arXiv:0906.1258v2 [hep-th] 13 Aug 2009

On the semiclassical mass of S^2-kinks

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One-loop mass shifts to the classical masses of stable kinks arising in a massive non-linear S^2-sigma model are computed. Ultraviolet divergences are controlled using the heat kernel/zeta function regularization method. A comparison between the results achieved from exact and high-temperature asymptotic heat traces is analyzed in depth.

PACS numbers: 11.15.Kc; 11.27.+d; 11.10.Gh

I. INTRODUCTION

In a seminal paper, Olive and Witten [1] linked extended supersymmetric theories to BPS solitons by showing that the classical mass of these stable lumps agreed exactly with the central charge of the extended SUSY algebra. The subsequent issue concerning BPS saturation at one-loop (rather than tree) level has proved to be extremely subtle, prompting a remarkable amount of work over the last twelve years. See, e.g., [2] and References quoted therein to find an in-depth report on these developments.

A new actor entered the stage when in [3] a Stony Brook/Wien group computed the one-loop mass shift of the supersymmetric CP^1-kink in a $N = (2;2)$ non-linear sigma model with twisted mass. Kinks of several types in massive non-linear sigma models were, however, discovered earlier, see [4], [5], [6], [7]. In Reference [8], three of us found several families of non-topological kinks in another non-linear sigma model: we chose S^2 as the target space and considered the case when the masses of the pseudo-Nambu-Goldstone particles were different. The $O(2)$-symmetry of the equal-mass case is explicitly broken to $Z_2 \times Z_2$ and the $SO(2)$-families of topological kinks of the former system are deformed to the four families of non-topological kinks arising in the second system. The boundary of the moduli space of non-topological kinks in the last model is formed by a pair of topological kinks of different energy. The analysis of kink stability in the massive non-linear S^2-sigma model performed in [9] allowed us to calculate the one-loop mass shifts for the topological kinks by using the Cahill-Comtet-Glauber formula [10]. These authors showed that the one-loop mass shift for static solitons can be read from the eigenvalues of the bound states of the kink second-order fluctuation operator and the threshold to the continuous spectrum when this operator is a transparent Schrodinger operator of the Posch-Teller type. This is the case of the topological kinks of the massive non-linear S^2-sigma model when a parallel frame to the kink orbits is chosen to refer to the fluctuations.

The aim of this paper is to offer another route for computing the one-loop kink mass shift in order to unveil some of the intricacies hidden in this subtle problem. We shall follow the method developed in References [12] and [13] based on heat kernel/zeta function regularization of ultraviolet divergences. See also the lectures [14], where full details can be found. Because the spectrum of small kink fluctuations in our system can be identified analytically, we are able to give the exact answer for the mass shifts. We shall also show, however, how to reach approximately the same result using the coefficients of the heat kernel asymptotic expansion. The interest of this calculation is that a formula belonging to the class of formulas shown in [17] will be derived. The importance of this type of formula lies in the fact that it can be applied to obtain the one-loop mass shifts of topological defects even when the spectrum of the second-order fluctuation operator is not known; for instance, in the case of two-component topological kinks: see [12], [13]. Similar formulas work even for Abelian gauge theories in (2+1)-dimensions and thus the mass shifts of self-dual Nielsen-Olesen vortices and semi-local strings can be calculated approximately, see [18], [19], and [20].

To end this brief Introduction we simply mention that interesting calculations have recently appeared addressing one-loop kink mass corrections and kink melting at finite temperatures in the sine-Gordon, CP^1, and ϕ^4 models in a purely bosonic setting, see [15].

The organization of the paper is as follows: In Section §II, we introduce the model and explain our conventions. In Section §III, the perturbative sector as well as the mass renormalization procedure are discussed. Section §IV is devoted to the analysis of the stable topological kinks in this system. The second-order kink fluctuation operator is obtained, placing special emphasis on its geometric properties. In Section §V, the one-loop mass shift is computed using the heat kernel/zeta function regularization method. Section §VI offers a comparison of the exact result obtained in §V with the approximation reached from the high-temperature asymptotic expansion. Finally, a summary and outlook are offered whereas two Appendices containing some technical material are included.

II. THE (1+1)-DIMENSIONAL MASSIVE NON-LINEAR S^2-SIGMA MODEL

The action governing the dynamics of the non-linear S^2-sigma model and the constraint on the scalar fields are:

$$S[\phi_1;\phi_2;\phi_3]=\int dt dx \left\{\frac{1}{2} g^{\mu\nu}\sum_{a=1}^{3}\frac{\partial\phi_a}{\partial x^\mu}\frac{\partial\phi_a}{\partial x^\nu}\right\} \quad ; \quad \phi_1^2+\phi_2^2+\phi_3^2=R^2 \quad . \tag{1}$$

The scalar fields are thus maps, $\phi_a(t;x)\in \mathrm{Maps}(\mathbb{R}^{1;1};S^2)$, $a=1;2;3$, from the (1+1)-dimensional Minkowski space-time to a S^2-sphere of radius R, which is the target manifold of this non-linear sigma model. Our conventions for $\mathbb{R}^{1;1}$ are as follows: $x^\mu\in\mathbb{R}^{1;1}$, $\mu=0;1$ with $x^0=t; x^1=x$ and $g^{\mu\nu}=\mathrm{diag}(1;-1)$. Then $x^\mu x_\mu=g_{\mu\nu}x^\mu x^\nu=t^2-x^2$ and

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}=g^{\mu\nu}\frac{\partial^2}{\partial x^\mu\partial x^\nu}=\Box=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2} \quad .$$

The infrared asymptotics forbids massless particles in (1+1)-dimensional scalar field theories, see [16]. We therefore include the simplest potential energy density that would be generated by quantum fluctuations [21]:

$$V(\phi_1;\phi_2;\phi_3)=\frac{1}{2}\left(\alpha_1^2\phi_1^2+\alpha_2^2\phi_2^2+\alpha_3^2\phi_3^2\right) \quad .$$

1. Solving ϕ_3 in favor of ϕ_1 and ϕ_2, $\mathrm{sg}(\phi_3)\phi_3=\sqrt{R^2-\phi_1^2-\phi_2^2}$, we find:

$$S=\frac{1}{2}\int dt dx\left\{\frac{\partial\phi_1}{\partial x^\mu}\frac{\partial\phi_1}{\partial x_\mu}+\frac{\partial\phi_2}{\partial x^\mu}\frac{\partial\phi_2}{\partial x_\mu}+\frac{(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}}\frac{(\phi_1\partial^\mu\phi_1+\phi_2\partial^\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}}-V_{S^2}[\phi_1;\phi_2]\right\} \quad ;$$

where

$$V_{S^2}(\phi_1;\phi_2)=\frac{1}{2}\left[(\alpha_1^2-\alpha_3^2)\phi_1^2+(\alpha_2^2-\alpha_3^2)\phi_2^2\right]+\mathrm{const.}\simeq\frac{\lambda^2}{2}\phi_1^2(t;x)+\frac{\gamma^2}{2}\phi_2^2(t;x) \quad ;$$

with $\lambda^2=(\alpha_1^2-\alpha_3^2)$, $\gamma^2=(\alpha_2^2-\alpha_3^2)$, $\lambda^2\geq\gamma^2$. The masses of the pseudo-Nambu-Goldstone bosons are respectively λ and γ.

2. Interactions, however, come from the geometry:

$$\frac{(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}}\frac{(\phi_1\partial^\mu\phi_1+\phi_2\partial^\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}} \simeq \frac{1}{R^2}\left[1+\frac{1}{R^2}(\phi_1^2+\phi_2^2)+\frac{1}{R^4}(\phi_1^2+\phi_2^2)^2+\cdots\right]\left(\phi_1\frac{\partial\phi_1}{\partial x^\mu}+\phi_2\frac{\partial\phi_2}{\partial x^\mu}\right)\left(\phi_1\frac{\partial\phi_1}{\partial x_\mu}+\phi_2\frac{\partial\phi_2}{\partial x_\mu}\right)$$

and $\frac{1}{R^2}$ is a non-dimensional coupling constant.

In the unit natural system, $\hbar=c=1$, the dimensions of fields, masses and coupling constants are respectively: $[\phi_a]=1=[R]$, $[\lambda]=M=[\gamma]$. We define non dimensional space-time coordinates and masses:

$$x^\mu \to \frac{x^\mu}{\lambda} \quad ; \quad \sigma^2=\frac{\alpha_2^2-\alpha_3^2}{\alpha_1^2-\alpha_3^2}=\frac{\gamma^2}{\lambda^2} \quad ; \quad 0<\sigma^2\leq 1$$

to write the action and the energy in terms of them:

$$S=\frac{1}{2}\int dt dx\left\{\frac{\partial\phi_1}{\partial x^\mu}\frac{\partial\phi_1}{\partial x_\mu}+\frac{\partial\phi_2}{\partial x^\mu}\frac{\partial\phi_2}{\partial x_\mu}+\frac{(\phi_1\partial_\mu\phi_1+\phi_2\partial_\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}}\frac{(\phi_1\partial^\mu\phi_1+\phi_2\partial^\mu\phi_2)}{\sqrt{R^2-\phi_1^2-\phi_2^2}}-\phi_1^2(t;x)-\sigma^2\phi_2^2(t;x)\right\} \tag{2}$$

$$E=\frac{\lambda}{2}\int dx\left\{\left(\frac{\partial\phi_1}{\partial t}\right)^2+\left(\frac{\partial\phi_2}{\partial t}\right)^2+\frac{(\phi_1\partial_t\phi_1+\phi_2\partial_t\phi_2)^2}{R^2-\phi_1^2-\phi_2^2}+\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_2}{\partial x}\right)^2+\frac{(\phi_1\partial_x\phi_1+\phi_2\partial_x\phi_2)^2}{R^2-\phi_1^2-\phi_2^2}\right.$$
$$\left.+\phi_1^2(t;x)+\sigma^2\phi_2^2(t;x)\right\} \quad .$$

There are two homogeneous minima of the action or vacua of our model: $\phi_1^V=\phi_2^V=0$; $\phi_3^V=\pm R$, the North and South Poles. Choice of one of the poles to quantize the system spontaneously breaks the $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry of the action (2), $\phi_a\to(-1)^{\delta_{ab}}\phi_b$; $a;b=1;2;3$, to: $\mathbb{Z}_2\times\mathbb{Z}_2$; $\phi_\alpha\to(-1)^{\delta_{\alpha\beta}}\phi_\beta$; $\alpha;\beta=1;2$. Therefore, the configuration space $\mathcal{C}=\{\mathrm{Maps}(\mathbb{R};S^2)/E<+\infty\}$ is the union of four disconnected sectors $\mathcal{C}=\mathcal{C}_{NN}\sqcup\mathcal{C}_{SS}\sqcup\mathcal{C}_{NS}\sqcup\mathcal{C}_{SN}$ labeled by the vacua reached by each configuration at the two disconnected components of the boundary of the real line: $x=\pm\infty$.

III. MASS RENORMALIZATION

The field equations

$$\Box\phi_1 + \partial_\mu\left[\frac{\phi_1\sum_{\alpha=1}^{2}\phi_\alpha\partial^\mu\phi_\alpha}{R^2-\sum_{\alpha=1}^{2}\phi_\alpha^2}\right] + \phi_1\left[\frac{\sum_{\alpha=1}^{2}\phi_\alpha\partial_\mu\phi_\alpha\sum_{\beta=1}^{2}\phi_\beta\partial^\mu\phi_\beta}{(R^2-\sum_{\alpha=1}^{2}\phi_\alpha^2)^{\frac{3}{2}}}\right] + 1 = 0$$

$$\Box\phi_2 + \partial_\mu\left[\frac{\phi_2\sum_{\alpha=1}^{2}\phi_\alpha\partial^\mu\phi_\alpha}{R^2-\sum_{\alpha=1}^{2}\phi_\alpha^2}\right] + \phi_2\left[\frac{\sum_{\alpha=1}^{2}\phi_\alpha\partial_\mu\phi_\alpha\sum_{\beta=1}^{2}\phi_\beta\partial^\mu\phi_\beta}{(R^2-\sum_{\alpha=1}^{2}\phi_\alpha^2)^{\frac{3}{2}}}\right] + {}^{2}\phi_2 = 0$$

become linear for small fluctuations, $G_\alpha(x^\mu) = \phi^V_\alpha + \delta G_\alpha(x^\mu)$, around the vacuum:

$$\Box G_1(t;\mathbf{x}) + G_1(t;\mathbf{x}) = O(\delta G\,\delta G) \qquad ; \qquad \Box G_2(t;\mathbf{x}) + {}^{2}G_2(t;\mathbf{x}) = O(\delta G\,\delta G) \qquad : \tag{3}$$

We shall need the Feynman rules only for the four-valent vertices. Besides the two propagators for the (pseudo) Nambu-Goldstone bosons there are three vertices with four external legs. The derivatives appearing in the interactions induce dependence on the momenta in the weights. This also affects the sign and the combinatorial factors. Naturally, there are many more vertices in this model, but we list only the vertices that contribute to the self-energy of the Nambu-Goldstone bosons up to one-loop order.

Table I: Propagators

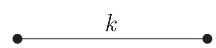
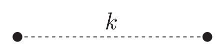
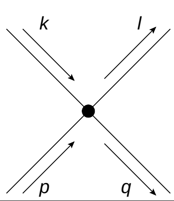
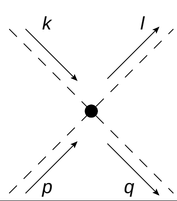
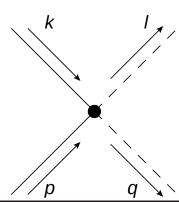
Particle	Field	Propagator	Diagram
Nambu-Goldstone	$G_1(x^\mu)$	$\frac{i}{k_0^2 - k^2 - 1 + i\varepsilon}$	
Nambu-Goldstone	$G_2(x^\mu)$	$\frac{i}{k_0^2 - k^2 - {}^{2} + i\varepsilon}$	

Table II: Fourth-order vertices

Vertex	Weight	Vertex	Weight	Vertex	Weight
	$2i\frac{k\,p}{R^2}$		$2i\frac{k\,p}{R^2}$		$2i\frac{k\,p}{R^2}$

A. Plane waves and vacuum energy

The general solution of the linearized field equations (3) governing the small fluctuations of the Nambu-Goldstone fields is:

$$G_1(x_0;\mathbf{x}) = \frac{1}{2}\sqrt{\frac{1}{l}}\sum_k \frac{1}{\sqrt{2\omega_1(k)}}\left[a_1(k)e^{-ik_0x_0+ikx} + a_1^\dagger(k)e^{ik_0x_0-ikx}\right]$$

$$G_2(x_0;\mathbf{x}) = \frac{1}{2}\sqrt{\frac{1}{l}}\sum_q \frac{1}{\sqrt{2\omega_2(q)}}\left[a_2(q)e^{iq_0x_0-iqx} + a_2^\dagger(q)e^{-iq_0x_0+iqx}\right] \qquad ;$$

where $k_0 = \omega_1(k) = \sqrt{k^2+1}$, $q_0 = \omega_2(q) = \sqrt{q^2 + {}^2}$, and the dispersion relations $k_0^2 - k^2 - 1 = 0$, $q_0^2 - q^2 - {}^2 = 0$ hold:

$$K_0 \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} = \omega_1^2(k) \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \quad ; \quad K_0 \begin{pmatrix} 0 \\ e^{iqx} \end{pmatrix} = \omega_2^2(q) \begin{pmatrix} 0 \\ e^{iqx} \end{pmatrix}$$

$$K_0 = \begin{pmatrix} K_{011} & 0 \\ 0 & K_{022} \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 & 0 \\ 0 & -\frac{d^2}{dx^2} + {}^2 \end{pmatrix} :$$

We have chosen a normalization interval of non-dimensional "length" $l = L$, $I = [-\frac{1}{2}; \frac{1}{2}]$, and we impose PBC on the plane waves so that: $k\, l = 2\pi n_1$, $q\, l = 2\pi n_2$ with $n_1; n_2 \in \mathbb{Z}$. Thus, K_0 acts on $L^2 = \bigoplus_{\alpha=1}^{2} L^2(S^1)$, and its spectral density at the $l \to \infty$ limit is: $\rho_{K_0}(k) = \begin{pmatrix} \frac{dn_1}{dk} & 0 \\ 0 & \frac{dn_2}{dq} \end{pmatrix} = \frac{l}{2\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

From the classical free (quadratic) Hamiltonian

$$H^{(2)} = \frac{}{2} \int dx \left[\left(\frac{\partial G_1}{\partial x_0} \frac{\partial G_1}{\partial x_0} + \frac{\partial G_1}{\partial x} \frac{\partial G_1}{\partial x} \right) + \left(\frac{\partial G_2}{\partial x_0} \frac{\partial G_2}{\partial x_0} + \frac{\partial G_2}{\partial x} \frac{\partial G_2}{\partial x} \right) \right.$$
$$\left. + G_1 G_1 + {}^2 G_2 G_2 \right] = \sum_k \sum_{\alpha=1}^{2} \frac{}{2} [\omega_\alpha(k)(a_\alpha(k) a_\alpha^*(k) + a_\alpha^*(k) a_\alpha(k))] \quad ;$$

one passes via canonical quantization to the quantum free Hamiltonian:

$$[\hat{a}_\alpha(k); \hat{a}_\beta^\dagger(q)] = \delta_{\alpha\beta}\delta_{kq} \quad ; \quad \hat{H}_0^{(2)} = \sum_k \left[\omega_1(k) \left(\hat{a}_1^\dagger(k)\hat{a}_1(k) + \frac{1}{2} \right) + \omega_2(k) \left(\hat{a}_2^\dagger(k)\hat{a}_2(k) + \frac{1}{2} \right) \right] :$$

The vacuum energy is:

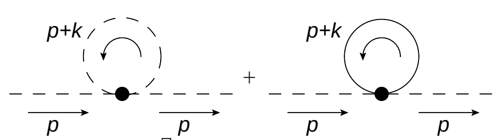
$$\hat{a}_\alpha(k)|0;V\rangle = 0\ ; \forall k; \forall\alpha \quad ; \quad E_0 = \langle 0;V|\hat{H}_0^{(2)}|0;V\rangle = \frac{\hbar}{2} \sum_k \omega_1(k) + \frac{\hbar}{2} \sum_k \omega_2(k) = \frac{\hbar}{2} \mathrm{Tr}_{L^2} K_0^{\frac{1}{2}}$$

B. One-loop mass renormalization counter-terms

There are four ultraviolet divergent graphs in one-loop order of the $\hbar$-expansion contributing to the $G_1(x)$ and $G_2(x)$ Nambu-Goldstone bosons self-energies:

Self-energy of G_2

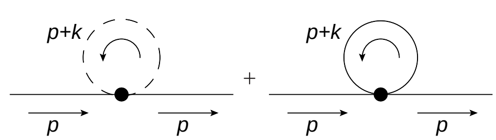
$$\frac{2i}{R^2} I(1) + \frac{2i}{R^2} I({}^2) =$$



$$= \frac{2i}{R^2} \int \frac{d^2k}{(2\pi)^2} \frac{i(p+k)p}{(p+k)k - 1 + i\varepsilon} + \frac{2i}{R^2} \int \frac{d^2k}{(2\pi)^2} \frac{i(p+k)p}{(p+k)k - {}^2 + i\varepsilon}$$
$$= \frac{2i}{R^2} \int \frac{d^2k}{(2\pi)^2} \frac{i}{k\,k - 1 + i\varepsilon} + \frac{2i}{R^2} \int \frac{d^2k}{(2\pi)^2} \frac{i}{k\,k - {}^2 + i\varepsilon} \quad ;$$

Self-energy of G_1

$$\frac{2i\ {}^2}{R^2} I(1) + \frac{2i\ {}^2}{R^2} I({}^2) =$$

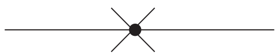
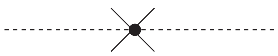


$$= \frac{2i\ {}^2}{R^2} \int \frac{dk}{4\pi} \frac{1}{\sqrt{k^2+1}} + \frac{2i\ {}^2}{R^2} \int \frac{dk}{4\pi} \frac{1}{\sqrt{k^2 + {}^2}} \quad ;$$

where we have computed the k_0 integrations by using the residue theorem. We only show this step explicitly in the computation of the self-energy of G_1 because it su ces to point out how to regularize these divergent integrals by means of spectral zeta functions. The regularization just mentioned will be performed later in Section x.V D.

The p p factor becomes constant when the momentum is put \on shell" in the external legs, p p = 1, p p = ^{2}. This process gives us the mass renormalization counter-terms. The Lagrangian density of counter-terms shown in Table III must be added to cancel the above divergences exactly. We also show the vertices generated at one-loop level.

Table III: One-loop counter-terms

Diagram	Weight
	$\frac{2i}{R^2}(I(1)+I(\ ^2))$
	$\frac{2i}{R^2}(I(1)+I(\ ^2))$

$$L_{C:T:} = \frac{1}{R^2}\quad I(1)+I(\ ^2)\quad {}^2_1(x\ )+\ {}^2\ {}^2_2(x\ )$$

IV. ISOTHERMAL COORDINATES AND TOPOLOGICAL KINKS

In this Section we shall use the isothermal coordinates in the chart S^2 f(0;0; R)g obtained via stereographic projection from the South Pole:

$$^1 = \frac{^1}{1+\frac{^3}{R}} = \frac{R\ ^1}{R + sg(\ _3)\ \sqrt{R^2\ \ ^2_1\ \ ^2_2}} \quad ; \quad ^2 = \frac{^2}{1+\frac{^3}{R}} = \frac{R\ ^2}{R + sg(\ _3)\ \sqrt{R^2\ \ ^2_1\ \ ^2_2}} \quad : \tag{4}$$

The geometric data of the sphere in this coordinate system are:

$$ds^2 = \frac{4R^4}{(R^2 + \ ^1\ ^1 + \ ^2\ ^2)^2} \quad ; \quad g_{11}(\ ^1;\ ^2) = g_{22}(\ ^1;\ ^2) = \frac{4R^4}{(R^2 + \ ^1\ ^1 + \ ^2\ ^2)^2}$$

$$^1_{11}(\ ^1;\ ^2) = \quad ^1_{22}(\ ^1;\ ^2) = \ ^2_{12}(\ ^1;\ ^2) = \ ^2_{21}(\ ^1;\ ^2) = \frac{2\ ^1}{R^2 + \ ^1\ ^1 + \ ^2\ ^2}$$

$$^2_{22}(\ ^1;\ ^2) = \quad ^2_{11}(\ ^1;\ ^2) = \ ^1_{12}(\ ^1;\ ^2) = \ ^1_{21}(\ ^1;\ ^2) = \frac{2\ ^2}{R^2 + \ ^1\ ^1 + \ ^2\ ^2}$$

$$R^1_{122}(\ ^1;\ ^2) = \quad R^1_{212}(\ ^1;\ ^2) = \quad R^2_{121}(\ ^1;\ ^2) = R^2_{211}(\ ^1;\ ^2) = \frac{4R^4}{(R^2 + \ ^1\ ^1 + \ ^2\ ^2)^2} \quad :$$

The kinetic and potential energy densities read:

$$T(\ ^1;\ ^2) = \frac{2R^4}{(R^2 + \ ^1\ ^1 + \ ^2\ ^2)^2}\quad @_t\ ^1 @_t\ ^1 + @_t\ ^2 @_t\ ^2$$

$$V(\ ^1;\ ^2) = \frac{2R^4}{(R^2 + \ ^1\ ^1 + \ ^2\ ^2)^2}\quad @_x\ ^1 @_x\ ^1 + @_x\ ^2 @_x\ ^2 + \ ^1\ ^1 + \ ^2\ ^2\ ^2 \quad :$$

From the action $S = \int d^2x\ [T \quad V]$ one derives the eld equations:

$$2\ ^i + \ ^i_{jk} @\ \ ^j @\ \ ^k + \ ^i_1\ ^1 + \ ^2\ ^i_2\ ^2 \quad 2(\ ^i_1\ ^1 + \ ^i_2\ ^2)\frac{^1\ ^1 + \ ^2\ ^2\ ^2}{R^2 + \ ^1\ ^1 + \ ^2\ ^2} = 0 \quad ; \tag{5}$$

which for static con gurations reduce to:

$$\frac{d^2\ ^i}{dx^2} \quad ^i_{jk}\frac{d\ ^j}{dx}\frac{\ ^k}{dx} + \ ^i_1\ ^1 + \ ^2\ ^i_2\ ^2 \quad 2(\ ^i_1\ ^1 + \ ^i_2\ ^2)\frac{^1\ ^1 + \ ^2\ ^2\ ^2}{R^2 + \ ^1\ ^1 + \ ^2\ ^2} = 0 \quad : \tag{6}$$

A . Topological K kinks

We try the $^{1} = 0$ orbit in (6) and reduce this ODE system to the single ODE:

$$\frac{d^2\ ^2}{dx^2} \quad \frac{2\ ^2}{R^2 + \ ^2\ ^2}\frac{d\ ^2}{dx}\frac{d\ ^2}{dx} = \ ^2\ ^2\ 1 \quad 2\frac{^2\ ^2}{R^2 + \ ^2\ ^2} \qquad : \tag{7}$$

$$^1_K(x) = 0 \qquad ; \qquad ^2_K(x) = \ \ R e^{\ (x \ \ x_0)} \qquad ; \tag{8}$$

are solutions of (7) of nite energy:

$$E[K] = \int_{1}^{1} dx \frac{R^2\ ^2}{\cosh^2(\ (x \quad x_0))} = 2\ R^2 \qquad : \tag{9}$$

In (8), x_0 is an integration constant that sets the kink center. The kink eld components in the original coordinates

$$^K_1(x) = 0 \qquad ; \qquad ^K_2(x) = \frac{R}{\cosh[\ (x \quad x_0)]} \qquad ; \qquad ^K_3(x) = \ \ R \tanh[\ (x \quad x_0)]$$

are either kink-shaped, $^K_3$, or bell-shaped, $^K_2$. It is clear that the four solutions (8) belong to the topological sectors C_{NS} or C_{SN} of the con guration space. Lorentz invariance tells us that

$$^1_K(x) = 0 \qquad ; \qquad ^2_K(x) = \ \ R \exp[\ \ (\frac{x \quad vt}{\sqrt{1 \quad v^2}} \quad x_0)] \tag{10}$$

are solitary wave solutions of the full eld equations (5).

B . Second-order uctuation operator

Let us consider small kink uctuations:

$$(x) = \ _K(x) + \ \ (x) \qquad ; \qquad (x) = (\ ^1(x);\ ^2(x)) \qquad :$$

Here, $_K(x) = (\ ^1_K(x);\ ^2_K(x))$ is the kink solution and $\ (x) = \ ^1(x)\frac{@}{@\ ^1} + \ ^2(x)\frac{@}{@\ ^2}\ 2 \ \ (TS^2)$ are vector elds along the kink orbit – expressed in the orthonormal basis $h\frac{@}{@\ ^i};\frac{@}{@\ ^j}i = \ ^{ij}$ of TS^2 – giving the small uctuations on the kink. From the tangent vector eld to the orbit $\ ^0_K(x) = \frac{d\ ^1_K}{dx}\frac{@}{@\ ^1} + \frac{d\ ^2_K}{dx}\frac{@}{@\ ^2}$, the covariant derivative, and the curvature tensor

$$r_{\ ^0_K}\ \ (x) = \ ^{0i}(x) + \ ^i_{jk}(\ _K)\ ^j(x)\ ^{0k}_K(x)\ \frac{@}{@\ ^i} \qquad ; \qquad R(\ ^0_K;\ )\ ^0_K = \ ^{0i}_K(x)\ ^j(x)\ ^{0k}(x) R^l_{ijk}(\ _K)\frac{@}{@\ ^l} \quad ;$$

we obtain the geodesic deviation operator:

$$\frac{D^2}{dx^2}(x) = r_{\ ^0_K} r_{\ ^0_K}\ \ (x) \qquad ; \qquad \frac{D^2}{dx^2}(x) + R(\ ^0_K;\ )\ ^0_K(x) \qquad :$$

We also need the Hessian of the \mechanical" potential $U(\ ^1;\ ^2) = \ \ V(\ ^1;\ ^2)$

$$r\ \ \mathrm{grad} U(x) = \ ^i(x)\ \ \frac{@^2 U}{@\ ^i @\ ^j}(\ _K) \quad \ ^k_{ij}(\ _K)\frac{@U}{@\ ^k}(\ _K)\ \ g^{jl}\frac{@}{@\ ^l}$$

The second-order uctuation operator around the kink $_K$ is:

$$(K\ )\ \ (x) = \ \ \frac{D^2}{dx^2}(x) + R(\ ^0_K;\ )\ ^0_K + r\ \ \mathrm{grad} U(x) \qquad : \tag{11}$$

C. Small fluctuations on K kinks

Application to the K kink $_K(x) = (^1_K(x) = 0;\ ^2_K(x) = R\, e^{\ x})$ gives:

$$(K) = \left[-\frac{d^2\ ^1}{dx^2} + 2\ (1 - \tanh\ x)\frac{d\ ^1}{dx} - \left(1 - 2\ ^2 + 2\ ^2 \tanh\ x\right)\ ^1 \right]\frac{@}{@\ ^1}$$
$$+\left[-\frac{d^2\ ^2}{dx^2} + 2\ (1 - \tanh\ x)\frac{d\ ^2}{dx} + \ ^2 (1 - 2\tanh\ x)\ ^2 \right]\frac{@}{@\ ^2} \quad : \qquad (12)$$

The second-order fluctuation operator in the orthonormal frame is a second-order differential operator that has first-order derivatives both in the direction of the kink orbit, $\frac{@}{@\ ^2}$, and the orthogonal direction to the orbit $\frac{@}{@\ ^1}$.

Alternatively, we can use a parallel frame, $(x) = \ ^1(x)\frac{@}{@\ ^1} + \ ^2(x)\frac{@}{@\ ^2}$, along the K kink orbit:

$$\frac{d\ ^i}{dx} + \ ^i_{jk}(\ _K)\ ^{0j}_K\ ^k = 0 \qquad \begin{cases} \frac{d\ ^1}{dx} + (1 - \tanh)\ ^1(x) = 0 & \Rightarrow \quad ^1(x) = 1 + e^{2\ x} \\ \frac{d\ ^2}{dx} + (1 - \tanh)\ ^2(x) = 0 & \Rightarrow \quad ^2(x) = 1 + e^{2\ x} \end{cases} :$$

In this parallel frame the vectors of the basis $^i(x)\frac{@}{@\ ^i}$ point in the same directions as $\frac{@}{@\ ^i}$ but their moduli vary along the kink orbit:

$$h\ ^1(x)\frac{@}{@\ ^1};\ ^1(x)\frac{@}{@\ ^1}i = h\ ^2(x)\frac{@}{@\ ^2};\ ^2(x)\frac{@}{@\ ^2}i = (1 + e^{2\ x})^2 \quad :$$

Writing the fluctuations in this frame, $(x) = \ ^1(x)\ ^1(x)\frac{@}{@\ ^1} + \ ^2(x)\ ^2(x)\frac{@}{@\ ^2}$, we find:

$$(K) = \ ^1(x)\left[-\frac{d^2\ ^1}{dx^2} + \left(1 - \frac{2\ ^2}{\cosh^2\ x}\right)\ ^1 \right]\frac{@}{@\ ^1} + \ ^2(x)\left[-\frac{d^2\ ^2}{dx^2} + \left(\ ^2 - \frac{2\ ^2}{\cosh^2\ x}\right)\ ^2 \right]\frac{@}{@\ ^2} \quad : \qquad (13)$$

In the parallel frame the second-order fluctuation operator is a transparent (reflection coefficient equal to zero) Posch–Teller Schrodinger operator both in the parallel and orthogonal directions to the kink orbit.

This analysis is deceptively simple: acting respectively on $^1(x) = (1 + e^{2\ x})\ ^1(x)$ and $^2(x) = (1 + e^{2\ x})\ ^2(x)$ the terms with first-order derivatives in (12) disappear and $(1 + e^{2\ x})$ factors out, leaving very well known Schrodinger operators acting respectively on $^1(x)$ and $^2(x)$. The key point is that the differential operators in (12) and (13) share the eigenvalues although their eigenfunctions differ by the $^i(x)$ factors. The spectral functions associated are thus identical and it seems wise to use the best known form. What we have shown here is the geometrical meaning of the $^i(x)$ factors: they provide a parallel frame along the kink orbit.

D. The spectrum of small kink fluctuations

Changing from vector to matrix notation,

$$\frac{@}{@\ ^1} \ ! \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \frac{@}{@\ ^2} \ ! \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ;$$

we now use the differential operators of formula (13) to write the linearized field equations satisfied by the small kink fluctuations in the parallel frame:

$$^1(t;x) = \ ^1_K(x) + \ ^1(x)\ K_1(t;x) \quad ; \quad ^2(t;x) = \ ^2_K(x) + \ ^2(x)\ K_2(t;x)$$
$$\frac{@^2\ K_1}{@t^2} + K_{11}\ K_1 = 0 \quad ; \quad \frac{@^2\ K_2}{@t^2} + K_{22}\ K_2 = 0 \quad :$$

Therefore, the eigenfunctions of the differential operator

$$K = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - \frac{2\ ^2}{\cosh^2\ x} & 0 \\ 0 & -\frac{d^2}{dx^2} + \ ^2 - \frac{2\ ^2}{\cosh^2\ x} \end{pmatrix} \qquad (14)$$

provide the general solution of the linearized equations via the separation ansatz: $K_1(t;x) = g_1(t)\ ^1(x)$, $K_2(t;x) = g_2(t)\ ^2(x)$. The eigenvalues and eigenfunctions of K are shown in the following Table:

Eigenvalues	Eigenfunctions	Eigenvalues	Eigenfunctions
$\omega^2_{1-\sigma^2} = 1-\sigma^2$	$\vec{f}_{1-\sigma^2}(x) = \begin{pmatrix} \frac{1}{\cosh^2 \sigma x} \\ 0 \end{pmatrix}$	$\omega^2_0 = 0$	$\vec{f}_0(x) = \begin{pmatrix} 0 \\ \frac{1}{\cosh^2 \sigma x} \end{pmatrix}$
$\omega^2_1(q) = \sigma^2 q^2 + 1$	$\vec{f}^1_q(x) = e^{iq\sigma x}(\tanh \sigma x - iq)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\omega^2_2(k) = \sigma^2(k^2+1)$	$\vec{f}^2_k(x) = e^{ik\sigma x}(\tanh \sigma x - ik)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The spectrum of K_{22} contains a bound state of zero eigenvalue –the translational mode– and a branch of the continuous spectrum, with the threshold at $\omega^2_2(0) = \sigma^2$. SpecK_{11} also is formed by a bound state of positive eigenvalue and a branch of the continuous spectrum starting at $\omega^2_1(0) = 1$. Periodic boundary conditions in the $[-\frac{l}{2}, \frac{l}{2}]$ interval require:

$$q\sigma l + \delta_1(q) = 2\pi n_1 \quad ; \quad k\sigma l + \delta_2(k) = 2\pi n_2 \quad ; \quad n_1, n_2 \in \mathbb{Z} \quad ;$$

such that the phase shifts and the induced spectral densities are:

$$\delta_1(q) = 2\arctan\frac{1}{q} = \delta(q) \quad ; \quad \delta_2(k) = 2\arctan\frac{1}{k} = \delta(k)$$

$$\rho_{K_{11}}(q) = \frac{1}{2\pi}\left(\sigma l + \frac{d\delta_1}{dq}(q)\right) \quad ; \quad \rho_{K_{22}}(k) = \frac{1}{2\pi}\left(\sigma l + \frac{d\delta_2}{dk}(k)\right) \quad . \qquad (15)$$

In sum, K also acts in the Hilbert space $L^2 = \bigoplus_{\alpha=1}^{2} L^2(S^1)$, and its spectral density in the limit of very large radius of the circle is:

$$\rho_K(k) = \begin{pmatrix} \frac{dn_1}{dk} & 0 \\ 0 & \frac{dn_2}{dq} \end{pmatrix} = \frac{1}{2\pi}\left(\sigma l + \frac{d\delta}{dk}(k)\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

V. ONE-LOOP SHIFT TO THE CLASSICAL K KINK MASSES IN THE MASSIVE NON-LINEAR S^2-SIGMA MODEL

A. Zero-point kink energy

The general solution of the linearized field equations governing the small kink fluctuations is:

$$\begin{aligned}
K_1(x_0;x) &= \frac{1}{2}\frac{1}{\sqrt{2\sqrt{1-\sigma^2}}}\left(A_{1-\sigma^2}\, e^{-i\sqrt{1-\sigma^2}x_0} + A^*_{1-\sigma^2}\, e^{i\sqrt{1-\sigma^2}x_0}\right) f_{1-\sigma^2}(x) \\
&+ \frac{1}{2}\sqrt{\frac{1}{\sigma l}}\sum_k \frac{1}{\sqrt{2\omega_1(k)}}\left\{A_1(k)e^{-i\omega_1(k)x_0} f^1_k(x) + A^*_1(k)e^{i\omega_1(k)x_0} f^{1*}_k(x)\right\} \\
K_2(x_0;x) &= \frac{1}{2}\sqrt{\frac{1}{\sigma l}}\sum_k \frac{1}{\sqrt{2\omega_2(k)}}\left\{A_2(k)e^{-i\omega_2(k)x_0} f^2_k(x) + A^*_2(k)e^{i\omega_2(k)x_0} f^{2*}_k(x)\right\} \quad .
\end{aligned}$$

Note that the zero mode is not included because only contribute to quantum corrections at two-loop order. In the orthogonal complement to the kernel of K_{22} in $\bigoplus_{\alpha=1}^{2} L^2(S^1)$, the eigenfunctions of the K operator satisfying PBC form a complete orthonormal system. Therefore, the classical free Hamiltonian

$$\begin{aligned}
H^{(2)} &= E + \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \left\{ \frac{1}{2}\sum_{\alpha=1}^{2}\left[\frac{\partial K_\alpha}{\partial x_0}\frac{\partial K_\alpha}{\partial x_0} + K_\alpha\, K_{\alpha\beta}\, K_\beta\right]\right\} \\
&= E + \frac{1}{2}\left[\sqrt{1-\sigma^2}\left(A^*_{1-\sigma^2}A_{1-\sigma^2} + A_{1-\sigma^2}A^*_{1-\sigma^2}\right) + \sum_k\sum_{\alpha=1}^{2}\left\{\omega_\alpha(k)\left(A^*_\alpha(k)A_\alpha(k) + A_\alpha(k)A^*_\alpha(k)\right)\right\}\right]
\end{aligned}$$

can be written in terms of the normal modes of the system in the quadratic approximation. From this expression, one passes via canonical quantization, $[\hat A_\alpha(k);\hat A^\dagger_\beta(q)]=\delta_{\alpha\beta}\delta_{kq}$, $[\hat A_{1-\gamma^2};\hat A^\dagger_{1-\gamma^2}]=1$, to the quantum free Hamiltonian:

$$\hat H^{(2)} = E + \sqrt{1-\gamma^2}\left(\hat A^\dagger_{1-\gamma^2}\hat A_{1-\gamma^2}+\frac{1}{2}\right) + \sum_k\sum_{\alpha=1}^{2}\varepsilon_\alpha(k)\left(\hat A^\dagger_\alpha(k)\hat A_\alpha(k)+\frac{1}{2}\right) \quad .$$

The kink ground state is a coherent state annihilated by all the destruction operators:

$$\hat A_\alpha(k)|0;K\rangle = \hat A_{1-\gamma^2}|0;K\rangle = 0\,;\forall k\,;\forall\alpha \quad ; \qquad \hat\phi_a(t;x)|0;K\rangle = \phi^K_a(x)|0;K\rangle \quad .$$

The kink ground state energy is:

$$E+\Delta E = \langle 0;K|\hat H^{(2)}|0;K\rangle = 2\pi R^2 + \frac{\hbar}{2}\sqrt{1-\gamma^2} + \frac{\hbar}{2}\sum_k\sum_{\alpha=1}^{2}\varepsilon_\alpha(k) = 2\pi R^2 + \frac{\hbar}{2}\mathrm{Tr}_{L^2}K^{\frac12} \quad . \tag{16}$$

B. Zeta function regularization and Casimir kink energy

Both $\mathrm{Tr}_{L^2}K_0^{\frac12}$ and $\mathrm{Tr}_{L^2}K^{\frac12}$ are ultraviolet divergent quantities: one sums over an infinite number of eigenvalues, and a regularization/renormalization procedure must be implemented to make sense of these formal expressions. We renormalize the zero-point kink energy by subtracting from it the vacuum energy to define the kink Casimir energy:

$$\Delta E^C = \Delta E - \Delta E_0 = \frac{\hbar}{2}\left[\mathrm{Tr}_{L^2}K^{\frac12} - \mathrm{Tr}_{L^2}K_0^{\frac12}\right] \quad .$$

The subtraction of these two divergent quantities is regularized by using the associated generalized zeta functions, i.e., we temporarily assign to ΔE^C the finite value:

$$\Delta E^C(s) = \frac{\hbar\mu}{2}\left(\frac{\mu^2}{m^2}\right)^s\left[\mathrm{Tr}_{L^2}K^{-s} - \mathrm{Tr}_{L^2}K_0^{-s}\right] = \frac{\hbar\mu}{2}\left(\frac{\mu^2}{m^2}\right)^s\left[\zeta_K(s)-\zeta_{K_0}(s)\right]$$

at a regular point of both $\zeta_K(s)$ and $\zeta_{K_0}(s)$. Here,

$$\zeta_K(s) = \sum_{\mathrm{Spec}K}\lambda^{-s} \quad ; \quad \zeta_{K_0}(s) = \sum_{\mathrm{Spec}K_0}\lambda_0^{-s} \quad ; \qquad \lambda\in\mathrm{Spec}K \;;\; \lambda_0\in\mathrm{Spec}K_0 \;;\; s\in\mathbb{C}$$

are the spectral zeta functions of K and K_0, which are meromorphic functions of the complex variable s. An auxiliary parameter μ with dimensions of inverse length is used to keep the physical dimension right and we shall go to the physical limit $\Delta E^C = \lim_{s\to-\frac12}\Delta E^C(s)$ at the end of the process.

C. Partition and generalized zeta functions

Because analytical information about the spectrum of K is only available at the the limit of large l (bound state energies, phase shifts and spectral densities) it is better to consider first the partition or heat functions:

$$\mathrm{Tr}_{L^2}e^{-\beta K_0} = \frac{l}{2\pi}\left[\int_{-\infty}^{\infty}dk\,e^{-\beta(\gamma^2k^2+1)} + \int_{-\infty}^{\infty}dk\,e^{-\beta\sigma^2(k^2+1)}\right] = \frac{l}{\sqrt{4\pi\beta}}\left(e^{-\beta}+e^{-\beta\sigma^2}\right) \quad ; \qquad \beta\in\mathbb{R}^+ \quad .$$

Note that here we have replaced k and q defined in Section §3.1 by γk and σq for a better comparison between the spectra of K_0 and K. The PBC spectral density of K_0 is thus obtained by replacing γ by σ. The K-heat function is also expressed in terms of integrals over the continuous spectrum at the $l=\infty$ limit, rather than infinite sums. The integrals, however, must be weighted with the PBC spectral densities:

$$\begin{aligned}\mathrm{Tr}_{L^2}e^{-\beta K} &= \mathrm{Tr}_{L^2}e^{-\beta K_0} + e^{-\beta(1-\gamma^2)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{d\delta}{dk}\left[e^{-\beta(\gamma^2k^2+1)}+e^{-\beta\sigma^2(k^2+1)}\right] \\ &= \mathrm{Tr}_{L^2}e^{-\beta K_0} + e^{-\beta(1-\gamma^2)}\,\mathrm{Erf}(\sqrt{\beta\gamma^2}) - \mathrm{Erfc}(\sqrt{\beta}) \quad .\end{aligned}$$

The error and complementary error functions of $\beta = \frac{1}{k_B T}$, a fictitious inverse temperature, arise and the asterisk means that we have not included the zero mode because zero modes do not enter the one-loop formula (16).

The generalized zeta functions are Mellin transforms of the heat functions:

$$\zeta_{K_0}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1}\,\mathrm{Tr}_{L^2}\, e^{-\beta K_0} = \frac{l}{2\pi}\int_{-\infty}^{\infty} dk \left\{\frac{1}{[\sigma^2k^2+1]^s} + \frac{1}{[\sigma^2k^2+\sigma^2]^s}\right\} = \frac{l}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(1+\frac{1}{\sigma^{2s-1}}\right)$$

$$\zeta_K(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1}\,\mathrm{Tr}_{L^2}\, e^{-\beta K} = \zeta_{K_0}(s) + \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(s)}\frac{2}{(1-\sigma^2)^{s+\frac{1}{2}}}\,{}_2F_1[\tfrac{1}{2}, s+\tfrac{1}{2}, \tfrac{3}{2}; -\tfrac{\sigma^2}{1-\sigma^2}] - \frac{1}{\sigma^{2s}} \quad .$$

We indeed find meromorphic functions of s with poles and residues determined from the poles and residues of Euler $\Gamma(s)$ and Gauss hypergeometric ${}_2F_1[a,b,c;z]$ functions [22].

In the APPENDIX I we show that the kink Casimir energy in the physical limit $s = -\frac{1}{2}$ is the divergent quantity:

$$\Delta E^C = -\frac{\hbar m}{2\pi}\lim_{\varepsilon\to 0}\left[\frac{2}{\varepsilon} + 2\ln\frac{\mu^2}{m^2} + \ln\frac{16}{\sigma^2(1-\sigma^2)} - 2 + {}_2F_1^{(0,1,0,0)}[\frac{1}{2},0,\frac{3}{2}; -\frac{\sigma^2}{1-\sigma^2}]\right] \quad , \tag{17}$$

where ${}_2F_1^{(0,1,0,0)}[\frac{1}{2},0,\frac{3}{2}; -\frac{\sigma^2}{1-\sigma^2}]$ is the derivative of the Gauss hypergeometric function with respect to the second argument.

D. Zeta function regularization of the self-energy graphs and kink mass renormalization

It remains to take the effect of mass renormalization into account. The contribution to the kink energy due to the mass renormalization counter-terms is:

$$4\Delta E^{MR} = \int dx\, \mathcal{L}_{C.T.}(\psi_1^{K_1}, \psi_2^{K_1}) = \frac{\hbar\lambda^2}{R^2}[I(1)+I(\sigma^2)]\int dx\, \psi_2^{K_1}(x)\psi_2^{K_1}(x) = 2\hbar[I(1)+I(\sigma^2)] \quad .$$

In the normalization interval of length l the integrals become infinite sums

$$I(1) = \frac{\hbar}{2}\int\frac{dk}{2\pi}\frac{1}{\sqrt{\sigma^2k^2+1}} = \frac{1}{2l}\sum_{n=-\infty}^{\infty}\frac{1}{(\sigma^2n^2+1)^{\frac{1}{2}}} \quad ; \quad I(\sigma^2) = \frac{\hbar}{2}\int\frac{dk}{2\pi}\frac{1}{\sqrt{\sigma^2k^2+\sigma^2}} = \frac{1}{2l}\sum_{n=-\infty}^{\infty}\frac{1}{(\sigma^2n^2+\sigma^2)^{\frac{1}{2}}}$$

that can be regularized by using zeta functions:

$$I(1) = \frac{1}{L}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{m^2}\right)^{s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_{K_{011}}(s+1) \quad ; \quad I(\sigma^2) = \frac{1}{L}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{m^2}\right)^{s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_{K_{022}}(s+1) \;;$$

such that [23]:

$$\Delta E^{MR}(s) = \frac{2\hbar m^2}{\sqrt{4\pi}}\left(\frac{\mu^2}{m^2}\right)^{s+1}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\left(1+\frac{1}{\sigma^{2s+1}}\right) \quad .$$

In the APPENDIX I it is proved that the physical limit $s = -\frac{1}{2}$ is also a pole of $\Delta E^{MR}(s)$:

$$\Delta E^{MR} = \frac{\hbar m}{2\pi}\lim_{\varepsilon\to 0}\left[\frac{2}{\varepsilon} + 2\ln\frac{\mu^2}{m^2} + 2(\ln 4 - 2) - \ln\sigma^2\right] \quad . \tag{18}$$

The divergent terms in ΔE^C (17) and ΔE^{MR} (22), as well as the μ-dependent terms, cancel each other exactly and the one-loop K kink mass shift is:

$$\Delta E = \frac{\hbar m}{2\pi}\left[2 + {}_2F_1^{(0,1,0,0)}[\frac{1}{2},0,\frac{3}{2}; -\frac{\sigma^2}{1-\sigma^2}] - \ln(1-\sigma^2)\right] = -\frac{\hbar m}{\pi}\left[2 - \frac{\sqrt{1-\sigma^2}}{\sigma}\arccos\sqrt{1-\sigma^2}\right] \quad . \tag{19}$$

In formula (19) we have also written the result found in our derivation a la Cahill-Comtet-Glauber of the quantum correction, see [9]. The heat kernel/zeta function result is $-\frac{\hbar m}{\pi}f(\sigma)$ whereas the CCH formula leads to $-\frac{\hbar m}{\pi}g(\sigma)$, where

$$f(\sigma) = 1 + \frac{1}{2}{}_2F_1^{(0,1,0,0)}[\frac{1}{2},0,\frac{3}{2}; -\frac{\sigma^2}{1-\sigma^2}] - \frac{1}{2}\ln(1-\sigma^2) \quad ; \quad g(\sigma) = 2 - \frac{\sqrt{1-\sigma^2}}{\sigma}\arccos\sqrt{1-\sigma^2} \quad .$$

Despite appearances, $f(\sigma)$ and $g(\sigma)$ are identical functions of $\sigma \in [0,1]$, as the Mathematica plots in the Figure 1 show. This is remarkable: there is no mention about the analytic identity between the functions $f(\sigma)$ and $g(\sigma)$ in the ample Literature on special functions. Nevertheless, they trace identical curves as functions of σ.

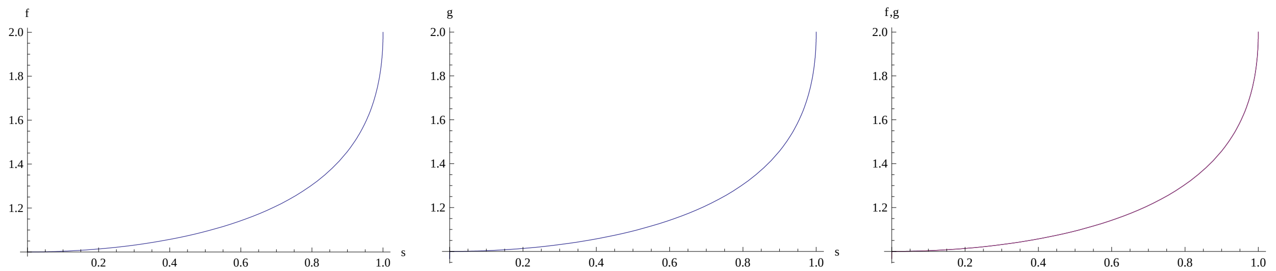


Figure 1: Graphics of $f(\sigma)$ (left), $g(\sigma)$ (center), and $f(\sigma)$ and $g(\sigma)$ plotted together (right). In the Figure, σ is labeled as s in the abscissa axis.

VI. HIGH-TEMPERATURE ASYMPTOTIC EXPANSION

The exact heat or partition function can be written in the form:

$$\mathrm{Tr}_{L^2} e^{-\beta K} = \mathrm{tr}\begin{pmatrix} \mathrm{Tr}_{L^2_1} e^{-\beta K_{11}} & 0 \\ 0 & \mathrm{Tr}_{L^2_2} e^{-\beta K_{22}} \end{pmatrix} = \left(\frac{1}{\sqrt{4\pi\beta}} + e^{\beta\sigma^2}\,\mathrm{Erf}[\sqrt{\beta}\,\sigma]\right)\mathrm{tr}\begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta\sigma^2} \end{pmatrix};$$

where "tr" means trace in the matrix sense. There is an alternative way of computing this quantity by means of a high-temperature asymptotic expansion. Although we have the exact formula in our system, we shall also perform the approximate calculation, which is the only one possible in other systems in order to gain control of this second approach in this favorable case.

In the APPENDIX II it is shown how the coefficients of the power expansion of the K-heat trace

$$\mathrm{Tr}\, e^{-\beta K} = \frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty} c_n(K)\,\beta^{n-\frac{1}{2}}\,\mathrm{tr}\begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta\sigma^2} \end{pmatrix}; \qquad (20)$$

the Seeley coefficients $c_n(K)$, are obtained through integration of the Seeley densities over the whole line. The densities satisfy recurrence relations tantamount to the heat kernel equation starting from a general potential $U(x)$. In our problem we must solve the recurrence relations between these densities for the potential $U(x) = -\frac{2\sigma^2}{\cosh^2\sigma x}$, essentially the same potential as for the sine-Gordon kink, see [11]. We list these coefficients up to the twentieth order in Table IV:

n	$c_n(K)/\sigma^{2n-1}$
1	4.
2	2.66667
3	1.06667
4	0.304762
5	0.0677249
6	0.0123136
7	1.8944×10^{-3}
8	2.52587×10^{-4}
9	2.97161×10^{-5}
10	3.12801×10^{-6}
11	2.97906×10^{-7}
12	2.59049×10^{-8}
13	2.072239×10^{-9}
14	1.5351×10^{-10}
15	1.05869×10^{-11}
16	6.83027×10^{-13}
17	4.13956×10^{-14}
18	2.36546×10^{-15}
19	1.27863×10^{-16}
20	6.55706×10^{-18}

Table IV: Seeley Coefficients

Write now the spectral zeta functions in the form:

$$\begin{aligned}\zeta_{K_0}(s) &= \zeta_{K_0}(s;b) + B_{K_0}(s;b) \\ &= \frac{1}{\Gamma(s)}\frac{1}{\sqrt{4\pi}}\left[\mathrm{tr}\int_0^b d\beta\,\beta^{s-\frac{3}{2}}\begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta\sigma^2} \end{pmatrix} + \mathrm{tr}\int_b^\infty d\beta\,\beta^{s-\frac{3}{2}}\begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta\sigma^2} \end{pmatrix}\right] \\ &= \frac{1}{\Gamma(s)}\frac{1}{\sqrt{4\pi}}\left[\mathrm{tr}\begin{pmatrix} \gamma[s-\frac{1}{2};b] & 0 \\ 0 & \frac{1}{\sigma^{2s-1}}\gamma[s-\frac{1}{2};\sigma^2 b] \end{pmatrix} + \mathrm{tr}\begin{pmatrix} \Gamma[s-\frac{1}{2};b] & 0 \\ 0 & \frac{1}{\sigma^{2s-1}}\Gamma[s-\frac{1}{2};\sigma^2 b] \end{pmatrix}\right]\end{aligned}$$

$$\zeta_K(s) = \gamma_K(s;b) + B_K(s;b)$$
$$= \frac{1}{\Gamma(s)\sqrt{4\pi}}\sum_{n=0}^{\infty} c_n(K)\,\mathrm{tr}\int_0^b d\beta\,\beta^{s+n-\frac{3}{2}}\begin{pmatrix} e^{-\beta} & 0\\ 0 & e^{-\beta\nu^2}\end{pmatrix} + \frac{1}{\Gamma(s)}\int_b^{\infty} d\beta\,\beta^{s-1}\,\mathrm{Tr}_{L^2}\, e^{-\beta K}$$
$$= \frac{1}{\Gamma(s)\sqrt{4\pi}}\sum_{n=0}^{\infty} c_n(K)\,\mathrm{tr}\begin{pmatrix}\gamma[s+n-\frac{1}{2};b] & 0\\ 0 & \nu^{-2(s+n)}\gamma[s+n-\frac{1}{2};\nu^2 b]\end{pmatrix} + \frac{1}{\Gamma(s)}\int_b^{\infty} d\beta\,\beta^{s-1}\,\mathrm{Tr}_{L^2}\, e^{-\beta K}:$$

The incomplete Euler Gamma functions $\gamma[z;a]$ are meromorphic functions of z whereas $B_{K_0}(s;b)$ and $B_K(s;b)$ are entire functions of s. The splitting point of the Mellin transform is usually taken at $b = 1$. We leave b as a free parameter for reasons to be explained later.

Neglecting the entire parts, the zero-point energy renormalization

$$\gamma_K(s;b) - \gamma_{K_0}(s;b) = \frac{1}{\Gamma(s)\sqrt{4\pi}}\sum_{n=1}^{\infty} c_n(K)\,\mathrm{tr}\begin{pmatrix}\gamma[s+n-\frac{1}{2};b] & 0\\ 0 & \nu^{-2(s+n)}\gamma[s+n-\frac{1}{2};\nu^2 b]\end{pmatrix}$$

gets rid of the $c_0(K)$ term. The contribution of $c_1(K)$

$$\triangle E^C_{(1)} = \frac{1}{\sqrt{\pi}}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{2}\right)^{s}\frac{\hbar}{\Gamma(s)}\,\mathrm{tr}\begin{pmatrix}\gamma[s+\frac{1}{2};b] & 0\\ 0 & \nu^{-1-2s}\gamma[s+\frac{1}{2};\nu^2 b]\end{pmatrix}$$

is exactly canceled by the mass renormalization counter-terms:

$$\triangle E^{MR} = -\frac{1}{\pi}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{2}\right)^{s+1}\frac{2\hbar}{\Gamma(s)}\,\mathrm{tr}\begin{pmatrix}\gamma[s+\frac{1}{2};b] & 0\\ 0 & \nu^{-2s+1}\gamma[s+\frac{1}{2};\nu^2 b]\end{pmatrix}:$$

We must now subtract the contribution of the zero mode:

$$\zeta^*_K(s;b) = \gamma_K(s;b) - \frac{1}{\Gamma(s)}\lim_{\varepsilon\to 0}\int_0^b d\beta\,\beta^{s-1}e^{-\beta\varepsilon}$$
$$= \gamma_K(s;b) - \frac{1}{\Gamma(s)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^s}\gamma[s;\varepsilon b] = \gamma_K(s;b) - \frac{b^s}{s\,\Gamma(s)}$$

Finally, the high-temperature one-loop correction to the K kink energy is:

$$\triangle E(b) = \frac{\hbar}{2}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{2}\right)^{s}\frac{1}{\Gamma(s)}\left(\frac{1}{\sqrt{4\pi}}\sum_{n=2}^{\infty} c_n(K)\,\mathrm{tr}\begin{pmatrix}\gamma[s+n-\frac{1}{2};b] & 0\\ 0 & \nu^{-2(s+n)}\gamma[s+n-\frac{1}{2};\nu^2 b]\end{pmatrix} - \frac{b^s}{s}\right):$$

In practice, truncation of the series is also necessary:

$$\triangle E(b;N_0) = -\frac{\hbar\mu}{\sqrt{4\pi}}\left[\frac{2}{\sqrt{b}} + \frac{1}{\sqrt{4\pi}}\sum_{n=2}^{N_0} c_n(K)\,\mathrm{tr}\begin{pmatrix}\gamma[n-1;b] & 0\\ 0 & \nu^{2-2n}\gamma[n-1;\nu^2 b]\end{pmatrix}\right]: \qquad (21)$$

Using formula (21) to calculate the one-loop kink mass shift, we admit an error of:

$$\triangle E - \triangle E(b;N_0) = \frac{\hbar}{2^3\nu\pi}\left(\sum_{n=2}^{N_0} c_n(K)\left(\gamma[n-1;b] + \frac{\gamma[n-1;\nu^2 b]}{\nu^{2(n-1)}}\right) + \sum_{n=N_0+1}^{\infty} c_n(K)\,\Gamma(n-1)\left(1+\frac{1}{\nu^{2(n-1)}}\right)\right):$$

We offer a Figure where formula (21) has been applied for $N_0 = 20$ and several values of ν. The very good precision of the asymptotic formula was achieved by adapting the parameter b to the value of ν. For instance, we have taken $b = 1000$ for $\nu = 0.1$, $b = 100$ for $\nu = 0.3$, $b = 50$ for $\nu = 0.5$, $b = 20$ for $\nu = 0.7$, $b = 10$ for $\nu = 0.9$, and $b = 10$ for $\nu = 1$. Physically, this means that the lighter the particle mass (ν^2) is, the longer the integration interval in the Mellin transform must be taken to minimize the error produced by the neglected entire parts. In practice, we have chosen b in each case at the frontier near the point $\beta_0 \in (0;1]$, where the asymptotic formula of the K-heat trace departs from its exact value.

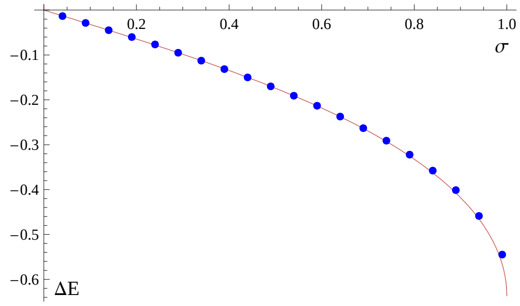


Figure 2: Points obtained using the asymptotic expansions for several values of plotted on the exact curve giving the one-loop mass shift as a function of .

VII. CONCLUSIONS AND FURTHER COMMENTS

In sum, we may draw the following conclusions:

1. We have obtained the one-loop mass shift to the classical mass of the stable topological kink that exists in a massive anisotropic non-linear S^2-sigma model.
2. In the isotropic case, = 1, our result agrees with the answer provided by other authors: the one-loop correction is twice (in modulus) the correction for the sine-Gordon kink, see [3] and [15].
3. Our procedure is based on the heat kernel/zeta function regularization method. The result is identical to the answer achieved by means of the Cahill-Comtet-Glauber formula.

   This is a remarkable fact: the CCH formula takes into account only the bound state eigenvalues and the thresholds to the two branches of the continuous spectrum of the Schrodinger operators that govern the eld small uctuations. It is essentially nite. Our computation involves in nite renormalizations. The criterion chosen to set nite renormalizations -no modi cation of the particle masses at the one-loop level, equivalent to the vanishing tadpole criterion in linear sigma models- does exactly the job.
4. We have also derived a high-temperature approximated formula for the mass shift, relying on the heat kernel asymptotic expansion. We stress that we have improved a former weakness of our method. The approximation to the exact result was poor for light masses -non-dimensional mass < 1- in the model studied in [9]. We have achieved a very good approximation in this paper even for light particles by enlarging the integration interval of the Mellin transform and considering an optimum number of Seeley coe cients. We believe that this is a general procedure, working also in models where the exact generalized zeta function is not available.

As a nal comment, we look forward to addressing the quantization procedure for: (a) Multi-solitons and breather modes of this model. (b) Stable topological kinks that may arise in other massive non-linear sigma models with di erent potentials, e.g., quartic, and/or di erent target manifolds, e.g., S^3.

VIII. ACKNOWLEDGEMENTS

We thank the Spanish Ministerio de Educacion y Ciencia and Junta de Castilla y Leon for partial nancial support under grants FIS2006-09417, GR224, and SA034A08. JMG thanks the ESF Research Network CASIMIR for providing excellent opportunities on the Casimir e ect and related topics like topological defect uctuations.

APPENDIX I: Kink Casimir energy and mass renormalization near the pole

The Casimir kink energy is, see Section x. V.C:

$$E^C = \lim_{s\to -\frac{1}{2}} E^C(s) = \lim_{s\to -\frac{1}{2}} \frac{}{2\sqrt{}} \frac{^2}{^2}^{s} \frac{(s+\frac{1}{2})}{(s)} \frac{2}{(1-^2)^{s+\frac{1}{2}}}\, {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; \frac{^2}{1-^2}] \frac{1}{s\,^{2s}}$$

$$= \frac{}{2\sqrt{}} \lim_{\varepsilon\to 0} \frac{^2}{^2}^{\varepsilon} \frac{(\varepsilon)}{(-\frac{1}{2}+\varepsilon)} \frac{2}{(1-^2)^{\varepsilon}}\, {}_2F_1[\tfrac{1}{2}; \varepsilon; \tfrac{3}{2}; \frac{^2}{1-^2}] \frac{1}{(-\frac{1}{2}+\varepsilon)\,^{2\varepsilon}} \quad ;$$

but $s = -\frac{1}{2}$ is a pole of $E^{C}(s)$. To find the residue, we expand this function in the neighborhood of the pole by using the following results

$$\frac{^2}{_2}^{\varepsilon}\frac{(\varepsilon)}{(-\frac{1}{2}+\varepsilon)} \simeq \frac{1}{2\sqrt{\ }}\left[\frac{1}{\varepsilon}+\ln\frac{^2}{_2}+\ln 4-2\right] \quad ; \quad \frac{1}{-\frac{1}{2}+\varepsilon}\frac{1}{2\varepsilon} \simeq -2-\varepsilon(4-2\ln{}^2)$$

$$\frac{2}{(1-{}^2)^{\varepsilon}}\,{}_2F_1[\tfrac{1}{2};\varepsilon;\tfrac{3}{2};\tfrac{^2}{1-{}^2}] \simeq 2-2\varepsilon\left(\ln(1-{}^2)-{}_2F_1^{(0;1;0;0)}[\frac{1}{2};0;\frac{3}{2};-\frac{^2}{1-{}^2}]\right)$$

where ${}_2F_1^{(0;1;0;0)}[\frac{1}{2};0;\frac{3}{2};-\frac{^2}{1-{}^2}]$ is the derivative of the Gauss hypergeometric function with respect to the second argument and we made use of the fact that ${}_2F_1[\frac{1}{2};0;\frac{3}{2};-\frac{^2}{1-{}^2}] = 1$.

The physical limit $s = -\frac{1}{2}$ is also a pole of $E^{MR}(s)$, see Section x.V.D:

$$E^{MR} = -\frac{2}{\sqrt{\ }4}\lim_{\varepsilon\to 0}\frac{^2}{_2}^{\varepsilon}\frac{(\varepsilon)}{(-\frac{1}{2}+\varepsilon)}\left[-\frac{1}{2\varepsilon}+1\right] = \frac{}{2}\lim_{\varepsilon\to 0}\left[1+\varepsilon\ln\frac{^2}{_2}\right]\left[\frac{1}{\varepsilon}+(1)-1-\varepsilon(\frac{1}{2})\right](2-\varepsilon\ln{}^2)$$

$$= \frac{}{2}\left[\lim_{\varepsilon\to 0}\frac{2}{\varepsilon}+2\ln\frac{^2}{_2}+2(\ln 4-2)-\ln{}^2\right] \quad : \qquad (22)$$

APPENDIX II: The heat kernel expansion

Consider the K_0- and K-heat kernels:

$$\left(\frac{\partial}{\partial}+K_0\right)K_{K_0}(x;y;\ ) = 0 \quad ; \quad K_{K_0}(x;y;0) = (x-y)$$

$$\left(\frac{\partial}{\partial}+K\right)K_{K}(x;y;\ ) = 0 \quad ; \quad K_{K}(x;y;0) = (x-y) \quad ; \qquad (23)$$

which provide an alternative way of writing the K_0- and K-heat traces:

$$\mathrm{Tr}_{L^2}e^{-K_0} = \lim_{l\to\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}}dx\,K_{K_0}(x;x;\ ) \quad ; \quad \mathrm{Tr}_{L^2}e^{-K} = \lim_{l\to\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}}dx\,K_{K}(x;x;\ ) \quad :$$

Note that the form of the K-heat equation (23), $\left(\frac{\partial}{\partial}+K_0-U(x)\right)K_K(x;y;\ ) = 0$, suggests a solution based on the K_0-heat kernel: $K_K(x;y;\ ) = C_K(x;y;\ )K_{K_0}(x;y;\ )$. The density $C_K(x;y;\ )$ satisfies the infinite temperature condition $C_K(x;y;0) = I_{N\times N}$ and the transfer equation:

$$\left(\frac{\partial}{\partial}+\frac{x-y}{}\frac{\partial}{\partial x}-\frac{\partial^2}{\partial x^2}\right)C_K(x;y;\ ) = -U(x)C_K(x;y;\ ) \quad : \qquad (24)$$

Next we seek a power series solution, $C_K(x;y;\ ) = \sum_{n=0}^{\infty}c_n(x;y)\ ^n$, of (24), which becomes tantamount to the recurrence relations:

$$nc_n(x;y)+(x-y)\frac{\partial c_n}{\partial x}(x;y) = \frac{\partial^2 c_{n-1}}{\partial x^2}(x;y)+U(x)c_{n-1}(x;y) \quad : \qquad (25)$$

In fact, only the densities at coincident points $x = y$ on the line are needed. We introduce the notation ${}^{(k)}C_n(x) = \lim_{x\to y}\frac{\partial^k c_n}{\partial x^k}(x;y)$ to write the recurrence relations for the Seeley densities (and their derivatives) in the abbreviated form:

$$^{(k)}C_n(x) = \frac{1}{n+k}\left[{}^{(k+2)}C_{n-1}(x)-\sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j U(x)}{\partial x^j}\,{}^{(k-j)}C_{n-1}(x)\right] \quad :$$

The (Seeley) coefficients $c_n(K)$ are the integrals over the infinite line of the densities $c_n(x;x)$, i.e., $c_n(K) = \int_{-\infty}^{\infty}dx\,c_n(x;x)$.

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[21] Without loss of generality, we choose the parameters such that: $^2_1$ $^2_2 >$ $^2_3$ 0.

[22] Strictly speaking, Mellin transforms are de ned in their fundamental strips, respectively Res > 1=2, Res > 0 in our problems. In the spirit of zeta function regularization, we extend the results of the Mellin transforms to the whole complex s-plane by analytic continuation.

[23] The di erential operators K_{011} and K_{022} are de ned in Page 4.