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Simplifying the variational iteration method: A new approach to obtain the Lagrange multiplier

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Abstract

The variational iteration method (VIM) has been in the last two decades, one of the most used semi-analytical techniques for approximating nonlinear differential equations. The notion of VIM is based on the identification of the Lagrange multiplier using the variational theory. The performance of the method is highly dependent on how the Lagrange multiplier is determined. In this paper, a novel method for calculating the Lagrange multiplier is provided, making the VIM more efficient in solving a variety of nonlinear problems. To illustrate the effectiveness of the new approach, a standard nonlinear oscillator problem is tested and the results demonstrate that only one iteration leads to an excellent outcome.

Keywords: Variational iteration method; Variational principle; Lagrange multiplier; Nonlinear oscillator

1. Introduction

The variational iteration approach proposed by He [9] for nonlinear differential equations has been employed to address complex problems involving seepage flow with fractional derivatives and a nonlinear oscillator [8]. Various researchers handled many real life application's including nonlinear heat transfer and porous media problems [6], the Korteweg–de Vries equation [21], Burgers equations [22], Riccati equations [2], Jaulent–Miodek equations [6], Helmholtz equation [18], KdV equations [24], evolution equations [25], Boussinesq equations [26], logarithmic Schrödinger equations [27], Lane–Emden problems [7], dispersive water wave phenomena [4], a SIR epidemic model [20] and others [3,10,15,19] using the VIM. Other mathematical developments concerning this method can be found in [11,12,20,25,27,29] and references therein.

The method necessitates knowledge of variational theory in order to identify the Lagrange multiplier which makes the identification process very tedious, hence computationally expensive. The success of VIM is heavily dependent on a reliable Lagrange multiplier identification process. Additionally, the computations for determining the multipliers are extremely complicated and ambitious for strongly nonlinear equations. This motivates us to

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devise an approach for estimating the identifier effectively and efficiently. An alternate approach based on the Laplace transform has been suggested for computing the multipliers using the concept of Laplace transform [1,5,17]. However, in some cases, it is a very cumbersome task [23] to use this approach due to the computation of inverse Laplace transform. Therefore, this article proposes a straightforward and effective procedure that does not require knowledge of the variational theory and Laplace transform. The new approach for identifying the multiplier makes the VIM more accessible.

2. Evaluation of the Lagrange multiplier

To illustrate the basic concept of the evaluation of the Lagrange multiplier in a straightforward manner, let us consider the following general nonlinear problem

$$Ł[z(t)] + \aleph[z(t)] = p(t), \qquad 0 \le t \le T, \tag{2.1}$$

where $Ł[z(t)] = \dfrac{d^n z(t)}{dt^n}$ is a linear operator, $\aleph$ is a nonlinear operator and $p(t)$ is a known continuous function.

First, let us recall the standard VIM methodology. The VIM's main feature is that it constructs the following correction functional for Eq. (2.1) given as follows:

$$z_{n+1}(t) = z_n(t) + \int_0^t \lambda(t,s)(Ł[z_n(s)] + \widetilde{\aleph}[z_n(s)] - p(s))ds, \quad n \ge 0, \tag{2.2}$$

where λ can be determined optimally using the theory of calculus of variations [13,14] and is known as the Lagrange multiplier, and $\widetilde{\aleph}$ is considered as a restricted variation, that is $\widetilde{\aleph}[z_n] = 0$. Different Lagrange multipliers and their interpretations are provided in detail by He [12].

Further, we shall show that identifying the Lagrange multiplier is considerably easier than using the approaches such as variational theory and Laplace transform. To achieve this, let us define

$$\Re = Ł[z(t)] + \aleph[z(t)] - p(t), \tag{2.3}$$

and a simple rearrangement of Eq. (2.3) yields

$$Ł[z(t)] = -\Re + Ł[z(t)]. \tag{2.4}$$

Now let us construct an associated integral representation of Eq. (2.4). Integrating Eq. (2.4) from 0 to t gives

$$Ł_1[z(t)] = \frac{d^{n-1}z(0)}{dt^{n-1}} + \int_0^t (-\Re + Ł[z(s)])ds, \quad \text{where} \;\; Ł_1 = \frac{d^{n-1}}{dt^{n-1}}. \tag{2.5}$$

By again performing integration on Eq. (2.5) from 0 to t gives

$$Ł_2[z(t)] = \frac{d^{n-2}z(0)}{dt^{n-2}} + \frac{d^{n-1}z(0)}{dt^{n-1}}t + \int_0^t \int_0^s (-\Re + Ł[z(\xi)])d\xi ds, \quad \text{where} \;\; Ł_2 = \frac{d^{n-2}}{dt^{n-2}}. \tag{2.6}$$

Changing the order of integration on the above integral provides the following relation:

$$Ł_2[z(t)] = \frac{d^{n-2}z(0)}{dt^{n-2}} + \frac{d^{n-1}z(0)}{dt^{n-1}}t + \int_0^t (t-s)(-\Re + Ł[z(s)])ds. \tag{2.7}$$

Proceeding repeatedly in this way, after integrating the above equation $(n-2)$ times, we arrive at

$$z(t) = \sum_{j=0}^{n-1} \frac{d^j z(0)}{dt^j}\frac{t^j}{j!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(-\Re + Ł[z(s)])ds. \tag{2.8}$$

Using the following identity

$$\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}Ł[z(s)]ds = z(t) - \sum_{j=0}^{n-1} \frac{d^j z(0)}{dt^j}\frac{t^j}{j!},$$

and replacing the value of $\Re$ in Eq. (2.8), the following operator form after rearrangement is obtained:

$$z(t) = T[z(t)], \tag{2.9}$$

where

$$T[z(t)] = z(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(Ł[z(s)] + \aleph[z(s)] - p(s))ds.$$

From Eq. (2.9), we formulate the iterative procedure

$$z_{n+1}(t) = T[z_n(t)],$$

or equivalently,

$$z_{n+1}(t) = z_n(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(Ł[z_n(s)] + \aleph[z_n(s)] - p(s))ds. \tag{2.10}$$

Rearrangement of Eq. (2.10) leads to

$$z_{n+1}(t) = z_n(t) + \int_0^t \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}(Ł[z_n(s)] + \aleph[z_n(s)] - p(s))ds. \tag{2.11}$$

By comparing the relations given in (2.2) and (2.11), we have

$$\lambda(t,s) = \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}. \tag{2.12}$$

We note that the Lagrange multiplier given in Eq. (2.12) for Eq. (2.2) is exactly the same as that obtained by He [12] via the variational theory and restricted variations. The new approach to identify the Lagrange multiplier is by far simpler than the existing one that uses the variational theory, restricted variations and Laplace transform.

3. Illustrative examples

This section is devoted to check the efficiency and accuracy on some problems including a real life application [5,16,28].

Example 3.1. Let us consider a general nonlinear problem of the form

$$z'(t) + \alpha z(t) + \aleph[z(t)] = p(t), \quad z(0) = \gamma, \tag{3.1}$$

where $Ł[z(t)] = z'(t) + \alpha z(t)$ and by means of the method of variation of parameters, Eq. (2.8) takes the following form corresponding to (3.1):

$$z(t) = e^{-\alpha t}z(0) + \int_0^t e^{\alpha(s-t)}(-\Re + Ł[z(s)])ds. \tag{3.2}$$

Using the identity

$$\int_0^t e^{\alpha(s-t)}Ł[z(s)]ds = z(t) - e^{-\alpha t}z(0),$$

we can easily get the following result

$$z_{n+1}(t) = z_n(t) - \int_0^t e^{\alpha(s-t)}(Ł[z_n(s)] + \aleph[z_n(s)] - p(s))ds. \tag{3.3}$$

Then, comparing Eqs. (2.2) and (3.3), it follows that

$$\lambda(t,s) = -e^{\alpha(s-t)}, \tag{3.4}$$

which coincides with the Lagrange multiplier given by He [12].

Example 3.2. Let us consider a nonlinear oscillator of the form

$$z''(t) + \omega^2 z(t) + \aleph[z(t)] = p(t), \quad z(0) = \gamma, \quad z'(0) = \xi, \tag{3.5}$$

where ω is the frequency to be evaluated and $Ł[z(t)] = z''(t) + \omega^2 z(t)$. Furthermore, the solution of the problem obtained through the method of variation of parameters

$$z(t) = z(0)\cos(wt) + z'(0)\frac{\sin w(t)}{w} + \int_0^t \frac{\sin w(t-s)}{w}(-\Re + Ł[z(s)])ds. \tag{3.6}$$

Using the following relation

$$\int_0^t \frac{\sin w(t-s)}{w} Ł[z(s)]ds = z(t) - z(0)\cos(wt) - z'(0)\frac{\sin w(t)}{w},$$

we obtain that

$$z_{n+1}(t) = z_n(t) - \int_0^t \frac{\sin w(t-s)}{w}(Ł[z_n(s)] + \aleph[z_n(s)] - p(s))ds. \tag{3.7}$$

Then, comparing (2.2) and (3.7), it is easy to see that

$$\lambda(t,s) = \frac{\sin w(s-t)}{w}, \tag{3.8}$$

which is the Lagrange multiplier obtained by the new approach for (3.5). This is the same result as the one obtained in He [12].

Example 3.3. Let us consider the following nonlinear oscillator given in [5,16,28]

$$(1 + \alpha z(t)^2)z''(t) + \alpha z(t)z'(t)^2 = z(t)(1 - z(t)^2), \quad z(0) = A, \quad z'(0) = 0. \tag{3.9}$$

This problem (3.9) is solved in the literature via the homotopy perturbation method [28].

To perform the new approach to this problem, let us rewrite Eq. (3.9) in the form of (2.1), which results in

$$z''(t) + \omega^2 z(t) + \aleph[z(t)] = 0, \tag{3.10}$$

where $\aleph[z(t)] = \alpha z(t)^2 z''(t) + \alpha z(t)z'(t)^2 - z(t)(1 - z(t)^2) - \omega^2 z(t)$. Using (3.7), the following expression is derived

$$z_{n+1}(t) = z_n(t) - \int_0^t \frac{\sin w(t-s)}{w}\Big((1 + \alpha z_n(s)^2)z_n''(s) + \alpha z_n(s)z_n'(s)^2 - z_n(s)(1 - z_n(s)^2)\Big)ds. \tag{3.11}$$

Begin with $z_0(t) = A\cos w(t)$, which satisfies the initial condition in (3.9). Thus, the first order approximation of (3.11) is written as

$$\begin{aligned} z_1(t) = {} & A\cos w(t) - \frac{1}{4}A^3\alpha\cos w(t)^3 + \frac{1}{8\omega^2}A^3\cos(wt)^3 - \frac{1}{\omega^2}A^3\cos(wt) \\ & - \Big(\frac{3}{8}\frac{A^3}{\omega} - \frac{1}{4}A^3\omega\alpha - \frac{1}{2}A\omega - \frac{1}{2}\frac{A}{\omega}\Big)t\sin w(t). \end{aligned} \tag{3.12}$$

In (3.12), imposing the vanishing of the no secular term leads to

$$\frac{1}{2\omega}\Big(\frac{3}{4}A^3 - \frac{1}{2}A^3\omega^2\alpha - A\omega^2 - A\Big) = 0, \tag{3.13}$$

from which we obtain that

$$\omega = \left(\frac{\frac{3}{4}A^2 - 1}{\frac{1}{2}\alpha A^2 + 1}\right)^{\frac{1}{2}}.$$

Note that this expression of the frequency is the same as that obtained by Anjum and He in Anjum and He [5] and Wu and He in Wu and He [28].

It can be seen that the current approach to compute the Lagrange multiplier is highly efficient compared to the approach using the variational theory, restricted variations, and Laplace transform.

4. Conclusions

This article focuses on the development of a simple and effective method for identifying the Lagrange multiplier, which is fundamental in the variational iterative method. In order to show the applicability of the new approach, some examples have been presented, including a standard nonlinear oscillator. Therefore, in formulating the VIM formula no specific expertise in the variational theory and Laplace transform is required. The results demonstrate that the new approach for identifying the Lagrange multiplier is very straightforward and easy to implement on a variety of nonlinear differential equations.

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