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A two-step hybrid block method with fourth derivatives for solving third-order boundary value problems

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ABSTRACT

This manuscript proposes an implicit two-step hybrid block method which incorporates fourth derivatives, for solving linear and non-linear third-order boundary value problems in ODEs. The derivation of the present method is based on collocation and interpolation techniques, and the convergence analysis of the new strategy is proved to be seventh-order convergent. The proposed approach produces discrete approximations at the grid points, obtained after solving an algebraic system of equations. Numerical experiments are studied to show the performance and viability of the proposed approach. The numerical results demonstrated that the new technique gives accurate approximations, which are better than some existing strategies in the available literature and also found to be in good agreement with known analytical solutions.

1. Introduction

This article considers the approximate solutions to third-order boundary value problems (BVPs) of ordinary differential equations (ODEs) of the form;

$$u'''(x) = w(x, u(x), u'(x), u''(x)), \qquad x \in [x_0, x_N] \subset \mathbb{R}, \tag{1}$$

subject to boundary conditions of the form

$$u(x_0) = u_0,\ \ u'(x_0) = u'_0,\ \ u(x_N) = u_N, \tag{2}$$

although one can consider general mixed boundary conditions of the form

$$\begin{aligned} &G_1(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = u_P\ ,\\ &G_2(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = u_Q\ ,\\ &G_3(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = u_R\ , \end{aligned} \tag{3}$$

where $u_0, u'_0, u_N, u_P, u_Q, u_R$ are real constants. We assume that the function w is continuous on $[x_0, x_N] \times \mathbb{R}^3$ and fulfils the Lipchitz's conditions to verify the uniqueness and existence theorem (see Agarwal [1]). The existence and uniqueness of a solution for the general problem in (1)–(3) has been established in [2] and [3].

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Problems of type (1) with boundary conditions (BCs) arise in the investigation of the obstacle, thin-film flow, sandwich boundary layer and laminar flow beam, fluid mechanics and dynamics, the deflection of a curved beam having a constant or varying cross-section, the motion of a rocket, the study of stellar interiors and draining and coating flows. These problems also have significant applications in different parts of engineering, pure and applied sciences (see [4–10] and references therein).

In most cases, the theoretical solution to the problem in (1) is complicated to obtain for arbitrary nonlinearities and BCs. Thus, a lot of research activities are carried out to develop numerical strategies for its solution.

There are three main types of approximation methods for solving boundary value problems of ODEs (see [11]). We emphasize here that most of the basic methodologies for solving (1) involve the reduction of the problem to an equivalent system of first-order ODEs and then solving the system by utilizing one of the numerical techniques for solving first-order problems. Prominent researchers such as those in [12–14], and [15] have discussed extensively reduction methods. Some of these scholars reported that due to the dimension of the resulting systems of first-order ODEs, the approach involves progressively more human efforts and CPU times.

Recent manuscripts on the approximation of solutions for solving (1) with particular boundary conditions have utilized different methods. Among the most commonly explored techniques, we can mention the simple Homotopy perturbation method in [16], the modified Adomian decomposition technique in [17], the automatic differentiation strategy presented by Asai [18], a non-polynomial spline technique in [19], the linear multistep method reported by Awoyemi [20] or the direct integrators proposed by Jator et al. [21] to mention but a few. In the present manuscript, we introduce a new strategy, considering a two-step hybrid block method with fourth derivatives for directly solving the problem (1) with boundary conditions (see [22–25] for other strategies using block methods for solving different types of BVPs).

This article is arranged as follows. In Section 2, we develop a two-step fourth derivative hybrid block method (2FDHBM) for solving (1)–(3). Convergence analysis is proved in Section 3, and some notes about the implementation and computational details of the proposed 2FDHBM are well explained in Section 4. In Section 5 we report some numerical test examples to show the productivity and reliability of the proposed methodology. Some conclusions are given in the final section.

2. Development of the two-step hybrid block method

In this section, we will derive the proposed hybrid block method for solving (1) subject to the BCs in (3). We are interested in obtaining approximations of the true solution $u(x)$ at the grid points $x_0 < x_1 < \cdots < x_N$ of the integration interval $[x_0, x_N]$, taking a constant step-size $h = x_{j+1} - x_j$, $j = 0, 1, \ldots, N-1$. To get the discrete formulas we consider that the true solution can be approximated on the interval $[x_n, x_{n+2}]$ by a polynomial $P(x)$, that is,

$$u(x) \simeq P(x) = \sum_{n=0}^{9} a_n x^n, \tag{4}$$

where $a_n \in \mathbb{R}$ are real unknown coefficients that will be determined by imposing collocation conditions at selected points (the degree of the polynomial is taken on the basis of those conditions). Consider the intermediate points $x_{n+r} = x_n + (1/2)h$, $x_{n+s} = x_n + (3/2)h$ on $[x_n, x_{n+2}]$ and the approximation in (4), its first and second derivatives applied to the point x_n, its third derivative applied to the points $x_n, x_{n+r}, x_{n+1}, x_{n+s}, x_{n+2}$, and its fourth derivative applied to the points x_n, x_{n+2}. In this way, we get a system of ten equations with ten unknowns a_n, $n = 0(1)9$, given by

$$\begin{aligned}
&P(x_n) = u_n\,,\ P'(x_n) = u'_n\,,\ P''(x_n) = u''_n,\\
&P'''(x_n) = w_n\,,\ P'''(x_{n+r}) = w_{n+r}\,,\ P'''(x_{n+1}) = w_{n+1}\,,\ P'''(x_{n+s}) = w_{n+s}\,,\ P'''(x_{n+2}) = w_{n+2},\\
&P''''(x_n) = g_n\,,\ P''''(x_{n+2}) = g_{n+2}\,,
\end{aligned} \tag{5}$$

where $u_{n+i}, u'_{n+i}, u''_{n+i}, w_{n+i} = w(x_{n+i}, u_{n+i}, u'_{n+i}, u''_{n+i}), g_{n+i} = g(x_{n+i}, u_{n+i}, u'_{n+i}, u''_{n+i})$ denote approximations of $u(x_{n+i})$, $u'(x_{n+i})$, $u''(x_{n+i})$, $w(x_{n+i}, u(x_{n+i}), u'(x_{n+i}), u''(x_{n+i}))$ and $g(x_{n+i}, u(x_{n+i}), u'(x_{n+i}), u''(x_{n+i}))$ respectively, with

$$g(x, u, u', u'') = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}u' + \frac{\partial w}{\partial u'}u'' + \frac{\partial w}{\partial u''}w(x, u, u', u'').$$

The system in (5) may be written in matrix form as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9\\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7\\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6\\
0 & 0 & 0 & 6 & 24x_{n+r} & 60x_{n+r}^2 & 120x_{n+r}^3 & 210x_{n+r}^4 & 336x_{n+r}^5 & 504x_{n+r}^6\\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6\\
0 & 0 & 0 & 6 & 24x_{n+s} & 60x_{n+s}^2 & 120x_{n+s}^3 & 210x_{n+s}^4 & 336x_{n+s}^5 & 504x_{n+s}^6\\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6\\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1830x_n^4 & 3024x_n^5\\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1830x_{n+2}^4 & 3024x_{n+2}^5
\end{pmatrix}
\begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\\ a_9 \end{pmatrix}
=
\begin{pmatrix} u_n\\ u'_n\\ u''_n\\ w_n\\ w_{n+r}\\ w_{n+1}\\ w_{n+s}\\ w_{n+2}\\ g_n\\ g_{n+2} \end{pmatrix}.$$

Solving the above system of equations, we obtain the values of the coefficients $a_n,\ n = 0(1)9$. After obtaining the values of these coefficients and changing the variable, $x = x_n + zh$, the polynomial in (4) may be written as

$$\begin{aligned} P(x_n+zh) &= \alpha_0(z)u_n + h\alpha_1(z)u_n' + h^2\alpha_2(z)u_n'' \\ &\quad + h^3(\beta_0(z)w_n + \beta_r(z)w_{n+r} + \beta_1(z)w_{n+1} + \beta_s(z)w_{n+s} + \beta_2(z)w_{n+2}) \\ &\quad + h^4(\gamma_0(z)g_n + \gamma_1(z)g_{n+2}), \end{aligned} \tag{6}$$

where h is the chosen step-size and $\alpha_0(z), \alpha_1(z), \alpha_2(z), \beta_0(z), \beta_r(z), \beta_1(z), \beta_s(z), \beta_2(z), \gamma_0(z), \gamma_1(z)$ are continuous coefficients.

2.1. Main formulas

The two step main formulas are obtained by substituting the values of $\alpha_0(z), \alpha_1(z), \alpha_2(z), \beta_0(z), \beta_r(z), \beta_1(z), \beta_s(z), \beta_2(z), \gamma_0(z), \gamma_1(z)$ into (6) and evaluating $P(x), P'(x), P''(x)$ at the point $x_{n+2} = x_n + 2h$ to get the following approximations for $u(x_{n+2}), u'(x_{n+2}), u''(x_{n+2})$.

$$\begin{aligned} u_{n+2} &= u_n + 2hu_n' + 2h^2u_n'' + \frac{4h^3}{2835}\left(64\left(7w_{n+r} + w_{n+s}\right) + 296w_n + 144w_{n+1} - 7w_{n+2}\right) \\ &\qquad + \frac{2h^4}{945}\left(14g_n + g_{n+2}\right), \\ u_{n+2}' &= u_n' + 2hu_n'' + \frac{h^2}{945}\left(30hg_n + 256\left(3w_{n+r} + w_{n+s}\right) + 419w_n + 432w_{n+1} + 15w_{n+2}\right), \\ u_{n+2}'' &= u_n'' + \frac{h}{945}\left(512w_{n+r} + 512w_{n+s} + 217w_n + 432w_{n+1} + 217w_{n+2}\right) + \frac{h^2}{63}\left(g_n - g_{n+2}\right). \end{aligned} \tag{7}$$

To obtain a two-step hybrid block method, we need to consider additional formulas. For this, we consider the evaluation of $P(x)$, $P'(x)$, and $P''(x)$ at the points $x_{n+r}, x_{n+1}, x_{n+s}$. In this way, we obtain a total of twelve formulas that form the proposed method. The rest of the formulas are as follows

$$\begin{aligned} u_{n+r} &= \frac{h^3\left(11586hg_n + 924hg_{n+2} + 63856w_{n+r} + 13648w_{n+s} + 194261w_n - 25020w_{n+1} - 4825w_{n+2}\right)}{11612160} + \frac{1}{8}h^2u_n'' + \frac{hu_n'}{2} + u_n, \\ u_{n+1} &= \frac{h^3\left(579hg_n + 51hg_{n+2} + 7456w_{n+r} + 736w_{n+s} + 8345w_n - 1152w_{n+1} - 265w_{n+2}\right)}{90720} + \frac{1}{2}h^2u_n'' + hu_n' + u_n, \\ u_{n+s} &= \frac{3h\left(3h\left(h\left(18h\left(14g_n + g_{n+2}\right) + 4656w_{n+r} + 272w_{n+s} + 3585w_n + 540w_{n+1} - 93w_{n+2}\right) + 17920u_n''\right) + 71680u_n'\right)}{143360} + u_n \end{aligned} \tag{8}$$

$$\begin{aligned} u_{n+r}' &= \frac{h^2\left(15h\left(781g_n + 75g_{n+2}\right) + 90720w_{n+r} + 16736w_{n+s} + 171562w_n - 31212w_{n+1} - 5886w_{n+2}\right)}{1935360} + \frac{hu_n''}{2} + u_n', \\ u_{n+1}' &= \frac{h^2\left(225hg_n + 15hg_{n+2} + 4288w_{n+r} + 192w_{n+s} + 3156w_n - 76w_{n+2}\right)}{15120} + hu_n'' + u_n' \\ u_{n+s}' &= \frac{3h^2\left(555hg_n + 45hg_{n+2} + 13152w_{n+r} + 1120w_{n+s} + 7778w_n + 5076w_{n+1} - 246w_{n+2}\right)}{71680} + \frac{3hu_n''}{2} + u_n' \end{aligned} \tag{9}$$

$$\begin{aligned} u_{n+r}'' &= \frac{h\left(5055hg_n + 615hg_{n+2} + 69728w_{n+r} + 9248w_{n+s} + 62923w_n - 17712w_{n+1} - 3227w_{n+2}\right)}{241920} + u_n'', \\ u_{n+1}'' &= \frac{h\left(225hg_n - 15hg_{n+2} + 8576w_{n+r} - 384w_{n+s} + 3381w_n + 3456w_{n+1} + 91w_{n+2}\right)}{15120} + u_n'' \\ u_{n+s}'' &= \frac{h\left(165hg_n + 45hg_{n+2} + 4512w_{n+r} + 2272w_{n+s} + 2177w_n + 4752w_{n+1} - 273w_{n+2}\right)}{8960} + u_n''. \end{aligned} \tag{10}$$

For each of the above formulas, we can obtain the local truncation error in the usual form: passing all the terms to the left, substituting the approximate values for the true ones, and expanding the resulting formula by Taylor series in powers of h. In this way, we obtain the following local truncation errors

$$\mathcal{L}[u(x_{n+r}), h] = -\frac{139h^{10}u^{(10)}(x_n)}{1625702400} + \mathcal{O}(h^{11})$$

$$\mathcal{L}[u(x_{n+1}), h] = -\frac{h^{10}u^{(10)}(x_n)}{1612800} + \mathcal{O}(h^{11})$$

$$\mathcal{L}[u(x_{n+s}),h] = -\frac{27h^{10}u^{(10)}(x_n)}{20070400} + \mathcal{O}(h^{11})$$

$$\mathcal{L}[u(x_{n+2}),h] = -\frac{h^{10}u^{(10)}(x_n)}{396900} + \mathcal{O}(h^{11})$$

$$\mathcal{L}[u'(x_{n+r}),h] = -\frac{3991h^{9}u^{(10)}(x_n)}{6502809600} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u'(x_{n+1}),h] = -\frac{h^{9}u^{(10)}(x_n)}{793800} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u'(x_{n+s}),h] = -\frac{153h^{9}u^{(10)}(x_n)}{80281600} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u'(x_{n+2}),h] = \frac{h^{9}u^{(10)}(x_n)}{396900} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u''(x_{n+r}),h] = -\frac{3h^{9}u^{(11)}(x_n)}{1146880} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u''(x_{n+1}),h] = \frac{h^{9}u^{(11)}(x_n)}{6350400} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u''(x_{n+s}),h] = -\frac{3h^{9}u^{(11)}(x_n)}{1146880} + \mathcal{O}(h^{10})$$

$$\mathcal{L}[u''(x_{n+2}),h] = -\frac{h^{9}u^{(11)}(x_n)}{3175200} + \mathcal{O}(h^{10}), \tag{11}$$

which show that each formula has an order of approximation of at least $\mathcal{O}(h^9)$.

To be applied for solving third-order boundary-value problems, the above formulas in (7)–(10) for $n = 0, 2, \ldots, N-2$ are considered altogether along with the two-step blocks together with the given boundary conditions, thus resulting in a global method that provides an approximate solution over all the grid points simultaneously. Considering the grid points $x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N$ with $N \in \mathbb{N}$, N an even positive integer, we obtain a system of $6N+3$ equations and boundary conditions. It is also clear that the number of unknowns is $6N+3$ (the approximate values of the solution and the first and second derivatives at the grid and intermediate points). Solving this system by an appropriate solver, usually a Newton-type procedure, provides the approximate values of the problem.

3. Convergence analysis

To be of any use, the numerical approximations obtained by a numerical method for solving a given differential equation must have a convergent behaviour. This subsection addresses the convergence theorem for the proposed global scheme for solving third order BVPs. Firstly, we shall state the definition of convergence of a numerical method for solving a BVP.

Definition 3.1. Let $u(x)$ be the true solution of a BVP of the form in (1) with given boundary conditions as in (2), and let $\{u_j\}_{j=0}^{N}$ be the numerical approximations of $u(x)$ at the corresponding grid points, obtained by the proposed global method. The numerical method is said to have pth-order of convergence if for sufficiently small stepsize h, there exists a constant K (independent of h) such that

$$\max_{0\le j\le N} \|u(x_j) - u_j\| \le Kh^p.$$

Note that from the above definition, we readily obtain that

$$\max_{0\le j\le N} \|u(x_j) - u_j\| \to 0 \text{ as } h \to 0.$$

Theorem 3.1 (*Convergence Theorem*)**.** *Let $u(x)$ be the true solution of the BVP in (1) with the boundary conditions in (2), and $\{u_j\}_{j=0}^{N}$ the discrete solution provided by the proposed global method. Then, the proposed method is convergent of seventh order.*

Proof. We note that the theorem can be proved similarly for other types of boundary conditions by making the appropriate changes.

Firstly, we assume that the given boundary conditions are exact values. Therefore, the unknowns in the global method are those corresponding to the solution, the first derivatives, and the second derivatives, as follows

$$\{u_r, u_1, u_s, u_2, u_{2+r}, u_3, u_{2+s}, u_4, \ldots, u_{N-1}, u_{N-2+s}\},$$

$$\left\{u'_r, u'_1, u'_s, u'_2, u'_{2+r}, u'_3, u'_{2+s}, u'_4, \ldots, u'_{N-2+s}, u'_N\right\},$$

$$\left\{u''_0, u''_r, u''_1, u''_s, u''_2, u''_{2+r}, u''_3, u''_{2+s}, u''_4, \ldots, u''_{N-2+s}, u''_N\right\}.$$

We consider the following matrices required in the subsequent steps of the proof. Let D be the $6N \times 6N$ matrix defined as

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix},$$

where the $D_{j1}, j = 1, 2, 3$, are sub-matrices of dimension $2N \times (2N-1)$ as follows

$$D_{11} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0
\end{pmatrix},$$

D_{21} and D_{31} are null sub-matrices. The sub-matrices $D_{j2}, j = 1, 2, 3$, have dimension $2N \times 2N$ and are given as follows

$$D_{12} = h\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_4 & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $\alpha_1 = -1/2, \alpha_2 = -1, \alpha_3 = -3/2, \alpha_4 = -2$;

$$D_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and D_{32} is a null sub-matrix. We note that removing the last column in the sub-matrix D_{22} we get the sub-matrix D_{11}.

The sub-matrices $D_{j3}, j = 1, 2, 3$ have dimension $2N \times (2N + 1)$ and are given as follows

$$D_{13} = h^2 \begin{pmatrix} \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\beta_1 = -1/8, \beta_2 = -1/2, \beta_3 = -9/8, \beta_4 = -2$,

$$D_{23} = h \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_4 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the α_i the same as in the sub-matrix D_{12}, and

$$D_{33} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Note that if we remove the first column in the sub-matrix D_{33} we get the sub-matrix D_{22}.

Further, let U be the matrix of dimension $6N \times (4N+2)$ given by

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{pmatrix},$$

where the U_{ij} are sub-matrices of dimensions $2N \times (2N+1)$ given by

$$U_{11} = h^2 \begin{pmatrix}
a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
d_1^1 & d_2^1 & d_3^1 & d_4^1 & d_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1^1 & d_2^1 & d_3^1 & d_4^1 & d_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^1 & d_2^1 & d_3^1 & d_4^1 & d_5^1
\end{pmatrix},$$

with $a_1^1 = -\frac{194261}{11612160}$, $a_2^1 = -\frac{3991}{725760}$, $a_3^1 = \frac{139}{64512}$, $a_4^1 = -\frac{853}{725760}$, $a_5^1 = \frac{965}{2322432}$, $b_1^1 = -\frac{1669}{18144}$, $b_2^1 = -\frac{233}{2835}$, $b_3^1 = \frac{4}{315}$, $b_4^1 = -\frac{23}{2835}$, $b_5^1 = \frac{53}{18144}$, $c_1^1 = -\frac{6453}{28672}$, $c_2^1 = -\frac{2619}{8960}$, $c_3^1 = -\frac{243}{7168}$, $c_4^1 = -\frac{153}{8960}$, $c_5^1 = \frac{837}{143360}$, $d_1^1 = -\frac{1184}{2835}$, $d_2^1 = -\frac{256}{405}$,

$d_3^1 = -\frac{64}{315}, d_4^1 = -\frac{256}{2835}, d_5^1 = \frac{4}{405};$

$$U_{21} = h\begin{pmatrix}
a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2
\end{pmatrix},$$

with $a_1^2 = -\frac{85781}{967680}, a_2^2 = -\frac{3}{64}, a_3^2 = \frac{289}{17920}, a_4^2 = -\frac{523}{60480}, a_5^2 = \frac{109}{35840}, b_1^2 = -\frac{263}{1260}, b_2^2 = -\frac{268}{945}, b_3^2 = 0, b_4^2 = -\frac{4}{315}, b_5^2 = \frac{19}{3780}, c_1^2 = -\frac{11667}{35840}, c_2^2 = -\frac{1233}{2240}, c_3^2 = -\frac{3807}{17920}, c_4^2 = -\frac{3}{64}, c_5^2 = \frac{369}{35840}, d_1^2 = -\frac{419}{945}, d_2^2 = -\frac{256}{315}, d_3^2 = -\frac{16}{35}, d_4^2 = -\frac{256}{945}, d_5^2 = -\frac{1}{63};$

$$U_{31} = \begin{pmatrix}
a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3
\end{pmatrix},$$

with $a_1^3=-\frac{8989}{34560}, a_2^3=-\frac{2179}{7560}, a_3^3=\frac{41}{560}, a_4^3=-\frac{289}{7560}, a_5^3=\frac{461}{34560}, b_1^3=-\frac{161}{720}, b_2^3=-\frac{536}{945}, b_3^3=-\frac{8}{35}, b_4^3=\frac{8}{315}, b_5^3=-\frac{13}{2160}, c_1^3=-\frac{311}{1280}, c_2^3=-\frac{141}{280}, c_3^3=-\frac{297}{560}, c_4^3=-\frac{71}{280}, c_5^3=\frac{39}{1280}, d_1^3=-\frac{31}{135}, d_2^3=-\frac{512}{945}, d_3^3=-\frac{16}{35}, d_4^3=-\frac{512}{945}, d_5^3=-\frac{31}{135}$;

$$U_{12}=h^3\begin{pmatrix}
t_1^1 & 0 & 0 & 0 & v_1^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_2^1 & 0 & 0 & 0 & v_2^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_3^1 & 0 & 0 & 0 & v_3^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_4^1 & 0 & 0 & 0 & v_4^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_1^1 & 0 & 0 & 0 & v_1^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_2^1 & 0 & 0 & 0 & v_2^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3^1 & 0 & 0 & 0 & v_3^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_4^1 & 0 & 0 & 0 & v_4^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_1^1 & 0 & 0 & 0 & v_1^1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_2^1 & 0 & 0 & 0 & v_2^1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_3^1 & 0 & 0 & 0 & v_3^1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_4^1 & 0 & 0 & 0 & v_4^1
\end{pmatrix},$$

with $t_1^1=-\frac{1931}{1935360}, v_1^1=-\frac{11}{138240}, t_2^1=-\frac{193}{30240}, v_2^1=-\frac{17}{30240}, t_3^1=-\frac{81}{5120}, v_3^1=-\frac{81}{71680}, t_4^1=-\frac{4}{135}, v_4^1=-\frac{2}{945}$;

$$U_{22}=h^2\begin{pmatrix}
t_1^2 & 0 & 0 & 0 & v_1^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_2^2 & 0 & 0 & 0 & v_2^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_3^2 & 0 & 0 & 0 & v_3^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_4^2 & 0 & 0 & 0 & v_4^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_1^2 & 0 & 0 & 0 & v_1^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_2^2 & 0 & 0 & 0 & v_2^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3^2 & 0 & 0 & 0 & v_3^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_4^2 & 0 & 0 & 0 & v_4^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_1^2 & 0 & 0 & 0 & v_1^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_2^2 & 0 & 0 & 0 & v_2^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_3^2 & 0 & 0 & 0 & v_3^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_4^2 & 0 & 0 & 0 & v_4^2
\end{pmatrix},$$

with $t_1^2=-\frac{781}{129024}$, $v_1^2=-\frac{25}{43008}$, $t_2^2=-\frac{5}{336}$, $v_2^2=-\frac{1}{1008}$, $t_3^2=-\frac{333}{14336}$, $v_3^2=-\frac{27}{14336}$, $t_4^2=-\frac{2}{63}$, $v_4^2=0$;

$$U_{32}=h\begin{pmatrix}
t_1^3 & 0 & 0 & 0 & v_1^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_2^3 & 0 & 0 & 0 & v_2^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_3^3 & 0 & 0 & 0 & v_3^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
t_4^3 & 0 & 0 & 0 & v_4^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_1^3 & 0 & 0 & 0 & v_1^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_2^3 & 0 & 0 & 0 & v_2^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3^3 & 0 & 0 & 0 & v_3^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_4^3 & 0 & 0 & 0 & v_4^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_1^3 & 0 & 0 & 0 & v_1^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_2^3 & 0 & 0 & 0 & v_2^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_3^3 & 0 & 0 & 0 & v_3^3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t_4^3 & 0 & 0 & 0 & v_4^3
\end{pmatrix},$$

with $t_1^3=-\frac{337}{16128}$, $v_1^3=-\frac{41}{16128}$, $t_2^3=-\frac{5}{336}$, $v_2^3=\frac{1}{1008}$, $t_3^3=-\frac{33}{1792}$, $v_3^3=-\frac{9}{1792}$, $t_4^3=-\frac{1}{63}$, $v_4^3=\frac{1}{63}$.

Now, let $u(x)$ be the true solution of the considered BVP, and define the $6N$-vector V as follows

$$V=\left(u(x_r),u(x_1),u(x_s),u(x_2),\dots,u(x_{N-2+s}),u'(x_r),u'(x_1),\dots,u'(x_N),u''(x_0),\dots,u''(x_N)\right)^T,$$

and the $(4N+2)$-vector F by

$$\begin{aligned}F=&\big(w(x_0,u(x_0),u'(x_0),u''(x_0)),w(x_r,u(x_r),u'(x_r),u''(x_r)),\dots,w(x_N,u(x_N),u'(x_N),u''(x_N)),\\&g(x_0,u(x_0),u'(x_0),u''(x_0)),g(x_r,u(x_r),u'(x_r),u''(x_r)),\dots,g(x_N,u(x_N),u'(x_N),u''(x_N))\big)^T.\end{aligned}$$

Using the above vector–matrix notation, the exact representation of the global system may be expressed as follows

$$D_{6N\times 6N}V_{6N}+hU_{6N\times(4N+2)}F_{4N+2}+C_{6N}=\mathcal{L}(h)_{6N}\,. \tag{12}$$

Note that in the above identity, the subscripts denote the corresponding dimensions of vectors and matrices. The vector C_{6N} contains the known values, provided by the given boundary conditions, that is,

$$C_{6N}=(-u_0-\frac{h}{2}u_0',-u_0-hu_0',-u_0-\frac{3h}{2}u_0',-u_0-2hu_0',0,\dots,0,u_N,-u_0',-u_0',-u_0',-u_0',0,\dots,0)^T,$$

and the vector $\mathcal{L}(h)_{6N}$ consists of the local truncation errors of the formulas, given by

$$\mathcal{L}(h)=\begin{pmatrix}
\mathcal{L}[u(x_r),h]\\
\mathcal{L}[u(x_1),h]\\
\mathcal{L}[u(x_s),h]\\
\mathcal{L}[u(x_2),h]\\
\dots\\
\mathcal{L}[u(x_N),h]\\
\mathcal{L}[u'(x_r),h]\\
\mathcal{L}[u'(x_1),h]\\
\mathcal{L}[u'(x_s),h]\\
\mathcal{L}[u'(x_2),h]\\
\dots\\
\mathcal{L}[u'(x_N),h]\\
\mathcal{L}[u''(x_r),h]\\
\mathcal{L}[u''(x_1),h]\\
\mathcal{L}[u''(x_s),h]\\
\mathcal{L}[u''(x_2),h]\\
\dots\\
\mathcal{L}[u''(x_N),h]
\end{pmatrix}.$$

Now, consider the system of approximate values of the problem expressed as follows

$$D_{6N\times 6N}\bar{V}_{6N} + hU_{6N\times(4N+2)}\bar{F}_{4N+2} + C_{6N} = 0, \tag{13}$$

where $\bar{V}_{6N}$ is used to denote the vector of approximate values of V_{6N}, that is,

$$\bar{V}_{6N} = (u_r, u_1, u_s, u_2, \ldots, u_{N-2+s}, u'_r, u'_1, \ldots, u'_N, u''_0, u''_r, \ldots, u''_N)^T,$$

and $\bar{F}_{4N+2}$ is given by

$$\bar{F}_{4N+2} = (w_0, w_r, w_1, w_s, w_2, \ldots, w_N, g_0, g_r, g_1, g_s, g_2, \ldots, g_N)^T.$$

By subtracting (13) from (12), after some simplifications we get

$$D_{6N\times 6N}\mathcal{E}_{6N} + hU_{6N\times(4N+2)}(F-\bar{F})_{4N+2} = \mathcal{L}(h)_{6N}, \tag{14}$$

where

$$\mathcal{E}_{6N} = V_{6N} - \bar{V}_{6N} = (E_r, E_1, E_s, E_2, \ldots, E_{N-2+s}, E'_r, \ldots, E'_N, E''_0, E''_r, \ldots, E''_N)^T$$

consists of the errors at off-steps and nodal points.

On the other hand, using the Mean Value Theorem, one can consider for $i = 0, r, 1, s, 2, 2+r, 3, 2+s, 4, \ldots, N$ the identities

$$\begin{aligned}
w(x_i, u(x_i), u'(x_i), u''(x_i)) - w(x_i, u_i, u'_i, u''_i) &= (u(x_i)-u_i)\frac{\partial w}{\partial u}(\xi_i) + (u'(x_i)-u'_i)\frac{\partial w}{\partial u'}(\xi_i)\\
&\quad + (u''(x_i)-u''_i)\frac{\partial w}{\partial u''}(\xi_i),\\
g(x_i, u(x_i), u'(x_i), u''(x_i)) - g(x_i, u_i, u'_i, u''_i) &= (u(x_i)-u_i)\frac{\partial g}{\partial u}(\eta_i) + (u'(x_i)-u'_i)\frac{\partial g}{\partial u'}(\eta_i)\\
&\quad + (u''(x_i)-u''_i)\frac{\partial g}{\partial u''}(\eta_i),
\end{aligned} \tag{15}$$

where ξ_i and η_i stand for intermediate points on the line segment joining $(x_i, u(x_i), u'(x_i), u''(x_i))$ to (x_i, u_i, u'_i, u''_i). Now, using the formulas in (15) we have

$$F-\bar{F} = \begin{pmatrix}
\frac{\partial w}{\partial u}(\xi_0) & \cdots & 0 & \frac{\partial w}{\partial u'}(\xi_0) & \cdots & 0 & \frac{\partial w}{\partial u''}(\xi_0) & \cdots & 0\\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
0 & \cdots & \frac{\partial w}{\partial u}(\xi_N) & 0 & \cdots & \frac{\partial w}{\partial u'}(\xi_N) & 0 & \cdots & \frac{\partial w}{\partial u''}(\xi_N)\\
\frac{\partial g}{\partial u}(\eta_0) & \cdots & 0 & \frac{\partial g}{\partial u'}(\eta_0) & \cdots & 0 & \frac{\partial g}{\partial u''}(\eta_0) & \cdots & 0\\
0 & \cdots & 0 & 0\ldots & 0 & 0 & 0\ldots & 0 & \\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
0 & \cdots & \frac{\partial g}{\partial u}(\eta_N) & 0 & \cdots & \frac{\partial g}{\partial u'}(\eta_N) & 0 & \cdots & \frac{\partial g}{\partial u''}(\eta_N)
\end{pmatrix}
\begin{pmatrix} E_0\\ E_r\\ .\\ E_N\\ E'_0\\ E'_r\\ .\\ E'_N\\ E''_0\\ E''_r\\ .\\ E''_N \end{pmatrix}$$

$$= \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \frac{\partial w}{\partial u''}(\xi_0) & \cdots & 0\\
\frac{\partial w}{\partial u}(\xi_r) & \cdots & 0 & \frac{\partial w}{\partial u'}(\xi_r) & \cdots & 0 & 0 & \cdots & 0\\
. & \cdots & . & . & \cdots & . & . & . & \\
. & \cdots & . & . & \cdots & . & . & . & \\
. & \cdots & . & . & \cdots & . & . & . & \\
0 & \cdots & \frac{\partial w}{\partial u}(\xi_{N-2+s}) & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \cdots & \frac{\partial w}{\partial u'}(\xi_N) & 0 & \cdots & \frac{\partial w}{\partial u''}(\xi_N)\\
0 & \cdots & 0 & 0 & \cdots & 0 & \frac{\partial g}{\partial u''}(\eta_0) & \cdots & 0\\
\frac{\partial g}{\partial u}(\eta_r) & \cdots & 0 & \frac{\partial g}{\partial u'}(\eta_r) & \cdots & 0 & 0 & \cdots & 0\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
. & \cdots & . & . & \cdots & . & . & \cdots & .\\
0 & \cdots & \frac{\partial g}{\partial u}(\eta_{N-2+s}) & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \cdots & \frac{\partial g}{\partial u'}(\eta_N) & 0 & \cdots & \frac{\partial g}{\partial u''}(\eta_N)
\end{pmatrix}
\begin{pmatrix} E_r\\ E_1\\ .\\ E_{N-1}\\ E_{N-2+s}\\ E'_r\\ .\\ E'_{N-2+s}\\ E'_N\\ E''_0\\ .\\ E''_{N-2+s}\\ E''_N \end{pmatrix}$$

$$= J_{(4N+2)\times 6N}\mathcal{E}_{6N}\,.$$

In the above expressions we have used the fact that the exact boundary conditions are known, that is, $E_0 = u(x_0) - u_0 = 0$, $E'_0 = u'(x_0) - u'_0 = 0$ and $E_N = u(x_N) - u_N = 0$. Finally, from Eq. (14), we have that

$$D_{6N\times 6N}\mathcal{E}_{6N} + hU_{6N\times(4N+2)}J_{(4N+2)\times 6N}\mathcal{E}_{6N} = \mathcal{L}(h)_{6N}. \tag{16}$$

The formula in (16) may be rewritten as

$$\left(D_{6N\times 6N} + hU_{6N\times(4N+2)}J_{(4N+2)\times 6N}\right)\mathcal{E}_{6N} = \mathcal{L}(h)_{6N}, \tag{17}$$

and putting $\mathcal{M} = D + hUJ$ we have

$$\mathcal{M}_{6N\times 6N}\mathcal{E}_{6N} = \mathcal{L}(h)_{6N}. \tag{18}$$

We want to prove that matrix $\mathcal{M}$ is invertible. To see this, we observe that D contains nonzero diagonal submatrices, as

$$D = [D_{ij}] = \begin{cases} \text{non-zero} : j = 1, 2, 3, i = 1, \\ \text{non-zero} : j = 2, 3, i = 2, \\ \text{non-zero} : j = 3, i = 3, \\ 0 : \text{otherwise.} \end{cases} \tag{19}$$

Having in mind that N must be even, that is $N = 2j$, using the abbreviate notation $D_j = D_{6N\times 6N}$ it is easy to prove that for $j = 1$ the determinant is $|D_1| = -2h^2$. Now, by induction, it can be proved that $|D_j| = -2j^2h^2$, and thus D has an inverse as long as $h > 0$.

We can now rewrite the matrix $\mathcal{M}$ as

$$\mathcal{M} = D + hUJ = (I - C)D$$

where I denotes the identity matrix of order $6N$, and $C = -hUJD^{-1}$. Hence, we have that $|\mathcal{M}| = |I - C|\,|D|$.

As $|\mu I - C| = \prod_{i=1}^{6N}(\mu - \mu_i)$ denotes the characteristic polynomial of C, to have that $|I - C| \neq 0$, if we take $\mu = 1$ and choose a small and sufficient value of h such that

$$h \notin \left\{1/\bar{\mu}_i : \bar{\mu}_i \text{ is an eigenvalue of } UJD^{-1}\right\},$$

then the matrix $\mathcal{M}$ is invertible. Thus, the equation in (18) may be rewritten as

$$\mathcal{E}_{6N} = \mathcal{M}^{-1}_{6N\times 6N}\mathcal{L}(h)_{6N}\,. \tag{20}$$

We consider the maximum norm in $\mathbb{R}^{6N}$, $\|\mathcal{E}\| = \max_i |E_i|$, and the corresponding induced matrix norm in $\mathbb{R}^{6N\times 6N}$. Then, expanding each term of $\mathcal{M}^{-1}_{6N\times 6N}$ in a series about h, it can be shown that after some tedious manipulations we have $\|\mathcal{M}^{-1}_{6N\times 6N}\| = \mathcal{O}(h^{-2})$. Finally, from Eq. (20) and assuming that $u(x)$ has enough bounded derivatives, we get

$$\|\mathcal{E}_{6N}\| \leq \|\mathcal{M}^{-1}_{6N\times 6N}\|\,\|\mathcal{L}(h)_{6N}\| = |\mathcal{O}(h^{-2})|\,|\mathcal{O}(h^9)| \leq Kh^7.$$

This completes the proof. □

Remark 3.1. We have shown that the global method exhibits a seventh order convergence. Nevertheless, in view of the form of the vector $\mathcal{L}(h)$ we see that, assuming sufficient smoothness of the solution, at the mesh points we obtain a superconvergence order (see [26]):

$$|E_j| = |u(x_j) - u_j| \leq |\mathcal{O}(h^{-2})|\,|\mathcal{O}(h^{10})| \leq Kh^8, \quad j = 1, 2, \ldots N,$$

This interesting behaviour will produce accurate results, as can be seen in the numerical examples.

4. Implementation and computational procedure of the proposed method

To give numerical solutions for the considered test problems, we must solve the system of $6N + 3$ equations with $6N + 3$ unknowns given in (7)–(10), for $n = 0, 2, \ldots, N - 2$, using any of the BCs in (2)–(3). We then consider the general BC in (3) and formulate the non-linear system as

$$\begin{cases} G_1(u(x_0), u'(x_0), u''(x_0)) - u_P = 0\,, \\ G_2(u(x_0), u'(x_0), u''(x_0)) - u_Q = 0\,, \\ G_3(u(x_N), u'(x_N), u''(x_N)) - u_R = 0\,, \\ W_0 = 0\,, \\ W_2 = 0\,, \\ \quad\ldots \\ W_{N-2} = 0\,, \end{cases} \tag{21}$$

where W_n, $n = 0, 2, \ldots, N-2$, contains the corresponding formulas in (7)–(10). We denote the system in (21) as $\boldsymbol{W(u) = 0}$ and the $6N + 3$ unknowns as

$$\tilde{\mathbf{U}} = (u_0, u'_0, u''_0, u_r, u'_r, u''_r, u_1, u'_1, u''_1, u_s, u'_s, u''_s, u_2, u'_2, u''_2, u_{2+r}, u'_{2+r}, u''_{2+r}, \ldots, u_N, u'_N, u''_N)\,.$$

If the function w in (1) is linear we utilize any available linear solver. If w is non-linear we use Modified Newton's method. The Modified Newton iteration is given by

$$\tilde{\mathbf{U}}^{i+1} = \tilde{\mathbf{U}}^{i} - \left(\mathbf{J}^{i}\right)^{-1}\mathbf{W}^{i},$$

where $\mathbf{J}$ represents the jacobian matrix of $\mathbf{W}$. In this case, we would require initial guesses close to the exact roots.We distinguish two cases as follows:

- In case of Robin boundary conditions in (2), the system is simplified to $6N$ equations with $6N$ unknowns. For this situation, one can consider linear interpolation to get the starting points.
- In case of general mixed boundary conditions in (3), we use a method similar to the one in Marasco and Romano [27], where a homotopy-type technique is utilized. We consider a set of non-linear BVPs $P_j, j = 0, 1, 2, \ldots, k$, to such an extent that for $j = 0$ the problem P_0 admits just the solution $u(x) = 0$, while when $j = k$ we obtain the original problem. Along these lines, we have a set of BVPs given by

$$P_j \equiv \begin{cases} u''' = w(x, u, u', u'') - w(x, 0, 0, 0) + \dfrac{j}{k}\, w(x, 0, 0, 0)\,, \\ G_1(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = \dfrac{j}{k}\, u_P\,, \\ G_2(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = \dfrac{j}{k}\, u_Q\,, \\ G_3(u(x_0), u'(x_0), u''(x_0), u(x_N), u'(x_N), u''(x_N)) = \dfrac{j}{k}\, u_R\,, \end{cases} \tag{22}$$

  for $j = 0, 1, \ldots, k$. Every one of these problems P_j for $j = 1, 2, \ldots, k$ is solved utilizing the proposed method, taking as initial guesses the values obtained after solving the previous problem P_{j-1}. For $j = k$ the non-linear algebraic system corresponding to the given BVP is solved taking as initial guesses the values obtained after solving the problem P_{k-1}. In our implementations, we have taken $k = 1$ except in problem (26), where we take $k = 6$ to display the effect of the homotopy strategy graphically. The stopping criterion used in the Newton procedure has been $\|\tilde{\mathbf{U}}^{i+1} - \tilde{\mathbf{U}}^{i}\| \leq 10^{-12}$, taking 50 as the maximum number of iterations.

This procedure has the sole reason for giving appropriate starting values for the Modified Newton's method. If one has another way of providing those values, it may be utilized.

4.1. Algorithm

This subsection outlines the proposed algorithm, which is given as follows

Data: Stepsize: h;
Number of steps: N;
Starting and final point of integration: $[x_0, x_N]$
Result: Approximations of the problem in (1)–(3) at selected grid points.
Set $n = 0, 2, 4, \ldots, N-2$, and define $h = (x_N - x_0)/(N)$;
Solve the system of equations in (7)–(10) for the above values of n to obtain the u_i;
Save the approximate solution $\{(x_i, u_i)\}_{i=0,1,2,3,\ldots,N}$
End

5. Numerical test problems and results

Here, we report the obtained numerical solutions for the problem (1)–(3) utilizing the proposed method in (7)–(10). Also, we will include comparisons with various numerical approaches available in the literature, to show the benefit of the proposed approach. The 2FDHBM is implemented by using the composed codes in Mathematica 11.0, considering the command FindRoot (that uses a modified Newton's method) for solving the systems. We used a personal computer with configuration i7-7500U, 8 GB memory and 64-bit Windows 10 operating system. The codes considered for comparisons are:

- 2FDHBM: A two-step fourth derivative hybrid block method derived in this paper.
- NPST: The non-polynomial spline technique in [28].
- CSM: The cubic spline method proposed in [29].
- PAM: The Padé approximation method presented in [7].
- FDBM: The fourth derivative eight-order block method in [30].

Table 1
Comparison of the maximum absolute error (MAXAE) for problem (23).

h	Methods	MAXAE
$\frac{1}{4}$	2FDHBM	5.14305×10^{-12}
$\frac{1}{16}$	NPST	2.38190×10^{-8}
$\frac{1}{16}$	CSM	9.75010×10^{-5}
$\frac{1}{8}$	2FDHBM	2.52021×10^{-14}
$\frac{1}{32}$	NPST	1.11840×10^{-9}
$\frac{1}{32}$	CSM	2.59650×10^{-5}
$\frac{1}{16}$	2FDHBM	1.38778×10^{-16}
$\frac{1}{64}$	NPST	6.30200×10^{-11}
$\frac{1}{64}$	CSM	6.6004×10^{-6}

Table 2
Comparison of the maximum absolute error (MAXAE) for problem (24).

h	Methods	MAXAE
$\frac{1}{4}$	2FDHBM	6.14072×10^{-11}
$\frac{1}{16}$	NPST	5.29920×10^{-7}
$\frac{1}{16}$	CSM	1.68610×10^{-3}
$\frac{1}{8}$	2FDHBM	2.58404×10^{-13}
$\frac{1}{32}$	NPST	2.61270×10^{-8}
$\frac{1}{32}$	CSM	4.45100×10^{-4}
$\frac{1}{16}$	2FDHBM	1.11022×10^{-16}
$\frac{1}{64}$	NPST	1.49990×10^{-9}
$\frac{1}{64}$	CSM	1.12930×10^{-4}

- QSM: The Quartic spline in [31].
- TOM6P: The higher-order method of algebraic order-six in [12].

5.1. Numerical test problem 1

We firstly consider the following linear BVP which was also considered in [28] and [29]

$$u'''(x) = (x-4)\sin(x) + (1-x)\cos(x) - u(x),\;\; u(0) = 0, u'(0) = -1, u'(1) = \sin(1), 0 \le x \le 1. \tag{23}$$

The exact solution is given as :

$$u(x) = \sin(x)(x-1).$$

5.2. Numerical test problem 2

As a second test problem, we consider the following BVP which was also considered in [28] and [29]

$$u'''(x) = xu(x) + \left(x^3 - 2x^2 - 5x - 3\right)\exp(x),\;\; u(0) = 0, u'(0) = 1, u(1) = 0, 0 \le x \le 1. \tag{24}$$

The exact solution is given as :

$$u(x) = x(1-x)\exp(x).$$

The numerical outcomes and the comparisons of 2FDHBM, NPST in [28], and CSM in [29] for the test problems (23) and (24) are summarized in Tables 1–2 . We note that for the step sizes, $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, the proposed 2FDHBM gives better accuracy compared with the step-sizes $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ used with the methods in [28] and [29].

5.3. Numerical test problem 3

In the third example, we tested the new proposed numerical method on a sandwich beam problem to determine its usefulness in a real-world application. This problem was also solved by Tirmizi et al. [7] but with lower accuracy. The problem is

$$u'''(x) - c^2 u'(x) + l = 0,\;\; u'(0) = 0, u'(1) = 0,\;\; u\left(\frac{1}{2}\right) = 0,\;\; 0 \le x \le 1, \tag{25}$$

Table 3
Comparison of the absolute errors (AE) for $l = 1, c = 5$ for problem (25).

x-value	AE with 2FDHBM	AE with PAM
0.0000	1.37308×10^{-11}	6.65300×10^{-5}
0.1000	1.29817×10^{-11}	6.50000×10^{-5}
0.2000	1.02789×10^{-11}	5.25400×10^{-5}
0.3000	6.81773×10^{-12}	3.63000×10^{-5}
0.4000	3.39005×10^{-12}	1.87500×10^{-5}
0.6000	3.39005×10^{-12}	1.73400×10^{-5}
0.7000	6.81773×10^{-12}	3.40500×10^{-5}
0.8000	1.02789×10^{-11}	4.98000×10^{-5}
0.9000	1.29817×10^{-11}	6.20100×10^{-5}
1.0000	1.37308×10^{-11}	6.34700×10^{-5}

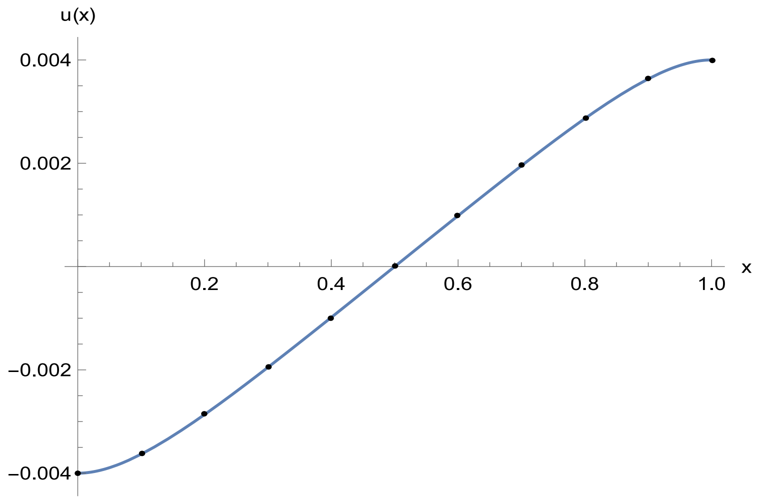

Fig. 1. Numerical results of the 2FDHBM and exact solution with $l = 1, c = 10$ for problem (25) taking $N = 10$.

where $c^2 = \dfrac{(G_u L^2)}{(D_u A_e)\left(C_2 A_e - C_1^2\right)}$, $\quad l = \dfrac{(C_1 L^3)}{(D_u A_e)}$, L is the span of the beam, u represents the shear wrapping, A_e represents the effective area of cross-section of the beam, C_1 and C_2 are shear parameters, D_u is the shear rigidity and G_u is the face shear moduli. According to [7] the exact solution is given by

$$u(x) = \left(\frac{l}{c^3}\right)\left[\left(\sinh\left(\frac{c}{2}\right) - \sinh(cx)\right) + c\left(x - \frac{1}{2}\right) + \tanh\left(\left(\cosh(cx) - \cosh\left(\frac{c}{2}\right)\right)\right)\right].$$

Problem (25) is solved using the 2FDHBM for $l = 1, c = 5$ and the numerical results are contrasted with the PAM method in [7] and the obtained absolute errors (AEs) are given in Table 3. Moreover, the exact and the numerical solution obtained with the 2FDHBM for $N = 10$ are depicted in Fig. 1. Fig. 2 displays the plot of the absolute errors on problem (25) for $l = 1, c = 10, h = \frac{1}{10}$. From Table 3 and Figs. 1–2, one can easily see that the proposed 2FDHBM produces accurate results.

5.4. Numerical test problem 4

We also consider the following non-linear third order boundary value problem which was also solved in [30]

$$u'''(x) = -2e^{-3u(x)} + \frac{4}{(1+x)^3}, \; u(0) = 0, u'(0) = 1, u(1) = \ln 2, 0 \le x \le 1. \tag{26}$$

The exact solution is given by

$$u(x) = \ln(1+x).$$

Table 4 and Fig. 3 show the good results obtained with the proposed method.

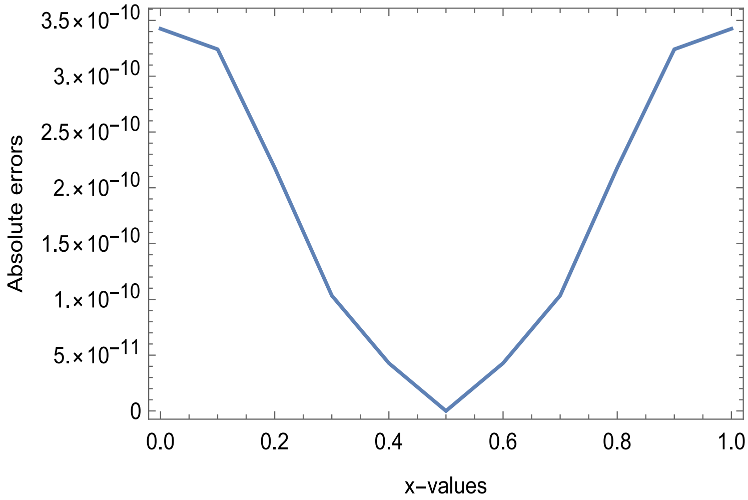

Fig. 2. Absolute errors of the 2FDHBM with $l = 1, c = 10$ for problem (25) taking $N = 10$.

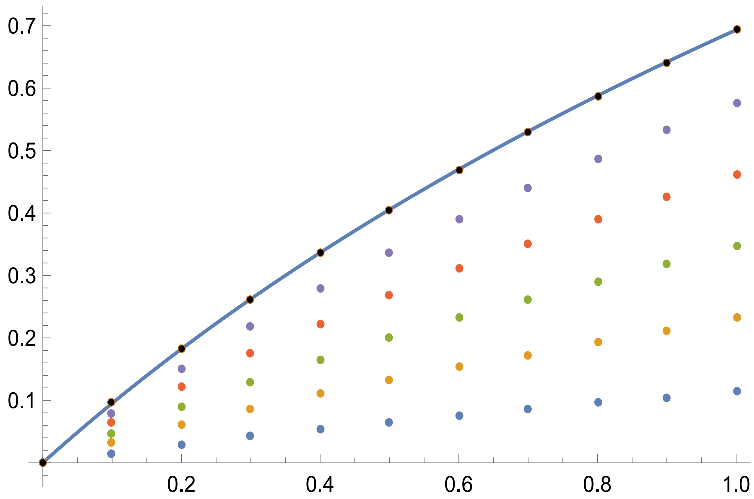

Fig. 3. Discrete and exact solutions with $N = 10$ for problem (26) using the homotopy approach with $k = 6$.

Table 4
Comparison of the maximum absolute errors (MAXAE) for problem (26).

N	2FDHBM	FDBM
14	4.12986×10^{-12}	2.39000×10^{-11}
28	1.80411×10^{-14}	9.5000×10^{-14}
56	1.66533×10^{-16}	3.62000×10^{-16}

5.5. Numerical test problem 5

We also consider the following BVP with mixed BCs

$$u'''(x) = \frac{u'(x)}{\sqrt{x+1}} - 2u(x) + q(x),\;\; u(0) = 0, u'(0) = 0, 3u(1) - u'(1) = 0, 0 \leq x \leq 1, \tag{27}$$

where $q(x)$ is obtained from the exact solution $u(x) = \frac{x^3}{2}$.

Table 5
Approximate solutions with 2FDHBM and exact solutions for Problem (27) at different grid points for $h = 0.1$.

x	2FDHBM solutions	Exact solutions	Absolute errors
0.1	0.000499999999999997	0.0005000000000000	3.03577×10^{-18}
0.2	0.003999999999999986	0.0040000000000000	1.38778×10^{-17}
0.3	0.013499999999999970	0.0135000000000000	2.94902×10^{-17}
0.4	0.031999999999999960	0.0320000000000000	4.16333×10^{-17}
0.5	0.062499999999999910	0.0625000000000000	9.02056×10^{-17}
0.6	0.107999999999999960	0.1080000000000000	4.16334×10^{-17}
0.7	0.171499999999999930	0.1715000000000000	8.32667×10^{-17}
0.8	0.255999999999999900	0.2560000000000000	1.11022×10^{-16}
0.9	0.364499999999999900	0.3645000000000000	1.11022×10^{-16}
1.0	0.500000000000000000	0.500000000000000	0.0000

Table 6
Comparison of the maximum absolute errors (MAXAE) for problem (28).

N	ϵ	Methods	MAXAE
10	$\frac{1}{16}$	2FDHBM	6.50521×10^{-19}
20	$\frac{1}{16}$	2FDHBM	2.71051×10^{-19}
10	$\frac{1}{16}$	QSM	1.30000×10^{-2}
20	$\frac{1}{16}$	QSM	1.10000×10^{-3}
10	$\frac{1}{32}$	2FDHBM	1.62630×10^{-19}
20	$\frac{1}{32}$	2FDHBM	9.75782×10^{-19}
10	$\frac{1}{32}$	QSM	3.20000×10^{-3}
20	$\frac{1}{32}$	QSM	2.70000×10^{-4}
10	$\frac{1}{64}$	2FDHBM	1.35525×10^{-19}
20	$\frac{1}{64}$	2FDHBM	6.43745×10^{-20}
10	$\frac{1}{64}$	QSM	3.40000×10^{-4}
20	$\frac{1}{64}$	QSM	2.20000×10^{-5}

In Table 5 we present the data for the 2FDHBM and exact solutions to show that the proposed technique is in good agreement with known analytical solutions.

5.6. Numerical test problem 6

We also consider the following BVP that contains a perturbation parameter,

$$\epsilon u'''(x) = q(x) + u(x), \;\; u(0) = 0, u(1) = 0, u''(0) = 0, 0 \le x \le 1, \tag{28}$$

where $q(x) = 6\epsilon(-1+x)^5x^3 - 36\epsilon^2(-1+x)^2(-1+18x-63x^2+56x^3)$ and the exact solution is $u(x) = 6\epsilon x^3(1-x)^5$.

In Table 6, the 2FDHBM is compared with the QSM method in Akram [31] who also solved (28) with the same parameters and step-size (h). It is observed that, the MAXAE 6.50521×10^{-19} obtained with the 2FDHBM is extremely smaller than 1.30000×10^{-2} obtained by Akram [31]. Consequently, the 2FDHBM performed better. Moreover, the graphical representation of absolute errors (AEs) for different values of x in Fig. 4 confirms that the solution provided by the 2FDHBM is very close to the exact one.

5.7. Numerical test problem 7

In the last numerical test problem, we consider the following non-linear boundary layer problem in [32]

$$\epsilon u'''(x) = \beta\left(u'(x)^2 - 1\right) - \alpha u(x)u''(x), \;\; u(0) = 0, u(1) = 0, u'(\infty) = 1, \tag{29}$$

where the exact solution (29) is unknown.

We solved (29) with the 2FDHBM scheme by fixing $\alpha = 1$ and taking different values of β for $h = \frac{1}{10}$. This problem is related to the Falkner–Skan equation, according to [32]. The numerical solutions obtained with the 2FDHBM for $\alpha = 1$, $\beta = 0, 0.5, 1, 2$ are displayed in Fig. 5. We note that the solutions $u(x)$ for $\alpha = 1, \beta = 1$ and $\alpha = 1, \beta = \frac{1}{2}$ in Fig. 5 are called Hiemenz flow and Homann axisymmetric stagnation flow, respectively. Also, the velocity profile solutions in Fig. 6 confirm that the solutions provided by the 2FDHBM are in good agreement with the results presented in [32,33], and [34].

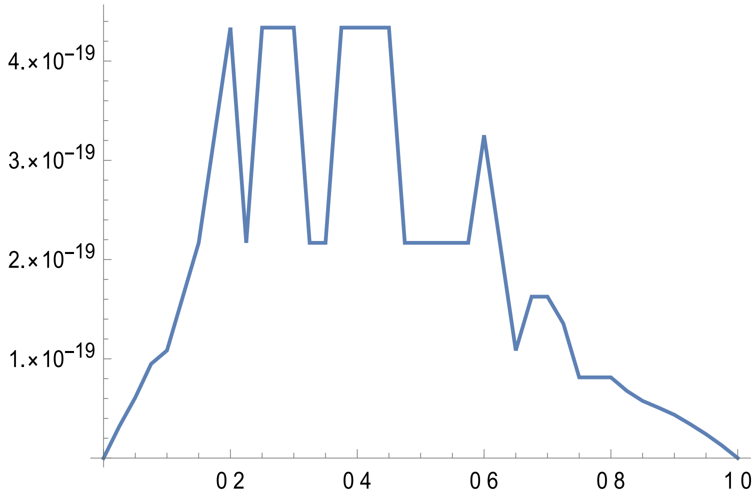

Fig. 4. Absolute errors of the 2FDHBM with $\epsilon = \frac{1}{16}, N = 40$ for problem (28).

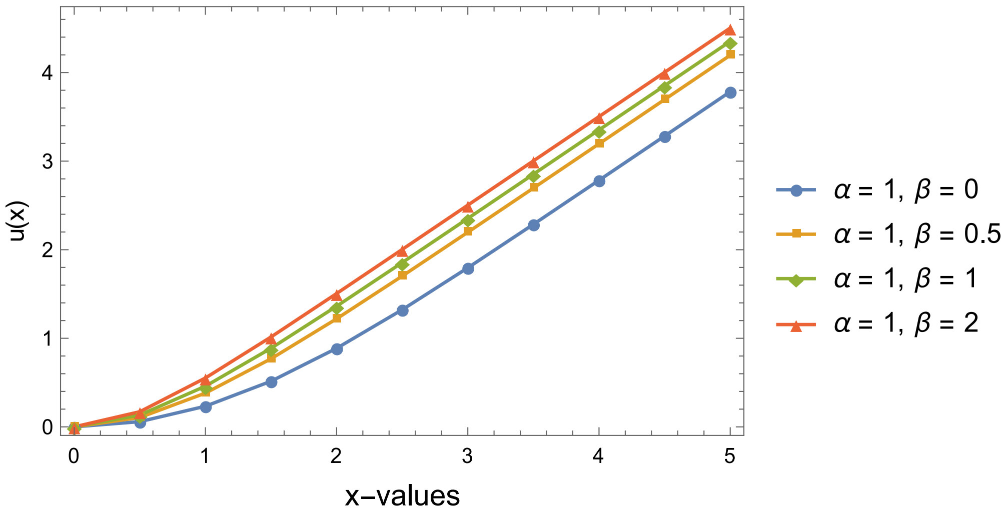

Fig. 5. Approximate solutions of 2FDHBM with $h = \frac{1}{10}, \alpha = 1$ for different values of β for problem (29).

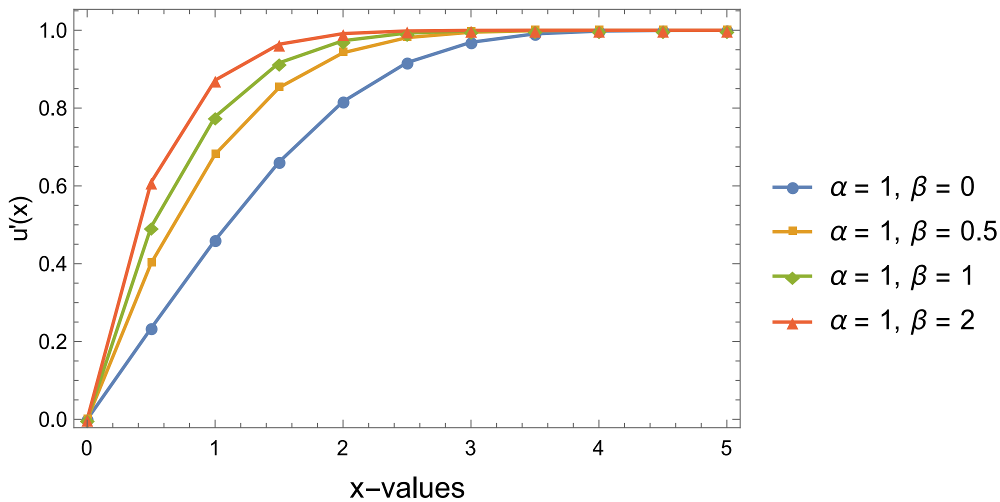

Fig. 6. Velocity profiles solutions of 2FDHBM with $h = \frac{1}{10}, \alpha = 1$ for different values of β for problem (29).

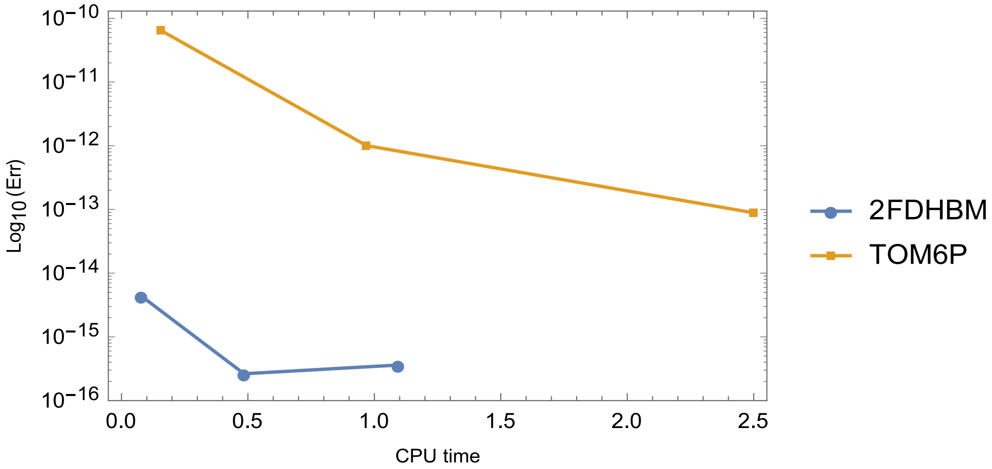

Fig. 7. Efficiency curves of the numerical schemes for the test problem (23).

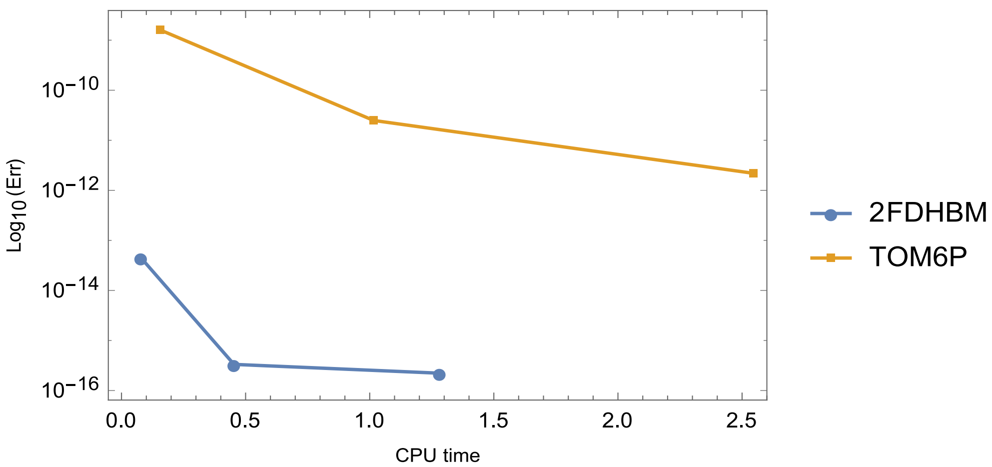

Fig. 8. Efficiency curves of the numerical schemes for the test problem (24).

5.8. *Efficiency curves*

In this subsection, we present some plots with the efficiency curves obtained with the proposed methodology (2FDHBM) for $N = 10, 20, 30$, and the TOM6P method for $N = 30, 60, 90$, exhibiting the maximum absolute errors (MAXAE) versus CPU time. The curves are given for the numerical test problems (23)–(26) in Figs. 7–10 respectively. These curves display that the newly proposed technique is effective in solving the type of problems considered.

6. Conclusions

This paper develops an efficient numerical methodology for solving problems of the type (1)–(3). The proposed 2FDHBM strategy is obtained by considering two off-grid points and two fourth derivatives evaluated appropriately. The newly 2FDHBM method gives the approximate solutions to the sandwich beam and the Falkner–Skan equation of boundary layer problems to determine its usefulness in real-world applications. The technique is exceptionally straightforward and accurate, and above all, the numerical approximations converged quickly to the solution of the considered problems, as we can see in the results presented in Tables 1–6 and Figs. 1–10.

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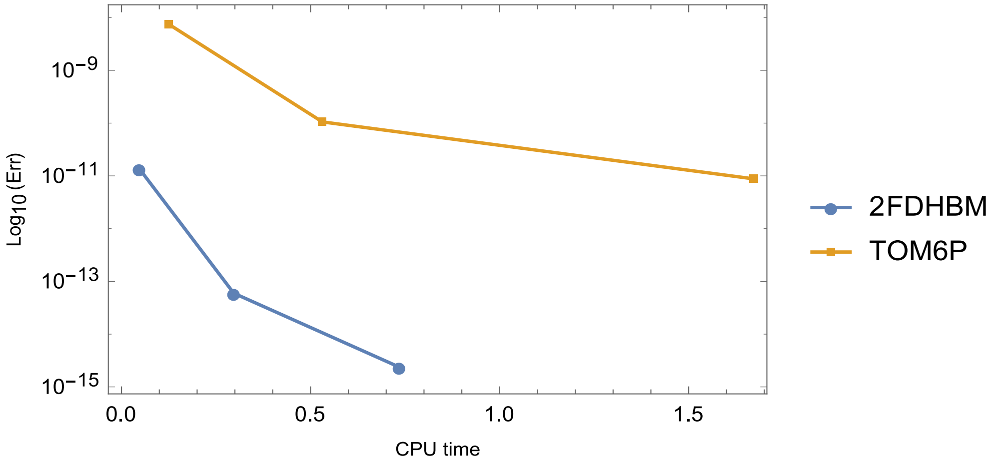

Fig. 9. Efficiency curves of the numerical schemes for the test problem (25).

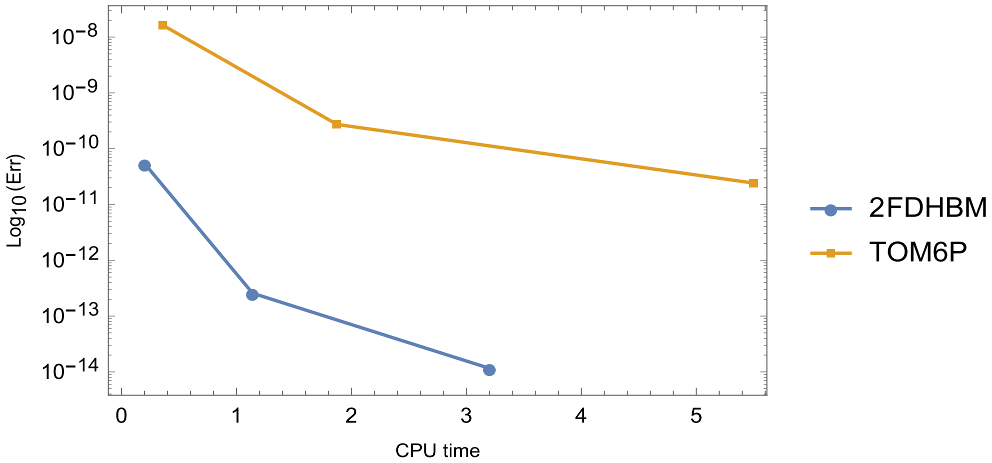

Fig. 10. Efficiency curves of the numerical schemes for the test problem (26).

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