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# Attributes reduction algorithms for *m*-polar fuzzy relation decision systems

Muhammad Akram [a], Ghous Ali [b], José Carlos R. Alcantud [c,*]

[a] *Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan*
[b] *Department of Mathematics, Division of Science & Technology University of Education, Lahore, Pakistan*
[c] *BORDA Research Unit and IME, University of Salamanca, 37007 Salamanca, Spain*

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ABSTRACT

Nowadays, attribute reduction has become a significant topic in relation decision systems. Their applications come from different domains of the computer sciences, including machine learning, data mining and pattern recognition, which often involve a large number of attributes in data. Several attribute reduction methods are presented in the literature in order to help solving decision-making problems efficiently. A common characterization for these approaches is still missing, that is, although attribute reduction methods of relation decision systems and fuzzy relation decision systems exist, a common generalization for them is still missing. This study presents a systematic discussion of attribute reduction based on *m*-polar fuzzy (*m*F, in short) relation systems and *m*F relation decision systems, which are respective extensions of fuzzy relation systems and fuzzy relation decision systems. This study provides mathematical results on the attribute reduction algorithms based upon *m*F relation systems and *m*F relation decision systems. Both are explained with numerical examples. The resulting algorithms permit to reinterpret the upshots of traditional reduction methods, providing them with larger generality and unification abilities. Afterwards, two real-life applications of the proposed attribute reduction approaches prove their validity and feasibility. Finally, the attribute reduction methods developed here are compared with some existing approaches to show their reliability.



## 1. Introduction

Zadeh [38] was the first who gave the idea of fuzzy sets to formalize the existence of partial truth among "absolutely false" and "absolutely true". The main contribution of this useful model is that in a fuzzy context, the objects belong to the universe with a possibly partial membership degree. After the production of this theory a number of researchers from almost every field were attracted by it [1,6]. They introduced many remarkable results, and produced plenty of applications. However in case of multi-polar information, the fuzzy set model falls short of formal ability.

There is no denying that multi-polarity exists in many real-world situations too, and it plays an effective role in different domains of technology and science. For example, in information technology, multi-polar technology can be used to operate large scale systems. In neurobiology, multi-polar neurons in the brain gather a great deal of information from other neurons.

* Corresponding author.
*E-mail addresses:* m.akram@pucit.edu.pk (M. Akram), ghous.ali@ue.edu.pk (G. Ali), jcr@usal.es (J.C.R. Alcantud).

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In a food web, breeds may be of multiple kinds such as: vegetarian, non-vegetarian; weak or strong predators; and preys may be digestive, harmful, or energetic. In social networking, the influence rate of distinct people may be multi-polar regarding trading relations, pro-activeness, or socialization. In the judgment of the market power of a company multiple characteristics may be involved, including price control of its product, annual profit, brand reputation, and product quality. Thus, uncertain multi-polar information exist in numerous real-world decision-making problems. So after the introduction of $m$F sets [10] which is an efficient extension of fuzzy sets that accounts for multi-polar data, many authors have produced research in order to solve different real-world problems involving multi-objects, multi-attribute, and multi-polar information. The fact that $m$F sets allow for more precise graphical representations of uncertain data facilitates a significantly better analysis in data incompleteness and relationships. In an $m$F set, the membership degree of an object belongs to $[0,1]^m$ which describes all the $m$ distinct characteristics of the object. A number of hybrid models have been proposed which involve $m$F sets as one of their components. For instance, Akram [3] implemented $m$F set theory on graphs and introduced several useful results. In addition, Akram et al. [5] presented the $m$F $N$-soft set model and discussed its decision-making applications. Mahapatra et al. [27] developed a new model called interval-valued $m$F sets as a natural extension of $m$F sets.

The other element that motivates our analysis is attribute reduction (AR, henceforth). This is a method by which dispensable attributes are deleted from a given data-set while preserving consistency [32]. Pawlak [29] initiated the idea of AR in information systems. Presently, the reduction of pattern dimensionality plays a vital role in areas such as pattern recognition, machine learning, data mining, decision aiding and knowledge representation. Attribute reduction of information systems removes redundant attributes while maintaining the optimal decision object as was with the whole set of attributes. In this regard, Skowron and Rauszer [33] were the first to present the idea of discernibility matrices, which converts the discernibility functions from their conjunctive normal forms (CNFs) into the disjunctive normal forms (DNFs). From the DNF of a discernibility function, minimal subset of attributes can be easily computed. In fact computational complexity was the major problem in the reduction methods. In order to compute the minimal reduction subset of the set of attributes, the existing procedures of transformation of discernibility functions from their CNFs to the DNFs were excessively time consuming. To resolve this issue, Borowik and Luba [9] presented a fast algorithm that converts the discernibility function from its CNF to its DNF. Several researchers [12,15,16] utilized this useful idea of discernibility matrix and proposed different types of AR algorithms. For instance, Chen et al. [11] proposed algorithms for parallel AR, and Greco et al. [14] studied the ordered properties of attributes using a novel dominance based rough set method. Jing et al. [18] proposed an AR approach for knowledge granular systems. Jia et al. [17] developed reduction methods of different kinds. Yang and Li [39] studied reduction in the context of covering generalized rough sets. Ziarko [41] introduced the concept of AR for the variable-precision rough set model and developed it to some extent. Afterwards, Mi et al. [26] presented an AR method based on variable-precision rough sets. Ren and Wei [30] extended the AR approaches to three-way concept lattices. Wang et al. [35,36] proposed the idea of relation decision systems and their AR methods. Nevertheless there was a flaw in their concept of the relation decision system, that is, the decision attribute must be an equivalence relation. Liu et al. [22] improved the concept of relation decision system (RDS) in which the decision attribute is free from the restriction of being an equivalence relation. For general relation decision systems, Liu and Hua [21] presented the concepts of partial AR methods. Feng et al. [13] presented AR approaches for fuzzy relation systems (FRSs) and fuzzy relation decision systems (FRDSs). A number of experts were attracted by different AR approaches because they are based on efficient mathematical proofs, for instance, the heuristic reduction approach or AR based discernibility matrix.

The authors of this paper have recently proposed the concept of bipolar FRDSs along with real-life applications [7]. The problem remains whether and to what extent the formal characteristics and practical performance of that approach hold true in a multi-polar environment. Indeed, despite the fact that the above mentioned researches have promoted the development of AR methods in information systems to a certain extent, no author has yet studied AR of $m$F sets and its hybrid structures, in particular, AR in $m$F relation decision systems. This motivated us to present the more general, useful concept of AR for $m$F relation decision systems.

The motivations of the novel AR methodologies of $m$FRSs are summarized as follows:

1. Chen et al. [10] initiated the concept of $m$F sets as an extension of bipolar fuzzy sets [42]. The reason for the existence of “multi-polar information” is that sometimes the data for real-world problems comes from $m$ agents ($m \geq 2$). For instance, the exact degree of human telecommunications security is a point within $[0,1]^m$ ($m = 7 \times 10^9$) as different people are monitored at different times. There exist many other examples like truthfulness of logical formulas based on $m$ logical implication operators ($m \geq 2$), the similarity between two logical formulas based on $m$ logical implication operators ($m \geq 2$), magazine ordering results, university ordering results, and rough set inclusion degrees. Consider the statement “China is a good country” as an example. This statement may be too simplistically realized if we only allow its ‘membership value’ to be a single real number in $[0,1]$: Becoming a good country has different constituents such as being good at economic power, and at public transportation systems, medical facilities, educational system, etc. Then each constituent can rightfully be a real number in $[0,1]$. If $j$ shows the number of such constituents/components under study, then the membership value of fuzzy sentences is a $j$-tuple of real numbers in $[0,1]$. The existence of multi-polar representations which are fuzzy in nature is a strong reason behind this study.
2. $m$F information systems provide an efficient way to express the complete data with all properties of elements under consideration.

3. As an important processing step, AR approaches are used to reduce the dimensionality of the data through the removal of dispensable attributes in a data-set. The existence of multi-polar inputs together with the standard defense of AR motivate the necessity of AR in such $m$F information systems.
4. If we wish to add new attributes after execution for a data-set, then AR of information systems is very significant because after adding new attributes the whole process of execution will start from the initial step. Therefore redundant attributes (if any) may increase computational time.
5. Very recently, Feng et al. [13] introduced certain AR algorithms for FRSs. However FRSs are not capable to deal with all $m$ properties of objects in a system, and AR methods for $m$FRDSs have been unattended. Thus, a generalized information system is needed.

This article is therefore a natural continuation of the approach to AR proposed for FRSs in [13].

In order to discuss the reduction methods for the systems having $m$F data, two novel concepts, namely, $m$F relation systems ($m$FRSs) and $m$F relation decision systems ($m$FRDSs) are introduced as further generalizations of FRSs and FRDSs. Then their AR approaches are considered. The algorithms that we propose produce a far greater unification and generality than the existing ones. These algorithms are executed with MATLAB software therefore our paper extends the scope of applicability of AR methods for multi-polar fuzzy information systems.

For other terminologies not stated in this paper, the readers are referred to [4,19,20,23–25,28,31,34,37,40].

The rest of the article is structured as follows. Section 2 recalls certain fundamental terms related to information systems, RDSs and $m$F sets. In Section 3, a novel discernibility matrix and a discernibility function based upon $m$F relation systems are proposed. They are explained with illustrative examples and followed by an algorithm. Section 4 provides the AR algorithm for $m$F relation decision systems and also provides illustrative numerical examples of its operations. Section 5 provides two real life applications of the proposed AR methods for $m$FRSs and $m$FRDSs. In Section 6, the reduction methods developed in this paper are compared with some existing models. Section 7 provides conclusions and future directions. An Appendix contains the MATLAB codes for (a) the two algorithms for attribute reduction, and (b) the transformation of information into the required formats, in order to facilitate the reproducibility of our results by the potential reader.

## 2. Preliminaries

This section recalls some fundamental notions, including information system, discernibility matrix, discernibility function, $m$F set, $m$F relation and some of their fundamental properties.

**Definition 2.1.** [33] An *information system* is a pair $(\mathcal{X}, \mathcal{E}')$, $\mathcal{X}$ is a nonempty universal set and $\mathcal{E}'$ is a set of attributes, i.e., $e : \mathcal{X} \to \mathcal{V}_e$, for $e \in \mathcal{E}'$ where $\mathcal{V}_e$ is said to be the value set of $\mathcal{E}'$.

**Definition 2.2.** [33] Let $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ be a universal set, $\mathcal{E}' = \{e_1, e_2, \dots, e_t\}$ be a set of attributes. For an information system $(\mathcal{X}, \mathcal{E}')$, an $n \times n$ matrix, denoted by $\mathcal{M}(\mathcal{E}') = (a_{jk})$, is said to be the *discernibility matrix* of $\mathcal{E}'$ where

$$a_{jk} = \{e \in \mathcal{E}' \mid e(x_j) \neq e(x_k),\ j, k = 1, 2, \dots, n\}.$$

**Definition 2.3.** [33] Let $(\mathcal{X}, \mathcal{E}')$ be an information system. Then a *discernibility function* $h_{\mathcal{E}'}$ is a Boolean function of $t$ variables(Boolean) $\hat{e}_1, \dots, \hat{e}_t$ with respect to the attributes $e_1, \dots, e_t$, respectively, which is given by

$$h_{\mathcal{E}'}(\hat{e}_1, \dots, \hat{e}_t) = \wedge\{\vee(a_{jk}) \mid 1 \le k < j \le n,\ c_{jk} \neq \emptyset\}.$$

**Definition 2.4.** [10] An *mF set* over a universal set $\mathcal{X}$ is a function $\mathcal{H} : \mathcal{X} \to [0, 1]^m$. Note that from now on, $[0, 1]^m$ is considered as a partially ordered set with point-wise order $\le$, where $m$ is a natural number. '$\le$' is given by $x \le y$ if and only if for any $i = 1, 2, ..., m$, $p_i(x) \le p_i(y)$, where $x, y \in [0, 1]^m$ and $p_i : [0, 1]^m \to [0, 1]$ is the $i^{th}$ projection mapping.

**Definition 2.5.** [10] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $m$F sets over a $\mathcal{X}$. Then, $\mathcal{H}_1 \cup \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2$ are also $m$F sets on $\mathcal{X}$ defined as follows: for $i = 1, 2, \dots, m$ and $x \in \mathcal{X}$, $p_i \circ (\mathcal{H}_1 \cup \mathcal{H}_2)(x) = \max\{p_i \circ \mathcal{H}_1(x), p_i \circ \mathcal{H}_2(x)\}$ and $p_i \circ (\mathcal{H}_1 \cap \mathcal{H}_2)(x) = \min\{p_i \circ \mathcal{H}_1(x), p_i \circ \mathcal{H}_2(x)\}$. Clearly $\mathcal{H}_1 \subseteq \mathcal{H}_2$ if and only if $p_i \circ \mathcal{H}_1(x) \le p_i \circ \mathcal{H}_2(x)$ and $\mathcal{H}_1 = \mathcal{H}_2$ if and only if $p_i \circ \mathcal{H}_1(x) = p_i \circ \mathcal{H}_2(x)$ for all $x \in \mathcal{X}$.

Just like fuzzy relations over a universe $\mathcal{X}$, which are designed by fuzzy subsets over the Cartesian product $\mathcal{X} \times \mathcal{X}$, $m$F relations over the set $\mathcal{X} \times \mathcal{X}$ are modeled as below:

**Definition 2.6.** [27] Let $\mathcal{H}$ be an $m$F set over a universe $\mathcal{X}$. An $m$F relation on $\mathcal{H}$ is an $m$F set $\mathcal{G}$ of $\mathcal{X} \times \mathcal{X}$ such that $\mathcal{G}(x, y) \le \min\big(\mathcal{H}(x), \mathcal{H}(y)\big)$ for all $x, y \in \mathcal{X}$, that is, for all $i = 1, 2, \dots, m$, $x, y \in \mathcal{X}$, $p_i \circ \mathcal{G}(x, y) \le \min\big(p_i \circ \mathcal{H}(x), p_i \circ \mathcal{H}(y)\big)$. An $m$F relation $\mathcal{G}$ on $\mathcal{X}$ is said to be symmetric if $\mathcal{G}(x, y) = \mathcal{G}(y, x)$, $\forall\ x, y \in \mathcal{X}$.

The presentation of an $m$F set carries ahead to the most recent idea. In an $m$F relation $\mathcal{G}$ on $\mathcal{X} \times \mathcal{X}$, $p_i \circ \mathcal{G}(x, y)$ where $i = 1, 2, \ldots, m$ are the membership degrees which represent the satisfaction degree of an element $(x, y)$ to all its properties corresponding to the $m$F set $\mathcal{G}$.

From now on we consider $p_i \circ \mathcal{G}(x, y) = (p_1 \circ \mathcal{G}(x, y), p_2 \circ \mathcal{G}(x, y), \ldots, p_m \circ \mathcal{G}(x, y))$ to describe this relation from $x$ to $y$. It is a more sensible way to describe knowledge than data expressed by an $m$F relation $\mathcal{R}$, in which the notions alternatively expressed by $(x, y) \in \mathcal{R}$, $x\mathcal{R}y$, or $\mathcal{R}(x, y)$ mean "$x$ and $y$ are related in terms of the $m$F relation $\mathcal{R}$".

Now we recall some more useful characteristics of $m$F relations:

**Definition 2.7.** [3] Let $\mathcal{G}$ be a $m$F relation over a universe $\mathcal{X}$. Then

1. $\mathcal{G}$ is said to be reflexive if $p_i \circ \mathcal{G}(x, x) = 1$ for all $x \in \mathcal{X}$, $i = 1, 2, \ldots, m$.
2. $\mathcal{G}$ is symmetric if $p_i \circ \mathcal{G}(x, y) = p_i \circ \mathcal{G}(v, u)$, $\forall\ x, y \in \mathcal{X}$, $i = 1, 2, \ldots, m$.
3. $\mathcal{G}$ is transitive if $p_i \circ \mathcal{G}(x, y) = p_i \circ \mathcal{G}(x, z)$ and $p_i \circ \mathcal{G}(x, z) = p_i \circ \mathcal{G}(y, z)$ then $p_i \circ \mathcal{G}(x, y) = p_i \circ \mathcal{G}(y, z)$ for all $x, y, z \in \mathcal{X}$, $i = 1, 2, \ldots, m$.

Further, intersection and union for $m$F relations are defined as below:

**Definition 2.8.** Let $\{\mathcal{G}_s\}_{s\in J}$ be a finite collection of $m$F relations on $\mathcal{X}$. Write $\mathcal{G}_s = (p_1 \circ \mathcal{G}_s, p_2 \circ \mathcal{G}_s, \ldots, p_m \circ \mathcal{G}_s)$ for each $s \in J$. Then $\cap_{s\in J}\mathcal{G}_s$ and $\cup_{s\in J}\mathcal{G}_s$ are the $m$F relations on $\mathcal{X}$ defined by:

1. $\cap_{s\in J}\mathcal{G}_s(x, y) = \big(\bigwedge_{s\in J}(p_1 \circ \mathcal{G}_s(x, y)), \bigwedge_{s\in J}(p_2 \circ \mathcal{G}_s(x, y)), \ldots, \bigwedge_{s\in J}(p_m \circ \mathcal{G}_s(x, y))\big)$ for all $x, y \in \mathcal{X}$.
2. $\cup_{s\in J}\mathcal{G}_s(x, y) = \big(\bigvee_{s\in J}(p_1 \circ \mathcal{G}_s(x, y)), \bigvee_{s\in J}(p_2 \circ \mathcal{G}_s(x, y)), \ldots, \bigvee_{s\in J}(p_m \circ \mathcal{G}_s(x, y))\big)$ for all $x, y \in \mathcal{X}$.

Also, when $\mathcal{G}, \mathcal{G}'$ are $m$F relations on $\mathcal{X}$, we write $\mathcal{G} \subseteq \mathcal{G}'$ if $p_i \circ \mathcal{G}(x, y) \le p_i \circ \mathcal{G}'(x, y)$ for all $x, y \in \mathcal{X}$, $i = 1, 2, \ldots, m$.

For an $m$F relation $\mathcal{G}$ over $\mathcal{X}$, the left $\mathcal{G}$-relative $m$F set $l_{\mathcal{G}}(x) = (p_1 \circ \mathcal{G}(x), p_2 \circ \mathcal{G}(x), \ldots, p_m \circ \mathcal{G}(x))$ and right $\mathcal{G}$-relative $m$F set $r_{\mathcal{G}}(x) = (p_1 \circ \mathcal{G}(x), p_2 \circ \mathcal{G}(x), \ldots, p_m \circ \mathcal{G}(x))$ of $x \in \mathcal{X}$ are respectively defined by

$$l_{\mathcal{G}}(x)(y) = (p_1 \circ \mathcal{G}(y, x), p_2 \circ \mathcal{G}(y, x), \ldots, p_m \circ \mathcal{G}(y, x)),$$
$$r_{\mathcal{G}}(x)(y) = (p_1 \circ \mathcal{G}(x, y), p_2 \circ \mathcal{G}(x, y), \ldots, p_m \circ \mathcal{G}(x, y)),$$

where $y \in \mathcal{X}$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $m$F relations on $\mathcal{X}$, then $r_{\mathcal{G}_1}(x) \subseteq r_{\mathcal{G}_2}(x)$ if and only if $p_i \circ \mathcal{G}_1(x, y) \le p_i \circ \mathcal{G}_2(x, y)$ for all $y \in \mathcal{X}$, $i = 1, 2, \ldots, m$. Note that, for an $m$F relation $\mathcal{G}$ on $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, its relational matrix $\mathcal{M}_{\mathcal{G}} = (s_{jk})_{n\times n}$ is given by $s_{jk} = \mathcal{G}(x_j, x_k) = (p_1 \circ \mathcal{G}(x_j, x_k), p_2 \circ \mathcal{G}(x_j, x_k), \ldots, p_m \circ \mathcal{G}(x_j, x_k))$.

## 3. Attribute reduction of *m*FRSs

In this section, the notion of $m$FRS and its AR algorithm are presented along with some formal mathematical results.

**Definition 3.1.** Let $\mathcal{X}$ be a finite universal set and let $\mathcal{E} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ be a collection of $m$F relations on $\mathcal{X}$, then $(\mathcal{X}, \mathcal{E})$ is said to be an *mF relation systems* or $m$FRS. Let $\mathcal{E} = \mathcal{S} \cup \mathcal{T}$ with $\mathcal{S} \cap \mathcal{T} = \emptyset$. Then we call $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ an *mF relation decision systems* or $m$FRDS, where the sets $\mathcal{S}$ and $\mathcal{T}$ are respectively containing condition and decision attributes.

In the following, we provide a remark which describes the importance of Definition 3.1 by connecting it to some existing related concepts:

**Remark 3.2.** The concepts explained in Definition 3.1 degenerate into certain well-known existing terminologies which are explained as below:

1. An $m$FRS $(\mathcal{X}, \mathcal{E})$ degenerates into an information system if $\mathcal{E}$ contains equivalence relations over $\mathcal{X} \times \mathcal{X}$.
2. An $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ degenerates into a decision table if $\mathcal{S}$ and $\mathcal{T}$ contain equivalence relations over $\mathcal{X} \times \mathcal{X}$.
3. An $m$FRS $(\mathcal{X}, \mathcal{E})$ degenerates into a relation system if $\mathcal{E}$ contains binary relations on $\mathcal{X}$.
4. An $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ degenerates into a relation decision system if $\mathcal{S}$ and $\mathcal{T}$ contains binary relations on $\mathcal{X}$.
5. An $m$FRS $(\mathcal{X}, \mathcal{E})$ degenerates into a FRS if $\mathcal{E}$ contains fuzzy relations over $\mathcal{X} \times \mathcal{X}$.
6. An $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ degenerates into a FRDS if $\mathcal{S}$ and $\mathcal{T}$ contain fuzzy relations over $\mathcal{X} \times \mathcal{X}$.
7. An $m$FRS $(\mathcal{X}, \mathcal{E})$ degenerates into a bipolar fuzzy relation system if $\mathcal{E}$ contains bipolar fuzzy relations on $\mathcal{X}$.
8. An $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ degenerates into a bipolar fuzzy-RDS if $\mathcal{S}$ and $\mathcal{T}$ contain bipolar fuzzy relations on $\mathcal{X}$.

Therefore, $m$FRSs and $m$FRDSs are respective generalizations of FRSs and FRDSs.

With the following definition we can easily convert all the information provided in an $m$FRS into a single $m$F relation.

**Definition 3.3.** Let $(\mathcal{X}, \mathcal{E})$ be an $m$FRS. For each $\emptyset \neq \mathcal{R} \subseteq \mathcal{E}$, we define an $m$F relation $\mathcal{G}_{\mathcal{R}}$ on $\mathcal{X}$ by

$$\mathcal{G}_{\mathcal{R}}(x_j, x_k) = \cap_{\mathcal{G}_s \in \mathcal{R}} \mathcal{G}_s = \Big( \bigwedge_{\mathcal{G}_s \in \mathcal{R}} p_1 \circ \mathcal{G}_s(x_j, x_k), \bigwedge_{\mathcal{G}_s \in \mathcal{R}} p_2 \circ \mathcal{G}_s(x_j, x_k), \ldots, \bigwedge_{\mathcal{G}_s \in \mathcal{R}} p_m \circ \mathcal{G}_s(x_j, x_k) \Big),$$

for all $x_j, x_k \in \mathcal{X}$, by Definition 2.8.

Definition 3.3 produces a single $m$F relation by amalgamating all the information given in an $m$FRS $(\mathcal{X}, \mathcal{E})$.

Using this useful concept at hand, the consistency of an $m$FRDS is defined as follows:

**Definition 3.4.** An $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ is called *consistent* if $\mathcal{G}_{\mathcal{S}} \subseteq \mathcal{G}_{\mathcal{T}}$, i.e., $p_i \circ \mathcal{G}_{\mathcal{S}}(x_j, x_k) \leq p_i \circ \mathcal{G}_{\mathcal{T}}(x_j, x_k)$ for all $x_j, x_k \in \mathcal{X}$, $i = 1, 2, \ldots, m$ (v., Definition 2.8). Otherwise, $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ is called inconsistent. We call $\mathcal{X}_{\mathcal{S}\mathcal{T}} = \{x \mid r_{\mathcal{G}_{\mathcal{S}}}(x) \subseteq r_{\mathcal{G}_{\mathcal{T}}}(x)\}$ the consistent part of $m$FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$.

One can easily observe from Definition 3.4 that $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ is consistent if and only if $\mathcal{X}_{\mathcal{S}\mathcal{T}} = \mathcal{X}$.

The aim of this study concerns the following new concept:

**Definition 3.5.** Let $(\mathcal{X}, \mathcal{E})$ be an $m$FRS and fix $\mathcal{R} \subseteq \mathcal{E}$. Then, the subset $\mathcal{R}$ is called AR of $\mathcal{E}$ if

1. $\mathcal{G}_{\mathcal{E}}(x_j, x_k) = \mathcal{G}_{\mathcal{R}}(x_j, x_k)$ for all $x_j,\ x_k \in \mathcal{X}$.
2. For each $\emptyset \neq \mathcal{R}' \subseteq \mathcal{R}$, $\mathcal{G}_{\mathcal{E}} \neq \mathcal{G}_{\mathcal{R}'}$.

In information systems, knowledge can be represented by discernibility matrices. We now extend this concept to an $m$FRS. Let $(\mathcal{X}, \mathcal{E})$ be an $m$FRS with $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$.

To find reductions, the discernibility matrices $\mathcal{M}^i = (b^i_{jk})_{n \times n}$ for all the membership poles are defined separately, and their cells are given by the expression as below: for every $j, k \in J = \{1, 2, \ldots, n\}$,

$$b^i_{jk} = \{\mathcal{G} \in \mathcal{E} \mid p_i \circ \mathcal{G}(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k)\}, \tag{1}$$

for all $i = 1, 2, \ldots, m$. Thus in every cell, this matrix collects the $m$F relations to summarize the precise information that we amalgamate from the attributes concerning with the objects. The discernibility functions for all the membership values are represented by $\hat{h}^i$, and their expression is defined as $\hat{h}^i = \bigwedge_{b^i_{jk} \neq \emptyset} (\bigvee b^i_{jk})$.

To construct an efficient AR algorithm for $m$FRSs, the following results will be very helpful.

**Lemma 3.6.** *Let $(\mathcal{X}, \mathcal{E})$ be an mFRS, then $b^i_{jk}$ for all $j, k \in J$, $i = 1, 2, \ldots, m$ is nonempty set.*

**Proof.** Since $\mathcal{E}$ is finite, for $x_j, x_k \in \mathcal{X}$, we have at least one $\mathcal{G}_s \in \mathcal{E}$ which implies $\mathcal{G}_s(x_j, x_k) = \mathcal{G}_{\mathcal{E}}(x_j, x_k)$. Hence $\mathcal{G}_s \in b^i_{jk}$ and $b^i_{jk} \neq \emptyset$, for all $i = 1, 2, \ldots, m$. □

**Theorem 3.7.** *Let $(\mathcal{X}, \mathcal{E})$ be an mFRS and $\mathcal{R}(\neq \emptyset) \subseteq \mathcal{E}$. Then, $p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{R}}(x_j, x_k)$ if and only if $b^i_{jk} \cap \mathcal{R} \neq \emptyset$, for all $j, k \in J$ and $i = 1, 2, \ldots, m$.*

**Proof.** We assume that $p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{R}}(x_j, x_k)$ for all $x_j, x_k \in \mathcal{X}$, and $i = 1, 2, \ldots, m$. From Definition 3.1, $\mathcal{E}$ is a finite set, so there is at least one $\mathcal{G}_s \in \mathcal{R}$ such that $p_i \circ \mathcal{G}_s(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{R}}(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k)$, thus $\mathcal{G}_s \in \mathcal{R} \cap b^i_{jk}$, which means that $b^i_{jk} \cap \mathcal{R} \neq \emptyset$.

Conversely, assume that $b^i_{jk} \cap \mathcal{R} \neq \emptyset$ for all $j, k \in J$, $i = 1, 2, \ldots, m$. Suppose $\mathcal{G}_s \in b^i_{jk} \cap \mathcal{R}$, from definition of $b^i_{jk}$, $p_i \circ \mathcal{G}_{\mathcal{R}}(x_j, x_k) = p_i \circ \mathcal{G}_s(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k)$. Hence, $p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k) = p_i \circ \mathcal{G}_{\mathcal{R}}(x_j, x_k)$. This completes the proof. □

**Theorem 3.8.** *Let $(\mathcal{X}, \mathcal{E})$ be an mFRS and $\emptyset \neq \mathcal{R} \subseteq \mathcal{E}$. Then, $\mathcal{G}_{\mathcal{E}}(x_j, x_k) = \mathcal{G}_{\mathcal{R}}(x_j, x_k)$ for all $j, k \in J$ if $\hat{h} = \mathcal{R}$.*

**Proof.** Its proof immediately follows from Theorem 3.7. □

**Corollary 3.9.** *Let $(\mathcal{X}, \mathcal{E})$ be an mFRS and $\mathcal{R}(\neq \emptyset) \subseteq \mathcal{E}$. Then, $\mathcal{R}$ is an AR of $\mathcal{E}$ $\iff$ $\mathcal{R}$ is a smallest subset such that $\hat{h} = \mathcal{R}$.*

**Proof.** Straightforward. □

**Algorithm 1:** Attribute reduction of an $m$FRS.

1. **Input**

   $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, a universal set.

   $\mathcal{E} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t\}$, a set of $m$F relations.

   $(\mathcal{X}, \mathcal{E})$, an $m$FRS.

2. Compute $\mathcal{M}_{\mathcal{E}}$ by Definition 3.3.

3. Find discernibility matrices $\mathcal{M}^{i*} = (b^{i*}_{jk})_{n\times n}$ for all the membership values.

4. Compute discernibility functions $\hat{h}^i = \bigwedge_{b^{i*}_{jk}\neq\emptyset}(\bigvee b^{i*}_{jk})$ with respect to all discernibility matrices.

5. Transform each discernibility function $\hat{h}^i$ from its CNF $\hat{h}^i = \bigwedge_{b^{i*}_{jk}\neq\emptyset}(\bigvee b^{i*}_{jk})$ into DNF $\hat{h}^i = \bigvee_{q=1}^{s}(\bigwedge \mathcal{C}_q)$ where $\mathcal{C}_q \subseteq \mathcal{E}$.

6. **Output**

   If discernibility functions with respect to all given poles are equal, that is, $\hat{h}^1 = \hat{h}^2 = \ldots = \hat{h}^m$, then $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$ are the feasible reductions of $\mathcal{E}$ from Definition 3.5 and $Core(\mathcal{E}) = \cap_{q=1}^{s}\mathcal{C}_q$.

Using Equation (1), one can easily see that the complexity in computations of the discernibility matrix $\mathcal{M}^i$ is $O(n^m)$. For this reason we intend to improve the definition of discernibility matrix $\mathcal{M}^i$ by replacing it with $\mathcal{M}^{i*}$ below, in which computations will be more effective and feasible. Therefore we propose to change the discernibility matrix $\mathcal{M}^i = (b^i_{jk})_{n\times n}$ with the novel discernibility matrix $\mathcal{M}^{i*} = (b^{i*}_{jk})_{n\times n}$ where for each $j, k \in \{1, \ldots, n\}$,

$$b^{i*}_{jk} = \begin{cases} \{\mathcal{G}_s \mid \mathcal{G}_s \in \mathcal{E},\ p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k) = p_i \circ \mathcal{G}_s(x_j, x_k)\}, & \text{if } p_i \circ \mathcal{G}_{\mathcal{E}}(x_j, x_k) \neq 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The new discernibility matrix $\mathcal{M}^{i*}$ minimizes the computational time with respect to calculations made on the original matrix $\mathcal{M}$. This concept is explained in detail in Example 3.10 given below.

From Corollary 3.9, we present the reduction Algorithm 1 in order to compute all reductions in an $m$FRS $(\mathcal{X}, \mathcal{E})$. As argued above, Algorithm 1 provides a more general structure for the reduction algorithms (cf. [13,22, section 3]). The Appendix provides the MATLAB code for Algorithm 1, which should facilitate the reproducibility of our results.

As mentioned in [22], the conversion of the discernibility function from CNF $\hat{h} = \bigwedge_{b^*_{jk}\neq\emptyset}(\bigvee b^*_{jk})$ to a DNF $\hat{h} = \bigvee_{b^*_{jk}\neq\emptyset}(\bigwedge b^*_{jk})$ is time-consuming. Therefore, Borowik and Luba [9] launched a fast algorithm to calculate this transformation. The computational complexity of the proposed Algorithm 1 is $O(n^2)$.

Now we explain Algorithm 1 with the following illustrative example.

**Example 3.10.** Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ be a universe of four objects and $\mathcal{E} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$ be a collection of 3F relations on $\mathcal{X}$. The relational matrices $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_4$ with respect to given 3F relations are respectively given below:

$$\mathcal{M}_1 = \begin{pmatrix} (0.2,0.7,0.3) & (0.3,0.6,0.7) & (0.1,0.7,0.8) & (0.5,0.4,0.6) \\ (0.1,0.6,0.5) & (0.4,0.9,0.5) & (0.9,0.5,0.4) & (0.5,0.3,0.5) \\ (0.3,0.6,0.3) & (0.1,0.4,0.2) & (0.2,0.1,0.3) & (0.8,0.6,0.2) \\ (0.4,0.3,0.5) & (0.2,0.3,0.3) & (0.2,0.4,0.5) & (0.3,0.3,0.5) \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} (0.6,0.5,0.4) & (0.2,0.5,0.6) & (0.3,0.4,0.3) & (0.7,0.5,0.3) \\ (0.4,0.6,0.4) & (0.6,0.8,0.4) & (0.4,0.3,0.4) & (0.5,0.6,0.5) \\ (0.6,0.5,0.4) & (0.4,0.8,0.2) & (0.4,0.1,0.6) & (0.7,0.3,0.2) \\ (0.3,0.3,0.7) & (0.3,0.4,0.5) & (0.3,0.7,0.4) & (0.1,0.4,0.4) \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} (0.2,0.4,0.5) & (0.1,0.8,0.2) & (0.5,0.4,0.3) & (0.8,0.9,0.1) \\ (0.3,0.4,0.8) & (0.3,0.5,0.2) & (0.6,0.1,0.7) & (0.9,0.4,0.7) \\ (0.1,0.5,0.9) & (0.5,0.3,0.8) & (0.2,0.4,0.6) & (0.3,0.7,0.2) \\ (0.6,0.9,0.3) & (0.1,0.6,0.7) & (0.1,0.2,0.3) & (0.7,0.5,0.4) \end{pmatrix},$$

$$\mathcal{M}_4 = \begin{pmatrix} (0.3, 0.4, 0.8) & (0.3, 0.5, 0.6) & (0.6, 0.4, 0.3) & (0.9, 0.4, 0.1) \\ (0.6, 0.5, 0.4) & (0.4, 0.8, 0.2) & (0.4, 0.1, 0.6) & (0.7, 0.9, 0.8) \\ (0.4, 0.8, 0.3) & (0.2, 0.3, 0.8) & (0.2, 0.4, 0.3) & (0.3, 0.3, 0.5) \\ (0.3, 0.6, 0.7) & (0.1, 0.4, 0.8) & (0.6, 0.5, 0.3) & (0.1, 0.6, 0.7) \end{pmatrix}.$$

Thus $(\mathcal{X}, \mathcal{E})$ is an 3FRS and the relational matrix $\mathcal{M}_{\mathcal{E}} = \big(\bigwedge_{s=1}^{4} p_i \circ \mathcal{M}_s(x_i, x_j)\big)$ of $\mathcal{G}_{\mathcal{E}}$ is given by

$$\mathcal{M}_{\mathcal{E}} = \begin{pmatrix} (0.2, 0.4, 0.3) & (0.1, 0.5, 0.2) & (0.1, 0.4, 0.3) & (0.5, 0.4, 0.1) \\ (0.1, 0.4, 0.4) & (0.3, 0.5, 0.2) & (0.4, 0.1, 0.4) & (0.5, 0.3, 0.5) \\ (0.1, 0.5, 0.3) & (0.1, 0.3, 0.2) & (0.2, 0.1, 0.3) & (0.3, 0.3, 0.2) \\ (0.3, 0.3, 0.3) & (0.1, 0.3, 0.3) & (0.1, 0.2, 0.3) & (0.1, 0.3, 0.4) \end{pmatrix}.$$

Therefore, the discernibility matrix $\mathcal{M}^{1*} = (b^{1*}_{jk})_{4\times 4}$ for the 1st pole is given by

$$\mathcal{M}^{1*} = \begin{pmatrix} \{\mathcal{G}_1, \mathcal{G}_3\} & \{\mathcal{G}_3\} & \{\mathcal{G}_1\} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} \\ \{\mathcal{G}_3\} & \{\mathcal{G}_1\} & \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} \\ \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_3\} & \{\mathcal{G}_2, \mathcal{G}_4\} \end{pmatrix}.$$

By simple calculations, the discernibility function $h^1$ for the 1st pole is given by $h^1 = \mathcal{G}_1 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_2 \vee \mathcal{G}_4) = (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_2) \vee (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4)$.

Similarly, the discernibility matrices $\mathcal{M}^{2*} = (b^{2*}_{jk})_{4\times 4}$ and $\mathcal{M}^{3*} = (b^{3*}_{jk})_{4\times 4}$ for the 2nd and 3rd poles are respectively given by

$$\mathcal{M}^{2*} = \begin{pmatrix} \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_4\} \\ \{\mathcal{G}_3\} & \{\mathcal{G}_3\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_2, \mathcal{G}_3\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_2, \mathcal{G}_4\} \\ \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_1\} \end{pmatrix},$$

$$\mathcal{M}^{3*} = \begin{pmatrix} \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} \\ \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_2\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} \\ \{\mathcal{G}_3\} & \{\mathcal{G}_1\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_3\} \end{pmatrix},$$

and the discernibility functions $h^2$ and $h^3$ for the 2nd and 3rd poles are computed as below:

$$h^2 = \mathcal{G}_1 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_2 \vee \mathcal{G}_4) = (\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge \mathcal{G}_3) \vee (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4),$$

and

$$h^3 = \mathcal{G}_1 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_2 \vee \mathcal{G}_4) = (\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge \mathcal{G}_3) \vee (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4).$$

Clearly, $h^1 = h^2 = h^3$. It can be easily observed that for $\mathcal{R}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} \subseteq \mathcal{E}$, $\mathcal{R}_2 = \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\} \subseteq \mathcal{E}$ we have $\mathcal{G}_{\mathcal{E}}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_1}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_2}(x_j, x_k)$ for all $x_j,\ x_k \in \mathcal{X}$. Thus, $\mathcal{R}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\mathcal{R}_2 = \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\}$ are the ARs of the $m$FRS $(\mathcal{X}, \mathcal{E})$. Clearly, $\{\mathcal{G}_1, \mathcal{G}_3\}$ is the core for the ARs. Note that the existing discernibility matrix based AR methods in the literature cannot handle this example.

Now we present another concept which enlarges the scope of applicability of $m$FRSs by converting set-valued information systems into $m$FRSs with the help of MATLAB code. A set-valued information system can be easily converted into $m$F relational matrices using the formula given by Equation (2) below. Note that it gives symmetrical values, that is, one just needs to compute membership values for the 1st pole and all the other poles' membership values will be similar.

$$\mathcal{G}_s(x_j, x_k) = \Big(\frac{|\mathcal{G}_s(x_j) \cap \mathcal{G}_s(x_k)|}{|\mathcal{G}_s(x_j) \cup \mathcal{G}_s(x_k)|}\Big), \tag{2}$$

where $|\mathcal{G}_s(x_j)|$ denotes the cardinality of the set $\mathcal{G}_s(x_j)$.

The following example explains the above new concept:

**Example 3.11.** Let $\mathcal{X} = \{x_1, x_2, \ldots, x_6\}$ be a set of six objects and $\mathcal{E} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$ a set of four attributes(relations) on $\mathcal{X}$. Then an information system $(\mathcal{X}, \mathcal{E})$ is provided by Table 1 given below.

Every $\mathcal{G}_s$ $(s = 1, 2, 3, 4)$ can be determined as a symmetric $m$F relation on $\mathcal{X}$ using Formula (2). Each symmetric $m$F relation $\mathcal{G}_s$ $(s = 1, 2, 3, 4)$ is provided in the form of relational matrices as:

**Table 1**
A set-valued information system.

| $\mathcal{X}$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_3$ | $\mathcal{G}_4$ |
|---|---|---|---|---|
| $x_1$ | {1, 2} | {1} | {2, 3} | {1, 3} |
| $x_2$ | {3} | {2, 3} | {1} | {2} |
| $x_3$ | {2} | {1, 2, 3} | {1, 2} | {2, 3} |
| $x_4$ | {2, 3} | {3} | {1, 3} | {1, 2} |
| $x_5$ | {1} | {2, 3} | {1, 2, 3} | {3} |
| $x_6$ | {2, 3} | {2} | {1, 2} | {1, 3} |

$$\mathcal{M}_1=\begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.33,0.33,0.33)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.5,0.5,0.5)\\ (0.5,0.5,0.5) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.5,0.5,0.5)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0)\\ (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_2=\begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.67,0.67,0.67) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.5,0.5,0.5)\\ (0.33,0.33,0.33) & (0.67,0.67,0.67) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (0.33,0.33,0.33)\\ (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.67,0.67,0.67) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.5,0.5,0.5)\\ (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_3=\begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (0.33,0.33,0.33)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.5,0.5,0.5)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (1.0,1.0,1.0)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.67,0.67,0.67) & (0.33,0.33,0.33)\\ (0.67,0.67,0.67) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (0.67,0.67,0.67) & (1.0,1.0,1.0) & (0.67,0.67,0.67)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_4=\begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.5,0.5,0.5) & (1.0,1.0,1.0)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.33,0.33,0.33)\\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33)\\ (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.5,0.5,0.5)\\ (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.5,0.5,0.5) & (1.0,1.0,1.0) \end{pmatrix}.$$

The relational matrix $\mathcal{M}_{\mathcal{E}}=\big(\bigwedge_{s=1}^{4} p_i\circ\mathcal{M}_s(x_j,x_k)\big)$ of $\mathcal{G}_{\mathcal{E}}$ is given by

$$\mathcal{M}_{\mathcal{E}}=\begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\ (0.33,0.33,0.33) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.33,0.33,0.33)\\ (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix}.$$

Thus $(\mathcal{X},\mathcal{E})$ is an 3FRS. The discernibility matrix $\mathcal{M}^{1*}=(b^{1*}_{jk})_{5\times 5}$ is given as:

$$\mathcal{M}^{1*}=\begin{pmatrix} \emptyset & \mathcal{E} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4\} & \{\mathcal{G}_2\} & \{\mathcal{G}_2\} & \{\mathcal{G}_2\}\\ \mathcal{E} & \emptyset & \{\mathcal{G}_1\} & \mathcal{E} & \{\mathcal{G}_1,\mathcal{G}_4\} & \{\mathcal{G}_4\}\\ \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4\} & \{\mathcal{G}_1\} & \emptyset & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4\} & \{\mathcal{G}_1\} & \{\mathcal{G}_2,\mathcal{G}_4\}\\ \{\mathcal{G}_2\} & \mathcal{E} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4\} & \emptyset & \{\mathcal{G}_1,\mathcal{G}_4\} & \{\mathcal{G}_2\}\\ \{\mathcal{G}_2\} & \{\mathcal{G}_1,\mathcal{G}_4\} & \{\mathcal{G}_1\} & \{\mathcal{G}_1,\mathcal{G}_4\} & \emptyset & \{\mathcal{G}_1\}\\ \{\mathcal{G}_2\} & \{\mathcal{G}_4\} & \{\mathcal{G}_2,\mathcal{G}_4\} & \{\mathcal{G}_2\} & \{\mathcal{G}_1\} & \emptyset \end{pmatrix}.$$

By simple calculations, it can be easily observed that $\mathcal{M}^{1*}=\mathcal{M}^{2*}=\mathcal{M}^{3*}$ and the discernibility function $h^1=\mathcal{G}_1\wedge\mathcal{G}_2\wedge\mathcal{G}_4=h^2=h^3$. Thus $\mathcal{R}=\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_4\}\subset\mathcal{E}$ is the reduction of 3FRS $(\mathcal{X},\mathcal{E})$ because it can be easily observed that $\mathcal{G}_{\mathcal{E}}(x_j,x_k)=\mathcal{G}_{\mathcal{R}_1}(x_j,x_k)$ for all $x_j,\ x_k\in\mathcal{X}$. Clearly, the set $\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_4\}$ is the core of AR set.

## 4. Attributes reduction of *m*FRDSs

Our main concept in this section is as follows:

**Definition 4.1.** Let $(\mathcal{X},\mathcal{S}\cup\mathcal{T})$ be an $m$FRDS and $\mathcal{L}(\neq\emptyset)\subseteq\mathcal{S}$. Then the subset $\mathcal{L}$ is called an AR of $(\mathcal{X},\mathcal{S}\cup\mathcal{T})$ if

1. $\mathcal{X}_{\mathcal{S}\mathcal{T}}=\mathcal{X}_{\mathcal{L}\mathcal{T}}$,

2. For every $\emptyset \neq \mathcal{L}' \subseteq \mathcal{L}$ with $\mathcal{X}_{\mathcal{S}\mathcal{T}} \neq \mathcal{X}_{\mathcal{L}'\mathcal{T}}$.

Thus, it is clear from Definition 4.1 that the reduction set $\mathcal{L}$ is smallest and $\mathcal{X}_{\mathcal{S}\mathcal{T}}$ is not changed. As we know, knowledge in information systems can be easily described by discernibility matrices. For this reason we proceed to export this approach to the case of $m$FRDSs. Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an $m$FRDS where $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$, $\mathcal{S} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_t\}$ and $\mathcal{T} = \{\mathcal{D}\}$. To determine the reduction algorithm based on $m$FRDS, the discernibility matrices $\mathcal{N}^i = (a^i_{jk})_{v\times n}$ are defined for all the membership poles '$i$' ($i = 1, 2, \dots, m$) where $v = |\mathcal{X}_{\mathcal{S}\mathcal{T}}|$ represents the cardinality of $\mathcal{X}_{\mathcal{S}\mathcal{T}}$, which are given by:

$$a^i_{jk} = \{\mathcal{G}_l \mid \mathcal{G}_l \in \mathcal{X},\, p_i \circ \mathcal{G}_l(x_j, x_k) \leq p_i \circ \mathcal{G}_{\mathcal{D}}(x_j, x_k)\}, \tag{3}$$

for $x_j, x_k \in \mathcal{X}_{\mathcal{S}\mathcal{T}}$. The discernibility functions for all the membership poles are given by $h^i = \wedge_{a^i_{jk}\neq\emptyset}(\vee a^i_{jk})$. One can easily see that the computations complexity of all discernibility matrices $\mathcal{N}^i$ is $O(vn)$ where $v = |\mathcal{X}_{\mathcal{S}\mathcal{T}}|$.

In order to justify the corresponding algorithm for the attributes reduction, some formal results are needed.

**Lemma 4.2.** *Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an mFRDS, if $x_i \in \mathcal{X}_{\mathcal{S}\mathcal{T}}$ then discernibility matrices $a^i_{jk}$ are nonempty sets, $\forall\ j, k \in J$ and $i = 1, 2, \dots, m$.*

**Proof.** Consider $x_j \in \mathcal{X}$, then $r_{\mathcal{G}_\mathcal{S}}(x_j) \subseteq r_{\mathcal{G}_\mathcal{T}}(x_j)$, specifically, $p_i \circ \mathcal{G}_\mathcal{S}(x_j, x_k) \leq p_i \circ \mathcal{G}_\mathcal{T}(x_j, x_k)$ for all $x_k \in \mathcal{X}$. For $a^i_{jk} = \emptyset$, we have $r_{\mathcal{G}_l}(x_j) \supset r_{\mathcal{G}_\mathcal{T}}(x_j)\ \forall\ \mathcal{G}_l \in \mathcal{S}$. Thus,

$$\Big(\bigwedge_{\mathcal{G}_l\in\mathcal{S}} p_i \circ \mathcal{G}_l(x_j, x_k)\Big) = \mathcal{G}_\mathcal{S}(x_j, x_k),$$

which implies $r_{\mathcal{G}_\mathcal{S}}(x_j) \supset r_{\mathcal{G}_\mathcal{T}}(x_j)$ for all $x_k \in \mathcal{X}$. This is the contradiction of our supposition. Hence, $a^i_{jk} \neq \emptyset$ for all $i = 1, 2, \dots, m$. □

**Theorem 4.3.** *Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an mFRDS and $\mathcal{L}(\neq\emptyset) \subseteq \mathcal{S}$. Then,*

$$\mathcal{X}_{\mathcal{S}\mathcal{T}} = \mathcal{X}_{\mathcal{L}\mathcal{T}} \iff a^i_{jk} \cap \mathcal{L} \neq \emptyset$$

*for every $a^i_{jk} \neq \emptyset$, $j, k \in J$ and $i = 1, 2, \dots, m$.*

**Proof.** We consider $\mathcal{X}_{\mathcal{S}\mathcal{T}} = \mathcal{X}_{\mathcal{L}\mathcal{T}}$ and $a^i_{jk} \neq \emptyset$. If $a^i_{jk} \cap \mathcal{L} = \emptyset$, the definition of $a^i_{jk}$ implies $r_\mathcal{G}(x_j) \supset r_{\mathcal{G}_\mathcal{T}}(x_j)\ \forall\ \mathcal{G} \in \mathcal{L}$. Therefore, $r_{\mathcal{G}_\mathcal{L}}(x_j) \supset r_{\mathcal{G}_\mathcal{T}}(x_j)$. As we know $x_j \in \mathcal{X}_{\mathcal{S}\mathcal{T}} = \mathcal{X}_{\mathcal{L}\mathcal{T}}$, $r_{\mathcal{G}_\mathcal{L}}(x_j) \subseteq r_{\mathcal{G}_\mathcal{T}}(x_j)$ for all $x_k \in \mathcal{X}$, which is the contradiction of our supposition. Thus $a^i_{jk} \cap \mathcal{L} \neq \emptyset$.

Conversely, suppose that $a^i_{jk} \cap \mathcal{L} \neq \emptyset$ for every $a^i_{jk} \neq \emptyset$. Since $\mathcal{L} \subseteq \mathcal{S}$, $r_{\mathcal{G}_\mathcal{L}}(x_j) \supseteq r_{\mathcal{G}_\mathcal{S}}(x_j)$, which implies $\mathcal{X}_{\mathcal{L}\mathcal{T}} \subseteq \mathcal{X}_{\mathcal{S}\mathcal{T}}$. We now prove that $\mathcal{X}_{\mathcal{S}\mathcal{T}} \subseteq \mathcal{X}_{\mathcal{L}\mathcal{T}}$.

Let $x_j \in \mathcal{X}_{\mathcal{S}\mathcal{T}}$, then $a^i_{jk} \neq \emptyset\ \forall\ k \in J$, from the Lemma 4.2, so, $a^i_{jk} \cap \mathcal{L} \neq \emptyset$. Consider $\mathcal{G} \in a^i_{jk} \cap \mathcal{L}$, then $r_\mathcal{G}(x_j) \subseteq r_{\mathcal{G}_\mathcal{T}}(x_j)$, which implies $r_{\mathcal{G}_\mathcal{L}}(x_j) \subseteq r_{\mathcal{G}_\mathcal{T}}(x_j)\ \forall\ x_k \in \mathcal{X}$. From the definition of $\mathcal{X}_{\mathcal{L}\mathcal{T}}$, $x_j \in \mathcal{X}_{\mathcal{L}\mathcal{T}}$. Thus, $\mathcal{X}_{\mathcal{S}\mathcal{T}} \subseteq \mathcal{X}_{\mathcal{L}\mathcal{T}}$, and this completes the proof. □

**Theorem 4.4.** *Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an mFRDS and $\mathcal{L}(\neq\emptyset) \subseteq \mathcal{S}$. Then, $\mathcal{X}_{\mathcal{S}\mathcal{T}} = \mathcal{X}_{\mathcal{L}\mathcal{T}}$ if $h^i = L$ for all $i = 1, 2, \dots, m$.*

**Proof.** Its proof is directly followed by Theorem 4.3. □

**Corollary 4.5.** *Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an mFRDS and $\mathcal{L}(\neq\emptyset) \subseteq \mathcal{S}$. Then, $\mathcal{L}$ is the reduction set of $(\mathcal{X}, \mathcal{S}\cup\mathcal{T}) \iff \mathcal{L}$ is smallest set such that $h^i = L$ for all $i = 1, 2, \dots, m$.*

**Proof.** Straightforward. □

Corollary 4.5 enables us to develop the reduction Algorithm 2 given below, which computes all reductions for an $m$FRDS $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$. Algorithm 2 provides a more general structure for the reduction algorithms (cf. [13,22, section 3]): The Appendix gives the MATLAB code for it, in order to facilitate the reproducibility of our results.

The computational complexity of the developed Algorithm 2 is $O(vn)$ where $v = |\mathcal{X}_{\mathcal{S}\mathcal{T}}|$. The following example explains the application of Algorithm 2.

**Example 4.6.** Let $(\mathcal{X}, \mathcal{S}\cup\mathcal{T})$ be an $m$FRDS, where $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$ is a universe of five objects and $\mathcal{S} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$ and $\mathcal{T} = \{\mathcal{D}\}$ are the collections of $m$F relations on $\mathcal{X}$, which are provided by corresponding relational matrices as below:

**Algorithm 2:** Attributes reduction of an $m$FRDS.

1. **Input**

   $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, a universal set.

   $\mathcal{S} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t\}$, a collection of $m$F relations in the form of conditional attributes

   $\mathcal{T}$, a set of decision attributes in the form of $m$F relations.

   $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$, an $m$FRDS.

4. Find $\mathcal{X}_{\mathcal{S}\mathcal{T}}$ by using Definition 3.4.

5. Construct discernibility matrices $N^i = (a^i_{jk})_{v\times n}$ for all membership poles, where $v = |\mathcal{X}_{\mathcal{S}\mathcal{T}}|$.

6. Find discernibility functions $h^i = \bigwedge_{a^i_{jk}\neq\emptyset}(\bigvee a^i_{jk})$ for all membership poles.

7. Transform the discernibility functions from their CNF $h^i = \bigwedge_{a^i_{jk}\neq\emptyset}(\bigvee a^i_{jk})$ into DNF
   $h^i = \bigvee_{y=1}^{s}(\bigwedge \mathcal{L}_y)$ where $\mathcal{L}_y \subseteq \mathcal{S}$.

8. **Output**

   If the discernibility functions with respect to all given poles are equal, that is, $h^1 = h^2 = \ldots = h^m$,
   then by using Definition 4.1, $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s$ are the possible reductions of $\mathcal{S}$ and
   $Core(\mathcal{S}) = \cap_{y=1}^{s}\mathcal{L}_s$.

$$\mathcal{M}_1 = \begin{pmatrix} (0.7,0.6,0.4) & (0.8,0.3,0.5) & (0.3,0.3,0.6) & (0.7,0.5,0.2) & (0.2,0.3,0.2) \\ (0.2,0.7,0.3) & (0.1,0.0,0.7) & (0.5,0.7,0.4) & (0.2,0.1,0.6) & (0.4,0.5,0.3) \\ (0.6,0.8,0.7) & (0.4,0.2,0.1) & (0.7,0.5,0.6) & (0.1,0.8,0.5) & (0.7,0.9,0.3) \\ (0.2,0.4,0.3) & (0.2,0.3,0.6) & (0.2,0.5,0.6) & (0.4,0.5,0.7) & (0.6,0.3,0.3) \\ (0.6,0.2,0.2) & (0.7,0.4,0.5) & (0.1,0.4,0.8) & (0.3,0.6,0.3) & (0.9,0.3,0.6) \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} (0.1,0.4,0.5) & (0.5,0.3,0.6) & (0.2,0.5,0.6) & (0.4,0.4,0.7) & (0.5,0.3,0.2) \\ (0.4,0.2,0.1) & (0.7,0.6,0.5) & (0.3,0.4,0.8) & (0.3,0.7,0.3) & (0.8,0.3,0.6) \\ (0.3,0.7,0.4) & (0.1,0.1,0.7) & (0.5,0.8,0.4) & (0.2,0.4,0.6) & (0.4,0.6,0.3) \\ (0.5,0.8,0.6) & (0.4,0.5,0.1) & (0.7,0.5,0.6) & (0.1,0.6,0.5) & (0.7,0.8,0.2) \\ (0.6,0.4,0.5) & (0.7,0.6,0.5) & (0.4,0.2,0.6) & (0.8,0.4,0.3) & (0.4,0.3,0.2) \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} (0.4,0.7,0.6) & (0.6,0.2,0.4) & (0.5,0.4,0.3) & (0.4,0.2,0.1) & (0.6,0.4,0.1) \\ (0.3,0.5,0.6) & (0.0,0.3,0.2) & (0.3,0.6,0.7) & (0.3,0.2,0.4) & (0.7,0.8,0.6) \\ (0.6,0.7,0.7) & (0.8,0.3,0.2) & (0.6,0.4,0.2) & (0.2,0.7,0.4) & (0.6,0.4,0.5) \\ (0.3,0.2,0.4) & (0.4,0.6,0.7) & (0.3,0.4,0.7) & (0.6,0.8,0.7) & (0.5,0.4,0.1) \\ (0.5,0.2,0.3) & (0.6,0.7,0.2) & (0.2,0.4,0.3) & (0.5,0.7,0.3) & (0.4,0.6,0.6) \end{pmatrix},$$

$$\mathcal{M}_4 = \begin{pmatrix} (0.8,0.5,0.3) & (0.6,0.2,0.4) & (0.3,0.2,0.4) & (0.6,0.5,0.3) & (0.1,0.0,0.1) \\ (0.1,0.3,0.5) & (0.6,1.0,0.8) & (0.4,0.6,0.3) & (0.4,0.2,0.5) & (0.6,0.4,0.5) \\ (0.5,0.4,0.6) & (0.3,0.3,0.1) & (0.5,0.4,0.7) & (0.2,0.7,0.4) & (0.6,0.8,0.2) \\ (0.1,0.5,0.2) & (0.3,0.4,0.5) & (0.6,0.4,0.6) & (0.7,0.3,0.2) & (0.5,0.4,0.3) \\ (0.5,0.3,0.1) & (0.6,0.8,0.4) & (0.3,0.4,0.7) & (0.2,0.5,0.1) & (0.8,0.7,0.5) \end{pmatrix},$$

$$\mathcal{M}_{\mathcal{D}} = \begin{pmatrix} (0.7,0.6,0.4) & (0.8,0.3,0.5) & (0.3,0.2,0.6) & (0.7,0.4,0.2) & (0.1,0.3,0.2) \\ (0.2,0.7,0.3) & (0.1,0.0,0.7) & (0.5,0.7,0.4) & (0.2,0.1,0.6) & (0.4,0.5,0.3) \\ (0.6,0.8,0.7) & (0.4,0.2,0.1) & (0.7,0.5,0.6) & (0.1,0.8,0.5) & (0.7,0.9,0.3) \\ (0.2,0.4,0.3) & (0.2,0.3,0.6) & (0.2,0.5,0.6) & (0.4,0.5,0.7) & (0.6,0.3,0.2) \\ (0.6,0.2,0.1) & (0.7,0.4,0.5) & (0.1,0.4,0.8) & (0.3,0.6,0.3) & (0.9,0.3,0.6) \end{pmatrix}.$$

The relational matrix $\mathcal{M}_{\mathcal{S}} = \big(\bigwedge_{s=1}^{4} p_i \circ \mathcal{M}_s(x_j, x_k)\big)$ of $\mathcal{G}_{\mathcal{S}}$ is given by

$$\mathcal{M}_{\mathcal{S}} = \begin{pmatrix} (0.1,0.4,0.3) & (0.5,0.2,0.4) & (0.2,0.2,0.3) & (0.4,0.2,0.1) & (0.1,0.0,0.1) \\ (0.1,0.2,0.1) & (0.0,0.0,0.2) & (0.3,0.4,0.3) & (0.2,0.1,0.3) & (0.4,0.3,0.3) \\ (0.3,0.4,0.4) & (0.1,0.1,0.1) & (0.5,0.4,0.2) & (0.1,0.4,0.4) & (0.4,0.4,0.2) \\ (0.1,0.2,0.2) & (0.2,0.3,0.1) & (0.2,0.4,0.6) & (0.1,0.3,0.2) & (0.5,0.3,0.1) \\ (0.5,0.2,0.1) & (0.6,0.4,0.2) & (0.1,0.2,0.3) & (0.2,0.4,0.1) & (0.4,0.3,0.2) \end{pmatrix}.$$

**Table 2**
An interval value decision table.

| $\mathcal{X}$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_3$ | $\mathcal{G}_4$ | $\mathcal{G}_5$ | $\mathcal{D}$ |
|---|---|---|---|---|---|---|
| $x_1$ | [3.37, 5.85] | [4.55, 7.21] | [7.33, 9.33] | [5.31, 6.05] | [4.65, 5.22] | [4.95, 5.25] |
| $x_2$ | [3.10, 3.82] | [3.20, 3.92] | [5.68, 7.66] | [3.77, 4.63] | [3.20, 3.94] | [4.98, 5.21] |
| $x_3$ | [4.72, 6.85] | [5.47, 7.21] | [9.44, 12.38] | [5.86, 7.80] | [5.53, 7.38] | [3.99, 4.14] |
| $x_4$ | [4.32, 5.17] | [4.53, 5.42] | [6.47, 13.36] | [4.76, 5.78] | [4.49, 5.30] | [3.05, 3.13] |
| $x_5$ | [4.61, 6.14] | [5.71, 7.42] | [9.22, 13.36] | [6.54, 9.01] | [5.16, 6.75] | [2.97, 3.25] |
| $x_6$ | [3.34, 6.85] | [3.45, 4.01] | [5.93, 6.38] | [4.23, 5.11] | [3.82, 4.44] | [5.05, 5.20] |

Thus $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ is a 3FRDS with $\mathcal{X}_{\mathcal{S}\mathcal{T}} = \{x_1, x_2, x_4, x_5\}$. Therefore, the discernibility matrices $\mathcal{N}^i = (a^i_{jk})_{4\times5}$ $(i = 1, 2, 3)$ are given by

$$\mathcal{N}^1 = \begin{pmatrix} \{\mathcal{G}_2, \mathcal{G}_3\} & \mathcal{S} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} & \mathcal{S} & \{\mathcal{G}_4\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_3\} & \mathcal{S} & \{\mathcal{G}_1\} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1\} & \{\mathcal{G}_1\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\} \\ \mathcal{S} & \mathcal{S} & \{\mathcal{G}_1\} & \{\mathcal{G}_1, \mathcal{G}_4\} & \mathcal{S} \end{pmatrix},$$

$$\mathcal{N}^2 = \begin{pmatrix} \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} & \mathcal{S} & \{\mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_3\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} \\ \mathcal{S} & \{\mathcal{G}_1\} & \mathcal{S} & \{\mathcal{G}_1\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} \\ \{\mathcal{G}_1, \mathcal{G}_3\} & \{\mathcal{G}_1\} & \mathcal{S} & \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_1, \mathcal{G}_3\} & \{\mathcal{G}_1\} & \mathcal{S} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} \end{pmatrix},$$

$$\mathcal{N}^3 = \begin{pmatrix} \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\} & \mathcal{S} & \{\mathcal{G}_1, \mathcal{G}_3\} & \mathcal{S} \\ \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} & \{\mathcal{G}_1, \mathcal{G}_4\} & \mathcal{S} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} & \mathcal{S} & \{\mathcal{G}_2, \mathcal{G}_3\} \\ \{\mathcal{G}_4\} & \mathcal{S} & \mathcal{S} & \mathcal{S} & \mathcal{S} \end{pmatrix},$$

and the discernibility functions $h^1$, $h^2$ and $h^3$ are computed as $h^1 = \mathcal{G}_1 \wedge \mathcal{G}_4 \wedge (\mathcal{G}_2 \vee \mathcal{G}_3) = (\mathcal{G}_1 \wedge \mathcal{G}_4 \wedge \mathcal{G}_2) \vee (\mathcal{G}_1 \wedge \mathcal{G}_4 \wedge \mathcal{G}_3) = h^2 = h^3$, so $\mathcal{L}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4\} \subset \mathcal{S}$ and $\mathcal{L}_2 = \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\} \subset \mathcal{S}$ are the reduction sets of 3FRDS $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ because $\mathcal{G}_{\mathcal{S}}(x_j, x_k) = \mathcal{G}_{\mathcal{L}}(x_j, x_k)$ for all $x_j$, $x_k \in \mathcal{X}$. Clearly, $\{\mathcal{G}_1, \mathcal{G}_4\}$ is the core of reductions.

To reduce the knowledge in interval-valued information systems, we use a MATLAB code for the conversion of an interval-valued decision table into an $m$FRDS. This enlarges the scope of applicability of $m$FRDSs. One can easily transform an interval-valued decision information system into an $m$FRDS using the formulas given below:

$$\mathcal{G}_s(x_j, x_k) = \Big(\frac{|\mathcal{G}_s(x_j) \cap \mathcal{G}_s(x_k)|}{|\mathcal{G}_s(x_j) \cup \mathcal{G}_s(x_k)|}\Big), \tag{4}$$

$$\mathcal{D}(x_j, x_k) = \Big(\frac{|\mathcal{D}(x_j) \cap \mathcal{D}(x_k)|}{|\mathcal{D}(x_j) \cup \mathcal{D}(x_k)|}\Big), \tag{5}$$

where $|\mathcal{G}_s(x_j)|$ and $|\mathcal{D}(x_j)|$ denote the lengths of sets $\mathcal{G}_s(x_j)$ and $\mathcal{D}(x_j)$, respectively.

Now an example is given below to explain this novel concept:

**Example 4.7.** Let $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ be an interval-valued decision information system, which is displayed by the Table 2, where $\mathcal{X} = \{x_1, x_2, \dots, x_6\}$, $\mathcal{S} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$ and $\mathcal{T} = \{\mathcal{D}\}$. With the help of formulas (6) and (7), every $\mathcal{G}_s$ $(s = 1, 2, \dots, 5)$ and $\mathcal{D}$ can be described as a 3F relation on $\mathcal{X}$, and their relational matrices are given as below:

$$\mathcal{M}_1 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.16, 0.16, 0.16) & (0.33, 0.33, 0.33) & (0.34, 0.34, 0.34) & (0.45, 0.45, 0.45) & (0.71, 0.71, 0.71) \\ (0.16, 0.16, 0.16) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.61, 0.61, 0.61) \\ (0.33, 0.33, 0.33) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.18, 0.18, 0.18) & (0.63, 0.63, 0.63) & (0.61, 0.61, 0.61) \\ (0.34, 0.34, 0.34) & (0.0, 0.0, 0.0) & (0.18, 0.18, 0.18) & (1.0, 1.0, 1.0) & (0.31, 0.31, 0.31) & (0.24, 0.24, 0.24) \\ (0.45, 0.45, 0.45) & (0.0, 0.0, 0.0) & (0.63, 0.63, 0.63) & (0.31, 0.31) & (1.0, 1.0, 1.0) & (0.44, 0.44, 0.44) \\ (0.71, 0.71, 0.71) & (0.13, 0.13, 0.13) & (0.61, 0.61, 0.61) & (0.24, 0.24, 0.24) & (0.44, 0.44) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.65, 0.65, 0.65) & (0.33, 0.33, 0.33) & (0.52, 0.52, 0.52) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.58, 0.58, 0.58) \\ (0.65, 0.65, 0.65) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.77, 0.77, 0.77) & (0.0, 0.0, 0.0) \\ (0.33, 0.33, 0.33) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.52, 0.52, 0.52) & (0.0, 0.0, 0.0) & (0.77, 0.77, 0.77) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.58, 0.58, 0.58) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.09, 0.09, 0.09) & (0.0, 0.0, 0.0) & (0.29, 0.29, 0.29) & (0.02, 0.02, 0.02) & (0.0, 0.0, 0.0) \\ (0.09, 0.09, 0.09) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.16, 0.16, 0.16) & (0.0, 0.0, 0.0) & (0.23, 0.23, 0.23) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.43, 0.43, 0.43) & (0.71, 0.71, 0.71) & (0.0, 0.0, 0.0) \\ (0.29, 0.29, 0.29) & (0.16, 0.16, 0.16) & (0.43, 0.43, 0.43) & (1.0, 1.0, 1.0) & (0.60, 0.60, 0.60) & (0.0, 0.0, 0.0) \\ (0.02, 0.02, 0.02) & (0.0, 0.0, 0.0) & (0.71, 0.71, 0.71) & (0.60, 0.60, 0.60) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.23, 0.23, 0.23) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_4=\begin{pmatrix}
(1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.08,0.08,0.08) & (0.36,0.36,0.36) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.30,0.30,0.30)\\
(0.08,0.08,0.08) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.40,0.40,0.40) & (0.0,0.0,0.0)\\
(0.36,0.36,0.36) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.23,0.23,0.23)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.40,0.40,0.40) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.30,0.30,0.30) & (0.0,0.0,0.0) & (0.23,0.23,0.23) & (0.0,0.0,0.0) & (1.0,1.0,1.0)
\end{pmatrix},$$

$$\mathcal{M}_5=\begin{pmatrix}
(1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.70,0.70,0.70) & (0.03,0.03,0.03) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.10,0.10,0.10)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.55,0.55,0.55) & (0.0,0.0,0.0)\\
(0.70,0.70,0.70) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.06,0.06,0.06) & (0.0,0.0,0.0)\\
(0.03,0.03,0.03) & (0.0,0.0,0.0) & (0.55,0.55,0.55) & (0.06,0.06,0.06) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.10,0.10,0.10) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0)
\end{pmatrix},$$

$$\mathcal{M}_{\mathcal{D}}=\begin{pmatrix}
(1.0,1.0,1.0) & (0.77,0.77,0.77) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.50,0.50,0.50)\\
(0.77,0.77,0.77) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.65,0.65,0.65)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.29,0.29,0.29) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.29,0.29,0.29) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\
(0.50,0.50,0.50) & (0.65,0.65,0.65) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0)
\end{pmatrix}.$$

The relational matrix $\mathcal{M}_{\mathcal{S}}=\big(\bigwedge_{s=1}^{5} p_i\circ M_s(x_j,x_k)\big)$ of $\mathcal{G}_{\mathcal{S}}$ is given by

$$\mathcal{M}_{\mathcal{S}}=\begin{pmatrix}
(1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.29,0.29,0.29) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.10,0.10,0.10)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.40,0.40,0.40) & (0.0,0.0,0.0)\\
(0.29,0.29,0.29) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.40,0.40,0.40) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0)\\
(0.0,0.0,0.0) & (0.10,0.10,0.10) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (-1.0,1.0)
\end{pmatrix}.$$

Hence $(\mathcal{X},\mathcal{S}\cup\mathcal{T})$ is a 3FRDS with $\mathcal{X}_{\mathcal{S}\mathcal{T}}=\{x_2,x_6\}$. The discernibility matrix $\mathcal{N}^1=(a^1_{jk})_{2\times 6}$ is given by

$$\mathcal{N}^1=\begin{pmatrix}
\mathcal{S} & \mathcal{S} & \mathcal{S} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_4,\mathcal{G}_5\} & \mathcal{S} & \mathcal{S}\\
\mathcal{S} & \mathcal{S} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4,\mathcal{G}_5\} & X
\end{pmatrix}.$$

Obviously, $\mathcal{N}^1=\mathcal{N}^2=\mathcal{N}^3$. Therefore, by simple calculations, the discernibility functions are computed as $h^1=(\mathcal{G}_1\vee\mathcal{G}_2\vee\mathcal{G}_4\vee\mathcal{G}_5)\wedge(\mathcal{G}_2\vee\mathcal{G}_3\vee\mathcal{G}_5)=\mathcal{G}_2\vee\mathcal{G}_5\vee(\mathcal{G}_1\wedge\mathcal{G}_3)\vee(\mathcal{G}_3\wedge\mathcal{G}_4)=h^2=h^3$. So, $\{\mathcal{G}_2\},\{\mathcal{G}_5\},\{\mathcal{G}_1,\mathcal{G}_3\}$ and $\{\mathcal{G}_3,\mathcal{G}_4\}$ are the reductions of $(\mathcal{X},\mathcal{S}\cup\mathcal{T})$ and the core of these reductions is an empty set.

**Remark 4.8.** Notice that attribute reductions in $m$FRSs for $m=1$, $m=2$, $m=3$ and $m=4$ are not generally similar. It is nonetheless true that in case of set-valued information systems and interval-valued information systems, similar values are obtained for each pole of information due to the symmetrical behavior of Formulas (2), (4) and (5).

We explain the above-mentioned remark with the following example:

**Example 4.9.** Let $\mathcal{X}=\{x_1,x_2,x_3,x_4\}$ be a universe of four objects and $\mathcal{E}=\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4\}$ be a collection of 3F relations on $\mathcal{X}$. The relational matrices $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3$, and $\mathcal{M}_4$ with respect to the 3F relations are respectively given below:

$$\mathcal{M}_1=\begin{pmatrix}
(0.3,0.7,0.2) & (0.7,0.6,0.4) & (0.8,0.7,0.6) & (0.6,0.4,0.5)\\
(0.5,0.6,0.7) & (0.5,0.9,0.3) & (0.4,0.5,0.7) & (0.5,0.3,0.7)\\
(0.3,0.6,0.5) & (0.2,0.4,0.6) & (0.3,0.1,0.6) & (0.2,0.6,0.4)\\
(0.5,0.3,0.1) & (0.3,0.3,0.4) & (0.5,0.4,0.3) & (0.5,0.3,0.7)
\end{pmatrix},$$

$$\mathcal{M}_2=\begin{pmatrix}
(0.4,0.5,0.6) & (0.6,0.5,0.4) & (0.3,0.4,0.2) & (0.3,0.5,0.9)\\
(0.4,0.6,0.7) & (0.4,0.8,0.7) & (0.4,0.3,0.6) & (0.5,0.6,0.6)\\
(0.4,0.5,0.3) & (0.2,0.8,0.4) & (0.6,0.1,0.5) & (0.2,0.3,0.1)\\
(0.7,0.3,0.4) & (0.5,0.4,0.1) & (0.4,0.7,0.3) & (0.4,0.4,0.5)
\end{pmatrix},$$

$$\mathcal{M}_3=\begin{pmatrix}
(0.5,0.4,0.3) & (0.2,0.8,0.1) & (0.3,0.4,0.5) & (0.1,0.9,0.2)\\
(0.8,0.4,0.7) & (0.2,0.5,0.5) & (0.7,0.1,0.6) & (0.7,0.4,0.8)\\
(0.9,0.5,0.6) & (0.8,0.3,0.3) & (0.6,0.4,0.5) & (0.2,0.7,0.4)\\
(0.3,0.9,0.2) & (0.7,0.6,0.4) & (0.3,0.2,0.5) & (0.4,0.5,0.6)
\end{pmatrix},$$

$$\mathcal{M}_4=\begin{pmatrix}
(0.8,0.4,0.7) & (0.6,0.5,0.2) & (0.3,0.4,0.6) & (0.1,0.4,0.7)\\
(0.4,0.5,0.2) & (0.2,0.8,0.6) & (0.6,0.1,0.7) & (0.8,0.9,0.7)\\
(0.3,0.8,0.4) & (0.8,0.3,0.6) & (0.3,0.4,0.2) & (0.5,0.3,0.8)\\
(0.7,0.6,0.1) & (0.8,0.4,0.6) & (0.3,0.5,0.3) & (0.7,0.6,0.8)
\end{pmatrix}.$$

**Table 3**
A set-valued information system about language proficiency.

| $\mathcal{X}$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_3$ | $\mathcal{G}_4$ | $\mathcal{G}_5$ |
|---|---|---|---|---|---|
| $x_1$ | $\{S, A, C\}$ | $\{E\}$ | $\{A, C\}$ | $\{E, A, C\}$ | $\{S, A\}$ |
| $x_2$ | $\{E, C\}$ | $\{E, A\}$ | $\{S\}$ | $\{E\}$ | $\{A, C\}$ |
| $x_3$ | $\{E\}$ | $\{A, C\}$ | $\{E\}$ | $\{E, A\}$ | $\{S, C\}$ |
| $x_4$ | $\{S, E, C\}$ | $\{S\}$ | $\{S, C\}$ | $\{S, E\}$ | $\{E\}$ |
| $x_5$ | $\{S, A\}$ | $\{E, A, C\}$ | $\{E, A\}$ | $\{E, C\}$ | $\{A\}$ |
| $x_6$ | $\{E, C\}$ | $\{A\}$ | $\{E, C\}$ | $\{S, E, C\}$ | $\{E, C\}$ |

Thus $(\mathcal{X}, \mathcal{E})$ is a 3FRS and the relational matrix $\mathcal{M}_{\mathcal{E}} = \big(\bigwedge_{s=1}^{4} p_i \circ \mathcal{M}_s(x_i, x_j)\big)$ of $\mathcal{G}_{\mathcal{E}}$ is given by

$$\mathcal{M}_{\mathcal{E}} = \begin{pmatrix} (0.3, 0.4, 0.2) & (0.2, 0.5, 0.1) & (0.3, 0.4, 0.2) & (0.1, 0.4, 0.2) \\ (0.4, 0.4, 0.2) & (0.2, 0.5, 0.3) & (0.4, 0.1, 0.6) & (0.5, 0.3, 0.6) \\ (0.3, 0.5, 0.3) & (0.2, 0.3, 0.3) & (0.3, 0.1, 0.2) & (0.2, 0.3, 0.1) \\ (0.3, 0.3, 0.1) & (0.3, 0.3, 0.1) & (0.1, 0.2, 0.3) & (0.1, 0.3, 0.5) \end{pmatrix}.$$

Therefore, the discernibility matrix $\mathcal{M}^{1*} = (b_{jk}^{1*})_{4\times 4}$ for the 1st pole is

$$\mathcal{M}^{1*} = \begin{pmatrix} \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} \\ \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_2\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} \\ \{\mathcal{G}_3\} & \{\mathcal{G}_1\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_3\} \end{pmatrix}.$$

By simple calculations, the discernibility function $h^1$ for the 1st pole is computed as $h^1 = \mathcal{G}_1 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_2 \vee \mathcal{G}_4) = (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_2) \vee (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4)$. Similarly, the discernibility matrices $\mathcal{M}^{2*} = (b_{jk}^{2*})_{4\times 4}$ and $\mathcal{M}^{3*} = (b_{jk}^{3*})_{4\times 4}$ for the 2nd and 3rd poles are respectively given by

$$\mathcal{M}^{2*} = \begin{pmatrix} \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_4\} \\ \{\mathcal{G}_3\} & \{\mathcal{G}_3\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1\} \\ \{\mathcal{G}_2, \mathcal{G}_3\} & \{\mathcal{G}_3, \mathcal{G}_4\} & \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_2, \mathcal{G}_4\} \\ \{\mathcal{G}_1, \mathcal{G}_2\} & \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_1\} \end{pmatrix},$$

$$\mathcal{M}^{3*} = \begin{pmatrix} \{\mathcal{G}_1\} & \{\mathcal{G}_3\} & \{\mathcal{G}_2\} & \{\mathcal{G}_3\} \\ \{\mathcal{G}_4\} & \{\mathcal{G}_1\} & \{\mathcal{G}_2, \mathcal{G}_3\} & \{\mathcal{G}_2\} \\ \{\mathcal{G}_2\} & \{\mathcal{G}_3\} & \{\mathcal{G}_4\} & \{\mathcal{G}_2\} \\ \{\mathcal{G}_1, \mathcal{G}_4\} & \{\mathcal{G}_2\} & \{\mathcal{G}_2, \mathcal{G}_4\} & \{\mathcal{G}_2\} \end{pmatrix},$$

and the discernibility functions $h^2$ and $h^3$ for the 2nd and 3rd poles are computed as below:

$$h^2 = \mathcal{G}_1 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_2 \vee \mathcal{G}_4) = (\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge \mathcal{G}_3) \vee (\mathcal{G}_1 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4),$$

and

$$h^3 = \mathcal{G}_2 \wedge \mathcal{G}_3 \wedge (\mathcal{G}_1 \vee \mathcal{G}_4) = (\mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \mathcal{G}_1) \vee (\mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4).$$

It can be easily observed that for $m = 2$, we have $\mathcal{R}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} \subseteq \mathcal{E},\ \mathcal{R}_2 = \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} \subseteq \mathcal{E}$ which satisfy $\mathcal{G}_{\mathcal{E}}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_1}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_2}(x_j, x_k)$ for all $x_j,\ x_k \in \mathcal{X}$. Hence, $\mathcal{R}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\mathcal{R}_2 = \{\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4\}$ are the ARs of the $m$FRS $(\mathcal{X}, \mathcal{E})$ when $m = 2$. Elsewhere, when $m = 3$, clearly, $h^1 = h^2 \neq h^3$, so the discernibility function $h^3$ provides different reductions with the variation in poles of information. This happens because all the poles in an $m$F set are independent. For this reason, it is quite possible in some examples that AR for $m = 2$ occurred, but when $m = 3$ or $m = 4$ there exists no reduction.

## 5. Applications

Till date, several real-world decision-making problems that occur in the form of set-valued information systems and interval-valued decision tables have been solved in intelligent decision-making and knowledge discovery from set-valued information systems with uncertain information and interval-valued decision tables. This section provides two real-world applications of the reduction approaches developed above.

**1.** Let $(\mathcal{X}, \mathcal{E})$ be an information system provided by Table 3, where $\mathcal{X} = \{x_1, x_2, \ldots, x_6\}$ is a set of six foreign students from different countries studying on scholarship in University of the Punjab, Lahore, Pakistan. Some research students of a modern languages course conduct a survey about the language abilities of the aforementioned students. They are concerned with a set of languages $L = \{\text{Spanish}, \text{English}, \text{Arabic}, \text{Chinese}\}$, and the skills regarding these languages are captured by the set $\mathcal{E} = \{\mathcal{G}_1 = \text{Spoken Language}, \mathcal{G}_2 = \text{Reading}, \mathcal{G}_3 = \text{Writing}, \mathcal{G}_4 = \text{Audition}, \mathcal{G}_5 = \text{Listening}\}$ as attributes. For simplicity in Table 3, we use $S, E, A$ and $C$ for Spanish, English, Arabic and Chinese, respectively. Our major assignment is to identify the redundant attributes (abilities) and remove them by transforming the set-valued information system into a 3FRS. Every $\mathcal{G}_s$ $(s = 1, 2, 3, 4)$ can be determined as a symmetric 3F relation on $\mathcal{X}$ using Formula (2). Each symmetric 3F relation $\mathcal{G}_s$ $(s = 1, 2, 3, 4)$ is provided in the form of their corresponding relational matrices as below:

$$\mathcal{M}_1 = \begin{pmatrix} (1.0,1.0,1.0) & (0.25,0.25,0.25) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.67,0.67,0.67) & (0.25,0.25,0.25) \\ (0.25,0.25,0.25) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.67,0.67,0.67) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \\ (0.0,0.0,0.0) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) \\ (0.5,0.5,0.5) & (0.67,0.67,0.67) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.25,0.25,0.25) & (0.67,0.67,0.67) \\ (0.67,0.67,0.67) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.25,0.25,0.25) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.25,0.25,0.25) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.67,0.67,0.67) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) \\ (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.67,0.67,0.67) & (0.5,0.5,0.5) \\ (0.0,0.0,0.0) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.67,0.67,0.67) & (0.5,0.5,0.5) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.33,0.33,0.33) & (0.67,0.67,0.67) & (0.67,0.67,0.67) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.33,0.33,0.33) \\ (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.33,0.33,0.33) \\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.5,0.5,0.5) \\ (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) \\ (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.33,0.33,0.33) \\ (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_4 = \begin{pmatrix} (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.67,0.67,0.67) & (0.25,0.25,0.25) & (0.67,0.67,0.67) & (0.5,0.5,0.5) \\ (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.33,0.33,0.33) \\ (0.67,0.67,0.67) & (0.5,0.5,0.5) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.25,0.25,0.25) \\ (0.25,0.25,0.25) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.67,0.67,0.67) \\ (0.67,0.67,0.67) & (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.67,0.67,0.67) \\ (0.5,0.5,0.5) & (0.33,0.33,0.33) & (0.25,0.25,0.25) & (0.67,0.67,0.67) & (0.67,0.67,0.67) & (1.0,1.0,1.0) \end{pmatrix},$$

$$\mathcal{M}_5 = \begin{pmatrix} (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.0,0.0,0.0) \\ (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.5,0.5,0.5) & (0.33,0.33,0.33) \\ (0.33,0.33,0.33) & (0.33,0.33,0.33) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.5,0.5,0.5) \\ (0.5,0.5,0.5) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.33,0.33,0.33) & (0.5,0.5,0.5) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix}.$$

The relational matrix $\mathcal{M}_{\mathcal{E}} = \big( \bigwedge_{s=1}^{5} p_i \circ \mathcal{M}_s(x_j, x_k) \big)$ of $\mathcal{G}_{\mathcal{E}}$ is given by

$$\mathcal{M}_{\mathcal{E}} = \begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.25,0.25,0.25) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.33,0.33,0.33) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix}.$$

Thus $(\mathcal{X}, \mathcal{E})$ is a 3FRS. The discernibility matrix $\mathcal{M}^{1*} = (b^{1*}_{jk})_{5\times 5}$ is given as:

$$\mathcal{M}^{1*} = \begin{pmatrix} \emptyset & \{\mathcal{G}_3\} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} & \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_3\} & \{\mathcal{G}_2,\mathcal{G}_5\} \\ \{\mathcal{G}_3\} & \emptyset & \{\mathcal{G}_3\} & \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_3\} & \{\mathcal{G}_3\} \\ \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} & \{\mathcal{G}_3\} & \emptyset & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_5\} & \{\mathcal{G}_4\} \\ \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \emptyset & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \\ \{\mathcal{G}_2,\mathcal{G}_3\} & \{\mathcal{G}_1,\mathcal{G}_3\} & \{\mathcal{G}_1,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \emptyset & \{\mathcal{G}_1,\mathcal{G}_5\} \\ \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_3\} & \{\mathcal{G}_5\} & \{\mathcal{G}_2\} & \{\mathcal{G}_1,\mathcal{G}_5\} & \emptyset \end{pmatrix}.$$

By simple calculations, we obtain that $\mathcal{M}^{1*} = \mathcal{M}^{2*} = \mathcal{M}^{3*}$ and the discernibility function $\hat{h}^1 = \mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4 \wedge (\mathcal{G}_1 \vee \mathcal{G}_5) = (\mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4 \wedge \mathcal{G}_1) \vee (\mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \mathcal{G}_4 \wedge \mathcal{G}_5) = \hat{h}^2 = \hat{h}^3$. Thus $\mathcal{R}_1 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\} \subset \mathcal{E}$ and $\mathcal{R}_2 = \{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\} \subset \mathcal{E}$ are the reductions of 3FRS $(\mathcal{X}, \mathcal{E})$ because one can readily see that $\mathcal{G}_{\mathcal{E}}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_1}(x_j, x_k) = \mathcal{G}_{\mathcal{R}_2}(x_j, x_k)$ for all $x_j,\ x_k \in \mathcal{X}$. Clearly, the set $\{\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$ is the core of reductions.

**2.** Let $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ be a decision system given by Table 5, where $\mathcal{X} = \{x_1, x_2, \ldots, x_6\}$ is a set of six patients and $\mathcal{E} = \{\mathcal{G}_1 = \text{Respiratory Problems}, \mathcal{G}_2 = \text{Fever}, \mathcal{G}_3 = \text{Headache}, \mathcal{G}_4 = \text{Muscle Pain}, \mathcal{G}_5 = \text{Sore Throat}\}$ is the set of symptoms diagnosed by different experts (doctors) of a country from different localities in these patients. They provide their diagnosis reports compiled in the form of the interval-valued decision Table 5, where each attribute is assigned interval value with respect to the criteria stated in Table 4, because each symptom has different stages in medical terms, that is, incubation, prodromal,

**Table 4**
Diagnostic Criteria.

| Symptom Stages | Interval values range |
|---|---|
| Incubation period | [1, 2.5] |
| Prodromal | (2.5, 5.0] |
| Illness | (5.0, 7.5] |
| Decline | (7.5, 10.0] |
| Convalescence | (10.0, 12.5] |

**Table 5**
An interval value decision table about patients health.

| $\mathcal{X}$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_3$ | $\mathcal{G}_4$ | $\mathcal{G}_5$ | $D$ |
|---|---|---|---|---|---|---|
| $x_1$ | [2.17, 2.86] | [5.32, 7.23] | [2.45, 2.96] | [2.22, 4.02] | [2.51, 4.04] | [4.23, 5.60] |
| $x_2$ | [3.37, 4.75] | [7.24, 10.47] | [3.43, 4.85] | [1.64, 3.75] | [2.52, 4.12] | [4.37, 6.27] |
| $x_3$ | [1.83, 2.70] | [7.23, 10.27] | [1.78, 2.98] | [1.75, 3.86] | [7.12, 11.26] | [6.95, 8.54] |
| $x_4$ | [1.35, 2.12] | [2.59, 3.93] | [1.42, 2.09] | [1.34, 4.91] | [1.0, 1.72] | [4.89, 6.04] |
| $x_5$ | [3.46, 5.35] | [6.37, 10.28] | [3.37, 5.11] | [1.20, 3.30] | [1.10, 1.82] | [5.88, 7.11] |
| $x_6$ | [2.22, 3.07] | [4.37, 7.05] | [2.43, 3.32] | [1.14, 3.21] | [3.58, 5.65] | [2.85, 4.15] |

illness, decline, convalescence (notice that these are all stages of any symptom). Our major assignment is to identify the redundant attributes (symptoms) and remove them by transforming the interval-valued information system into a 3FRDS.

With the help of formulas (4) and (5), every $\mathcal{G}_s$ $(s = 1, 2, \ldots, 5)$ and $\mathcal{D}$ can be described as 3F relations on $\mathcal{X}$, and their relational matrices are given as below:

$$\mathcal{M}_1 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.52, 0.52, 0.52) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.71, 0.71, 0.71) \\ (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.65, 0.65, 0.65) & (0.0, 0.0, 0.0) \\ (0.52, 0.52, 0.52) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.22, 0.22, 0.22) & (0.0, 0.0, 0.0) & (0.39, 0.39, 0.39) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.22, 0.22) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.65, 0.65, 0.65) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) \\ (0.71, 0.71, 0.71) & (0.0, 0.0, 0.0) & (0.39, 0.39, 0.39) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_2 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.17, 0.17) & (0.61, 0.61, 0.61) \\ (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.94, 0.94, 0.94) & (0.0, 0.0, 0.0) & (0.74, 0.74, 0.74) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.94, 0.94, 0.94) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.78, 0.78, 0.78) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.17, 0.17, 0.17) & (0.74, 0.74, 0.74) & (0.78, 0.78, 0.78) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.12, 0.12, 0.12) \\ (0.61, 0.61, 0.61) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.12, 0.12) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.43, 0.43, 0.43) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.57, 0.57, 0.57) \\ (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.82, 0.82, 0.82) & (0.0, 0.0, 0.0) \\ (0.43, 0.43, 0.43) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.20, 0.20, 0.20) & (0.0, 0.0, 0.0) & (0.36, 0.36, 0.36) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.20, 0.20, 0.20) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.82, 0.82, 0.82) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) \\ (0.57, 0.57, 0.57) & (0.0, 0.0, 0.0) & (0.36, 0.36, 0.36) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_4 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.64, 0.64, 0.64) & (0.72, 0.72, 0.72) & (0.50, 0.50, 0.50) & (0.38, 0.38, 0.38) & (0.34, 0.34, 034) \\ (0.64, 0.64, 0.64) & (1.0, 1.0, 1.0) & (0.90, 0.90, 0.90) & (0.59, 0.59, 0.59) & (0.65, 0.65, 0.65) & (0.60, 0.60, 0.60) \\ (0.72, 0.72, 0.72) & (0.90, 0.90, 0.90) & (1.0, 1.0, 1.0) & (0.59, 0.59, 0.59) & (0.58, 0.58, 0.58) & (0.54, 0.54, 0.54) \\ (0.50, 0.50, 0.50) & (0.59, 0.59, 0.59) & (0.59, 0.59, 0.59) & (1.0, 1.0, 1.0) & (0.53, 0.53, 0.53) & (0.50, 0.50, 0.50) \\ (0.38, 0.38, 0.38) & (0.65, 0.65, 0.65) & (0.58, 0.58, 0.58) & (0.53, 0.53, 0.53) & (1.0, 1.0, 1.0) & (0.93, 0.93, 0.93) \\ (0.34, 0.34, 0.34) & (0.60, 0.60, 0.60) & (0.54, 0.54, 0.54) & (0.50, 0.50, 0.50) & (0.93, 0.93, 0.93) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_5 = \begin{pmatrix} (1.0, 1.0, 1.0) & (0.94, 0.94, 0.94) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.15, 0.15, 0.15) \\ (0.94, 0.94, 0.94) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.17, 0.17, 0.17) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) & (0.76, 0.76, 0.76) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.76, 0.76, 0.76) & (1.0, 1.0, 1.0) & (0.0, 0.0, 0.0) \\ (0.15, 0.15, 0.15) & (0.17, 0.17, 0.17) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 1.0, 1.0) \end{pmatrix},$$

$$\mathcal{M}_{\mathcal{D}} = \begin{pmatrix} (1.0,1.0,1.0) & (0.60,0.60,0.60) & (0.0,0.0,0.0) & (0.39,0.39,0.39) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.60,0.60,0.60) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.61,0.61,0.61) & (0.14,0.14,0.14) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.06,0.06,0.06) & (0.0,0.0,0.0) \\ (0.39,0.39,0.39) & (0.61,0.61,0.61) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.07,0.07,0.07) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.14,0.14,0.14) & (0.06,0.06,0.06) & (0.07,0.07,0.07) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix}.$$

The relational matrix $\mathcal{M}_{\mathcal{S}} = \big(\bigwedge_{s=1}^{5} p_i \circ \mathcal{M}_s(x_j, x_k)\big)$ of $\mathcal{G}_{\mathcal{S}}$ is given by

$$\mathcal{M}_{\mathcal{S}} = \begin{pmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.15,0.15,0.15) \\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.15,0.15,0.15) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{pmatrix},$$

Thus $(\mathcal{X}, \mathcal{S} \cup \mathcal{T})$ is a 3FRDS with $\mathcal{X}_{\mathcal{ST}} = \{x_2, x_3, x_4, x_5\}$. The discernibility matrix $\mathcal{N}^1 = (a^i_{jk})_{4\times 6}$ is given by

$$\mathcal{N}^1 = \begin{pmatrix} \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} & \mathcal{S} & \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} & \mathcal{S} & \{\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} \\ \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} & \mathcal{S} & \{\mathcal{G}_2,\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} & \{\mathcal{G}_2,\mathcal{G}_5\} \\ \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} & \mathcal{S} & \{\mathcal{G}_2,\mathcal{G}_5\} & \mathcal{S} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_5\} \\ \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} & \{\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} & \{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\} & \mathcal{S} & \{\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_5\} \end{pmatrix}.$$

Clearly, $\mathcal{N}^1 = \mathcal{N}^2 = \mathcal{N}^3$. Therefore, by simple computations, the discernibility functions are given by $h^1 = \mathcal{G}_5 \wedge (\mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{G}_3) = (\mathcal{G}_1 \wedge \mathcal{G}_5) \vee (\mathcal{G}_2 \wedge \mathcal{G}_5) \vee (\mathcal{G}_3 \wedge \mathcal{G}_5) = h^2 = h^3$. So, $\{\mathcal{G}_1, \mathcal{G}_5\}, \{\mathcal{G}_2, \mathcal{G}_5\}$ and $\{\mathcal{G}_3, \mathcal{G}_5\}$, are the reductions and the core is $\{\mathcal{G}_5\}$.

## 6. Discussion

Due to the existence of multi-polar data in almost every field of life ranging from engineering to medical sciences, multi-polar fuzzy set theory has a rich potential for applications. The reason behind the importance and capabilities of this theory is that it covers most possible properties or characteristics of an element in a universe. Thus, better results are obtained as compared to existing models since no loss in data occurs in case of a modelization by an multi-polar fuzzy set.

A critical inspection of bipolar FRDSs [7] and the multi-polar FRDSs developed in this paper leads to an important conclusion: both these structures have abilities to tackle different modes of information. Bipolar fuzzy relation decision systems consider a piece of information in two dimensional format. It is presumed that its first dimension is for the membership function regarding certain property, while the second dimension is for the membership function regarding some implicit counter property. In contrast, multi-polar FRDSs consider a piece of information in $m$-dimensional format, and each of its dimensions enables us to represent the membership function with respect to a sub-characteristic of a certain property. For example, consider $(0.7, -0.6)$ as a bipolar fuzzy representation of the property "yearly earning profits" of a company. Then 0.7 is the belongingness value for the company regarding profit while $-0.6$ is bound to be the belongingness value regarding loss. However in a multi-polar fuzzy environment, a 3-tuple like $(0.6, 0.2, 0.5)$ enables us to capture more subtle information: 0.6 can be used to represent the belongingness value with respect to profits due to investments in hotel industry, 0.2 can be used to represent the belongingness value with respect to profits due to investments in the stock market, and 0.5 can represent the belongingness value for profits from investments in film industry. Another important fact is that for $m = 1$, the multi-polar FRSs developed in this paper boil down to the FRSs [13]; and when $m = 2$ our proposed multi-polar FRSs and multi-polar FRDSs become bipolar FRSs [7] and bipolar FRDSs [7], respectively. Note that for $m = 2$, the developed multi-polar FRDSs and the existing bipolar fuzzy RDSs are cryptomorphic mathematical notions. From the above discussion, it is clear now that if we compare multi-polar FRDSs with the existing FRSs or bipolar FRDSs, then our developed multi-polar FRDSs and their reduction approaches are more precise and flexible as compared to both FRSs [13] and bipolar FRDSs [7].

Several structures have been developed to represent data in an appropriate way like relation systems, relation decision systems, FRSs and FRDSs. A relation system is an efficient extension of a typical information system which only considers the information in binary form, while a FRS has ability to tackle information in both binary and fuzzy environments. AR of these structures are very helpful to minimize the computational time in order to reach the final decision by deleting irrelevant attributes (by this we mean those attributes whose exclusion does not affect the final decision). Specifically, ARs of FRSs [13] and FRDSs [13] have been recently presented. Multi-polar fuzzy relations are playing a significant role in many areas to solve decision-making problems, especially in graph theory[3]. Nevertheless, for a multi-polar FRDS, no AR algorithm that is capable to compute possible reductions is available in the literature. For this reason this paper presents multi-polar

**Table 6**
Comparison Table.

| AR Methods | Reduction | Dependence | Variation in reduction |
|---|---|---|---|
| FRSs [13] | $\{R_1, R_2, R_3\}$ when $X = \{1.3\}$ | Yes (on $X$) | Yes (with variation in $X$) |
| Proposed $m$FRDSs | $\{R_1, R_2, R_3\}$ | No | No |

FRDSs and their AR methods as a natural and direct extension of FRSs [13]. Notice that the AR technique used in the FRSs studied in [13] is different from the approach developed herein because it was based on lower and upper approximations of a subset $X$ of the universal set. Thus the reduction changes with the variation in $X$. But to verify the effectiveness of our approach to AR, we have applied the reduction Algorithm 1 to Example 1 of [13], and we have obtained similar reduction results which are shown by Table 6. We conclude that our AR methods, namely, multi-polar FRSs and multi-polar FRDSs, are more effective and cogent than the existing procedures.

## 7. Conclusion

In the real-world, databases comprise a wide range of attributes and data. Attribute reduction is one of the key procedures whereby irrelevant attributes are deleted from a given database of knowledge while preserving its consistency. AR is used for dimensionality reduction of an information system and the major concern is to choose a most suitable subset having unique features of the whole data-set. During the last few decades, AR methods in both rough set and soft set theories have been developed by many computer scientists and mathematicians who made an active subject of research of attribute reduction. The concepts of FRSs and their AR algorithms are studied in [13]. In many real-world problems, data sometimes come from two or more agents and so multi-polar information exists. This information cannot be well represented by means of fuzzy sets or bipolar fuzzy sets. The existence of multi-polar fuzzy relations in everyday life motivated us to consider the reduction problem for multi-polar FRSs. Thus in this paper, we have presented a systematic discussion about AR based on multi-polar FRSs and multi-polar FRDSs, which are extensions of existing structures such as FRSs and bipolar FRDSs. With the assistance of mathematical results and their formal proofs, we have provided AR algorithms and solved different problems based upon both multi-polar FRSs and multi-polar FRDSs. Afterwards, we have explored two real-life applications of the proposed AR algorithms, to prove their validity and reliability. The reduction methods that we have proposed can be regarded as a unification of certain existing reduction approaches in [13,22]. Finally, the new AR methods are compared with some existing approaches to show their reliability over them. Our contribution may enhance alternative concepts and set new objectives for AR in the context of our AR methods, or even promote knowledge reduction in existing extensions of multi-polar fuzzy systems including multi-polar fuzzy soft sets [2], hesitant multi-polar fuzzy sets [8], and multi-polar fuzzy $N$-soft sets [5].

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A

This Appendix contains the MATLAB codes for (a) the two algorithms for attribute reduction, namely, Algorithm 1 (for multi-polar fuzzy relation systems) and Algorithm 2 (for multi-polar fuzzy relation decision systems), plus (b) the transformation of information into the required formats.

**Algorithm:** MATLAB code for Algorithm 1.

```
Start
n =input('insert the number of objects in the universe 𝒳:');
t =input('insert the number of mF relations on the universe 𝒳:');
Q =input('enter the poles of mF relations separately as a matrix of order n∗m × n')
[k,n] = size(Q); m = k/n; M = ones(n,n); C = zeros(n,n); O = zeros(m,n^2); p = 0;
for k = 1 : n∗m
        l = mod(k,n);
        if l == 0
                l = n;
        end
        for i = 1 : n
                C(l,i) = Q(k,i);
                M(l,i) = min(M(l,i), C(l,i));
        end
        if mod(k,n) == 0
                p = p + 1;
                C = C';
                O(p,:) = C(:)';
        end
end
M
M = M'; K = M(:)';
L = zeros(n,n∗m); q = zeros(m,n^2);
for v = 1 : m
        for w = 1 : n^2
                if K(1,w) == O(v,w)
                        q(v,w) = 1;
                end
        end
end
q
N = q';
for i = 1 : n^2
        l = mod(i,n);
        c = 0;
        fprintf('{');
        for j = 1 : m
                if N(i,j) == 1
                        fprintf('r_%c',int2str(j));
                end
        end
        fprintf('}');
        if l == 0
                fprintf('\n')
        end
        fprintf(' ');
end
N = q'
S = N; k = 1 : m;
for i = 1 : n^2
        if N(i,k) == zeros(1,m)
                S(i,k) = ones(1,m);
        end
end
S
k = 1 : m;
for i = 1 : n∗n
        for j = 1 : n∗n
                if min(S(i,k), S(j,k)) == S(j,k)
                        S(i,k) = S(j,k);
                end
        end
end
S
z = S'; B = zeros(m,m);
for i = 1 : m
        for j = 1 : m
                for k = 1 : n^2
                        if S(k,i) == 1
                                B(i,j) = max(z(j,k), S(k,j));
                        end
                end
```

```
        end
end
B
L = zeros(m, 1);
for i = 1 : m
        for j = 1 : m
                if B(i, j) == 1
                        L(i) = L(i) + B(i, j);
                end
        end
end
L;
fprintf('\n the reduction is obtained by transforming this CNF into DNF by using fast
                algorithm [9]')
for i = m : −1 : 1
        if L(i) == 1
                for j = 1 : m
                        if B(i, j) == 1
                            fprintf('r_%c',int2str(j));
                        end
                end
        end
end
for i = m : −1 : 1
        if L(i) > 1
                c = 0;
                for j = 1 : m
                        if B(i, j) == 1
                                c = c + 1;
                                if c == 1
                                    fprintf('(')
                                end
                                fprintf('r%c',int2str(j));
                                if c >= 1&&j < m
                                    z = 0;
                                    for k = c + 1 : m
                                            if B(i, k) == 1
                                                    z = z + 1;
                                                    if z == 1
                                                        fprintf('+');
                                                    end
                                            end
                                    end
                                end
                        end
                        if j == m
                                fprintf(')')
                        end
                end
        end
end
fprintf('\n the core is')
for i = m : −1 : 1
        if L(i) == 1
                for j = 1 : m
                        if B(i, j) == 1
                                fprintf('r_%c',int2str(j));
                        end
                end
        end
end
Stop
```

**Algorithm:** MATLAB code for the transformation of set value information systems into *m*FRS.

```
Start
n =input('insert the number of objects in the universe 𝒳:');
t =input('insert the number of mF relations on the universe 𝒳:');
S=input('insert the data in a row having order n*t for each attribute t in binary form with
        respect to relations, that is, {1, 2} is represented as "1 1 0 0", for four mF relations')
y = zeros(n, t); A_n = zeros(n*t, n); B_n = zeros(n*t, n); V = zeros(n*t, n);
for l = 1 : t
    fprintf('Insert the data for attribute %d as a row having order %d',l, n*t);
    A = S(l, :);%input('insert the data');
    r = 0; c = 0;
    for f = 1 : n*t
        if mod(f, t) == 1
            r = r + 1;
        end
        h = mod(f, t);
        if h == 0
            h = t;
        end
        y(r, h) = A(1, f);
    end
    y;
    R = y';
    s = zeros(n, n); q = zeros(n, n);
    for b = 1 : n
        for j = 1 : n
            for k = 1 : t
                if min(y(b, k), R(k, j)) == 1
                    s(b, j) = s(b, j) + 1;
                end
                if max(y(b, k), R(k, j)) == 1
                    q(b, j) = q(b, j) + 1;
                    if q(b, j) == 0
                        q(b, j) = q(b, j) + 1;
                    end
                end
            end
        end
        for x = n*l − (n − 1) : n*l
            h = mod(x, n);
            k = 1 : n;
            c = k;
            if h == 0
            h = n;
            end
            A_n(x, c) = s(h, k);
            B_n(x, c) = q(h, k);
        end
    end
end
A_n
B_n
V = A_n./B_n
Stop
```

**Algorithm:** MATLAB code for Algorithm 2.

```
Start
Same steps as Code for Algorithm 1 to calculate M.
M
D =input('enter the corresponding pole of mF relation decision matrix of order n*m × n')
P = zeros(n, n);
for i = 1 : n
    for j = 1 : n
        if M(i, j) <= D(i, j)
            P(i, j) = 1;
        end
    end
end
for i = 1 : n
    if P(i, :) == ones(1, n)
        P(i, :) = P(i, :);
    else
        P(i, :) = zeros(1, n);
    end
end
P
C = zeros(n, n); N = zeros(n, n); O = zeros(m, n^2); p = 0;
for k = 1 : n*m
    l = mod(k, n);
    if l == 0
        l = n;
    end
    for i = 1 : n
        if P(l, i) == 0
            C(l, i) = 0;
        else
            C(l, i) = Q(k, i);
        end
    end
    if mod(k, n) == 0
        N = C';
        p = p + 1;
        O(p, :) = N(:)';
    end
end
D = D'; K = D(:)'; q = zeros(m, n^2);
for v = 1 : m
    for w = 1 : n^2
        if O(v, w) == 0
            q(v, w) = 0;
        else
            if K(1, w) >= O(v, w)
                q(v, w) = 1;
            end
        end
    end
end
q
After computing q, the discernibility matrix, discernibility function, reduction sets and core
 will be calculated by using same steps as Algorithm 1.
Stop
```

**Algorithm:** MATLAB code for the transformation of interval-valued decision tables into $m$FRDS.

```
Start
n =input('insert the number of objects in the universe 𝒳:');
t =input('insert the number of mF relations on the universe 𝒳:');
J =input('insert the data for each attribute t as a row having order n * t')
[t, k] = size(J); n = k/2;
for i = 1 : t
        fprintf('insert the data for attribute %d as a row having order %d',i, 2 * n);
        A = J(i, :);%input('insert the data');
        q = 0; y = zeros(n, 2);
        for e = 1 : 2 * n
                if mod(e, 2) == 1
                        q = q + 1;
                end
                r = mod(e, 2);
                if r == 0
                        r = 2;
                end
                y(q, r) = A(1, e);
        end
        y; R = y'; o = zeros(n, 2 * n); Q = zeros(n, n); S = zeros(n, n);
        for a = 1 : n
                for b = 1 : n
                        j = 2 * b − 1;
                        if y(a, 1) >= R(1, b)
                                if y(a, 1) <= R(2, b)
                                        o(a, j) = y(a, 1);
                                        if y(a, 2) >= R(1, b)
                                                if y(a, 2) <= R(2, b)
                                                        o(a, j + 1) = y(a, 2);
                                                elseif y(a, 2) >= R(2, b)
                                                        o(a, j + 1) = R(2, b);
                                                end
                                        end
                                end
                        elseif R(1, b) >= y(a, 1)
                                if R(1, b) <= y(a, 2)
                                        o(a, j + 1) = R(1, b);
                                        if R(2, b) >= y(a, 1)
                                                if R(2, b) <= y(a, 2)
                                                        o(a, j) = R(2, b);
                                                elseif R(2, b) >= y(a, 2)
                                                        o(a, j) = y(a, 2);
                                                end
                                        end
                                end
                        end
                        Q(a, b) = abs(abs(o(a, j + 1) − o(a, j)));
                        S(a, b) = abs(abs(y(a, 2) − y(a, 1)) + abs(R(2, b) − R(1, b)) − Q(a, b));
                end
        end
        for w = n * i − (n − 1) : n * i
                r = mod(w, n); k = 1 : n; c = 2 * k − 1;
                if r == 0
                        r = n;
                end
                A_n(w, c) = Q(r, k); A_n(w, c + 1) = Q(r, k); B_n(w, c) = S(r, k); B_n(w, c + 1) = S(r, k);
        end
end
T = A_n./B_n
Stop
```

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