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A computational method for a two-parameter singularly perturbed elliptic problem with boundary and interior layers

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Abstract

In this article, we investigate a two-dimensional (2-D) singularly perturbed convection–reaction–diffusion elliptic type problem where two parameters ϵ and μ multiply the diffusion and convection terms, respectively. Furthermore, we assume that jump discontinuities exist in the source term along the x- and y-axis. Due to the presence of perturbation parameters, the solutions to such problems show boundary and corner layers. Moreover, the discontinuity in the source term adds the interior layers to the solution whose suitable numerical approach is the important goal of this article. A numerical approach is carried out using an upwind finite-difference technique that includes an appropriate layer-adapted piecewise uniform Shishkin mesh. Some examples are presented which show the good performance of the proposed method and the agreement with the theoretical analysis.

Keywords: Discontinuous source term; Finite-difference method; Shishkin mesh; Elliptic equation; Two singular perturbation parameters; Two dimensional space

1. Introduction

A singularly perturbed differential equation has a small parameter that makes the behavior of the solutions very different than when it is set to zero. In other words, the use of asymptotic expansions cannot make the problem-solution uniformly approximated. The solution of a singularly perturbed problem has boundary (and/or interior) layers, where the solution has an extremely high gradient. A boundary (and/or interior) layer of regular or parabolic type may appear away from any corner of the domain. If the solutions of the reduced equation corresponding to ϵ =0 are very close to the boundary layer then it is said to be of parabolic type and in another case, it is said to be of the regular type. A boundary (and/or interior) layer that is located near a corner is called a corner type boundary layer.

Singularly perturbed elliptic problems are widespread in mathematical modeling. This article considers an equation of convection–reaction–diffusion type; Morton [17] gives several examples that are modeled using similar equations, ranging from simulation of oil and gas reservoirs, as well as magnetohydrodynamic flow, to chemical flow

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reactor theory (see, e.g. [19]), to boundary layers influenced by suction (or blowing) of some fluid (see, e.g. [27]). In all circumstances, if the model has to be realistic, the equations are too difficult to solve exactly, so the use of numerical methods is necessary. Classical approaches, on the other hand, can completely fail in the presence of layers (see, e.g. [26,30]), therefore this is a particularly active topic of research for numerical analysts.

The literature on parameter-uniform numerical methods is very extensive (see, [26]). Here we survey those most relevant to this work. Butuzov [4] investigated the asymptotic structure of a solution to a problem like the one considered here. In that paper, a substantial relationship between the ordering of small parameters was observed. When $\mu = O(\epsilon^{1/2})$, we are getting close to the reaction–diffusion situation with the layer structure. In 2-D reaction–diffusion problems, solutions have parabolic boundary layers along all the edges of the unit square (see, [5,15,23]). Gracia and Clavero [8] explain a class of 2-D singularly perturbed problems with reaction–diffusion type, using a compact finite-difference technique considering a piecewise uniform mesh to get more reliable estimates. When $\mu = 1$, we are getting close to the convection–diffusion situation with an exponential layer near the outflow boundary, and corner layers near the boundary layer junctions. In a 2-D elliptic convection–diffusion problem, solutions may have boundary and interior layers. Thus, the norms on derivatives of the solution in [13] demonstrate that the solution contains exponential type boundary layers on two sides of the domain under consideration, certain bounds in [28] show a parabolic type boundary layer in the solution and the bounds in [25] reveal both parabolic and exponential type boundary layers in the solution. The asymptotic character of singularly perturbed problems of convection–diffusion type was investigated in [29] to better understand their solutions and the difficulties associated with their numerical approaches. Lin et al. [12] handle an efficient approach based on a local discontinuous Galerkin (LDG) discretization for the weak form of 2-D singularly perturbed problems of convection–diffusion type. Andreev and Belukhina [3] explain the solution decomposition for a 2-D singularly perturbed problem of convection–diffusion type in a square domain. Nhan and Vulanovic [18] propose a complete finite-difference technique of positive type to solve a class of 2-D convection–diffusion singularly perturbed problems using a Bakhvalov mesh to get more reliable estimates. In [24], they solve the 2-D singularly perturbed convection–reaction–diffusion problem including boundary and interior layers, using the finite difference method (FDM) on a piecewise uniform Shishkin mesh. In the case when μ increases away from $\epsilon^{1/2}$ and remains small compared to $\mu = 1$, we have a completely different layer structure.

In a 2-D singularly perturbed two-parameter elliptic problem Li [11] considers an approach based on a finite-element method (FEM) with bilinear trial functions. O'Riordan et al. [20,22] consider the upwind finite difference method on suitable Shishkin meshes for a singularly perturbed two-parameter convection–diffusion problem over a two-dimensional domain with smooth boundaries. Zhang and Lv [32] handle an efficient approach based on the FEM using a Bakhvalov mesh for the 2-D singularly perturbed convection–diffusion two-parameter problems.

In this paper, motivated by the work of O'Riordan et al. [20,22], Shanthi et al. [1] and Rao et al. [24], we are introducing a finite-difference technique for solving the 2-D singularly perturbed convection–diffusion problem with two parameters where f is discontinuous along two lines, one in each coordinate direction, and demonstrating its robust convergence. Also, an upwind FDM is used but adjusted at the discontinuities.

Let us consider a two-parameter singularly perturbed elliptic convection–reaction–diffusion equation

$$L_{\epsilon,\mu}u(x, y) = f(x, y), \quad \forall(x, y) \in \Omega, \qquad u(x, y) = q(x, y), \quad \forall(x, y) \in \partial\Omega, \tag{1.1a}$$

where the differential operator is represented by

$$L_{\epsilon,\mu}u(x, y) = \epsilon^2\left(\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2}\right) + \mu^2\left(a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y}\right) - c(x, y)u(x, y), \tag{1.1b}$$

and

$$f(x, y) = \begin{cases} f_1(x, y) \text{ if } (x, y) \in \Omega_1 \\ f_2(x, y) \text{ if } (x, y) \in \Omega_2 \\ f_3(x, y) \text{ if } (x, y) \in \Omega_3 \\ f_4(x, y) \text{ if } (x, y) \in \Omega_4 \end{cases}$$

with f_i sufficiently differentiable in their respective domains.

The two small perturbation parameters satisfy $0 < \epsilon, \mu \ll 1$. The problem's domain is $\Omega = \bigcup_{k=1}^{4} \Omega_k$, being $\Omega_1 = (0, d) \times (0, d)$, $\Omega_2 = (d, 1) \times (0, d)$, $\Omega_3 = (0, d) \times (d, 1)$ and $\Omega_4 = (d, 1) \times (d, 1)$. Let $\Gamma_1 = \{(d, y) : 0 \le y \le 1,\ y \ne d\}$, $\Gamma_2 = \{(x, d) : 0 \le x \le 1,\ x \ne d\}$, with d any point in (0, 1).

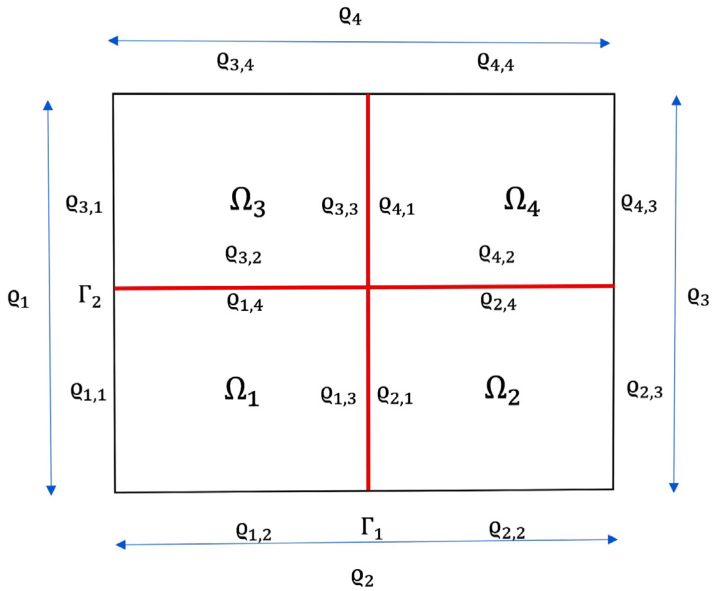


Fig. 1. Notation for subregions and domain boundaries.

The convection and reaction coefficients are positive, and bounded as follows

$$a(x, y) \geq \alpha_1 > 0, \qquad b(x, y) \geq \alpha_2 > 0 \qquad \text{and} \qquad c(x, y) > 0, \tag{1.2}$$

for some constants α_1, α_2. We use the notations $\alpha = \min(\alpha_1, \alpha_2)$, $\lambda = \min_{\bar{\Omega}}\left\{\frac{c}{2a}, \frac{c}{2b}\right\}$, and assume that the reaction and convection coefficients $a(x, y), b(x, y), c(x, y)$ belong to the Hölder space $C^{4,\gamma}(\bar{\Omega})$ and the given boundary function $q(x, y) \in C^{4,\gamma}(\partial\Omega)$, for some $\gamma \in (0, 1]$. We also assume that there are enough compatibility and regularity conditions [10,14] such that $u \in C^{4,\gamma}(\Omega_k)$, $k = 1, 2, 3, 4$.

Furthermore, we assume that the source term $f(x, y)$ has a jump discontinuity at both lines $x = d$ and $y = d$. As usual, we denote the jump discontinuities in any function w at a point $(x, y) \in \Omega$ along the lines $x = d$ or $y = d$ as $[w](d, y) = w(d^+, y) - w(d^-, y)$ and $[w](x, d) = w(x, d^+) - w(x, d^-)$, respectively. The average of any function w at those points are denoted as $[\bar{w}](d, y) = \frac{1}{2}[w(d^+, y) + w(d^-, y)]$ and $[\bar{w}](x, d) = \frac{1}{2}[w(x, d^+) + w(x, d^-)]$, respectively. Also, we use the notation $[\bar{w}_1, \bar{w}_2](d, d) = \frac{1}{2}([\bar{w}_1](d, d) + [\bar{w}_2](d, d))$, where $\bar{w}_1$ and $\bar{w}_2$ are the averages along $x = d$ and $y = d$, respectively.

The boundaries are defined using the following symbols:

$$\partial\Omega = \begin{cases} \varrho_1 = \{(0, y) \mid (0 \leq y \leq 1)\}, & \varrho_3 = \{(1, y) \mid (0 \leq y \leq 1)\}, \\ \varrho_2 = \{(x, 0) \mid (0 \leq x \leq 1)\}, & \varrho_4 = \{(x, 1) \mid (0 \leq x \leq 1)\}, \end{cases}$$

and $\varrho = \varrho_1 \cup \varrho_2 \cup \varrho_3 \cup \varrho_4$. Recalling from (1.1) that $u = q$ on the boundary, we denote by q_i the restriction of q onto ϱ_i, $i = 1, 2, 3, 4$.

We further denote the continuous subsets of the boundaries and the interior line segments of the discontinuity as $\varrho_{k,j}$, $c_{k,j}$, where $j = 1, 2, 3, 4$ indicates the edges and corners of Ω_k, respectively (see Fig. 1).

The article is structured as follows. In Section 2, we derive the minimum principle, stability estimate, and bounds of the continuous solution and its partial derivatives exhibiting their dependence on the singular perturbation parameters. Section 3 explores the numerical approach of the standard 5-point finite-difference scheme built on a Shishkin mesh. In Section 4, we derive the error estimation. In Section 5, some test problems are provided to verify the theoretical results, and Section 6 ends the article with some conclusions.

2. *A priori* bounds on the solution and its derivatives

The present Section contains the minimum principle, a stability estimate, and some useful bounds for the derivatives of the true solution. In addition, we obtain some bounds of the regular, singular, and corner layer components of the solution.

Lemma 2.1. *(Minimum principle): Let $L_{\epsilon,\mu}$ be the differential operator given in (1.1). If $\phi(x, y) \geq 0$ on $\partial\Omega$, $L_{\epsilon,\mu}\phi(x, y) \leq 0$ for all $(x, y) \in \Omega$, $\left[\frac{\partial\phi}{\partial x}\right](d, y) \leq 0$ on Γ_1, $\left[\frac{\partial\phi}{\partial y}\right](x, d) \leq 0$ on Γ_2, and $\left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right](d, d) \leq 0$, then it is $\phi(x, y) \geq 0$ for all $(x, y) \in \bar{\Omega}$.*

Proof. Consider the function ω on $\bar{\Omega}$ defined through $\phi(x, y) = \omega(x, y)\psi(x, y)$, with the function

$$\psi(x, y) = \exp\left(\frac{\mu^2(d - x)\alpha_1}{2\epsilon^2} + \frac{\mu^2(d - y)\alpha_2}{2\epsilon^2}\right), \quad (x, y) \in \bar{\Omega}.$$

Let be $\omega(x^*, y^*) = \min_{(x,y)\in\bar{\Omega}}\{\omega(x, y)\}$. If $\omega(x^*, y^*) \geq 0$, then there is nothing to prove. Suppose $\omega(x^*, y^*) < 0$. At the point (x^*, y^*) it is $\dfrac{\partial\omega}{\partial x}(x^*, y^*) = \dfrac{\partial\omega}{\partial y}(x^*, y^*) = 0$ and $\dfrac{\partial^2\omega}{\partial x^2}(x^*, y^*) \geq 0$, $\dfrac{\partial^2\omega}{\partial y^2}(x^*, y^*) \geq 0$. By the assumption on the boundary values, either the point $(x^*, y^*) \in \Omega$ or $(x^*, y^*) \in \Gamma_1 \cup \Gamma_2 \cup \{(d, d)\}$. Let us consider the two cases.

Case(i): Firstly, assume that $(x^*, y^*) \in \Omega$.

Since, by our assumption it is $\omega(x^*, y^*) < 0$, then we have

$$L_{\epsilon,\mu}\phi(x^*, y^*) = \psi(x^*, y^*)\Bigg(\epsilon^2\Delta\omega + \left(\frac{\mu^4\alpha_1}{2\epsilon^2}\left(\frac{\alpha_1}{2} - a(x^*, y^*)\right) + \frac{\mu^4\alpha_2}{2\epsilon^2}\left(\frac{\alpha_2}{2} - b(x^*, y^*)\right)\right)\omega(x^*, y^*)$$
$$- c\omega(x^*, y^*)\Bigg) > 0,$$

which contradicts the hypothesis. Thus, it must be $w(x^*, y^*) \geq 0$, and therefore the desired result is obtained.

Case(ii): $(x^*, y^*) \in \Gamma_1 \cup \Gamma_2 \cup \{(d, d)\}$.

Let us assume $(x^*, y^*) = (d, y^*)$. Since ω takes minimum value at (x^*, y^*), this implies that $\frac{\partial\omega}{\partial x}(d^+, y^*) \geq 0$ and $\frac{\partial\omega}{\partial x}(d^-, y^*) \leq 0$. Then, it is evident that $[\frac{\partial\omega}{\partial x}](d, y^*) \geq 0$. Now, since $\omega(d, y^*) < 0$, it follows that

$$\left[\frac{\partial\phi}{\partial x}\right](d, y^*) = \exp\left(\frac{\mu^2\alpha_2(d - y^*)}{2\epsilon^2}\right)\left(\left[\frac{\partial\omega}{\partial x}\right](d, y^*)\right) > 0,$$

which contradicts the hypothesis $[\frac{\partial\phi}{\partial x}](x, y) \leq 0,\ \forall(x, y) \in \Gamma_1$. The remaining two cases can be proved similarly. This completes the proof. □

A consequence of this minimum principle is the parameter uniform boundedness of the solution of (1.1) given below.

Lemma 2.2. *(Stability result): Let $u(x, y)$ be the continuous solution of (1.1). Then, it holds*

$$\|u\|_{\bar{\Omega}} \leq \frac{1}{\alpha}\|f\|_{\bar{\Omega}} + \max\left\{\|u\|_{\varrho_1}, \|u\|_{\varrho_2}, \|u\|_{\varrho_3}, \|u\|_{\varrho_4}\right\},$$

where $\|.\|$ represents the pointwise maximum norm.

Proof. We define the barrier functions

$$\phi^{\pm}(x, y) = \begin{cases} M + \frac{\|f\|_{\bar{\Omega}}}{\alpha}\left(\frac{1}{2} + \frac{x}{8} + \frac{y}{8} - \frac{d}{4}\right) \pm u(x, y), & (x, y) \in [0, d] \times [0, d], \\ M + \frac{\|f\|_{\bar{\Omega}}}{\alpha}\left(\frac{1}{2} - \frac{x}{4} + \frac{y}{8} + \frac{d}{8}\right) \pm u(x, y), & (x, y) \in (d, 1] \times [0, d], \\ M + \frac{\|f\|_{\bar{\Omega}}}{\alpha}\left(\frac{1}{2} + \frac{x}{8} - \frac{y}{4} + \frac{d}{8}\right) \pm u(x, y), & (x, y) \in [0, d] \times (d, 1], \\ M + \frac{\|f\|_{\bar{\Omega}}}{\alpha}\left(\frac{1}{2} - \frac{x}{4} - \frac{y}{4} + \frac{d}{2}\right) \pm u(x, y), & (x, y) \in (d, 1] \times (d, 1], \end{cases}$$

where $M = \max\{\|u\|_{\varrho_1}, \|u\|_{\varrho_2}, \|u\|_{\varrho_3}, \|u\|_{\varrho_4}\}$.

Then, clearly $\phi^{\pm}(x, y) \geq 0$, $\forall(x, y) \in \partial\Omega$.

For each $(x, y) \in \Omega$, we have

$$L_{\epsilon,\mu}\phi^{\pm}(x, y) \leq 0.$$

Since $u(x, y) \in C(\bar{\Omega}) \cap C^2(\Omega)$, we have

$$\left[\frac{\partial \phi^\pm}{\partial x}\right](d, y) = \frac{-3\|f\|_{\bar{\Omega}}}{8\alpha} \pm \left[\frac{\partial u^\pm}{\partial x}\right](d, y) \le 0,$$
$$\left[\frac{\partial \phi^\pm}{\partial y}\right](x, d) = \frac{-3\|f\|_{\bar{\Omega}}}{8\alpha} \pm \left[\frac{\partial u^\pm}{\partial y}\right](x, d) \le 0,$$
$$\left[\frac{\partial \phi^\pm}{\partial x}, \frac{\partial \phi^\pm}{\partial y}\right](d, d) \le 0.$$

It follows from Lemma 2.1 that $\phi^\pm(x, y) \ge 0, \ \forall (x, y) \in \bar{\Omega}$, which allows to get the bound on $\|u(x, y)\|_{\bar{\Omega}}$. □

The derivatives of the solution satisfy the parameter-explicit bound shown below.

Lemma 2.3. *Let u be the continuous solution of (1.1). Then, for $1 \le i + j \le 4$, if $\alpha\mu^2 \le \lambda\epsilon$, it holds*

$$\left\|\frac{\partial^{i+j} u}{\partial x^i \partial y^j}\right\|_{\Omega_k} \le C\epsilon^{-(i+j)}, \tag{2.1}$$

and if $\alpha\mu^2 > \lambda\epsilon$, then it is

$$\left\|\frac{\partial^{i+j} u}{\partial x^i \partial y^j}\right\|_{\Omega_k} \le C\left(\frac{\epsilon}{\mu}\right)^{-2(i+j)}, \tag{2.2}$$

where C denotes a generic positive constant which is independent of the parameters ϵ, μ.

Proof. It can be easily obtained using standard procedures, as in [16,31]. □

Now, we decompose the continuous solution $u(x, y)$ into regular and singular components. The regular components $v_k(x, y)$, $k = 1, 2, 3, 4$, are obtained as the solution of the problem

$$L_{\epsilon,\mu} v_k = f, \ \forall (x, y) \in \Omega, \tag{2.3a}$$

with the following boundary conditions, respectively, according to the values of k,

$$v_k(x, y) = q_1(y), \quad \forall (x, y) \in \varrho_1, \ k = 1, 3, \quad v_k(x, y) = q_3(y), \quad \forall (x, y) \in \varrho_3, \ k = 2, 4, \tag{2.3b}$$
$$v_k(x, y) = q_2(x), \quad \forall (x, y) \in \varrho_2, \ k = 1, 2, \quad v_k(x, y) = q_4(x), \quad \forall (x, y) \in \varrho_4, \ k = 3, 4, \tag{2.3c}$$
$$[v_k](x, y) = 0, \quad [\bar{v}_k](x, y) = 0, \quad \forall (x, y) \in \Gamma_1 \cup \Gamma_2, \tag{2.3d}$$

and

$$[\bar{v}_{k_1}, \bar{v}_{k_2}](d, d) = 0. \tag{2.3e}$$

Lemma 2.4. *The regular components $v_k(x, y)$ at (2.3) and their derivatives satisfy the bounds:*

$$\left\|\frac{\partial^{i+j} v_k}{\partial x^i \partial y^j}\right\|_{\Omega} \le C\left(1 + \epsilon^{2-(i+j)}\right), \qquad \textit{for} \quad 1 \le i + j \le 4, \quad \textit{if} \quad \alpha\mu^2 \le \lambda\epsilon,$$
$$\left\|\frac{\partial^{i+j} v_k}{\partial x^i \partial y^j}\right\|_{\Omega} \le C\left(1 + \left(\frac{\epsilon}{\mu}\right)^{4-2(i+j)}\right), \quad \textit{for} \quad 1 \le i + j \le 4, \quad \textit{if} \quad \alpha\mu^2 > \lambda\epsilon.$$

Proof. Let us consider the two cases.

Case(i): Firstly, assume that $\alpha\mu^2 \le \lambda\epsilon$.

Suppose the regular components $v_k(x, y)$, $k = 1, 2, 3, 4$, can be decomposed as

$$v_k(x, y) = v_k^0(x, y) + \epsilon v_k^1(x, y, \epsilon, \mu) + \epsilon^2 v_k^2(x, y, \epsilon, \mu), \tag{2.4}$$

where

$$-cv_k^0 = f, \quad cv_k^1 = \epsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v_k^0 + \frac{\mu^2}{\epsilon}\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)v_k^0,$$

$$L_{\epsilon,\mu}v_k^2 = -\epsilon\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_k^1 - \frac{\mu^2}{\epsilon}\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)v_k^1, \quad v_k^2 = 0,\ \forall(x,y)\in\partial\Omega_k,\ k=1,\ 2,\ 3,\ 4.$$

Since v_k^0 and v_k^1 satisfy zero order differential equations and there are no compatibility issues, the term v_k^2 denotes the solution of the elliptic problem on the domain Ω_k. Since, $v_k^0 \in C^{4,\gamma}(\bar{\Omega}_k)$, we get $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_k^0 \in C^{2,\gamma}(\bar{\Omega}_k)$, $k = 1,\ 2,\ 3,\ 4$.

Applying Lemmas 2.2 and 2.3 to the problem (2.3), it results that $v_k \in C^{4,\gamma}(\Omega_k)$ and

$$\left\|\frac{\partial^{i+j}v_k}{\partial x^i\partial y^j}\right\| \le C(1+\epsilon^{2-(i+j)}), \quad 1\le i+j\le 4,\ k=1,\ 2,\ 3,\ 4. \tag{2.5}$$

Case(ii): Now, consider that $\alpha\mu^2 > \lambda\epsilon$.

Suppose the regular components $v_k(x, y)$, $k = 1,\ 2,\ 3,\ 4$, can be decomposed as

$$v_k(x,y) = v_k^0(x,y,\mu) + \epsilon^2 v_k^1(x,y,\mu) + \epsilon^4 v_k^2(x,y,\epsilon,\mu),$$

where

$$L_\mu v_k^0 = f, \quad L_\mu v_k^1 = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_k^0,$$

$$L_{\epsilon,\mu}v_k^2 = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_k^1, \quad v_k^2 = 0,\ \forall(x,y)\in\partial\Omega_k,\ k=1,\ 2,\ 3,\ 4.$$

Applying Lemmas 2.3 and 2.6 to the problem (2.3) we get

$$\left\|\frac{\partial^{i+j}v_k}{\partial x^i\partial y^j}\right\| \le C\left(1+\left(\frac{\epsilon}{\mu}\right)^{4-2(i+j)}\right), \quad 1\le i+j\le 4,\ k=1,\ 2,\ 3,\ 4. \tag{2.6}$$

□

Now, let us consider the first order problem

$$L_\mu u(x,y) = \mu^2\left(a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}\right) - c(x,y)u = f(x,y),\ \forall(x,y)\in\Omega, \tag{2.7a}$$

$$u(x,y) = q_i,\ (x,y)\in\varrho_i,\ i=3,4. \tag{2.7b}$$

Note that L_μ satisfies the following comparison principle:

Lemma 2.5. *Let L_μ be the differential operator given in (2.7). If $\phi(x,y)\ge 0$ on ϱ_i, $i=3,4$, $L_\mu\phi(x,y)\le 0$ for all $(x,y)\in\Omega$, $\left[\frac{\partial\phi}{\partial x}\right](d,y)\le 0$ on Γ_1, $\left[\frac{\partial\phi}{\partial y}\right](x,d)\le 0$ on Γ_2, and $\left[\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right](d,d)\le 0$, then it is $\phi(x,y)\ge 0$ for all $(x,y)\in\bar{\Omega}$.*

Proof. The proof is similar to the one of Lemma 2.1. □

Lemma 2.6. *Let $u(x, y)$ be the continuous solution of problem (2.7). Then, it holds the stability estimate*

$$\|u\|_{\bar{\Omega}} \le \frac{1}{\alpha}\|L_\mu u\|_\Omega + \max\left\{\|u\|_{\varrho_3}, \|u\|_{\varrho_4}\right\}.$$

Proof. This Lemma can be proved similarly to Lemma 2.2. □

Corresponding to the edge $x = 0$ in Ω_1 (see Fig. 1), a layer function w_{R_1} exists that is determined by:

$$L_{\epsilon,\mu}w_{R_1} = 0, \quad \forall(x,y)\in\Omega, \tag{2.8a}$$

$$w_{R_1}(x,y) = u - v_1, \quad \forall(x,y)\in\varrho_{1,1}, \tag{2.8b}$$

$$w_{R_1}(x,y) = 0, \quad \forall(x,y)\in\varrho_{3,1}\cup\varrho_2\cup\varrho_3\cup\varrho_4, \tag{2.8c}$$

$$[w_{R_1}](x,y) = 0, \quad [\bar{w}_{R_1}](x,y) = 0, \quad \forall(x,y)\in\Gamma_1\cup\Gamma_2, \tag{2.8d}$$

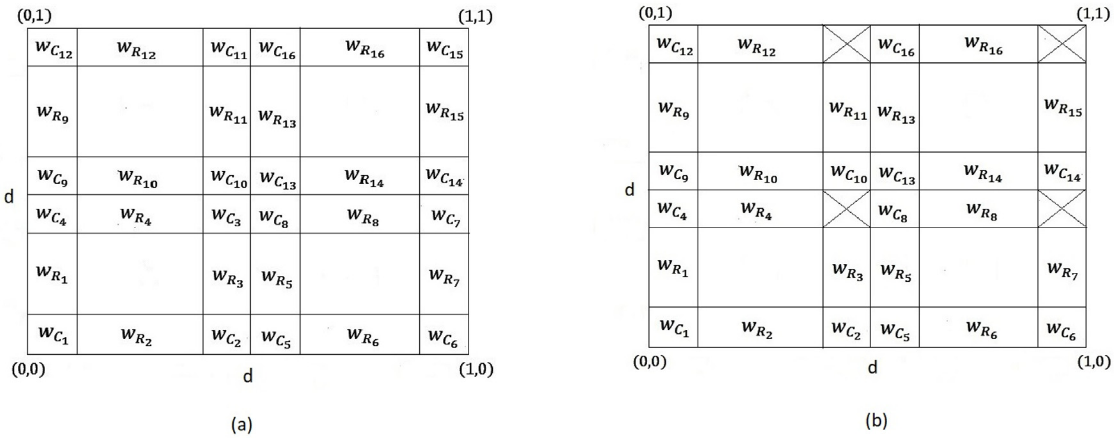


Fig. 2. Situation of the layer functions in the domain Ω: (a) when $\alpha\mu^2 \le \lambda\epsilon$; (b) when $\alpha\mu^2 > \lambda\epsilon$.

$$[(\bar{w}_{R_1})_1, (\bar{w}_{R_1})_2](x, y) = 0, \quad \forall (x, y) = (d, d). \tag{2.8e}$$

The following Lemmas give some bounds on the derivatives of the layer components, which are necessary for the convergence analysis.

Lemma 2.7. *Let w_{R_1} be the boundary layer component satisfying the equations in (2.8). If $\alpha\mu^2 \le \lambda\epsilon$, then it holds*

$$|w_{R_1}(x, y)| \le C \exp\left(-\frac{\alpha\lambda}{\epsilon}x\right), \quad \left\|\frac{\partial^i w_{R_1}}{\partial x^i}\right\| \le C(1 + \epsilon^{1-i}), \quad i = 1, 2, 3, 4.$$

If $\alpha\mu^2 > \lambda\epsilon$, then it holds

$$|w_{R_1}(x, y)| \le C \exp\left(-\frac{\alpha\mu^2}{\epsilon^2}x\right), \quad \left\|\frac{\partial^i w_{R_1}}{\partial x^i}\right\| \le C\left(1 + \left(\frac{\epsilon}{\mu}\right)^{2-2i}\right), \quad i = 1, 2, 3, 4.$$

Proof. It can be referred from the works by O'Riordan et al. [20,22]. □

Similarly, as has been done for w_{R_1}, for the different edges of Ω_k, $k = 1, 2, 3, 4$, (see Fig. 1) we can consider the corresponding boundary layer components w_{R_i}, $i = 2, 3, \dots, 16$, (see Fig. 2) for which we can obtain similar bounds as in Lemma 2.7.

Related to the corner at $c_{1,1} = (0, 0)$ in Ω_1, we consider the corner layer component w_{C_1}, which is determined by

$$L_{\epsilon,\mu} w_{C_1} = 0, \quad \forall (x, y) \in \Omega, \tag{2.9a}$$

$$w_{C_1} = -w_{R_1}, \ \forall (x, y) \in \varrho_{1,1}, \quad w_{C_1} = -w_{R_2}, \ \forall (x, y) \in \varrho_{1,2}, \tag{2.9b}$$

$$w_{C_1}(x, y) = 0, \ \forall (x, y) \in \varrho_{2,2} \cup \varrho_{3,1} \cup \varrho_3 \cup \varrho_4, \tag{2.9c}$$

$$[w_{C_1}](x, y) = 0, \quad [\bar{w}_{C_1}](x, y) = 0, \quad \forall (x, y) \in \Gamma_1 \cup \Gamma_2, \tag{2.9d}$$

$$[(\bar{w}_{C_1})_1, (\bar{w}_{C_1})_2](x, y) = 0, \quad \forall (x, y) = (d, d). \tag{2.9e}$$

Lemma 2.8. *Let w_{C_1} be the corner layer component satisfying the equations in (2.9). If $\alpha\mu^2 \le \lambda\epsilon$, then it holds*

$$|w_{C_1}(x, y)| \le C \exp\left(-\frac{\alpha\lambda}{\epsilon}x\right) \exp\left(-\frac{\alpha\lambda}{\epsilon}y\right), \quad \left\|\frac{\partial^{i+j} w_{C_1}}{\partial x^i \partial y^j}\right\| \le C(1 + \epsilon^{-(i+j)}), \quad 1 \le i + j \le 4.$$

If $\alpha\mu^2 > \lambda\epsilon$, then it holds

$$|w_{C_1}(x,y)| \le C\exp\left(-\frac{\alpha\mu^2}{\epsilon^2}x\right)\exp\left(-\frac{\alpha\mu^2}{\epsilon^2}y\right), \quad \left\|\frac{\partial^{i+j}w_{C_1}}{\partial x^i\partial y^j}\right\| \le C\left(\frac{\epsilon}{\mu}\right)^{-2(i+j)}, \quad 1 \le i+j \le 4.$$

Proof. It can be referred from the works by O'Riordan et al. [20,22]. □

Similarly, we can describe other corner layer components w_{C_k}, $k = 2, 3, \ldots, 16$, corresponding to the different corners of Ω_k, $k = 1, 2, 3, 4$, which verify similar bounds as the ones in Lemma 2.8.

Finally, from the above lemmas we can establish the following theorem.

Theorem 2.9. *The continuous solution $u(x, y)$ of (1.1) may be written as*

$$u = \sum_{k=1}^{4} v_k + \sum_{j=1}^{16} w_{R_j} + \sum_{j=1}^{16} w_{C_j},$$

where

$$L_{\epsilon,\mu}v_k = f, \quad L_{\epsilon,\mu}w_{R_j} = 0, \quad L_{\epsilon,\mu}w_{C_j} = 0,\ k = 1,2,3,4,\ j = 1,2,3,\ldots,16.$$

$$\begin{cases} w_{C_k} \neq 0, & \text{if } \alpha\mu^2 \le \lambda\epsilon,\ k = 3,7,11,15,\\ w_{C_k} = 0, & \text{if } \alpha\mu^2 > \lambda\epsilon,\ k = 3,7,11,15. \end{cases}$$

Furthermore, the regular and singular components and their derivatives satisfy the following bounds

$$\begin{cases} \left\|\dfrac{\partial^{i+j}v_k}{\partial x^i\partial y^j}\right\| \le C(1+\epsilon^{2-(i+j)}), & 1 \le i+j \le 4,\ k=1,2,3,4,\ \text{if } \alpha\mu^2 \le \lambda\epsilon,\\ \left\|\dfrac{\partial^{i+j}v_k}{\partial x^i\partial y^j}\right\| \le C(1+(\frac{\epsilon}{\mu})^{4-2(i+j)}), & 1 \le i+j \le 4,\ k=1,2,3,4,\ \text{if } \alpha\mu^2 > \lambda\epsilon, \end{cases}$$

$$\begin{cases} |w_{R_1}(x,y)| & \le Ce^{-\theta_1 x};\\ |w_{R_2}(x,y)| & \le Ce^{-\theta_1 y};\\ |w_{R_3}(x,y)| & \le Ce^{-\theta_2(d-x)};\\ |w_{R_4}(x,y)| & \le Ce^{-\theta_2(d-y)}; \end{cases} \quad \begin{cases} |w_{C_1}(x,y)| & \le Ce^{-\theta_1 x}e^{-\theta_1 y},\\ |w_{C_2}(x,y)| & \le Ce^{-\theta_2(d-x)}e^{-\theta_1 y},\\ |w_{C_3}(x,y)| & \le Ce^{-\frac{\lambda\alpha}{2\epsilon}(d-x)}e^{-\frac{\lambda\alpha}{2\epsilon}(d-y)},\\ |w_{C_4}(x,y)| & \le Ce^{-\theta_1 x}e^{-\theta_2(d-y)}, \end{cases}$$

$$\begin{cases} |w_{R_5}(x,y)| & \le Ce^{-\theta_1(x-d)};\\ |w_{R_6}(x,y)| & \le Ce^{-\theta_1 y};\\ |w_{R_7}(x,y)| & \le Ce^{-\theta_2(1-x)};\\ |w_{R_8}(x,y)| & \le Ce^{-\theta_2(d-y)}; \end{cases} \quad \begin{cases} |w_{C_5}(x,y)| & \le Ce^{-\theta_1(x-d)}e^{-\theta_1 y},\\ |w_{C_6}(x,y)| & \le Ce^{-\theta_2(1-x)}e^{-\theta_1 y},\\ |w_{C_7}(x,y)| & \le Ce^{-\frac{\lambda\alpha}{2\epsilon}(1-x)}e^{-\frac{\lambda\alpha}{2\epsilon}(d-y)},\\ |w_{C_8}(x,y)| & \le Ce^{-\theta_1(x-d)}e^{-\theta_2(d-y)}, \end{cases}$$

$$\begin{cases} |w_{R_9}(x,y)| & \le Ce^{-\theta_1 x};\\ |w_{R_{10}}(x,y)| & \le Ce^{-\theta_1(y-d)};\\ |w_{R_{11}}(x,y)| & \le Ce^{-\theta_2(d-x)};\\ |w_{R_{12}}(x,y)| & \le Ce^{-\theta_2(1-y)}; \end{cases} \quad \begin{cases} |w_{C_9}(x,y)| & \le Ce^{-\theta_1 x}e^{-\theta_1(y-d)},\\ |w_{C_{10}}(x,y)| & \le Ce^{-\theta_2(d-x)}e^{-\theta_1(y-d)},\\ |w_{C_{11}}(x,y)| & \le Ce^{-\frac{\lambda\alpha}{2\epsilon}(d-x)}e^{-\frac{\lambda\alpha}{2\epsilon}(1-y)},\\ |w_{C_{12}}(x,y)| & \le Ce^{-\theta_1 x}e^{-\theta_2(1-y)}, \end{cases}$$

$$\begin{cases} |w_{R_{13}}(x,y)| & \le Ce^{-\theta_1(x-d)};\\ |w_{R_{14}}(x,y)| & \le Ce^{-\theta_1(y-d)};\\ |w_{R_{15}}(x,y)| & \le Ce^{-\theta_2(1-x)};\\ |w_{R_{16}}(x,y)| & \le Ce^{-\theta_2(1-y)}; \end{cases} \quad \begin{cases} |w_{C_{13}}(x,y)| & \le Ce^{-\theta_1(x-d)}e^{-\theta_1(y-d)},\\ |w_{C_{14}}(x,y)| & \le Ce^{-\theta_2(1-x)}e^{-\theta_1(y-d)},\\ |w_{C_{15}}(x,y)| & \le Ce^{-\frac{\lambda\alpha}{2\epsilon}(1-x)}e^{-\frac{\lambda\alpha}{2\epsilon}(1-y)},\\ |w_{C_{16}}(x,y)| & \le Ce^{-\theta_1(x-d)}e^{-\theta_2(1-y)}, \end{cases}$$

$$\text{where } \theta_1 = \begin{cases} \frac{\lambda\alpha}{\epsilon}, & \text{if } \alpha\mu^2 \le \lambda\epsilon,\\ \frac{\alpha\mu^2}{\epsilon^2}, & \text{if } \alpha\mu^2 > \lambda\epsilon, \end{cases} \qquad \theta_2 = \begin{cases} \frac{\lambda\alpha}{2\epsilon}, & \text{if } \alpha\mu^2 \le \lambda\epsilon,\\ \frac{\lambda}{2\mu^2}, & \text{if } \alpha\mu^2 > \lambda\epsilon, \end{cases} \tag{2.10}$$

$$\begin{cases} \left\| \dfrac{\partial^i w_{R_k}}{\partial x^i} \right\| \le C(1+\epsilon^{(2-i)}), & \textit{where } k=2,4,6,8,10,12,14,16 \textit{ and } 1 \le i \le 4, \quad \textit{if } \alpha\mu^2 \le \lambda\epsilon, \\ \left\| \dfrac{\partial^i w_{R_k}}{\partial x^i} \right\| \le C\left(1+\left(\dfrac{\epsilon}{\mu}\right)^{2-2i}\right), & \textit{where } k=2,6,10,14 \textit{ and } 1 \le i \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon, \\ \left\| \dfrac{\partial^i w_{R_k}}{\partial x^i} \right\| \le C\left(1+\left(\dfrac{1}{\mu}\right)^{2i-2}\right), & \textit{where } k=4,8,12,16 \textit{ and } 1 \le i \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon, \\ \left\| \dfrac{\partial^j w_{R_k}}{\partial y^j} \right\| \le C(1+\epsilon^{(2-j)}), & \textit{where } k=1,3,5,7,9,11,13,15 \textit{ and } 1 \le j \le 4, \quad \textit{if } \alpha\mu^2 \le \lambda\epsilon, \\ \left\| \dfrac{\partial^j w_{R_k}}{\partial y^j} \right\| \le C\left(1+\left(\dfrac{\epsilon}{\mu}\right)^{2-2j}\right), & \textit{where } k=1,5,9,13 \textit{ and } 1 \le j \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon, \\ \left\| \dfrac{\partial^j w_{R_k}}{\partial y^j} \right\| \le C\left(1+\left(\dfrac{1}{\mu}\right)^{2j-2}\right), & \textit{where } k=3,7,11,15 \textit{ and } 1 \le j \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon, \end{cases}$$

$$\begin{cases} \max\left\{ \left\| \dfrac{\partial^{i+j} w_{R_k}}{\partial x^i \partial y^j} \right\|, \left\| \dfrac{\partial^{i+j} w_{C_k}}{\partial x^i \partial y^j} \right\| \right\} \le C\epsilon^{-(i+j)}, & 1 \le i+j \le 4, \quad \textit{if } \alpha\mu^2 \le \lambda\epsilon, \\ \left\| \dfrac{\partial^{i+j} w_{C_k}}{\partial x^i \partial y^j} \right\| \le C\left(\dfrac{\epsilon}{\mu}\right)^{-2(i+j)}, & 1 \le i+j \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon,\ k=1,5,9,13, \\ \left\| \dfrac{\partial w_{C_k}}{\partial x} \right\| \le C\left(\dfrac{1}{\mu^2}\right), \left\| \dfrac{\partial^2 w_{C_k}}{\partial x^2} \right\| \le C\left(\dfrac{1}{\epsilon^2}\right), & \textit{if } \alpha\mu^2 > \lambda\epsilon,\ k=2,6,10,14, \\ \left\| \dfrac{\partial^3 w_{C_k}}{\partial x^3} \right\| \le C\left(\dfrac{\mu^2}{\epsilon^4}\right), \left\| \dfrac{\partial^4 w_{C_k}}{\partial x^4} \right\| \le C\left(\dfrac{\mu^4}{\epsilon^6}\right), & \textit{if } \alpha\mu^2 > \lambda\epsilon,\ k=2,6,10,14, \\ \left\| \dfrac{\partial^i w_{C_k}}{\partial x^i} \right\| \le C\left(\dfrac{\epsilon}{\mu}\right)^{-2i}, & 1 \le i \le 4, \quad \textit{if } \alpha\mu^2 > \lambda\epsilon,\ k=4,8,12,16. \end{cases} \qquad \square$$

The continuous solution $u(x, y)$ of (1.1) can be determined as follows

$$u(x,y) = \begin{cases} (v_1 + w_{R_1} + w_{R_2} + w_{R_3} + w_{R_4} + w_{C_1} + w_{C_2} + w_{C_3} + w_{C_4})(x,y), & \forall (x,y) \in \Omega_1, \\ (v_2 + w_{R_5} + w_{R_6} + w_{R_7} + w_{R_8} + w_{C_5} + w_{C_6} + w_{C_7} + w_{C_8})(x,y), & \forall (x,y) \in \Omega_2, \\ (v_3 + w_{R_9} + w_{R_{10}} + w_{R_{11}} + w_{R_{12}} + w_{C_9} + w_{C_{10}} + w_{C_{11}} + w_{C_{12}})(x,y), & \forall (x,y) \in \Omega_3, \\ (v_4 + w_{R_{13}} + w_{R_{14}} + w_{R_{15}} + w_{R_{16}} + w_{C_{13}} + w_{C_{14}} + w_{C_{15}} + w_{C_{16}})(x,y), & \forall (x,y) \in \Omega_4, \\ [(v + w_R + w_C)](x,y) = 0, & \forall (x,y) \in \Gamma_1, \\ [(v + w_R + w_C)](x,y) = 0, & \forall (x,y) \in \Gamma_2, \\ ([\bar{v}_{1^*}, \bar{v}_{2^*}] + [\bar{w}_{R_1^*}, \bar{w}_{R_2^*}] + [\bar{w}_{C_1^*}, \bar{w}_{C_2^*}])(x,y) = 0, & (x,y) = (d,d), \end{cases}$$

where, $v = \sum_{i=1}^{4} v_i,\ w_R = \sum_{k=1}^{16} w_{R_k},\ w_C = \sum_{k=1}^{16} w_{C_k}$,

$$\begin{cases} w_{C_k} \ne 0, & \text{if } \alpha\mu^2 \le \lambda\epsilon,\ k=3,7,11,15, \\ w_{C_k} = 0, & \text{if } \alpha\mu^2 > \lambda\epsilon,\ k=3,7,11,15, \end{cases}$$

and $\bar{v}_{1^*}, \bar{w}_{R_1^*}, \bar{w}_{C_1^*}$ are the averages along the y-axis and $\bar{v}_{2^*}, \bar{w}_{R_2^*}, \bar{w}_{C_2^*}$ are the averages along the x-axis.

3. Discretization of the problem

3.1. Shishkin Mesh

In this Section, we construct a Shishkin mesh for problem (1.2) and use the finite-difference technique on this mesh to get a numerical solution. To make an appropriate fitted piecewise uniform mesh, we first subdivide the unit interval in x-direction into six subintervals as

$$[0,1]=[0,\sigma_1]\cup[\sigma_1,d-\sigma_2]\cup[d-\sigma_2,d]\cup[d,d+\sigma_1]\cup[d+\sigma_1,1-\sigma_2]\cup[1-\sigma_2,1]. \tag{3.1}$$

Similarly, we can subdivide the unit interval in y-direction into six subintervals, where the transition points σ_1 and σ_2 are expressed as

$$\sigma_1=\min\left\{\frac{d}{4},\frac{2}{\theta_1}\ln N\right\},\ \sigma_2=\min\left\{\frac{d}{4},\frac{2}{\theta_2}\ln N\right\},$$

where θ_1 and θ_2 are defined in (2.10) and N is the total number of subintervals determined by the grid points.

The grid points of the space variables are defined by

$$x_i=\begin{cases} ih_1 & \text{if}\quad 0\le i\le N/8,\\ \sigma_1+(i-N/8)H_1 & \text{if}\quad N/8\le i\le 3N/8,\\ (d-\sigma_2)+(i-3N/8)h_2 & \text{if}\quad 3N/8\le i\le N/2,\\ d+(i-N/2)h_1 & \text{if}\quad N/2\le i\le 5N/8,\\ d+\sigma_1+(i-5N/8)H_2 & \text{if}\quad 5N/8\le i\le 7N/8,\\ (1-\sigma_2)+(i-7N/8)h_2 & \text{if}\quad 7N/8\le i\le N,\end{cases}$$

and the y_j are described similarly, where the step sizes are given by

$$h_1=k_1=\frac{8\sigma_1}{N},\quad H_1=K_1=\frac{4(d-\sigma_1-\sigma_2)}{N},\quad h_2=k_2=\frac{8\sigma_2}{N},\quad H_2=K_2=\frac{4(1-\sigma_1-\sigma_2-d)}{N},$$

$$h_{i+1}=x_{i+1}-x_i,\quad k_{j+1}=y_{j+1}-y_j,\ 0\le i,j\le N-1,$$

$$\text{and}\quad \bar{h}_i=\frac{h_{i+1}+h_i}{2},\quad \bar{k}_j=\frac{k_{j+1}+k_j}{2},\ 1\le i,j\le N-1.$$

The interior regions of the mesh are defined by $\Omega^N=\bigcup_{k=1}^4\Omega_k^N$, where the subdomains are given by

$$\Omega_1{}^N=\left\{(x_i,y_j):1\le i\le\frac{N}{2}-1,1\le j\le\frac{N}{2}-1\right\};\quad \Omega_2{}^N=\left\{(x_i,y_j):\frac{N}{2}+1\le i\le 1,1\le j\le\frac{N}{2}-1\right\},$$
$$\Omega_3^N=\left\{(x_i,y_j):1\le i\le\frac{N}{2}-1,\frac{N}{2}+1\le j\le N-1\right\};$$
$$\Omega_4^N=\left\{(x_i,y_j):\frac{N}{2}+1\le i\le 1,\frac{N}{2}+1\le j\le N-1\right\}.$$

The boundaries and interior line segments of these subdomains are denoted as

$$\varrho_1^N=\left\{(0,y_j)\,\middle|\,(0\le j\le N)\right\},\quad \varrho_2^N=\left\{(x_i,0)\,\middle|\,(0\le i\le N)\right\},$$
$$\varrho_3^N=\left\{(1,y_j)\,\middle|\,(0\le j\le N)\right\},\quad \varrho_4^N=\left\{(x_i,1)\,\middle|\,(0\le i\le N)\right\},$$
$$\Gamma_1^N=\left\{(x_{\frac{N}{2}},y_j):1\le j\le N,\ j\ne\frac{N}{2}\right\},\quad \Gamma_2^N=\left\{(x_i,y_{\frac{N}{2}}):1\le i\le N,\ i\ne\frac{N}{2}\right\},$$

and $\varrho^N=\varrho_1^N\cup\varrho_2^N\cup\varrho_3^N\cup\varrho_4^N$.

3.2. Finite difference method (FDM)

On the piecewise-uniform mesh $\bar{\Omega}^N$, we discretize the problem in (1.1) as follows

$$\begin{cases} L_{\epsilon,\mu}^N U(x_i,y_j)=F(x_i,y_j),\quad \forall(x_i,y_j)\in\Omega^N\cup\Gamma_1^N\cup\Gamma_2^N\cup(x_{N/2},y_{N/2}),\\ U(x_i,y_j)=q_1(y_j),\ (x_i,y_j)\in\varrho_1^N,\quad U(x_i,y_j)=q_2(x_i),\ (x_i,y_j)\in\varrho_2^N,\\ U(x_i,y_j)=q_3(y_j),\ (x_i,y_j)\in\varrho_3^N,\quad U(x_i,y_j)=q_4(x_i),\ (x_i,y_j)\in\varrho_4^N,\end{cases} \tag{3.2}$$

where

$$L^N_{\epsilon,\mu}U(x_i,y_j)\equiv\begin{cases}L^N_{u,\epsilon,\mu}U(x_i,y_j), & \forall(x_i,y_j)\in\Omega^N,\\ L^N_{m,\epsilon,\mu}U(x_i,y_j), & \forall(x_i,y_j)\in\Gamma_1^N\cup\Gamma_2^N,\\ L^N_{m,\epsilon,\mu}U(x_i,y_j), & \forall(x_i,y_j)=(x_{N/2},y_{N/2}),\end{cases}\tag{3.3}$$

$$F(x_i,y_j)\equiv\begin{cases}f(x_i,y_j), & \forall(x_i,y_j)\in\Omega^N,\\ \bar f(x_i,y_j), & \forall(x_i,y_j)\in\Gamma_1^N\cup\Gamma_2^N,\\ \hat f(x_i,y_j), & \forall(x_i,y_j)=(x_{N/2},y_{N/2}),\end{cases}\tag{3.4}$$

$$\begin{cases}L^N_{u,\epsilon,\mu}U(x_i,y_j)=\epsilon^2(\delta^2_{xx}+\delta^2_{yy})U(x_i,y_j)+\mu^2(a_{ij}D^+_x+b_{ij}D^+_y)U(x_i,y_j)-c_{ij}U(x_i,y_j)\\ =f(x_i,y_j),\ \forall(x_i,y_j)\in\Omega^N,\\ L^N_{m,\epsilon,\mu}U(x_i,y_j)=\epsilon^2(\delta^2_{xx}+\delta^2_{yy})U(x_i,y_j)+\mu^2(\bar a_{ij}D^+_x+\bar b_{ij}D^+_y)U(x_i,y_j)-\bar c_{ij}\bar U(x_i,y_j)\\ =\bar f(x_i,y_j),\forall(x_i,y_j)\in(\Gamma_1^N\cup\Gamma_2^N),\\ L^N_{m,\epsilon,\mu}U(x_i,y_j)=\epsilon^2(\delta^2_{xx}+\delta^2_{yy})U(x_i,y_j)+\mu^2(\hat a_{ij}D^+_x+\hat b_{ij}D^+_y)U(x_i,y_j)-\hat c_{ij}\hat U(x_i,y_j)\\ =\hat f(x_i,y_j),\ (x_i,y_j)=(x_{N/2},y_{N/2}),\end{cases}\tag{3.5}$$

where

$$\begin{cases}\bar a(x_i,y_j)=a\left(\frac{x_{i+1}+x_{i-1}}{2},y_j\right), & (x_i,y_j)\in\Gamma_1^N,\\ \bar a(x_i,y_j)=a\left(x_i,\frac{y_{j+1}+y_{j-1}}{2}\right), & (x_i,y_j)\in\Gamma_2^N,\end{cases}\qquad \begin{cases}\bar f(x_i,y_j)=f\left(\frac{x_{i+1}+x_{i-1}}{2},y_j\right), & (x_i,y_j)\in\Gamma_1^N,\\ \bar f(x_i,y_j)=f\left(x_i,\frac{y_{j+1}+y_{j-1}}{2}\right), & (x_i,y_j)\in\Gamma_2^N,\end{cases}$$

$$\hat f(x_i,y_j)=\left\{f\left(\frac{x_{i+1}+x_{i-1}}{2},\frac{y_{j+1}+y_{j-1}}{2}\right),\ (x_i,y_j)=(d,d),\right.$$

and $\bar b$, $\bar c$, $\hat a$, $\hat b$ and $\hat c$ can be defined similarly on Γ_1^N, Γ_2^N and (d,d), respectively.

Further, the discrete differential operators in (3.5) are defined as follows

$$\begin{cases}D^-_xU(x_i,y_j)=\frac{U(x_i,y_j)-U(x_{i-1},y_j)}{h_i}, & D^-_yU(x_i,y_j)=\frac{U(x_i,y_j)-U(x_i,y_{j-1})}{k_j},\\ D^+_xU(x_i,y_j)=\frac{U(x_{i+1},y_j)-U(x_i,y_j)}{h_{i+1}}, & D^+_yU(x_i,y_j)=\frac{U(x_i,y_{j+1})-U(x_i,y_j)}{k_{j+1}},\\ \delta^2_{xx}U(x_i,y_j)=\frac{1}{\bar h_i}(D^+_xU(x_i,y_j)-D^-_xU(x_i,y_j)), & \delta^2_{yy}U(x_i,y_j)=\frac{1}{\bar k_j}(D^+_yU(x_i,y_j)-D^-_yU(x_i,y_j)).\end{cases}$$

The following Lemmas demonstrate that the discrete operator above yields a stable numerical solution.

Lemma 3.1 (*Discrete minimum principle*). *Let $L^N_{\epsilon,\mu}$ be the discrete operator given in (3.2). If $\phi(x_i,y_j)\geq 0$ on ϱ^N, $L^N_{u,\epsilon,\mu}\phi(x_i,y_j)\leq 0$, $\forall(x_i,y_j)\in\Omega^N$ and $L^N_{m,\epsilon,\mu}\bar\phi(x_i,y_j)\leq 0$, $\forall(x_i,y_j)\in\Gamma_1^N\cup\Gamma_2^N$, $L^N_{m,\epsilon,\mu}\hat\phi(x_i,y_j)\leq 0$, $(x_i,y_j)=(x_{N/2},y_{N/2})$, then $\phi(x_i,y_j)\geq 0$, $\forall(x_i,y_j)\in\bar\Omega^N$.*

Proof. It can be proved using the procedure adopted from [1] and suitable barrier functions. □

Lemma 3.2 (*Discrete stability result*). *Let $U(x_i,y_j)$ be the discrete solution of (3.2). Then it holds the stability estimate*

$$\|U(x_i,y_j)\|_{\bar\Omega^N}\leq\frac{1}{\alpha}\|f\|_{\Omega^N}+\max\left\{\|U\|_{\varrho^N}\right\},$$

where $\|.\|$ denotes the pointwise maximum norm.

Proof. It can be proved using Lemma 3.1. □

4. Error analysis

Lemma 3.2 will be used to prove the uniform convergence. Using standard techniques, the local truncation error may be readily bounded as

$$|L^N_{\epsilon,\mu}(U-u)(x_i,y_j)| \le \begin{cases} C\epsilon^2\left(\bar{h}_i\left\|\frac{\partial^3 u}{\partial x^3}\right\| + \bar{k}_j\left\|\frac{\partial^3 u}{\partial y^3}\right\|\right) + C\mu^2\left(h_{i+1}\left\|\frac{\partial^2 u}{\partial x^2}\right\| + k_{j+1}\left\|\frac{\partial^2 u}{\partial y^2}\right\|\right), \\ \quad \text{if } x_i = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \text{ or } y_j = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \\ C\epsilon^2\left({h_i}^2\left\|\frac{\partial^4 u}{\partial x^4}\right\| + {k_j}^2\left\|\frac{\partial^4 u}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 u}{\partial x^2}\right\| + k_j\left\|\frac{\partial^2 u}{\partial y^2}\right\|\right), \quad \text{otherwise.} \end{cases} \tag{4.1}$$

To get suitable bounds of this error, we decompose the discrete solution as

$$U = \sum_{k=1}^{4} V_k + \sum_{l=1}^{16} W_{R_l} + \sum_{m=1}^{16} W_{C_m},$$

where

$$\begin{cases} W_{C_m} \neq 0, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \ m = 3, 7, 11, 15, \\ W_{C_m} = 0, & \text{if } \alpha\mu^2 > \lambda\epsilon, \ m = 3, 7, 11, 15, \end{cases}$$

and V_k, $k = 1, 2, 3, 4$ are the discrete smooth components, W_{R_l}, $l = 1, \ldots, 16$ are the discrete boundary and interior layer components, W_{C_m}, $m = 1, \ldots, 16$ are the discrete corner layer components. These components are the solutions of the following problems, respectively:

$$\begin{cases} L^N_{\epsilon,\mu} V_k(x_i,y_j) = f(x_i,y_j), & \forall (x_i,y_j) \in \Omega^N, \quad k = 1,2,3,4, \\ V_k(x_i,y_j) = v_k(x_i,y_j), & \forall (x_i,y_j) \in \varrho^N, \\ [V_k](x_i,y_j) = 0, \quad [\bar{V}_k](x_i,y_j) = 0, & \forall (x_i,y_j) \in \Gamma_1^N \cup \Gamma_2^N, \\ [\bar{V}_{k_1}, \bar{V}_{k_2}](x_i,y_j) = 0, & (x_i,y_j) = (x_{N/2}, y_{N/2}). \end{cases} \tag{4.2}$$

$$\begin{cases} L^N_{\epsilon,\mu} W_{R_l}(x_i,y_j) = 0, & \forall (x_i,y_j) \in \Omega^N, \quad l = 1,\ldots,16, \\ W_{R_l}(x_i,y_j) = w_{R_l}(x_i,y_j), & \forall (x_i,y_j) \in \varrho^N, \\ [W_{R_l}](x_i,y_j) = 0, \quad [\bar{W}_{R_l}](x_i,y_j) = 0, & \forall (x_i,y_j) \in \Gamma_1^N \cup \Gamma_2^N, \\ [(\bar{W}_{R_l})_1, (\bar{W}_{R_l})_2](x_i,y_j) = 0, & (x_i,y_j) = (x_{N/2}, y_{N/2}). \end{cases} \tag{4.3}$$

$$\begin{cases} L^N_{\epsilon,\mu} W_{C_m}(x_i,y_j) = 0, & \forall (x_i,y_j) \in \Omega^N, \quad m = 1,\ldots,16, \\ W_{C_m}(x_i,y_j) = w_{C_m}(x_i,y_j), & \forall (x_i,y_j) \in \varrho^N, \\ [W_{C_m}](x_i,y_j) = 0, \quad [\bar{W}_{C_m}](x_i,y_j) = 0, & \forall (x_i,y_j) \in \Gamma_1^N \cup \Gamma_2^N, \\ [(\bar{W}_{C_m})_1, (\bar{W}_{C_m})_2](x_i,y_j) = 0, & (x_i,y_j) = (x_{N/2}, y_{N/2}). \end{cases} \tag{4.4}$$

Using the result (2.5), from (2.3) and (4.2) we get the following straightforward estimate

$$|L^N_{\epsilon,\mu}(V_k - v_k)(x_i,y_j)| \le \begin{cases} C\epsilon N^{-1}, & \text{if } x_i = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \text{ or } y_j = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \\ CN^{-2}, & \text{otherwise.} \end{cases}$$

Following [5,16], we consider the barrier function

$$\Psi(x_i,y_j) = CN^{-2}(\phi(x_i) + \phi(y_j)) + CN^{-2},$$

where $\phi(z)$ represents the piecewise-linear polynomial

$$\phi(z) = \begin{cases} 1, & 0 \le z \le \sigma_1, \\ 1 - \frac{z-\sigma_1}{2(d-\sigma_1-\sigma_2)}, & \sigma_1 \le z \le d-\sigma_2, \\ \frac{d-z}{2\sigma_2}, & d-\sigma_2 \le z \le d, \\ \frac{z-d}{2\sigma_1}, & d \le z \le d+\sigma_1, \\ 1 - \frac{z-d-\sigma_1}{2(1-d-\sigma_1-\sigma_2)}, & d+\sigma_1 \le z \le 1-\sigma_2, \\ \frac{1-z}{2\sigma_2}, & 1-\sigma_2 \le z \le 1. \end{cases}$$

Noting that $1/\sigma_2 \ge 4$, we have that

$$\epsilon^2 \delta_x^2 \Psi(x_i) = \begin{cases} O(-N^{-1}\epsilon^2), & x_i = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \\ 0, & \text{otherwise}, \end{cases}$$
$$\epsilon^2 \delta_y^2 \Psi(y_j) = \begin{cases} O(-N^{-1}\epsilon^2), & y_j = \sigma_1, d-\sigma_2, d+\sigma_1, 1-\sigma_2, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$D_x^+ \Psi(x_i) \le 0, \;\; D_y^+ \Psi(y_j) \le 0.$$

Combining this with Lemma 3.2 we get

$$\|V_k - v_k\| \le CN^{-2}, \quad k = 1, 2, 3, 4, \tag{4.5}$$

which shows a suitable bound for the errors of the regular components.

We utilize evidence-based on suitable barrier functions to show ϵ-uniform bounds of the errors related to the corner and edge components. We consider the barrier functions as shown below:

$$\begin{cases} B_{w_{R_1};i} = \begin{cases} \prod_{a=1}^{i}(1+h_a\theta_1)^{-1}, & i \neq 0,\ 1 \le i < \frac{N}{2}, \\ 1, & i = 0, \end{cases} & B_{w_{R_5};i} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{i}(1+h_a\theta_1)^{-1}, \\ \quad i \neq N,\ \frac{N}{2}+1 \le i < N, \\ 1, & i = N, \end{cases} \\ B_{w_{R_2};j} = \begin{cases} \prod_{a=1}^{j}(1+k_a\theta_1)^{-1}, & j \neq 0,\ 1 \le j < \frac{N}{2}, \\ 1, & j = 0, \end{cases} & B_{w_{R_6};j} = \begin{cases} \prod_{a=1}^{j}(1+k_a\theta_1)^{-1}, \\ \quad j \neq \frac{N}{2},\ 1 \le j < \frac{N}{2}, \\ 1, & j = \frac{N}{2}, \end{cases} \\ B_{w_{R_3};i} = \begin{cases} \prod_{a=i+1}^{\frac{N}{2}}(1+h_a\theta_2)^{-1}, & i \neq \frac{N}{2},\ 0 \le i < \frac{N}{2}, \\ 1, & i = \frac{N}{2}, \end{cases} & B_{w_{R_7};i} = \begin{cases} \prod_{a=i+1}^{N}(1+h_a\theta_2)^{-1}, \\ \quad i \neq N,\ \frac{N}{2}+1 \le i < N, \\ 1, & i = N, \end{cases} \\ B_{w_{R_4};j} = \begin{cases} \prod_{a=j+1}^{\frac{N}{2}}(1+k_a\theta_2)^{-1}, & j \neq \frac{N}{2},\ 0 \le j < \frac{N}{2}, \\ 1, & j = \frac{N}{2}, \end{cases} & B_{w_{R_8};j} = \begin{cases} \prod_{a=j+1}^{\frac{N}{2}}(1+k_a\theta_2)^{-1}, \\ \quad j \neq \frac{N}{2},\ 0 \le j < \frac{N}{2}, \\ 1, & j = \frac{N}{2}, \end{cases} \end{cases}$$

$$\begin{cases} B_{w_{R_9};i} = \begin{cases} \prod_{a=1}^{i}(1+h_a\theta_1)^{-1}, & i \neq 0,\ 1 \le i < \frac{N}{2}, \\ 1, & i = 0, \end{cases} & B_{w_{R_{13}};i} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{i}(1+h_a\theta_1)^{-1}, & i \neq N,\ \frac{N}{2}+1 \le i < N, \\ 1, & i = N, \end{cases} \\ B_{w_{R_{10}};j} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{j}(1+k_a\theta_1)^{-1}, & j \neq N,\ \frac{N}{2}+1 \le j < N, \\ 1, & j = N, \end{cases} & B_{w_{R_{14}};j} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{j}(1+k_a\theta_1)^{-1}, & j \neq N,\ \frac{N}{2}+1 \le j < N, \\ 1, & j = N, \end{cases} \\ B_{w_{R_{11}};i} = \begin{cases} \prod_{a=i+1}^{\frac{N}{2}}(1+h_a\theta_2)^{-1}, & i \neq \frac{N}{2},\ 1 \le i < \frac{N}{2}, \\ 1, & i = \frac{N}{2}, \end{cases} & B_{w_{R_{15}};i} = \begin{cases} \prod_{a=i+1}^{N}(1+h_a\theta_2)^{-1}, & i \neq N,\ \frac{N}{2}+1 \le i < N, \\ 1, & i = N, \end{cases} \\ B_{w_{R_{12}};j} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{j}(1+k_a\theta_2)^{-1}, & j \neq N,\ \frac{N}{2}+1 \le j < N, \\ 1, & j = N, \end{cases} & B_{w_{R_{16}};j} = \begin{cases} \prod_{a=\frac{N}{2}+1}^{j}(1+k_a\theta_2)^{-1}, & j \neq N,\ \frac{N}{2}+1 \le j < N, \\ 1, & j = N. \end{cases} \end{cases}$$

The above functions depict first-order Taylor estimates of the exponential functions associated with the singular components of problem (1.1). For all j, we have $\exp(-\theta_1 x_i) = \prod_{a=1}^{i}\exp(-\theta_1 h_a) \le B_{w_{R_1,i}}$, and for $\sigma_1 < \frac{d}{8}$ and $N/8 \le i \le N/2$ we have

$$B_{w_{R_1,i}} \le B_{w_{R_1,N/8}} = \left(1 + \frac{16\ \ln N}{N}\right)^{-N/8} \le CN^{-2}, \tag{4.6}$$

$$L^N_{\epsilon,\mu} B_{w_{R_1,i}} \le (\epsilon^2\theta_1^2 - \mu a(x_i, y_j)\theta_1 - c(x_i, y_j))B_{w_{R_1,i}}. \tag{4.7}$$

Similar bound may be obtained for the remaining edge functions.

Lemma 4.1. *If w_{R_l} and W_{R_l} are the solutions of* (2.8) and (4.3)*, respectively, then, for $l = 1, 2, \ldots, 16$,*

$$|w_{R_l}(x_i, y_j) - W_{R_l}(x_i, y_j)| \le C(N^{-2}\ln^2 N), \quad \text{if } \alpha\mu^2 \le \lambda\epsilon.$$

Proof. If $\sigma_1 = d/4$ the proof can be obtained using standard techniques by taking into account that $\epsilon^{-2} \le C(\ln N)^2$. Thus, we will assume that $\sigma_1 < d/4$. Here we merely provide the specifications pertaining to the edge layer function w_{R_1}. Similar results are valid for the remaining boundary layer components. From (4.3) and Theorem 2.9, we have

$$|W_{R_1}(x_i, y_j)| = |w_{R_1}(x_i, y_j)| \le Ce^{-\theta_1 x_i} \le CB_{w_{R_1,i}}, \quad (x_i, y_j) \in \varrho^N. \tag{4.8}$$

Further, for all internal grid points $(x_i, y_j) \in \Omega_1^N$, from (4.3), (4.7), and the discrete minimum principle, we have

$$|W_{R_1}(x_i, y_j)| \le B_{w_{R_1,i}}. \tag{4.9}$$

After applying (4.9) and Theorem 2.9, we conclude that

$$|w_{R_1}(x_i, y_j) - W_{R_1}(x_i, y_j)| \le |w_{R_1}(x_i, y_j)| + |W_{R_1}(x_i, y_j)| \le CB_{w_{R_1,i}}$$

Finally, from (4.6), we have

$$|w_{R_1}(x_i, y_j) - W_{R_1}(x_i, y_j)| \le CN^{-2}, \qquad N/8 \le i \le N/2,\ 0 \le j \le N/2. \tag{4.10}$$

To get appropriate bounds of the error in the region $\Omega^N_{1,1} = \{(x_i, y_j) \mid 0 < i < N/8,\ 0 < j < N/2\}$, we proceed as follows. Applying Taylor expansions, we get

$$|L^N_{\epsilon,\mu}[W_{R_1} - w_{R_1}](x_i, y_j)| \leq \begin{cases} C\epsilon^2\left({h_i}^2\left\|\dfrac{\partial^4 w_{R_1}}{\partial x^4}\right\| + \bar{k}_j\left\|\dfrac{\partial^3 w_{R_1}}{\partial y^3}\right\|\right) + C\mu^2\left(h_i\left\|\dfrac{\partial^2 w_{R_1}}{\partial x^2}\right\| + \bar{k}_j\left\|\dfrac{\partial^2 w_{R_1}}{\partial y^2}\right\|\right), \\ \text{if } j = \frac{N}{8},\ \frac{3N}{8}, \\ C\epsilon^2\left({h_i}^2\left\|\dfrac{\partial^4 w_{R_1}}{\partial x^4}\right\| + {k_j}^2\left\|\dfrac{\partial^4 w_{R_1}}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\dfrac{\partial^2 w_{R_1}}{\partial x^2}\right\| + k_j\left\|\dfrac{\partial^2 w_{R_1}}{\partial y^2}\right\|\right), \\ \text{otherwise.} \end{cases} \tag{4.11}$$

If $\alpha\mu^2 \leq \lambda\epsilon$, from Theorem 2.9, we have

$$|L^N_{\epsilon,\mu}[W_{R_1}(x_i, y_j) - w_{R_1}(x_i, y_j)]| \leq \begin{cases} CN^{-2}\ln^2 N + C\mu^2(N^{-1} + N^{-1}\ln^2 N), & \text{if } j = \frac{N}{8},\ \frac{3N}{8}, \\ CN^{-2}\ln^2 N + C\mu^2 N^{-1}\ln^2 N, & \text{otherwise,} \end{cases}$$

and using the discrete minimum principle and a suitable barrier function on $\bar{\Omega}^N_{1,1}$, we obtain

$$|w_{R_1}(x_i, y_j) - W_{R_1}(x_i, y_j)| \leq CN^{-2}\ln^2 N, \quad (x_i, y_j) \in \bar{\Omega}^N_{1,1}. \tag{4.12}$$

The result follows easily from (4.10) and (4.12).

Similar results can be obtained for the remaining boundary and interior layer components w_{R_l}, $l = 2, 3, \ldots, 16$. □

Lemma 4.2. *If w_{R_l} and W_{R_l} are the solutions of* (2.8) and (4.3)*, respectively, then, for $\alpha\mu^2 > \lambda\epsilon$ it holds*

$$|w_{R_l}(x_i, y_j) - W_{R_l}(x_i, y_j)| \leq \begin{cases} CN^{-1}\ln^2 N, & \text{if } l = 1, 2, 5, 6, 9, 10, 13, 14, \\ CN^{-1}\ln N, & \text{if } l = 3, 4, 7, 8, 11, 12, 15, 16. \end{cases}$$

Proof. If $\sigma_2 = d/4$ the proof can be obtained using standard techniques by taking into account that $(\frac{1}{\mu})^2 \leq C\ln N$ and $\left(\frac{\mu}{\epsilon}\right)^2 \leq C\ln N$. Thus, we will assume that $\sigma_2 < d/4$. Here we merely provide the specifications pertaining to the edge layer function w_{R_7}. Similar results can be obtained for the remaining boundary layer components. From (4.3) and Theorem 2.9, we have

$$|W_{R_7}(x_i, y_j)| = |w_{R_7}(x_i, y_j)| \leq Ce^{-\theta_2(1-x_i)} \leq CB_{w_{R_7,i}}, \quad (x_i, y_j) \in \varrho^N. \tag{4.13}$$

Further, for all internal grid points $(x_i, y_j) \in \Omega^N_2$, from (4.3), (4.7), and the discrete minimum principle, we have

$$|W_{R_7}(x_i, y_j)| \leq B_{w_{R_7,i}}. \tag{4.14}$$

Therefore, applying Theorem 2.9 and (4.14), we conclude that

$$|w_{R_7}(x_i, y_j) - W_{R_7}(x_i, y_j)| \leq |w_{R_7}(x, y_j)| + |W_{R_7}(x, y_j)| \leq CB_{w_{R_7,i}}.$$

Therefore, from the corresponding result as in (4.6), we have

$$|w_{R_7}(x_i, y_j) - W_{R_7}(x_i, y_j)| \leq CN^{-2}, \qquad 0 \leq i \leq 7N/8,\ 0 \leq j \leq d. \tag{4.15}$$

To get appropriate bounds for the error in the region $\Omega^N_{2,1} = \{(x_i, y_j) \mid 7N/8 < i < N,\ 0 < j < N/2\}$, we proceed as follows . Applying Taylor series, we get

$$|L^N_{\epsilon,\mu}[W_{R_7} - w_{R_7}](x_i, y_j)| \leq \begin{cases} C\epsilon^2\left({h_i}^2\left\|\dfrac{\partial^4 w_{R_7}}{\partial x^4}\right\| + \bar{k}_j\left\|\dfrac{\partial^3 w_{R_7}}{\partial y^3}\right\|\right) + C\mu^2\left(h_i\left\|\dfrac{\partial^2 w_{R_7}}{\partial x^2}\right\| + \bar{k}_j\left\|\dfrac{\partial^2 w_{R_7}}{\partial y^2}\right\|\right), \\ \text{if } j = \frac{N}{8},\ \frac{3N}{8}, \\ C\epsilon^2\left({h_i}^2\left\|\dfrac{\partial^4 w_{R_7}}{\partial x^4}\right\| + {k_j}^2\left\|\dfrac{\partial^4 w_{R_7}}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\dfrac{\partial^2 w_{R_7}}{\partial x^2}\right\| + k_j\left\|\dfrac{\partial^2 w_{R_7}}{\partial y^2}\right\|\right), \\ \text{otherwise.} \end{cases}$$

If $\alpha\mu^2 > \lambda\epsilon$, from Theorem 2.9, it follows that

$$|L^N_{\epsilon,\mu}[W_{R_7}(x_i, y_j) - w_{R_7}(x_i, y_j)]| \leq \begin{cases} CN^{-1}\ln N, & \text{if } j = \frac{N}{8}, \frac{3N}{8}, \\ CN^{-1}\ln N, & \text{otherwise.} \end{cases}$$

Therefore, the discrete minimum principle, only on $\bar{\Omega}^N_{2,1}$, leads to

$$|w_{R_7}(x_i, y_j) - W_{R_7}(x_i, y_j)| \leq CN^{-1}\ln N, \quad (x_i, y_j) \in \bar{\Omega}^N_{2,1}. \tag{4.16}$$

The result follows easily from (4.15) and (4.16).

Likewise, we can prove the corresponding bounds for the other boundary and interior layer components w_{R_l}, $l = 3, 4, 7, 8, 11, 12, 15, 16$.

If $\alpha\mu^2 > \lambda\epsilon$, we examine the boundary layer function w_{R_1}. From (4.6)–(4.11) and Theorem 2.9, we have

$$|L^N_{\epsilon,\mu}[W_{R_1}(x_i, y_j) - w_{R_1}(x_i, y_j)]| \leq \begin{cases} C\mu^4\epsilon^{-2}(N^{-1}\ln N + N^{-1}), & \text{if } j = \frac{N}{8}, \frac{3N}{8}, \\ C\mu^4\epsilon^{-2}(N^{-1}\ln N), & \text{otherwise.} \end{cases}$$

After using the barrier function $\mu^2\epsilon^{-2}(\sigma_1 - x_i)$ to get a feasible bound on the error in the layer region $\Omega^N_{1,1}$ the application of the discrete minimum principle on $\Omega^N_{1,1}$, gives

$$|w_{R_1}(x_i, y_j) - W_{R_1}(x_i, y_j)| \leq CN^{-1}ln^2N, \quad (x_i, y_j) \in \bar{\Omega}^N_{1,1}. \tag{4.17}$$

Finally, the result follows from (4.10) and (4.17).

We can proceed similarly to get appropriate bounds for the remaining boundary and interior layer functions w_{R_l}, $l = 2, 5, 6, 9, 10, 13, 14$. □

Lemma 4.3. *If w_{C_m} and W_{C_m} are the solutions of* (2.9) and (4.4), *respectively, then for $m = 1, \ldots, 16$, it holds*

$$|w_{C_m}(x_i, y_j) - W_{C_m}(x_i, y_j)| \leq \begin{cases} C(N^{-2}\ln^2 N), & \text{if } \alpha\mu^2 \leq \lambda\epsilon, \\ C(N^{-1}\ln^2 N), & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases} \tag{4.18}$$

Proof. We merely provide the proof of (4.18) for the corner layer component w_{C_1} and in case of $\sigma_1 < \frac{d}{4}$. Proceeding similarly as in Lemma 4.1, we get

$$|W_{C_1}(x_i, y_j)| \leq C\min\{B_{w_{R_2,j}}, B_{w_{R_1,i}}\}, \quad \text{if } (x_i, y_j) \in \varrho^N,$$

$$|w_{C_1}(x_i, y_j) - W_{C_1}(x_i, y_j)| \leq C\min\{B_{w_{R_2,j}}, B_{w_{R_1,i}}\}, \quad \text{if } (x_i, y_j) \in \Omega^N, \quad 0 < i, j < N/2.$$

Then, applying (4.6) we conclude that

$$|w_{C_1}(x_i, y_j) - W_{C_1}(x_i, y_j)| \leq CN^{-2}, \qquad (x_i, y_j) \in \Omega^N \backslash \Omega^N_{1,2}, \tag{4.19}$$

where, $\Omega^N_{1,2} = \{(x_i, y_j) \mid 0 < i, j < N/8\}$. Ultimately, in $\Omega^N_{1,2}$ the truncation error satisfies

$$|L^N_{\epsilon,\mu}[W_{C_1}(x_i, y_j) - w_{C_1}(x_i, y_j)]| \leq C\epsilon^2\left(h_i{}^2\left\|\frac{\partial^4 w_{C_1}}{\partial x^4}\right\| + k_j{}^2\left\|\frac{\partial^4 w_{C_1}}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 w_{C_1}}{\partial x^2}\right\| + k_j\left\|\frac{\partial^2 w_{C_1}}{\partial y^2}\right\|\right).$$

If $\alpha\mu^2 \leq \lambda\epsilon$, from Theorem 2.9 it follows that

$$|L^N_{\epsilon,\mu}[W_{C_1}(x_i, y_j) - w_{C_1}(x_i, y_j)]| \leq C(N^{-2}\ln^2 N) + C\mu^2(N^{-1}\ln^2 N),$$

and using a suitable barrier function and Lemma 3.1 on $\bar{\Omega}^N_{1,2}$ we get

$$|w_{C_1}(x_i, y_j) - W_{C_1}(x_i, y_j)| \leq CN^{-2}\ln^2 N, \quad (x_i, y_j) \in \bar{\Omega}^N_{1,2}. \tag{4.20}$$

The result follows from (4.19) and (4.20).

If $\alpha\mu^2 > \lambda\epsilon$, from Theorem 2.9, we have

$$|L^N_{\epsilon,\mu}[W_{C_1}(x_i, y_j) - w_{C_1}(x_i, y_j)]| \leq C\mu^4\epsilon^{-2}(N^{-1}\ln N).$$

Using the barrier function $\mu^2\epsilon^{-2}(\sigma_1 - x_i)$ to attain a feasible bound on the error in the layer region $\Omega^N_{1,2}$, and the discrete minimum principle on $\bar{\Omega}^N_{1,2}$, we obtain

$$|w_{C_1}(x_i, y_j) - W_{C_1}(x_i, y_j)| \le CN^{-1}\ln^2 N, \quad (x_i, y_j) \in \bar{\Omega}^N_{1,2}. \tag{4.21}$$

The result follows from (4.19) and (4.21).

We can proceed similarly to get appropriate bounds for the remaining corner layer components w_{C_m}, $m = 2, \dots, 16$. □

The discrete solution $U(x_i, y_j)$ of (3.2) can be written as

$$U(x,y) = \begin{cases} (V_1 + W_{R_1} + W_{R_2} + W_{R_3} + W_{R_4} + W_{C_1} + W_{C_2} + W_{C_3} + W_{C_4})(x,y), & \forall (x,y) \in {\Omega_1}^N, \\ (V_2 + W_{R_5} + W_{R_6} + W_{R_7} + W_{R_8} + W_{C_5} + W_{C_6} + W_{C_7} + W_{C_8})(x,y), & \forall (x,y) \in {\Omega_2}^N, \\ (V_3 + W_{R_9} + W_{R_{10}} + W_{R_{11}} + W_{R_{12}} + W_{C_9} + W_{C_{10}} + W_{C_{11}} + W_{C_{12}})(x,y), & \forall (x,y) \in {\Omega_3}^N, \\ (V_4 + W_{R_{13}} + W_{R_{14}} + W_{R_{15}} + W_{R_{16}} + W_{C_{13}} + W_{C_{14}} + W_{C_{15}} + W_{C_{16}})(x,y), & \forall (x,y) \in \Omega_4^N, \\ [(V + W_R + W_C)](x,y) = 0, & \forall (x,y) \in \Gamma_1^N, \\ [(V + W_R + W_C)](x,y) = 0, & \forall (x,y) \in \Gamma_2^N, \\ ([\bar{V}_{1^*}, \bar{V}_{2^*}] + [\bar{W}_{R_1^*}, \bar{W}_{R_2^*}] + [\bar{W}_{C_1^*}, \bar{W}_{C_2^*}])(x,y) = 0, & (x,y) = (x_{\frac{N}{2}}, y_{\frac{N}{2}}). \end{cases}$$

with $V = \sum_{k=1}^{4} V_k,\ W_R = \sum_{l=1}^{16} W_{R_l},\ W_C = \sum_{m=1}^{16} W_{C_m}$,

$$\begin{cases} W_{C_m} \neq 0, & \text{if } \alpha\mu^2 \le \lambda\epsilon,\ m = 3, 7, 11, 15, \\ W_{C_m} = 0, & \text{if } \alpha\mu^2 > \lambda\epsilon,\ m = 3, 7, 11, 15, \end{cases}$$

where $\bar{V}_{1^*}, \bar{W}_{R_1^*}, \bar{W}_{C_1^*}$ are the averages along the y-axis, and $\bar{V}_{2^*}, \bar{W}_{R_2^*}, \bar{W}_{C_2^*}$ are the averages along the x-axis.

Lemma 4.4. *At the discontinuities $(x_i, y_j) \in \Gamma_1^N \cup \Gamma_2^N \cup (d,d)$, we have the following estimates:*

$$|U(x_i, y_j) - u(x_i, y_j)| \le \begin{cases} CN^{-1}\ln N, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ CN^{-1}\ln^2 N, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases}$$

Proof. We follow a similar methodology as shown by De Falco and O'Riordan in [6],

Let h_1 and h_2 be the mesh interval sizes to the right and left sides of the point $x = d$, $\bar{h} = \frac{h_1+h_2}{2}$ and $h = \max\{h_1, h_2\}$. Now the error at $(d, y_j) \in \Gamma_1^N$ is denoted by

$$e(d, y_j) = u(d, y_j) - U(d, y_j).$$

Thus, we have

$$\begin{aligned} L^N_{\epsilon,\mu}\bar{e}(d,y_j) =& L^N_{\epsilon,\mu}\left(\bar{U}(d,y_j) - \bar{u}(d,y_j)\right) \\ \le& \frac{1}{h_2}\int_{t=d}^{d+h_2}\int_{s=d}^{t} \epsilon^2 \frac{\partial^2 u(s,y_j)}{\partial x^2} ds\, dt + \frac{1}{h_1}\int_{t=d-h_1}^{d}\int_{s=d}^{t} \epsilon^2 \frac{\partial^2 u(s,y_j)}{\partial x^2} ds\, dt \\ &+ \frac{1}{h_2}\int_{t=d}^{d+h_2}\int_{s=d}^{t} \mu^2 a \frac{\partial u(s,y_j)}{\partial x} ds\, dt + \frac{1}{h_1}\int_{t=d-h_1}^{d}\int_{s=d}^{t} \mu^2 a \frac{\partial u(s,y_j)}{\partial y} ds\, dt \\ &+ \epsilon^2 \bar{k}_j \left\| \frac{\partial^3 u(d,y_j)}{\partial x^3} \right\| + \mu^2 k_{j+1} \left\| \frac{\partial^2 u(d,y_j)}{\partial y^2} \right\| + (\bar{f} + cu)(d,y_j) \\ \le& \frac{1}{h_2}\int_{t=d}^{d+h_2}\int_{s=d}^{t} \left(\epsilon^2 \frac{\partial^2 u(s,y_j)}{\partial x^2} + \mu^2 a \frac{\partial u(s,y_j)}{\partial x} \right) ds\, dt \\ &+ \frac{1}{h_1}\int_{t=d-h_1}^{d}\int_{s=d}^{t} \left(\epsilon^2 \frac{\partial^2 u(s,y_j)}{\partial x^2} + \mu^2 a \frac{\partial u(s,y_j)}{\partial x} \right) ds\, dt + \epsilon^2 \bar{k}_j \left\| \frac{\partial^3 u(d,y_j)}{\partial y^3} \right\| \\ &+ \mu^2 k_{j+1} \left\| \frac{\partial^2 u(d,y_j)}{\partial y^2} \right\| + (\bar{f} + cu)(d,y_j) \\ \le& \frac{1}{h_2}\int_{t=d}^{d+h_2}\int_{s=d}^{t} \Big((\bar{f} + cu)(s,y_j) \Big) ds dt + \frac{1}{h_1}\int_{t=d-h_1}^{d}\int_{s=d}^{t} \Big((\bar{f} + cu)(s,y_j) \Big) ds dt \end{aligned}$$

Table 1
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.1.

$\epsilon = 10^{-3}$ N/μ	64	128	256	512	1024
1.00e−03	2.957e−02	1.203e−02	4.431e−03	1.481e−03	4.923e−04
	1.2975	1.4409	1.5811	1.5890	–
1.00e−04	2.946e−02	1.196e−02	4.392e−03	1.460e−03	4.813e−04
	1.3005	1.4453	1.5889	1.6010	–
1.00e−05	2.946e−02	1.196e−02	4.392e−03	1.460e−03	4.812e−04
	1.3005	1.4453	1.5889	1.6013	–
1.00e−06	2.946e−02	1.196e−02	4.392e−03	1.460e−03	4.812e−04
	1.3005	1.4453	1.5889	1.6013	–
1.00e−07	2.946e−02	1.196e−02	4.392e−03	1.460e−03	4.812e−04
	1.3005	1.4453	1.5889	1.6013	–
1.00e−08	2.946e−02	1.196e−02	4.392e−03	1.460e−03	4.812e−04
	1.3005	1.4453	1.5889	1.6013	–
E^N	2.957e−02	1.203e−02	4.431e−03	1.481e−03	4.923e−04
Q^N	1.2975	1.4409	1.5811	1.5890	–

$$
\begin{aligned}
&+ \epsilon^2 \bar{k}_j \left\| \frac{\partial^3 u(d, y_j)}{\partial y^3} \right\| + \mu^2 k_{j+1} \left\| \frac{\partial^2 u(d, y_j)}{\partial y^2} \right\| + (\bar{f} + cu)(d, y_j) \\
\leq & \frac{1}{h_2} \int_{t=d}^{d+h_2} \int_{s=d}^{t} \int_{p=s}^{d+h_2} \left(\frac{\partial(\bar{f} + cu)}{\partial x}(p, y_j) \right) dp\, ds\, dt \\
&+ \frac{1}{h_1} \int_{t=d-h_1}^{d} \int_{s=d}^{t} \int_{p=d-h_1}^{s} \left(\frac{\partial(\bar{f} + cu)}{\partial x}(p, y_j) \right) dp\, ds\, dt \\
&+ \epsilon^2 \bar{k}_j \left\| \frac{\partial^3 u(d, y_j)}{\partial y^3} \right\| + \mu^2 k_{j+1} \left\| \frac{\partial^2 u(d, y_j)}{\partial y^2} \right\| + \frac{h_2 c(d+h_2)}{2} \int_{t=d}^{d+h_2} \frac{\partial u(t, y_j)}{\partial x} dt \\
&+ \frac{h_1 c(d-h_1)}{2} \int_{t=d-h_1}^{d} \frac{\partial u(t, y_j)}{\partial x} dt.
\end{aligned} \tag{4.22}
$$

When $\alpha\mu^2 \leq \lambda\epsilon$, using the derivative bounds in Theorem 2.9 on (4.22), we get

$$|L^N_{m,\epsilon,\mu}(\bar{U} - \bar{u})(d, y_j)| \leq CN^{-1} \ln N.$$

When $\alpha\mu^2 > \lambda\epsilon$, using the bounds on the derivatives in Theorem 2.9 and methodology given in O'Riordan et al. [21] and Gracia et al. [9], we obtain

$$|L^N_{m,\epsilon,\mu}(\bar{U} - \bar{u})(d, y_j)| \leq CN^{-1} \ln^2 N.$$

We can obtain similar bounds for $(x_i, d) \in \Gamma_2^N$ and $(d, d) = (x_{N/2}, y_{N/2})$. Thus, $\forall (x_i, y_j) \in \Gamma_1^N \cup \Gamma_2^N \cup \{(d, d)\}$, we get

$$|U(x_i, y_j) - u(x_i, y_j)| \leq \begin{cases} CN^{-1} \ln N, & \text{if } \alpha\mu^2 \leq \lambda\epsilon, \\ CN^{-1} \ln^2 N, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases} \quad \square$$

Theorem 4.5. *Let u be the continuous solution of problem (1.1) and U the discrete solution of (3.2) on the constructed piecewise-uniform Shishkin mesh. Then the error at the mesh points $(x_i, y_j) \in \bar{\Omega}^N$ satisfies*

$$|U(x_i, y_j) - u(x_i, y_j)| \leq \begin{cases} CN^{-2} \ln^2 N, & \text{if } \alpha\mu^2 \leq \lambda\epsilon, \\ CN^{-1} \ln^2 N, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases}$$

Proof. When $\alpha\mu^2 \leq \lambda\epsilon$, we adopt a similar methodology as shown by De Falco and O'Riordan [6].

Consider $\sigma_1 < \frac{1}{4}$ and $\sigma_2 < \frac{1}{4}$.

Table 2
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.1.

$\mu = 10^{-1}$ N/ϵ	64	128	256	512	1024
1.00e−3	9.166e−02	6.874e−02	4.580e−02	2.868e−02	1.696e−02
	0.4151	0.5858	0.6753	0.75791	–
1.00e−4	9.208e−02	6.901e−02	4.604e−02	2.880e−02	1.703e−02
	0.4160	0.5840	0.6767	0.75799	–
1.00e−5	9.209e−02	6.902e−02	4.604e−02	2.880e−02	1.703e−02
	0.4160	0.5840	0.6767	0.75799	–
1.00e−6	9.209e−02	6.902e−02	4.604e−02	2.880e−02	1.703e−02
	0.4160	0.5840	0.6767	0.75799	–
1.00e−7	9.208e−02	6.902e−02	4.604e−02	2.880e−02	1.703e−02
	0.4160	0.5840	0.6768	0.75799	–
1.00e−8	9.248e−02	6.883e−02	4.597e−02	2.878e−02	1.690e−02
	0.4259	0.5825	0.6755	0.76804	–
E^N	9.248e−02	6.902e−02	4.604e−02	2.880e−02	1.703e−02
Q^N	0.4160	0.5840	0.6768	0.75799	–

Table 3
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.2.

$\epsilon = 10^{-3}$ N/μ	64	128	256	512	1024
1.00e−03	2.958e−02	1.200e−02	4.450e−03	1.480e−03	4.735e−04
	1.3016	1.4312	1.5882	1.6442	–
1.00e−04	2.945e−02	1.192e−02	4.391e−03	1.448e−03	4.541e−04
	1.3049	1.4408	1.6005	1.6730	–
1.00e−05	2.945e−02	1.192e−02	4.391e−03	1.448e−03	4.540e−04
	1.3049	1.4408	1.6005	1.6733	–
1.00e−06	2.945e−02	1.192e−02	4.391e−03	1.448e−03	4.540e−04
	1.3049	1.4408	1.6005	1.6733	–
1.00e−07	2.945e−02	1.192e−02	4.391e−03	1.448e−03	4.540e−04
	1.3049	1.4408	1.6005	1.6733	–
1.00e−08	2.945e−02	1.192e−02	4.391e−03	1.448e−03	4.540e−04
	1.3049	1.4408	1.6005	1.6733	–
E^N	2.958e−02	1.200e−02	4.450e−03	1.480e−03	4.735e−04
Q^N	1.3016	1.4312	1.5882	1.6442	–

Let $\phi(z)$ be a piecewise-linear polynomial defined by

$$\phi(z) = \begin{cases} \frac{z}{\sigma_1}, & 0 \le z \le \sigma_1, \\ 1, & \sigma_1 \le z \le d - \sigma_2, \\ \frac{d-z}{\sigma_2}, & d - \sigma_2 \le z \le d, \\ \frac{z-d}{\sigma_1}, & d \le z \le d + \sigma_1, \\ 1, & d + \sigma_1 \le z \le 1 - \sigma_2, \\ \frac{1-z}{1-\sigma_2}, & 1 - \sigma_2 \le z \le 1. \end{cases}$$

From Lemma 4.4, we have

$$|U(x_i, y_j) - u(x_i, y_j)| \le CN^{-1} \ln N, \ \ \forall (x_i, y_j) \in \Gamma_1^N \cup \Gamma_2^N \cup \{(d, d)\}.$$

Now, choose the barrier function $\xi^{\pm}$ given by

$$\xi^{\pm}(x_i, y_j) = CN^{-2} \ln^2 N \Big(1 + \phi(x_i) + \phi(y_j) \Big) \pm e(x_i, y_j),$$

Table 4
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.2.

$\mu = 10^{-1}$ N/ϵ	64	128	256	512	1024
1.00e−3	1.231e−01	9.449e−02	6.735e−02	4.409e−02	2.711e−02
	0.3818	0.4885	0.6112	0.70163	–
1.00e−4	1.234e−01	9.465e−02	6.747e−02	4.416e−02	2.715e−02
	0.3825	0.4883	0.6115	0.70179	–
1.00e−5	1.234e−01	9.465e−02	6.747e−02	4.416e−02	2.715e−02
	0.3825	0.4883e	0.6115	0.70179	–
1.00e−6	1.234e−01	9.465e−02	6.747e−02	4.416e−02	2.715e−02
	0.3825	0.4883	0.6115	0.70179	–
1.00e−7	1.234e−01	9.465e−02	6.748e−02	4.416e−02	2.715e−02
	0.3824	0.4882	0.6116	0.70179	–
1.00e−8	1.240e−01	9.448e−02	6.760e−02	4.401e−02	2.697e−02
	0.3918	0.4829	0.6194	0.70648	–
E^N	1.240e−01	9.465e−02	6.760e−02	4.416e−02	2.715e−02
Q^N	0.38967	0.48558	0.61428	0.70179	–

Table 5
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.3.

$\epsilon = 10^{-3}$ N/μ	64	128	256	512	1024
1.00e−03	6.548e−02	2.483e−02	8.784e−03	3.025e−03	9.684e−04
	1.3990	1.4991	1.5379	1.6433	–
1.00e−04	6.487e−02	2.443e−02	8.562e−03	2.921e−03	9.124e−04
	1.4089	1.5126	1.5515	1.6787	–
1.00e−05	6.487e−02	2.442e−02	8.560e−03	2.920e−03	9.119e−04
	1.4095	1.5124	1.5516	1.6790	–
1.00e−06	6.487e−02	2.442e−02	8.560e−03	2.920e−03	9.119e−04
	1.4095	1.5124	1.5516	1.6790	–
1.00e−07	6.487e−02	2.442e−02	8.560e−03	2.920e−03	9.119e−04
	1.4095	1.5124	1.5516	1.6790	–
1.00e−08	6.487e−02	2.442e−02	8.560e−03	2.920e−03	9.119e−04
	1.4095	1.5124	1.5516	1.6790	–
E^N	6.548e−02	2.483e−02	8.784e−03	3.025e−03	9.684e−04
Q^N	1.3990	1.4991	1.5379	1.6433	–

where $e(x_i, y_j) = U(x_i, y_j) - u(x_i, y_j)$. Therefore, applying the discrete minimum principle to $\xi^{\pm}(x_i, y_j)$, we get that

$$|U(x_i, y_j) - u(x_i, y_j)| \leq CN^{-2} \ln^2 N, \ \forall (x_i, y_j) \in \bar{\Omega}^N.$$

When $\alpha\mu^2 > \lambda\epsilon$, from Lemmas 4.1–4.4, the required result follows. □

5. Numerical experiments

We have considered three examples of the type in (1.1) to illustrate the performance of the method developed in the previous sections.

Example 5.1. $\epsilon^2(u_{xx}(x, y) + u_{yy}(x, y)) + 3\mu^2(u_x(x, y) + u_y(x, y)) - 3u(x, y) = f(x, y)$, $\forall (x, y) \in \Omega$, $f_1(x, y) = 0.5$; $f_2(x, y) = -1$; $f_3(x, y) = 3$; $f_4(x, y) = -2.5$; $d = 0.5$, where each f_k is defined over Ω_k, $k = 1, 2, 3, 4$, with boundary conditions

$$u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0.$$

Table 6
Maximum point-wise errors E^N and orders of convergence Q^N for Example 5.3.

$\mu = 10^{-1}$ N/ϵ	64	128	256	512	1024
1.00e−3	2.055e−01	1.428e−01	9.840e−02	6.138e−02	3.709e−02
	0.5256	0.5370	0.6809	0.72674	–
1.00e−4	2.059e−01	1.429e−01	9.855e−02	6.146e−02	3.713e−02
	0.5267	0.5365	0.6812	0.72706	–
1.00e−5	2.059e−01	1.429e−01	9.855e−02	6.146e−02	3.713e−02
	0.5267	0.5365	0.6812	0.72706	–
1.00e−6	2.059e−01	1.429e−01	9.855e−02	6.146e−02	3.713e−02
	0.5267	0.5365	0.6812	0.72745	–
1.00e−7	2.059e−01	1.429e−01	9.856e−02	6.146e−02	3.712e−02
	0.5267	0.5365	0.6813	0.72745	–
1.00e−8	2.066e−01	1.425e−01	9.837e−02	6.141e−02	3.670e−02
	0.5356	0.5349	0.6799	0.74269	–
E^N	2.066e−01	1.429e−01	9.856e−02	6.146e−02	3.712e−02
Q^N	0.53183	0.5365	0.6813	0.72745	–

Example 5.2. $\epsilon^2(u_{xx}(x,y)+u_{yy}(x,y))+\mu^2((3+x)u_x(x,y)+(5+xy)u_y(x,y))-(2+x)u(x,y) = f(x,y),\ \forall(x,y) \in \Omega,\ f_1(x,y) = 2x+y = f_4(x,y);\ f_2(x,y) = -(x+y) = f_3(x,y);\ d = 0.5$, where each f_k is defined over $\Omega_k,\ k = 1,2,3,4$, with boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$

Although in the previous pages we have considered the same point of discontinuity in the two axis, two different values could be considered, one for each axis. In the following example we consider different points of discontinuities ($d = d_1 \neq d_2$) on the x- and y-axis, respectively.

Example 5.3. $\epsilon^2(u_{xx}(x,y)+u_{yy}(x,y))+\mu^2((3+\exp(x))u_x(x,y)+(4+\exp(y))u_y(x,y))-(2+xy/2)u(x,y) = f(x,y),\ \forall(x,y) \in \Omega,\ f_1(x,y) = 4\cos(\pi x/2)\cos(\pi y/2);\ f_2(x,y) = -2\cos(\pi(x-d_1)/2)\cos(\pi y/2);\ f_3(x,y) = -6\cos(\pi x/2)\cos(\pi(y-d_2)/2);\ f_4(x,y) = 5\cos(\pi(x-d_1)/2)\cos(\pi(y-d_2)/2);\ d_1 = 0.5,\ d_2 = 0.3$, where each f_k is defined over $\Omega_k,\ k = 1,2,3,4$, with boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$

The exact solutions of these problems are not known. Therefore, we use the double mesh principle explained in [7] to estimate the maximum point-wise error. It is given by

$$E^N_{\epsilon,\mu} = \max_{(x_i,y_j)\in\bar{\Omega}^N} |U^{2N}(x_{2i},y_{2j}) - U^N(x_i,y_j)|$$

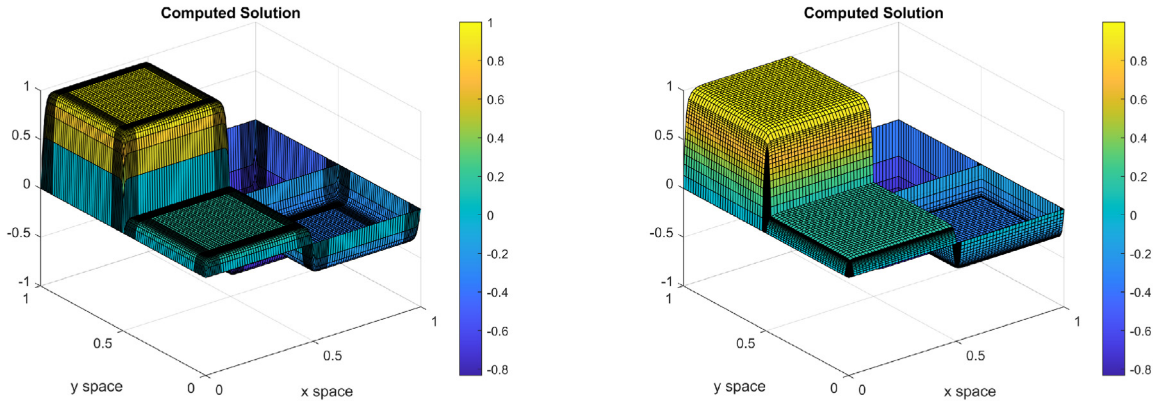
where $U^{2N}(x_{2i},y_{2j})$ represents the approximate solution on a mesh with $2N$ subintervals. The parameter uniform maximum point-wise errors are determined applying the formula

$$E^N = \max_{\epsilon,\mu} E^N_{\epsilon,\mu}.$$

The numerical order of convergence is given by

$$Q^N = log_2\left(\frac{E^N}{E^{2N}}\right).$$

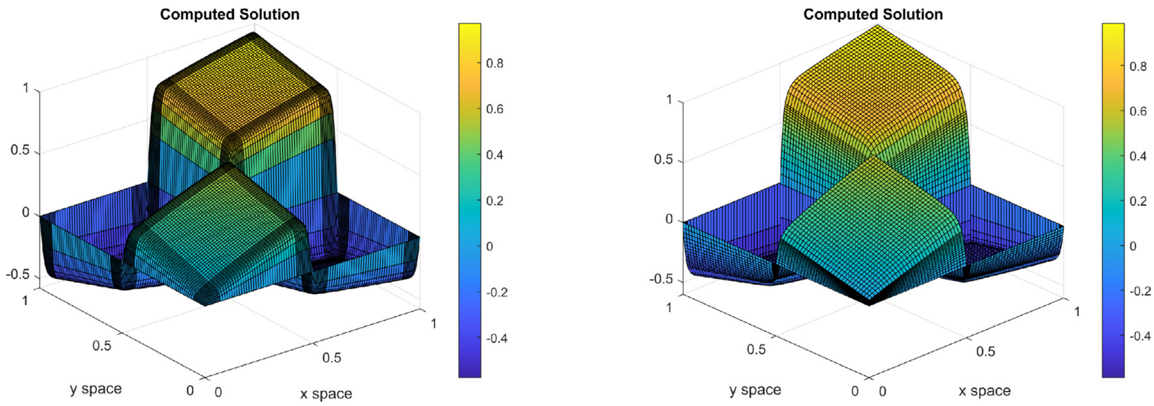
In Tables 1, 3 and 5 we have taken $\epsilon = 10^{-3}$ and μ values varying from 10^{-3} to 10^{-8}. These tables show the maximum point-wise errors and orders of convergence corresponding to Examples 5.1 and 5.2. Further, from these tables, it is clear that our numerical scheme has an almost second-order convergence as $\mu \to 0$, which is the rate required (see, e.g., [2]) when dealing with a reaction–diffusion problem. In Tables 2, 4 and 6 we have taken $\mu = 10^{-1}$ and ϵ values varying from 10^{-3} to 10^{-8}. From these tables it is clear that our numerical scheme has an almost first-order convergence as $\alpha\mu^2 > \lambda\epsilon$ or $\mu = 1$ when dealing with a convection–diffusion problem.



(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$, $d = 0.5$;

(b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$, $d = 0.5$;

Fig. 3. Surface graphs of the numerical solution U for Example 5.1.



(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$, $d = 0.5$;

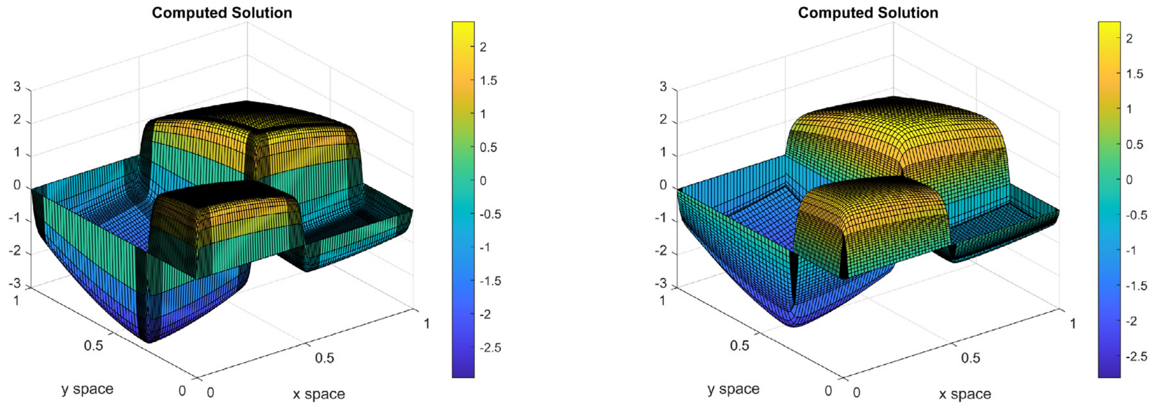
(b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$, $d = 0.5$;

Fig. 4. Surface graphs of the numerical solution U for Example 5.2.

Figs. 3(a), 3(b) and 4(a), 4(b) show the numerical solution when $\epsilon = 10^{-2}$, $\mu = 10^{-4}$ and $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$, $d = 0.5$ for Examples 5.1 and 5.2, respectively. Figs. 5(a) and 5(b) show the numerical solution for Example 5.3 when $\epsilon = 10^{-2}$, $\mu = 10^{-4}$ and $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$, $d_1 = 0.5$, $d_2 = 0.3$, respectively. Figs. 6 and 7 show the layer appearance (boundary, corner and interior layers) of the numerical solution U for Examples 5.2 and 5.3, taking $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$ and $\epsilon = 10^{-3}$, $\mu = 10^{-1}$, $N = 128$, respectively. The boundary layers associated with the boundaries $\varrho_{k,j}$ interact at the corners $c_{k,j}$ giving rise to the corner layers shown in Fig. 2. Further, Fig. 8 shows the error graph of the numerical solution U for Example 5.2.

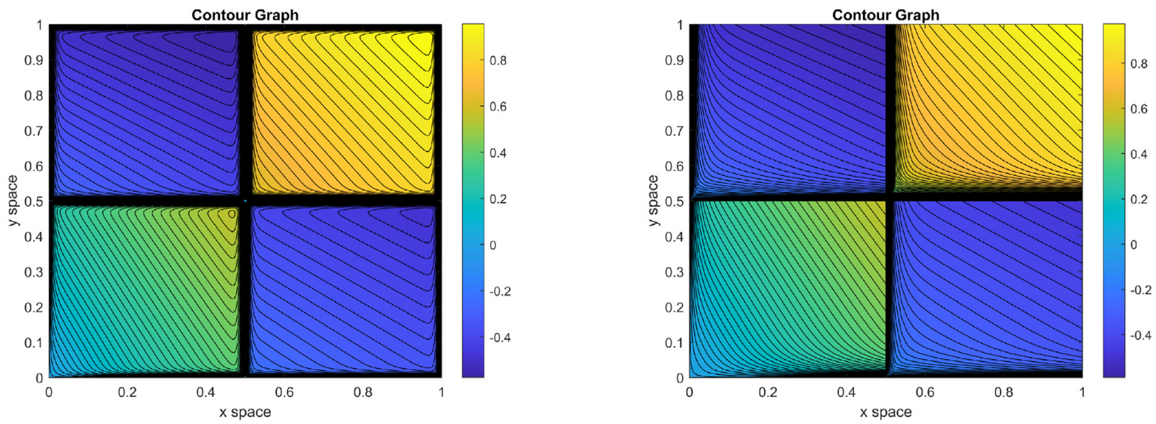
6. Conclusion

This paper deals with two-parameter singularly perturbed elliptic convection–reaction–diffusion problems with non-smooth data. A parameter-uniform discrete solution is obtained using a finite-difference scheme that produces almost second-order convergence approximations when $\mu \to 0$ or $\alpha\mu^2 \leq \lambda\epsilon$ and almost first-order convergence when $\alpha\mu^2 > \lambda\epsilon$ or $\mu = 1$. To address the numerical analysis and obtain finer bounds, the analytical and discrete solutions are split into a sum of regular, singular, and corner components. The presented numerical experiments provide results that agree with the theoretical analysis.



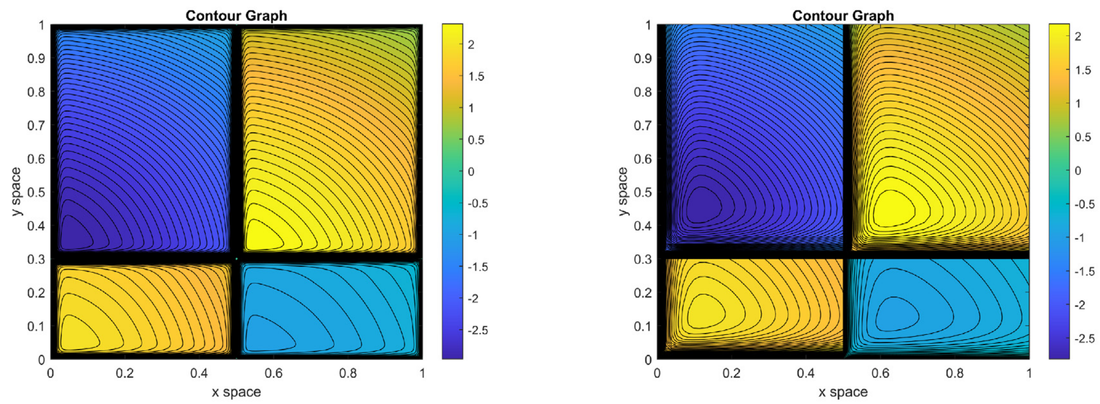
(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$, $d_1 = 0.5$, $d_2 = 0.3$;   (b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$, $d_1 = 0.5$, $d_2 = 0.3$;

Fig. 5. Surface graphs of the numerical solution U for Example 5.3.



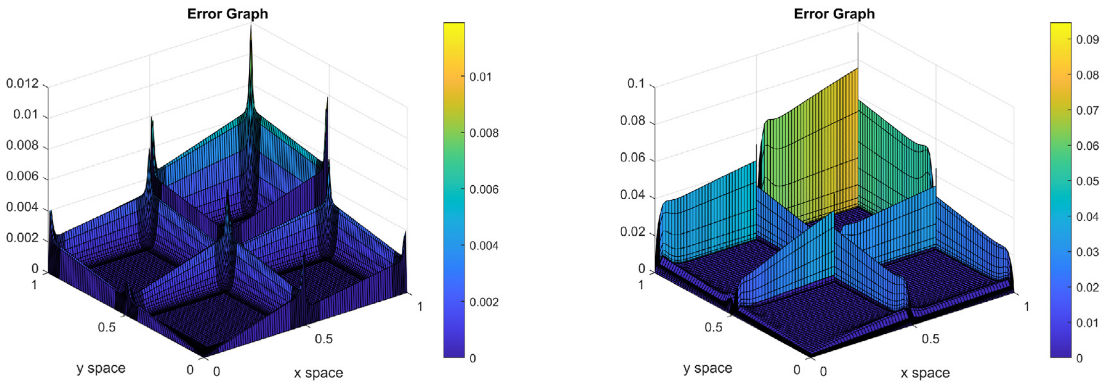
(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$;   (b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$;

Fig. 6. Contour graphs of the numerical solution U for Example 5.2.



(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$;   (b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$;

Fig. 7. Contour graphs of the numerical solution U for Example 5.3.



(a) When $\epsilon = 10^{-2}$, $\mu = 10^{-4}$, $N = 128$;

(b) When $\epsilon = 10^{-4}$, $\mu = 10^{-1}$, $N = 128$;

Fig. 8. Error graphs of the numerical solution U for Example 5.2.

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CRediT authorship contribution statement

Ram Shiromani: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft. **Vembu Shanthi:** Conceptualization, Methodology, Formal analysis, Investigation, Supervision, Writing – review & editing. **Higinio Ramos:** Conceptualization, Methodology, Formal analysis, Investigation, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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