

Adelic and idelic pairings related to Weil reciprocity on algebraic curves

Luis Manuel Navas Vicente and Francisco J. Plaza Martín

ABSTRACT. In this paper we present an overview of some recent results concerning the properties of a function field embedded in the adèles. Specifically, for an algebraic curve over a perfect field, Weil reciprocity allows us to define a subgroup of the idele class group whose topology encodes arithmetic properties of the base field and geometric properties of the curve. In particular we establish a relationship with the Picard group. The construction is inspired by Tate's results on the character group of the adèles over a global field.

1. Introduction

Our aim is to begin a study of the function field Σ_X of an algebraic curve X over an arbitrary perfect field k , using the machinery of the geometric adèles and ideles. This means studying the topological embedding of the field in the additive and multiplicative settings and how it reflects certain arithmetic and geometric data. With this in mind, let us give, by way of introduction, a list of examples that serve as inspiration and motivation.

1.1. Tate's thesis. In the idelic formulation of Class Field Theory over a global field Σ , one considers the idele class group \mathbb{I}/Σ^* and more specifically \mathbb{I}^1/Σ^*N where \mathbb{I}^1 are the ideles of norm 1 and N the norm group. This is one of the standard approaches, whereby one studies local class field theory via the local reciprocity map and then puts together these local data to construct the global versions.

In his well-known thesis ([5, Chap. XV, Tate's Thesis], [23]), Tate reformulated Hecke's proof of the analytic continuation and functional equation of the zeta and L -functions of an algebraic number field using the Fourier transform and the associated Pontryagin duality at each place, archimedean and non-archimedean. This approach necessitates studying characters on the adèles and ideles, highlighting some interesting relationships between the additive and multiplicative structures, for example the appearance of gamma factors, which combine additive and multiplicative characters (over $\mathbb{R} = \mathbb{Q}_\infty$ this is clear in the well-known definition of the gamma function as an integral).

2020 *Mathematics Subject Classification.* Primary 14H05; Secondary 19F15, 11R37.

Key words and phrases. Weil reciprocity, class field theory, algebraic curves, function fields, residues.

Research of both authors supported by grants PGC2018-099599-B-I00, MTM2015-66760-P, J416/463AC03 and of the second author also by PGC2018-096504-B-C32.

In particular, self-duality at the local level results in self-duality of the adèles \mathbb{A}_Σ , in the following sense: on the rational adèles $\mathbb{A}_\mathbb{Q}$ there is a standard additive character $\chi : \mathbb{A}_\mathbb{Q} \rightarrow S^1$ projecting onto the exponential map $e^{-2\pi ix}$ at the archimedean place. We then obtain the standard additive character $\chi_\Sigma(\alpha) = \chi_\mathbb{Q}(\text{tr}_{\Sigma/\mathbb{Q}}(\alpha))$ on \mathbb{A}_Σ , and the pairing $(\alpha, \beta) \mapsto \chi(\alpha\beta)$ induces an isomorphism of \mathbb{A}_Σ with its continuous character group $\widehat{\mathbb{A}}_\Sigma = \text{Hom}(\mathbb{A}_\Sigma, S^1)$.

In addition, since χ is trivial on Σ , the field Σ is a totally isotropic subspace. The previous result, along with Pontryagin duality and standard facts about the adelic topology, show that Σ^\perp/Σ is discrete and compact, hence trivial, so that the pairing above also induces an isomorphism of the character group of the quotient \mathbb{A}_Σ/Σ with the ground field Σ .

An exposition of these results in the context of the Langlands Program, of which this can be said to be the $\text{Gl}(1)$ case, can be found in [10].

In the geometric context of a curve over a perfect field, we will no longer have the machinery of integration on locally compact topological groups which allowed Tate, and later Weil in the distributional sense, to use Pontryagin duality. However, focusing on the isomorphism between the character group of \mathbb{A}_Σ and Σ , we can look for other pairings which may lead to similar results in the geometric context, ultimately finding a suitable analog in Weil reciprocity.

1.2. The cross-ratio. As an illustration of how Weil Reciprocity characterizes the adèles corresponding to a function, let us consider the case when X is the projective line $X = \mathbb{P}_k^1$. Fix a function z having a unique zero and pole, so that the function field of the curve is $\Sigma_X = k(z)$. Assuming that k is algebraically closed, z may be thought of as an affine coordinate. Recall that the cross-ratio of four distinct points in affine coordinates is given by

$$(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

The cross-ratio satisfies the symmetry relation $(z_1, z_2, z_3, z_4) = (z_3, z_4, z_1, z_2)$ and in general its properties can be used to prove many classical theorems in Projective Geometry, such as those of Menealus, Ceva, Carnot, etc.

Now, note that the rational functions

$$F(z) = \frac{z - z_1}{z - z_2}, \quad G(z) = \frac{z - z_3}{z - z_4}$$

have divisors of zeros and poles equal to $z_1 - z_2$ and $z_3 - z_4$ respectively. Hence the symmetry of the cross-ratio can be written as

$$F(z_3)F(z_4)^{-1} = G(z_1)G(z_2)^{-1}.$$

If $F, G \in \Sigma_{\mathbb{P}_k^1}^*$ have disjoint supports, then this can be generalized to

$$\prod F(x)^{v_x(G)} = \prod G(x)^{v_x(F)} \quad \forall F, G \in \Sigma_{\mathbb{P}_k^1}^*,$$

where the product is over closed points. It can be easily seen that the restriction on the supports can be avoided by writing the above relation instead as

$$(1.1) \quad \prod_{x \in \mathbb{P}_k^1} (-1)^{v_x(F)v_x(G)} \frac{F^{v_x(G)}}{G^{v_x(F)}}(x) = 1 \quad \forall F, G \in \Sigma_{\mathbb{P}_k^1}^*.$$

This leads to the insight that for \mathbb{P}^1 , the symmetry of the cross-ratio is equivalent to the Weil Reciprocity Law (we have seen one direction; the converse can be proved using resultants, among other methods). By using that any 0 degree divisor on \mathbb{P}^1 (over an algebraically closed field) is the divisor of a rational function, one finds easily that in this case reciprocity characterizes the function field inside the ideles:

PROPOSITION. For $X = \mathbb{P}^1$ over k algebraically closed, given an idele $(\alpha_x) \in \mathbb{I}_X$ such that over closed points

$$\prod_{x \in X} (-1)^{v_x(\alpha_x)v_x(G)} \left(\frac{\alpha_x(x)^{v_x(G)}}{G(x)^{v_x(\alpha_x)}} \right) = 1$$

for all $G \in \Sigma_X^*$, there exists $F \in \Sigma_X^*$ such that $F(x) = \alpha_x(x)$ for all $x \in X$.

1.3. Residues. From another point of view, namely, by looking at the infinitesimal level, one finds a suitable pairing in terms of residues. Let X be a complete, non-singular, irreducible algebraic curve over $k = \mathbb{C}$ and x a closed point of X . For integers $m < n$, consider the exact sequence of sheaves on X

$$0 \rightarrow \mathcal{O}_X(mx) \rightarrow \mathcal{O}_X(nx) \rightarrow \mathcal{O}_X(nx)/\mathcal{O}_X(mx) \rightarrow 0,$$

inducing the long exact sequence in cohomology:

$$\dots \rightarrow H^0(X, \mathcal{O}_X(nx)) \rightarrow H^0(X, \mathcal{O}_X(nx)/\mathcal{O}_X(mx)) \rightarrow H^1(X, \mathcal{O}_X(mx)) \rightarrow \dots$$

Taking direct and inverse limits in n and m respectively (noting $H^0(X, \mathcal{O}_X(mx)) = H^1(X, \mathcal{O}_X(nx)) = 0$ for $m \ll 0$ and $n \gg 0$), we obtain the exact sequence

$$0 \rightarrow H^0(X \setminus \{x\}, \mathcal{O}_X) \rightarrow K_x = (\widehat{\mathcal{O}}_{X,x})_{(0)} \rightarrow \varprojlim_m H^1(X, \mathcal{O}_X(mx)) \rightarrow 0.$$

Denote the canonical bundle by ω_X . Then by the Riemann-Roch Theorem and Serre duality, we find that

$$\varprojlim_m H^1(X, \mathcal{O}_X(mx)) = \left(\varinjlim_m H^0(X, \omega_X(-mx)) \right)^* = H^0(X \setminus \{x\}, \omega_X)^*$$

and the map $K_x \rightarrow H^0(X \setminus \{x\}, \omega_X)^*$ sends α to the linear form $\omega \mapsto \text{Res}_x \alpha \omega$. Hence the subspace $H^0(X \setminus \{x\}, \mathcal{O}_X)$ is totally isotropic with respect to the residue pairing on K_x :

$$(f, g) \mapsto \text{Res}_x(f dg) : K_x \times K_x \longrightarrow k.$$

We may repeat this argument in the ring of adèles \mathbb{A}_X over Σ_X , which contains Σ_X via the diagonal embedding. With x replaced by an effective divisor D and taking the limit, we find that Σ_X is totally isotropic with respect to the residue pairing on adèles

$$(1.2) \quad (\alpha, \beta) \mapsto \text{Res}_x(\alpha d\beta) : \mathbb{A}_X \times \mathbb{A}_X \longrightarrow k.$$

Furthermore, this will allow us to characterize the function field in the adèles in the sense that for a fixed $\alpha \in \mathbb{A}_X$, if $\sum_x \text{Res}_x(\alpha dg) = 0$ for all $g \in \Sigma_X$, then in fact $\alpha \in \Sigma_X + \prod_{x \in X} k$ (see Theorem 6.4). The key point is the following. The Weil Reciprocity Law allows us to introduce a bilinear pairing on the group of ideles $\mathbb{I}_X = \mathbb{A}_X^*$ and the pairing (1.2) is exactly the pairing induced on the level of Lie algebras (this approach is carried out in [13]). Thus, once more, we see how the Weil Reciprocity Law might be useful in characterizing algebraic functions, differentials, etc. inside the adèle ring.

1.4. Goals and the main results. In this paper we aim at giving a unified approach to the facts pointed out in §1.2 and §1.3. This goal will be attained by introducing certain subgroups and quotients of the ideles which reflect some properties of the curve X (such as the genus) and of the base field k . At the end of the paper, we will present some evidence on how these objects may be useful for studying algebraic extensions of Σ_X and, hopefully, in furthering the geometric Langlands program. A second line of future research consists of reinterpreting our results in terms of K -theory, along the lines established in [4, 20] for local and global fields.

Let us now briefly describe the main construction and principal results of this paper.

For a smooth, complete and connected curve X over a perfect field k , the multiplicative group of its function field Σ_X can be embedded as a subgroup of the norm-1 idele group \mathbb{I}_X^1 . The pairing which we will use to study duality will be the global symbol arising from the local multiplicative symbols constructed in ([12, 17]). This pairing leads us to consider the corresponding “orthogonal” which we will denote $(\Sigma_X^*)^\perp$. The Weil Reciprocity Law is equivalent to the existence of an induced inclusion $\Sigma_X^* \subseteq (\Sigma_X^*)^\perp$.

Based on the ideas of Tate and Weil, it is natural to study the topology of the quotient $(\Sigma_X^*)^\perp/\Sigma_X^*$ which, as can be expected, is related to the arithmetic properties of the base field (e.g. we have results like Proposition 4.1). Geometrically, the relation between this quotient and the curve is summarized in one of our main results, Theorem 5.2, which proves the exactness of

$$(1.3) \quad 0 \rightarrow \text{Hom}_{\text{gr}}(\text{Pic}^0(X), k^*) \rightarrow (\Sigma_X^*)^\perp/\Sigma_X^* \rightarrow \text{Pic}^0(X)$$

when k is algebraically closed k , as well as an analog for finite fields. As a corollary we have that $(\Sigma_X^*)^\perp/\Sigma_X^* = 0$ for $X = \mathbb{P}_1$ over an algebraically closed field, but note that the quotient need not be trivial for other curves or other base fields.

1.5. Notation and set-up. All through this article we will be working with a smooth, complete and connected curve X over a perfect field k . To simplify the technical details, we also assume that k is algebraically closed in the function field of X , as well as the existence of a k -rational point. We use the following notation:

- Σ_X is the function field of X .
- When we simply write $x \in X$, it is implied that x is a closed point of X , with associated valuation v_x , local ring $\mathcal{O}_{X,x}$, and maximal ideal \mathfrak{m}_x .
- Let $\widehat{\mathcal{O}}_{X,x}$ be the v_x -completion of $\mathcal{O}_{X,x}$, $\widehat{\mathfrak{m}}_x$ its maximal ideal, and K_x its quotient field.
- Denote the residue field at $x \in X$ by $k(x) = \widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x$. Since k is assumed perfect, $k(x)/k$ is a finite separable extension of degree $\deg(x) = \dim_k k(x)$.
- In general, for a ring R , we let R^* denote the group of invertible elements.
- For $x \in X$, let $N_{k(x)/k}$ denote the norm map $k(x)^* \rightarrow k^*$ and let N_x be its kernel; i.e. the subgroup of $k(x)^*$ consisting of elements of norm 1.

REMARK 1.1. In fact, the hypothesis that k is algebraically closed in the function field of X is redundant, as it follows from the other assumptions (smoothness, completeness, connectedness, perfect base field) and the existence of a k -rational point. Under these assumptions, X is geometrically integral by [18, Lemma 0CDW],

and therefore geometrically irreducible. From [8, Proposition 4.5.9], it follows that the separable algebraic closure of k in Σ_X is k and, since k is perfect, this is the algebraic closure.

2. Adeles, ideles, and the function field

Cassels, Serre and Weil ([5, Chp. II] and [17, 21]) established the list of basic topological properties of the adèles and ideles over global fields. In the geometric case, however, we have not found a similar comprehensive account in the literature. Here we will give a summary of the main results needed for our objectives.

REMARK 2.1. The details regarding the properties of the adelic and idelic topologies are rather technical and in some cases straightforward, so we will omit the proofs. The abstract theory of topological groups and vector spaces is enough for deducing most of the claims of this section. See, for example [6, 9].

2.1. **Adeles.** The ring of adèles \mathbb{A}_X of Σ_X/k is the restricted direct product

$$\mathbb{A}_X = \prod'_{x \in X} K_x = \left\{ (\alpha_x)_{x \in X} \mid \alpha_x \in \widehat{\mathcal{O}}_{X,x} \text{ for almost all } x \in X \right\}$$

over closed points. Let \mathbb{A}_X^+ be the subring $\mathbb{A}_X^+ = \prod_{x \in X} \widehat{\mathcal{O}}_{X,x}$. The *idele group* \mathbb{I}_X is the group \mathbb{A}_X^* of invertible elements of \mathbb{A}_X . It is the restricted product of K_x^* with respect to the unit groups $\widehat{\mathcal{O}}_{X,x}^*$. For $x \in X$ and an adèle $\alpha = (\alpha_x)_{x \in X}$ with $v_x(\alpha_x) \geq 0$, let $\alpha(x)$ be the image in the residue field $k(x)$ of the x -component α_x .

The adelic topology is the restriction of the direct product topology induced by the valuation topology at each (closed) point x . A neighborhood basis at (0) for this adelic topology is given by either of the following collections:

- (1) The k -vector subspaces $V \subseteq \mathbb{A}_X$ commensurable with \mathbb{A}_X^+ , i.e. such that $\dim_k(V + \mathbb{A}_X^+) / (V \cap \mathbb{A}_X^+) < \infty$.
- (2) For a divisor $D = \sum_x n_x x$, which may be assumed to be effective, the sets $U_D = \{ \alpha \in \mathbb{A}_X \mid v_x(\alpha_x) \geq n_x \ \forall x \in X \}$, equal to the products $\prod_x \widehat{\mathfrak{m}}_x^{n_x}$ with $n_x \geq 0$ for all x and $n_x = 0$ for almost all x . Note $\mathbb{A}_X^+ = U_0$.

It is easy to check that the U_D are clopen subgroups, satisfying $\mathbb{A}_X = \bigcup_D U_D$ and $(0) = \bigcap_D U_D$ where D runs over $\text{Div}(X)$. These facts are frequently referred to.

2.2. **Ideles.** Recall that for a topological ring R , the subspace topology on the group of units R^* does not in general make R^* into a topological group. Instead one uses the topology induced by the map $x \mapsto (x, x^{-1}) : R^* \rightarrow R \times R$. This is also the case for the ideles $\mathbb{I}_X = \mathbb{A}_X^*$. A neighborhood basis at 1 is given by the sets

$$(2.1) \quad V_D = \prod_{x \in \text{supp } D} (1 + \widehat{\mathfrak{m}}_x^{n_x}) \times \prod_{x \notin \text{supp } D} \widehat{\mathcal{O}}_{X,x}^*$$

where $D = \sum_x n_x x$ is an effective divisor on X . As in the additive context, the sets V_D are clopen subgroups,

$$\mathbb{I}_X \cap \mathbb{A}_X^+ = V_0 = \bigcup_D V_D,$$

$\bigcap_D V_D = (1)$, where D runs over effective divisors, and $V_D \subseteq V_{D'}$ if $D \leq D'$. In addition,

$$(2.2) \quad V_D \subseteq (1 + U_D) \cap \mathbb{I}_X,$$

for an effective divisor D , which implies that the inclusion $\mathbb{I}_X \hookrightarrow \mathbb{A}_X$ is continuous.

There is a natural notion of degree on the ideles, which is the additive version of the content or norm commonly used over global fields. Given an idele α , since $v_x(\alpha) = 0$ for almost all x , there is a homomorphism

$$(2.3) \quad \alpha \mapsto D(\alpha) = \sum_{x \in X} v_x(\alpha_x) x : \mathbb{I}_X \rightarrow \text{Div}(X)$$

from ideles to divisors. Its kernel is the neighborhood $V_0 = \prod_x \widehat{\mathcal{O}}_{X,x}^*$. The subgroup \mathbb{I}_X^1 consisting of ideles α with $\deg D(\alpha) = \sum_x \deg(x)v_x(\alpha_x) = 0$ corresponds to the norm one ideles for global fields. It plays a central role throughout this discussion.

As also happens over global fields, one can check that the adelic and idelic topologies induce the same topology on \mathbb{I}_X^1 . In particular, since $V_0 = \mathbb{I}_X^1 \cap \mathbb{A}_X^+$, we see that \mathbb{I}_X^1 is an open subgroup of \mathbb{I}_X . Its basic properties are summarized in the following list.

THEOREM 2.2.

- (1) \mathbb{I}_X^1 is a Hausdorff topological group.
- (2) \mathbb{I}_X^1 is locally compact if and only if k is finite.
- (3) \mathbb{I}_X^1 is not locally connected.

PROOF. See [11, Theorem 2.1]. □

2.3. The function field. Assigning to a rational function its germ at a point induces an injective morphism $\Sigma_X \hookrightarrow K_x$, leading to a diagonal embedding $\Sigma_X \hookrightarrow \mathbb{A}_X$. We of course want to know how Σ_X “sits inside” \mathbb{A}_X . From the general theory of linear topological vector spaces (see [9]), a subspace V inherits the discrete topology if and only if its intersection with an open neighborhood is finite-dimensional, in which case V is also closed.

PROPOSITION 2.3. *The function field Σ_X , embedded diagonally in \mathbb{A}_X , is discrete and closed in the adelic topology.*

PROOF. An argument similar to that given in §1.3 can be used to prove that the kernel and cokernel of $\Sigma_X \rightarrow \mathbb{A}_X/\mathbb{A}_X^+$ are $H^0(X, \mathcal{O})$ and $H^1(X, \mathcal{O})$ respectively (this means that $\Sigma_X \in \text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$, the infinite Grassmannian studied in [12, 16]). In particular, recalling that $\mathbb{A}_X^+ = U_0$ is open, it follows that $\mathbb{A}_X^+ \cap \Sigma_X = H^0(X, \mathcal{O}_X)$ is finite-dimensional (by the hypotheses on the curve) and open. By the remark above, the conclusion follows. □

On the other hand, removing a point from the restricted direct product yields an embedding of Σ_X with dense image. This is a well-known result for global fields, known as the Strong Approximation Theorem [5]. The geometric version is also a known result, but necessitates a different proof since Σ_X is not a global field if k is infinite.

THEOREM 2.4 (Geometric Strong Approximation Theorem). *For a closed point $x_0 \in X$, the diagonal embedding $\Sigma_X \rightarrow \prod'_{x \neq x_0} K_x$ of the function field Σ_X in the restricted product of $(K_x, \widehat{\mathcal{O}}_{X,x})$ is injective with dense image in the adelic topology.*

This means that, having fixed principal parts $\alpha_x \in K_x/\widehat{\mathcal{O}}_{X,x}$ for $x \neq x_0$, we can find a function F with principal part α_x at $x \neq x_0$.

PROOF. A proof using the Riemann-Roch Theorem is given in [11]. □

Turning to the multiplicative context, recall that the idele class group $\mathbb{I}_X^1/\Sigma_X^*$ is one of the central objects in Class Field Theory over a global field, and its compactness in that case is in itself an important result. In our case, however, compactness is going to be the exception rather than the rule.

THEOREM 2.5.

- (1) Σ_X^* is closed and discrete in \mathbb{I}_X^1 .
- (2) The quotient $\mathbb{I}_X^1/\Sigma_X^*$ is compact if and only if k is finite.

PROOF. The first part follows from the corresponding additive result and the equality of the adelic and idelic topologies on \mathbb{I}_X^1 . Compactness for finite fields is a known result. The converse may be proved using Theorem 2.2. See [11, Theorem 2.4]. □

In particular, for algebraically closed base fields k we lose the local compactness of \mathbb{I}_X^1 and with it the compact/discrete relationship provided by Pontryagin duality, which is central in Tate’s approach via adelic and idelic character groups. There is no longer a Haar measure and consequently no local or global analytic Fourier transforms.

3. Reciprocity pairing on ideles

In order to find a suitable analog to the duality results of Tate which were discussed in §1.1, we need a pairing on the ideles with respect to which Σ_X^* is totally isotropic. This requirement leads naturally to the local and global symbols that arise from Weil Reciprocity, as we now see.

3.1. The local and global symbols. Recall the discussion of the case $X = \mathbb{P}^1$ in §1.2 and (1.1). This is a special case of a fundamental result [22].

THEOREM 3.1 (The Weil Reciprocity Law). *Under the stated hypotheses on the curve X , we have:*

$$\prod_{x \in X} (-1)^{\deg(x)v_x(F)v_x(G)} N_{k(x)/k} \left(\frac{F^{v_x(G)}}{G^{v_x(F)}}(x) \right) = 1 \quad (F, G \in \Sigma_X^*).$$

To work with ideles, we need to generalize Serre’s definition of the local symbol in [17], as a pairing $\Sigma_X^* \times \Sigma_X^* \rightarrow k^*$ for algebraically closed k . Under our hypotheses, we may define the local symbol at a closed point $x \in X$ as the pairing $K_x^* \times K_x^* \rightarrow k^*$ given by

$$(3.1) \quad (f, g)_x = (-1)^{\deg(x) \cdot v_x(f) \cdot v_x(g)} N_{k(x)/k} \left(\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right) \quad (f, g \in K_x^*).$$

It is immediate that this reduces to Serre’s definition in [17]. Next, we define the global symbol $\langle \cdot, \cdot \rangle : \mathbb{I}_X \times \mathbb{I}_X \rightarrow k^*$ as

$$(3.2) \quad \langle \alpha, \beta \rangle = \prod_{x \in X} (\alpha_x, \beta_x)_x = \prod_{x \in X} (-1)^{\deg(x) \cdot v_x(\alpha_x) \cdot v_x(\beta_x)} N_{k(x)/k} \left(\frac{\alpha^{v_x(\beta_x)}}{\beta^{v_x(\alpha_x)}}(x) \right).$$

As detailed in [12, 15], these symbols and Weil Reciprocity can be derived from the study of the infinite Grassmannian $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$ given by subspaces $U \subseteq \mathbb{A}_X$ such that the kernel and cokernel of $U \rightarrow \mathbb{A}_X/\mathbb{A}_X^+$ are finite-dimensional, which in particular contains the point Σ_X . The action of \mathbb{I}_X^1 on $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$ yields a central extension with an associated commutator pairing which yields precisely the global symbol (3.2). The corresponding action of K_x^* on $\text{Gr}(K_x, \widehat{\mathcal{O}}_{X,x})$ yields the local symbol (3.1).

This approach also gives an alternative proof of the Weil Reciprocity Law. Indeed, since the above mentioned central extension becomes trivial when restricted to the subgroup $\Sigma_X^* \subset \mathbb{I}_X^1$, the global pairing on Σ_X^* is accordingly trivial also; that is,

$$(3.3) \quad \langle f, g \rangle = \prod_{x \in X} (f, g)_x = 1 \quad (f, g \in \Sigma_X^*),$$

which, after substituting (3.1), takes on the classical form of the Weil Reciprocity Law.

As could be expected, the global pairing is locally constant, which amounts to continuity in the idelic topology. Indeed, every pair of ideles (α, β) has a clopen neighborhood on which $\langle \cdot, \cdot \rangle$ is constant.

The interpretation of the global symbol as a commutator pairing allows us to easily deduce that it is a symbol in the sense of algebraic K -theory as in [20], namely, it satisfies

$$\begin{aligned} \langle \alpha_1 \alpha_2, \beta \rangle &= \langle \alpha_1, \beta \rangle \cdot \langle \alpha_2, \beta \rangle, & \langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle &= 1, \\ \langle \alpha, -\alpha \rangle &= 1, & \langle \alpha, \alpha \rangle &= \langle \alpha, -1 \rangle, \\ \langle \alpha, 1 - \alpha \rangle &= 1, \quad \text{if } \alpha, 1 - \alpha \in \mathbb{I}_X, & \langle \alpha, \alpha \rangle &= 1, \quad \text{if } \alpha \in \mathbb{I}_X^1. \end{aligned}$$

REMARK 3.2. In a manner similar to the method Tate ([19]) used to deduce the residue theorem from the properties of a cocycle, we now rely on the properties of local symbols to obtain our results. The consequences of local symbols have been widely studied ([1, 2]) and, among them, the above-mentioned proof of the Weil reciprocity law given in [12]. Indeed, the ideas of [12] together with the techniques of [16] might be used to study analogues for arithmetic curves. Furthermore, Artin and Whaples ([3]) characterized certain fields in terms of a *product formula* for valuations; however, they need archimedean valuations in the case of function fields, while we have only non-archimedean valuations. Finally, the relation of our results to the classical approach of class field theory in terms of K -theory (e.g. [20]) deserves further research.

3.2. The radical and orthogonal complements. The radicals of the global symbol on \mathbb{I}_X and \mathbb{I}_X^1 will be denoted respectively by

$$\begin{aligned} R &= \{ \alpha \in \mathbb{I}_X \mid \langle \alpha, \beta \rangle = 1 \ \forall \beta \in \mathbb{I}_X \} \\ R^1 &= \{ \alpha \in \mathbb{I}_X^1 \mid \langle \alpha, \beta \rangle = 1 \ \forall \beta \in \mathbb{I}_X^1 \}. \end{aligned}$$

We have the following characterizations (recall the definition of the neighborhoods V_D in (2.1)):

THEOREM 3.3. *The radical of the global reciprocity pairing on the group of norm one ideles \mathbb{I}_X^1 satisfies:*

(1) $k^* \cdot \prod_{x \in X} ((1 + \widehat{\mathfrak{m}}_x)N_x) = R^1 \cap V_0$, where N_x is the kernel of the norm,

$$N_x = \{\lambda \in k(x) \mid N_{k(x)/k}(\lambda) = 1\}.$$

(2) If k is infinite, then $R^1 \subseteq V_0$.

(3) k is algebraically closed if and only if $R^1 = k^* \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x)$.

The radical on the total group of ideles \mathbb{I}_X similarly satisfies:

(1) $\prod_{x \in X} ((1 + \widehat{\mathfrak{m}}_x)N_x) = R \cap V_0$.

(2) If k is infinite, then $R \subseteq V_0$.

(3) k is algebraically closed if and only if $R = \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x)$.

PROOF. The reasoning for \mathbb{I}_X is similar to that for \mathbb{I}_X^1 . The inclusion in R^1 is easily checked, but one needs a k -rational point (stated in our main hypotheses) for equality. For the reverse implication in the third part we use the general characterization of the kernel of the norm maps given in [14]. See [11, Theorem 3.1] for details. □

Note that for a finite field k , the complete radical R may contain elements of non-zero degree.

Just as one takes the quotient of the idele class group by the norm group in Class Field Theory, it is natural to consider

$$\overline{\mathbb{I}}_X^1 = \mathbb{I}_X^1 / \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x)$$

with the quotient topology. The quotient map is open but not necessarily closed since the above product need not be compact.

We list its basic topological properties.

THEOREM 3.4. Consider the quotient group $\overline{\mathbb{I}}_X^1 = \mathbb{I}_X^1 / \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x)$. Then:

- (1) $\overline{\mathbb{I}}_X^1$ is a Hausdorff and totally disconnected topological group.
- (2) $\overline{\mathbb{I}}_X^1$ is profinite if and only if k is finite.

PROOF. The argument relies on showing that a neighborhood basis of $\overline{\mathbb{I}}_X^1$ at 1 is given by the subgroups $\overline{V}_S = \prod_{x \in S} \{1\} \times \prod_{x \notin S} k(x)^*$ where $S \subset X$ runs over finite sets of closed points. The profinite topological groups are precisely those which are Hausdorff, compact and totally disconnected. See for example [6] for these aspects of topological groups. □

With this result in mind, we proceed to study the pairing induced on $\overline{\mathbb{I}}_X^1$ by the global pairing on \mathbb{I}_X^1 . We will use the same symbol for both pairings. We will take “orthogonal complements” of subsets $G \subset \mathbb{I}_X^1$ with respect to the global symbol, denoted by

$$G^\perp = \{\alpha \in \mathbb{I}_X^1 \mid \langle \alpha, g \rangle = 1 \ \forall g \in G\},$$

as well as complements of subsets $G \subseteq \overline{\mathbb{I}}_X^1$ with respect to the induced pairing on the quotient, denoted by

$$G^\perp = \{\alpha \in \overline{\mathbb{I}}_X^1 \mid \langle \alpha, g \rangle = 1 \ \forall g \in G\}.$$

PROPOSITION 3.5. The orthogonal complement satisfies the following properties:

- (1) For any subset $G \subseteq \mathbb{I}_X^1$, the orthogonal complement G^\perp is a closed subgroup of \mathbb{I}_X^1 containing the radical R^1 .

- (2) $\mathbb{I}_X^1 \subseteq (k^*)^\perp$, with equality when k is infinite. In the latter case, if $k^* \subseteq G \subseteq \mathbb{I}_X$ and $\alpha \in \mathbb{I}_X$ satisfies $\langle \alpha, g \rangle = 1 \ \forall g \in G$, then actually $\alpha \in \mathbb{I}_X^1$.
- (3) If $p : \mathbb{I}_X^1 \rightarrow \bar{\mathbb{I}}_X^1$ is the projection map, then for any subset $G \subseteq \bar{\mathbb{I}}_X^1$ we have $p^{-1}(G^\perp) = p^{-1}(G)^\perp$, and for any subset $G \subseteq \mathbb{I}_X^1$, we have $p(G)^\perp = p(G^\perp)$.
- (4) For any subset $G \subseteq \bar{\mathbb{I}}_X^1$, the orthogonal complement G^\perp is a closed and totally disconnected subgroup of $\bar{\mathbb{I}}_X^1$.

PROOF. This is not hard to check. See [11, Proposition 3.2]. □

4. The Main Quotient $(\Sigma_X^*)^\perp / \Sigma_X^*$ – Topology

The global symbol and its associated complement allows us to finally study an analogy of Tate duality as was discussed in the introduction (§1.1), namely, the quotients $\bar{\mathbb{I}}_X^1$ and, foremost, $(\Sigma_X^*)^\perp$. Here we will see how the latter encodes properties of the base field and, in the next section, how it depends on the curve.

Since $\Sigma_X^* \cap \prod(1 + \widehat{\mathfrak{m}}_x) = \{1\}$, we may consider Σ_X^* as a subset $\Sigma_X^* \subseteq \bar{\mathbb{I}}_X^1$. Furthermore, since Σ_X^* is totally isotropic with respect to the global reciprocity symbol (3.3), we have in fact

$$\Sigma_X^* \subseteq (\Sigma_X^*)^\perp \subseteq \bar{\mathbb{I}}_X^1.$$

By Proposition 3.5 we can then conclude that

$$(4.1) \quad (\Sigma_X^*)^\perp = p((\Sigma_X^*)^\perp) \simeq (\Sigma_X^*)^\perp / \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x).$$

As usual, we summarize the main topological properties of these embeddings and the corresponding quotients.

THEOREM 4.1.

- (1) Σ_X^* is closed in $\bar{\mathbb{I}}_X^1$.
- (2) $(\Sigma_X^*)^\perp$ is closed, Hausdorff, totally disconnected and with empty interior in $\bar{\mathbb{I}}_X^1$.
- (3) $(\Sigma_X^*)^\perp / \Sigma_X^*$ is closed in $\bar{\mathbb{I}}_X^1 / \Sigma_X^*$. Both are Hausdorff totally disconnected topological groups.
- (4) $\bar{\mathbb{I}}_X^1 / \Sigma_X^*$ is profinite if and only if k is finite.
- (5) If k is finite, then $(\Sigma_X^*)^\perp / \Sigma_X^*$ is profinite.

$(\Sigma_X^*)^\perp$ is not necessarily a discrete subgroup of $\bar{\mathbb{I}}_X^1$. In fact, the next result implies that this can only happen for algebraically closed k (Proposition 4.2).

PROPOSITION 4.2. Consider the subgroup $\prod_{x \in X} N_x \subset \bar{\mathbb{I}}_X^1$. Then

$$(4.2) \quad \prod_{x \in X} N_x \subseteq (\Sigma_X^*)^\perp / \Sigma_X^*.$$

In addition, if $(\Sigma_X^*)^\perp / \Sigma_X^*$ is discrete, then k must be algebraically closed.

PROOF. The characterization of the radical given in Theorem 3.3 implies the inclusion. If k is not algebraically closed then there are infinitely many closed points x over \bar{k} which are not k -rational. Using the basic facts about the norm map in general separable extensions given in [14], we see that $N_x \neq \{1\}$ for these points. This implies that $(\prod_{x \in X} N_x) \cap \bar{V}_S \neq \{1\}$ in $\bar{\mathbb{I}}_X^1$ for any finite $S \subset X$ and also that $p(\prod_{x \in X} N_x) \cap p(\bar{V}_S) \neq \{1\}$ where $p : \bar{\mathbb{I}}_X^1 \rightarrow \bar{\mathbb{I}}_X^1 / \Sigma_X^*$ is the projection. Thus

$p(\prod_{x \in X} N_x)$ is not discrete, and the inclusion (4.2) implies that $(\Sigma_X^*)^\perp / \Sigma_X^*$ is not discrete either. \square

COROLLARY 4.3. *Let $X = \mathbb{P}_1$. If k is infinite, then*

$$\prod_{x \in X} N_x = (\Sigma_X^*)^\perp / \Sigma_X^*,$$

and hence $(\Sigma_X^*)^\perp / \Sigma_X^* = 0$ if and only if k is algebraically closed.

PROOF. When $X = \mathbb{P}_1$, given $\alpha \in (\Sigma_X^*)^\perp$, then $D(\alpha) = D(f)$ for some $f \in \Sigma_X^*$. It is enough to show that $f^{-1} \cdot \alpha \in \prod_{x \in X} N_x$. Fix a k -rational point x_0 and let x_1 be any other closed point. Let $\beta \in \Sigma_X^*$ have divisor $\deg(x_1)x_0 - x_1$. Then the relation $\langle \alpha, \beta \rangle = 1$ implies that $N_{k(x_1)/k}(f^{-1}\alpha) = 1$. \square

5. The Main Quotient $(\Sigma_X^*)^\perp / \Sigma_X^*$ – Geometry

In this final section we explore some relationships between the quotient $(\Sigma_X^*)^\perp / \Sigma_X^*$ and the Picard group of the curve X . In particular we have non-triviality when X is an elliptic curve over \mathbb{C} . First, note that the map from degree 0 ideles \mathbb{I}_X^1 to $\text{Div}^0(X)$ defined by (2.3) induces a map

$$(\Sigma_X^*)^\perp / \Sigma_X^* \longrightarrow \text{Pic}^0(X).$$

The study of this map sheds light on the structure of $(\Sigma_X^*)^\perp / \Sigma_X^*$.

The global reciprocity pairing on \mathbb{I}_X^1 defines a map $\mathbb{I}_X^1 \rightarrow \text{Hom}_{\text{gr}}(\mathbb{I}_X^1, k^*)$ which in turn, by (3.3) and (4.1), factors through the quotient, yielding a map

$$\begin{aligned} \Phi : (\Sigma_X^*)^\perp &\longrightarrow \text{Hom}_{\text{gr}}(\bar{\mathbb{I}}_X^1 / \Sigma_X^*, k^*) \\ \alpha &\longmapsto (\beta \mapsto \langle \alpha, \beta \rangle). \end{aligned}$$

Note that here we are considering only group homomorphisms Hom_{gr} , discarding the previous topological structure or equivalently, giving all groups the discrete topology. In addition, the global pairing (3.2) also induces a map

$$(5.1) \quad \begin{aligned} \Psi : \text{Div}^0(X) &\longrightarrow \text{Hom}_{\text{gr}}(\bar{V}_0 / k^*, k^*) \\ &\sum_{x \in X} n_x x \longmapsto \left(\alpha \mapsto \prod_{x \in X} N_{k(x)/k}(\alpha(x))^{n_x} \right) \end{aligned}$$

where $\bar{V}_0 = \prod_x \widehat{\mathcal{O}}_{X,x}^* / \prod_x (1 + \widehat{\mathfrak{m}}_x) \subseteq \bar{I}_X^1$. The maps Φ and Ψ are injective if k is algebraically closed (see Theorem 3.3).

Now, there is an exact sequence

$$0 \rightarrow \bar{V}_0 / k^* \rightarrow \bar{\mathbb{I}}_X^1 / \Sigma_X^* \rightarrow \bar{\mathbb{I}}_X^1 / \bar{V}_0 \cdot \Sigma_X^* \rightarrow 0$$

and we can identify

$$\bar{\mathbb{I}}_X^1 / V_0 \simeq \bar{\mathbb{I}}_X^1 / \bar{V}_0 \simeq \text{Div}^0(X), \quad \bar{\mathbb{I}}_X^1 / \Sigma_X^* V_0 \simeq \bar{\mathbb{I}}_X^1 / \Sigma_X^* \bar{V}_0 \simeq \text{Pic}^0(X).$$

It is then not hard to check that the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & (\Sigma_X^*)^\perp \cap \mathbb{A}_X^+ & \longrightarrow & (\Sigma_X^*)^\perp & \longrightarrow & \text{Div}^0(X) \\ & & \downarrow & & \downarrow \Phi & & \downarrow \Psi \\ 0 & \longrightarrow & \text{Hom}_{\text{gr}}(\text{Pic}^0(X), k^*) & \longrightarrow & \text{Hom}_{\text{gr}}(\bar{\mathbb{I}}_X^1 / \Sigma_X^*, k^*) & \longrightarrow & \text{Hom}_{\text{gr}}(\bar{V}_0 / k^*, k^*). \end{array}$$

The Picard group appears when we consider the corresponding map defined by the following diagram:

$$\begin{array}{ccc}
 (\Sigma_X^*)^\perp & \longrightarrow & \text{Div}^0(X) \\
 \downarrow & & \downarrow \\
 (\Sigma_X^*)^\perp / \Sigma_X^* & \longrightarrow & \text{Pic}^0(X).
 \end{array}$$

PROPOSITION 5.1. *There are maps ι, π such that the following sequence is exact:*

$$0 \rightarrow \prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x) N_x \xrightarrow{\iota} (\Sigma_X^*)^\perp \cap \mathbb{A}_X^+ \xrightarrow{\pi} \text{Hom}_{gr}(\text{Pic}(X), k^*).$$

As a consequence, there is an induced exact sequence

$$0 \rightarrow k^* \prod_{x \in X} N_x \rightarrow ((\Sigma_X^*)^\perp \cap \bar{V}_0) \xrightarrow{\bar{\pi}} \text{Hom}_{gr}(\text{Pic}^0(X), k^*).$$

The maps π and $\bar{\pi}$ are surjective if k is either finite or algebraically closed.

PROOF. ι is the inclusion since by Theorem 3.3, we have

$$\prod_{x \in X} (1 + \widehat{\mathfrak{m}}_x) N_x \subseteq \mathbb{A}_X^+ \cap (\Sigma_X^*)^\perp = \ker((\Sigma_X^*)^\perp \rightarrow \text{Div}(X)).$$

To define π on an idele λ , consider the standard action of ideles on divisors,

$$(5.2) \quad D = \sum_{x \in X} n_x x \mapsto \lambda(D) = \prod_{x \in X} N_{k(x)/k}(\lambda(x))^{n_x} : \text{Div}(X) \rightarrow k^*.$$

One checks easily that $\lambda(D(f)) = 1$ for $\lambda \in \ker((\Sigma_X^*)^\perp \rightarrow \text{Div}(X))$ and $f \in \Sigma_X^*$, hence λ induces a map $\text{Pic}(X) = \text{Div}(X)/\Sigma_X^* \rightarrow k^*$, which is $\pi(\lambda)$. The details regarding the maps in the second sequence and exactness of both are given in [11]. □

We are finally in a position to state the main result.

THEOREM 5.2. *Let k be either finite or algebraically closed. There is an exact sequence*

$$(5.3) \quad 0 \rightarrow \text{Hom}_{gr}(\text{Pic}^0(X), k^*) \rightarrow (\Sigma_X^*)^\perp / (\Sigma_X^* \prod_{x \in X} N_x) \rightarrow \text{Pic}^0(X).$$

Moreover, for $k = \mathbb{C}$, the last arrow is surjective.

PROOF. We give a brief sketch of the proof. First, apply the Snake Lemma to

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma_X^* & \longrightarrow & (\Sigma_X^*)^\perp & \longrightarrow & (\Sigma_X^*)^\perp / \Sigma_X^* \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Sigma_X^* / k^* & \longrightarrow & \text{Div}^0(X) & \longrightarrow & \text{Pic}^0(X) \longrightarrow 0.
 \end{array}$$

Bearing in mind the other results in this section, we obtain the exactness of the sequence

$$0 \rightarrow k^* \rightarrow \ker((\Sigma_X^*)^\perp \rightarrow \text{Div}^0(X)) \rightarrow (\Sigma_X^*)^\perp / \Sigma_X^* \rightarrow \text{Pic}^0(X)$$

and we apply Proposition 5.1.

To prove the surjectivity for $k = \mathbb{C}$, we have relied on the following facts: (1) any rational function is a quotient of products of prime forms [7], (2) the divisors

of the form $a - b$ generate $\text{Div}^0(X)$ and (3) the explicit form of the reciprocity symbols which we discussed in §3. □

COROLLARY 5.3. *If $k = \mathbb{C}$ and X is an elliptic curve, then*

$$(\Sigma_X^*)^\perp / \Sigma_X^* \neq 0.$$

PROOF. We may write X as a Tate curve $X \simeq \mathbb{C}^* / q^{\mathbb{Z}}$ with $|q| < 1$. Then the map

$$z \longmapsto e^{2\pi i \frac{\log |z|}{\log |q|}} : \mathbb{C}^* / q^{\mathbb{Z}} \rightarrow \mathbb{C}^*$$

is a non-trivial group homomorphism

$$\text{Pic}^0(X) \simeq X \simeq \mathbb{C}^* / q^{\mathbb{Z}} \rightarrow \mathbb{C}^*$$

from which one deduces that $\text{Hom}_{\text{gr}}(\text{Pic}^0(X), \mathbb{C}^*) \neq 0$. □

6. Applications

6.1. Study of field extensions. One application of these results is to study field extensions of Σ_X in the adèles. On the one hand, we have first an example of the kind of results one may expect to obtain via topological characterizations.

THEOREM 6.1. *For a field extension Ω / Σ_X inside the adèles, $\Sigma_X \subset \Omega \subset \mathbb{A}_X$, the following conditions are equivalent:*

- (1) $\dim_k \Omega \cap \mathbb{A}_X^+ < \infty$.
- (2) Ω is discrete (and therefore closed).
- (3) $\Omega^* \subset \mathbb{I}_X^1$.
- (4) $\Omega \cap \mathbb{A}_X^+ = k$.
- (5) $\Omega^* \cap \mathbb{A}_X^+ = k^*$.

PROOF. See [11, Theorem 2.5]. □

REMARK 6.2. If any of the conditions in the theorem hold, then it also follows that Ω^* is discrete in \mathbb{I}_X . However, the converse is false and is incorrectly stated in the above reference.

On the other hand, the following result shows how the group $(\Sigma_X^*)^\perp$, defined by orthogonality with respect to the global reciprocity symbol, is also relevant for such a study.

THEOREM 6.3. *Let k be algebraically closed and let Ω be a field with $\Sigma_X^* \subset \Omega^* \subset (\Sigma_X^*)^\perp$. Then*

- (1) Ω^* is discrete in \mathbb{I}_X .
- (2) The natural map $\Omega^* \rightarrow (\Sigma_X^*)^\perp$ is injective and its image is discrete.
- (3) There is a natural injection

$$\Omega^* / \Sigma_X^* \hookrightarrow \text{Pic}^0(X).$$

PROOF. See [11, Theorem 5.2]. □

6.2. Residues. Let us finish by sketching how the discussion in §1.3 is connected to our general approach. We refer the reader to [13] for the details.

Recalling that the group \mathbb{A}_X is defined as a restricted product, we may set: $\text{Lie } \mathbb{I}_X = \{1 + \epsilon\alpha_x \in k[\epsilon]/(\epsilon^2) \otimes_k K_x \text{ s.t. } 1 + \epsilon\alpha_x \in k[\epsilon]/(\epsilon^2) \otimes_k \widehat{\mathcal{O}}_{X,x} \text{ for almost all } x\}$ and there is thus a canonical isomorphism $\mathbb{A}_X \simeq \text{Lie } \mathbb{I}_X$. Accordingly, the global symbol (3.2) yields a pairing

$$\langle \cdot, \cdot \rangle_{\text{Lie}} : \mathbb{A}_X \times \mathbb{A}_X \longrightarrow k$$

where $\langle \cdot, \cdot \rangle_{\text{Lie}}$ is defined as follows. For $\alpha, \beta \in \mathbb{A}_X$, $\langle \alpha, \beta \rangle_{\text{Lie}}$ is defined by the relation

$$1 + \epsilon_1 \epsilon_2 \langle \alpha, \beta \rangle_{\text{Lie}} = [1 + \epsilon_1 \alpha, 1 + \epsilon_2 \beta],$$

where the second bracket is the commutator. The explicit computation in [13, Thm 4.10] yields

$$\langle \alpha, \beta \rangle_{\text{Lie}} = \sum_{x \in X} \text{tr}_{k(x)/k} \text{Res}_x \alpha \, d\beta \quad (\alpha, \beta \in \mathbb{A}_X)$$

where $\text{tr}_{k(x)/k}$ is the trace and $\text{Res}_x : K_x \simeq k(x)((z_x)) \rightarrow k(x)$ maps $\alpha_x = \sum_i a_i z_x^i \in k(x)((z_x))$ to a_{-1} (this does not depend on the choice of a formal parameter z_x at x).

In particular, via this mechanism, the Weil Reciprocity Law now implies the Residue Theorem, which is the statement that Σ_X is totally isotropic for $\langle \cdot, \cdot \rangle_{\text{Lie}}$.

We finish with the following result, which can be thought of as an algebraicity criterion for adèles based on the Residue Theorem.

THEOREM 6.4. *Under the hypotheses on the curve X and the field k in §1.5 and $\text{char } k = 0$, we have*

$$\Sigma_X^\perp = \Sigma_X + \prod_{x \in X} k(x).$$

PROOF. The Residue Theorem implies the reverse inclusion $\Sigma_X + \prod_{x \in X} k(x) \subseteq \Sigma_X^\perp$. Now, let $\alpha \in \Sigma_X^\perp$. Then the linear map

$$\bar{\alpha} : \mathbb{A}_X \longrightarrow k \quad : \quad \beta \longmapsto \langle \alpha, \beta \rangle_{\text{Lie}} = \sum_{x \in X} \text{tr}_{k(x)/k} \text{Res}_x \alpha \, d\beta$$

factors through the quotient to give a map $\mathbb{A}_X/\Sigma_X \rightarrow k$. It is also straightforward that it factors through

$$\mathbb{A}_X/(\Sigma_X + U_D) \longrightarrow k$$

where D is an effective divisor on X such that $D + D(\alpha) = D + \sum_{x \in X} v_x(\alpha)x \geq 0$. The following diagram provides a geometrical interpretation of $\mathbb{A}_X/(\Sigma_X + U_D)$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(X, \mathcal{O}_X(-D)) & \longrightarrow & \Sigma_X & \longrightarrow & H^0(X, \Sigma_X/\mathcal{O}_X(-D)) & \longrightarrow & H^1(X, \mathcal{O}_X(-D)) & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & U_D \cap \Sigma_X & \longrightarrow & \Sigma_X & \longrightarrow & \mathbb{A}_X/U_D & \longrightarrow & \mathbb{A}_X/(U_D + \Sigma_X) & \longrightarrow & 0. \end{array}$$

Since X is smooth, the dualizing sheaf is given by the canonical sheaf ω_X . Thus, the linear map $\bar{\alpha}$ is given by a meromorphic differential $\omega \in H^0(X, \omega_X(D)) \simeq H^1(X, \mathcal{O}_X(-D))^*$ via the relation $\bar{\alpha}(\beta) = -\sum_{x \in X} \text{Res}_x \beta \omega$, that is to say, $d\alpha = \omega$.

Now, let us consider a non-empty open subset $U \subset X$ and a rational function $f \in \Sigma_X$ such that $df = \omega$. Since $f \in \Sigma_X^\perp$, it follows that also $\alpha - f \in \Sigma_X^\perp$, which is tantamount to assuming that there is a non-empty open subset U such that

$d(\alpha - f)|_U = 0$. Since $\prod_{x \in X} k(x) \in \Sigma_X^\perp$, we may assume without loss of generality that $\alpha|_U = 0$. Accordingly, for such an α , the condition $\alpha \in \Sigma_X^\perp$ means that

$$\sum_{x \in X \setminus U} \operatorname{tr}_{k(x)/k} \operatorname{Res}_x \alpha dg = 0 \quad \forall g \in \Sigma_X,$$

but this implies that $\alpha_x \in k(x)$ for all $x \in X \setminus U$ and the conclusion follows. \square

Acknowledgments

We would like to dedicate this paper to José María Muñoz Porras, Professor at the University of Salamanca, who passed away on December 31st, 2018. He was our friend and a brilliant mathematician, who actively participated in several meetings of “Geometry at the Frontier” of which he had very pleasant memories. He had in mind the study of algebraic extensions in terms of adèles, and many ideas of this paper have been inspired by him.

In particular, the results presented here are a detailed version of the talk titled *Algebraic Curves, Residues and Weil Reciprocity Law* given at the *Geometry at the Frontier III* conference in November 2018.

References

- [1] G. W. Anderson and F. Pablos Romo, *Simple proofs of classical explicit reciprocity laws on curves using determinant groupoids over an Artinian local ring*, *Comm. Algebra* **32** (2004), no. 1, 79–102, DOI 10.1081/AGB-120027853. MR2036223
- [2] E. Arbarello, C. De Concini, and V. G. Kac, *The infinite wedge representation and the reciprocity law for algebraic curves*, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), *Proc. Sympos. Pure Math.*, vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 171–190, DOI 10.1007/bf01228409. MR1013132
- [3] E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, *Bull. Amer. Math. Soc.* **51** (1945), 469–492, DOI 10.1090/S0002-9904-1945-08383-9. MR13145
- [4] J.-L. Brylinski and P. Deligne, *Central extensions of reductive groups by \mathbf{K}_2* , *Publ. Math. Inst. Hautes Études Sci.* **94** (2001), 5–85, DOI 10.1007/s10240-001-8192-2. MR1896177
- [5] *Algebraic number theory*, Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union. Edited by J. W. S. Cassels and A. Fröhlich, Academic Press, London; Thompson Book Co., Inc., Washington, D.C., 1967. MR0215665
- [6] D. Dikranjan, *Introduction to Topological Groups*, <https://users.dimi.uniud.it/~dikran.dikranjan/ITG.pdf>.
- [7] J. D. Fay, *Theta functions on Riemann surfaces*, *Lecture Notes in Mathematics*, Vol. 352, Springer-Verlag, Berlin-New York, 1973. MR0335789
- [8] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV* (French), *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 361. MR238860
- [9] G. Köthe, *Topological Vector Spaces*, *Grundlehren der mathematischen Wissenschaften*, Springer-Verlag Berlin Heidelberg, 1983.
- [10] S. S. Kudla, *Tate’s thesis*, An introduction to the Langlands program (Jerusalem, 2001), Birkhäuser Boston, Boston, MA, 2003, pp. 109–131. MR1990377
- [11] J. M. Muñoz Porras, L. M. Navas Vicente, F. Pablos Romo, and F. J. Plaza Martín, *An idelic quotient related to Weil reciprocity and the Picard group*, *Collect. Math.* **71** (2020), no. 1, 151–171, DOI 10.1007/s13348-019-00252-7. MR4047704
- [12] J. M. Muñoz Porras and F. Pablos Romo, *Generalized reciprocity laws*, *Trans. Amer. Math. Soc.* **360** (2008), no. 7, 3473–3492, DOI 10.1090/S0002-9947-08-04554-6. MR2386233
- [13] J. M. Muñoz Porras and F. J. Plaza Martín, *Weil Reciprocity Law and the Theorem of Residues*, <http://arxiv.org/abs/1905.11774>

- [14] T. Ono, *On algebraic groups defined by norm forms of separable extensions*, Nagoya Math. J. **11** (1957), 125–130. MR85229
- [15] F. Pablos Romo, *On the tame symbol of an algebraic curve*, Comm. Algebra **30** (2002), no. 9, 4349–4368, DOI 10.1081/AGB-120013323. MR1936475
- [16] F. J. Plaza Martín, *Arithmetic infinite Grassmannians and the induced central extensions*, Collect. Math. **61** (2010), no. 1, 107–129, DOI 10.1007/BF03191229. MR2604862
- [17] J.-P. Serre, *Groupes algébriques et corps de classes* (French), Publications de l’institut de mathématique de l’université de Nancago, VII. Hermann, Paris, 1959. MR0103191
- [18] *The Stacks project*, <https://stacks.math.columbia.edu> (2019).
- [19] J. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) **1** (1968), 149–159. MR227171
- [20] J. Tate, *Symbols in arithmetic*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 201–211. MR0422212
- [21] A. Weil, *Basic number theory*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition. MR1344916
- [22] A. Weil, *Généralisation des fonctions abéliennes*, J. Math. Pures et Appl. **17** (1938), pp. 47–87.
- [23] A. Weil, *Sur la théorie du corps de classes* (French), J. Math. Soc. Japan **3** (1951), 1–35, DOI 10.2969/jmsj/00310001. MR44569

DEPARTAMENTO DE MATEMÁTICAS AND IUFFYM, UNIVERSIDAD DE SALAMANCA, PLAZA DE LA MERCED 1-4, 37008 SALAMANCA, SPAIN
Email address: `navas@usal.es`

DEPARTAMENTO DE MATEMÁTICAS AND IUFFYM, UNIVERSIDAD DE SALAMANCA, PLAZA DE LA MERCED 1-4, 37008 SALAMANCA, SPAIN
Email address: `fplaza@usal.es`