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OWA aggregation operators and multi-agent decisions with N-soft sets

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A R T I C L E I N F O

Dataset link: https://github.com/gsantosgarcia/N-soft-sets

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A B S T R A C T

This work improves the theoretical basis of N-soft sets and insists on their practical applications in a multi-agent context. N-soft sets characterize the alternatives in terms of multinary descriptions by attributes. We argue that unlike the basic example of soft set theory, N-soft sets are amenable to various interesting approaches in relation to aggregation. However the literature has not provided mechanisms for the aggregation of N-soft sets. And there exists no *multi-agent* decision making procedure based on N-soft sets either.

Therefore, the goal of this work is twofold. We present a novel theory of aggregation of N-soft sets. The fundamental tools for this purpose are OWA operators. We produce the first scores of alternatives defined by multi-agent, multinary evaluations. Then we apply this theory to *multi-agent* decision-making. In this way we produce the first algorithms for multi-agent decisions based on N-soft sets. We support our findings with real examples that illustrate the application and versatility of these flexible approaches.

1. Introduction

In this work we contribute to the theoretical development of N-soft sets. This model captures the idea that a universe can be described by the satisfaction of various attributes, in one of various ranked degrees. Put shortly, we speak of a multinary parameterized description of the universe of alternatives, henceforth extending the binary case (i.e., the soft set model).

Soft set theory was conceived of as an alternative to popular approaches to uncertain and vague knowledge like those of probability and statistics, fuzzy set theory and rough set theory. Fuzzy set theory (Zadeh, 1965) allows for gradual assessments of the memberships of the elements in a set. Extensions include intuitionistic fuzzy sets (Atanassov, 1986, 1989), interval-valued fuzzy sets (Gorzałczany, 1987), Pythagorean fuzzy sets (Yager, 2014), Fermatean fuzzy sets (Senapati & Yager, 2020), *et cetera*. Rough set theory (Pawlak, 1982) is based on the notion of indistinguishable elements, which produces formal approximations of the conventional or crisp sets. The ethos of soft set theory, by contrast, consists of providing alternative "approximate descriptions" of the elements in a set. These descriptions are made by a parameterized binary choice: for every relevant attribute or parameter, a given element either satisfies it or not.

These disciplines have not followed independent paths. Yao (1998) gave a comparative study of fuzzy sets and rough sets. Soft sets and rough sets were combined in Feng et al. (2011). Soft sets and fuzzy sets merged into fuzzy soft sets in Maji et al. (2001a), and intuitionistic fuzzy soft sets were defined too (Maji et al., 2001b). Rough fuzzy sets and fuzzy rough sets were defined in Dubois and Prade (1990).

In addition to the theoretical development of these models, applications to decision making were soon available in many areas. Let us focus on medicine and clinical diagnosis for the moment. Then we can list applications of soft sets (Alcantud et al., 2015, 2019; Sreedevi et al., 2016), fuzzy soft sets (Çelik & Yamak, 2013; Xiao, 2018), intuitionistic fuzzy soft sets (Muthukumar & Sai Sundara Krishnan, 2016), fuzzy sets (Adlassnig, 1986), intuitionistic fuzzy sets (De et al., 2001), and rough sets (Pawlak et al., 1986; Stefanowski & Słowiński, 1997). But of course, the decision making practice with these models, their further generalizations, and hybrid structures is not limited to medicine. Other applied contributions include Ahmed et al. (2019), Alcantud and Calle (2017), Alcantud et al. (2017), Chiang et al. (2014), Kahraman and Kaya (2010), Pozna and Precup (2014).

Let us now return to the main point at issue: examples abound where the parameterized description of the universe of alternatives is not binary but multinary. Several real examples that fit into the N-soft set model were shown in the founding Fatimah et al. (2017) and afterwards in Alcantud et al. (2020). For completeness we proceed to give two more real examples of this classification scheme:

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PROFICIENT USER	C2	Can understand with ease virtually everything heard or read. Can summarise information from different spoken and written sources, reconstructing arguments and accounts in a coherent presentation. Can express him/herself spontaneously, very fluently and precisely, differentiating finer shades of meaning even in more complex situations.
	C1	Can understand a wide range of demanding, longer texts, and recognise implicit meaning. Can express him/herself fluently and spontaneously without much obvious searching for expressions. Can use language flexibly and effectively for social, academic and professional purposes. Can produce clear, well-structured, detailed text on complex subjects, showing controlled use of organisational patterns, connectors and cohesive devices.
INDEPENDENT USER	B2	Can understand the main ideas of complex text on both concrete and abstract topics, including technical discussions in his/her field of specialisation. Can interact with a degree of fluency and spontaneity that makes regular interaction with native speakers quite possible without strain for either party. Can produce clear, detailed text on a wide range of subjects and explain a viewpoint on a topical issue giving the advantages and disadvantages of various options.
	B1	Can understand the main points of clear standard input on familiar matters regularly encountered in work, school, leisure, etc. Can deal with most situations likely to arise whilst travelling in an area where the language is spoken. Can produce simple connected text on topics which are familiar or of personal interest. Can describe experiences and events, dreams, hopes & ambitions and briefly give reasons and explanations for opinions and plans.
BASIC USER	A2	Can understand sentences and frequently used expressions related to areas of most immediate relevance (e.g. very basic personal and family information, shopping, local geography, employment). Can communicate in simple and routine tasks requiring a simple and direct exchange of information on familiar and routine matters. Can describe in simple terms aspects of his/her background, immediate environment and matters in areas of immediate need.
	A1	Can understand and use familiar everyday expressions and very basic phrases aimed at the satisfaction of needs of a concrete type. Can introduce him/herself and others and can ask and answer questions about personal details such as where he/she lives, people he/she knows and things he/she has. Can interact in a simple way provided the other person talks slowly and clearly and is prepared to help.

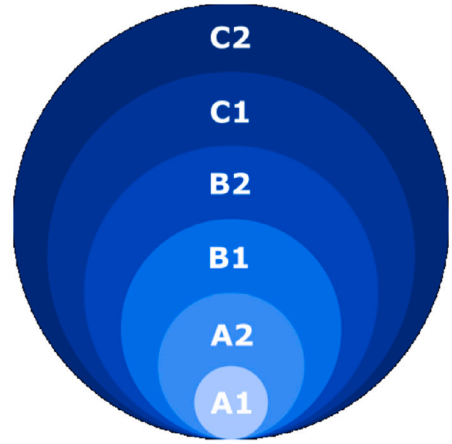

Fig. 1. Global scale (Common Reference Levels) in the Common European Framework of Reference for Languages (Council of Europe homepage, 2020).
Source: Council of Europe.

(1) The *Common European Framework of Reference for Languages: Learning, teaching, assessment* or CEFR uses an ordered scale (Council of Europe homepage, 2020). Quoting from the Council of Europe webpage, CEFR "was designed to provide a transparent, coherent and comprehensive basis for the elaboration of language syllabuses and curriculum guidelines, the design of teaching and learning materials, and the assessment of foreign language proficiency." Fig. 1 describes its global scale which consists of six ordered levels of proficiency in any foreign language.

(2) For measuring generic health status, EuroQol's EQ-5D questionnaire uses five dimensions, namely, mobility, self-care, usual activities, pain/discomfort, and anxiety/depression (EuroQol Research Foundation, 2020). In the original EQ-5D-3L version, three rated levels codify: "having no problems" with 1, "having some problems" with 2, and "having extreme problems" with 3. Then it defines a person's health status by a 5-digit number that ranges from 11111 (no problems in all dimensions) to 33333 (extreme problems in all dimensions). Currently, a five-level scale is used to correct the reported ceiling effects for the EQ-5D-3L. The scale in EQ-5D-5L represents "having no problems", "having slight problems", "having moderate problems", "having severe problems", and "being unable to do/having extreme problems". The five dimensions were kept unaltered.

Once we have motivated the model, the first goal of this work is to introduce the problem of aggregating information that comes in the form of multinary descriptions of a common universe. Formally speaking, in this paper we first raise and tackle the problem of aggregation of N-soft sets. This issue is of paramount importance because multi-agent information is typically condensed in a single group dataset as a first step in multi-agent decision making. Aggregation theory is well developed, and specific methods abound in other models of vague knowledge: fuzzy sets, hesitant fuzzy sets (Bedregal et al., 2014), intuitionistic fuzzy sets (Liao & Xu, 2014), Fermatean fuzzy sets (Senapati & Yager, 2019), necessary and possible hesitant fuzzy sets (Alcantud & Giarlotta, 2019), and so on. However, to the best of our knowledge, this theory has been neglected in the analysis of soft sets, in all likelihood because the problem is too naive.[1]

We will show here that by contrast to that simple case, the idea of aggregating N-soft sets is far-reaching and yields a rich setting for future research. One of the reasons is its versatility, which owes to the fact that N-soft sets have been extended to account for fuzziness (fuzzy N-soft sets), hesitancy (hesitant N-soft sets), and other variations of uncertain knowledge (Akram et al., 2018, 2019a, 2019b, 2019c; Chen et al., 2020; Kamacı & Petchimuthu, 2020; Liu et al., 2020; Zhang et al., 2020). In particular, OWA (for ordered weighted averaging) operators defined by Yager (1988) produce aggregation mechanisms for aggregating N-soft sets both when they embody purely ordinal information or a proper measure of the degree of satisfaction of the attributes. We shall use a real case study to justify their relevance in this particular setting too. Another reason is that there exist prospective applications of a theory of aggregation to decision-making, which owes to the fact that this field has already produced some advances in these models. Actually, here we avail ourselves of our novel tool in order to produce three flexible and adaptable procedures for multi-agent decision-making. These proposals are then applied to practical situations for illustration.

The outline of this work is as follows. Section 2 gives preliminaries concerning both the models that we use and some technical findings about the extension of OWA operators to complete lattices. Section 3 establishes the three basic settings for aggregating N-soft sets and discusses their scopes of application. They depend upon the structure of the output, namely, it might be an N'-soft set, a fuzzy N-soft set, or a hesitant N-soft set. We develop a full theory of OWA operators in the first instance (i.e., the aggregation of N-soft sets by another N-soft set or by an N'-soft set with $N' \neq N$). And we shall define a novel OWA-inspired score that produces a global evaluation of each alternative directly from the multi-agent multinary information: we call it the weighted average OWA (or WAOWA) score. The other two instances are only sketched and left for future theoretical development. A running made up example illustrates the practical possibilities of our proposal. But we also use a real case study to explain how the theory can be adapted to down-to-earth uses. A discussion of our achievements and future extensions of the aggregation results ends this section. Section 4 establishes the three adaptable algorithms for multi-agent decision making with N-soft sets that we put forward. They are different in the structure of their aggregate output, namely, we first model it by another N-soft set (Section 4.1), then we take advantage of the novel WAOWA scores (Section 4.2), and finally we proceed when we model the output by a hesitant N-soft set (Section 4.3). We recall our running example in order to prove the feasibility and adaptability of these algorithms. Then Section 4.4 illustrates the techniques in Section 4.2 with a real-life example. Section 5 offers some concluding remarks, as well as lines for future research. An Appendix contains tables corresponding

[1] Other extended soft set models allow for a worthy mathematical treatment of aggregation because they provide a richer environment. Çagman et al. (2011) pose the problem of aggregating fuzzy soft sets into fuzzy soft sets. Hayat et al. (2020) refer to aggregation of group-based generalized intuitionistic fuzzy soft sets.

Table 1
Tabular form of a general N-soft set.

(F,T,N)	t_1	t_2	$\ldots\ldots$	t_q
o_1	r_{11}	r_{12}	$\ldots\ldots$	r_{1q}
o_2	r_{21}	r_{22}	$\ldots\ldots$	r_{2q}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	r_{p1}	r_{p2}	$\ldots\ldots$	r_{pq}

to the real case study in Section 4.4. The datasets of this example are publicly available at the Github repository to facilitate reproducibility and replicability of our numerical exercise.

2. Preliminaries

Henceforth, $\mathcal{F}(X)$ denotes the set of all fuzzy sets on the crisp set X. As is standard, X^k denotes the Cartesian product of k copies of the set X, and $\mathscr{P}(X)$ denotes the set of all subsets of X; whereas $\mathscr{P}^*(X)$ stands for the set of all non-empty subsets of X.

The fundamental definitions and facts that we need to state our contribution concern two separate fields. First, in Section 2.1 we brief the reader about the characterization of alternatives by multinary evaluations of their attributes. Secondly, in Section 2.2 we recall some facts about the construction of OWA operators on complete lattices.

2.1. Definitions concerning the model

This subsection presents the models that we focus on. We fix the following common elements: O is the universe of elements under consideration, T is a set of attributes (possibly $T \subseteq P$, with P a 'universal' set of attributes) and $G = \{0, 1, 2, \ldots, N-1\}$ is a set of ordered grades with $N \in \{2, 3, \ldots\}$.[2]

Most of our contribution concerns the notion of N-soft set. They generalize soft sets (which appear when $N = 2$):

Definition 1 (*Fatimah et al., 2017*)**.** A triple (F,T,N) is an N-soft set on O if F is mapping from T to $2^{O\times G} = \mathscr{P}(O \times G)$ with the property that for each $t \in T$ and $o \in O$, there exists a unique $(o, g_t) \in O \times G$ such that $(o, g_t) \in F(t)$, $g_t \in G$.

The set of all N-soft sets on O will be denoted as $\mathcal{NS}(O)$.

An N-soft set captures a multinary parameterized description of the universe of alternatives. Suppose that we have (F,T,N), an N-soft set on O. Then, for any attribute $t \in T$, every object o from O receives exactly one evaluation from the space of grades G. It is the unique g_t for which $(o, g_t) \in F(t)$. Then, it is customary to write $F(t)(o) = g_t$ to mean that $(o, g_t) \in F(t)$.

When both $O = \{o_i : i = 1, 2, \ldots, p\}$ and $T = \{t_j : j = 1, 2, \ldots, q\}$ are finite, one can represent the N-soft set (F,T,N) in tabular form. This is expressed by Table 1 where we write down every possible $r_{ij} = F(t_j)(o_i) \in G$.

Basic examples of N-soft sets include two extreme cases:

(1) The N-soft set with the property $(o, 0) \in F(t)$, for each $o \in O$, $t \in T$ is denoted $\mathbf{0}_{T,N}$. It is the N-soft set whose elements are always parameterized by the attainment of 0 grade for every attribute in T. In this sense $\mathbf{0}_{T,N}$ is the worst possible assessment (if the attributes in T are all beneficial).

(2) The N-soft set with the property $(o, N-1) \in F(t)$, for each $o \in O$, $t \in T$ is denoted $\mathbf{1}_{T,N}$. It is the N-soft set whose elements are always parameterized by the attainment of the highest $N-1$ grade for every attribute in T. Therefore, $\mathbf{1}_{T,N}$ is the best possible assessment (if the attributes in T are all beneficial).

In order to quantify the value of each alternative, Fatimah et al. (2017) define the extended weighted choice value (EWCV) of o_i in (F,T,N). Suppose that we have a vector of weights $(w_1, \ldots, w_q)$ associated with the attributes in T, i.e., $w_i \geqslant 0$ for each i and $w_1 + \cdots + w_q = 1$. Then we use:

$$\sigma_i^w = \sum_{j=1}^{q} w_j r_{ij}. \qquad (1)$$

In contexts with a multiplicity of N-soft sets this score will be generalized with the help of OWA operators (cf., Section 3.2 below).

The next remark will be useful in the analysis of the multi-agent setting:

Remark 1. Suppose that we have a finite collection where (F_1, T, N_1) is an N_1-soft set, …, (F_k, T, N_k) is an N_k-soft set. They can be regarded as a collection of N-soft sets, where $N = \max\{N_1, \ldots, N_k\}$ (cf., Fatimah et al., 2017, Remark 2). So, with finite families of triples that satisfy Definition 1, we do not lose generality if we consider them as members of a common collection of N-soft sets $\mathcal{NS}(O)$.

Some N-soft sets with common option and attribute sets assign better assessments on every possible alternative, whatever the option, than other N-soft sets. When $\{(F_1,T,N),(F_2,T,N)\} \subseteq \mathcal{NS}(O)$, we write $(F_1,T,N) \leqslant (F_2,T,N)$ if and only if for each $t \in T$ and $o \in O$, $(o, g_t) \in F_1(t)$ and $(o, g_t') \in F_2(t)$ implies $g_t \leqslant g_t'$.

Akram et al. (2018, 2019a) define fuzzy and hesitant extensions of Definition 1. The set of all fuzzy N-soft sets on O will be denoted as $\mathcal{FN}(O)$. The set of all hesitant N-soft sets on O will be denoted as $\mathcal{HN}(O)$. For any $(H,T,N) \in \mathcal{HN}(O)$, the evaluation of o_i in terms of the attribute t_j is an ordered list h_{ij} of $\ell(ij)$ different ratings from G. These ordered lists are called hesitant N-tuples or HNTs (Akram et al., 2019a).

Table 2
Tabular representation of a fuzzy N-soft set.

$(\mu,(F,T,N))$	t_1	t_2	$\ldots\ldots$	t_q
o_1	$\langle r_{11}, \mu_{11}\rangle$	$\langle r_{12}, \mu_{12}\rangle$	$\ldots\ldots$	$\langle r_{1q}, \mu_{1q}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	$\langle r_{p1}, \mu_{p1}\rangle$	$\langle r_{p2}, \mu_{p2}\rangle$	$\ldots\ldots$	$\langle r_{pq}, \mu_{pq}\rangle$

Tables 2 and 3 respectively show the standard tabular representations of a fuzzy N-soft set and a hesitant N-soft set with its constituent HNTs. We refer to Akram et al. (2018, 2019a) for the formal definitions of these two models.

In addition, fuzzy N-soft sets and hesitant N-soft sets admit an alternative functional representation. These facts are explained and illustrated in Akram et al. (2018, 2019a), which also give details about the intuitive interpretations of these two models.

In order to compare HNTs by their performance, their scores are a natural tool which replicates the important role of scores in hesitant fuzzy sets (Xu & Xia, 2011) and extended settings (Alcantud & Giarlotta, 2019):

Definition 2 (*Akram et al., 2019a*)**.** A score for hesitant N-tuples is a mapping $s : \mathcal{P}^*(G) \longrightarrow \mathbb{R}^+$, with the following properties:

(1) $s(\{0\}) = 0$, $s(\{1\}) = 1$, and $s(\{N-1\}) = N-1$;
(2) *Boundedness*: for all $h \in \mathcal{P}^*(G)$, we have $s(\min(h)) \leqslant s(h) \leqslant s(\max(h))$.

[2] The figures or numbers $0, 1, 2, \ldots, N-1$ are just symbols that we use for notational convenience. They may be any other ordered scale, e.g., $A_1, A_2, B_1, B_2, C_1, C_2$ in CEFR, or stars in hotels or movie ratings (Akram et al., 2018, 2019a; Alcantud et al., 2020).

Table 3
Tabular representation of a hesitant N-soft set. Each h_{ij} is a HNT.

(H,T,N)	t_1	t_2	…	t_q
o_1	$h_{11}=\{\eta^1_{11},\eta^2_{11},\ldots,\eta^{\ell(11)}_{11}\}$	$h_{12}=\{\eta^1_{12},\eta^2_{12},\ldots,\eta^{\ell(12)}_{12}\}$	…	$h_{1q}=\{\eta^1_{1q},\eta^2_{1q},\ldots,\eta^{\ell(1q)}_{1q}\}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	$h_{p1}=\{\eta^1_{p1},\eta^2_{p1},\ldots,\eta^{\ell(p1)}_{p1}\}$	$h_{p2}=\{\eta^1_{p2},\eta^2_{p2},\ldots,\eta^{\ell(p2)}_{p2}\}$	…	$h_{pq}=\{\eta^1_{pq},\eta^2_{pq},\ldots,\eta^{\ell(pq)}_{pq}\}$

Example 1. Examples of scores for HNTs include the Min, Max, Arithmetic and Geometric scores (Akram et al., 2019a). To define them, let us fix the ordered list $h=\{\eta^{(1)},\eta^{(2)},\ldots,\eta^{(\ell)}\}\in\mathcal{P}^*(G)$:

(1) The Min score uses the formula $s_m(h)=\eta^{(1)}$.
(2) The Max score uses the formula $s_M(h)=\eta^{(\ell)}$.
(3) The Arithmetic score uses the formula $s_a(h)=(\eta^{(1)}+\eta^{(2)}+\cdots+\eta^{(\ell)})/\ell$.
(4) The Geometric score uses the formula $s_g(h)=(\eta^{(1)}\cdot\eta^{(2)}\cdot\ldots\cdot\eta^{(\ell)})^{1/\ell}$.

Now we proceed to recall some basic facts about aggregation operators. We are especially concerned with the OWA operator beyond the standard numerical case that we recall below in Section 3.2.

We recall that Z^k denotes the k-ary Cartesian power of the set Z. Thus for example, $\mathcal{N}\mathcal{S}(O)^k=\mathcal{N}\mathcal{S}(O)\times\overset{k}{\ldots}\times\mathcal{N}\mathcal{S}(O)$.

2.2. Definitions concerning the aggregators: OWA operators on complete lattices

Lizasoain and Moreno (2013) extend the idea of the OWA operator (Yager, 1988) that will be recalled in Section 3.1, to complete lattices endowed with a t-norm and a t-conorm. Their approach is quite general; however, we only need to study a special case because we are concerned with $Z=G$ which is linearly ordered. This restriction will permit us to be much more focused on the features of the extension of the OWA concept. The details are as follows.

Let us fix Z, a linearly ordered set with top element 1_Z and bottom element 0_Z. A k-ary aggregation function on Z is a mapping $M: Z^k\longrightarrow Z$ such that:

(1) $M(z_1,\ldots,z_k)\leqslant M(z'_1,\ldots,z'_k)$ when $z_i\leqslant z'_i$ for each $i=1,\ldots,k$.
(2) $M(0_Z,\overset{k}{\ldots},0_Z)=0_Z$ and $M(1_Z,\overset{k}{\ldots},1_Z)=1_Z$.

In addition, M is idempotent if $M(z,\ldots,z)=z$ for each $z\in Z$, and it is symmetric if $M(z_1,\ldots,z_k)=M(z_{\sigma(1)},\ldots,z_{\sigma(k)})$ for every $(z_1,\ldots,z_k)\in Z^k$ and every permutation σ of $\{1,\ldots,k\}$.

Let us now fix a t-norm $\mathbf{T}$ (commutative, associative, increasing in each component, and with a neutral element 1_Z), and a t-conorm $\mathbf{S}$ on Z (commutative, associative, increasing in each component, and with a neutral element 0_Z).

Lizasoain and Moreno (2013) first proceed with the construction of a linearly ordered vector associated with every $(z_1,\ldots,z_k)\in Z^k$. When $\{z_1,\ldots,z_k\}$ is linearly ordered, that vector coincides with $(z_{\sigma(1)},\ldots,z_{\sigma(k)})$ where σ is a permutation of $\{1,\ldots,k\}$ for which $z_{\sigma(1)}\geqslant\ldots\geqslant z_{\sigma(k)}$, by Lizasoain and Moreno (2013, Lemma 3.1). Thus in our case of application, this first step replicates Yager's construction of an ordered rearrangement of $(z_1,\ldots,z_k)$.

Secondly, Lizasoain and Moreno (2013, Definition 3.3) define a distributive weighting vector $(\alpha_1,\ldots,\alpha_k)\in Z^k$. It is a vector that satisfies two properties:

(i) $\mathbf{S}(\alpha_1,\ldots,\alpha_k)=1_Z$, and
(ii) $z=\mathbf{T}(z,\mathbf{S}(\alpha_1,\ldots,\alpha_k))=\mathbf{S}(\mathbf{T}(z,\alpha_1),\ldots,\mathbf{T}(z,\alpha_k))$ for any $z\in Z$.

Importantly, when Z is a complete lattice then any $(\alpha_1,\ldots,\alpha_k)\in Z^k$ for which $\alpha_i=1_Z$ for some i is a distributive weighting vector (Lizasoain & Moreno, 2013, Remark 3.4). This property will allow us to define several OWA operators without effort.

Table 4
A list of k N-soft sets $\{(F_1,T,N),\ldots,(F_k,T,N)\}$ on $O=\{o_1,\ldots,o_p\}$ with a common set of attributes $T=\{t_1,\ldots,t_q\}$. They are represented in tabular form.

(F_1,T,N)	t_1	……	t_q		(F_k,T,N)	t_1	……	t_q
o_1	r^1_{11}	……	r^1_{1q}		o_1	r^k_{11}	……	r^k_{1q}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	……	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	r^1_{p1}	……	r^1_{pq}		o_p	r^k_{p1}	……	r^k_{pq}

Finally, Lizasoain and Moreno (2013, Definition 3.5) define the k-ary aggregation operator F_α associated with a distributive weighting vector $\alpha=(\alpha_1,\ldots,\alpha_k)\in Z^k$. It takes advantage of the concepts defined above for the linearly ordered case:

$$\begin{aligned} F_\alpha:\ Z^k &\longrightarrow Z\\ (z_1,\ldots,z_k) &\longmapsto \mathbf{S}(\mathbf{T}(z_{\sigma(1)},\alpha_1),\ldots,\mathbf{T}(z_{\sigma(k)},\alpha_k)) \end{aligned} \tag{2}$$

Then F_α is a symmetric, idempotent k-ary aggregation function on Z called k-ary OWA operator (Lizasoain & Moreno, 2013, Proposition 3.7).

3. New concepts and results about OWA operators and aggregation

A great deal has been written about the issue of aggregating multi-agent information on a problem into a collective output. Although there seems to be no attempt to aggregate soft sets because their structure is too simplistic, the idea of aggregation of N-soft sets is less trivial and it has not been studied before.

The purpose of the current section is to establish a formal context for the aggregation of information when it comes in the form of n-ary parameterized descriptions of a universe of objects O, in terms of a set of attributes T. We will show that this problem is quite challenging and that it can inspire future research on the topic.

It is important to clarify on the outset that, in a context of multi-attribute and multi-agent decision-making, we can assume that the set of attributes T is common to all N-soft sets. In addition, we do not lose generality if we assume that N, which defines the set of available grades, is also fixed. First, because it is also natural in a context of multi-agent decision-making where the advisors must have a common language to express their opinions. Secondly, because Remark 1 gives formal support to this claim.

In conclusion, it is just natural to assume that throughout this section, the input in our aggregation problem has the form of the data in Table 4.

Let us give a real example of this situation. It will be thoroughly reconsidered in Section 4.4.

Example 2. In competitive diving, the scoring system of individual events works as follows. There are $k=7$ judges that award points from 0 (completely fail) to 10. Judges score in whole or half points thus $G=\{0,0.5,1,1.5,\ldots,9.5,10\}$, that we can for simplicity transform into $G=\{0,1,2,\ldots,19,20\}$ (we just divide by 0.5 each score). Thus $N=21$ which means a fine grading. The judges score a dive based on four elements, namely, approach to the dive, take-off from the platform or springboard, execution (flight through the air) and entry into water. Each diver performs a fixed number of dives.

A thornier problem is the choice of the form of the output, because N-soft sets have been expanded in different reasonable directions. The next subsections study four different alternatives, namely, the output is also an N-soft set (Section 3.1), a general N'-soft set with possibly $N' \neq N$ (Section 3.2), a fuzzy N-soft set (Section 3.3), and a hesitant N-soft set (Section 3.4). Then, we discuss their roles in this theory (Section 3.5).

Remark 2. Henceforth, we are imposing one reasonable constraint in this approach: the global assessment of the degree of satisfaction of one attribute, for a given alternative, is independent of the evaluations of other attributes and alternatives. In other words, for each alternative $o_i \in O$ and attribute $t_j \in T$, the assessments $r^1_{ij}, \dots, r^k_{ij} \in G$ are all the information that we need to produce the collective assessment of the performance of o_i in terms of t_j. Any other information in the inputs is disregarded for the purpose of computing this rating.

3.1. OWA operators defined on N-soft sets

This section explores the aggregation of the data in Table 4 by another N-soft set (F, T, N). This output is expressed in tabular form as in Table 1, thus we intend to produce $r_{ij} = F(t_j)(o_i) \in G$ for each possible i, j as a result of the aggregation of the elements in Table 4. We incorporate the constraint in Remark 2.

Put shortly: for each alternative $o_i \in O$ and each attribute $t_j \in T$, we input k assessments $r^1_{ij}, \dots, r^k_{ij} \in G$ and *exactly with this information*, we need to produce an aggregate assessment $r_{ij} \in G$.

The main concept that we introduce to model the aggregation of several N-soft sets by an N-soft set is our next definition:

Definition 3. A k-ary aggregation function on N-soft sets (with a common set of attributes T) is a mapping $M : \mathcal{NS}(O)^k \longrightarrow \mathcal{NS}(O)$ such that:

(1) $M((F_1, T, N), \dots, (F_k, T, N)) \leqslant M((F'_1, T, N), \dots, (F'_k, T, N))$ if $(F_i, T, N) \leqslant (F'_i, T, N)$ for each $i = 1, \dots, k$.
(2) $M(\mathbf{0}_{T,N}, \overset{k}{\dots}, \mathbf{0}_{T,N}) = \mathbf{0}_{T,N}$ and $M(\mathbf{1}_{T,N}, \overset{k}{\dots}, \mathbf{1}_{T,N}) = \mathbf{1}_{T,N}$.

In addition, M is idempotent if $M((F, T, N), \dots, (F, T, N)) = (F, T, N)$ for each $(F, T, N) \in \mathcal{NS}(O)$. M is symmetric if $M((F_{\sigma(1)}, T, N), \dots, (F_{\sigma(k)}, T, N)) = M((F_1, T, N), \dots, (F_k, T, N))$ when $(F_1, T, N), \dots, (F_k, T, N) \in \mathcal{NS}(O)$ and σ is a permutation of $\{1, \dots, k\}$.

This interesting case will benefit from the developments in Section 2.2. It will allow us to define an OWA operator on N-soft sets by the recourse to their tabular forms. Let us therefore fix a t-norm $\mathbf{T}$ and a t-conorm $\mathbf{S}$ on the set of grades $G = \{0, 1, 2, \dots, N-1\}$, which is linearly ordered and has a top element $1_G = N-1$ and a bottom element $0_G = 0$. For example, the t-norm might be the meet or $\wedge$ operator, and the t-conorm might be the join or $\vee$ operator. Also, let us fix a distributive weighting vector $(\alpha_1, \dots, \alpha_k) \in G^k$. For example, it could be any vector from G^k such that at least one of its components is $1_G = N - 1$.

With all these premises, Section 2.2 recalls the k-ary aggregation operator F_α associated with $\alpha = (\alpha_1, \dots, \alpha_k) \in G^k$: when for every $(g_1, \dots, g_k) \in G^k$, σ is a permutation of $\{1, \dots, k\}$ for which $g_{\sigma(1)} \geqslant \dots \geqslant g_{\sigma(k)}$,

$$\begin{aligned} F_\alpha :\ & G^k \longrightarrow G \\ & (g_1, \dots, g_k) \longmapsto \mathbf{S}(\mathbf{T}(g_{\sigma(1)}, \alpha_1), \dots, \mathbf{T}(g_{\sigma(k)}, \alpha_k)). \end{aligned} \tag{3}$$

We can apply the operator F_α cell-by-cell to our input (which is given by the data in Table 4) in order to define an OWA operator M_α on N-soft sets. By the properties of F_α, we can assure that M_α is a symmetric, idempotent k-ary aggregation function on N-soft sets. We call it an OWA operator on N-soft sets.

Its application to our problem uses the practical expression: when for each $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$, σ is a permutation of $\{1, \dots, k\}$ with $r^{\sigma(1)}_{ij} \geqslant \dots \geqslant r^{\sigma(k)}_{ij}$

$$F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \mathbf{S}(\mathbf{T}(r^{\sigma(1)}_{ij}, \alpha_1), \dots, \mathbf{T}(r^{\sigma(k)}_{ij}, \alpha_k)). \tag{4}$$

Remember that for the sake of applicability we look at the tabular form of the N-soft sets, and then $(r^1_{ij}, \dots, r^k_{ij})$ are the k degrees of attainment of attribute t_j by alternative o_i.

Also for the sake of applicability, our next example gives several noteworthy expressions for particular OWA operators on N-soft sets:

Example 3. Suppose that in the construction in this section, the t-norm is the meet or $\wedge$ operator, and the t-conorm is the join or $\vee$ operator.

Then for any distributive weighting vector $(\alpha_1, \dots, \alpha_k) \in G^k$, M_α derives from $F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \max\left(\min(r^{\sigma(1)}_{ij}, \alpha_1), \dots, \min(r^{\sigma(k)}_{ij}, \alpha_k)\right)$ for every k-tuple $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$, with $r^{\sigma(1)}_{ij} \geqslant \dots \geqslant r^{\sigma(k)}_{ij}$ for the permutation σ of $\{1, \dots, k\}$.

We obtain some remarkable instances as a function of the vector α:

(1) When $\alpha = (N-1, 0, \dots, 0)$, M_α aggregates the N-soft sets according to the maximum evaluations that they attain (for each fixed option and attribute):
$F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \max\left(\min(r^{\sigma(1)}_{ij}, N-1), \min(r^{\sigma(2)}_{ij}, 0), \dots, \min(r^{\sigma(k)}_{ij}, 0)\right) = \max(r^{\sigma(1)}_{ij}, 0, \dots, 0) = r^{\sigma(1)}_{ij} = \max\{r^1_{ij}, \dots, r^k_{ij}\}$, for every $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$.
(2) When $\alpha = (0, \dots, 0, N-1)$, M_α aggregates the N-soft sets according to the minimum evaluation that they attain (for each fixed option and attribute):
$F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \max\left(\min(r^{\sigma(1)}_{ij}, 0), \dots, \min(r^{\sigma(k-1)}_{ij}, 0), \min(r^{\sigma(k)}_{ij}, N-1)\right) = \max(0, \dots, 0, r^{\sigma(k)}_{ij}) = r^{\sigma(k)}_{ij} = \min\{r^1_{ij}, \dots, r^k_{ij}\}$, for every $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$.
(3) When $\alpha = (0, N-1, 0, \dots, 0)$, M_α aggregates the N-soft sets by taking the second best evaluation that they attain (for each fixed option and attribute). This means that we truncate the top evaluation, and then we take the best evaluation of the remaining figures. To prove it, observe: $F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \max\left(\min(r^{\sigma(1)}_{ij}, 0), \min(r^{\sigma(2)}_{ij}, N-1), \min(r^{\sigma(3)}_{ij}, 0), \dots, \min(r^{\sigma(k)}_{ij}, 0)\right)$ $= \max(0, r^{\sigma(2)}_{ij}, 0, \dots, 0) = r^{\sigma(2)}_{ij}$ for every k-tuple $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$.
(4) When $\alpha = (0, \dots, 0, N-1, 0)$, M_α aggregates the N-soft sets by taking the second worst evaluation that they attain (for each fixed option and attribute). This means that we truncate the bottom evaluation, and then we take the worst evaluation of the remaining figures. To prove it, observe:
$F_\alpha(r^1_{ij}, \dots, r^k_{ij}) = \max\left(\min(r^{\sigma(1)}_{ij}, 0), \dots, \min(r^{\sigma(k-2)}_{ij}, 0), \min(r^{\sigma(k-1)}_{ij}, N-1), \min(r^{\sigma(k)}_{ij}, 0)\right) = \max(0, \dots, 0, r^{\sigma(k-1)}_{ij}, 0) = r^{\sigma(k-1)}_{ij}$ for every k-tuple $(r^1_{ij}, \dots, r^k_{ij}) \in G^k$.
(5) When k is odd, $\alpha = (0, \overset{(k-1)/2}{\dots}, 0, N-1, 0, \overset{(k-1)/2}{\dots}, 0)$ evaluates the alternatives with the median value of their evaluations for each attribute.

An example clarifies the application of the OWA operators on N-soft sets:

Example 4. Five agents evaluate the alternatives $O = \{o_1, o_2, o_3, o_4\}$ in terms of their degree of satisfaction of the attributes $T = \{t_1, t_2, t_3\}$. They can use four possible ordered degrees ($N = 4$).

Table 5 displays the data that they submit.

In order to aggregate this information, we place ourselves in the case of Example 3. Several outputs arise, depending on the choice of the vector α:

(1) When $\alpha = (3, 0, 0, 0, 0)$, the OWA operator M_α aggregates the 4-soft sets by $(M^1_\alpha, T, 4)$ in Table 6.

Table 5
Tabular representation of the 4-soft sets $(F_i, T, 4)$, $i = 1, \ldots, 5$, in the input of Example 4.

F_1	t_1	t_2	t_3
o_1	1	3	2
o_2	1	2	0
o_3	2	0	1
o_4	1	1	3
F_2	t_1	t_2	t_3
o_1	1	2	2
o_2	2	2	2
o_3	2	0	0
o_4	1	2	3
F_3	t_1	t_2	t_3
o_1	2	2	3
o_2	2	1	1
o_3	3	1	2
o_4	0	1	2
F_4	t_1	t_2	t_3
o_1	2	3	1
o_2	1	2	0
o_3	3	1	1
o_4	1	1	2
F_5	t_1	t_2	t_3
o_1	1	3	2
o_2	2	3	1
o_3	3	1	1
o_4	0	1	3

Table 6
Tabular representation of the 4-soft sets $(M^i_\alpha, T, 4)$, $i = 1, \ldots, 5$, in the respective outputs of Example 4.

M^1_α	t_1	t_2	t_3
o_1	2	3	3
o_2	2	3	2
o_3	3	1	2
o_4	1	2	3
M^2_α	t_1	t_2	t_3
o_1	1	2	1
o_2	1	1	0
o_3	2	0	0
o_4	0	1	2
M^3_α	t_1	t_2	t_3
o_1	2	3	2
o_2	2	2	1
o_3	3	1	1
o_4	1	1	1
M^4_α	t_1	t_2	t_3
o_1	1	2	2
o_2	1	2	0
o_3	2	0	1
o_4	0	1	2
M^5_α	t_1	t_2	t_3
o_1	1	3	2
o_2	2	2	1
o_3	3	1	1
o_4	1	1	3

(2) When $\alpha = (0, 0, 0, 0, 3)$, the OWA operator M_α aggregates the 4-soft sets by $(M^2_\alpha, T, 4)$ in Table 6.
(3) When $\alpha = (0, 3, 0, 0, 0)$, the OWA operator M_α aggregates the 4-soft sets by $(M^3_\alpha, T, 4)$ in Table 6.
(4) When $\alpha = (0, 0, 0, 3, 0)$, the OWA operator M_α aggregates the 4-soft sets by $(M^4_\alpha, T, 4)$ in Table 6.
(5) When $\alpha = (0, 0, 3, 0, 0)$, the OWA operator M_α aggregates the 4-soft sets by $(M^5_\alpha, T, 4)$ in Table 6.

3.2. Aggregation of N-soft sets by OWA operators: The general case and a novel score

In Section 3.1 we have set forth the aggregation of N-soft sets that produce merely ordinal information. Then the natural output is another N-soft set. We have taken advantage of the developments presented in Section 2.2. However OWA operators are much more powerful a tool than the ordinal formulation that has been used for that case. And in fact we need to resort to the standard application of OWA operators or closely related procedures in examples like sports competitions (e.g., Example 2) where the grades convey measurable and comparable information.

To see why and how we can do this, let us now take another look at the aggregation procedure in the real case of Example 2. The rules and regulations of diving establish a particular formal mechanism that can be implemented via some modified OWA operators.

The definition of the OWA operator is as follows (Yager, 1988). Suppose that $\mathbf{w} = (w_1, \ldots, w_k) \in [0,1]^k$ is a weighting vector such that $\sum_{i=1}^k w_i = 1$. The OWA operator associated with $\mathbf{w}$ is the function $F^{\mathbf{w}} : \mathbb{R}^k \longrightarrow \mathbb{R}$ defined by

$$F^{\mathbf{w}}(r_1, \ldots, r_k) = \sum_{i=1}^{k} w_i\, b_i \text{ for each } (r_1, \ldots, r_k) \in \mathbb{R}^k, \quad (5)$$

where b_i is the ith largest element in the collection of (possibly repeated) values $\{r_1, \ldots, r_k\}$.

We can now use (a trivial modification of) OWAs to justify the aggregation mechanism in Example 2 as follows:

Example 5. In competitive diving, the scoring system of individual events inputs the scores from $k = 7$ judges. We have interpreted these scores as grades from $G = \{0, 1, 2, \ldots, 19, 20\}$.

For each dive, the top two scores and the bottom two scores are discarded. Then the remaining three scores are added together. Afterwards they are multiplied by the dive's difficulty rating, which is known as the degree of difficulty (DD) or 'tariff'. This correction is totally objective, it is independent of the aggregation mechanism of subjective information, therefore we can postpone its application. At any rate, we would only need to multiply the output by the corresponding DD or tariff. Rules and Regulations can be obtained from the Encyclopædia Britannica article (Grannan, 2021). Other references for diving can be found at Borg and Love (2018), Emerson et al. (2009), Kramer (2017), Wnuk and Soatto (2010).

It is apparent that the sum of the three scores of each dive produces an aggregate 61-soft set. The aggregate score of each dive is a grade that can be computed by an OWA operator as in Eq. (5) with the collection of weights $(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$, multiplied by 3. We would also multiply by the corresponding DD in order to obtain the final score. Scores have been 'normalized' so we also need to divide each final score by 2 if we want to retrieve the standard score that we observe in the competition. OWAs are scale invariant, so this back and forth conversion does not alter the output of the aggregate.

With this motivation in mind, we are ready to define a novel concept that combines general OWA operators and weighted averages in order to produce a score of each alternative, *directly from the multi-agent information* that we assumed in Table 4:

Definition 4. Suppose that $\mathbf{w} = (w_1, \ldots, w_k) \in [0,1]^k$ is a weighting vector such that $\sum_{i=1}^k w_i = 1$, and $\omega = (\omega_1, \ldots, \omega_q) \in [0,1]^q$ is another weighting vector such that $\sum_{i=1}^q \omega_i = 1$.

The WAOWA score of $o_i \in O$ at $\{(F_1, T, N), \ldots, (F_k, T, N)\}$ is defined by

$$Y_i^{\mathbf{w},\omega} = \sum_{j=1}^{q} \omega_j \cdot F^{\mathbf{w}}(r^1_{ij}, \ldots, r^k_{ij}) \quad (6)$$

Table 7
Tabular representation of the aggregation of the 4-soft sets in Example 4 (see Table 5), by the $(F,4)$-soft set achieved by top and frequency.

$(\mu_t,(F_t,T,4))$	t_1	t_2	t_3
o_1	$\langle 2,0.4\rangle$	$\langle 3,0.6\rangle$	$\langle 3,0.2\rangle$
o_2	$\langle 2,0.6\rangle$	$\langle 3,0.2\rangle$	$\langle 2,0.2\rangle$
o_3	$\langle 3,0.6\rangle$	$\langle 1,0.6\rangle$	$\langle 2,0.2\rangle$
o_4	$\langle 1,0.6\rangle$	$\langle 2,0.4\rangle$	$\langle 3,0.6\rangle$

The rationale of Definition 4 is as follows. WAOWA evaluates each alternative in two steps. First an OWA operator gives respective assessments of the alternative in terms of each attribute. Then a weighted average of these values serves as a global evaluation, by inspiration of the notion of EWCV defined by Eq. (1).

Definition 4 clearly coincides with the EWCV of o_i in a single-agent N-soft setting (i.e., when $k=1$). This property justifies the validity of WAOWAs within a unified theory of scoring for N-soft sets.

3.3. Aggregation of N-soft sets by fuzzy N-soft sets

The fuzzy extension of N-soft sets is designed to express uncertainty about the degrees that are provided. This concept may be a good tool to capture the uncertainty that a collection of evaluations imposes on the collective assessments. For this reason, we believe that in some circumstances, it is sensible to consider $\mathcal{FN}(O)$ as a reasonable realm for the output of the aggregation of N-soft sets.

The main concept that we introduce to model the aggregation of several N-soft sets by a fuzzy N-soft set is a mapping $M : \mathcal{NS}(O)^k \longrightarrow \mathcal{FN}(O)$. We can mimic the formal requirements imposed on Definition 3 in order to request some technical conditions (like monotonicity) on this concept.

In this tentative approach, we just give some examples that prove the feasibility of the proposal. These solutions incorporate the constraint in Remark 2. For practical purposes, we recall that in case of finiteness the input is given by data as in Table 4 whereas the output is expected to be in the form of Table 2.

3.3.1. Aggregation by top and frequency

Let us fix an alternative t_j and an option o_i. We may produce the aggregate of the evaluations $\{r^1_{ij},\ldots,r^k_{ij}\}$ by taking the best evaluation submitted by the agents, which means an optimistic assessment by $\max\{r^1_{ij},\ldots,r^k_{ij}\}$. In each case, the degree of membership is the proportion of agents that submitted it, namely,

$$\frac{\left|\{a=1,\ldots,k \mid r^a_{ij}=\max\{r^1_{ij},\ldots,r^k_{ij}\}\}\right|}{k}. \tag{7}$$

This procedure returns the output in Table 7 for the agents of Example 4.

3.3.2. Aggregation by bottom and frequency

We can be more conservative than in the procedure above by taking $\min\{r^1_{ij},\ldots,r^k_{ij}\}$, i.e., the worst evaluation submitted by the agents. Again, its degree of membership is the proportion of agents that submitted it, namely,

$$\frac{\left|\{a=1,\ldots,k \mid r^a_{ij}=\min\{r^1_{ij},\ldots,r^k_{ij}\}\}\right|}{k}. \tag{8}$$

This procedure returns the output in Table 8 for the agents of Example 4.

3.3.3. Aggregation by other operators

We can use any other operators on N-soft sets, like the OWA operators that we studied in Section 3.1, and complement the evaluations by their frequencies like we did in the case of the top and bottom evaluations. Example 3 can inspire several procedures, the first two of which we have explained above in this section.

Table 8
Tabular representation of the aggregation of the 4-soft sets in Example 4, by an $(F,4)$-soft set given by bottom and frequency.

$(\mu_b,(F_b,T,4))$	t_1	t_2	t_3
o_1	$\langle 1,0.6\rangle$	$\langle 2,0.4\rangle$	$\langle 1,0.2\rangle$
o_2	$\langle 1,0.4\rangle$	$\langle 1,0.2\rangle$	$\langle 0,0.4\rangle$
o_3	$\langle 2,0.4\rangle$	$\langle 0,0.4\rangle$	$\langle 0,0.2\rangle$
o_4	$\langle 0,0.4\rangle$	$\langle 1,0.8\rangle$	$\langle 2,0.4\rangle$

Table 9
Aggregation of Example 4 by union (top) and by top and bottom (bottom).

(H_u,T,N)	t_1	t_2	t_3
o_1	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
o_2	$\{1,2\}$	$\{1,2,3\}$	$\{0,1,2\}$
o_3	$\{2,3\}$	$\{0,1\}$	$\{0,1,2\}$
o_4	$\{0,1\}$	$\{1,2\}$	$\{2,3\}$

(H_{tb},T,N)	t_1	t_2	t_3
o_1	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$
o_2	$\{1,2\}$	$\{1,3\}$	$\{0,2\}$
o_3	$\{2,3\}$	$\{0,1\}$	$\{0,2\}$
o_4	$\{0,1\}$	$\{1,2\}$	$\{2,3\}$

3.4. Aggregation of N-soft sets by hesitant N-soft sets

The hesitant extension of N-soft sets is designed to express hesitation about the grades that are provided. This concept may also be a suitable tool to capture the uncertainty associated with a collection of evaluations in some contexts. For this reason, we suggest that it is useful to take into consideration $\mathcal{HN}(O)$ as another feasible domain for the output of the aggregation of N-soft sets.

We model the aggregation of several N-soft sets by a hesitant N-soft set with the assistance of a mapping $M : \mathcal{NS}(O)^k \longrightarrow \mathcal{HN}(O)$. Also, in this case we can mimic the formal requirements imposed on Definition 3 in order to request additional technical conditions on this concept.

In order to prove the feasibility of this scheme, in this tentative approach we give some fundamental examples that incorporate the constraint in Remark 2. We remind the reader that in case of finiteness, the input is given by data as in Table 4 whereas the output is expected to be in the form of Table 3.

3.4.1. Aggregation by union

This procedure combines all the grades submitted by the various agents, for each fixed alternative and attribute. It returns the output in Table 9 (top) for the agents of Example 4. Put succinctly, in each cell we take the collection of all the evaluations appearing at that cell in any of the tables submitted by the agents.

3.4.2. Aggregation by top and bottom

This procedure uses the best and worst grades submitted by the agents, for each fixed alternative and attribute. In this way, every evaluation informs us of the span of evaluations that the agents submitted. It returns the output in Table 9 (bottom) for the agents of Example 4.

3.5. Discussion

Beyond the soft set model, N-soft set theory arises as an efficient tool to handle the uncertainty of multinary information in crisp form. We have shown real examples that do not fit into the soft set model. Prior to this paper, works like Fatimah et al. (2017) and Alcantud et al. (2020) showed other sources of real data that fit into the N-soft set model but are not soft sets. Some of these sources were in fact aggregators: Metacritic, Trivago or customer reviews of products in Amazon (Alcantud et al., 2020), 'Tomatometer rating' at the Rottentomatoes website or scores of electronic devices at the Kimovil website (Fatimah et al., 2017). Therefore we can claim that aggregation in N-soft set theory has a wider range of real-world applications than in the case of soft sets.

We have been able to import OWA operators into this practical framework both when the grades have a purely ordinal content or represent a measurable rate of fulfillment. OWA operators produce either an N-soft set (ordinal case) or another N'-soft set with possibly different N' (measurable, intercomparable case). But we have shown that it is also natural to benefit from the fuzzy N-soft set approach instead, which allows for fuzziness on top of the requirements of N-soft sets. Thus the aggregation of N-soft sets by fuzzy N-soft sets provides a richer environment for research than the previous case; for this reason, a more extensive treatment is beyond the scope of this paper. Actually one can even go beyond these facts and use the more productive structure of the intuitionistic fuzzy N-soft set model (Akram et al., 2019c) where both membership and non-membership values inform us of the collective opinion. We can only leave these questions for future research.

On a different front, we have only started a brief discussion of the aggregation of N-soft sets with the help of hesitant N-soft sets. It must be possible to embed our findings on aggregation by N-soft sets into this case as well. This is another challenge for the future development of this area.

4. Making decisions with multi-source N-soft sets

Our setting in this section is like that of Section 3: it presumes that we have $(F_1, T, N), \ldots, (F_k, T, N)$ at our disposal. In words, we have one N-soft set on O submitted by each of the k agents. In practical terms, this input is comprised of k individual tabular forms of the N-soft sets produced by the agents. As in Section 3, we give them in Table 4, which summarizes every possible $r^a_{ij} = F_a(t_j)(o_i) \in G$ from our setting.

The literature lacks a procedure to select an optimal alternative from this information. In order to solve the multi-agent problem that arises, two classes of flexible procedures will be stated in this section. In both cases we specify which elements (or parameters) are decided by the practitioner. These elements grant flexibility and adaptability on the side of the practitioner.

Two algorithms will be given that belong to the first class. Both share a *merge-then-decide* pattern. First, we aggregate the multi-source information, either into a collective N-soft set or into a collective hesitant N-soft set. Then, we apply the corresponding decision-making procedure for the combined output. At this stage we avail ourselves of the respective contributions in Akram et al. (2019a) and Fatimah et al. (2017). The second class of procedures take advantage of WAOWA scores in order to give a direct ranking of the alternatives.

Therefore, the structure of these classes of parametric procedures bears comparison with well-established methodologies. We find a comparable approach in the analysis of hesitant fuzzy sets in Xia and Xu (2011). In the extended setting of necessary and possible hesitant fuzzy sets, it has been used by Alcantud and Giarlotta (2019). A distant antecedent in decision-making by soft sets is Çağman and Enginoğlu (2010), although these authors bypass the (rather naive) problem of aggregating soft sets by the recourse to their product.

We proceed to formulate our algorithms and then a case study in the real context of Examples 2 and 5 ends this section.

4.1. Aggregation by an N-soft set

In our first proposal, we begin by combining the multi-source information by the recourse to the OWA operator in Section 3.1. This process generates another aggregate or collective N-soft set. We can then take advantage of the proposal in Fatimah et al. (2017) in order to select the best alternatives from this collective output. To that purpose Fatimah et al. (2017) suggests using the extended weighted choice values of the alternatives, which presumes the allocation of weights to the attributes. We give the steps in Algorithm 1.

Algorithm 1 - The algorithm of weighted choice values with N-soft sets.

Elements of the algorithm: aggregation methodology as in Section 3.1, i.e., an instance of Eq. (4), like the particular cases in Example 3; and weights w of attributes.

1: Input objects $O = \{o_1, o_2, \cdots, o_p\}$, and attributes $T = \{t_1, t_2, \cdots, t_q\}$.

2: Input $G = \{0, 1, 2, \cdots, N-1\}$ with $N \in \{2, 3, \cdots\}$.
Then for each $o_i \in O$ and $t_j \in T$, input the evaluation $r^a_{ij} \in G$, provided by each expert $a = 1, \ldots, k$.

3: Compute the aggregate N-soft set (F, T, N). For each $o_i \in O$ and $t_j \in T$, its tabular form attaches the global evaluation $r_{ij} \in G$.

4: For each $o_i \in O$ in (F, T, N), compute its EWCV $\sigma^w_i = \sum_{j=1}^{q} w_j \cdot r_{ij}$ as in Eq. (1).

We rank O by the EWCV of its members.
Any of the alternatives for which $\sigma^w_l = \max_{i=1,\ldots,p} \sigma^w_i$ can be chosen.

Our next example illustrates the application of Algorithm 1 with various aggregation operators that implement different attitudes. The first case is fully developed. Routine computations are avoided in the subsequent variations.

Example 6 (*Application of Algorithm 1*). Let us use this procedure to make a decision in the situation of Example 4. Thus, the input consists of the five 4-soft sets in Table 5. The weights of the attributes are $w = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. So the first and second attributes are equally important, and they are half as important as the third.

1. Suppose that we select the OWA operator with $\alpha = (3, 0, 0, 0, 0)$. Example 3 explains that this operator takes the most optimistic reading of all submitted evaluations (for every alternative and attribute).

At Step 3, the OWA operator M_α aggregates the 4-soft sets by $(M^1_\alpha, T, 4)$ in Table 6. Then at Step 4, the EWCV σ^w_i of each $o_i \in O$ is:

$$\sigma^w_1 = \tfrac{1}{4} \cdot 2 + \tfrac{1}{4} \cdot 3 + \tfrac{1}{2} \cdot 3 = \tfrac{11}{4}; \quad \sigma^w_2 = \tfrac{1}{4} \cdot 2 + \tfrac{1}{4} \cdot 3 + \tfrac{1}{2} \cdot 2 = \tfrac{9}{4};$$
$$\sigma^w_3 = \tfrac{1}{4} \cdot 3 + \tfrac{1}{4} \cdot 1 + \tfrac{1}{2} \cdot 2 = \tfrac{8}{4}; \quad \sigma^w_4 = \tfrac{1}{4} \cdot 1 + \tfrac{1}{4} \cdot 2 + \tfrac{1}{2} \cdot 3 = \tfrac{9}{4}.$$

Therefore, the ranking is $o_1 \succ o_2 \sim o_4 \succ o_3$, and o_1 is recommended.

2. Suppose that we select the OWA operator with $\alpha = (0, 0, 0, 0, 3)$. Example 3 explains that this operator takes the most conservative estimate of all submitted evaluations (for every alternative and attribute). At Step 3, the OWA operator M_α aggregates the 4-soft sets by $(M^2_\alpha, T, 4)$ in Table 6. The ranking is $o_1 \sim o_4 \succ o_2 \sim o_3$, and either o_1 or o_4 can be recommended.

3. Suppose that we select the OWA operator with $\alpha = (0, 3, 0, 0, 0)$. Example 3 explains that this operator takes the second best reading of all submitted evaluations (for every alternative and attribute). At Step 3, the OWA operator M_α aggregates the 4-soft sets by $(M^3_\alpha, T, 4)$ in Table 6. The ranking is $o_1 \succ o_2 \sim o_3 \succ o_4$, and o_1 is recommended.

4. Suppose that we select the OWA operator with $\alpha = (0, 0, 0, 3, 0)$. Example 3 explains that this operator implements a cautious attitude because it takes the second worst reading of all submitted evaluations (for every alternative and attribute). At Step 3, the OWA operator M_α aggregates the 4-soft sets by $(M^4_\alpha, T, 4)$ in Table 6. The ranking is $o_1 \succ o_4 \succ o_3 \succ o_2$, and o_1 is recommended.

5. Suppose that we select the OWA operator with $\alpha = (0, 0, 3, 0, 0)$. Example 3 explains that this operator takes the median all submitted evaluations (for every alternative and attribute). At Step 3, the OWA operator M_α aggregates the 4-soft sets by $(M^5_\alpha, T, 4)$ in Table 6. The ranking is $o_1 \sim o_4 \succ o_2 \sim o_3$, and o_1 is recommended.

Table 10 summarizes the rankings that Algorithm 1 endorses and the interpretation of the procedures of aggregation. The attitudes that

Table 10
Application of Algorithm 1 in Example 6: results with aggregation made by 5 respective OWA operators, and a common vector of weights.

	Interpretation	Ranking
$\alpha = (3,0,0,0,0)$	Best evaluation — optimistic attitude	$o_1 \succ o_2 \sim o_4 \succ o_3$
$\alpha = (0,0,0,0,3)$	Worst evaluation — convervative attitude	$o_1 \sim o_4 \succ o_2 \sim o_3$
$\alpha = (0,3,0,0,0)$	Second best evaluation — moderately positive attitude	$o_1 \succ o_2 \sim o_3 \succ o_4$
$\alpha = (0,0,0,3,0)$	Second worst evaluation — moderately cautious attitude	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\alpha = (0,0,3,0,0)$	Median evaluation — prudent attitude	$o_1 \sim o_4 \succ o_2 \sim o_3$

Table 11
Computations at Step 4 of Algorithm 3: aggregation by union plus arithmetic score (top), and aggregation by top and bottom plus geometric score (bottom). The weights are $\omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$and $\omega' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, respectively.

	t_1	t_2	t_3
o_1	$s_a(h^u_{11}) = \frac{1+2}{2} = 1.5$	$s_a(h^u_{12}) = \frac{2+3}{2} = 2.5$	$s_a(h^u_{13}) = \frac{1+2+3}{3} = 2$
o_2	$s_a(h^u_{21}) = \frac{1+2}{2} = 1.5$	$s_a(h^u_{22}) = \frac{1+2+3}{3} = 2$	$s_a(h^u_{23}) = \frac{0+1+2}{3} = 1$
o_3	$s_a(h^u_{31}) = \frac{2+3}{2} = 2.5$	$s_a(h^u_{32}) = \frac{0+1}{2} = 0.5$	$s_a(h^u_{33}) = \frac{0+1+2}{3} = 1$
o_4	$s_a(h^u_{41}) = \frac{0+1}{2} = 0.5$	$s_a(h^u_{42}) = \frac{1+2}{2} = 1.5$	$s_a(h^u_{43}) = \frac{2+3}{2} = 2.5$

	t_1	t_2	t_3
o_1	$s_g(h^{tb}_{11}) = \sqrt{1\cdot 2} = \sqrt{2}$	$s_g(h^{tb}_{12}) = \sqrt{2\cdot 3} = \sqrt{6}$	$s_g(h^{tb}_{13}) = \sqrt{1\cdot 3} = \sqrt{3}$
o_2	$s_g(h^{tb}_{21}) = \sqrt{1\cdot 2} = \sqrt{2}$	$s_g(h^{tb}_{22}) = \sqrt{1\cdot 3} = \sqrt{3}$	$s_g(h^{tb}_{23}) = \sqrt{0\cdot 2} = 0$
o_3	$s_g(h^{tb}_{31}) = \sqrt{2\cdot 3} = \sqrt{6}$	$s_g(h^{tb}_{32}) = \sqrt{0\cdot 1} = 0$	$s_g(h^{tb}_{33}) = \sqrt{0\cdot 2} = 0$
o_4	$s_g(h^{tb}_{41}) = \sqrt{0\cdot 1} = 0$	$s_g(h^{tb}_{42}) = \sqrt{1\cdot 2} = \sqrt{2}$	$s_g(h^{tb}_{43}) = \sqrt{2\cdot 3} = \sqrt{6}$

Table 12
Final results in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Diver	Team	Place	Score
Brandon Loschiavo	PURD	1	464.00
David Dinsmore	MIAF	2	461.25
Steele Johnson	PURD	3	425.35
Benjamin Bramley	PURD	4	397.00
Tyler Downs	RIP	5	392.70
Maxwell Weinrich	MDC	6	376.60
Jacob Cornish	TEX	7	362.70
Quentin Henninger	MILE	8	357.80
Chase Lane	U	9	354.40
Zach Cooper	MIAF	10	353.65
Jordan Rzepka	OHIO	11	352.75
Jacob Siler	U	12	289.85

these mechanisms implement vary from very optimistic (first) to very conservative (second). However in all cases o_1 can be recommended.

4.2. Aggregation by OWA operators: a variation of Algorithm 1

Our first solution inspires a natural variation when we prefer to combine the multi-source information by the recourse to a general OWA operator. We have argued that this process produces a novel WAOWA score for each alternative (cf., Definition 4).

We give the steps of the alternative procedure that arises in Algorithm 2. As WAOWA scores coincide with EWCVs in the case of a single-agent problem, Algorithm 2 extends the proposal in Fatimah et al. (2017) much like Algorithm 1 does.

Algorithm 2 - The algorithm of WAOWA scores.

Elements of the algorithm: OWA aggregator $F^{\mathbf{w}}$ given by Eq. (5), and weights ω of the attributes.

1: Input objects $O = \{o_1, o_2, \cdots, o_p\}$, and attributes $T = \{t_1, t_2, \cdots, t_q\}$.

2: Input $G = \{0, 1, 2, \cdots, N-1\}$ with $N \in \{2, 3, \cdots\}$.
Then for each $o_i \in O$ and $t_j \in T$, input the evaluation $r^a_{ij} \in G$, provided by each expert $a = 1, \ldots, k$.

3: Compute the WAOWA score of each $o_i \in O$ at $\{(F_1, T, N), \ldots, (F_k, T, N)\}$, namely, $Y^{\mathbf{w},\omega}_i$, by Eq. (6).

We rank O by the WAOWA scores of its members.
Any of the alternatives for which $Y^{\mathbf{w},\omega}_l = \max_{i=1,2,\cdots,p} Y^{\mathbf{w},\omega}_i$ can be chosen.

We postpone the practical illustration of Algorithm 2 until the case study in Section 4.4.

4.3. Aggregation by a hesitant N-soft set

Alternatively to Section 4.1, we can combine the multi-source information in the form of a hesitant N-soft set. The techniques developed in Section 3.1 are helpful in this first step. We can then take advantage of Akram et al. (2019a) in order to select the best alternatives from the resulting hesitant N-soft set. To that purpose Akram et al. (2019a) suggests to use their weighted scores (cf., Section 2.1), which also presumes the allocation of weights to the attributes. We give these steps in Algorithm 3.

Algorithm 3 - The algorithm of weighted scores with HNSSs.

Elements of the algorithm: aggregation methodology from Section 3.4, score function, and weights ω of the attributes.

1: Input objects $O = \{o_1, o_2, \cdots, o_p\}$, and attributes $T = \{t_1, t_2, \cdots, t_q\}$.

2: Input $G = \{0, 1, 2, \cdots, N-1\}$ with $N \in \{2, 3, \cdots\}$. Then for each $o_i \in O$ and $t_j \in T$, input the evaluation $r^a_{ij} \in G$, provided by each expert $a = 1, \ldots, k$.

3: Compute the aggregate HNSS (H, T, N). Its tabular form is as in Table 3.

4: Compute the scores $s(h_{ij})$ of its HNTs, for all $o_i \in O$ and $t_j \in T$ (cf., Example 1).

5: Compute the weighted score of each $o_i \in O$, namely, $\varrho^{\omega}_i = \sum_{j=1}^{q} \omega_j \cdot s(h_{ij})$.

We rank O by the weighted scores of its members.
Any of the alternatives for which $\varrho^{\omega}_l = \max_{i=1,2,\cdots,p} \varrho^{\omega}_i$ can be chosen.

Our next example illustrates the application of Algorithm 3.

Example 7 (*Application of Algorithm 3*). Let us use this procedure to make a decision in the situation of Example 4. Again, the input is the data in Table 5.

First, we select the aggregation by union operator, the arithmetic score for HNTs, and weights $\omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Step 3 produces (H_u, T, N) in Table 9 (top).

Table 11 (top) summarizes the application of Step 4 of the algorithm in this case. The computation of the elements in Step 5 gives
$\varrho^{\omega}_1 = \frac{1}{4}\cdot 1.5 + \frac{1}{4}\cdot 2.5 + \frac{1}{2}\cdot 2 = 2$; $\varrho^{\omega}_2 = \frac{1}{4}\cdot 1.5 + \frac{1}{4}\cdot 2 + \frac{1}{2}\cdot 1 = 1.375$;
$\varrho^{\omega}_3 = \frac{1}{4}\cdot 2.5 + \frac{1}{4}\cdot 0.5 + \frac{1}{2}\cdot 1 = 1.25$; $\varrho^{\omega}_4 = \frac{1}{4}\cdot 0.5 + \frac{1}{4}\cdot 1.5 + \frac{1}{2}\cdot 2.5 = 1.75$.
Therefore, the ranking is $o_1 \succ o_4 \succ o_2 \succ o_3$, and o_1 is recommended.

Table 13
The computations for the application of Algorithm 2 to the example in Section 4.4.

	Diver	OWA aggregation for round:						WAOWAs	Rank
		1	2	3	4	5	6		
1	Brandon Loschiavo	16.67	14	17	13	14	13.67	**14.72**	1
2	David Dinsmore	16.33	16.33	14.33	11.67	13	16	**14.51**	3
3	Steele Johnson	13	15	15	13.33	15.67	16	**14.67**	2
4	Benjamin Bramley	16	14	8	12.67	16	16	**13.78**	5
5	Tyler Downs	13.67	16.33	11.67	15	10.67	14	**13.56**	6
6	Maxwell Weinrich	14	15	10.67	16	14	14	**13.94**	4
7	Jacob Cornish	15	14	11	9.67	11	14.33	**12.50**	8
8	Quentin Henninger	10.33	13	13.67	8.67	15.67	16	**12.89**	7
9	Chase Lane	13	9.33	12.67	13.33	13.67	11.67	**12.28**	9
10	Zach Cooper	14.67	8.33	10.67	11	12.67	11	**11.39**	11
11	Jordan Rzepka	11	13.67	10.33	11.33	12.67	9.67	**11.44**	10
12	Jacob Siler	10	8.67	12.33	4.33	11	13.67	**10.00**	12

Table 14
Final results for **Brandon J Loschiavo** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	407C	10 M	Inward 3 1/2 Somersault Tuck	25.00	3.20	80.00	1
2	6245D	10 M	Armstand Back 2 Somersault 2 1/2 Twist Free	21.00	3.60	75.60	2
3	307C	10 M	Reverse 3 1/2 Somersault Tuck	25.50	3.40	86.70	1
4	207B	10 M	Back 3 1/2 Somersault Pike	19.50	3.60	70.20	3
5	109C	10 M	Forward 4 1/2 Somersault Tuck	21.00	3.70	77.70	2
6	5255B	10 M	Back 2 1/2 Somersault 2 1/2 Twist Pike	20.50	3.60	73.80	5
			Totals	132.50	21.10	**464.00**	

Table 15
Final results for David A Dinsmore in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	407C	10 M	Inward 3 1/2 Somersault Tuck	24.50	3.20	78.40	2
2	6245D	10 M	Armstand Back 2 Somersault 2 1/2 Twist Free	24.50	3.60	88.20	1
3	307C	10 M	Reverse 3 1/2 Somersault Tuck	21.50	3.40	73.10	3
4	207B	10 M	Back 3 1/2 Somersault Pike	17.50	3.60	63.00	6
5	109C	10 M	Forward 4 1/2 Somersault Tuck	19.50	3.70	72.15	3
6	5255B	10 M	Back 2 1/2 Somersault 2 1/2 Twist Pike	24.00	3.60	86.40	1
			Totals	131.50	21.10	**461.25**	

Table 16
Final results for Steele A Johnson in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	19.50	3.20	62.40	8
2	407C	10 M	Inward 3 1/2 Somersault Tuck	22.50	3.20	72.00	4
3	207C	10 M	Back 3 1/2 Somersault Tuck	22.50	3.30	74.25	2
4	107B	10 M	Forward 3 1/2 Somersault Pike	20.00	3.00	60.00	8
5	307C	10 M	Reverse 3 1/2 Somersault Tuck	23.50	3.40	79.90	1
6	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	24.00	3.20	76.80	2
			Totals	132.00	19.30	**425.35**	

Table 17
Final results for Benjamin Bramley in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	24.00	3.20	76.80	3
2	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	21.00	3.20	67.20	6
3	207C	10 M	Back 3 1/2 Somersault Tuck	12.00	3.30	39.60	12
4	307C	10 M	Reverse 3 1/2 Somersault Tuck	19.00	3.40	64.60	4
5	107B	10 M	Forward 3 1/2 Somersault Pike	24.00	3.00	72.00	4
6	407C	10 M	Inward 3 1/2 Somersault Tuck	24.00	3.20	76.80	2
			Totals	124.00	19.30	**397.00**	

Secondly, we select the aggregation by top and bottom operator, the geometric score for HNTs, and equal weights $w' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Step 3 produces (H_{tb}, T, N) in Table 9 (bottom).

Table 11 (bottom) summarizes the application of Step 4 of the algorithm in this case. The computation of the elements in Step 5 gives $\varrho_1^{w'} = \frac{1}{3} \cdot \sqrt{2} + \frac{1}{3} \cdot \sqrt{6} + \frac{1}{3} \cdot \sqrt{3} \approx 1.832$; $\varrho_2^{w'} = \frac{1}{3} \cdot \sqrt{2} + \frac{1}{3} \cdot \sqrt{3} + \frac{1}{3} \cdot 0 \approx 1.049$;

Table 18
Final results for Tyler M Downs in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	407C	10 M	Inward 3 1/2 Somersault Tuck	20.50	3.20	65.60	6
2	107B	10 M	Forward 3 1/2 Somersault Pike	24.50	3.00	73.50	3
3	207C	10 M	Back 3 1/2 Somersault Tuck	17.50	3.30	57.75	7
4	626C	10 M	Armstand Back 3 Somersault Tuck	22.50	3.30	74.25	2
5	307C	10 M	Reverse 3 1/2 Somersault Tuck	16.00	3.40	54.40	10
6	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	21.00	3.20	67.20	7
			Totals	122.00	19.40	**392.70**	

Table 19
Final results for Maxwell R Weinrich in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	107B	10 M	Forward 3 1/2 Somersault Pike	21.00	3.00	63.00	7
2	407C	10 M	Inward 3 1/2 Somersault Tuck	22.50	3.20	72.00	4
3	305C	10 M	Reverse 2 1/2 Somersault Tuck	16.00	2.80	44.80	11
4	207C	10 M	Back 3 1/2 Somersault Tuck	24.00	3.30	79.20	1
5	614B	10 M	Armstand Forward 2 Somersault Pike	21.00	2.40	50.40	12
6	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	21.00	3.20	67.20	7
			Totals	125.50	17.90	**376.60**	

Table 20
Final results for Jacob R Cornish in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	22.50	3.20	72.00	4
2	107B	10 M	Forward 3 1/2 Somersault Pike	21.00	3.00	63.00	8
3	407C	10 M	Inward 3 1/2 Somersault Tuck	16.50	3.20	52.80	9
4	207C	10 M	Back 3 1/2 Somersault Tuck	14.50	3.30	47.85	11
5	307C	10 M	Reverse 3 1/2 Somersault Tuck	16.50	3.40	56.10	9
6	5237D	10 M	Back 1 1/2 Somersault 3 1/2 Twist Free	21.50	3.30	70.95	6
			Totals	112.50	19.40	**362.70**	

Table 21
Final results for Quentin R Henninger in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	201B	10 M	Back Dive Pike	15.50	1.80	27.90	12
2	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	19.50	3.20	62.40	9
3	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	20.50	3.20	65.60	4
4	305C	10 M	Reverse 2 1/2 Somersault Tuck	19.50	2.80	54.60	10
5	107B	10 M	Forward 3 1/2 Somersault Pike	23.50	3.00	70.50	5
6	407C	10 M	Inward 3 1/2 Somersault Tuck	24.00	3.20	76.80	2
			Totals	122.50	17.20	**357.80**	

Table 22
Final results for Chase G Lane in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	107B	10 M	Forward 3 1/2 Somersault Pike	19.50	3.00	58.50	9
2	307C	10 M	Reverse 3 1/2 Somersault Tuck	14.00	3.40	47.60	10
3	207C	10 M	Back 3 1/2 Somersault Tuck	19.00	3.30	62.70	5
4	407C	10 M	Inward 3 1/2 Somersault Tuck	20.00	3.20	64.00	5
5	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	20.50	3.20	65.60	7
6	5253B	10 M	Back 2 1/2 Somersault 1 1/2 Twist Pike	17.50	3.20	56.00	11
			Totals	110.50	19.30	**354.40**	

$o_3^{w'} = \frac{1}{3}\cdot\sqrt{6}+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot 0 \approx 0.817; \qquad o_4^{w'} = \frac{1}{3}\cdot 0+\frac{1}{3}\cdot\sqrt{2}+\frac{1}{3}\cdot\sqrt{6} \approx 1.288.$
Therefore, the ranking is $o_1 \succ o_4 \succ o_2 \succ o_3$, and o_1 is recommended.

4.4. A case study: a re-examination of Example 2

We now proceed to make a numerical simulation with real data extracted from the 2019 USA Diving Senior National Diving Championships held from May 19, 2019 to May 26, 2019 (TeamUSA, USA Diving, The US Olympic & Paralympic Committee, 2019). A total of $p = 12$ divers performed $q = 6$ dives at the final of the Senior Men Platform competition. Examples 2 and 5 gave an explanation of the scoring system.

Table 12 shows the official final results of the 2019 USA Diving Senior National Diving Championships in the Senior Men Platform category.

The name of the divers is in the first column. The team is in the second column. For example, Brandon Loschiavo belongs to the PURD team (Purdue Diving of West Lafayette, IN, United States). The place he

Table 23
Final results for Zach T Cooper in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	407C	10 M	Inward 3 1/2 Somersault Tuck	22.00	3.20	70.40	5
2	307C	10 M	Reverse 3 1/2 Somersault Tuck	12.50	3.40	42.50	12
3	6245D	10 M	Armstand Back 2 Somersault 2 1/2 Twist Free	16.00	3.60	57.60	8
4	109C	10 M	Forward 4 1/2 Somersault Tuck	16.50	3.70	61.05	7
5	207C	10 M	Back 3 1/2 Somersault Tuck	19.00	3.30	62.70	8
6	5255B	10 M	Back 2 1/2 Somersault 2 1/2 Twist Pike	16.50	3.60	59.40	10
			Totals	102.50	20.80	**353.65**	

Table 24
Final results for Jordan N Rzepka in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	407C	10 M	Inward 3 1/2 Somersault Tuck	16.50	3.20	52.80	10
2	6243D	10 M	Armstand Back 2 Somersault 1 1/2 Twist Free	20.50	3.20	65.60	7
3	207C	10 M	Back 3 1/2 Somersault Tuck	15.50	3.30	51.15	10
4	307C	10 M	Reverse 3 1/2 Somersault Tuck	17.00	3.40	57.80	9
5	109C	10 M	Forward 4 1/2 Somersault Tuck	19.00	3.70	70.30	6
6	5156B	10 M	Forward 2 1/2 Somersault 3 Twist Pike	14.50	3.80	55.10	12
			Totals	103.00	20.60	**352.75**	

Table 25
Final results for Jacob A Siler in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

Dive	Number	Height	Description	Net score	DD	Score	Round place
1	5154B	10 M	Forward 2 1/2 Somersault 2 Twist Pike	15.00	3.30	49.50	11
2	207C	10 M	Back 3 1/2 Somersault Tuck	13.00	3.30	42.90	11
3	626C	10 M	Armstand Back 3 Somersault Tuck	18.50	3.30	61.05	6
4	307C	10 M	Reverse 3 1/2 Somersault Tuck	6.50	3.40	22.10	12
5	407C	10 M	Inward 3 1/2 Somersault Tuck	16.50	3.20	52.80	11
6	107B	10 M	Forward 3 1/2 Somersault Pike	20.50	3.00	61.50	9
			Totals	90.00	19.50	**289.85**	

Table 26
Final results by judges for **Brandon J Loschiavo** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		407C	6245D	307C	207B	109C	5255B		
1	Goodman, Ed D	8.50	6.50	8.50	6.00	7.00	7.50	44.0	7.33
2	Johnson, Jonathan	8.00	7.00	8.50	7.00	7.00	7.00	44.5	7.42
3	Sorgi, Erica	8.50	6.50	8.50	6.50	7.00	7.00	44.0	7.33
4	Ahlering, Julie	8.50	7.00	8.00	7.00	6.50	6.50	43.5	7.25
5	Richmond, Emily M	8.00	7.00	8.50	6.50	7.00	6.50	43.5	7.25
6	Krug, Dorothy A	8.00	7.00	8.50	6.50	7.00	6.50	43.5	7.25
7	Faber, Marc	9.00	7.00	8.50	6.50	7.00	7.00	45.0	7.50

Table 27
Final results by judges for **David A Dinsmore** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		407C	6245D	307C	207B	109C	5255B		
1	Goodman, Ed D	8.50	7.50	7.50	6.00	6.00	8.00	43.5	7.25
2	Johnson, Jonathan	8.00	8.00	7.00	6.00	6.50	8.00	43.5	7.25
3	Sorgi, Erica	8.50	8.50	7.50	5.00	6.50	8.50	44.5	7.42
4	Ahlering, Julie	7.50	8.00	7.00	5.50	6.50	8.50	43.0	7.17
5	Richmond, Emily M	8.00	8.50	7.00	6.00	6.50	8.00	44.0	7.33
6	Krug, Dorothy A	8.50	8.00	7.00	6.00	6.50	7.50	43.5	7.25
7	Faber, Marc	8.00	8.50	8.00	5.50	6.00	8.00	44.0	7.33

obtained in this competition is in the next column and the final score is in the last column. Continuing with the example, Loschiavo took first place with 464.00 points.

The Appendix shows the scores received by each diver at each round.

Table 14 shows the official results for Brandon Loschiavo, who placed first.

Tables 15–25 show the results for the other divers. The score of each jump is the product of its net score and degree of difficulty (DD).

Our numerical exercise will compare the results of the 2019 final with the output of Algorithm 2. This means that we shall discard the

Table 28
Final results by judges for **Steele A Johnson** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		6243D	407C	207C	107B	307C	5253B		
1	Goodman, Ed D	6.50	7.50	7.50	6.50	8.00	8.50	44.5	7.42
2	Johnson, Jonathan	6.50	7.50	7.00	7.00	8.00	8.50	44.5	7.42
3	Sorgi, Erica	6.50	8.00	7.50	6.50	8.00	8.00	44.5	7.42
4	Ahlering, Julie	6.00	7.50	7.00	6.50	8.00	8.00	43.0	7.17
5	Richmond, Emily M	6.50	7.00	7.50	7.00	7.50	8.00	43.5	7.25
6	Krug, Dorothy A	7.00	7.50	8.00	7.00	7.00	8.00	44.5	7.42
7	Faber, Marc	7.00	7.50	7.50	6.50	7.50	8.00	44.0	7.33

Table 29
Final results by judges for **Benjamin Bramley** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		5253B	6243D	207C	307C	107B	407C		
1	Goodman, Ed D	8.00	7.00	4.00	6.50	8.00	7.50	41.0	6.83
2	Johnson, Jonathan	8.00	7.50	4.00	6.50	7.50	7.50	41.0	6.83
3	Sorgi, Erica	8.00	7.00	3.50	5.50	8.00	8.00	40.0	6.67
4	Ahlering, Julie	7.50	6.50	4.00	7.00	8.00	8.00	41.0	6.83
5	Richmond, Emily M	8.50	7.00	4.00	6.00	7.50	8.00	41.0	6.83
6	Krug, Dorothy A	7.50	7.00	4.00	6.00	8.00	8.50	41.0	6.83
7	Faber, Marc	8.50	7.00	3.50	6.50	8.00	8.00	41.5	6.92

Table 30
Final results by judges for **Tyler M Downs** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		407C	107B	207C	626C	307C	5253B		
1	Goodman, Ed D	7.50	8.00	6.00	7.50	5.50	7.50	42.0	7.00
2	Johnson, Jonathan	7.00	8.50	5.50	7.00	5.50	7.00	40.5	6.75
3	Sorgi, Erica	6.50	8.50	6.00	7.50	5.00	7.50	41.0	6.83
4	Ahlering, Julie	6.50	8.50	6.00	7.50	5.50	7.00	41.0	6.83
5	Richmond, Emily M	6.00	7.50	5.50	7.50	5.50	7.00	39.0	6.50
6	Krug, Dorothy A	7.00	8.00	6.00	7.00	4.50	7.00	39.5	6.58
7	Faber, Marc	7.00	8.00	5.50	7.50	5.00	6.50	39.5	6.58

Table 31
Final results by judges for **Maxwell R Weinrich** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		107B	407C	305C	207C	614B	5253B		
1	Goodman, Ed D	7.00	7.50	5.50	7.00	6.50	7.00	40.5	6.75
2	Johnson, Jonathan	7.00	7.50	5.50	7.50	7.00	7.50	42.0	7.00
3	Sorgi, Erica	7.00	7.50	6.00	8.00	7.00	7.00	42.5	7.08
4	Ahlering, Julie	7.00	7.50	5.50	8.00	7.00	7.00	42.0	7.00
5	Richmond, Emily M	7.00	7.50	5.00	8.00	7.00	7.00	41.5	6.92
6	Krug, Dorothy A	7.00	7.50	4.50	8.50	7.00	7.00	41.5	6.92
7	Faber, Marc	7.50	8.00	4.50	8.00	7.00	7.50	42.5	7.08

Table 32
Final results by judges for **Jacob R Cornish** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		6243D	107B	407C	207C	307C	5237D		
1	Goodman, Ed D	7.50	6.50	5.50	5.50	6.00	7.00	38.0	6.33
2	Johnson, Jonathan	7.50	6.50	5.50	5.50	5.50	7.50	38.0	6.33
3	Sorgi, Erica	7.50	7.00	5.50	4.50	5.50	7.50	37.5	6.25
4	Ahlering, Julie	7.50	7.00	5.00	4.50	5.00	7.00	36.0	6.00
5	Richmond, Emily M	7.00	7.00	5.50	5.50	5.50	7.00	37.5	6.25
6	Krug, Dorothy A	6.50	7.00	6.00	4.50	5.50	7.00	36.5	6.08
7	Faber, Marc	7.50	7.00	5.50	4.00	4.50	7.50	36.0	6.00

information provided by the tariff (DD in the tables) in the computation of the final scores. To do this we need the scoring given by each judge.

Table 26 in the Appendix shows the score sheet for the diver Brandon Loschiavo. The names of the 7 judges that evaluate his dives

Table 33
Final results by judges for **Quentin R Henninger** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		201B	6243D	5253B	305C	107B	407C		
1	Goodman, Ed D	5.00	6.50	6.50	7.00	8.00	8.00	41.0	6.83
2	Johnson, Jonathan	4.50	7.00	7.00	6.00	8.00	8.50	41.0	6.83
3	Sorgi, Erica	5.00	6.50	7.00	6.00	8.50	8.00	41.0	6.83
4	Ahlering, Julie	5.50	7.00	5.50	6.50	7.50	8.00	40.0	6.67
5	Richmond, Emily M	5.50	6.50	6.50	6.50	8.00	8.00	41.0	6.83
6	Krug, Dorothy A	5.50	6.50	7.00	6.50	7.50	7.50	40.5	6.75
7	Faber, Marc	4.50	6.00	7.00	7.00	7.50	8.00	40.0	6.67

Table 34
Final results by judges for **Chase G Lane** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		107B	307C	207C	407C	6243D	5253B		
1	Goodman, Ed D	7.50	5.00	7.00	7.00	7.00	6.00	39.5	6.58
2	Johnson, Jonathan	6.50	5.00	6.00	7.00	6.50	5.50	36.5	6.08
3	Sorgi, Erica	6.50	4.50	6.50	6.50	6.50	5.50	36.0	6.00
4	Ahlering, Julie	6.00	4.50	6.50	7.00	7.00	6.00	37.0	6.17
5	Richmond, Emily M	6.50	4.50	6.00	6.00	7.00	6.00	36.0	6.00
6	Krug, Dorothy A	6.50	4.50	6.50	6.50	7.00	5.50	36.5	6.08
7	Faber, Marc	7.00	5.00	6.00	6.50	6.50	6.50	37.5	6.25

Table 35
Final results by judges for **Zach T Cooper** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		407C	307C	6245D	109C	207C	5255B		
1	Goodman, Ed D	7.50	5.00	5.50	6.50	7.00	6.50	38.0	6.33
2	Johnson, Jonathan	7.00	4.00	5.50	5.50	6.00	5.50	33.5	5.58
3	Sorgi, Erica	6.50	4.00	5.00	6.00	6.50	5.50	33.5	5.58
4	Ahlering, Julie	7.00	4.50	5.00	5.50	6.00	5.50	33.5	5.58
5	Richmond, Emily M	7.50	4.50	5.50	5.50	6.50	6.00	35.5	5.92
6	Krug, Dorothy A	8.00	4.00	5.50	5.50	6.00	5.50	34.5	5.75
7	Faber, Marc	8.00	4.00	5.00	5.50	6.50	5.50	34.5	5.75

Table 36
Final results by judges for **Jordan N Rzepka** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		407C	6243D	207C	307C	109C	5156B		
1	Goodman, Ed D	2.00	6.50	5.50	5.50	6.00	4.50	30.0	5.00
2	Johnson, Jonathan	7.00	7.00	5.00	6.00	6.50	3.00	34.5	5.75
3	Sorgi, Erica	2.00	6.50	5.50	5.50	6.50	5.00	31.0	5.17
4	Ahlering, Julie	6.50	7.00	5.50	6.00	6.00	5.50	36.5	6.08
5	Richmond, Emily M	4.50	7.00	4.50	6.00	6.50	5.00	33.5	5.58
6	Krug, Dorothy A	5.50	6.50	5.00	5.00	7.00	5.00	34.0	5.67
7	Faber, Marc	8.00	7.00	5.00	5.50	6.00	4.50	36.0	6.00

Table 37
Final results by judges for **Jacob A Siler** in Senior Men Platform: 2019 USA Diving Senior National Diving Championships.
Source: Dive Meets (DiveMeets, 2019).

	Judge	1	2	3	4	5	6	Tot	Avg
		5154B	207C	626C	307C	407C	107B		
1	Goodman, Ed D	4.50	4.50	6.00	3.50	6.00	7.00	31.5	5.25
2	Johnson, Jonathan	5.00	4.00	6.50	2.00	6.00	6.50	30.0	5.00
3	Sorgi, Erica	5.50	4.50	6.50	2.00	5.50	7.00	31.0	5.17
4	Ahlering, Julie	5.00	4.50	6.00	3.50	5.50	7.00	31.5	5.25
5	Richmond, Emily M	5.00	5.00	6.00	2.00	5.50	7.00	30.5	5.08
6	Krug, Dorothy A	6.00	4.00	6.00	2.50	5.50	6.50	30.5	5.08
7	Faber, Marc	5.00	4.00	6.50	2.00	5.50	6.50	29.5	4.92

are in the first column. The next 6 columns show each judge's score for each of the diver jumps. The final two columns give the total and average scores.

Tables 27–37 in the Appendix show the score sheets of the remaining divers. Scores that are discarded have red (bottom) and blue (top) colors for easier identification.

Table 38
$(F_1, T, 21)$: the normalized scores submitted by Judge Ed D Goodman.

F_1	t_1	t_2	t_3	t_4	t_5	t_6
o_1	17	13	17	12	14	15
o_2	17	15	15	12	12	16
o_3	13	15	15	13	16	17
o_4	16	14	8	13	16	15
o_5	15	16	12	15	11	15
o_6	14	15	11	14	13	14
o_7	15	13	11	11	12	14
o_8	10	13	13	14	16	16
o_9	15	10	14	14	14	12
o_{10}	15	10	11	13	14	13
o_{11}	4	13	11	11	12	9
o_{12}	9	9	12	7	12	14

Table 39
$(F_2, T, 21)$: the normalized scores submitted by Judge Jonathan Johnson.

F_2	t_1	t_2	t_3	t_4	t_5	t_6
o_1	16	14	17	14	14	14
o_2	16	16	14	12	13	16
o_3	13	15	14	14	16	17
o_4	16	15	8	13	15	15
o_5	14	17	11	14	11	14
o_6	14	15	11	15	14	15
o_7	15	13	11	11	11	15
o_8	9	14	14	12	16	17
o_9	13	10	12	14	13	11
o_{10}	14	8	11	11	12	11
o_{11}	14	14	10	12	13	6
o_{12}	10	8	13	4	12	13

Table 40
$(F_3, T, 21)$: the normalized scores submitted by Judge Erica Sorgi.

F_3	t_1	t_2	t_3	t_4	t_5	t_6
o_1	17	13	17	13	14	14
o_2	17	17	15	10	13	17
o_3	13	16	15	13	16	16
o_4	16	14	7	11	16	16
o_5	13	17	12	15	10	15
o_6	14	15	12	16	14	14
o_7	15	14	11	9	11	15
o_8	10	13	14	12	17	16
o_9	13	9	13	13	13	11
o_{10}	13	8	10	12	13	11
o_{11}	4	13	11	11	13	10
o_{12}	11	9	13	4	11	14

Table 41
$(F_4, T, 21)$: the normalized scores submitted by Judge Julie Ahlering.

F_4	t_1	t_2	t_3	t_4	t_5	t_6
o_1	17	14	16	14	13	13
o_2	15	16	14	11	13	17
o_3	12	15	14	13	16	16
o_4	15	13	8	14	16	16
o_5	13	17	12	15	11	14
o_6	14	15	11	16	14	14
o_7	15	14	10	9	10	14
o_8	11	14	11	13	15	16
o_9	12	9	13	14	14	12
o_{10}	14	9	10	11	12	11
o_{11}	13	14	11	12	12	11
o_{12}	10	9	12	7	11	14

Table 42
$(F_5, T, 21)$: the normalized scores submitted by Judge Emily M Richmond.

F_5	t_1	t_2	t_3	t_4	t_5	t_6
o_1	16	14	17	13	14	13
o_2	16	17	14	12	13	16
o_3	13	14	15	14	15	16
o_4	17	14	8	12	15	16
o_5	12	15	11	15	11	14
o_6	14	15	10	16	14	14
o_7	14	14	11	11	11	14
o_8	11	13	13	13	16	16
o_9	13	9	12	12	14	12
o_{10}	15	9	11	11	13	12
o_{11}	9	14	9	12	13	10
o_{12}	10	10	12	4	11	14

Table 43
$(F_6, T, 21)$: the normalized scores submitted by Judge Dorothy A Krug.

F_6	t_1	t_2	t_3	t_4	t_5	t_6
o_1	16	14	17	13	14	13
o_2	17	16	14	12	13	15
o_3	14	15	16	14	14	16
o_4	15	14	8	12	16	17
o_5	14	16	12	14	9	14
o_6	14	15	9	17	14	14
o_7	13	14	12	9	11	14
o_8	11	13	14	13	15	15
o_9	13	9	13	13	14	11
o_{10}	16	8	11	11	12	11
o_{11}	11	13	10	10	14	10
o_{12}	12	8	12	5	11	13

Table 44
$(F_7, T, 21)$: the normalized scores submitted by Judge Marc Faber.

F_7	t_1	t_2	t_3	t_4	t_5	t_6
o_1	18	14	17	13	14	14
o_2	16	17	16	11	12	16
o_3	14	15	15	13	15	16
o_4	17	14	7	13	16	16
o_5	14	16	11	15	10	13
o_6	15	16	9	16	14	15
o_7	15	14	11	8	9	15
o_8	9	12	14	14	15	16
o_9	14	10	12	13	13	13
o_{10}	16	8	10	11	13	11
o_{11}	16	14	10	11	12	9
o_{12}	10	8	13	4	11	13

We now normalize this information in compliance with the formal structure of our problem.

Tables 38–44 in the Appendix capture $k = 7$ tabular forms of 21-soft sets, one submitted by each judge (after normalization). The jump scores $(t_1, \ldots, t_6)$ of the 12 divers $(o_1, \ldots, o_{12})$ have been transformed to the conventional graded scale $\{0, 1, 2, \ldots, 20\}$, multiplying by 2.

In order to proceed with Algorithm 2 we need to select a suitable OWA operator and weights of the attributes. Thus the rules of competitive diving stipulate that we must fix $\mathbf{w} = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ and $\omega = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. We are now ready to apply the WAOWA scores corresponding to this problem, i.e., Eq. (6) in order to rank the divers. The computations are shown in Table 13. At each column (i.e., for every fixed round of dives), the figure represents the successive OWA scores. Our explanation ensures that they are 2/3 of the "net score" corresponding to the same dive and diver: We multiplied by 2 in order to fit the 21-soft set model; and now we divide by 3 (the three middle scores are just summed up in the championship, but we do their averages because the components of $\mathbf{w}$ must sum up to 1). The "net score" is the figure that is multiplied by DD in the championship.

Algorithm 2 recommends to use an average of each round's figures. The next column "WAOWAs" gives these averages, i.e., the successive values of $Y_i^{\mathbf{w},\omega}$. The far right column displays the position that Algorithm 2 assigns to each diver.

The conclusion of this hypothetical exercise is that the removal of the tariff would change the positions of the divers, albeit not strikingly so. This is of course a purely theoretical exercise, as the divers use DD strategically along the competition and its *ex-post* removal would be plainly unfair.

5. Conclusions and future research lines

This paper gives persuasive arguments to establish that the aggregation of N-soft sets is worthy of investigation. It is possible to aggregate N-soft sets into various different structures; we have given fundamental definitions of three natural possibilities with respective aggregation operators and illustrative examples. Importantly, we have been able to expand the OWA spirit to the aggregation of N-soft sets by another N'-soft set. Particular statements yield some noteworthy evaluations (e.g., by the best, worst, or median values of the grades). A novel WAOWA score extends the rationale of EWCVs from single-agent to multi-agent problems, thus contributing to a uniform view of scores for N-soft sets with the assistance of OWA operators.

All in all, this preliminary analysis of the aggregation issue paves the way for subsequent studies about: (i) alternative aggregators for N-soft sets in the context of Sections 3.1 and 3.2, (ii) more sophisticate aggregators in the context of Sections 3.3 and 3.4, (iii) the extension of the OWA approach to more general frameworks like the cases in these two sections or bipolar N-soft sets (Kamacı & Petchimuthu, 2020), and (iv) related theories with other types of outputs like intuitionistic fuzzy N-soft sets (Akram et al., 2019c).

It is technically possible to withdraw the restriction in Remark 2 so that for example, the individual evaluations about one attribute can influence the global assessment of another attribute. We have not pursued this avenue here. Real examples may be required to validate this idea.

Furthermore, the suitable combination of aggregation methodologies and existing results on individual decision-making with information provided by models of vague knowledge (Akram et al., 2018, 2019a) has allowed us to give the first *multi-agent* decision-making approaches in the context of N-soft sets.

Two adaptable methodologies have been developed. They are reasonable as long as they import the benefits of known methodologies from other settings in soft computing. Particularly, we take advantage of *single-agent* decision-making approaches in the context of N-soft sets and hesitant N-soft sets. In each case, we have clearly disclosed the elements that the practitioner can select in order to suit the goals of each problem. Examples have illustrated all the computations required to produce the final assessment of the alternatives. They confirm that the methodologies are feasible and versatile.

CRediT authorship contribution statement

José Carlos R. Alcantud: Conceptualization, Methodology, Formal analysis, Writing – original draft. **Gustavo Santos-García:** Conceptualization, Methodology, Formal analysis, Writing – original draft. **Muhammad Akram:** Methodology, Formal analysis, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Datasets of the diving example are available from the public repository https://github.com/gsantosgarcia/N-soft-sets. Its supporting information files are licensed under the GNU General Public License v3.0 and can be distributed under GNU General Public License v3.0.

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Appendix. Tables for the case study in Section 4.4

See Tables 14–44.

References

Adlassnig, K. (1986). Fuzzy set theory in medical diagnosis. *IEEE Transactions on Systems, Man, and Cybernetics*, *16*, 260–265. http://dx.doi.org/10.1109/TSMC.1986.4308946.

Ahmed, M. U., Brickman, S., Dengg, A., Fasth, N., Mihajlovic, M., & Norman, J. (2019). A machine learning approach to classify pedestrians' events based on imu and gps. *International Journal of Artificial Intelligence*, *17*, 154–167, http://www.ceser.in/ceserp/index.php/ijai/article/view/6260/6207.

Akram, M., Adeel, A., & Alcantud, J. C. R. (2018). Fuzzy N-soft sets: A novel model with applications. *Journal of Intelligent & Fuzzy Systems*, *35*, 4757–4771. http://dx.doi.org/10.3233/JIFS-18244.

Akram, M., Adeel, A., & Alcantud, J. C. R. (2019a). Group decision-making methods based on hesitant N-soft sets. *Expert Systems with Applications*, *115*, 95–105. http://dx.doi.org/10.1016/j.eswa.2018.07.060.

Akram, M., Adeel, A., & Alcantud, J. R. (2019b). Hesitant fuzzy N-soft sets: A new model with applications in decision-making. *Journal of Intelligent & Fuzzy Systems*, *36*, 6113–6127. http://dx.doi.org/10.3233/JIFS-181972.

Akram, M., Ali, G., & Alcantud, J. C. R. (2019c). New decision-making hybrid model: intuitionistic fuzzy N-soft rough sets. *Soft Computing*, *23*, 9853–9868. http://dx.doi.org/10.1007/s00500-019-03903-w.

Alcantud, J. C. R., & Calle, R. A. (2017). The problem of collective identity in a fuzzy environment. *Fuzzy Sets and Systems*, *315*, 57–75. http://dx.doi.org/10.1016/j.fss.2016.02.011.

Alcantud, J. C. R., Cruz, S., & Torrecillas, M. M. (2017). Valuation fuzzy soft sets: A flexible fuzzy soft set based decision making procedure for the valuation of assets. *Symmetry*, *9*(253).

Alcantud, J. C. R., Feng, F., & Yager, R. R. (2020). An N-soft set approach to rough sets. *IEEE Transactions on Fuzzy Systems*, *28*, 2996–3007. http://dx.doi.org/10.1109/TFUZZ.2019.2946526.

Alcantud, J. C. R., & Giarlotta, A. (2019). Necessary and possible hesitant fuzzy sets: A novel model for group decision making. *Information Fusion*, *46*, 63–76.

Alcantud, J. C. R., Santos-García, G., & Hernández-Galilea, E. (2015). Advances in artificial intelligence: 16th conference of the spanish association for artificial intelligence, caepia 2015 albacete, spain. In *Chapter glaucoma diagnosis: A soft set based decision making procedure* (pp. 49–60). Springer.

Alcantud, J. C. R., Varela, G., Santos-Buitrago, B., Santos-García, G., & Jiménez, M. F. (2019). Analysis of survival for lung cancer resections cases with fuzzy and soft set theory in surgical decision making. *PLOS ONE*, *14*, 1–17. http://dx.doi.org/10.1371/journal.pone.0218283.

Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*, 87–96.

Atanassov, K. T. (1989). More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *33*, 37–45.

Bedregal, B., Reiser, R., Bustince, H., López-Molina, C., & Torra, V. (2014). Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms. *Information Sciences*, *255*, 82–99.

Borg, E., & Love, C. (2018). A demonstration of the borg centimax® scale (CR100) for performance evaluation in diving. *Nordic Psychology*, *70*, 228–244.

Çagman, N., Enginoglu, S., & Çitak, F. (2011). Fuzzy soft set theory and its applications. *Iranian Journal of Fuzzy Systems*, *8*, 137–147. http://dx.doi.org/10.22111/IJFS.2011.292.

Çağman, N., & Enginoğlu, S. (2010). Soft set theory and uni–int decision making. *European Journal of Operational Research*, *207*, 848–855. http://dx.doi.org/10.1016/j.ejor.2010.05.004.

Çelik, Y., & Yamak, S. (2013). Fuzzy soft set theory applied to medical diagnosis using fuzzy arithmetic operations. *Journal of Inequalities and Applications*, *2013*, 82. http://dx.doi.org/10.1186/1029-242X-2013-82.

Chen, Y., Liu, J., Chen, Z., & Zhang, Y. (2020). Group decision-making method based on generalized vague N-soft sets. In *2020 Chinese control and decision conference* (pp. 4010–4015).

Chiang, H.-S., Shih, D.-H., Lin, B., & Shih, M.-H. (2014). An APN model for arrhythmic beat classification. *Bioinformatics*, *30*, 1739–1746. http://dx.doi.org/10.1093/bioinformatics/btu101.

Council of Europe homepage (2020). Common European framework of reference for languages (CEFR). http://www.coe.int/lang-CEFR. Accessed 13 2020.

De, S. K., Biswas, R., & Roy, A. R. (2001). An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets and Systems*, *117*, 209–213.

DiveMeets (2019). 2019 Usa diving senior national diving championships. may (2019) 19–26, indianapolis, indiana https://secure.meetcontrol.com/divemeets/system/eventresultsext.php?meetnum=5950&eventnum=1130&eventtype=9# Online; accessed 29 2021.

Dubois, D., & Prade, H. (1990). Rough fuzzy sets and fuzzy rough sets. *International Journal of General Systems*, *17*, 191–209. http://dx.doi.org/10.1080/03081079008935107.

Emerson, J. W., Seltzer, M., & Lin, D. (2009). Assessing judging bias: An example from the 2000 olympic games. *The American Statistician*, *63*, 124–131.

EuroQol Research Foundation (2020). Euroqol homepage. https://euroqol.org Accessed 13 2020.

Fatimah, F., Rosadi, D., Hakim, R. B. F., & Alcantud, J. C. R. (2017). N-Soft sets and their decision making algorithms. *Soft Computing*, http://dx.doi.org/10.1007/s00500-017-2838-6.

Feng, F., Liu, X., Leoreanu-Fotea, V., & Jun, Y. B. (2011). Soft sets and soft rough sets. *Information Sciences*, *181*, 1125–1137. http://dx.doi.org/10.1016/j.ins.2010.11.004.

Gorzałczany, M. B. (1987). A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems*, *21*, 1–17. http://dx.doi.org/10.1016/0165-0114(87)90148-5.

Grannan, C. (2021). How is diving scored? Encyclopædia britannica. https://www.britannica.com/story/how-is-diving-scored Online; accessed 29 2021.

Hayat, K., Ali, M. I., Karaaslan, F., Cao, B.-Y., & Shah, M. H. (2020). Design concept evaluation using soft sets based on acceptable and satisfactory levels: An integrated topsis and shannon entropy. *Soft Computing*, *24*, 2229–2263. http://dx.doi.org/10.1007/s00500-019-04055-7.

Kahraman, C., & Kaya, I. (2010). A fuzzy multicriteria methodology for selection among energy alternatives. *Expert Systems with Applications*, *37*, 6270–6281.

Kamacı, H., & Petchimuthu, S. (2020). Bipolar N-soft set theory with applications. *Soft Computing*, *24*, 16727–16743. http://dx.doi.org/10.1007/s00500-020-04968-8.

Kramer, R. S. (2017). Sequential effects in olympic synchronized diving scores. *Royal Society Open Science*, *4*, Article 160812.

Liao, H., & Xu, Z. (2014). Intuitionistic fuzzy hybrid weighted aggregation operators. *International Journal of Intelligent Systems*, *29*, 971–993. http://dx.doi.org/10.1002/int.21672.

Liu, J., Chen, Y., Chen, Z., & Zhang, Y. (2020). Multi-attribute decision making method based on neutrosophic vague N-soft sets. *Symmetry*, *12*(853), http://dx.doi.org/10.3390/sym12050853.

Lizasoain, I., & Moreno, C. (2013). OWA operators defined on complete lattices. *Fuzzy Sets and Systems*, *224*, 36–52. http://dx.doi.org/10.1016/j.fss.2012.10.012.

Maji, P., Biswas, R., & Roy, A. (2001a). Fuzzy soft sets. *Journal of Fuzzy Mathematics*, *9*, 589–602.

Maji, P., Biswas, R., & Roy, A. (2001b). Intuitionistic fuzzy soft sets. *Journal of Fuzzy Mathematics*, *9*, 677–692.

Muthukumar, P., & Sai Sundara Krishnan, G. (2016). A similarity measure of intuitionistic fuzzy soft sets and its application in medical diagnosis. *Applied Soft Computing*, *41*, 148–156. http://dx.doi.org/10.1016/j.asoc.2015.12.002.

Pawlak, Z. (1982). Rough sets. *International Journal of Parallel Programming*, *11*, 341–356. http://dx.doi.org/10.1007/BF01001956.

Pawlak, Z., Słowiński, K., & Słowiński, R. (1986). Rough classification of patients after highly selective vagotomy for duodenal ulcer. *International Journal of Man-Machine Studies*, *24*, 413–433.

Pozna, C., & Precup, R.-E. (2014). Applications of signatures to expert systems modelling. *Acta Polytechnica Hungarica*, *1*, 21–39. http://dx.doi.org/10.12700/aph.11.02.2014.02.2.

Senapati, T., & Yager, R. R. (2019). Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods. *Engineering Applications of Artificial Intelligence*, *85*, 112–121. http://dx.doi.org/10.1016/j.engappai.2019.05.012.

Senapati, T., & Yager, R. R. (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, *11*, 663–674. http://dx.doi.org/10.1007/s12652-019-01377-0.

Sreedevi, S., Mathew, T. J., & Sherly, E. (2016). Computerized classification of malignant and normal microcalcifications on mammograms: Using soft set theory. In *IEEE international conference on information science, ICIS 2016, Kochi, India, Aug (2016) 12-13* (pp. 131–137).

Stefanowski, J., & Słowiński, K. (1997). Rough set theory and rule induction techniques for discovery of attribute dependencies in medical information systems. In J. Komorowski, & J. Zytkow (Eds.), *LNCS*: *vol. 1263*, *Lect notes artif int* (pp. 36–46). Springer.

TeamUSA, USA Diving, The US Olympic & Paralympic Committee (2019). 2019 Usa diving senior national diving championships. may (2019) 19–26, indianapolis, indiana https://www.teamusa.org/USA-Diving/Events/2019/May/19/2019-USA-Diving-Senior-National-Championships Online; accessed 29 2021.

Wnuk, K., & Soatto, S. (2010). Analyzing diving: A dataset for judging action quality. In *Asian conference on computer vision* (pp. 266–276). Springer.

Xia, M., & Xu, Z. (2011). Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning*, *52*, 395–407. http://dx.doi.org/10.1016/j.ijar.2010.09.002.

Xiao, F. (2018). A hybrid fuzzy soft sets decision making method in medical diagnosis. *IEEE Access*, *6*, 25300–25312. http://dx.doi.org/10.1109/ACCESS.2018.2820099.

Xu, Z., & Xia, M. (2011). Distance and similarity measures for hesitant fuzzy sets. *Information Sciences*, *181*, 2128–2138. http://dx.doi.org/10.1016/j.ins.2011.01.028.

Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics*, *18*, 183–190. http://dx.doi.org/10.1109/21.87068.

Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, *22*, 958–965. http://dx.doi.org/10.1109/TFUZZ.2013.2278989.

Yao, Y. (1998). A comparative study of fuzzy sets and rough sets. *Information Sciences*, *109*, 227–242.

Zadeh, L. (1965). Fuzzy sets. *Information and Control*, *8*, 338–353.

Zhang, H., Jia-Hua, D., & C (2020). Multi-attribute group decision-making methods based on pythagorean fuzzy N-soft sets. *IEEE Access*, *8*, 62298–62309. http://dx.doi.org/10.1109/ACCESS.2020.2984583.