\tfrac{1}{{}^2_l}]\ \ \frac{(s+\ \frac{1}{2})}{(s)}:$$

The mass renormalization counter-terms are :

$$_2{}^{"K} = \lim_{s!\ \frac{1}{2}}\lim_{L!\ 1}\ \frac{2\sim}{L}\ \ \frac{2\ {}^2}{m^{\,2}}^{s+\frac{1}{2}}\ \ \frac{(s+\ 1)}{(s)}[3\ _{V_{11}}(s+\ 1)+\sum_{l=2}^{N}\ _{V_{ll}}(s+\ 1)]:$$

Therefore,

$$
\begin{aligned}
{}_1\Gamma^K + {}_2\Gamma^K &= \frac{\tilde{m}}{2\sqrt{2}}\left[4 + \ln\frac{4}{3} + {}_2F_1{}^0[\tfrac{1}{2};0;\tfrac{3}{2};-\tfrac{1}{3}] + \sum_{l=2}^{N} \ln\left(\frac{\mu_l^2}{\mu_l^2-1}\right) + {}_2F_1{}^0[\tfrac{1}{2};0;\tfrac{3}{2};\tfrac{1}{\mu_l^2-1}]\right] \\
&= \frac{\tilde{m}}{\sqrt{2}}\left(3-\sqrt{12}\right) + \sum_{l=2}^{N}\left(1 - \sqrt{\mu_l^2-1}\,\arcsin\frac{1}{\mu_l}\right) : \qquad (12)
\end{aligned}
$$

The TK1 kink is a bona fide quantum state only if $\mu_1^2 > 1{,}81\ldots$.

4.2 Quantum masses of two-component topological kinks: asymptotic expansion

In the $\mu_2^2 < 1$ regime only the TK2$_2$ kink is stable. In this sub-section, therefore, we shall present the calculation of the one-loop quantum correction to the TK2$_2$ kink mass. Since $\mathrm{Tr}\,e^{-\tau H^{(2)}} = \mathrm{Tr}\,e^{-\tau H^{(2)}_{ND}} + \mathrm{Tr}\,e^{-\tau H^{(2)}_{D}}$, we encounter two old friends: the heat functions arising respectively in connection with the TK2 kink in the MSTB model and the soliton of the sine-Gordon model. Using the information collected in previous Sections, we find:

Casimir energy :

$$
\begin{aligned}
{}_1\Gamma^K &= \frac{\tilde{m}}{2\sqrt{2}}\lim_{s\to-\frac{1}{2}}\left(\frac{2}{m^2}\right)^{s}\left[\frac{1}{\zeta(s)} + \sum_{a=1}^{2}\sum_{n=1}^{N_a}\frac{a_n^{aa}(H^{(2)}_{ND})}{v_a^{2s+1}}\frac{\Gamma[s+n-\frac{1}{2};v_a^2]}{\sqrt{\pi}\,\Gamma(s)}\right] + \\
&+ \frac{\tilde{m}}{2\sqrt{2}}\lim_{s\to-\frac{1}{2}}\left(\frac{2}{m^2}\right)^{s}\sum_{l=3}^{N}\frac{2}{(\mu_l^2-\mu_2^2)^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2};s+\tfrac{1}{2};\tfrac{3}{2};\tfrac{\mu_1^2-\mu_2^2}{\mu_l^2-\mu_2^2}]\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,\zeta(s)} :
\end{aligned}
$$

Mass renormalization energy :

$$
\begin{aligned}
{}_2\Gamma^K &= \frac{\tilde{m}}{\sqrt{2}}\lim_{s\to-\frac{1}{2}}\left(\frac{2}{m^2}\right)^{s+\frac{1}{2}}\left[\frac{2+\mu_2^2}{4^{s+\frac{1}{2}}}\frac{\Gamma[s+\frac{1}{2};4]}{\Gamma(s)} + \frac{2-3\mu_2^2}{2^{2s+1}}\frac{\Gamma[s+\frac{1}{2};\mu_2^2]}{\Gamma(s)}\right. \\
&\left. + \mu_2^2\sum_{l=3}^{N}\frac{\Gamma[s+\frac{1}{2};\mu_l^2]}{\mu_l^{2s+1}\,\Gamma(s)}\right] :
\end{aligned}
$$

We finally obtain the answer:

$$
\begin{aligned}
M_{TK2_2}(\mu_2;\mu_1) &= \frac{\tilde{m}}{2\sqrt{2}}\left[1 + \frac{1}{4\sqrt{\pi}}\sum_{a=1}^{2}\sum_{n=1}^{N_a}[a_n]_{aa}(H^{(2)}_{ND})\frac{\Gamma[n-1;v_a^2]}{v_a^{2n-2}}\right] \\
&\frac{\tilde{m}}{2\sqrt{2}}\sum_{l=3}^{N}\left[\log\frac{\mu_l^2-\mu_2^2}{\mu_l^2} + {}_2F_1{}^0[\tfrac{1}{2};0;\tfrac{3}{2};\tfrac{\mu_1^2-\mu_2^2}{\mu_l^2-\mu_2^2}]\right] ;
\end{aligned}
$$

or

$$
\begin{aligned}
M_{TK2_2}(\mu_2;\mu_1) &= \frac{\tilde{m}}{2\sqrt{2}}\left[1 + \frac{1}{4\sqrt{\pi}}\sum_{a=1}^{2}\sum_{n=1}^{N_a}[a_n]_{aa}(H^{(2)}_{ND})\frac{\Gamma[n-1;v_a^2]}{v_a^{2n-2}}\right] \\
&\frac{\tilde{m}}{\sqrt{2}}\sum_{l=3}^{N}\sqrt{(\mu_l^2-\mu_2^2)}\arcsin\frac{\mu_2}{\mu_1} :
\end{aligned}
$$

In both formulas the contribution coming from $H^{(2)}_{ND}$ can be read from the information on the quantum correction to the TK2 kink mass in the MSTB model, collected in the Tables and the Figure of sub-Section x.3.2 . A similar formula would show us that M_{TK2_l}, $8l \quad 2$, receives an imaginary contribution any $TK2_l$ quantum kink state is therefore a resonance.

5 Further comments

The family of deformations that we have treated admits a $N = 1$ super-symmetric extension. The super-potential $W(\ )$ in the $N = 2$ case is:

$$W(\ ) = \quad \sqrt{(\ _1^2 \quad)^2 + \ _2^2}\,[\frac{1}{3}(\ _1^2 + \ _2^2 \quad \ _1 + \ ^2) \quad 1] \ :$$

For the special value $= 2$, this system also admits the super-potential mentioned at the end of sub-Section x3.2 because it becomes the Wess-Zumino model. If $N \quad 3$, it is very difficult to write the super-potential in Cartesian coordinates in the R^N internal space; nevertheless, passing to elliptic coordinates one obtains easy expressions for the super-potential, see [18].

The super-symmetric extensions also include Majorana spinor fields:

$$\sim(x\ ) = \begin{matrix} \sim_1(x\ ) \\ \sim_2(x\ ) \end{matrix} \quad ; \quad \sim\ = \sim \quad ; \quad = 1;2$$

Choosing the Majorana representation $^0 = \ ^2; \ ^1 = i\ ^1; \ ^5 = \ ^3$ of the Clifford algebra $f\ ; \ g = 2g$ and defining the Majorana adjoint $\sim = \sim^t\ ^0$, the action of the super-symmetric model is:

$$S = \frac{1}{2c_d^2} \int dx^2 \left\{ @\ \sim@\ \sim + i\sim \ @\sim \quad \tilde{f}W\ \tilde{f}W \quad \sim\tilde{W}\sim \right\} \ ;$$

$$\tilde{f}\hat{W}(x\ ) = \sum_{a=1}^{2} \frac{@\hat{W}}{@\ _a}(x\ )e_a \ ; \quad \tilde{}\hat{W} = \tilde{f} \quad \tilde{f}\hat{W} = \sum_{a=1}^{2}\sum_{b=1}^{2} e_a \quad e_b \frac{@^2\hat{W}}{@\ _a@\ _b} :$$

The $N = 1$ super-symmetry transformation is generated on the space of classical configurations by the Hamiltonian spinor function :

$$Q = \int dx \left\{ \ ^0\sim@\ \sim + i\ ^0\sim\tilde{f}W \right\} :$$

The components of the Majorana spinorial charge Q close the super-symmetry algebra :

$$fQ \ ; Q\ g = 2(\quad ^0)\ P \quad 2i\ ^1\ T: \tag{13}$$

Their (anti)-Poisson bracket is given in (13) in terms of the momentum P and the topological central charge $T = |\int dW| = |\int \tilde{f}W\ d\tilde{}|$.

The chiral projections $Q \quad = \frac{1 \quad ^5}{2}Q$ and $\quad = \frac{1 \quad ^5}{2}$ provide a very special combination of the super-symmetric charges:

$$Q_+ + Q \quad = \int dx \quad (\sim_+ \quad \sim)\frac{d\sim}{dx} \quad (\sim \ + \sim_+)\tilde{f}W \quad :$$

$Q_+ + Q_-$ is zero for the classical configurations that satisfy $\frac{d\tilde{}}{dx} = \tilde{f}'W$ and $\tilde{} = 0$ which are thus classical BPS states. In Appendix B it is proved that the stable TK2 kinks are such BPS states and besides the small bosonic fluctuations one must take into account the small fermionic fluctuations around the kink in order to compute the quantum correction to the kink mass in the extended super-symmetric system. The fermionic fluctuations around the kink configuration lead to other solutions of the field equations if the $N \times N$ matrix Dirac equation

$$\left\{ i\partial_t + \tilde{W}'(\tilde{}_K)\,{}_F\tilde{}(x,t) = 0 \right\}$$

is satisfied. We multiply this equation for the adjoint of the Dirac operator:

$$\left\{ i\partial_t + \tilde{W}'(\tilde{}_K) \right\} \left\{ i\partial_t + \tilde{W}'(\tilde{}_K) \right\} {}_F\tilde{}(x,t) = 0 \quad ;$$

and, due to the time-independence of the kink background, look for solutions of the form: ${}_F\tilde{}(x,t) = \tilde{f}_F(x,!)e^{i!t}$. This is tantamount to solving the spectral problem

$$\frac{d^2}{dx^2} + \tilde{W}'(\tilde{}_K)\tilde{W}'(\tilde{}_K) \quad i\,{}^1\tilde{f}'W'(\tilde{}_K)\tilde{f} \quad \tilde{W}'(\tilde{}_K) \quad \tilde{f}_F(x,!) = !^2\tilde{f}_F(x,!):$$

Projecting onto the eigen-spinors of $i\,{}^1$,

$$\tilde{f}_F^{(1)}(x,!) = \frac{1 + i\,{}^1}{2}\tilde{f}_F(x,!) = \frac{1}{2} \begin{pmatrix} f_F^+(x,!) & \tilde{f}_F^-(x,!) \\ \tilde{f}_F^+(x,!) & f_F^-(x,!) \end{pmatrix}$$

we end with the spectral problem:

$$\frac{d^2}{dx^2} + \tilde{W}'(\tilde{}_K)\tilde{W}'(\tilde{}_K) \quad \tilde{f}'W'(\tilde{}_K)\tilde{f} \quad \tilde{W}'(\tilde{}_K) \quad \tilde{f}_F^{(1)}(x,!) = K\tilde{f}_F^{(1)}(x,!) = !^2\tilde{f}_F^{(1)}(x,!)$$

for the same Schrodinger operator that governs the bosonic fluctuations.

Therefore, generalized zeta function methods can also be used in super-symmetric models to compute the quantum corrections to the mass of BPS kinks. Great care, however, is needed in choosing the boundary conditions on the fermionic fluctuations without spoiling super-symmetry. We look forward to extending this research in this direction.

A Appendix: the matrix heat kernel expansion

In this Appendix we describe how to find the coefficients of the asymptotic expansion of the heat kernel associated with K and show the explicit expressions for them up to third order; for a more complete treatment see [15]. Formula (6) in the text tells us that the matrix elements of $A(x,x^0;\ )$ satisfy the N^2 coupled PDE:

$$\frac{\partial}{\partial} + \frac{x \quad x^0}{}\frac{\partial}{\partial x} \quad \frac{\partial^2}{\partial x^2} \quad [A\ ]_{ab}(x,x^0;\ ) = \sum_{c=1}^{N} V_{ac}(x)[A\ ]_{cb}(x,x^0;\ ) + (v_b^2 \quad v_a^2)[A\ ]_{ab}(x,x^0;\ );$$

with the initial condition: $A_{ab}(x;x^0;0) = \ _{ab}$. Plugging the power expansion of $A_{ab}(x;x^0; \ )$ into this system of equations we find the recurrence relations:

$$n\,[a_n\,]_{ab}(x;x^0) + (x \quad x^0)\frac{@\,[a_n\,]_{ab}}{@x}(x;x^0) \ = \ \frac{@^2\,[a_{n\ 1}\,]_{ab}}{@x^2}(x;x^0) + \sum_{c=1}^{N} V_{ac}(x))[a_{n\ 1}\,]_{cb}(x;x^0)$$
$$+ \ (v_b^2 \quad v_a^2)[a_{n\ 1}\,]_{ab}(x;x^0) \ :$$

In order to take the $x^0 \ ! \ x$ limit properly, we introduce the notation:

$$^{(k)}[A_n\,]_{ab}(x) = \lim_{x^0 ! \ x} \frac{@^k\,[a_n\,]_{ab}}{@x^k}(x;x^0) \ :$$

Then, the recurrence relations in the $x^0 = x$ limit become:

$$[a_n\,]_{ab}(x;x) = \frac{1}{n}\left(\ ^{(2)}[A_{n\ 1}\,]_{ab}(x) + \sum_{c=1}^{N} V_{ac}(x)[a_{n\ 1}\,]_{cb}(x;x) + (v_b^2 \quad v_a^2)[a_{n\ 1}\,]_{ab}(x;x)\right)$$

We also need the secondary recurrence relations among the $^{(k)}[A_n\,]_{ab}(x)$ derived from k-times differentiation of the primary recurrence relations above:

$$^{(k)}[A_n\,]_{ab}(x) = \frac{1}{n+k}\left(\ ^{(k+2)}[A_{n\ 1}\,]_{ab}(x) + \sum_{c=1}^{N}\sum_{j=0}^{k} \binom{k}{j} \frac{@^j V_{ac}}{@x^j} \ ^{(k\ j)}[A_{n\ 1}\,]_{cb}(x) + (v_b^2 \quad v_a^2)\ ^{(k)}[A_{n\ 1}\,]_{ab}(x)\right) \ :$$

Notice that $^{(k)}[A_0\,]_{ab}(x) = \lim_{x^0 ! \ x} \frac{@^k\,[a_0\,]_{ab}}{@x^k} = \ ^{k0}{}_{ab}$. Thus, the $^{(k)}[A_n\,]_{ab}(x)$ and, hence, the $[a_n\,]_{ab}(x;x)$ can be generated recursively. The three first coefficients are:

$$\begin{aligned}
[a_1\,]_{ab}(x) \ &= \ V_{ab}(x) \\
[a_2\,]_{ab}(x) \ &= \ \frac{1}{6}V_{ab}^{(2)}(x) + \frac{1}{2} \ V^2 \ _{ab}(x) + \frac{1}{2}(v_b^2 \quad v_a^2)V_{ab}(x) \\
[a_3\,]_{ab}(x) \ &= \ \frac{1}{60}V_{ab}^{(4)}(x) + \frac{1}{12} \ V^{(2)}(x)V \ (x) \ _{ab} + \frac{1}{12} \ V \ (x)V^{(2)}(x) \ _{ab} + \frac{1}{12} \ V^{(1)}(x)V^{(1)}(x) \ _{ab} \\
&+ \ \frac{1}{6} \ V^3 \ _{ab}(x) + \frac{1}{12}(v_b^2 \quad v_a^2) \ \left\{ V_{ab}^{(2)}(x) + 2 \ V^2 \ _{ab}(x)\right\} \\
&+ \ \frac{1}{6}(v_b^2 \quad v_a^2)^2 V_{ab}(x) + \frac{1}{6}\sum_{c=1}^{N}(v_b^2 \quad v_c^2)V_{ac}(x)V_{cb}(x) \ :
\end{aligned}$$

We mention that, as happens in the scalar case [19], the diagonal terms $[a_n\,]_{aa}(x;x)$ can be interpreted as the densities giving the infinite conserved charges of a matrix Korteweg-de Vries equation; namely:

$$\frac{@V}{@t} + 3[V\frac{@V}{@x} + \frac{@V}{@x}V\,] + \frac{@^3V}{@x^3} = 0; \qquad (14)$$

where now the matrix potential evolves in "time" t, $V = V(x;t)$. The reason is that (14) can be written as a Lax equation

$$L_t + [L;M\,] = 0$$

for the operators

$$L = \frac{@^2}{@x^2} \quad V \tag{15}$$

$$M = 4\frac{@^3}{@x^3} \quad 3V\frac{@}{@x} \quad 3\frac{@}{@x}V + B(t) \ : \tag{16}$$

with $B(t)$ arbitrary. Therefore, standard arguments [20] guarantee that the time evolution ruled by (14) produces an uniparametric isospectral transformation of the Schodinger operator (15). Because the integrals $[a_n]_{aa}$ are determined by the spectrum of (15), their invariance follows.

B Appendix: BPS and non-BPS kinks

This Appendix is devoted to characterize respectively the TK1 and TK2 kinks as non-BPS and BPS states in the $N = 2$ case, see Reference [18]. We have seen in Section x5 that the model admits four super-potentials. If $_a = 0;1$; $a = 1;2$, the four super-potentials

$$W^{(1;2)}(_1;_2) = (1)^1 \sqrt{(_1 + (1)^2)^2 + _2^2}[\frac{1}{3}(^2_1 + ^2_2 \quad (1)^2 \quad _1 + ^2) \quad 1] \ ; \tag{17}$$

satisfy:

$$\frac{1}{2}\sum_{a=1}^{2}\frac{@W^{(1;2)}}{@_a}\frac{@W^{(1;2)}}{@_a} = \frac{1}{2}(^2_1 + ^2_2 \quad 1)^2 + \frac{^2}{2}^2_2 \ :$$

The energy for static configurations reads:

$$E = \frac{1}{2}\int dx \sum_{a=1}^{2}\left(\frac{d_a}{dx} \quad \frac{@W^{(1;2)}}{@_a}\right)^2 + \int dx \sum_{a=1}^{2}\frac{d_a}{dx}\frac{@W^{(1;2)}}{@_a} \ :$$

The BPS kinks are the solutions of the ODE first-order system :

$$\frac{d_1}{dx} = \frac{@W^{(1;2)}}{@_1} = \frac{(1)^1(1 + ^2_1)(_1 + (1)^2) + _1^2_2}{\sqrt{^2_2 + (_1 + (1)^2)^2}} \tag{18}$$

$$\frac{d_2}{dx} = \frac{@W^{(1;2)}}{@_2} = \frac{(1)^1_2[^2 + ^2_2 + _1(_1 + (1)^2)]}{\sqrt{^2_2 + (_1 + (1)^2)^2}} \tag{19}$$

From the values of the super-potential at the vacuum points

$$W^{(1;2)}(1;0) = (1)^1(1 \quad (1)^2)(1 + \frac{1}{3}(1 \quad (1)^2 + ^2))$$

we calculate the Bogomolny bound

$$E^{BPS} = \int dW^{(1;2)}| = |W^{(1;2)}(1;0) \quad W^{(1;2)}(1;0)| = 2 \quad 1 \quad \frac{^2}{3} \ ; \tag{20}$$

which is saturated by the solutions of (18,19), the BPS kinks.

In the derivation of (20) we have used Stokes's theorem. The foci $(\ ^F_1 = \ (\ 1)^{\ 2}\ ;\ ^F_2 = 0)$ of the ellipse $\ ^2_1 + \frac{1}{\ 2}\ _2 = 1$ in the R^2 "internal" space are branching points of $W^{(\ _1;\ _2)}$. Therefore, we are legitimated to use Stokes's theorem – and the Bogomolny bound is reached– only if the kink configuration does not cross any of the two foci above mentioned. The TK2 kinks live on the semi-ellipses $\ ^{TK2}_2 = \ \sqrt{1\ \ (\ ^{TK2}_1)^2}$ and everything is fine: they are BPS kinks. The TK1 solutions are more involved. If $\ _2 = 0$ (18) reduces to:

$$\frac{d\ _1}{dx} = \ (\ 1)^{\ 1}(1\ \ ^2_1)\frac{\ _1 + (\ 1)^{\ 2}}{j\ _1 + (\ 1)^{\ 2}\ j} = \ (\ 1)^{\ 1}(1\ \ ^2_1)\ \ (\ _1 + \ (\ 1)^{\ 2})\ \ (\ _1\ \ (\ 1)^{\ 2})\ \ ;$$

where (u) is the Heaviside step function. Thus, $\ ^{TK1}_1(x) = \tanh x$ is not solution of the first-order equations on the whole real line; if $\ _1 + (\ 1)^{\ 2} < 0$, we must choose $\ _1 = 0$, and $\ _1 = 1$ otherwise. On the half-line $x\ 2\ (\ 1\ ;\ \text{arctanh}[(\ 1)^{\ 2}\ ]]$ the TK1 solution is the flow line of $\text{grad}W^{(0;\ 2)}$ but it becomes the flow line of $\text{grad}W^{(1;\ 2)}$ on $x\ 2\ [\ \text{arctanh}[(\ 1)^{\ 2}\ ];1\ )$. One can easily check that the TK1 kink is a proper solution of the second-order equations with energy given by a piece-wise application of Stokes's theorem:

$$E^{\ TK1} = \ jW^{\ (0;\ 2)}(\ 1;0)\ \ W^{\ (0;\ 2)}(\ \ (\ 1)^{\ 2}\ ;0)j + \ jW^{\ (1;\ 2)}(\ \ (\ 1)^{\ 2}\ ;0)\ \ W^{\ (1;\ 2)}(1;0)j = \frac{4}{3}$$

In a super-symmetric extension of this model the corresponding state would be not annihilated by any combination of the super-symmetry generators built from one of the $W^{\ (\ _1;\ _2)}$ super-potentials and, therefore, the TK1 kink is a non-BPS state in this system if $\ ^2 < 1$.

It is shown in Reference [18] that only the stable TK2 kinks are BPS kinks in the $N\ \ 3$ models; the proof has been performed using elliptic coordinates for the field variables.

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