

Arithmetic infinite Grassmannians and the induced central extensions

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ABSTRACT

The construction of families of Sato Grassmannians, their determinant line bundles and the extensions induced by them are given. The base scheme is an arbitrary scheme.

1. Introduction

Since infinite Grassmannians were introduced by Sato ([22]) as a classifying space for the solutions of the KP hierarchy, their structures and properties have been successfully applied in a wide range of topics, such as microlocal analysis ([6]), loop groups ([21]), conformal and quantum field theories and string theory ([24, 12, 19]), moduli problems on algebraic curves ([15, 23, 18]), representation theory of infinite dimensional lie algebras ([11]), abelian and non-abelian reciprocity laws on curves ([1, 17]), supersymmetric analogues ([4, 16]), etc.

Some of the above-cited works are strongly based on the algebraic structure of Sato Grassmannians ([2]). For instance, once a Sato Grassmannian has been endowed with an algebraic structure it makes sense to apply algebraic geometry to study line bundles on it and automorphisms of it or of certain line bundles. It is worth to mention that the functor of infinite dimensional Grassmannian has been studied over Grassmann algebras ([4, 16]) as well as on the category of noetherian schemes over a field, see [20] (but, unfortunately, it fails to be representable in this category). However, up to

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now only the representability of Sato Grassmannians of suitable infinite dimensional k -vector spaces on the category of k -schemes, where k is a field, were known.

This paper aims at offering a construction of Sato Grassmannians of certain \mathcal{O}_S -modules in the category of S -schemes where S is an arbitrary scheme; i.e. the base field is no longer required. Note that, from the algebro-geometric point of view, this new object is the natural framework within which to study the variation of coefficients in the KP-hierarchy. Besides its interest in its own right, let us say few words on some issues that could profit from this construction. First, let us recall that some main characters of the geometric Langlands program, such as the moduli space of pointed curves, the moduli space of G -bundles, etc. have been constructed with the help of Sato Grassmannians over the field of complex numbers. Now, such constructions can be generalized for other base schemes (allowing families of moduli spaces, function fields as base schemes, etc.). Indeed, this new object has already been applied in [10], where S parametrizes k -algebra structures in the k -vector space. For the interested reader, let us recommend Frenkel's survey [8]. Second, the Sato Grassmannian presented here would also provide a geometric formulation of the approach to reciprocity laws given in [3] (see also [1] where the base is the spectrum of a local artinian ring). Note that the case of valuation rings of unequal characteristic is also allowed.

A second main result are the constructions of the determinant line bundle (recall that its space of global sections is deeply connected to Fock spaces [11]) and of a central extension that includes the transformation groups considered in several works (e.g. [1, 17, 19],...) that will be suitable for explicit computations. Let us remark that our determinants and extensions are deeply related with those of §2 of [3] and this relationship deserves further study. In the future, we shall apply these results to some moduli and arithmetic problems for families of arithmetic curves; more precisely, in the problem of a reciprocity law for families of arithmetic curves defined over \mathbb{Z} and over p -adic numbers.

Let us briefly explain how the paper is organized. Section 2 is devoted to the construction of the Sato Grassmannian of an S -module V (under certain hypotheses) as an S -scheme. The main steps of this construction are inspired by [2] and by Grothendieck's constructions of Grassmannians ([7]), although some arguments are now quite technical and delicate. This section ends with explicit computations of two spaces of global sections, which will be required later.

Section 3 focuses on the construction of extensions with the help of Sato Grassmannians. Contrary to the case of a base field, now we must deal with monoids rather than with groups. Thus, Leech's ([14]) results on extension of groups by monoids will be extensively used. Indeed, once the determinant line bundle has been constructed (§3.A), the central extension shows up as the set of certain transformations of it endowed with the composition law (§3.B) and a criterion for this extension to be trivial is proven. Nevertheless, it turns out that, in the case of families (e.g. S arbitrary), the defining conditions may be too strong and, consequently, the monoid too small. In §3.C we propose a method to overcome this difficulty with the help of K -theory. Due to the close relationship between these two monoids, we expect that explicit computations can be carried out.

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2. Arithmetic Sato Grassmannians

2.A Commensurableness

Henceforth, we consider an arbitrary scheme, S , and a pair of sheaves, (V, V^+) , of flat quasicoherent \mathcal{O}_S -modules such that $V^+ \subset V$ and V/V^+ are flat.

DEFINITION 2.1 Let $A, B \subseteq V$ be quasicoherent \mathcal{O}_S -submodules such that $V/A, V/B$ are flat.

We say that A and B are commensurable, $A \sim B$, if $V/A + B$ and $V/A \cap B$ are flat and $A + B/A \cap B$ is locally free of finite type (l.f.f.t.).

Observe that the family $\{A \subseteq V \mid A \sim V^+\}$ canonically induces a topology on V and also topologies in submodules and quotients of V ; these topologies will be called V^+ -topologies.

Furthermore, it will be assumed that the following three conditions hold

- $V = \bigcup_{A \sim V^+} A$;
- separateness $(0) = \bigcap_{A \sim V^+} A$;
- completeness $V = \varprojlim_{A \sim V^+} V/A$.

Observe that $V/A = \hat{V}/\hat{A} = \varprojlim_{B \sim A} V/(A + B)$.

Let $T \rightarrow S$ be an S -scheme and W a module endowed with the V^+ -topology. Then we write

$$\hat{W}_T := W \hat{\otimes}_{\mathcal{O}_S} \mathcal{O}_T$$

where $\hat{\otimes}$ denotes the completion w.r.t. the V^+ -topology. Note that for a morphism $R \rightarrow T$ of S -schemes one has that $(\hat{W}_T)_R = \hat{W}_R$.

Given (V, V^+) and an S -scheme, T , we may construct a new pair satisfying the above requirements; namely, (\hat{V}_T, \hat{V}_T^+) .

Lemma 2.2

Let $B \subseteq A \subseteq V$ be \mathcal{O}_S -submodules such that A is quasicoherent and V/A is flat. If A/B is l.f.f.t., then B is quasicoherent, V/B is flat and $A \sim B$.

Proof. From the hypothesis, we know that A/B and V/A are quasicoherent and flat. Therefore, the exactness of the sequence

$$0 \rightarrow A/B \rightarrow V/B \rightarrow V/A \rightarrow 0$$

implies that V/B is quasicoherent and flat. Finally, B is also quasicoherent since it is the kernel of $V \rightarrow V/B$, which is a morphism of quasicoherent sheaves. \square

Theorem 2.3

Commensurableness is an equivalence relation and it is preserved under base change.

Proof. To say that it is preserved under base change means that $A \sim B$ implies $\hat{A}_T \sim \hat{B}_T$ for every morphism $T \rightarrow S$ and every A, B , as in Definition 2.1. This is straightforward.

The first part of the statement will follow easily if we can prove that

$$A \sim B \quad \implies \quad A \sim A \cap B \text{ and } A \sim A + B.$$

From the hypothesis we know that $A + B$ is quasicohherent and that $V/A + B$ is flat. By Lemma 2.2 it suffices to show that $A + B/A$ is l.f.f.t. . Observe that the exact sequence

$$0 \rightarrow A/A \cap B \rightarrow A + B/A \cap B \rightarrow A + B/A \rightarrow 0$$

shows that $A + B/A$ is of finite type. Recall that $V, V/A, V/B, V/A + B, V/A \cap B$ are flat and that, therefore, $A, B, A + B$ and $A \cap B$ are flat, and hence so are the three terms in the sequence. Using $A + B/A \simeq B/A \cap B \subseteq A + B/A \cap B$, flatness and Nakayama's Lemma one has that $A + B/A$ is l.f.f.t. .

Similarly, one proves that $A/A \cap B \simeq A + B/B$ is locally free of finite type. \square

Remark 1. Let A, B be as in Definition 2.1 and $A \sim B$. Note that the subset of S given by $\{s \in S \mid A_{k(s)} \subseteq B_{k(s)}\}$ coincides with the set of points where $A + B/B$ has rank 0. Since the latter is locally free, that subset consists of certain connected components of S .

Lemma 2.4

Let (V, V^+) be as in the beginning of this section and let T be an S -scheme.

Thus, the linear topology given by the submodules $\{A_T \mid A \sim V^+\}$ coincides with the topology given by $\{\mathcal{A} \subset \hat{V}_T \mid \mathcal{A} \sim \hat{V}_T^+\}$.

In particular, if $W \rightarrow W'$ is a morphism of \mathcal{O}_S -modules continuous w.r.t. the V^+ -topology, then the induced morphism $\hat{W}_T \rightarrow \hat{W}'_T$ is also continuous w.r.t. the \hat{V}_T^+ -topology.

Proof. Let \mathcal{A} be a submodule of \hat{V}_T such that $\mathcal{A} \sim \hat{V}_T^+$ and let $t \in T$ be a closed point. It suffices to show that there exists an open neighborhood R of t in T and submodules $B, C \subset V$ commensurable with V^+ such that $B|_R \subseteq \mathcal{A}|_R \subseteq C|_R$. Indeed, since $V = \bigcup_{C \sim V^+} C$ it follows that $V_{k(t)} = \bigcup_{C \sim V^+} C_{k(t)}$. Consider $C \sim V^+$ such that $\mathcal{A}_{k(t)} \subset C_{k(t)}$. The previous proposition shows that this inclusion also holds in a neighborhood R of t in T . The existence of B is proved similarly using the fact that $(0) = \bigcap_{B \sim V^+} B$. \square

2.B Grassmannian

Let T be an S -scheme, $L \subset \hat{V}_T$ a quasicohherent \mathcal{O}_T -submodule, and $A \sim V^+$. We say that (L, A) satisfies the condition $(*)$ if

$$(*) \quad \hat{V}_T/L + \hat{A}_T = (0) \quad \text{and} \quad L \cap \hat{A}_T \text{ is l.f.f.t. .}$$

Observe that if (L, A) satisfies $(*)$, then $L \cap A_T = L \cap \hat{A}_T$ and $L = \hat{L}$.

Lemma 2.5

Let $A, B \subseteq V$ be quasicoherent \mathcal{O}_S -submodules such that $V/A, V/B$ are flat. Let T be an S -scheme and $L \subseteq \hat{V}_T$ a quasicoherent \mathcal{O}_T -submodule such that \hat{V}_T/L is flat. Assume that (L, A) fulfills $(*)$. It then holds that

- (1) $(L \cap \hat{A}_T)_{\hat{k}(t)} = L_{k(t)} \cap \hat{A}_{k(t)}$ for all $t \in T$;
- (2) if $A \subset B$, then (L, B) satisfies $(*)$;
- (3) if $\hat{V}_T/L + \hat{B}_T = 0$, then (L, B) satisfies $(*)$;
- (4) $\hat{V}_T/L + \hat{B}_T$ is an \mathcal{O}_T -module of finite presentation;
- (5) if $L \cap \hat{B}_T = 0$, then $\hat{V}_T/L + \hat{B}_T$ is l.f.f.t. .

Proof. Before proving the statements, let us note the following. Let $C \subseteq D \subseteq V$ be arbitrary \mathcal{O}_S -submodules; then, Snake's Lemma yields the following exact sequence

$$\begin{aligned} 0 \rightarrow L \cap \hat{C}_T \rightarrow L \cap \hat{D}_T \rightarrow (D/C)_T \rightarrow \\ \rightarrow \hat{V}_T/L + \hat{C}_T \rightarrow \hat{V}_T/L + \hat{D}_T \rightarrow 0. \end{aligned} \quad (2.6)$$

Let us proceed with the proofs.

- (1) This follows from the flatness of \hat{V}_T/L and the sequence

$$0 \rightarrow L \cap \hat{A}_T \rightarrow \hat{A}_T \rightarrow \hat{V}_T/L \rightarrow 0.$$

- (2) Sequence (2.6) implies that $\hat{V}_T/L + \hat{B}_T = (0)$ and the exactness of

$$0 \rightarrow L \cap \hat{A}_T \rightarrow L \cap \hat{B}_T \rightarrow (B/A)_T \rightarrow 0.$$

Bearing in mind that $L \cap \hat{A}_T$ and $(B/A)_T$ are l.f.f.t., the claim follows.

- (3) Point (2) implies that $(L, A + B)$ satisfies $(*)$. The sequence (2.6) for B and $A + B$ reads

$$0 \rightarrow L \cap \hat{B}_T \rightarrow L \cap (A + B)_T \rightarrow (A + B/B)_T \rightarrow 0.$$

Since the middle and the right hand side terms are l.f.f.t. one concludes.

- (4) The sequence (2.6) for B and $A + B$ gives

$$L \cap (A + B)_T \rightarrow (A + B/B)_T \rightarrow \hat{V}_T/L + \hat{B}_T \rightarrow 0$$

where the middle and the left hand side terms are l.f.f.t. .

- (5) The sequence (2.6) for B and $A + B$ is

$$0 \rightarrow L \cap (A + B)_T \rightarrow (A + B/B)_T \rightarrow \hat{V}_T/L + \hat{B}_T \rightarrow 0$$

where the middle and the left hand side terms are l.f.f.t. Further, the second part of this Lemma shows that $(L, A + B)$ satisfies $(*)$ and, using the first part, one sees that

$$(L \cap (A + B)_T)_{\hat{k}(t)} = \hat{L}_{k(t)} \cap (A + B)_{k(t)} \hookrightarrow (A + B/B)_{k(t)} = (A_{k(t)} + B_{k(t)})/B_{k(t)}$$

or, what amounts to the same, $\hat{V}_T/L + \hat{B}_T$ is flat. These two facts allows one to prove that $\hat{V}_T/L + \hat{B}_T$ is l.f.f.t. . \square

Lemma 2.7

Let $A \subseteq V$ be a quasicoherent \mathcal{O}_S -submodule such that V/A is flat and $A \sim V^+$. Let T be an S -scheme and $L \subseteq \hat{V}_T$ a quasicoherent \mathcal{O}_T -submodule such that \hat{V}_T/L is flat. Assume that (L, A) satisfies $(*)$.

Thus, for every $t \in T$ there exists a neighborhood $R \subseteq T$ such that

- there exists $B \subset V$ such that $L_R \cap \hat{B}_R = 0$ and $B \sim A$ is quasicoherent and V/B flat;
- there exists $C \subset V$ such that $L_R \oplus \hat{C}_R = V_R$ and $C \sim A$ is quasicoherent and V/C flat;

Proof. Let us see the first part. For any $B \subset A$ such that A/B is l.f.f.t. one has the exact sequence

$$0 \rightarrow L \cap \hat{B}_T \rightarrow L \cap \hat{A}_T \rightarrow (A/B)_T$$

where the middle and right hand side terms are l.f.f.t. . Accordingly, let R be a neighborhood of t such that $(L \cap \hat{A}_T)_R = L_R \cap \hat{A}_R$ is a free \mathcal{O}_R -module of finite rank. Comparing the above sequence for $B' \subset B$ with A/B' , A/B l.f.f.t. and recalling that $\bigcap_{B \sim V^+} B = (0)$, the result follows.

Let us prove the second claim. Let B and R be as in the first part. We therefore obtain the following exact sequence

$$0 \rightarrow L_R \cap \hat{A}_R \rightarrow (A/B)_R \rightarrow \hat{V}_R/L_R + \hat{B}_R \rightarrow 0$$

where all terms are l.f.f.t. \mathcal{O}_R -modules. It follows that $L_R \cap \hat{A}_R$ is an R -valued point of the Grassmannian of A/B . Thus, shrinking R if necessary, there exists a l.f.f.t. \mathcal{O}_S -module, $\bar{C} \subseteq A/B$, such that $(A/B)/\bar{C}$ is l.f.f.t. and

$$(L_R \cap \hat{A}_R) \oplus \bar{C} = (A/B)_R. \quad (2.8)$$

Let us check that $C := \pi^{-1}(\bar{C}) \subseteq V$, where $\pi : A \rightarrow A/B$, satisfies the requirements of the statement. Indeed, equation (2.8) shows that the dashed arrow in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & L_R \cap \hat{C}_R & \rightarrow & L_R \cap \hat{B}_R = (0) & \rightarrow & C_R/B_R & \rightarrow & \hat{V}_R/L_R + \hat{B}_R & \rightarrow & \hat{V}_R/L_R + \hat{C}_R & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & \searrow & \parallel & & \downarrow & & \\ & & 0 & \longrightarrow & L_R \cap \hat{A}_R & \longrightarrow & A_R/B_R & \rightarrow & \hat{V}_R/L_R + \hat{B}_R & \longrightarrow & 0 & & \end{array}$$

is an isomorphism. Hence, the kernel and cokernel of the dashed arrow vanish; i.e., $L_R \cap \hat{C}_R = (0)$ and $\hat{V}_R/L_R + \hat{C}_R = (0)$.

Finally, since $A/C \simeq (A/B)/\bar{C}$ is l.f.f.t., we conclude by Lemma 2.2. \square

Lemma 2.9

Let T be an S -scheme and $L \subseteq \hat{V}_T$ be a quasicoherent \mathcal{O}_T -submodule such that \hat{V}_T/L is flat.

If there exists $B \sim V^+$ with V/B flat such that $L \cap \hat{B}_T = (0)$ and $\hat{V}_T/L + \hat{B}_T$ is l.f.f.t., then for every point t there exists a neighborhood R and a submodule $A \subseteq B$ such that $A \sim V^+$, V/A is flat and (L_R, \hat{A}_R) satisfies $(*)$.

Proof. Let us consider a submodule $A \subset V$ such that $A \sim V^+$, V/A flat and $B \subseteq A$. Let us consider the sequence

$$A/B \rightarrow V/B \rightarrow V/L + B.$$

Since $\cup_{A \sim V^+} = V$, there exists a neighborhood R of t and A such that the above composition is surjective on R . We therefore have

$$0 \rightarrow L_R \cap \hat{A}_R \rightarrow (A/B)_R \rightarrow \hat{V}_R/L_R + \hat{B}_R \rightarrow 0$$

and, hence, $L_R \cap \hat{A}_R$ is l.f.f.t. . □

Theorem 2.10

Let $V^+ \subset V$ be sheaves of \mathcal{O}_S -modules as in the beginning of this section. Accordingly, the functor from the category of S -schemes to the category of sets defined by

$$\mathrm{Gr}(V)(T) := \left\{ \begin{array}{l} \text{quasicoherent } \mathcal{O}_T\text{-submodules } L \subset \hat{V}_T \text{ s.t. } \hat{V}_T/L \text{ is flat and} \\ \text{s.t. for each point of } T \text{ there exists a neighborhood } R \text{ and} \\ \text{an } A \sim V^+ \text{ with } \hat{V}_R/(L_R + \hat{A}_R) = 0 \text{ and } L_R \cap \hat{A}_R \text{ is l.f.f.t.} \end{array} \right\}$$

is representable by an S -scheme that will be denoted by $\mathrm{Gr}(V)$ and will be called the Grassmannian of V (if V^+ is understood).

Proof. From the very definition of the functor, one deduces that it is a sheaf and that, if can therefore be assumed that S is an affine scheme. Let us consider the functors on the category of S -schemes

$$F_A(T) := \left\{ \begin{array}{l} \text{quasicoherent } \mathcal{O}_T\text{-submodules} \\ L \subset \hat{V}_T \text{ such that } L \oplus \hat{A}_T = \hat{V}_T \end{array} \right\}.$$

We shall prove that F_A is representable for all $A \sim V^+$ and that $\{F_A\}_{A \sim V^+}$ is a covering of $\mathrm{Gr}(V)$ by open subfunctors.

Consider $B \sim V^+$ such that $B \subset A$. For each $C \sim V^+$ with $A \subset C$ the functor on the category of S -schemes defined by

$$T \rightsquigarrow F_{C/A/B}(T) := \left\{ \begin{array}{l} \text{quasicoherent } \mathcal{O}_T\text{-modules } L \\ \text{such that } L \oplus (A/B)_T = (C/B)_T \end{array} \right\}$$

is representable by an S -scheme affine over S (which is an open subscheme of the Grassmannian of C/B). Observe that, as C varies, these affine S -schemes form an inverse system whose inverse limit represents the functor

$$T \rightsquigarrow F_{A/B} := \left\{ \begin{array}{l} \text{quasicoherent } \mathcal{O}_T\text{-modules } L \\ \text{such that } L \oplus (A/B)_T = (V/B)_T \end{array} \right\} = \varprojlim_{C \supseteq A} F_{C/A/B}$$

in the category of S -schemes. Let us denote this S -scheme by $F_{A/B}$ and let us observe that the map $F_{A/B} \rightarrow S$ is an affine morphism. Note that as B varies the schemes $\{F_{A/B}\}$ are an inverse system, and since the maps to S are affine, there exists its inverse limit in the category of S -schemes. Finally, it is clear that

$$F_A = \varprojlim_{B \subseteq A} F_{A/B}. \quad (2.11)$$

For the sake of clarity, note that there exists a correspondence between $\text{Hom}(L, \hat{A})$ and F_A (L being an S -valued point of F_A), which assigns its graph to a map.

Secondly, Lemma 2.7 shows that the schemes $\{F_A\}_{A \sim V^+}$ is a covering of $\text{Gr}(V)$.

It remains to show that for a S -scheme T and a functor homomorphism $T \rightarrow \text{Gr}(V)$ the map $F_A \times_{\text{Gr}(V)} T \rightarrow T$ is representable by an open subscheme of T or, what amounts to the same, that the set of points $t \in T$ s.t. $L_{k(t)}$ belongs to $F_A(k(t))$ is open in T (here L is the module corresponding to the map $T \rightarrow \text{Gr}(V)$).

Let t be such that $L_{k(t)} \in F_A(k(t))$. Then, $\hat{V}_{k(t)}/(L_{k(t)} + \hat{A}_{k(t)}) = (0)$. From Lemma 2.5, part 4, we know that $\hat{V}_T/L + \hat{A}_T$ is of finite presentation and it therefore follows that $(\hat{V}_T/L + \hat{A}_T)_t = (0)$. Let R_0 be the open set consisting of those t such that $(\hat{V}_T/L + \hat{A}_T)_t = (0)$. Thus, in R_0 the module $(L \cap A)_{R_0}$ is l.f.f.t. and $L_t \cap \hat{A}_t = (L \cap \hat{A}_T)_t$ for all $t \in R_0$. Therefore, the subset of those points $t \in R_0$ such that $L_t \cap \hat{A}_t = (0)$ is the desired open subscheme. \square

2.C First properties

Theorem 2.12

Let T be an S -scheme. It then holds that $\text{Gr}(\hat{V}_T) = \text{Gr}(V) \times_S T$. In particular, for a closed point $s \in S$ one has that $\text{Gr}(V)_{k(s)} = \text{Gr}(k(s)((t)))$.

Proof. It is obvious that the canonical map $\text{Gr}(V) \times_S T \hookrightarrow \text{Gr}(\hat{V}_T)$ is an open immersion. It therefore suffices to show that $\text{Gr}(\hat{V}_{k(t)}) = \text{Gr}(V) \times_S \text{Spec } k(t)$ for a closed point $t \in T$.

Let L be a point in $\text{Gr}(\hat{V}_{k(t)})$ and $\mathcal{A} \subset \hat{V}_{k(t)}$ with $\mathcal{A} \sim \hat{V}_{k(t)}^+$ and $L \oplus \mathcal{A} = \hat{V}_{k(t)}$. Since $\mathcal{A} + \hat{V}_{k(t)}^+/\hat{V}_{k(t)}^+$ is l.f.f.t. and $\cup_{A \supseteq V^+} A = V$, there exists $A \sim V^+$ such that $\mathcal{A} \subseteq A_{k(t)}$. Now, the second item of Lemma 2.5 for \mathcal{A} and $A_{k(t)}$ implies that L is a $k(t)$ -valued point of $\text{Gr}(V) \times_S \text{Spec } k(t)$. \square

Proposition 2.13

The morphism $\text{Gr}(V) \rightarrow S$ is separated.

Proof. We follow [7, I.§5.5]. We must show that the diagonal map $\text{Gr}(V) \rightarrow \text{Gr}(V) \times_S \text{Gr}(V)$ is a closed immersion. This condition is local, so it suffices to consider $A, B \sim V^+$ and an open subset $T \subset S$ such that $(A/A \cap B)_T, (B/A \cap B)_T$ are free of finite rank, and to show that

$$\mathcal{O}_{F_A} \otimes_{\mathcal{O}_T} \mathcal{O}_{F_B} \longrightarrow \mathcal{O}_{F_A \times_{\text{Gr}(V)} F_B} \quad (2.14)$$

is surjective.

Let us describe $\mathcal{O}_{F_A \times_{\text{Gr}(V)} F_B}$ in an explicit way. Let $L \in \text{Gr}(V)$ be the universal submodule. Thus, the canonical map $L \oplus \hat{A}_{\text{Gr}(V)} \rightarrow \hat{V}_{\text{Gr}(V)}$ induces a map

$$(A/A \cap B)_{\text{Gr}(V)} \xrightarrow{\delta_{AB}} \hat{V}_{\text{Gr}(V)}/L + (A \cap B)_{\text{Gr}(V)}$$

that, when restricted to F_A , is an isomorphism between l.f.f.t. \mathcal{O}_{F_A} -modules of the same rank (since $A \sim B$ and part 5 of Lemma 2.5).

Accordingly, on F_A we consider the following composition of morphisms

$$(B/A \cap B)_{F_A} \xrightarrow{\delta_{BA}} \hat{V}_{F_A}/L_{F_A} + (A \cap B)_{F_A} \xrightarrow{\delta_{AB}^{-1}} (A/A \cap B)_{F_A}$$

where all terms are l.f.f.t. \mathcal{O}_{F_A} -modules. Clearly, it holds that the points of $F_A \times_{\text{Gr}(V)} F_B$ are those points of F_A where $\delta_{AB}^{-1} \circ \delta_{BA}$ is an isomorphism. Hence, $\mathcal{O}_{F_A \times_{\text{Gr}(V)} F_B}$ is the localization of \mathcal{O}_{F_A} by $\det(\delta_{BA})$ or, what is tantamount, the localization of \mathcal{O}_{F_B} by $\det(\delta_{AB})$. And the map (2.14) is surjective. \square

Theorem 2.15

Let S be connected and let $\text{Gr}^0(V)$ denote a connected component of $\text{Gr}(V)$. Let $\pi: \text{Gr}^0(V) \rightarrow S$.

It therefore holds that

- (1) the canonical morphism $\mathcal{O}_S \rightarrow \pi_* \mathcal{O}_{\text{Gr}^0(V)}$ is an isomorphism;
- (2) $H^0(\text{Gr}^0(V), \mathcal{O}_{\text{Gr}^0(V)}) = H^0(S, \mathcal{O}_S)$;
- (3) $H^0(\text{Gr}^0(V), \mathcal{O}_{\text{Gr}^0(V)}^*) = H^0(S, \mathcal{O}_S^*)$.

Proof. Recall that $\pi_* \mathcal{O}_{\text{Gr}^0(V)}$ is the sheaf of \mathcal{O}_S -modules associated to the presheaf

$$T \rightsquigarrow H^0(\pi^{-1}(T), \mathcal{O}_{\text{Gr}^0(V)}) = H^0(\text{Gr}^0(V_T), \mathcal{O}_{\text{Gr}^0(V_T)})$$

where $T \subseteq S$ is an open subscheme. Therefore, if we show that

$$H^0(\text{Gr}^0(V), \mathcal{O}_{\text{Gr}^0(V)}) = H^0(S, \mathcal{O}_S)$$

holds for S affine, then (1) holds and therefore so does (2).

Let us assume that S is the spectrum of a local ring, $S = \text{Spec } \mathcal{O}_S$. Let $A \sim V^+$ be a submodule such that $F_A \subset \text{Gr}^0(V)$. Let $L \in \text{Gr}^0(V)(S)$ be a point of the Grassmannian such that $L \oplus \hat{A} = \hat{V}$; that is, $L \in F_A(S)$. Let us fix elements $\omega \in \text{Hom}_{\mathcal{O}_S}(L, \mathcal{O}_S)$ and $a \in A$.

Now, the pair (ω, a) defines an affine line inside F_A as the image of

$$\begin{aligned} j: \mathbb{A}_S^1 &\hookrightarrow \text{Hom}_{\mathcal{O}_S}(L, A) \simeq F_A \\ \lambda &\mapsto (l \mapsto \lambda \omega(l)a). \end{aligned}$$

Indeed, this procedure allows us to construct a projective line inside the Grassmannian that, when intersected with F_A , coincides with the previous construction. Indeed, bearing in mind that $(L + \langle a \rangle) / \ker \omega$ has rank 2, we have

$$\begin{aligned} \bar{j}: \mathbb{P}_S^1 &= \mathbb{P}((L + \langle a \rangle) / \ker \omega) \hookrightarrow \text{Gr}^0(V) \\ \bar{L} &\mapsto p^{-1}(\bar{L}) \end{aligned}$$

where $p: (L + \langle a \rangle) \rightarrow (L + \langle a \rangle) / \ker \omega$. Note that $\ker \omega \in \text{Gr}(V)$.

The restriction homomorphisms yield a commutative diagram

$$\begin{array}{ccc} H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}) & \longrightarrow & H^0(F_A, \mathcal{O}_{F_A}) \simeq \mathcal{O}_S[\{x_i\}] \\ \downarrow \bar{j}^* & & \downarrow j^* \\ H^0(\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}) = \mathcal{O}_S & \longrightarrow & H^0(\mathbb{A}_S^1, \mathcal{O}_{\mathbb{A}_S^1}) \simeq \mathcal{O}_S[y] \end{array}$$

($\{x_i\}$ being a family of indeterminates).

Let a global section $s \in H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)})$ be given. It then holds that $(\bar{j}^*(s))|_{\mathbb{A}_S^1} = j^*(s|_{F_A})$ and, in particular, $j^*(s|_{F_A}) \in \mathcal{O}_S$; i.e., it does not depend on y .

From expression (2.11) and the very definition of j , one sees that for each indeterminate x_i of the family $\{x_i\}$ there exists a pair ω, a such that $j^*(x_i)$ depends on y . We conclude that $s|_{F_A}$ cannot depend on x_i for all i ; that is, $s|_{F_A} \in \mathcal{O}_S$.

Since the above argument holds for all $A \sim V^+$ (with $F_A \subset \mathrm{Gr}^0(V)$) and since $\{F_A\}$ cover $\mathrm{Gr}^0(V)$, it follows that $s \in \mathcal{O}_S$. And the claim is proved.

For the third statement it suffices to show that

$$H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}^*) = H^0(S, \mathcal{O}_S^*)$$

for any affine scheme S . This can be shown by applying similar arguments to those provided above to the diagram

$$\begin{array}{ccc} H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}^*) & \longrightarrow & H^0(F_A, \mathcal{O}_{F_A}^*) \simeq (\mathcal{O}_S[\{x_i\}])^* \\ \downarrow \bar{j}^* & & \downarrow j^* \\ H^0(\mathbb{P}_S^1, \mathcal{O}_{\mathbb{P}_S^1}^*) = \mathcal{O}_S^* & \longrightarrow & H^0(\mathbb{A}_S^1, \mathcal{O}_{\mathbb{A}_S^1}^*) \simeq (\mathcal{O}_S[y])^* . \end{array} \quad \square$$

3. The determinant and central extensions

3.A The determinant

Theorem 3.1

Let T be an S -scheme and let $L \in \mathrm{Gr}(V)(T)$ be a T -valued point of $\mathrm{Gr}(V)$. Let $A \subseteq \hat{V}_T$ be a quasicoherent submodule such that \hat{V}_T/A is flat and $A \sim \hat{V}_T^+$.

Thus, the complex of \mathcal{O}_T -modules

$$\mathcal{C}_A^\bullet(L) \equiv \cdots \rightarrow 0 \rightarrow L \oplus A \xrightarrow{\delta_A} \hat{V}_T \rightarrow 0 \rightarrow \cdots$$

(δ_A being the addition map) is a perfect complex ([13]).

Proof. The claim is local in T so it suffices to show that given t there exists a neighborhood R of t in T such that $\mathcal{C}_A^\bullet(L)|_R = \mathcal{C}_{\hat{A}_R}^\bullet(L_R)$ is a perfect complex. Given t , take R such that $\hat{A}_R + \hat{V}_R^+/\hat{V}_R^+$ and $\hat{A}_R + \hat{V}_R^+/\hat{A}_R$ are free of finite rank. Observe that there

is a quasiisomorphism of complexes (written vertically)

$$\begin{array}{ccc} L_R \oplus \hat{A}_R & \xrightarrow{p} & L_R \\ \downarrow & & \downarrow \\ \hat{V}_R & \longrightarrow & \hat{V}_R / \hat{A}_R \end{array}$$

where p denotes the projection.

Let us now, consider a quasicoherent submodule $B \subset V$ with V/B flat such that $B \sim V^+$ and $\hat{A}_R \subseteq \hat{B}_R$ (its existence follows from Lemma 2.4). Shrinking R if necessary, we assume that $\hat{V}_R / (L_R + \hat{B}_R) = (0)$ and that $L_R \cap \hat{B}_R$ is l.f.f.t. . Therefore, the exactness of the sequence (2.6) means that the morphism of complexes

$$\begin{array}{ccc} L_R \cap \hat{B}_R & \longrightarrow & L_R \\ \downarrow & & \downarrow \\ \hat{B}_R / \hat{A}_R & \longrightarrow & \hat{V}_R / \hat{A}_R \end{array}$$

is a quasiisomorphism. Since \sim is an equivalence relation that is compatible with pullbacks, we have that $\hat{B}_R \sim \hat{A}_R$, and the result follows. \square

Recalling the theory of determinant of complexes of [13], we have the following

Corollary 3.2

- The determinant of the complex $\mathcal{C}_A^\bullet(L)$ changes base; that is, for a morphism of S -schemes $f: R \rightarrow T$ one has a canonical isomorphism

$$f^* \text{Det}(\mathcal{C}_A^\bullet(L)) \xrightarrow{\sim} \text{Det}(\mathcal{C}_{A_R}^\bullet(f^*L)).$$

- For $A, B \sim V_T^+$, there is a canonical isomorphism

$$\text{Det}(\mathcal{C}_A^\bullet(L)) \otimes_{\mathcal{O}_T} \wedge(A/A \cap B)^* \xrightarrow{\sim} \text{Det}(\mathcal{C}_B^\bullet(L)) \otimes_{\mathcal{O}_T} \wedge(B/A \cap B)^*.$$

In particular, $\text{Det}(\mathcal{C}_A^\bullet(L))$ defines a class on the relative Picard group that does not depend on the choice of A .

From the second item of the previous corollary, we give the following

DEFINITION 3.3 The determinant bundle on $\text{Gr}(V)$ is the class of $\text{Det}(\mathcal{C}_A^\bullet(\mathcal{L}))$ in the relative Picard group; that is, up to pullbacks of line bundles on S . A canonical representative of this class is $\text{Det}(\mathcal{C}_{V^+}^\bullet(\mathcal{L}))$, which, abusing notation, will also be called the determinant bundle and will be denoted by Det_V .

Remark 2. Let L be a T -valued point of $\text{Gr}(V)$. The Euler-Poincaré characteristic of the complex $\mathcal{C}_A^\bullet(L)$

$$\begin{array}{ccc} T & \longrightarrow & \mathbb{Z} \\ t & \longmapsto & \chi(\mathcal{C}_A^\bullet(L) \otimes k(t)) \end{array}$$

which, for closed points is given by $\dim_{k(t)} L_{k(t)} \cap A_{k(t)} - \dim V_{k(t)} / (L_{k(t)} + A_{k(t)})$, is a locally constant function. Therefore, we have a map

$$\begin{aligned} \mathrm{Gr}(V) &\longrightarrow H^0(S, \mathbb{Z}) \\ L &\longmapsto \chi(\mathcal{C}_A^\bullet(L)) \end{aligned}$$

and $H^0(S, \mathbb{Z})$ can be used to label the connected components of $\mathrm{Gr}(V)$.

DEFINITION 3.4 Let T be a S -scheme and $g : \mathrm{Gr}(V)_T \rightarrow \mathrm{Gr}(V)_T$ a homomorphism of T -schemes. We say that g preserves the determinant if there exists a line bundle on T , which will be denoted by $\mathcal{O}_T(g)$, such that

$$g^* p_1^* \mathrm{Det}_V \simeq p_1^* \mathrm{Det}_V \otimes p_2^* \mathcal{O}_T(g) \quad (3.5)$$

on $\mathrm{Gr}(V)_T := \mathrm{Gr}(V) \times_S T$, where \mathcal{L} is the universal submodule and p_i is the projection of $\mathrm{Gr}(V) \times_S T$ onto its i -th factor.

Theorem 3.6

Let T be an S -scheme and let $g : \hat{V}_T \rightarrow \hat{V}_T$ be a homomorphism of \mathcal{O}_T -modules such that

- g is injective;
- $\hat{V}_T/g(\hat{V}_T^+)$ is flat and $\hat{V}_T^+ \sim g(\hat{V}_T^+)$;
- $\hat{V}_T/g(\hat{V}_T)$ is flat and $\hat{V}_T \sim g(\hat{V}_T)$.

Then, g induces a homomorphism of T -schemes

$$g : \mathrm{Gr}(V)_T \rightarrow \mathrm{Gr}(V)_T$$

which sends L to $g(L)$ and preserves the determinant bundle.

Proof. Let L be a point of $\mathrm{Gr}(V)_T$. We must check that $g(L)$ also belongs to $\mathrm{Gr}(V)_T$. Observe that the left and right hand side terms of the exact sequence

$$0 \rightarrow \hat{V}_T/L \xrightarrow{g} \hat{V}_T/g(L) \rightarrow \hat{V}_T/g(\hat{V}_T) \rightarrow 0$$

are quasicohherent flat \mathcal{O}_T -modules, the one in the middle also does.

On the other hand, since the condition $L \in \mathrm{Gr}(V)(T)$ is local in T , we may assume that there exists $A \sim V^+$ such that $L \oplus \hat{A}_T = \hat{V}_T$. Then, $g(L) \oplus g(\hat{A}_T) = g(\hat{V}_T)$ and, therefore, $g(L) \cap g(\hat{A}_T) = (0)$ and

$$\hat{V}_T/g(L) + g(\hat{A}_T) = \hat{V}_T/g(\hat{V}_T)$$

is l.f.f.t. since $\hat{V}_T \sim g(\hat{V}_T)$. Recalling that $g(\hat{A}_T) \sim g(\hat{V}_T^+) \sim \hat{V}_T^+$ and Lemma 2.9, we conclude that $g(L) \in \mathrm{Gr}(V)_T$.

Let us now check that it preserves the determinant bundle. Let us now consider the following two exact sequences of complexes (written vertically)

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathcal{L}_T \oplus \hat{V}_T^+ & \xrightarrow{g} & g(\mathcal{L}_T) \oplus (\hat{V}_T^+ + g(\hat{V}_T^+)) & \rightarrow & (\hat{V}_T^+ + g(\hat{V}_T^+))/g(\hat{V}_T^+) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \hat{V}_T & \xrightarrow{g} & \hat{V}_T & \rightarrow & \hat{V}_T/g(\hat{V}_T) \rightarrow 0
 \end{array}$$

and

$$\begin{array}{ccccccc}
 0 & \rightarrow & g(\mathcal{L}_T) \oplus \hat{V}_T^+ & \rightarrow & g(\mathcal{L}_T) \oplus (\hat{V}_T^+ + g(\hat{V}_T^+)) & \rightarrow & (\hat{V}_T^+ + g(\hat{V}_T^+))/\hat{V}_T^+ \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \hat{V}_T & \rightarrow & \hat{V}_T & \rightarrow & 0 \rightarrow 0.
 \end{array}$$

Observe the following. The determinant of the complex on the l.h.s. of the first sequence is the determinant bundle, Det_V . The complex on the r.h.s. of the first sequence is perfect since $\hat{V}_T^+ \sim g(\hat{V}_T^+)$ and $\hat{V}_T \sim g(\hat{V}_T)$, and its determinant is the pullback of a line bundle on T by the projection $\text{Gr}(V)_T \rightarrow T$.

The complex in the middle of both sequences is perfect by Theorem 3.1. The determinant of the complex on the l.h.s. of the second sequence is $g^* \text{Det}_V$, and the determinant of the complex on the r.h.s. of the second sequence is the pullback of a line bundle on T .

Now, the properties of the determinants give a canonical isomorphism

$$g^* p_1^* \text{Det}_V \simeq p_1^* \text{Det}_V \otimes \wedge(\hat{V}_T^+ + g(\hat{V}_T^+))/\hat{V}_T^+ \otimes \wedge(\hat{V}_T^+ + g(\hat{V}_T^+))/g(\hat{V}_T^+) \otimes \wedge(\hat{V}_T/g(\hat{V}_T))$$

(where \wedge denotes the exterior algebra of highest degree) and the result follows. \square

Remark 3. Let T be an S -scheme and let $g : \hat{V}_T \rightarrow \hat{V}_T$ be an injective homomorphism of \mathcal{O}_T -modules. Therefore, the set of points where the second and third conditions are satisfied is an open subscheme.

3.B The extension

DEFINITION 3.7 Let $\text{Gr}(V) \rightarrow S$ be the Grassmannian of V . For each S -scheme T , we define the sets

$$\mathcal{Q}_S(T) := \left\{ \begin{array}{l} \mathcal{O}_T\text{-module monomorphisms } \hat{V}_T \rightarrow \hat{V}_T \\ \text{such that the conditions of Theorem 3.6 hold} \end{array} \right\}$$

and

$$\tilde{\mathcal{Q}}_S(T) := \left\{ \begin{array}{l} \text{pairs } (\tilde{g}, g) \text{ where } g \in \mathcal{Q}(T) \\ \text{and } \tilde{g} \in \text{Aut}_{\mathcal{O}_T} \mathcal{O}_T(g) \end{array} \right\}$$

where

$$\mathcal{O}_T(g) := \wedge(\hat{V}_T^+ + g(\hat{V}_T^+))/\hat{V}_T^+ \otimes \wedge(\hat{V}_T^+ + g(\hat{V}_T^+))/g(\hat{V}_T^+) \otimes \wedge(\hat{V}_T/g(\hat{V}_T)).$$

Theorem 3.8

With the above notation, it holds that

- (1) $\mathcal{Q}_S(T)$ is a monoid.
- (2) The assignment $T \rightsquigarrow \mathcal{Q}_S(T)$ is a contravariant functor from the category of S -schemes to the category of monoids,
- (3) The map $T \rightsquigarrow \mathcal{Q}_S(T)$, where T is an open subset of S , defines a sheaf on S , which will be denoted by \mathcal{Q}_S .

The same properties hold for $\tilde{\mathcal{Q}}_S$.

Proof. (1) Observe that the identity $\hat{V}_T \rightarrow \hat{V}_T$ belongs to $\mathcal{Q}_S(T)$ and that if $g, h \in \mathcal{Q}_S(T)$ satisfy the conditions of the Theorem, then so does $g \circ h$. The only non-trivial part is that $\hat{V}_T/(g \circ h)(\hat{V}_T)$ is flat. Note that the left and right hand side terms of the exact sequence

$$0 \rightarrow \hat{V}_T/h(\hat{V}_T) \xrightarrow{g} \hat{V}_T/(g \circ h)(\hat{V}_T) \rightarrow \hat{V}_T/g(\hat{V}_T) \rightarrow 0$$

are flat. So is the one in the middle.

Let us detail explicitly the composition law of $\tilde{\mathcal{Q}}_S$. Let (f, \tilde{f}) and (g, \tilde{g}) be two elements of $\tilde{\mathcal{Q}}_S(T)$. Since $\tilde{g} \in \text{Aut}_{\mathcal{O}_S} \mathcal{O}_T(g)$, it follows that $f^*(\tilde{g}) \in \text{Aut}_{\mathcal{O}_T} f^* \mathcal{O}_T(g)$. Note that

$$\begin{aligned} g^* \mathcal{O}_T(f) \otimes \mathcal{O}_T(g) &\simeq g^*(f^* p_1^* \text{Det}_V \otimes (p_1^* \text{Det}_V)^{-1}) \otimes g^* p_1^* \text{Det}_V \otimes (p_1^* \text{Det}_V)^{-1} \\ &\simeq (gf)^* p_1^* \text{Det}_V \otimes (p_1^* \text{Det}_V)^{-1} \end{aligned}$$

and therefore $g^*(\tilde{f}) \otimes \tilde{g} \in \text{Aut}_{\mathcal{O}_T}(\mathcal{O}_T(gf))$. Summing up, the composition law is

$$(g, \tilde{g})(f, \tilde{f}) := (gf, g^*(\tilde{f}) \otimes \tilde{g})$$

(2) Let T be an S -scheme and let g be an element in $\mathcal{Q}_S(T)$. If $R \rightarrow T$ is a morphism of S -schemes, then the morphism $g \otimes 1 : \hat{V}_T \hat{\otimes}_{\mathcal{O}_T} \mathcal{O}_R \rightarrow \hat{V}_T \hat{\otimes}_{\mathcal{O}_T} \mathcal{O}_R$ satisfies the conditions of Theorem 3.6. And the homomorphism of R -schemes induced by $g \otimes 1 : \text{Gr}(V)_R \rightarrow \text{Gr}(V)_R$ coincides with the one induced by $g : \text{Gr}(V)_T \rightarrow \text{Gr}(V)_T$ by $R \rightarrow T$. This means that there is a canonical map $\mathcal{Q}_S(T) \rightarrow \mathcal{Q}_S(R)$, which is an homomorphism of monoids.

(3) Straightforward. □

We shall now study the map $\tilde{\mathcal{Q}}_S \rightarrow \mathcal{Q}_S$ from the point of view of Leech's theory of *extensions of groups by monoids* (see [14]). We shall follow his notations and results. Recall that a *normal extension* of the group K by the monoid Q is a sequence $K \xrightarrow{i} \tilde{Q} \xrightarrow{\pi} Q$, where \tilde{Q} is a monoid, K is a subgroup of the group of units of \tilde{Q} such that $qK = Kq$ for all $q \in \tilde{Q}$, i is an injective morphism of monoids, and p is a surjective morphism of monoids such that it induces an isomorphism of the quotient monoid of left cosets \tilde{Q}/K with Q .

Theorem 3.9

For an S -scheme T , there is a normal extension of monoids

$$H^0(T, \mathcal{O}_T^*) \longrightarrow \tilde{\mathcal{Q}}_S(T) \xrightarrow{\pi_T} \mathcal{Q}_S(T) \quad (3.10)$$

Proof. By the very definition the map $\tilde{\mathcal{Q}}_T \rightarrow \mathcal{Q}_T$ is surjective morphism of monoids. The fiber of any element of \mathcal{Q}_T is isomorphic to $H^0(T, \mathcal{O}_T^*)$ by Theorems 3.6 and 2.15. \square

In particular, one could understand the above normal extension as a normal extension of the sheaf on monoids \mathcal{Q}_S by the sheaf on groups $\mathbb{G}_{m,S}$ (the relative multiplicative group).

Remark 4. The above definition of $\tilde{\mathcal{Q}}_S$ is closer to that employed in [1] than to that of [17]. However, let us see that essentially both coincide. Let us denote by $\bar{\mathcal{Q}}_S$ the extension of the latter reference, which consists of pairs (g, \bar{g}) where \bar{g} is an isomorphism $g^* p_1^* \text{Det}_V \simeq (p_1^* \text{Det}_V) \otimes \mathcal{O}_T(g)$. Let us fix any set-theoretic section of $\bar{\mathcal{Q}}_S \rightarrow \mathcal{Q}_S$, say σ . Accordingly, the map

$$\begin{aligned} \tilde{\mathcal{Q}}_S &\longrightarrow \bar{\mathcal{Q}}_S \\ (g, \bar{g}) &\longmapsto (g, (1 \otimes \bar{g}) \circ \sigma(g)) \end{aligned}$$

can be used to identify both central extensions as follows

$$\begin{array}{ccccccc} & & & \tilde{\mathcal{Q}}_S & & & \\ & & \nearrow & \downarrow \wr & \searrow & & \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & & \mathcal{Q}_S & \longrightarrow & 0 . \\ & & \searrow & \bar{\mathcal{Q}}_S & \nearrow & & \end{array}$$

Following Leech's paper, we know that the extension

$$H^0(T, \mathcal{O}_T^*) \longrightarrow \tilde{\mathcal{Q}}_T \xrightarrow{\pi_T} \mathcal{Q}_T$$

arises from two functors F_T (from the category given by the \mathcal{L} -quasiorder on Q to the category of groups) and G_T (from the category given by the \mathcal{R} -quasiorder on Q to the category of groups) and a factor system

$$\alpha_T : \mathcal{Q}_T \times \mathcal{Q}_T \longrightarrow \bigcup_{x \in \mathcal{Q}_T} F_T(x).$$

Therefore, the extension $\tilde{\mathcal{Q}}_T$ is the set $\bigcup_{x \in \mathcal{Q}_T} \{x\} \times F_T(x)$ endowed with the following composition law

$$(x, a) \cdot (y, b) := (xy, G_x^{xy}(a) \alpha_T(x, y) F_y^{xy}(b))$$

where G_x^{xy} denotes the image of the morphism $x \geq_{\mathcal{R}} xy$ by the functor G_T ; i.e. $G_x^{xy} : G_T(x) \rightarrow G_T(xy)$, and analogously for F_y^{xy} since $y \geq_{\mathcal{L}} xy$.

From the very construction of $\tilde{\mathcal{Q}}_T$ we deduce the following properties

- since $\tilde{\mathcal{Q}}_T$ is associative, it holds that $\alpha_T(1, x) = \alpha_T(x, 1) = 1_{F_T(1)}$.
- $F_T(x) = \pi^{-1}(x) = H^0(T, \mathcal{O}_T^*)$ (Theorems 3.6 and 2.15); that is F_T is the constant functor $x \rightsquigarrow H^0(T, \mathcal{O}_T^*)$, which is a commutative group (same for G_T).

In light of this latter observation, and by Theorem 2.13 of [14], we know that our extension *splits* (i.e. there exists a section of π_T that is a morphism of monoids) if and only if there exists $\phi \in \prod_{x \in \mathcal{Q}_T} F_T(x)$ with $\phi(1) = 1$, such that

$$\alpha_T(x, y) = (\delta\phi)(x, y) := G_x^{xy}(\phi(x)) \phi(xy)^{-1} F_y^{xy}(\phi(y)).$$

Let $f : T \rightarrow S$ be given. We can therefore consider the transformation of the extension $H^0(S, \mathcal{O}_S^*) \rightarrow \mathcal{Q}_S \rightarrow \mathcal{Q}_S$ by $H^0(S, \mathcal{O}_S^*) \rightarrow H^0(T, \mathcal{O}_T^*)$. This extension, which we denote by $f^* \tilde{\mathcal{Q}}_S$, is that associated with the constant functors $F = G = H^0(T, \mathcal{O}_T^*)$ and the factor system

$$f^* \alpha_S : \mathcal{Q}_S \times \mathcal{Q}_S \xrightarrow{\alpha_S} \bigcup_{x \in \mathcal{Q}_S} H^0(S, \mathcal{O}_S^*) \longrightarrow \bigcup_{x \in \mathcal{Q}_S} H^0(T, \mathcal{O}_T^*)$$

and is therefore of the following type

$$H^0(T, \mathcal{O}_T^*) \rightarrow f^* \tilde{\mathcal{Q}}_S \rightarrow \mathcal{Q}_S.$$

On the other hand, we can transform $H^0(T, \mathcal{O}_T^*) \rightarrow \tilde{\mathcal{Q}}_T \rightarrow \mathcal{Q}_T$ by $f^* : \mathcal{Q}_S \rightarrow \mathcal{Q}_T$. This extension, which we denote by $f_* \tilde{\mathcal{Q}}_T$, is that associated with the constant functors $F = G = H^0(T, \mathcal{O}_T^*)$ and the factor system

$$\begin{aligned} f_* \alpha_T : \mathcal{Q}_S \times \mathcal{Q}_S &\longrightarrow \bigcup_{x \in \mathcal{Q}_S} H^0(T, \mathcal{O}_T^*) \\ (x, y) &\longmapsto \alpha_T(f^*(x), f^*(y)) \end{aligned}$$

and it has the form

$$H^0(T, \mathcal{O}_T^*) \rightarrow f_* \tilde{\mathcal{Q}}_T \rightarrow \mathcal{Q}_S.$$

The explicit expressions show that both extensions coincide; that is, $f^* \tilde{\mathcal{Q}}_S = f_* \tilde{\mathcal{Q}}_T$, or, what is tantamount, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{Q}_S \times \mathcal{Q}_S & \xrightarrow{\alpha_S} & \bigcup_{x \in \mathcal{Q}_S} H^0(S, \mathcal{O}_S^*) \\ f^* \times f^* \downarrow & & \downarrow \\ \mathcal{Q}_T \times \mathcal{Q}_T & \xrightarrow{\alpha_T} & \bigcup_{x \in \mathcal{Q}_T} H^0(T, \mathcal{O}_T^*). \end{array}$$

Now, if we consider T varying in the open subschemes of S , we have proved the following

Theorem 3.11

The factor system defines a sheaf homomorphism

$$\mathcal{Q}_S \times \mathcal{Q}_S \xrightarrow{\alpha} \mathbb{G}_{m,S}.$$

Inspired by [17] we give the following criterion for splitting.

Theorem 3.12

Let T be an S -scheme and $\mathcal{A} \in \text{Gr}(V)(T)$ be a point. Let $\mathcal{Q}_T^{\mathcal{A}}$ be the submonoid of \mathcal{Q}_T consisting of those g such that $g(\mathcal{A}) = \mathcal{A}$.

Thus, the normal extension of $\mathcal{Q}_T^{\mathcal{A}}$ induced by (3.10)

$$H^0(T, \mathcal{O}_T^*) \xrightarrow{\iota} \tilde{\mathcal{Q}}_T^{\mathcal{A}} \rightarrow \mathcal{Q}_T^{\mathcal{A}}$$

splits.

Proof. First, let us note that the inclusion $\mathcal{Q}_T^A \hookrightarrow \mathcal{Q}_T$ gives rise to the above normal extension.

Since $\tilde{\mathcal{Q}}_T^A$ consists of pairs (g, \tilde{g}) such that $g(\mathcal{A}) = \mathcal{A}$, it follows that it acts on $\text{Det}_V \times_{\text{Gr}(V)_T} \{\mathcal{A}\}$, which is a locally free \mathcal{O}_T -module of rank 1. Thus, we obtain a homomorphism of monoids

$$\tilde{\mu}: \tilde{\mathcal{Q}}_T^A \longrightarrow H^0(T, \mathcal{O}_T^*).$$

The very construction shows that the composition

$$H^0(T, \mathcal{O}_T^*) \xrightarrow{\iota} \tilde{\mathcal{Q}}_T^A \xrightarrow{\tilde{\mu}} H^0(T, \mathcal{O}_T^*)$$

is the identity.

Let σ be a section of π as sets and let ϕ be defined by the relation $\sigma(x) = (x, \phi(x)) \in \{x\} \times F_T(x)$ for $x \in \mathcal{Q}_T$. We normalize σ by the condition $\phi(1) = 1$. Let c denote the cocycle associated with σ which is the only element $c(x, y) \in H^0(T, \mathcal{O}_T^*)$ satisfying the following identity

$$\sigma(x)\sigma(y) = c(x, y)\sigma(xy).$$

We claim that $G_1^{xy}(c(x, y)) = (\alpha + \delta\phi)(x, y)$, where the symbol $+$ on the r.h.s. is the addition of factor systems, which is defined pointwise. Indeed, this follows from the equality of the following two identities in $\tilde{\mathcal{Q}}_T$

$$\begin{aligned} \sigma(x)\sigma(y) &= (x, \phi(x)) \cdot (y, \phi(y)) = (xy, G_x^{xy}(\phi(x))\alpha_T(x, y)F_y^{xy}(\phi(y))) \\ c(x, y)\sigma(xy) &= (xy, G_1^{xy}(c(x, y))\phi(xy)). \end{aligned}$$

Consider the following section of π

$$\sigma'(x) := \tilde{\mu}(\sigma(x))^{-1} \cdot \sigma(x).$$

Let us compute the cocycle, c' , associated with σ' . Recall that c' satisfies

$$\sigma'(x)\sigma'(y) = c'(x, y)\sigma'(xy)$$

and applying $\tilde{\mu}$ we obtain

$$\tilde{\mu}(\sigma(x))^{-1}\sigma(x)\tilde{\mu}(\sigma(y))^{-1}\sigma(y) = c'(x, y)\tilde{\mu}(\sigma(xy))^{-1}\sigma(xy)$$

and therefore

$$c(x, y)\sigma(xy)\tilde{\mu}(c(x, y))^{-1}\tilde{\mu}(\sigma(xy))^{-1} = c'(x, y)\tilde{\mu}(\sigma(xy))^{-1}\sigma(xy).$$

Simplifying, we have

$$c(x, y)\tilde{\mu}(c(x, y))^{-1} = c'(x, y).$$

Recalling that $\tilde{\mu}$ is the identity on the elements of $H^0(T, \mathcal{O}_T^*)$, it follows that $c' \equiv 1$; or, what is tantamount, that the section σ' is a morphism of monoids. The statement is proved. \square

3.C Enlargement

Note that the conditions for a monomorphism to belong to \mathcal{Q}_S are rather strong and that the origin of these conditions is Theorem 3.6. Motivated by Remark 3, we wonder whether they could be weakened; that is, if the monoid \mathcal{Q}_S could be enlarged. Indeed, we shall show that there are cases in which this can be done.

Henceforth, S is a non-singular curve over an algebraically closed field (not necessarily complete). Then, [9, Example 6.11, Chapter II] shows that there is an isomorphism

$$(\text{Det}, \text{rk}): K(S) \xrightarrow{\sim} \text{Pic}(S) \times \mathbb{Z}$$

where $K(S)$ denotes the Grothendieck group of coherent sheaves of \mathcal{O}_S -modules. Furthermore, this is also valid for any non-empty open subset of S .

Remark 5. Another case in which this approach holds (because of the structure of its Grothendieck group) is the case of Dedekind domains. In a future work we plan to study this case and its applications to arithmetic (e.g. towards a reciprocity law for families of curves generalizing the ideas of [1, 17]).

Let $[M] \in K(S)$ denote the class of a coherent sheaf M in $K(S)$. Let us consider the category whose objects are complexes

$$M_\bullet \equiv \dots \rightarrow 0 \rightarrow M_0 \rightarrow M_1 \rightarrow 0 \rightarrow \dots$$

where M_i is i -th degree part of M_\bullet ; M_i is a quasicohherent \mathcal{O}_S -submodule of V ; $M_0 \subseteq M_1$ and the map $M_0 \rightarrow M_1$ is the inclusion; and the cokernel, M_1/M_0 , is coherent. The morphisms are the standard morphisms of complexes of \mathcal{O}_S -modules.

To each object M_\bullet on this category we may attach an element of the Grothendieck group, namely, the class $[M_1/M_0] \in K(S)$. Note that an exact sequence $0 \rightarrow M'_\bullet \rightarrow M_\bullet \rightarrow M''_\bullet \rightarrow 0$ induces an exact sequence between the cokernels and, hence, yields the identity $[M_1/M_0] = [M'_1/M'_0] + [M''_1/M''_0]$ in $K(S)$. Similarly, if $M_\bullet \rightarrow N_\bullet$ is a quasi-isomorphism, then M_\bullet and N_\bullet give rise to the same class $[M_1/M_0] = [N_1/N_0]$.

Let us introduce the sheaf on S

$$\bar{\mathcal{Q}}_S(T) := \left\{ \begin{array}{l} \mathcal{O}_T\text{-module monomorphisms } g : \hat{V}_T \rightarrow \hat{V}_T \text{ such that} \\ \hat{V}_T/g\hat{V}_T^+ \text{ is quasicohherent and } (\hat{V}_T^+ + g\hat{V}_T^+)/\hat{V}_T^+, \\ (\hat{V}_T^+ + g\hat{V}_T^+)/g\hat{V}_T^+, \hat{V}_T/g\hat{V}_T \text{ are coherent} \end{array} \right\}$$

for $T \subseteq S$, an open subscheme.

Observe that this monoid contains the *restricted general linear group* of [1] and that it could therefore be used to reformulate their results in our setup.

Theorem 3.13

$\bar{\mathcal{Q}}_S(T)$ is a sheaf on monoids.

Proof. We must show that given $f, g \in \bar{\mathcal{Q}}_S(T)$, then $gf \in \bar{\mathcal{Q}}_S(T)$. First, note that all sheaves involved are quasicohherent since V and V^+ are too and that it therefore suffices to show that certain (quasicohherent) sheaves are coherent.

Note that the hypotheses imply that the r.h.s. and l.h.s. terms of the exact sequence

$$0 \longrightarrow \hat{V}_T/f\hat{V}_T \xrightarrow{g} \hat{V}_T/gf\hat{V}_T \longrightarrow \hat{V}_T/g\hat{V}_T \longrightarrow 0$$

are quasicoherent and, hence, we have that $\hat{V}_T/gf\hat{V}_T$ is also coherent.

Let us show that $(\hat{V}_T^+ + gf\hat{V}_T^+)/\hat{V}_T^+$ is coherent. The module

$$(\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + g\hat{V}^+)$$

is coherent since there is a surjection

$$(\hat{V}^+ + f\hat{V}^+)/\hat{V}^+ \xrightarrow{g} (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + g\hat{V}^+) .$$

Thus, from the exact sequence

$$\begin{aligned} 0 &\rightarrow (\hat{V}^+ + g\hat{V}^+)/\hat{V}^+ \rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/\hat{V}^+ \\ &\rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + g\hat{V}^+) \rightarrow 0 \end{aligned}$$

we have that the middle term is also coherent since both sides are too. Note that $(\hat{V}^+ + f\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+)$ is coherent since

$$(\hat{V}^+ + f\hat{V}^+)/f\hat{V}^+ \xrightarrow{g} (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+) .$$

The latter two facts says that the middle and r.h.s. terms of the sequence

$$\begin{aligned} 0 &\rightarrow (\hat{V}^+ + gf\hat{V}^+)/\hat{V}^+ \rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/\hat{V}^+ \\ &\rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+) \rightarrow 0 \end{aligned}$$

are coherent, and hence so is the l.h.s. term; that is, $(\hat{V}^+ + gf\hat{V}^+)/\hat{V}^+$ is coherent.

It remains to show that $(\hat{V}_T^+ + gf\hat{V}_T^+)/gf\hat{V}_T^+$ is coherent. The arguments are similar to those given above and use the following facts. The module

$$(\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + g\hat{V}^+)$$

is coherent since it is a quotient of $(\hat{V}^+ + g\hat{V}^+)/g\hat{V}^+$. The module

$$(\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+)$$

is coherent since it is a quotient of $(\hat{V}^+ + g\hat{V}^+)/\hat{V}^+$. There are exact sequences

$$\begin{aligned} 0 &\rightarrow (\hat{V}^+ + f\hat{V}^+)/f\hat{V}^+ \xrightarrow{g} (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/gf\hat{V}^+ \\ &\rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\rightarrow (\hat{V}^+ + gf\hat{V}^+)/gf\hat{V}^+ \rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/gf\hat{V}^+ \\ &\rightarrow (\hat{V}^+ + g\hat{V}^+ + gf\hat{V}^+)/(\hat{V}^+ + gf\hat{V}^+) \rightarrow 0 . \end{aligned}$$

□

DEFINITION 3.14 We associate the element of $K(T)$ defined by

$$[f] := - \left[(\hat{V}_T^+ + f\hat{V}_T^+)/\hat{V}_T^+ \right] + \left[(\hat{V}_T^+ + f\hat{V}_T^+)/f\hat{V}_T^+ \right] - \left[\hat{V}_T/f\hat{V}_T \right] \in K(T)$$

to an element $f \in \bar{\mathcal{Q}}_S(T)$.

Given an element $g \in \bar{\mathcal{Q}}_S$ and an object M_\bullet , we may consider the object $g_*M_\bullet := gM_0 \rightarrow gM_1$. Therefore, the map $M_\bullet \rightarrow g_*M_\bullet$ gives rise, by linearity, to a transformation defined on the elements of the above type

$$[f] \rightsquigarrow g_*[f]$$

Theorem 3.15

Let $g, f \in \bar{\mathcal{Q}}_S(T)$. It holds that

$$g_*[f] = [gf] - [g]$$

Proof. Let $A \subseteq \hat{V}_T$ be a submodule such that $\hat{V}_T^+, f\hat{V}_T^+, g\hat{V}_T^+, gf\hat{V}_T^+ \subseteq A$ and such that the middle terms of sequences (2) to (7) below are coherent. Note that the previous result shows that $A = \hat{V}_T^+ + f\hat{V}_T^+ + g\hat{V}_T^+ + gf\hat{V}_T^+$ does the job.

Let us consider the following set of exact sequences

$$0 \longrightarrow \hat{V}_T/f\hat{V}_T \xrightarrow{g} \hat{V}_T/gf\hat{V}_T \longrightarrow \hat{V}_T/g\hat{V}_T \longrightarrow 0 \quad (1)$$

$$0 \longrightarrow \hat{V}_T^+ + f\hat{V}_T^+/f\hat{V}_T^+ \xrightarrow{g} A/gf\hat{V}_T^+ \longrightarrow A/(g\hat{V}_T^+ + gf\hat{V}_T^+) \longrightarrow 0 \quad (2)$$

$$0 \longrightarrow \hat{V}_T^+ + gf\hat{V}_T^+/gf\hat{V}_T^+ \longrightarrow A/gf\hat{V}_T^+ \longrightarrow A/(\hat{V}_T^+ + gf\hat{V}_T^+) \longrightarrow 0 \quad (3)$$

$$0 \longrightarrow \hat{V}_T^+ + g\hat{V}_T^+/g\hat{V}_T^+ \longrightarrow A/g\hat{V}_T^+ \longrightarrow A/(\hat{V}_T^+ + g\hat{V}_T^+) \longrightarrow 0 \quad (4)$$

$$0 \longrightarrow \hat{V}_T^+ + f\hat{V}_T^+/\hat{V}_T^+ \xrightarrow{g} A/g\hat{V}_T^+ \longrightarrow A/(g\hat{V}_T^+ + gf\hat{V}_T^+) \longrightarrow 0 \quad (5)$$

$$0 \longrightarrow \hat{V}_T^+ + gf\hat{V}_T^+/\hat{V}_T^+ \longrightarrow A/\hat{V}_T^+ \longrightarrow A/(\hat{V}_T^+ + gf\hat{V}_T^+) \longrightarrow 0 \quad (6)$$

$$0 \longrightarrow \hat{V}_T^+ + g\hat{V}_T^+/\hat{V}_T^+ \longrightarrow A/\hat{V}_T^+ \longrightarrow A/(\hat{V}_T^+ + g\hat{V}_T^+) \longrightarrow 0 \quad (7).$$

Observe, furthermore, that all sheaves are quasicoherent and that by hypothesis and by the previous results all terms on left hand sides are coherent and all middle terms are coherent sheaves by the choice of A . Thus, all terms on the right hand sides are coherent too.

The class $g_*[f]$ is, by definition of g_* , equal to

$$g_*[f] = - \left[(g\hat{V}_T^+ + gf\hat{V}_T^+)/g\hat{V}_T^+ \right] + \left[(g\hat{V}_T^+ + gf\hat{V}_T^+)/gf\hat{V}_T^+ \right] - \left[g\hat{V}_T/gf\hat{V}_T \right].$$

Accordingly, computing the first term with the help of the sequence (5), the second one with the sequence (2) and the last one with the sequence (1), one has that

$$\begin{aligned} g_*[f] &= -[A/g\hat{V}_T^+] + [A/(g\hat{V}_T^+ + gf\hat{V}_T^+)] + [A/gf\hat{V}_T^+] - [A/(g\hat{V}_T^+ + gf\hat{V}_T^+)] \\ &\quad - [\hat{V}_T/gf\hat{V}_T] + [\hat{V}_T/g\hat{V}_T] \\ &= -[A/g\hat{V}_T^+] + [A/gf\hat{V}_T^+] - [\hat{V}_T/gf\hat{V}_T] + [\hat{V}_T/g\hat{V}_T]. \end{aligned}$$

Now, plug in the relations given by sequences (4) and (3) into the last expression

$$\begin{aligned} g_*[f] &= -[\hat{V}_T^+ + g\hat{V}_T^+/g\hat{V}_T^+] - [A/(\hat{V}_T^+ + g\hat{V}_T^+)] + [\hat{V}_T^+ + gf\hat{V}_T^+/gf\hat{V}_T^+] \\ &\quad + [A/(\hat{V}_T^+ + gf\hat{V}_T^+)] - [\hat{V}_T/gf\hat{V}_T] + [\hat{V}_T/g\hat{V}_T]. \end{aligned}$$

Finally, apply the relations given by sequence (7) to the second term of the last expression and by sequence (6) to the fourth one

$$\begin{aligned} g_*[f] &= -[\hat{V}_T^+ + g\hat{V}_T^+/g\hat{V}_T^+] - [A/\hat{V}_T^+] + [\hat{V}_T^+ + g\hat{V}_T^+/\hat{V}_T^+] + [\hat{V}_T^+ + gf\hat{V}_T^+/gf\hat{V}_T^+] \\ &\quad + [A/\hat{V}_T^+] - [\hat{V}_T^+ + gf\hat{V}_T^+/\hat{V}_T^+] - [\hat{V}_T/gf\hat{V}_T] + [\hat{V}_T/g\hat{V}_T] \\ &= -[\hat{V}_T^+ + g\hat{V}_T^+/g\hat{V}_T^+] + [\hat{V}_T^+ + g\hat{V}_T^+/\hat{V}_T^+] + [\hat{V}_T^+ + gf\hat{V}_T^+/gf\hat{V}_T^+] \\ &\quad - [\hat{V}_T^+ + gf\hat{V}_T^+/\hat{V}_T^+] - [\hat{V}_T/gf\hat{V}_T] + [\hat{V}_T/g\hat{V}_T] \\ &= [gf] - [g] \end{aligned}$$

and the claim is proven. \square

Corollary 3.16

- $[1] = 0 \in K(S)$.
- $g_*[1] = [1]$.
- $1_*[g] = [g]$.
- $h_*g_*[f] = (hg)_*[f]$.

Further, g yields a quasi-isomorphism $M_\bullet \xrightarrow{g} g_*M_\bullet$ and, hence, an identification $[M_\bullet] = [g_*M_\bullet]$ in $K(S)$, which will be denoted by g . In particular, conjugation by it yields a group isomorphism

$$g^* : \text{Aut}(\text{Det}([M_\bullet])) \xrightarrow{\sim} \text{Aut}(\text{Det}([g_*M_\bullet]))$$

which is closely related to the *natural transformations* appearing in [1, §2.4.4.]

DEFINITION 3.17 Let us introduce

$$\widehat{\mathcal{Q}}_S := \left\{ \begin{array}{l} \text{pairs } (f, \tilde{f}) \text{ such that } f \in \bar{\mathcal{Q}}_S \text{ and} \\ \tilde{f} \text{ is an automorphism of } \text{Det}([f]) \end{array} \right\}.$$

Given $(g, \hat{g}), (f, \hat{f})$, two elements of $\widehat{\mathcal{Q}}_S$, observe that $g^*(\hat{f}) \in \text{Aut}(\text{Det}(g_*[f]))$ since $\hat{f} \in \text{Aut}(\text{Det}([f]))$. Therefore, one has that $g^*(\hat{f}) \otimes \hat{g}$ belongs to $\text{Aut}(\text{Det}(g_*[f]) \otimes \text{Det}([g])) \simeq \text{Aut}(\text{Det}([gf]))$. Summing up, one has

Proposition 3.18

The composition law

$$(g, \hat{g}) * (f, \hat{f}) := (gf, g^*(\hat{f}) \otimes \hat{g})$$

endows $\hat{\mathcal{Q}}_S$ with a monoid structure such that the forgetful map

$$\hat{\mathcal{Q}}_S \longrightarrow \bar{\mathcal{Q}}_S$$

is a morphism of monoids.

Further, since all the previous constructions are compatible with restrictions to non-empty open subsets of S , they can be stated in terms of sheaves on S . Observing that the very definitions of \mathcal{Q}_S and $\bar{\mathcal{Q}}_S$ imply that

$$\mathcal{Q}_S \hookrightarrow \bar{\mathcal{Q}}_S$$

the following theorem is straightforward.

Theorem 3.19

There is a central extension of sheaves on monoids on S

$$0 \rightarrow \mathbb{G}_{m,S} \longrightarrow \hat{\mathcal{Q}}_S \longrightarrow \bar{\mathcal{Q}}_S \rightarrow 0$$

that, when restricted to \mathcal{Q}_S , coincides with the extension of Theorem 3.11.

Let us finish with a sketch of an arithmetic application of our results that will be implemented in a future paper. The idea is to apply this machinery to study analogues of reciprocity laws in the case of families of curves $C \rightarrow S$ (including the non-abelian case) defined over \mathbb{Z} and over p -adic numbers.

Note that, as a corollary of the above theorem, the factor system

$$\bar{\mathcal{Q}}_S \times \bar{\mathcal{Q}}_S \xrightarrow{\bar{\alpha}_S} \mathbb{G}_{m,S}$$

associated with the extension $\hat{\mathcal{Q}}_S$ (Theorem 3.19) is equivalent to the factor system of $\tilde{\mathcal{Q}}_S$ (Theorem 3.11) when restricted to $\mathcal{Q}_S \times \mathcal{Q}_S$.

However, what is even more relevant is that for any $g \in \bar{\mathcal{Q}}_S(T)$ there exists an open subset $R \subseteq T \subseteq S$ such that $g|_R \in \mathcal{Q}_S(R)$ (see Remark 3). In particular, the stalks of \mathcal{Q}_S and $\bar{\mathcal{Q}}_S$ at the generic point of S coincide.

Finally, explicit expressions for the cases of curves C over a field and over local artinian rings are well known ([5, 1, 17]). These arguments will be the key points for obtaining explicit expressions for the factor system $\bar{\alpha}_S$.

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