On the derivatives of generalized Gegenbauer polynomials

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Abstract

We prove some new formulae for the derivatives of the generalized Gegenbauer polynomials associated to the Lie algebra A_2.

As it is well known [1], the classical Gegenbauer polynomials $C_m^\lambda(z)$ suffer, when differentiated in z, a shift in the parameter λ, namely

$$\frac{dP_m^\lambda}{dz} = m\,P_{m-1}^{\lambda+1};$$
$$P_m^\lambda(z) = \frac{m!}{(\lambda)_m}C_m^\lambda\left(\frac{z}{2}\right); \qquad (\lambda)_m = \lambda(\lambda+1)\ldots(\lambda+m-1)$$

The classical Gegenbauer polynomials are (up to a factor) the eigenfunctions of the simplest quantum Calogero–Sutherland Hamiltonian [2],[3], that related to the Lie algebra A_1. It is the purpose of this note to show that the same shift in λ takes place in the derivatives of the generalized Gegenbauer polynomials $P^\lambda_{m,n}(z_1,z_2)$ giving the quantum eigenfuctions of the Calogero–Sutherland system with Lie algebra A_2:

$$\begin{aligned}
\Delta P^\lambda_{m,n} &= \varepsilon_{m,n}(\lambda)P^\lambda_{m,n}(z_1,z_2);\\
P^\lambda_{m,n} &= z_1^m z_2^n + \text{lower terms};\\
\Delta &= (z_1^2-3z_2)\partial^2_{z_1} + (z_2^2-3z_1)\partial^2_{z_2} + (z_1z_2-9)\partial_{z_1}\partial_{z_2} + (3\lambda+1)(z_1\partial_{z_1}+z_2\partial_{z_2})\\
\varepsilon_{m,n}(\lambda) &= m^2+n^2+mn+3\lambda(m+n);
\end{aligned}$$

see [4],[5],[6],[7],[8]. Specifically, we will prove the following formulae:

$$\frac{\partial P^\lambda_{m,n}}{\partial z_1} = m\,P^{\lambda+1}_{m-1,n} + A_{m,n}(\lambda)P^{\lambda+1}_{m-2,n-1} + B_{m,n}(\lambda)P^{\lambda+1}_{m,n-2} \qquad (1)$$

$$\frac{\partial P^\lambda_{m,n}}{\partial z_2} = n\,P^{\lambda+1}_{m,n-1} + A_{n,m}(\lambda)P^{\lambda+1}_{m-1,n-2} + B_{n,m}(\lambda)P^{\lambda+1}_{m-2,n}; \qquad (2)$$

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where

$$
\begin{aligned}
A_{m,n}(\lambda) &= \frac{m(m-1)n(m+n+\lambda-1)(m+n+\lambda)}{(m+\lambda-1)(m+\lambda)(m+n+2\lambda-1)(m+n+2\lambda)} \\
B_{m,n}(\lambda) &= \frac{n(n-1)(m+n+\lambda)}{(n+\lambda-1)(n+\lambda)}.
\end{aligned}
\tag{3}
$$

Consider first (1). The proof of this formula proceeds by induction on the second quantum number. The generating function for the Jack polynomials $P_{m,0}$ is known to be [9]

$$
(1-z_1t+z_2t^2-t^3)^{-\lambda} = \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} P_{m,0} t^m . \tag{4}
$$

Differentiation of this expression shows the validity of (1) when $n=0$. On the other hand, we can use the recurrence relations for the generalized Gegenbauer polynomials [7] to express $P_{m,n}$ in terms of polynomials with lower n:

$$
P_{m,n} = z_2 P_{m,n-1} - a_{m,n-1}(\lambda) P_{m-1,n-1} - c_{n-1}(\lambda) P_{m+1,n-2} \tag{5}
$$

with

$$
\begin{aligned}
a_{m,n}(\lambda) &= \frac{m(n+m+\lambda)(m-1+2\lambda)(n+m-1+3\lambda)}{(m+\lambda)(m-1+\lambda)(n+m+2\lambda)(n+m-1+2\lambda)}; \\
c_n(\lambda) &= \frac{n(n-1+2\lambda)}{(n+\lambda)(n-1+\lambda)}.
\end{aligned}
\tag{6}
$$

Differentiating (5) with respect to z_1 under the assumption that (1) is valid when the second quantum number is lower than n, and applying the recurrence relation (5) to get rid of the remaining z_2 factors, we obtain:

$$
\begin{aligned}
\frac{\partial P_{m,n}}{\partial z_1} &= m P^{\lambda+1}_{m-1,n} \\
&+ [A_{m,n-1}(\lambda) a_{m-2,n-2}(\lambda+1) - A_{m-1,n-1}(\lambda) a_{m,n-1}(\lambda)] P^{\lambda+1}_{m-3,n-2} \\
&+ [A_{m,n-1}(\lambda) + m a_{m-1,n-1}(\lambda+1) - (m-1) a_{m,n-1}(\lambda)] P^{\lambda+1}_{m-2,n-1} \\
&+ [B_{m,n-1}(\lambda) - (m+1) c_{n-1}(\lambda) + m c_{n-1}(\lambda+1)] P^{\lambda+1}_{m,n-2} \\
&+ [B_{m,n-1}(\lambda) c_{n-3}(\lambda+1) - B_{m+1,n-2}(\lambda) c_{n-1}(\lambda)] P^{\lambda+1}_{m+1,n-4} \\
&+ [-a_{m,n-1}(\lambda) B_{m-1,n-1}(\lambda) + a_{m,n-3}(\lambda+1) B_{m,n-1}(\lambda) \\
&+ A_{m,n-1}(\lambda) c_{n-2}(\lambda+1) - A_{m+1,n-2}(\lambda) c_{n-1}(\lambda)] P^{\lambda+1}_{m-1,n-3}
\end{aligned}
\tag{7}
$$

and by explicit use of (3) and (6), we find:

$$
\begin{aligned}
A_{m,n-1}(\lambda) a_{m-2,n-2}(\lambda+1) - A_{m-1,n-1}(\lambda) a_{m,n-1}(\lambda) &= 0 \\
B_{m,n-1}(\lambda) c_{n-3}(\lambda+1) - B_{m+1,n-2}(\lambda) c_{n-1}(\lambda) &= 0 \\
-a_{m,n-1}(\lambda) B_{m-1,n-1}(\lambda) + a_{m,n-3}(\lambda+1) B_{m,n-1}(\lambda) & \\
+ A_{m,n-1}(\lambda) c_{n-2}(\lambda+1) - A_{m+1,n-2}(\lambda) c_{n-1}(\lambda) &= 0
\end{aligned}
\tag{8}
$$

and

$$A_{m,n\ 1}(\ )+m\,a_{m\ 1,n\ 1}(\ +1)\quad(m\quad 1)a_{m,n\ 1}(\ )\ =\ A_{m,n}(\ )$$
$$B_{m,n\ 1}(\ )\quad(m+1)c_{n\ 1}(\ )+m\,c_{n\ 1}(\ +1)\ =\ B_{m,n}(\ ) \qquad (9)$$

which establishes the desired result. The proof of (2) takes advantage of the twin recurrence relation to (5), see [7], and is completely analogous. In conclusion we would like to mention that the approach of this note may be used also for the A_n case. We hope to return to this problem in the future.

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