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Numerical integration of stiff problems using a new time-efficient hybrid block solver based on collocation and interpolation techniques

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ABSTRACT

In this study, an optimal $\mathcal{L}$-stable time-efficient hybrid block method with a relative measure of stability is developed for solving stiff systems in ordinary differential equations. The derivation resorts to interpolation and collocation techniques over a single step with two intermediate points, resulting in an efficient one-step method. The optimization of the two off-grid points is achieved by means of the local truncation error (LTE) of the main formula. The theoretical analysis shows that the method is consistent, zero-stable, seventh-order convergent for the main formula, and $\mathcal{L}$-stable. The highly stiff systems solved with the proposed and other algorithms (even of higher-order than the proposed one) proved the efficiency of the former in the context of several types of errors, precision factors, and computational time.

1. Introduction

Initial value problems (IVPs) in ordinary differential equations (ODEs) of the following form are the most frequently used problems in several fields of science and engineering:

$$s'(x) = g(x, s(x)),\ s(x_0) = s_0, x \in [x_0, x_N] \subset \mathbb{R}, \text{ and } s(x),\ g(x, s(x)) \in \mathbb{R}^d. \tag{1}$$

It is common knowledge that numerous problems of the form (1) do not have analytic solutions; consequently, numerical methods remain salient. Finding numerical solutions to stiff systems has been a significant challenge for numerical analysts. A potentially good numerical method for solutions of stiff systems must possess certain qualities in terms of its region of absolute stability and accuracy [16].

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Despite the several methods in the literature which had been implemented in a step-by-step fashion for the solution of (1), block methods for providing a numerical solution of (1) at more than one point have become a reasonable alternative to gain computational efficiency and numerical accuracy. These methods are based on multistep methods formulated to produce the continuous form. The continuous form of the multistep methods has the advantage of producing the primary and complementary methods combined with the main methods to produce the required block scheme. Furthermore, they have the capacity to provide error control (see [30]).

Since the introduction of the block methods by Milne [18], which were used only for providing starting values for predictor–corrector methods, and further improved by Rosser [29], many block methods, including hybrid ones, have been developed and implemented on different classes of problems. For instance, Ramos, et al. in [24], constructed a family of adapted block Falkner methods that were frequency-reliant for the direct numerical solution of second-order IVPs with oscillatory solutions. A family of stiffly stable second derivative block methods for solving first-order stiff ODEs was presented by Ajayi et al. in [4]. Furthermore, the second derivative trigonometrically fitted block backward differentiation formula whose coefficients rely on the frequency and step size was presented by Abdulganiy et al. in [2] for solving oscillatory problems, to mention but a few. To inquire more about block methods, one can refer [3,5,9,10,19,20,23,34] and the references therein.

Hybrid methods, which are the adapted form of the k-step multistep methods, have been developed by introducing intra-step points in the derivation process. When implemented in a step-by-step fashion, these methods are more laborious because of the inclusion of the intra-step points as they increase the amount of predictors needed to execute them. However, with the emergence of block hybrid methods, this difficulty has been overcome since block methods have the superiority of being self-starting methods. Few block hybrid methods have been presented in the literature for the numerical integration of the IVP (1) as can be found in [1,15,25]. In this paper, we propose a one-step $\mathcal{L}$-stable, optimized second derivative block method (OLSBM) with two intra-step points, which provides the solution of (1) without using predictors as discussed in [12,39].

The present article is organized as follows: In Section 2, the mathematical formulation of the proposed block method is discussed. Section 3 is devoted to the theoretical analysis, which presents the qualitative properties of the proposed numerical algorithm. Section 5 contains numerical simulations to illustrate the method's performance, and a comparative analysis is also presented in this same section. Finally, Section 6 presents the conclusion with future remarks.

2. Mathematical formulation

The purpose of this section is to develop an implicit second derivative one-step block method for solving (1) efficiently. We assume that g is an enough smooth function and $d = 1$ in order to simplify the method's derivation. After that, the method could be applied to systems using a component-wise procedure. The approximate solution on the specific sub-interval $[x_n, x_{n+1}]$, where $x_{n+1} = x_n + \Delta x$ and Δx is the step-size, is obtained by approximating $s(x)$ locally by a polynomial $q(x)$ of the form:

$$s(x) \approx q(x) = \sum_{j=0}^{5} \psi_j x^j, \tag{2}$$

where $\psi_j \in \mathbb{R}$ represent real unknown parameters. When Eq. (2) is differentiated, we obtain

$$s'(x) \approx q'(x) = \sum_{j=1}^{5} j\psi_j x^{j-1}, \tag{3}$$

$$s''(x) \approx q''(x) = \sum_{j=2}^{5} j(j-1)\psi_j x^{j-2}. \tag{4}$$

Consider two intra-step points, $x_{n+u} = x_n + u\Delta x, x_{n+v} = x_n + v\,\Delta x$ with $0 < u < v < 1$, that will be used to calculate the approximate solution of (1) at point x_{n+1}, assuming that $s_n = s(x_n)$. To commence the methodology, suppose the estimation in (2) computed at x_n, its first-order derivative (g) computed at the points $x_n, x_{n+u}, x_{n+v}, x_{n+1}$, and its second-order derivative (γ) computed at the point x_{n+1}. In this case, we have a square linear system with six equations and six unknowns (coefficients) ψ_j, $j = 0, 1, \ldots, 5$, where $\psi_j \in \mathbb{R}$:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+u} & 3x_{n+u}^2 & 4x_{n+u}^3 & 5x_{n+u}^4 \\ 0 & 1 & 2x_{n+v} & 3x_{n+v}^2 & 4x_{n+v}^3 & 5x_{n+v}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix} = \begin{pmatrix} s_n \\ g_n \\ g_{n+u} \\ g_{n+v} \\ g_{n+1} \\ \gamma_{n+1} \end{pmatrix}. \tag{5}$$

The solution of the square linear system above produces values of the six undetermined coefficients $\psi_j, j = 0, 1, \ldots, 5$. For the sake of brevity, the values of the obtained coefficients are not mentioned herein. However, substituting these coefficients in (2) with the change of variable $x = x_n + t\Delta x$, we reach the following:

$$q(x_n + t\Delta x) = \psi_0 s_n + \Delta x\Big(\nu_0 g_n + \nu_u g_{n+u} + \nu_v g_{n+v} + \nu_1 g_{n+1}\Big) + \Delta x^2(\zeta_1 \gamma_{n+1}), \tag{6}$$

where

$$\psi_0 = 1,$$

$$v_0 = \frac{\left(3/5\, t^4 + (-3/4\, u - 3/4\, v - 3/2)\, t^3 + ((v+2)\, u + 2\, v + 1)\, t^2 + ((-3\, v - 3/2)\, u - 3/2\, v)\, t + 3\, vu\right) t}{3vu},$$

$$v_u = -\frac{t^2\left(15\, vt^2 - 12\, t^3 - 40\, vt + 30\, t^2 + 30\, v - 20\, t\right)}{60\,(u-1)^2\,(u-v)\, u},$$

$$v_v = \frac{t^2(15\, ut^2 - 12\, t^3 - 40\, ut + 30\, t^2 + 30\, u - 20\, t)}{60\,(v-1)^2(u-v)v}, \tag{7}$$

$$v_1 = -\frac{t^2\left(t\left(5u^2\left(4v^2 - 3vt + 6(t-2)\right) + u\left(-15v^2t + 6v\left(2t^2 + 5t - 10\right) - 24t^2 + 80\right) + 30v^2(t-2) + v\left(80 - 24t^2\right) + 12t(3t-5)\right) - 60uv\left(u\left(v - \frac{3}{2}\right) - \frac{3v}{2} + 2\right)\right)}{60(u-1)^2(v-1)^2},$$

$$\zeta_1 = \frac{t^2\left(20\, uvt - 15\, ut^2 - 15\, vt^2 + 12\, t^3 - 30\, vu + 20\, ut + 20\, vt - 15\, t^2\right)}{60\,(v-1)\,(u-1)}.$$

To obtain the new block scheme, we evaluate $q(x_n + t\Delta x)$ at the collocation points x_{n+u}, x_{n+v}, and x_{n+1}, that is, we take $t = u, v, 1$. This results in the three formulas:

$$s_{n+u} = \frac{\Delta x^2 u^2\left(-3\, u^3 + 5\, u^2 v + 5\, u^2 - 10\, vu\right)\gamma_{n+1}}{(60\, v - 60)\,(u-1)} + \left(\frac{\left(-3\, u^4 + 5\, u^3 v + 10\, u^3 - 20\, u^2 v - 10\, u^2 + 30\, vu\right) g_n}{60\, v} - \frac{u\left(-12\, u^3 + 15\, u^2 v + 30\, u^2 - 40\, vu - 20\, u + 30\, v\right) g_{n+u}}{60\,(u-1)^2\,(u-s)} + \frac{u^2\left(3\, u^3 - 10\, u^2 + 10\, u\right) g_{n+v}}{60\,(v-1)^2\,(u-v)\, v} - \frac{u^2\left(-3\, u^4 v + 5\, u^3 v^2 + 6\, u^4 + 6\, u^3 v - 30\, v^2 u^2 - 24\, u^3 + 30\, u^2 v + 30\, uv^2 + 20\, u^2 - 40\, vu\right) g_{n+1}}{60\,(v-1)^2\,(u-1)^2}\right)\Delta x + s_n, \tag{8}$$

$$s_{n+v} = \frac{\Delta x^2 v^2\left(5\, uv^2 - 3\, v^3 - 10\, vu + 5\, v^2\right)\gamma_{n+1}}{(60\, v - 60)\,(u-1)} + \left(\frac{\left(5\, uv^3 - 3\, v^4 - 20\, uv^2 + 10\, v^3 + 30\, vu - 10\, v^2\right) g_n}{60\, u} - \frac{v^2\left(3\, v^3 - 10\, v^2 + 10\, v\right) g_{n+u}}{60\,(u-1)^2\,(u-v)\, u} + \frac{v\left(15\, uv^2 - 12\, v^3 - 40\, vu + 30\, v^2 + 30\, u - 20\, v\right) g_{n+v}}{60\,(v-1)^2\,(u-v)} - \frac{v^2\left(5\, u^2 v^3 - 3\, uv^4 - 30\, v^2 u^2 + 6\, uv^3 + 6\, v^4 + 30\, u^2 v + 30\, uv^2 - 24\, v^3 - 40\, vu + 20\, v^2\right) g_{n+1}}{60\,(v-1)^2\,(u-1)^2}\right)\Delta x + s_n, \tag{9}$$

$$s_{n+1} = \frac{\Delta x^2\,(-10\, uv + 5\, u + 5\, v - 3)\,\gamma_{n+1}}{(60\, v - 60)\,(u-1)} + \left(\frac{(20\, uv - 5\, u - 5\, v + 2)\, g_n}{60\, uv} - \frac{(5\, v - 2)\, g_{n+u}}{60\,(u-1)^2\,(u-v)\, u} + \frac{(5\, u - 2)\, g_{n+v}}{60\,(v-1)^2\,(u-v)\, v} - \frac{\left(-40\, u^2 v^2 + 75\, u^2 v + 75\, uv^2 - 30\, u^2 - 138\, uv - 30\, v^2 + 56\, u + 56\, v - 24\right) g_{n+1}}{60\,(v-1)^2\,(u-1)^2}\right)\Delta x + s_n, \tag{10}$$

where $s_{n+i} \simeq s(x_n + i\Delta x)$, are approximate solutions of the exact ones, $g_{n+i} = g(x_{n+i}, s_{n+i})$, for $i = u, v, 1$, and $\gamma_{n+1} = \gamma(x_{n+1}, s_{n+1})$. In the approximations found above, the two unknown parameters u, v are related to the two intra-step points x_u, x_v. In order to determine the parameters' values, we equate the first two terms of the LTE of s_{n+1} to zero. In this way, the parameters' values are optimized, and at the end of the sub-interval $[x_n, x_{n+1}]$, the value s_{n+1} is what is needed for advancing the integration on the next sub-interval. Henceforth, the local truncation error of Eq. (10) is taken into consideration as follows:

$$LTE(s(x_{n+1}); \Delta x) = \frac{((5\, v - 2)\, u - 2\, v + 1)\, s^{(6)}(x_n)\Delta x^6}{7200} + \frac{\left((35\, v - 14)\, u^2 + \left(35\, v^2 + 56\, s - 28\right) u - 14\, v^2 - 28\, u + 18\right) s^{(7)}(x_n)\Delta x^7}{302400} + \mathcal{O}(\Delta x^8). \tag{11}$$

After equating to zero the coefficients of Δx^6 and Δx^7 in (11) and solving the resulting system, the optimized version of the parameters' values are achieved as given below:

$$u = \frac{3}{7} - \frac{\sqrt{2}}{7}, \quad v = \frac{3}{7} + \frac{\sqrt{2}}{7}. \tag{12}$$

Substituting these values in the LTE of the main formula, we obtain:

$$LTE(s(x_{n+1}); \Delta x) = \frac{\Delta x^8}{8!}\,\frac{1}{735}\, s^{(8)}(x_n) + \mathcal{O}(\Delta x^9) = \frac{\Delta x^8 s^{(8)}(x_n)}{29635200} + \mathcal{O}(\Delta x^9). \tag{13}$$

With the above-computed parameters, the following one-step optimized $\mathcal{L}$-stable block method with two intra-step points is proposed:

$$\begin{aligned} s_{n+u} =& s_n + \Delta x\left(\frac{2649+328\sqrt{2}}{36015}g_n + \frac{680-89\sqrt{2}}{3360}g_{n+u} + \frac{189592-169889\sqrt{2}}{1152480}g_{n+v} + \frac{-171+316\sqrt{2}}{14406}g_{n+1}\right) \\ &+ \Delta x^2\left(\frac{411-928\sqrt{2}}{288120}\gamma_{n+1}\right), \\ s_{n+v} =& s_n + \Delta x\left(\frac{2649-328\sqrt{2}}{36015}g_n + \frac{-32714-45725\sqrt{2}}{164640\left(-3+\sqrt{2}\right)}g_{n+u} + \frac{-91238+20237\sqrt{2}}{164640\left(-3+\sqrt{2}\right)}g_{n+v} + \frac{-171-316\sqrt{2}}{14406}g_{n+1}\right) \\ &+ \Delta x^2\left(\frac{356-1356\sqrt{2}}{164640\left(-3+\sqrt{2}\right)}\gamma_{n+1}\right), \\ s_{n+1} =& s_n + \Delta x\left(\frac{196}{2940}g_n + \frac{9016-539\sqrt{2}}{23520}g_{n+u} + \frac{9016+539\sqrt{2}}{23520}g_{n+v} + \frac{98}{588}g_{n+1}\right) - \frac{49}{5880}\Delta x^2\gamma_{n+1}. \end{aligned} \tag{14}$$

Even though the above-mentioned one-step optimized $\mathcal{L}$-stable block approach with two intra-step points was developed with a fixed-stepsize (Δx), it is easily adaptable to a variable-stepsize mode that has been done in one of the forthcoming sections.

3. Theoretical analysis

Any numerical scheme's qualitative properties are crucial and determine its efficiency. In this section, the stability, consistency, and by extension, the convergence of the proposed scheme are analyzed. These concepts often play a fundamental role in the selection of an efficient numerical algorithm for finding the solution of the IVPs of the type (1).

3.1. Local truncation error and consistency

The one-step optimized block scheme (14) can be rewritten using the matrix notation as follows [23]

$$\mathcal{I}^0 S_{n+1} = \mathcal{C}^1 S_n + \Delta x(\mathcal{D}G_{n+1} + \mathcal{B}^0 G_n + \Delta x\mathcal{B}^1\bar{G}_{n+1}), \tag{15}$$

where $\mathcal{I}^0$, $\mathcal{C}^1$, $\mathcal{B}^0$, $\mathcal{B}^1$, and $\mathcal{D}$ stand for 3×3 matrices as given below.

$$\mathcal{I}^0 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \mathcal{C}^1 = \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix},\quad \mathcal{B}^0 = \begin{bmatrix}0&0&\frac{2649+328\sqrt{2}}{36015}\\0&0&\frac{2649-328\sqrt{2}}{36015}\\0&0&\frac{196}{2940}\end{bmatrix}, \mathcal{B}^1 = \begin{bmatrix}0&0&\frac{411-928\sqrt{2}}{288120}\\0&0&\frac{356-1356\sqrt{2}}{164640\left(-3+\sqrt{2}\right)}\\0&0&-\frac{49}{5880}\end{bmatrix}, \tag{16}$$

$$\mathcal{D} = \begin{bmatrix}\frac{680-89\sqrt{2}}{3360} & \frac{189592-169889\sqrt{2}}{1152480} & \frac{-171+316\sqrt{2}}{14406}\\ \frac{-32714-45725\sqrt{2}}{164640\left(-3+\sqrt{2}\right)} & \frac{-91238+20237\sqrt{2}}{164640\left(-3+\sqrt{2}\right)} & \frac{-171-316\sqrt{2}}{14406}\\ \frac{9016-539\sqrt{2}}{23520} & \frac{9016+539\sqrt{2}}{23520} & \frac{98}{588}\end{bmatrix}, \tag{17}$$

and

$$\begin{aligned} S_n &= (s_{n-1+u},\, s_{n-1+v},\, s_n)^T, \\ S_{n+1} &= (s_{n+u},\, s_{n+v},\, s_{n+1})^T, \\ G_n &= (g_{n-1+u},\, g_{n-1+v},\, g_n)^T, \\ G_{n+1} &= (g_{n+u},\, g_{n+v},\, g_{n+1})^T, \\ \bar{G}_{n+1} &= (\gamma_{n+u},\, \gamma_{n+v},\, \gamma_{n+1})^T. \end{aligned} \tag{18}$$

Consider the associated linear operator $\mathcal{Y}$ for the proposed block method (14) as follows:

$$\mathcal{Y}[J(x);\Delta x] = \sum_{k=0,u,v,1}\left[\bar{\gamma}_k J(x_n+k\Delta x) - \Delta x\bar{\eta}_k J'(x_n+k\Delta x) - \Delta x^2\bar{\zeta}_k J''(x_n+k\Delta x)\right], \tag{19}$$

where $\bar{\gamma}_k$, $\bar{\eta}_k$ and $\bar{\zeta}$ are column vectors of the matrices $\mathcal{B}^0$, $\mathcal{B}^1$ and $\mathcal{I}^0$, respectively. Given above, the symbol $J(x)$ stands for any arbitrary test function, which must be suitably differentiable on the integration interval. The suggested optimized $\mathcal{L}$-stable block method (14) has at least order r if after expanding the terms $J(x_n + k\Delta x)$, $J'(x_n + k\Delta x)$ and $J''(x_n + k\Delta x)$ in the Taylor expansion around x_n, and collecting the coefficients of Δx, we obtain the equation as shown below:

$$Y[J(x);\Delta x] = \bar{Q}_0 J(x_n) + \bar{Q}_1 \Delta x J'(x_n) + \bar{Q}_2 \Delta x^2 J''(x_n) + \cdots + \bar{Q}_r \Delta x^r J^{(r)}(x_n) + \ldots, \tag{20}$$

with $\bar{Q}_0 = \bar{Q}_1 = \cdots = \bar{Q}_r = 0$ and $\bar{Q}_{r+1} \neq 0$. The coefficients $\bar{Q}_j$ are vectors and $\bar{Q}_{r+1}$ is said to be the vector of error constants. For the Eq. (14), we obtain $\bar{Q}_0 = \bar{Q}_1 = \cdots = \bar{Q}_5 = 0$, whereas the error constant is obtained as follows:

$$\bar{Q}_6 = \left(\frac{11 + 92\sqrt{2}}{21176820}, \frac{11 - 92\sqrt{2}}{21176820}, 0 \right)^T. \tag{21}$$

We note that although the internal stages have accuracies of $\mathcal{O}(\Delta x^6)$ the main formula to advance the solution has accuracy of $\mathcal{O}(\Delta x^8)$, as given in (13). Hence, it has been proved from the above discussion that the main formula s_{n+1} of the Eq. (14) with two intra-step points does possess a seventh order, while the formulas of the two intermediate stages are of fifth order of convergence.

3.2. Zero stability and convergence

In the famous work of Rutishauser in [31], it was noted that the solution obtained from a numerical scheme characterized by small local error and high order of accuracy may still be unstable when applied to the IVP (1), under a minimum step size. Todd [36] also noted this fact when he applied specific difference methods to second-order differential equations [13]. The concept of zero-stability deals with the behavior of the solutions of the difference system in (15) when $\Delta x \to 0$. If $\Delta x \to 0$, then the method in (14) gives the following system of equations

$$\begin{aligned} s_{n+u} &= s_n, \\ s_{n+v} &= s_n, \\ s_{n+1} &= s_n. \end{aligned} \tag{22}$$

Using the matrix formalism, this may be rewritten as $\mathcal{I}^0 S_{n+1} - \mathcal{C}^1 S_n = 0$, with S_{n+1}, S_n and $\mathcal{C}^1$ as before, and $\mathcal{I}^0$ is the identity matrix of the third-order. The proposed block method is said to be zero stable when the roots λ_j of the first characteristic polynomial $\Omega(\lambda)$ given by $\Omega(\lambda) = |\mathcal{I}^0\lambda - \mathcal{C}^1|$ satisfy $|\lambda_j| \leq 1$, and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 1 (see Lambert [17]). Since $\Omega(\lambda) = \lambda^2(\lambda - 1)$, the block method put forward in (15) is zero-stable. As discussed by Henrici in [14], the convergence of the proposed block scheme given in (14) can be claimed since zero-stability+consistency = convergence.

3.3. Stability analysis

Aside from the fact that the proposed numerical algorithm is zero stable, there is another concept of stability established by Dahlquist [8]. It has to do with linear stability, and it guarantees a certain value of $\Delta x > 0$, whether the numerical scheme would yield good results. We discuss the linear stability of the newly developed method considering the standard Dahlquist's test equation $w' = \sigma w$, $Re(\sigma) < 0$ taken from [7]. After applying the numerical algorithm (14) to this equation, we obtain:

$$w_{n+1} = \mathcal{M}(z) w_n, \quad z = \sigma \Delta x, \tag{23}$$

where σ is a complex parameter and $\mathcal{M}(z)$ is known as the stability matrix of the numerical method, which can be expressed as:

$$\mathcal{M}(z) = (\mathcal{I}^0 - z\mathcal{D} - z^2\mathcal{B}^1)^{-1}(\mathcal{C}^1 + z\mathcal{B}^0). \tag{24}$$

The stability matrix $\mathcal{M}(z)$ has eigenvalues $\left\{ 0, 0, \dfrac{4z^3 + 60z^2 + 360z + 840}{z^4 - 16z^3 + 120z^2 - 480z + 840} \right\}$.

The eigenvalues of this matrix usually determine the behavior of the numerical solution, being the stability property of the method which is characterized by the spectral radius $\rho[\mathcal{M}(z)]$. Thus, the region of absolute stability is given as

$$\mathcal{S} = \{ z \in \mathbb{C} : |\rho[\mathcal{M}(z)]| \leq 1 \}. \tag{25}$$

Fig. 1 draws the stability region, showing that the proposed method is $\mathcal{A}-$ stable, since the entire left-half complex plane is included in the stability domain. That is, $\rho[\mathcal{M}(z)]$ is power-bounded for every z in the left half complex plane. In addition, the proposed scheme is $\mathcal{L}-$stable [21], since it is $\mathcal{A}-$stable and

$$\lim_{z\to\infty} (\rho[\mathcal{M}(z)]) = 0.$$

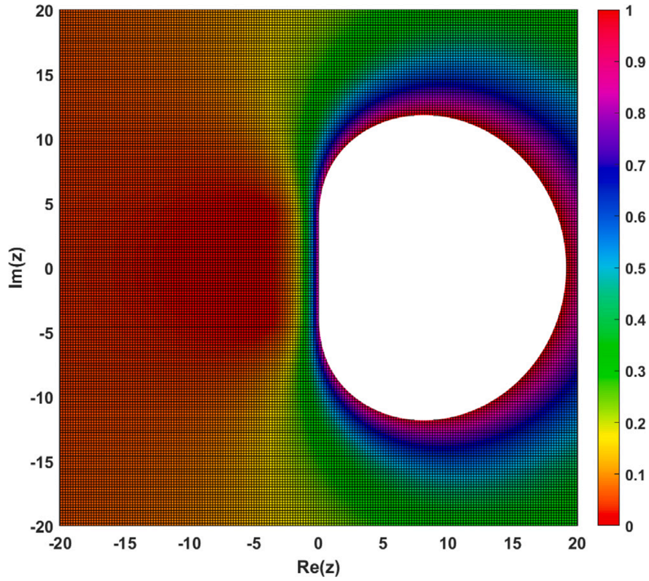

Fig. 1. Plot of the absolute stability region of the proposed method given in (14).

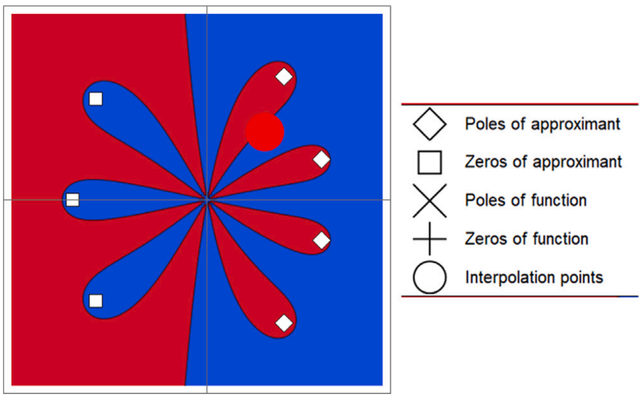

Fig. 2. Order stars for the proposed $\mathcal{L}$-stable optimal hybrid method given in (14). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3.4. Relative measure of stability

The $\mathcal{A}$-stability property also implies that the sufficient conditions for $\mathcal{A}$-stability of a numerical integrator are that the stability function $\rho[\mathcal{M}(z)]$ of the numerical algorithm must have no poles in the left half-plane and the magnitude of the stability function $|\rho[\mathcal{M}(z)]|$ must be bounded by 1, for z on the imaginary axis [6]. If $\rho[\mathcal{M}(z)]$ is multiplied by $\exp(-z)$, the sufficient conditions for $\mathcal{A}$-stability above will still not change as the factor $\exp(-z)$ will not add nor remove from the set of poles and its magnitude remains 1, where $Re(z) = 0$.

Consequently, the plot of $|\rho[\mathcal{M}(z)]\exp(-z)| > 1$ in Fig. 2 shows that our proposed scheme (14) is $\mathcal{A}$-stable. This plot is called order stars by their inventors [37]. From Fig. 2, the dual, also called the order star, is the interior of the red region, while the relative stability region, is the interior of the blue region.

4. Variable stepsize mode

Here, we formulate the one-step optimized $\mathcal{L}$-stable block approach with two intra-step points as a variable-step-size solver employing an embedded-type procedure. This process involves the simultaneous execution of a combination of two processes, one of order k_1 and the other of order k_2 ($k_2 < k_1$). The procedure in (14) is treated here as the higher-order procedure with $k_1 = 7$. We need an estimate of the local error, for which, in order to get an economical implementation, the second formula will use values that have been previously computed. To get a reliable estimate of the local error we adopt a similar strategy as the one considered by L. F. Shampine et al. [33].

Consider the approximation provided by the implicit trapezoidal rule [6], which has order $k_2 = 2$, and denote it by s^*_{n+1}, that is

$$s^*_{n+1} = s_n + \frac{\Delta x}{2}(g_n + g_{n+1}).$$

The local error is given by

$$Y_{\mathrm{LE}} = s(x_n + \Delta x) - s^*_{n+1},$$

where $s(x)$ denotes the exact solution of the problem given in (1). According to (13) the approximate solution with the method in (14) verifies $s(x_{n+1}) - s_{n+1} = \mathcal{O}(\Delta x^{k_1+1})$, and thus we have

$$Est = s_{n+1} - s^*_{n+1} \tag{26}$$

$$= \left(s(x_n + \Delta x) - s^*_{n+1}\right) - \left(s(x_n + \Delta x) - s_{n+1}\right) \tag{27}$$

$$= Y_{\mathrm{LE}} - \mathcal{O}(\Delta x^{k_1+1}). \tag{28}$$

Since Y_{LE} dominates in (26), assuming that Δx is small enough, we have that Est is a computable estimation of Y_{LE}.

If $|Y_{\mathrm{LE}}| \le tol$, where tol stands for the tolerance predefined by the user, then we accept the obtained results and select the next stepsize as $\Delta x_{\mathrm{new}} = 2 \times \Delta x_{\mathrm{old}}$, to minimize the computational burden and continue the integration process with Δx_{var} with the assumption that $\Delta x_{min} \le \Delta x_{\mathrm{var}} \le \Delta x_{max}$. However, if $|Y_{\mathrm{LE}}| > tol$, then we reject the achieved results by reducing it and repeat the calculations with the new step as follows:

$$\Delta x_{\mathrm{new}} = \eta \Delta x_{\mathrm{old}} \left(\frac{tol}{\|Est\|}\right)^{\frac{1}{k_2+1}}. \tag{29}$$

Here, the order of the lower order technique is $k_2 = 2$, while the value $0 < \eta (= 0.95) < 1$ denotes a safety factor whose purpose is to circumvent the steps that were unsuccessful. In the numerical examples, we have taken into account not only a very modest initial step size but also a strategy for modifying the step size that, if necessary, will cause the algorithm to change the step size. This was done so that we could demonstrate the behavior of the program under a variety of conditions.

5. Numerical simulations

In this section, we attempt to use the proposed optimal $\mathcal{L}$-stable block method (OLSBM) given in (14) on the basis of accuracy via error distributions (absolute maximum global error = $\max_{1\le n\le N} \|s(x_n) - s_n\|$, and root mean square error = $\sqrt{\frac{1}{N}\sum_{1\le n\le N}(s(x_n) - s_n)^2}$, precision factor (scd = $-\log_{10}\left[\max_{1\le n\le N} \|s(x_n) - s_n\|\right]$), and time-efficiency (CPU time measured in seconds)). In addition, for Problem 1 we have shown the performance of OLSBM by computing the percent difference in the error distributions including the percent difference in the number of steps (NS). For the calculation of the percent difference, the formula $\frac{|A-B|}{\frac{A+B}{2}} * 100$, where A shows the observation (say, MaxErr) in one of the selected methods and B (say, MaxErr) is selected from the proposed method.

Both fixed and variable stepsize approaches have been employed to solve the differential systems. Several numerical experiments are chosen in the form of stiff differential models and subsequently solved with the proposed method while choosing the following methods for comparison:

- TDBHM: An $\mathcal{L}$-stable seventh-order convergent hybrid block method based on third-order derivative proposed in [1].
- OSBIM: A seventh-order absolutely stable block method proposed recently in [12].
- EBM: An $\mathcal{L}$-stable eighth-order block method proposed in [28].
- ASHBM: An efficient $\mathcal{A}$-stable optimized hybrid block method with sixth-order of convergence proposed in [15].
- LobIIIB: Fully-implicit Lobatto type sixth-order method appeared in [6]
- RadIIA: Fully-implicit RK type fifth-order method appeared in [6]

It may also be noted that the pseudocodes for both fixed and variable step-size versions of the proposed optimal block method are given in Appendices A and B, respectively. The *FindRoot* command, which comes with Mathematica, has been used to implement the Newton–Raphson method. It is crucial to note that Mathematica 12.1, which is installed on a personal computer running Windows OS and equipped with an Intel(R) Core(TM) i7-1065G7 CPU @ 1.30 GHz and 1.50 GHz and 24.0 GB of installed RAM, is used to perform all of the numerical computations.

Problem 1. Consider the following nonlinear stiff model for the kinetic behavior of biosorption [22,26]:

$$\sigma s'(x) = s(x) - s(x)^3,\ s(0) = \sigma,\ 0 \le x \le 1/2, \tag{30}$$

with exact solution $s(x) = \dfrac{1}{\sqrt{99\exp(-2x/\sigma) + 1}}$, where $\sigma = 10^{-2}$.

Table 1
Error distributions and precision factor (scd) for Problem 1 with a number of steps = 10^2.

Method	MaxErr	RMSE	scd
OLSBM	3.5781×10^{-8}	3.9675×10^{-9}	7.44
TDBHM	1.0696×10^{-5}	1.4589×10^{-6}	4.97
OSBIM	1.9416×10^{-3}	3.8996×10^{-4}	2.71
EBM	8.1242×10^{-4}	1.0152×10^{-4}	3.09
ASHBM	1.0804×10^{-7}	1.3210×10^{-8}	6.96
LobIIIB	2.8076×10^{-7}	4.8473×10^{-8}	6.55
RadIIA	5.7372×10^{-6}	8.9133×10^{-7}	5.24

Table 2
Error distributions and precision factor (scd) for Problem 1 with a number of steps = 10^3.

Method	MaxErr	RMSE	scd
OLSBM	3.4633×10^{-15}	3.7132×10^{-16}	14.46
TDBHM	3.3394×10^{-13}	3.6439×10^{-14}	12.47
OSBIM	8.1279×10^{-11}	7.5344×10^{-12}	10.09
EBM	3.9248×10^{-12}	3.5881×10^{-13}	11.40
ASHBM	9.6686×10^{-14}	1.1989×10^{-14}	13.01
LobIIIB	2.8675×10^{-13}	4.8183×10^{-14}	12.54
RadIIA	5.9543×10^{-11}	8.9345×10^{-12}	10.23

Table 3
Error distributions and precision factor (scd) for Problem 1 with a number of steps = 10^4.

Method	MaxErr	RMSE	scd
OLSBM	3.4885×10^{-22}	3.7408×10^{-23}	21.45
TDBHM	3.3932×10^{-20}	3.6607×10^{-21}	19.46
OSBIM	8.9120×10^{-19}	7.4605×10^{-20}	18.05
EBM	3.5047×10^{-20}	3.7006×10^{-21}	19.45
ASHBM	9.6671×10^{-20}	1.1985×10^{-20}	19.01
LobIIIB	2.8676×10^{-19}	4.8200×10^{-20}	18.54
RadIIA	5.9887×10^{-16}	8.9520×10^{-17}	15.22

Table 4
Numerical results for Problem 1 with variable stepsize approach taking the tolerance = 10^{-6} and $\Delta x_{mini} = 10^{-3}$ over the interval $[0, \frac{1}{2}]$.

Method	MaxErr	RMSE	NS	CPU
OLSBM	3.620×10^{-11}	3.742×10^{-12}	177	2.266×10^{-1}
TDBHM	6.566×10^{-11}	5.753×10^{-12}	182	1.487×10^{-1}
OSBIM	4.956×10^{-9}	5.226×10^{-10}	204	7.546×10^{-2}
EBM	4.573×10^{-9}	3.674×10^{-10}	200	1.045×10^{-1}
ASHBM	5.503×10^{-11}	5.721×10^{-12}	177	2.274×10^{-1}
LobIIIB	7.132×10^{-11}	6.507×10^{-12}	239	10.150
RadIIA	2.584×10^{-10}	6.865×10^{-11}	239	9.588

Table 5
Percent difference (PD) in error distributions and in number of steps for Problem 1 with variable stepsize approach taking the tolerance = 10^{-6} and $\Delta x_{\text{mini}} = 10^{-3}$ over the interval $[0, \frac{1}{2}]$.

Method	$\text{MaxErr}_{\text{PD}}$	RMSE_{PD}	NS_{PD}
TDBHM	57.847	42.359	2.8
OSBIM	197.100	197.156	14
EBM	196.858	195.967	12
ASHBM	41.294	41.828	0
LobIIIB	65.3293	139.052	30
RadIIA	150.847	61.8993	30

Numerical results for Problem 1 displayed in Tables 1–3 with the fixed stepsize approach show the error distributions and the precision factor (scd) with the number of steps = $10^2, 10^3$ and 10^4, respectively. The proposed OLSBM performs better than the other four methods regarding all the error distributions and precision factors. The variable stepsize mode has been applied for Problem 1 in the Table 4 while taking the tolerance of 10^{-6} and $\Delta x_{mini} = 10^{-3}$ over the interval $[0, \frac{1}{2}]$. These results show that the proposed approach (OLSBM) yields the smallest amount of error distribution while taking considerably fewer steps (NS) with reasonably acceptable CPU time measured in seconds. In this specific problem, a fixed step size is used in Tables 1–3, considering powers on base 10. This is done so to see the decreasing pattern in the absolute errors. It has been noted that increasing the step size by one

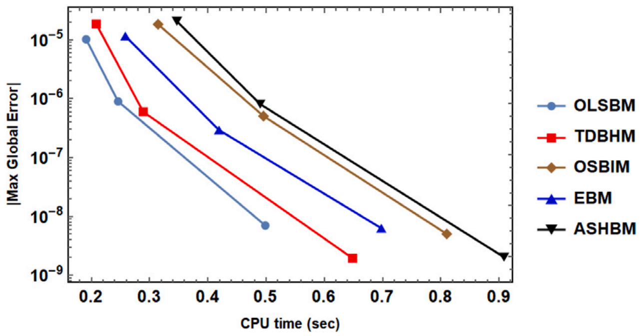

Fig. 3. Efficiency curves for Problem 1 with methods under consideration while the tolerance for the maximum global error is taken to 10^{-i}, where $i = 5, 7, 9$.

Table 6
Error distributions and precision factor (scd) for Problem 2 with a number of steps = 2^6.

Method	RMS	Mean	scd
OLSBM	1.3958×10^{-3}	1.3958×10^{-3}	2.85
TDBHM	1.8614×10^{-2}	1.8614×10^{-2}	1.72
OSBIM	1.8343×10^{0}	1.8343×10^{0}	−0.26
EBM	7.8813×10^{-1}	7.8784×10^{-1}	0.09
ASHBM	1.7850×10^{-3}	1.7850×10^{-3}	2.74
LobIIIB	1.1200×10^{-2}	1.1200×10^{-2}	1.95
RadIIA	1.0465×10^{-1}	1.0464×10^{-1}	0.01

Table 7
Error distributions and precision factor (scd) for Problem 2 with number of steps = 2^8.

Method	RMS	Mean	scd
OLSBM	9.5646×10^{-8}	9.5643×10^{-8}	7.01
TDBHM	1.2896×10^{-6}	1.2896×10^{-6}	5.88
OSBIM	3.5813×10^{-4}	3.5813×10^{-4}	3.44
EBM	1.5825×10^{-5}	1.5825×10^{-5}	4.80
ASHBM	5.6618×10^{-7}	5.6616×10^{-7}	6.24
LobIIIB	3.4072×10^{-6}	3.4071×10^{-6}	5.46
RadIIA	1.2276×10^{-4}	1.2276×10^{-4}	3.91

order of magnitude with base 10 results in a decrease of the greatest absolute error by a factor of seven. This provides the empirical seventh order of convergence.

The first row of Table 5 suggests that the absolute maximum global error, norm, root mean square error, and number of steps are differed by about 58%, 44%, 42%, and 3%, respectively for the $\mathcal{L}$-stable seventh-order convergent hybrid block method (TDBHM) when compared with OLSBM. Similarly, the better performance of OLSBM can be interpreted for the remaining rows of Table 5. Furthermore, from the efficiency curves shown in Fig. 3, the proposed approach performs better than the other four methods concerning machine time (in seconds).

Problem 2. Consider the following stiff system from [38] whose first component is slowly varying in the specified interval while the second component decays rapidly in the transient phase:

$$\begin{aligned} &s_1'(x) = -10^{-5} s_1(x) + 10^2 s_2(x), \\ &s_2'(x) = -10^2 s_1(x) - 10^{-5} s_2(x), \\ &s_1(0) = 0, \quad s_2(0) = 1, \quad 0 \le x \le 1, \end{aligned} \tag{31}$$

with exact solution $s_1(x) = \exp(-10^{-5}x)\sin(100x)$, $s_2(x) = \exp(-10^{-5}x)\cos(100x)$.

Tables 6–8 show the results of the error distributions and precision factor (scd) for Problem 2 with number of steps $2^6, 2^8$ and 2^{10} respectively. The OLSBM performs better than the other four methods in terms of the infinity norm, root mean square (RMS), mean errors, and precision factor (scd). Moreover, from the efficiency curves shown in Fig. 4, the OLSBM has an efficiency curve that distinguished it as the most efficient method from all the other four methods taken for comparison.

Table 8
Error distributions and precision factor (scd) for Problem 2 with number of steps = 2^{10}.

Method	RMS	Mean	scd
OLSBM	5.8881×10^{-12}	5.8880×10^{-12}	11.22
TDBHM	7.8430×10^{-11}	7.8427×10^{-11}	10.10
OSBIM	6.8581×10^{-9}	6.8579×10^{-9}	8.16
EBM	3.2548×10^{-10}	3.2546×10^{-10}	9.48
ASHBM	1.4061×10^{-10}	1.4061×10^{-10}	9.84
LobIIIA	8.4382×10^{-10}	8.4380×10^{-10}	9.07
RadIIA	1.2101×10^{-7}	1.2100×10^{-7}	6.92

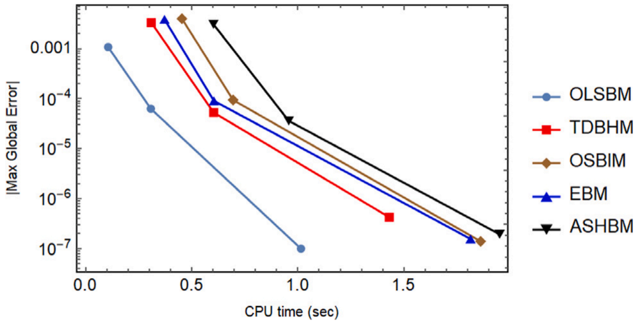

Fig. 4. Efficiency curves for Problem 2 with methods under consideration while the tolerance for the maximum global error is set to 10^{-i}, where $i = 3, 5, 7$.

Table 9
Error distributions at the final grid point in the state variable $s_1(x)$ over the integration interval $[0, 0.55139]$ for Problem 3 with number of steps = 2^i, $i = 2, 4, 8$.

Method	2^2	2^4	2^8
OLSBM	4.559×10^{-9}	3.975×10^{-13}	1.776×10^{-15}
TDBHM	2.691×10^{-7}	4.710×10^{-11}	1.776×10^{-15}
OSBIM	3.435×10^{-1}	7.200×10^{-2}	4.319×10^{-3}
EBM	1.514×10^{-1}	1.513×10^{-1}	8.660×10^{-3}
ASHBM	2.890×10^{-9}	8.893×10^{-13}	1.776×10^{-15}
LobIIIB	3.783×10^{-7}	1.022×10^{-10}	1.332×10^{-15}
RadIIA	7.045×10^{-7}	9.22×10^{-10}	2.220×10^{-15}

Table 10
Error distributions at the final grid point in the state variable $s_2(x)$ over the integration interval $[0, 0.55139]$ for Problem 3 with number of steps = 2^i, $i = 2, 4, 8$.

Method	2^2	2^4	2^8
OLSBM	6.762×10^{-8}	5.801×10^{-12}	2.665×10^{-15}
TDBHM	3.190×10^{-7}	7.097×10^{-11}	2.665×10^{-15}
OSBIM	6.247×10^{-1}	9.311×10^{-2}	5.173×10^{-3}
EBM	2.151×10^{-1}	2.151×10^{-1}	1.042×10^{-2}
ASHBM	7.608×10^{-8}	2.198×10^{-11}	2.665×10^{-15}
LobIIIB	5.153×10^{-6}	1.391×10^{-9}	3.109×10^{-15}
RadIIA	1.036×10^{-5}	1.330×10^{-8}	1.621×10^{-14}

Problem 3. Consider the following Van Der Pol System taken from [15]:

$$\begin{aligned} s_1'(x) &= s_2(x), \qquad s_1(0) = 2, \\ s_2'(x) &= \frac{(1 - s_1(x)^2)s_2(x) - s_1(x)}{\beta}, \qquad s_2(0) = -\frac{2}{3} + \frac{10}{81}\beta - \frac{292}{2187}\beta^2 - \frac{1814}{19683}\beta^3, \end{aligned} \tag{32}$$

where $\beta = 10^{-1}$. The reference solution at the final grid point over the integration interval $[0, 0.55139]$ is as follows

$$s_1(x_N) = 1.56337394423009, \quad s_2(x_N) = -1.00002083185427.$$

The Tables 9 and 10 represent the absolute errors in both state variables $s_1(x)$ and $s_2(x)$ respectively, obtained at the final grid point over the integration interval $[0, 0.55139]$ for Problem 3 with number of steps = 2^i, $i = 2, 4, 8$. It is observed that the proposed optimized method (OLSBM) performs better than rest of the methods with ASHBM being the most comparable method when the number of steps are 2^8. The OLSBM has an efficiency curve in Fig. 5 that distinguished it as the most efficient method from all the other four methods taken for comparison.

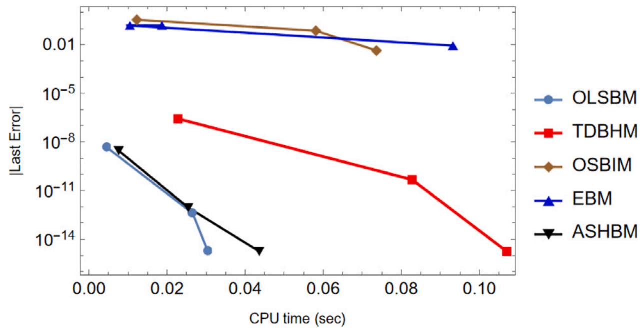

Fig. 5. Efficiency curves of the state variable $s_1(x)$ for Problem 3 with methods under consideration while the number of steps is 2^i, where $i = 2, 4, 8$.

Table 11
Error distributions and precision factor (scd) for Problem 4 with number of steps $= 2^6$.

Method	RMS	Mean	scd
OLSBM	2.2700×10^{-18}	1.8534×10^{-18}	17.73
TDBHM	2.9312×10^{-17}	2.3933×10^{-17}	16.62
OSBIM	8.3580×10^{-16}	6.8251×10^{-16}	15.17
EBM	4.1999×10^{-17}	3.4298×10^{-17}	16.47
ASHBM	1.7017×10^{-16}	1.3894×10^{-16}	15.86
LobIIIB	1.0210×10^{-15}	8.3364×10^{-16}	15.08
RadIIA	4.5496×10^{-13}	3.7148×10^{-13}	12.43

Table 12
Error distributions and precision factor (scd) for Problem 4 with number of steps $= 2^8$.

Method	RMS	Mean	scd
OLSBM	1.3896×10^{-22}	1.1346×10^{-22}	21.95
TDBHM	1.8380×10^{-21}	1.5007×10^{-21}	20.82
OSBIM	1.3043×10^{-20}	1.0653×10^{-20}	19.97
EBM	6.2302×10^{-22}	5.0884×10^{-22}	21.29
ASHBM	4.1543×10^{-20}	3.3920×10^{-20}	19.47
LobIIIB	2.4926×10^{-19}	2.0352×10^{-19}	18.69
RadIIA	4.4607×10^{-16}	3.6421×10^{-16}	15.44

Table 13
Error distributions and precision factor (scd) for Problem 4 with number of steps $= 2^{10}$.

Method	RMS	Mean	scd
OLSBM	8.4875×10^{-27}	6.9301×10^{-27}	26.16
TDBHM	1.1294×10^{-25}	9.2217×10^{-26}	25.04
OSBIM	2.0018×10^{-25}	1.6350×10^{-25}	24.79
EBM	9.5834×10^{-27}	7.8275×10^{-27}	26.11
ASHBM	1.0142×10^{-23}	8.2811×10^{-24}	23.08
LobIIIB	6.0853×10^{-23}	4.9687×10^{-23}	22.30
RadIIA	4.3605×10^{-19}	3.5603×10^{-19}	18.45

Problem 4. Consider the following linear stiff system [32] :

$$\begin{aligned} s_1'(x) &= -21 s_1(x) + 19 s_2(x) - 20 s_3(x), \\ s_2'(x) &= 19 s_1(x) - 21 s_2(x) + 20 s_3(x), \\ s_3'(x) &= 40 s_1(x) - 40 s_2(x) - 40 s_3(x), \\ s_1(0) &= 1, \quad s_2(0) = 0, \quad s_3(0) = -1, \quad 0 \le x \le 1, \end{aligned} \tag{33}$$

with exact solution $s_1(x) = \frac{1}{2}(\exp(-2x) + \exp(-40x)(\cos(40x) + \sin(40x)))$, $s_2(x) = \frac{1}{2}(\exp(-2x) - \exp(-40x)(\cos(40x) + \sin(40x)))$, $s_3(x) = \exp(-40x)(\cos(40x) + \sin(40x))$.

The numerical results in Tables 11–13 show that error distributions and precision factor (scd) for Problem 4 with number of steps $2^6, 2^8$ and 2^{10} respectively. The OLSBM performs better than the other four methods in terms of the infinity norm, root mean square (RMS), mean errors, and precision factor (scd). The OLSBM has an efficiency curve in Fig. 6 that distinguished it as the most efficient method from all the other four methods taken for comparison.

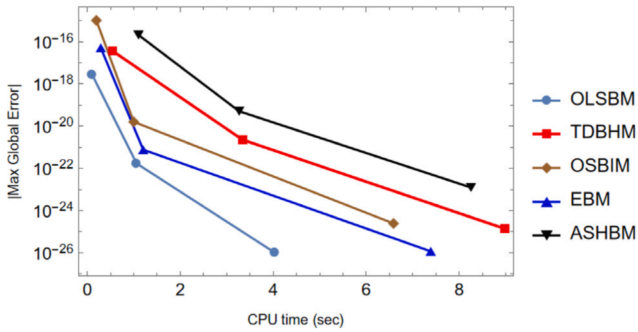

Fig. 6. Efficiency curves for Problem 4 with methods under consideration while the number of steps is 10^i, where $i = 6, 8, 10$.

Table 14
Error distributions and precision factor (scd) for Problem 5 with number of steps = 2^6.

Method	RMS	Mean	scd
OLSBM	9.4747×10^{-17}	8.1112×10^{-17}	15.79
TDBHM	1.4350×10^{-15}	1.2207×10^{-15}	14.66
OSBIM	5.3503×10^{-14}	4.7889×10^{-14}	13.10
EBM	4.9310×10^{-16}	4.5973×10^{-16}	15.17
ASHBM	1.0086×10^{-14}	9.1218×10^{-15}	13.80
LobIIIB	3.0921×10^{-13}	3.0808×10^{-13}	12.47
RadIIA	2.1929×10^{-11}	1.9150×10^{-11}	10.44

Table 15
Error distributions and precision factor (scd) for Problem 5 with number of steps = 2^8.

Method	RMS	Mean	scd
OLSBM	5.8056×10^{-21}	4.9822×10^{-21}	20.00
TDBHM	8.5703×10^{-20}	7.3083×10^{-20}	18.89
OSBIM	7.9472×10^{-19}	7.1643×10^{-19}	17.93
EBM	7.9655×10^{-21}	7.5230×10^{-21}	19.98
ASHBM	2.4628×10^{-18}	2.2273×10^{-18}	17.42
LobIIIB	7.5486×10^{-17}	7.5209×10^{-17}	16.08
RadIIA	2.1526×10^{-14}	1.8845×10^{-14}	13.45

Problem 5. A nonlinear two-body system [11] is considered below:

$$\begin{aligned} s_1''(x) &= \frac{-s_1(x)}{r^3}, \quad s_1(0) = 1, \quad s_1'(0) = 0, \\ s_2''(x) &= \frac{-s_2(x)}{r^3}, \quad s_2(0) = 0, \quad s_2'(0) = 1, \\ r &= \sqrt{s_1(x)^2 + s_2(x)^2}, \quad 0 \le x \le 2, \end{aligned} \tag{34}$$

with closed form solution $s_1(x) = \cos(x)$, $s_2(x) = \sin(x)$,

The numerical results in Tables 14–16 show the error distributions and precision factor (scd) for the four-dimensional nonlinear two-body system given in Problem 5 with number of steps 2^6, 2^8 and 2^{10} respectively. The OLSBM performs better than the other four methods in terms of the infinity norm, root mean square (RMS), mean errors, and precision factor (scd). It may be noted that the scd for the eighth-order method (EBM) gets a little higher than the proposed approach when $n = 2^{10}$. The OLSBM has an efficiency curve that distinguished it as the most efficient method from all the other four methods taken for comparison (see Fig. 7).

Problem 6. Finally, consider the periodic orbital system taken from [27,35]:

$$\begin{aligned} s_1''(x) &= -s_1(x) + \frac{\cos(x)}{1000}, \quad s_1(0) = 1, \quad s_1'(0) = 0, \\ s_2''(x) &= -s_2(x) + \frac{\sin(x)}{1000}, \quad s_2(0) = 1, \quad s_2'(0) = \frac{9995}{10000}, \end{aligned} \tag{35}$$

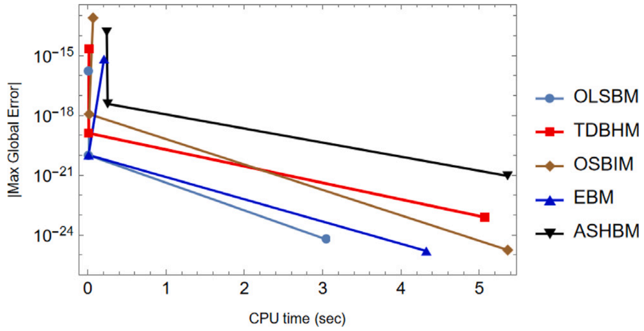

Fig. 7. Efficiency curves for Problem 5 with methods under consideration while the number of steps is 10^i, where $i = 6, 8, 10$.

Table 16
Error distributions and precision factor (scd) for Problem 5 with number of steps $= 2^{10}$.

Method	RMS	Mean	scd
OLSBM	3.5470×10^{-25}	3.0457×10^{-25}	24.22
TDBHM	5.2019×10^{-24}	4.4387×10^{-24}	23.10
OSBIM	1.2032×10^{-23}	1.0864×10^{-23}	22.75
EBM	1.2188×10^{-25}	1.1421×10^{-25}	24.80
ASHBM	6.0127×10^{-22}	5.4378×10^{-22}	21.03
LobIIIB	1.8429×10^{-20}	1.8362×10^{-20}	19.69
RadIIA	2.1049×10^{-17}	1.8438×10^{-17}	16.46

Table 17
Error distributions and precision factor (scd) for Problem 6 with number of steps $= 2^{6}$.

Method	RMS	Mean	scd
OLSBM	1.3984×10^{-11}	1.3928×10^{-11}	10.81
TDBHM	1.8395×10^{-10}	1.8320×10^{-10}	9.69
OSBIM	2.6458×10^{-8}	2.6305×10^{-8}	7.53
EBM	1.2120×10^{-9}	1.2073×10^{-9}	8.88
ASHBM	2.0918×10^{-10}	2.0834×10^{-10}	9.64
LobIIIB	1.2579×10^{-9}	1.2528×10^{-9}	8.86
RadIIA	1.1225×10^{-7}	1.1180×10^{-7}	6.91

Table 18
Error distributions and precision factor (scd) for Problem 6 with number of steps $= 2^{8}$.

Method	RMS	Mean	scd
OLSBM	8.5571×10^{-16}	8.5229×10^{-16}	15.03
TDBHM	1.1389×10^{-14}	1.1343×10^{-14}	13.90
OSBIM	4.1730×10^{-13}	4.1571×10^{-13}	12.34
EBM	1.9335×10^{-14}	1.9259×10^{-14}	13.67
ASHBM	5.1138×10^{-14}	5.0933×10^{-14}	13.25
LobIIIB	3.0737×10^{-13}	3.0614×10^{-13}	12.48
RadIIA	1.1001×10^{-10}	1.0957×10^{-10}	9.92

where the exact solution over the interval $[0, 10]$ is given as follows:

$$\begin{aligned} s_1(x) &= \cos(x) + \frac{x\sin(x)}{2000}, \\ s_2(x) &= \sin(x) - \frac{x\cos(x)}{2000}. \end{aligned} \tag{36}$$

The numerical results in Tables 17–19 show the error distributions and precision factor (scd) for the four-dimensional periodic orbit system given in Problem 6 with number of steps $2^6, 2^8$ and 2^{10} respectively. The OLSBM performs better than the other four methods in terms of the infinity norm, root mean square (RMS), mean errors, and precision factor (scd). It may be noted that all types of error and the scd for the proposed approach are more competitive than the rest of the methods used. The OLSBM has an efficiency curve in Fig. 8 that distinguished it as the most efficient method from all the other four methods taken for comparison.

Table 19
Error distributions and precision factor (scd) for Problem 6 with number of steps = 2^{10}.

Method	RMS	Mean	scd
OLSBM	5.2254×10^{-20}	5.2045×10^{-20}	19.24
TDBHM	6.9579×10^{-19}	6.9301×10^{-19}	18.12
OSBIM	6.3508×10^{-18}	6.3281×10^{-18}	17.16
EBM	2.9453×10^{-19}	2.9335×10^{-19}	18.49
ASHBM	1.2486×10^{-17}	1.2436×10^{-17}	16.86
LobIIIB	7.5049×10^{-17}	7.4748×10^{-17}	16.09
RadIIA	1.0750×10^{-13}	1.0707×10^{-13}	12.93

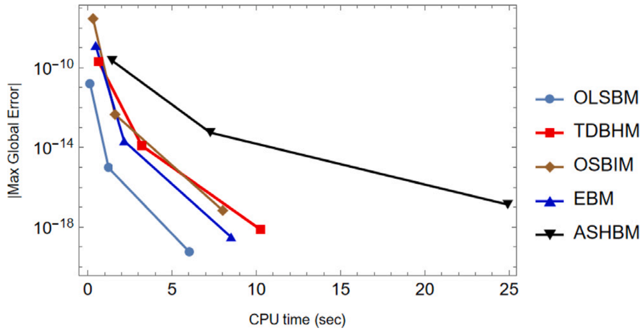

Fig. 8. Efficiency curves for Problem 6 with methods under consideration while the number of steps is 10^i, where $i = 6, 8, 10$.

6. Conclusion with future directions

A new efficient and optimized $\mathcal{L}$-stable one-step hybrid block method is developed in this paper. The optimization comes after imposing the vanishing of the first two leading terms of the local truncation error of the main formula. The proposed method, derived via interpolation and collocation concepts, is found to have a seventh-order of convergence for the main formula, zero-stability, absolute stability, consistency, and $\mathcal{L}$-stability features. Moreover, when applied to some highly stiff systems, the proposed method performs much better than several robust algorithms devised for the same purpose. Furthermore, the technique is better for accuracy and time-efficient as shown by the various efficiency curves. The adaptive step size approach of the proposed method will be investigated in the future. The future work will also devise strategies to modify the technique for solving partial differential equations.

CRediT authorship contribution statement

Sania Qureshi: Conceptualization, Formal analysis, Investigation, Software, Writing – original draft, Writing – review & editing. **Higinio Ramos:** Conceptualization, Validation, Writing – review & editing. **Amanullah Soomro:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Olusheye Aremu Akinfenwa:** Conceptualization, Formal analysis, Investigation, Software, Visualization, Writing – original draft, Writing – review & editing. **Moses Adebowale Akanbi:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

The data and code that support the findings of this study are available from the corresponding author, upon reasonable request.

Appendix A

Algorithm 1: Pseudo-code for the one-step optimized $\mathcal{L}$-stable block method with two intra-step points given in (14) under fixed stepsize approach.

Data: x_0, X (integration interval), N (number of steps), s_{00},
(initial values), g, $\frac{dg}{dx}$.
Result: sol (discrete approximate solution of the IVP (1))
1 Let $n = 0$, $\Delta x = \frac{X - x_0}{N}$.
2 Let $x_n = x_0, s_n = s_{00}$.
3 Let **sol** $= \{(x_n, s_n)\}$.
4 Solve (14) to obtain s_{n+k} where $k = u, v, 1$.
5 Let **sol** = **sol** $\cup\{(x_{n+k}, s_{n+k})\}_{k=u,v,1}$.
6 Let $x_n = x_n + \Delta x, s_n = s_{n+1}$.
7 Let $n = n + 1$,
8 **if** $n = N$ **then**
9 | go to 13
10 **else**
11 | go to 4;
12 **end**
13 **End**

Appendix B

Algorithm 2: Pseudo-code for the one-step optimized $\mathcal{L}$-stable block method with two intra-step points given in (14) under variable stepsize approach.

Data: Initial stepsize: $\Delta x = \Delta x_0 = \Delta x_{\text{old}}$, $x_m := x_0, s_m := s_0$; Integration interval: $[x_0, X_N]$;
Total number of steps in the main formula: $N - 1$; Initial value: x_0, s_0; Function s: $g(x, s(x))$; Given tolerance: tol;
Final point of the integration interval: X_N
Result: Approximations of the problem in (1) at selected points.
1 Introduce $g(x, u(x))$ and the initial values x_0, s_0;
2 **if** $x_m \geq X_N$ **then**
3 **end**
4 **if** $x_m + \Delta x > X_N$, $\Delta x = X_N - x_m$ **then**
5 **end**
6 **while** $x_m < X_N$, *then solve system of equations in* (14) *to get the values* s_{n+1} **do**
7 | compute s^*_{n+1} to get Y_{LE}.
8 **end**
9 **if** $|Y_{LE}| \leq$ *tol then accept the results and substitute* $\Delta x_{new} = 2 \times \Delta x_{old}$ **then**
10 **end**
11 Set $x_n = x_n + \Delta x, n = n + 1$ and use the formula in (29) to determine the new stepsize.
12 **if** $|Y_{LE}| >$ *tol, then reject the results and repeat the calculations using* (29) *and go to step (6)* **then**
13 **end**

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