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ALGORITHMIC ERRORS. COGNITIVE PROCESSES AND EDUCATIONAL ACTIONS

Errores algorítmicos. Procesos cognitivos y acciones educativas

Erreurs algorithmiques. Processus cognitifs et actions éducatives

Ana B. Sánchez García
College of Arts & Sciences. Appalachian State University. 520 Sanford Hall, ASU Box 32063. 225 Locust Street. Boone, NC 28608-2063-USA. Correo-e: asg@usal.es

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RESUMEN

En este trabajo definimos el espacio cognitivo de la sustracción e incidimos en el control del procedimiento y en los procesos que tiene que potenciar el marco educativo para su correcta adquisición. Describimos la teoría que subyace a la adquisición del error. Para ello, tomamos como referencia el análisis de los procesos de transferencia negativa inducidos desde el contexto educativo. El análisis se inscribe en la intersección entre la teoría de la educación y las teorías cognitivas sobre el aprendizaje algorítmico.

Palabras clave: aprendizaje algorítmico, error procedimental, errores en la sustracción, transferencia negativa, procesos cognitivos, mediación comunicacional.

SUMMARY

In this paper we define the cognitive space of subtraction and place emphasis on procedural control and on the processes that need to be improved by the educational framework for proper acquisition. We describe the theory behind error acquisition. To do this, we consider the analysis of negative transfer processes induced from the educational context. The analysis is inscribed within the intersection between educational theory and cognitive theories of algorithmic learning.

Key words: algorithmic learning, procedural error, errors in subtraction, negative transfer, cognitive processes, communicational mediation.

SOMMAIRE

Dans cet article, on définie l'espace cognitif de la soustraction en mettant l'accent sur le contrôle des mécanismes pour son correcte acquisition et sur les différents processus devant être mis en place par le cadre éducatif. La théorie sous-jacente à l'acquisition de l'erreur y est décrite. Pour ce faire, on tient compte de l'analyse des processus de transfert négatif induits par le contexte éducatif. L'analyse s'inscrit au point de rencontre de la théorie de l'éducation et des théories cognitives sur l'apprentissage algorithmique.

Mots clés: apprentissage algorithmique, erreur de procédure, erreurs dans la soustraction, transfert négatif, processus cognitifs, médiation communicative.

Our aim in this paper is to approach the cognitive processes and educational actions involved in the generation of algorithmic errors during the teaching-learning process. To do so we analyze the processes induced from the pedagogical contexts that are the cause of these errors as well as the conceptual, procedural and attitudinal knowledge that are at the basis of these errors. As a corollary to these analyses, we study the types of knowledge and styles of teaching-learning that the educator should foster so that these algorithmic structures can be acquired by students without interference by errors.

With this goal in mind, the present article[1] takes as its starting point the theories that in our view have contributed the most to the interpretation and analysis of these processes. It was Resnick and Omanson (1987) who established some of the postulates that marked the beginning of a solid theory based on the basic principles necessary for instruction that would limit or even eliminate the production of

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errors in algorithm acquisition. Our contribution is thus situated within the field of Theory of Education linked to the study of the processes involved in algorithm learning, and our ultimate aim is to understand the processes involved in learning subtraction. That is, our reflection is located in the crossroads where theory of education, algorithmic education and cognitive psychology meet. It is within this educational space that, as pointed out by García Carrasco and García del Dujo (2001), a solid theory of education should describe the processes involved in the understanding of these algorithms and, fundamentally, that type of «comprehension that is related to the adequate understanding of the rules underlying (the algorithms) and to a correct identification of the components of the situation as the information to which the rule must be applied» (García Carrasco and García del Dujo, 2001, 262). However, until we can reach that understanding, we need to answer questions such as the following: What kind of results or performance outcomes do we obtain when instruction is based on getting the learner to memorize the algorithmic processes? And with such attitudes how do students extrapolate the data and procedures that they already know and are part of their mental architecture? What *types* of learning should the pedagogical context offer when teaching arithmetic? What kind of pedagogical context facilitates and sets in place the best scaffolding for the process of acquiring such a skill? In order to answer these questions, the present study comprises an analysis of the educational processes that determine learning in the area of arithmetic and the comprehension and of concepts and procedures that sustain the errors made in a specific educational space.

We assume that algorithmic learning processes are socially mediated and exist in an area of social construction in which communicational mediators that favor the transfer processes intervene either negatively or positively. The central focus of this study is thus this communicational mediation in which both teachers and learners take part, together with other resources that support this learning process, such as textbooks.

In this context, the model we present is the conceptual representation of the real processes that take place during algorithmic learning. Specifically, we take the subtraction algorithm as a conceptual simulation that, the same as any other learning process, is socially mediated or inserted in that space of social construction in which communicational mediators intervene and from whom the leaner internalizes, assimilates and apprehends algorithmic systems. Thus, the Vygostkian contribution to the nature of communicational mediation will not be out of place with respect to the origin of errors, as we shall gradually show in the following sections. In short, we are attempting to find the conceptual representation of the genesis of algorithmic error, starting from the idea that communicational mediation is its possible point of origin.

1. Errors relating to the subtraction algorithm

In an attempt to find models for the conceptual representation of errors, in this section we review the pertinent literature. Analysis of this literature reveals the existence of different research lines that have approached this topic. First of all, we have the Repair Theory formulated by VanLenh, which highlights the importance of the procedural mechanisms that govern the algorithm as the central hub of error generation (Brown and Burton, 1978; Brown and VanLehn, 1982; VanLehn, 1982, 1983, 1986, 1987, 1990; VanLehn and Brown, 1980; Young and O'Shea, 1981). These authors essentially present the learning process of the procedures involved in the subtraction algorithm and, based on this process, give a plausible answer to why learners make mistakes. In their theoretical formulation they defend the idea that students learn the procedures through mechanisms of induction and that the errors committed stem from the examples the teacher gives in class. They thus position themselves on the side of the importance of communicational mediators, in this case the teacher's examples, as the source of errors.

A second theoretical approach can be found in research studies that consider that a lack of knowledge of the concepts and principles underlying the learning of the procedure is basic to understanding the errors committed by learners (Fuson, 1986, 1992; Fuson and Briars, 1990; Hiebert and Lefevre, 1986; Hiebert and Warne, 1996; Ohlsson and Rees, 1991; Resnick, 1982, 1983; Resnick and Omanson, 1987; Steffe and Cobb, 1988). These authors evaluate the relation between conceptual understanding and mathematical skill during instruction (Baroody and Ginsburg, 1986; Cockbourn and Littler, 2008; Hiebert and Wearne, 1996; Kamii, 1985).

The third approach, in our view too important to overlook, is Fischbein's theory, which focuses attention on the role played by intuitive models in the acquisition of algorithmic knowledge (Fichbein, 1987, 1994, 1999). «According to Fischbein, intuitive knowledge is a type of immediate, implicit, and self-evident cognition that leads in a coercive manner to generalization» (Tirosh and Tsamir, 2004, 537). It likewise considers the interaction among three components of mathematical knowledge –intuitive, formal and algorithmic, «the formal aspect refers to axioms, definitions, theorems and demonstrations. The algorithmic aspect refers to solving techniques and standard strategies. Finally, the intuitive aspect refers to the degree of subjective and direct acceptance by an individual of a notion, theorem, or solution» (Fischbein, 1994, 244)– organized in tacit models that Fischbein (1987) describes as imperfect mediators which lead to incorrect interpretations of the algorithm. In this sense, analogies play an important role in the construction of models. They are a font of resources that feed the model and can occasionally be responsible for false conceptions (Fischbein, 1987).

Thus, sometimes during algorithmic learning a conflict takes place between the intuitive aspect and the formal interpretation of the procedure. At the educational level, then, algorithmic error can be predicted by this incorrect interpretation of the procedure; «in this case, the teacher has to identify the intuitive tendencies of the student and try to explain their resources» (Fischbein, 1999, 24). In the same vein, Tirosh, Tsamir and Hershkovitz (2008) affirm that errors made by students in subtraction can be explained by the influence of a series of intuitive rules. This affirmation comes from the Theory of Intuitive Rules (Stavy and Tirosh, 1996, 2000; Tsamir, 2005; Tirosh and Stavy, 1996, 1999), which explains and predicts unsuitable answers to a wide variety of tasks in mathematics. With this theory they confirm that many learners react by giving similar answers in tasks that may share a component or trait in common but which are conceptionally unrelated. They conclude that many of these inadequate answers can be explained by the influence of three rules: «More A – More B», «Same A – Same B» and «Everything can be divided». They believe that these rules are intuitive, because the learners feel as if their explanations are evident and sufficient (Stavy *et al*., 2006). Likewise, they possess attributes of universality because the students tend to repeat them in different situations, and they are coercive, since the alternatives are often excluded as unacceptable (Stavy *et al*., 2006).

Tirosh and Stavy report (1999) that the answers students gave on a variety of tasks of comparative mathematics in which the two objects or systems being compared were equal as regards one quality or quantity (A1 = A2), but different in regard to another (B1 = B2), were influenced by the intuitive rule «Same A – Same B». In some of the tasks, the equality in quantity is directly perceived; in other cases, it could be deduced logically. An incorrect answer common to all the tasks was $B_1 = B_2$; because $A_1 = A_2$ (Tirosh and Stavy, 1999, 62). This rule could be activated by perceptual or logical mechanisms and these authors suggest it could be innate, since it is reasonable to assume that these learners generalize their experiences in a universal maxim «Same A – Same B», as this rule is often applied in different situations in the educational context, a fact that promotes its generalization.

In light of this theory, we suggest that algorithmic reasoning is affected by intuitive rules (Sánchez, 2012). More specifically, we believe that the «Same A – Same B» rule plays a major role in this affirmation. In research outcomes (Sánchez, 2012), we refer explicitly to this rule when we analyze the incorrect generalizations made by children in relation to understanding the concept of zero as an empty set, overlooking its positional value. Other authors have also informed about the intuitive components of algorithmic knowledge: Baroody (1988); Gelman and Gallistel (1978); Huttenlocher *et al.* (1994); Resnick (1987); and Sander (2001). We thus formulate the following question: what type of transference takes place when a student generalizes an intuitive rule acquired in a context in which communicational mediators are key, as is the case in the elementary teaching of algorithms? Or to be more precise, what are the cognitive processes that sustain the transfer of mathematical knowledge?

According to Hiebert and Lefevre (1986), transfer processes, and generalization in particular, facilitate the relation between conceptual and procedural knowledge. In this context of relation between the two types of knowledge it is common for teachers to use a textbook to teach algorithms. This can interfere with the child's ability to make generalizations. And like VanLehn (1990), we believe that it is not only the textbook examples that can interfere with generalization processes, but also the examples given by the teacher or classmates or other persons close to the student. Again it is the communicational mediators available for the learning process, be they textbooks or speech that can interfere in the algorithmic learning process and become an unsuitable resource that gives rise to errors. These resources that feed the algorithmic transfer process can be induced in educational contexts. That is, intuitions are especially sensitive to the influence of the context (Fischbein, 1999; Tirosh and Stavy, 1999).

We must therefore not ignore the methods and language used in algorithmic learning situations, given that they have a decisive influence on the acquisition of the skill in question. In this regard there is a collection of research studies in the field of mathematics education that approaches the epistemological structure of the understanding of mathematical knowledge. Contributions such as that by Bromme and Steinbring (1994) and Steinbring (1997, 2006) assume that the quality of classroom interaction is an essential component in the initial comprehension of mathematical concepts. We believe that instruction plays a role in the generation of errors by means of memorization processes highly influenced by a type of teaching that does not foster the conceptual understanding of the algorithm (Baroody, 1988; Fuson, 1986, 1992; Kamii, 1985; Resnick, 1982, 1983; Resnick and Omanson, 1987). Thus, classroom interaction processes determine the choice of resources that in turn determine the understanding of the skill (Bromme and Steinbring, 1994). Like Fischbein (1987) and Sander (2001), we maintain that the analogical transfer resources used by learners are not adequate. In other words, the behavior shown by children is characterized by a failure to comprehend the formal component of multidigit subtraction and also by the intuitive interpretation of this component (Fischbein, 1999) that results in the generation of errors based on analogical transfer processes. The student executes one by one all the elements that make up the set of arithmetic skills basing herself on the superficial characteristics of the examples used during instruction. These are then transferred to new subtractions with different structures from the ones learned initially, which were useful as example prototypes during instruction and are essentially configured as basic resources of transference under the influence of the template rule «Same A – Same B», mentioned above.

In summary, taking into account the contributions of previous research, in the following sections we attempt to analyze the conceptual model of error and locate the learning principles and steps that the educational framework should promote for the correct acquisition of the algorithm. To do so, we first present the formal architecture of the algorithm, to then establish the formal and algorithmic knowledge that should be fostered in education. Finally, we end with the description of the different stages of the conceptual model of error based on the negative analogical transfer and the importance that communicational mediators imprint on that process.

2. Procedural architecture of the subtraction algorithm

Olhsson and Rees (1991) constructed a cognitive functioning simulation model of learning the subtraction algorithm. They represent the steps of procedural acquisition of the conceptual structures from the beginning of learning the number system, or how to count, a prior step to the comprehensive learning of the subtraction algorithm. In their research they insist that it is conceptual knowledge that provides the codes needed for learning the procedure, stressing the semantics of the principles governing the objects and operations in this arithmetic domain. They need to know the justification under which the procedure operates, that is, its syntax. In their attempt to specify the levels of procedural acquisition of the algorithm, they construct a syntax of the procedure and provide it with a language and a space. This model is composed of types of entities, properties, and the relations that are established among the elements comprising the procedure. Subsequently, they define the procedural space of numerical or counting acquisition. And finally, they describe the rules governing the process.

Olhsson and Rees (1991) analyze the architecture of the procedure by differentiating conceptual knowledge from procedural knowledge. The former would be knowledge of objects and how they relate to the operations within the procedure, but analysis of this type of knowledge falls outside the scope of this paper, which is devoted to the acquisition of procedural knowledge.

Procedural knowledge refers to the task to be done, which comprises an initial situation with certain goals, some operators that act on objects, and a result. It is a kind of automatic compilation of conceptual knowledge which contributes information relating to the attainment of goals. Taking the contributions of these authors as a reference, we present below the procedural space of subtraction from our point of view.

Once we have presented the elements that comprise the conceptual understanding and the relations among them, as well as the space where the actions are placed to reach the goal or solve the algorithm, we can represent the learning process taking into account the different stages that comprise it. A child will successfully solve the task only if she applies her previous conceptual understanding to the context of the algorithm, and therefore she must think of each of the stages of the algorithm in terms of rules that govern how the task should be carried out. Hence, she must supervise each action in relation to the arithmetic principles that govern the execution of multi-column subtraction. If this does not take place, errors can be generated that would constitute a kind of «violation» (Resnick, 1987) or transgression of these principles.

A positive change in the way the algorithm is taught would be dominated by insisting that the child have a solid acquisition of the comprehensive relations between the actions and the contextual structure of the algorithm, in other words, strengthening of the concepts and their relations in the execution of the procedure, or what Ohlson and Rees (1991) call *knowledge of the domain in which the* *procedure operates*. Further, the learner has to know the actions through which such a procedure operates. However, for the learner to acquire the knowledge of the domain in which the procedure operates, the educational context must foster certain processes that we present below.

FIGURE 1. PROCEDURAL SPACE OF MULTI-COLUMN SUBTRACTION

INITIAL STATE COMPRISED OF

Elements that make up the problem: numbers, columns, and rows.

Initial conceptual background: «left», «right», «before», «underneath», being in (a row or column), etc. «the current column» «minuend, subtrahend and difference». Principles governing the counting procedure and the rules that allow these concepts to be related when executing the algorithm.

RULES

Declare the first column on the right as the first column to be acted on.

Next column: also at the beginning, take it as a reference.

Decrement (number, column), subtract.

Add 10 (number, column).

Find the difference (number 1, number 2, column). Write the value.

Find the top part-(number, column).

Skip the zero (column).

Change to the left (column).

Assign value

OPERATORS

First column: Take one column to start with and declare that it is the first column.

Next column: Take into account as a reference also at the beginning and pay simultaneous attention to the relations between the first column and the next one.

Decrement: (number, column). Take the number in the top row of the column and replace it with the result of subtracting the number of the subtrahend from it. Place it in the row of the difference and remember if a regrouping (borrowing) has taken place.

Increase: Start out at the position that was increased during regrouping; write the new value for the increased digit and remember that the increase has taken place.

Find the difference: specify the value of the difference by writing it in the difference row.

Skip the zero (column): Take a zero in the top row of a column and decide to write nothing in the difference row of that column.

Mark the regrouping column: Take the column of regrouping and mark it.

Move to the left (column): Take one column to start with and then take again the column immediately to its left, designating it as an increased column.

Write the number.

CONCLUSION STAGE

The algorithm has been executed when the answer is placed in all the columns comprising the algorithm.

3. What processes should the educational framework promote for correct acquisition and control of the structure of the subtraction algorithm?

Three consecutive stages interact in the acquisition of the algorithm. In the *initial stage* of learning, children try to apprehend the conceptual or domain knowledge without applying it. It is a stage that forms part of the strengthening process of the conceptual framework of numbers and the actions that can be taken with them. In the *intermediate stage*, children try to solve the algorithm, and to do so they often take as a reference the examples shown by the teacher, or the ones that come in their textbook. At this stage, they now have knowledge of the domain that can be applied, but they have yet to acquire the whole conceptual frame necessary. It is a stage that could be called heuristic, because it is based on doing an experimental search in order to strengthen performance of the procedure.

At this point, it is very important at the instructive level to teach the principles underlying subtraction, which are taught through the use of examples. The students use analogies as a means of learning. Applying the previous principle or example facilitates the process of recalling it, by making each of the parts of the example correspond to each part of the problem. Once this stage is reached, learners generalize the knowledge, modifying their understanding of the example, in turn allowing them to apply it to more problems.

As pointed out earlier, mastering the algorithm requires learning not only one principle, but several. Thus, if the task requires multiple principles, we need to attain not only generalization, but also transfer of learning to other situations that may seem similar but are not.

Transfer in learning is an adaptive system characterized by a kind of economy in task learning, which is the result of prior training in a different task. It consists of using knowledge in circumstances different from the ones in which it was acquired (Sánchez and López, 2011). For Hiebert and Lefevre (1986), transfer makes the connection between conceptual knowledge and procedural knowledge. This relation is what facilitates the effective use of the procedure.

The *third stage* is based on continued practice, which provides speed in the execution of the algorithm and greater accuracy in the answers. Each of these stages is related to a series of procedures that the educational context must foster. We address these procedures in the following section, in which we analyze the importance of negative transfer in the generation of errors.

4. The importance of negative transfer in the interpretation of errors

We consider the role played by transfer in analogical learning as the place where systematic errors originate. We thus locate ourselves in the definition of negative transfer in relation to learning, by means of negative analogies (Sánchez, 2005, 2012).

To lay the foundations of this conjecture and to define the terms clearly, in this section we analyze two of the significant processes that the educational context should promote. At this point we review what was explained earlier to guide us in developing the levels of comprehension and transgression of the learning principles of subtraction.

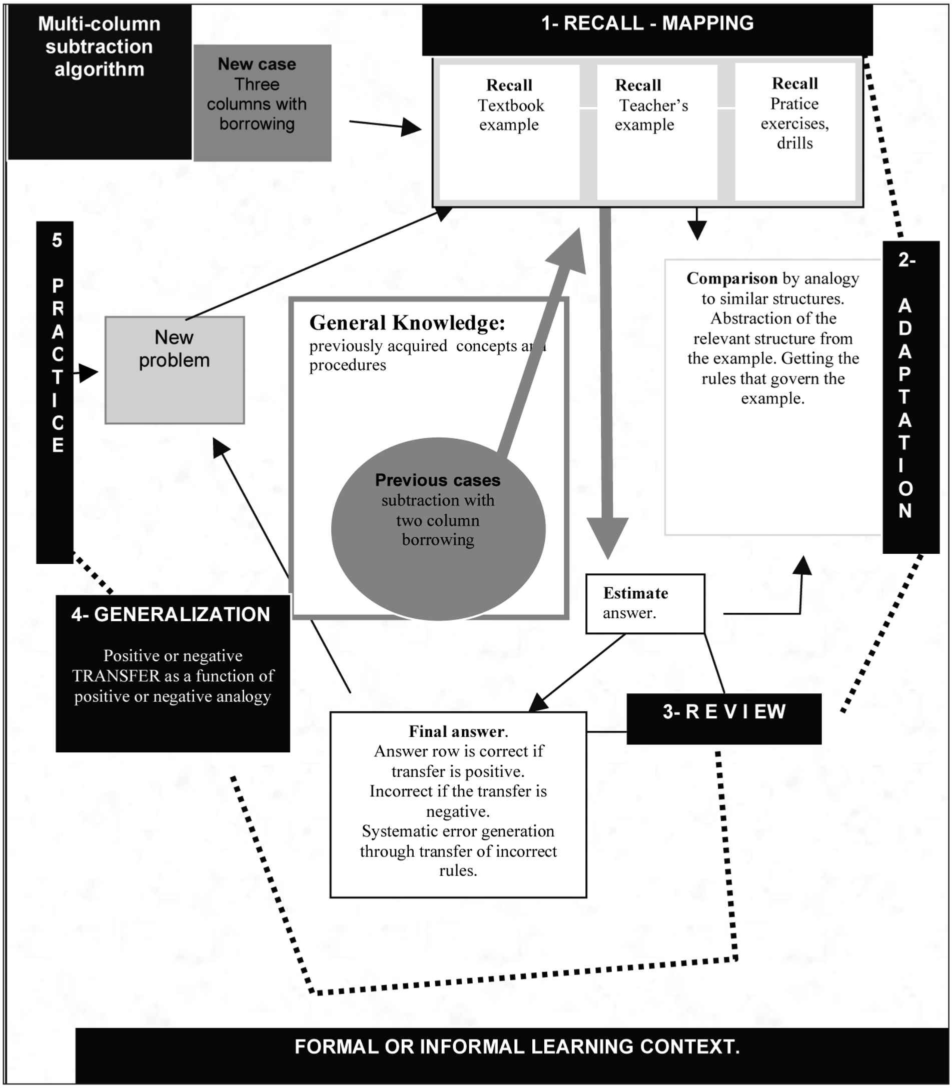
In the previous section we defended the existence of three different stages in the correct acquisition of the algorithm: i) acquisition of the conceptual knowledge underlying it, ii) acquisition of the procedural knowledge, and iii) continued practice. To develop this idea, we take as a reference the last two stages, in which we believe incorrect rules are generated that give rise to errors.

Although in this section we do not address the first stage of learning in which children try to learn the conceptual or domain knowledge without applying it, we consider it to be a very important stage in the subsequent learning of the algorithm. However, it is in the intermediate stage, in which children try to apply the algorithm, where the transgressions in the procedure have their origin. Subsequently, owing to the phenomenon of negative transfer, these transgressions become generalized and, as we shall explain below, turn into a source of systematic error.

But what do we mean when we speak of learning by analogy? To explain this type of learning we have to refer to example-based learning systems. Starting with examples, the student solves problems by adapting the solutions given previously to similar problems. In 1983, Kolodner formulated a model called Case-based Reasoning (CBR), which she applied to the expert systems of artificial intelligence. This model is based on the use of material learned in previous experiences to solve subsequent learning situations. To a certain extent it coincides with the Intuitive Rules theory by Tirosh and Stavy that we saw earlier, especially as regards the standard rule «Same A – Same B». In our opinion this model can be applied to arithmetic learning, as summarized in Figure 2 below, which is located in the second and third stages of algorithm learning.

In this figure we can see the five stages of the second process in procedural acquisition. The most salient actions are the experimental heuristic search, which strengthens the performance of the algorithm, and the third transfer process, which is based on attaining the relation between conceptual knowledge and procedural knowledge through continued practice. This practice provides a diversity of situations to facilitate learners' comprehensive understanding of this relation as well as acquisition of speed and accuracy. This relation forms an integral part of these processes that define the actions of recall by mapping, adaptation by analogy, and transfer, which we shall explain below.

FIGURE 2. CBR SYSTEM ADAPTED TO LEARNING OF THE SUBTRACTION ALGORITHM BY EXTRAPOLATION FROM EXAMPLES



Recall can be of two types; on one hand, spontaneous recall, and on the other, deliberate recall. The latter is more important in regard to instruction, since it is intentionally directed by the teacher and can most successfully activate the deep structural relations between the examples and the problem to be solved. It requires learners to recognize the structure of the examples solved correctly on other occasions. This action allows them to use the structural relations by describing a mapping of the parts that make up the example, in order subsequently to apply them to a new problem. We call this behavior learning by analogy. Its general objective it to transfer a system of relations from a familiar field or domain to another that is less familiar. With this system we help learners to understand new information under conditions of familiar information to make it easier for them to relate it to a structure of knowledge that they already possess (Beall, 1999; Glynn, 1991; Simons, 1984; Thiele and Treagust, 1991; Venville and Treagust, 1997 cited by Bransford and Schwartz, 2001).

In regard to analogies, some authors defend the idea that knowledge is constructed in the mind of the learner in the attempt to give meaning to the information being learned (Bodner, 1986). Moreover, the comparative nature of analogies promotes this meaningful learning, since children have to select the relations between the new knowledge and the concepts and propositions they already know (Ausubel, Novak, Hanesian, 1983). Thus, and in connection with the concept of «mapping» developed by Novack in 1983, analogies are a technique that allows learners to acquire the pertinent structures in a particular domain and is based on Ausubel's theory of meaningful learning. That is, new knowledge depends on existing knowledge. According to Novack (1983), the new concepts are acquired either by discovery or by reception, the latter being the type of learning most common in school children. The problem with reception learning is that the students memorize definitions of concepts, and in our case, algorithms, but do not acquire the meanings of the concepts that would allow them to solve tasks with full understanding. In this last case, children use the analogy that consists of relating familiar information to non-familiar information, making a correspondence between two different models that have the same or similar conceptual relations between them (Bassok, 1998). That is, they apply the «Same A – Same B» rule mentioned earlier.

In line with the above, analogies can play several roles in promoting meaningful learning, either by establishing the necessary relations of knowledge generated by the action of recall or mapping, or by generating errors in learning through the use of negative analogies, leading to a negative effect. This takes place when children resort to using analogies mechanically, without considering whether the analogical information is meaningful (Orgill and Bodner, 2003) For example, mechanical use of an analogy takes place when learners are trying to remember examples that have been useful in the past, something we call «efficient prototypes in the past»; which then lead them to resort to a familiar analogy that they transfer to the new concept. This may provide a mechanical answer that in turn is incorrect.

The mechanical use of an analogy can also be the result of students' inability to apply pertinent concepts of the analogical domain to the target domain in a correct way. In this way they can develop erroneous concepts regarding the target domain (Brown and Clement, 1989; Duit, 1991; Zook, 1991; Zook and Digesta, 1991; Thagard, 1992; Clement, 1993). Erroneous concepts that are developed as a result of an analogy are difficult to remedy, and in our case, they turn into systematic errors, since they depend on the superficial structure of the example and not its deep structure because the child has apprehended the non-basic principles that sustain the example.

Finally, even though one of the purposes of an analogy is to help students learn a concept in a meaningful way, the use of analogy can also limit their ability to develop a deeper understanding of that concept (Orgill and Bodner, 2003). When only one analogy is used to provide information on a particular topic, students may accept the analogical explanation from their teacher as the only one possible for that topic (Spiro, Feltovich, Coulson and Anderson, 1989). For example, there is one systematic error that could be explained as follows: children are often taught multi-column subtraction with borrowing through examples of two column subtractions. This behavior is taught by the teacher or else analyzed by the children in textbook examples. When they are then confronted with three or more columns, often only the new aspects of the concept of borrowing are addressed by analogical comparison with the structural mapping of the previous example of two-column subtraction. Thus, they only borrow from the most leftwards column and commit the *always-borrow-left*[2] error.

As pointed out earlier, mastery of an algorithm requires learning not only one principle, but several. In this case, if the task requires multiple principles or the process can be subdivided into parts, not only would we have to count on generalization, but also with transfer of learning to other situations that can seem similar, but are not.

In the action of transfer, we can include the concept of negative transfer, which we define as an incorrect adaptation of a known rule to solve a new problem that occurs when a training task or practice interferes with the learning of the transfer task and reduces the speed of learning. By transfer we understand a positive effect of transmission from one action to another. That is, an exercise for improving one factor may show a certain influence on the development of others (Sánchez, Cabrero and Llorente, 2012). This influence, which we call transfer, can be positive (favorable), negative (unfavorable), or neutral (indifferent) according to whether or not there exists some relation between the exercise or task and the

2. Always-borrow-left: the learner borrows from the digit farthest to the left instead of the digit immediately to the left. Example: 733 – 216 = 427. Description taken from VanLehn (1990).

other factors. Negative transfer is said to take place when the learning or execution of one task interferes with the learning or execution of a second task.

Thus, applying this theoretical foundation, we can conclude that in learning the subtraction algorithm children go through five important processes when they confront a new case of multi-column subtraction: recall, adaptation, review, generalization and transfer through practice (see Figure 3). To carry out the first process they recall previous examples through mapping, which consists in comparing through analogies the similar structures in the prior example and the new case. This behavior permits abstraction of the relevant structure from the example and facilitates the move to the second process, that of adaptation or apprehension of the deep structure or rules to the new case. After these two processes, the learner initiates a review, which consists in deciding whether the first approach to the answer is the correct one or not, although often the answer is mechanical. The final answer row will be correct if the transfer of structures is positive, and incorrect if the transfer is negative; that is, an error is generated by appropriation of the superficial structure of the first example considered through the transfer of incorrect rules to the target case. The fourth stage, called generalization, consists of a positive transfer when the previous steps were correct, and a negative transfer when meaningful learning has not mediated along the whole process. This transfer is reinforced through practice. And it is in the practice where most children manage to overcome the systematic errors through intervention from the educational context.

In the next section, we present the different models of acquisition of incorrect rules or errors in the educational context. Although we have already put forward some important concepts such as negative transfer or negative analogy, our objective is find a basis for the error using a procedural theoretical approach and the intervention of communicational mediators in such a process.

5. Educational models of incorrect rule acquisition from the procedural perspective

From the procedural perspective, errors can be divided into three categories: i) those having their origin in an incorrect choice when selecting a technique to extrapolate from prior examples, ii) errors that reflect poor conceptual knowledge, and iii) errors called «bugs» during the execution of a procedure. In this section we describe the first category which includes the cognitive extrapolation techniques that the learner uses when confronting a new problem. In this case, if she does not know the rule to be applied, she will be forced to find some way to mediate between the rules she knows and the unfamiliar problems to be solved. This action is what VanLehn (1983) called a *patch*, which is an indirect path for arriving at the solution, because the learner first has to decide how to adapt the old rule to a new situation, something that according to Sleeman (1986) would provoke a cognitive dissonance that would in turn activate the best solution to the algorithm. The cognitive extrapolation techniques most frequently used by students in primary education are linear extrapolation and generalization. When students learn the rules of arithmetic, in general the pattern rules can be replaced in a one-to-one correspondence (standard rule: Same A – Same B). In an arithmetic expression of the example to be solved, each simple sub-process encloses a simple sub-process. This equivalence can be defined as a high level rule from which the pattern rule can be projected within the object as a whole. That is, it accommodates itself to a new situation in which one can relate a procedure in question with a sub-procedure or have a single high-level correspondence that involves the understanding of the procedure, since one possesses the complete pattern. Insistence on replacing sub-procedure by sub-procedure obliges us to see the rules of the algorithm strictly as material of replacement patterns. This action is due to a purely rote learning process, in which children do not perceive the rules as schemes that act to build a new pattern. The difference between these two ways of interpreting the rules reflects whether the patterns are considered as chains of behavior or as expression trees (hierarchical processes) and they often separate experienced students from beginners when it comes to solving the algorithm (Chi, Basssok, Leewis, Reimann and Glaser, 1989). The experts view the models as descriptions of expression trees and they equate the patterns with sub-expressions. The perspective of perceiving the pattern rules as chains of characters leads to most of the extrapolations being incorrect.

The execution of procedures whose essential characteristic is linearity is very common in some learners, since most of their previous experience developed under the control of linear processes and produced positive results. For example, the number of times the students use the distributive property in the addition algorithm may consolidate the linear behavior that they will subsequently apply to the subtraction algorithm. Moreover, these reiterated applications in subtractions are evident in the use of routines learned by rote that are inclined to generate a variety of errors.

The first class of errors, then, are the errors of reiterated applications, or what VanLehn (1986) calls «strings» which are acquired by rote when solving examples during learning and constitute the initial models for the acquisition of incorrect rules that are the result of generalized induction. An example would be the bug «always-borrow-left» which we explained in the previous section. At this point, we must emphasize the importance of communicational mediators, or the examples used by the teacher in the learning context that contribute to the generation of bugs. The educational context that fosters these types of strings is characterized by poor and inefficient communication, since the sequence of the procedures to be acquired is not presented to the learners in a way that they can move up through complexity levels by surpassing all the lower objectives until they reach a greater level of difficulty in the task. That is, we cannot go from a subtraction without multicolumn borrowing to a subtraction with multicolumn borrowing without having defined both the conceptual and procedural meaning that defines its architecture. Thus, the context, and the communicational mediators used within it, mark and delimit the possibility of acquiring the correct rule, or in contrast, the incorrect rule that can be acquired in the same way and in the same educational context. Hence, linearity has a basis in some previous efficient working prototype, but is the result of a poor definition of the sequence of the learning process. Doing exercises that the teacher sets provides learners with the tools they need for extrapolating from previous generalized learning, but the reciprocal interpretation is not exact here. We can speak of a negative transfer as discussed earlier on. This error of reciprocity in turn involves another two errors, an inadequate interpretation of a prototype that was efficient in the past and a false concept of extrapolation in which the operator acts like the original prototype but the reciprocity is false (Maza, 1991).

The second extrapolation technique is generalization, which serves as a bridge between the known rules and their application to unfamiliar problems through review of a rule and the accommodation of the operators and numbers that appear in it to a new situation. This requires an effort of comparison and abstraction of the sequence of actions in which the learner has already been instructed. For example, the learner has been instructed in some situations of problems with borrowings, but not in all, and she acts by applying the rules she knows, which can be called «marcorules», to all the (Matz, 1982). Thus, if a child finds a column in the form of 0 – N, she may consider the 0 as an empty set, because she has overgeneralized the induced premise that 0 – N = N, based on the prototype that served as an example: N – 0 = N.

Instruction that is concerned with eliminating these erroneous extrapolation techniques must foster the establishment of differences between a problem type or a basic rule and a new problem. This ability would be the final result of good teaching that allows learners to relate all the concepts taught previously with the new ones to be acquired.

6. Conclusions

In this paper we have approached the description of cognitive processes and educational actions that involve the generation of algorithmic error during the teaching-learning process. To achieve this aim, we first reviewed the pertinent literature in theory of education that describes the comprehension processes of algorithmic structures, and in particular that which addresses comprehension in relation to the inadequate interpretation of the rules underlying them. Subsequently we represented the procedural architecture of the algorithm and the learning principles behind it. In this analysis, which we locate in the space where theory of education, algorithmic education, and cognitive psychology intersect, we consider of special interest the processes induced from the pedagogical context as the primary cause of error. These processes are socially mediated and come together in a space of social construction in which communicational mediators intervene who foster transfer processes in either a positive or negative direction. The essential thrust of this contribution thus addresses this communicational mediation in which teachers and learners participate along with other teaching-learning resources such as textbooks.

In this context, we have presented a model that constitutes a conceptual representation of the real processes taking place during algorithmic learning. More specifically, we take the subtraction algorithm as a conceptual simulation which, as in any other learning process, is socially mediated and is located in that area of social construction in which communicational mediators intervene, and from which the learner then internalizes, assimilates and apprehends algorithmic systems. Therefore, the Vygostkian contribution to the nature of communicational mediation is not out of place in relation to the origin of algorithmic errors.

In light of this theory, and based on research, we suggest that algorithmic reasoning is affected by intuitive rules. In particular, we believe that the rule «Same A – Same B» plays a major role in this affirmation. We have thus attempted to find what kind of transfer takes place when a learner generalizes an intuitive rule acquired in a context in which communicational mediators predominate, as is the case with the elementary teaching of algorithms, and in particular, what types of cognitive processes sustain the transfer of mathematical knowledge. In our view, it is these transfer processes, specifically generalization, that make the relation between conceptual knowledge and procedural knowledge possible. In this framework of the relation between the two types of knowledge it is common for teachers to use mediators such as textbooks. This can interfere with a child's ability to make generalizations. However, it is not only textbook examples that can interfere in these processes, but also the examples presented by the teacher, classmates or other persons close to the learner. Thus, these resources that feed the algorithmic transfer process can be induced from the educational context. We therefore feel that the methods and language used in learning situations cannot be ignored, since they can have a decisive effect on skill acquisition. The quality of classroom interaction is an essential component in the initial comprehension of mathematical concepts. We consider that instruction intervenes in the generation of errors through memorization processes highly influenced by a type of teaching that does not foster the conceptual understanding of algorithms. Such a model of instruction in the interaction processes affects the choice of the resources that will determine whether the skill is understood or not.

A positive change in the teaching of algorithms would be presided by insisting that the learner acquire a solid understanding of the comprehensive relations between actions and their contextual structure. That is, understanding of the concepts and their relation to the execution of the procedure should be strengthened, in other words what Ohlson and Rees (1991) call knowledge of the domain in which the procedure operates. Nonetheless, if the learner is to acquire knowledge of the domain in which the procedure operates, then the educational context must promote certain transfer processes. We consider the role played by transfer based on analogical learning as the origin of systematic errors. To support this conjecture we have conceptualized learning through analogy. We stress the possibility that the use of one analogy is another element limiting the deep comprehension of the concept, since when only one analogy carries the information on a certain topic, learners may accept the analogical explanation from their teacher as the only one possible, and this type of behavior can be tendered by the teacher.

Thus, the learning context and the communicational mediators used within it can either mark out and delimit the possibility of correct rule acquisition, or otherwise incorrect rules can be acquired in the same way and in the same educational context.

In summary, we have demonstrated the importance of the interpretations and resources used by children in solving algorithms because they have a decisive influence on conceptual knowledge and on the relation the child must establish between conceptual and procedural knowledge.

It is therefore the context of teacher-student interaction and the methodology used which specifies the space where algorithmic knowledge is constructed. This space of construction can define and determine an efficient learning process if the methodology and interaction are adequate and scientifically based, or, on the contrary, it can constitute a basis for the generation and consolidation of errors.

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