

PhD Thesis

The Galactic Center as a laboratory for dark matter and theories of modified gravity

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**VNiVERSiDAD
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Publications

This PhD thesis is based on the work carried out over the course of the last three years in the Department of Fundamental Physics of the University of Salamanca, Spain. The work resulted in a total of 14 publications in high-impact journals of which: the 12 that we detail below are directly correlated to the subject of the thesis:

- [1]: Ivan De Martino, Riccardo della Monica, and Mariafelicia De Laurentis. ‘f(R) gravity after the detection of the orbital precession of the S2 star around the Galactic Center massive black hole’. In: *Phys. Rev. D* 104.10, L101502 (2021). arXiv: [2106.06821 \[gr-qc\]](#).
- [2]: Riccardo Della Monica, Ivan de Martino, and Mariafelicia de Laurentis. ‘Orbital precession of the S2 star in Scalar-Tensor-Vector Gravity’. In: *MNRAS* 510.4 (2022). arXiv: [2105.12687 \[gr-qc\]](#).
- [3]: Riccardo Della Monica and Ivan de Martino. ‘Unveiling the nature of SgrA* with the geodesic motion of S-stars’. In: *J. Cosmology Astropart. Phys.* 2022.3, 007 (2022). arXiv: [2112.01888 \[astro-ph.GA\]](#).
- [4]: Riccardo Della Monica and Ivan de Martino. ‘Narrowing the allowed mass range of ultralight bosons with the S2 star’. In: *A&A* 670, L4 (2023). arXiv: [2206.03980 \[gr-qc\]](#).
- [5]: R. Della Monica, I. de Martino, D. Vernieri, and M. de Laurentis. ‘Testing horndeski gravity with S2 star orbit’. In: *MNRAS* 519.2 (2023). arXiv: [2212.05082 \[gr-qc\]](#).
- [6]: Riccardo Della Monica, Ivan de Martino, and Mariafelicia de Laurentis. ‘Response to: Comment on “Orbital precession of the S2 star in scalar-tensor-vector gravity”’. In: *MNRAS* (2023). arXiv: [2302.12296 \[gr-qc\]](#).
- [7]: M. Cadoni, M. De Laurentis, I. De Martino, R. Della Monica, M. Oi, and A. P. Sanna. ‘Are nonsingular black holes with super-Planckian hair ruled out by S2 star data?’ In: *Phys. Rev. D* 107.4, 044038 (2023). arXiv: [2211.11585 \[gr-qc\]](#).
- [8]: Rebeca Fernández Fernández, Riccardo Della Monica, and Ivan de Martino. ‘Constraining an Einstein-Maxwell-dilaton-axion black hole at the Galactic Center with the orbit of the S2 star’. In: *J. Cosmology Astropart. Phys.* 2023.8, 039 (2023). arXiv: [2306.06937 \[gr-qc\]](#).
- [9]: Mariafelicia De Laurentis, Ivan de Martino, and Riccardo Della Monica. ‘The Galactic Center as a laboratory for theories of gravity and dark matter’. In: *Reports on Progress in Physics* 86.10, 104901 (2023). arXiv: [2211.07008 \[astro-ph.GA\]](#).

- [10]: Riccardo Della Monica and Ivan de Martino. ‘Bounding the mass of ultralight bosonic dark matter particles with the motion of the S2 star around Sgr A*’. In: *Phys. Rev. D* 108.10, L101303 (2023). arXiv: [2305.10242 \[gr-qc\]](#).
- [11]: Riccardo Della Monica, Ivan De Martino, and Mariafelicia De Laurentis. ‘Testing space-time geometries and theories of gravity at the Galactic centre with pulsar’s time delay’. In: *MNRAS* 524.3 (2023). arXiv: [2305.18178 \[gr-qc\]](#).
- [12]: Ivan De Martino, Riccardo Della Monica, and Diego Rubiera-García. ‘Constraints on metric-affine gravity black holes from the stellar motion at the Galactic Center’. In: *Phys. Rev. D* 109.2, 024016 (2024). arXiv: [2311.08031 \[gr-qc\]](#).

Additionally, 2 publications, while belonging to the same macro-area, are focused on a different subject and therefore do not form part of the scientific content of the thesis:

- [13]: Riccardo Della Monica, Ivan de Martino, and Tom Broadhurst. ‘Explaining the oblate morphology of dwarf spheroidals with Wave Dark Matter perturbations’. In: *arXiv e-prints*, arXiv:2308.14664 (2023). arXiv: [2308.14664 \[astro-ph.GA\]](#),
- [14]: I. De Martino, R. Della Monica, and D. Rubiera-García. ‘Optical appearance of a nonsingular de Sitter core black hole geometry under several thin disk emissions’. In: *Phys. Rev. D* 108.12, 124054 (2023). arXiv: [2310.11039 \[gr-qc\]](#).

Finally, 2 additional works, developed over the same period, are in preparation or in the process of peer review:

- [15]: Riccardo Della Monica and Richard Brito. ‘Environmental effects of ultralight axions on compact binaries mergers’. In: *in preparation* (2024),
- [16]: Riccardo Della Monica, Ivan de Martino, and Mariafelicia de Laurentis. ‘Testing fifth force effects in the Algol three-body system’. In: *submitted* (2024).

Awards

- An oral presentation based on the article [11] and presented at the conference EREP2023 (Spanish and Portuguese Relativity Meetings) in Bilbao, Spain, has been awarded the first price for the best contribution, issued by the Spanish Society for Gravitation and General Relativity (SEGRE).

Academic Visits

Two academic visits have been carried out during the PhD studies, to work in collaboration with international experts on the subject of this thesis:

- **Institution:** Università degli Studi di Napoli - “Federico II”
Supervisor: Mariafelicia de Laurentis
Dates: 29-05-2022 / 04-06-2022
Subject: “3-body problem in Yukawa-like potential”
- **Institution:** Center for Astrophysics and Gravitation (CENTRA), Instituto Superior Técnico, Lisboa
Supervisor: Richard Brito
Dates: 15-09-2023 / 15-12-2023
Subject: “Environmental effects of ultralight axions on compact binaries merger”

Declaration

Yo, **Dr. Ivan de Martino**, Profesor Permanente Laboral del área de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca, director de la presente tesis

y

Yo, **Prof. Dr. Fernando Atrio Barandela**, Profesor Catedrático del área de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca, director del trabajo de investigación

CERTIFICAMOS

Que el trabajo de investigación desarrollado en la presente tesis doctoral por el investigador **D. Riccardo Della Monica** para optar al Título de Doctor por la Universidad de Salamanca con la Mención de Doctorado Internacional cumple con los requisitos formales, académicos y de calidad requeridos, autorizando por tanto su presentación.

Firmado electronicamente por I. de Martino y F. Atrio Barandela en Salamanca, el 30 de mayo de 2024

Abstract

The theory of General Relativity, formulated a little over a century ago by Albert Einstein, implied a paradigm shift in our understanding of gravity and the space we live in, and still today it resists with astounding predictive power to the numerous experimental tests it undergoes. In this context, the Galactic Center of the Milky Way has stood out in recent years as a unique and powerful laboratory at the edge of fundamental physics and astrophysics. Numerous observational pieces of evidence indicate that compact radio source Sagittarius A*, harbored in this region, is a four-million-solar-masses supermassive black hole, as described by General Relativity. As such, high resolution observations of this object and its surroundings not only open a unique new avenue to test the basic predictions of General Relativity but also to falsify it against theories that have been proposed to modify or extend it, to test different paradigms of dark matter and to place constraints on the very fundamental nature of Sagittarius A*.

In this thesis, we explore this fascinating possibility. First, we develop a numerical methodology and the appropriate modeling required to produce fully relativistic observables for stars and pulsars in the Galactic Center. We then use current data, mock catalogues for future observations of both known objects and putative ones, and Bayesian inference techniques to derive novel constraints on a variety of different scenarios, including extended dark matter components surrounding Sagittarius A*, black hole mimicker models and modifications to the underlying theory of gravity. Beyond the importance of the constraints derived for these models, which often represent complementary probes to those obtained at different astrophysical or cosmological scales in the literature, our work demonstrates the powerful ability of observations at the Galactic Center to further strengthen the central role of General Relativity and the black hole paradigm and the methodologies we have developed will represent a valuable tool for future research in this field.

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Introduction

In 1915 Albert Einstein published the Theory of General Relativity¹, drastically changing the way we intend gravity and the space we live in. For the first time, gravity was not anymore regarded as a force between any pair of massive objects, an idea that had withstood for centuries, since the formulation of Newton's law of universal gravitation in 1687. It was instead conceived as an emergent effect of the geometrical properties of four-dimensional spacetime that surrounds us. Spacetime itself, on the other hand, ceased to be considered as a mere stage on which physical events take place, with a fixed geometrical structure and eternal in time, and was replaced by a dynamical entity. Its geometry being not fixed *a priori*, but rather ever-evolving and determined by the matter and energy that it contains. Using the words of John Archibald Wheeler: "*Spacetime tells matter how to move, matter tells spacetime how to curve*"².

Many have acknowledged General Relativity as "*the most beautiful of all existing physical theories*"³ due to its elegant abstract mathematical formulation, intrinsically built upon differential geometry, and to its internal coherence and consistency. So much so, that Einstein himself would regard the problem of the experimental confirmation of General Relativity as of secondary importance and would jokingly answer to his assistant Ilse Rosenthal-Schneider, asking him what he would do if the theory would prove wrong, that he would "[...] *feel sorry for the good Lord. The theory is correct*"⁴.

Truth be told, apart from its mathematical elegance, General Relativity comes with a remarkably rich phenomenology when compared to classical Newtonian gravity, which has allowed to devise, shortly after its publication, a series of experimental tests that could falsify the theory in the Solar System. These experiments have become known as the *classical tests of General Relativity*. Each of the classical tests is based on a distinctive prediction of Einstein's theory of gravity that unambiguously discriminates it from the Newtonian model:

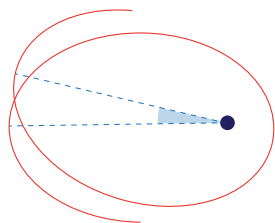
- Freely falling test particles in the gravitational field of a massive body experience geodesic motion, whereas in Newtonian gravity they follow Kepler's laws. One of

¹Einstein, A., *Die Feldgleichungen der Gravitation*, 1915 [17]

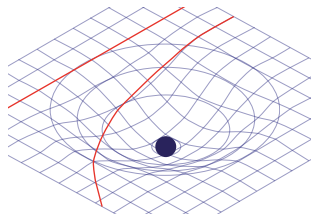
²Wheeler, J. A., *Geons, black holes, and quantum foam: A life in physics*, 1998 [18]

³Landau, L. D. and Lifshitz, E. M., *The Classical Theory of Fields*, 1975 [19]

⁴Rosenthal-Schneider, I., *Reality and Scientific Truth: Discussions with Einstein*, 1980 [20]

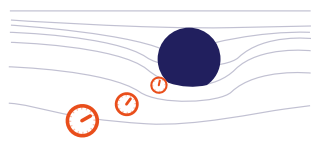


The perihelion precession of test particles.



Lensing of photons by the central mass.

⁵ Dyson, F. W., Eddington, A. S. and Davidson, C., *A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919* [21]



Gravitational time dilation.

⁶ Pound, R. V. and Rebka, G. A., *Variation with Temperature of the Energy of Recoil-Free Gamma Rays from Solids*, 1959 [22]

the distinctive differences is that relativistic orbits are no longer closed ellipses, but rather precessing ellipses about the center of mass of the system. Strikingly, this prediction quantitatively matched with perfect agreement the anomalous rate of *precession of the perihelion of Mercury's orbit*, first recognized in 1859 as a problem in celestial mechanics, by Urbain Le Verrier, and still unexplained at the time of the formulation of General Relativity.

- As a consequence of the equivalence principle, massless particles (such as photons) also experience geodesic motion and thus, rather than propagating in a straight line as Newtonian gravity predicts, they bend, accommodating the curvature of spacetime: a phenomenon known as *gravitational lensing*. Photons emitted by distant stars and grazing the surface of the Sun would be deflected by a detectable amount, which motivated several expeditions to observe the light of fixed stars during a total solar eclipse. The famous expedition of Sir Arthur Stanley Eddington in 1919⁵ provided the first experimental confirmation of this phenomenon, adding an important piece to the puzzle.
- The curvature of spacetime in the presence of a massive body implies that the flow of time is not universal, not even for stationary observers. Clocks closer to a gravitational source would experience a *gravitational time dilation* and any light that they emit would experience a *gravitational redshift* when collected by a more distant observer. This effect was first confirmed experimentally with the Pound-Rebka experiment⁶, conducted in 1959 within Earth's gravitational field and is now verified on a daily basis, allowing the Global Positioning System (GPS) to properly work.

The successful outcome of the classical tests provided early confirmation for the theory and has been of paramount importance for the imposition of General Relativity as one of the building blocks of modern physics. The realization that, in spite of the cumbersome mathematics involved in the theory and the profound conceptual paradigm shift that it implied, General Relativity offers indeed a more complete description of the gravitational interaction than Newtonian gravity, ignited the field, motivating numerous theoretical studies in the first half of past century.

With the technological advancement in more recent years and an astounding scientific effort, often involving large collaborations, such theoretical predictions have been systematically proved correct, both qualitatively and quantitatively. From the detection of gravitational waves⁷, the discovery that the seemingly paradoxical black holes are an actual physical reality⁹, to the attainment of the so-called “Golden Age of Cosmology”⁹, the central role of General Relativity has been consolidating even further.

In recent years, thanks to a series of fruitful observational campaigns, the Galactic Center of the Milky Way has gained in increasing popularity, sitting at the intersection of fundamental physics and astrophysics. The discovery that a supermassive compact source, Sagittarius A* (Sgr A*), is harbored in this peculiar environment and the ability to probe such small spatial scales with high angular resolution instrumentation, has led to a wealth of scientific discoveries in the last three decades, culminating in the 2020 Nobel prize for physics awarded to Roger Penrose, Reinhard Genzel and Andrea Ghez¹⁰. High-resolution observations in the Galactic Center shed new light on the classical tests of General Relativity, which find fresh confirmation in a new and completely different environment. The discovery of a very abundant population of rapidly moving stars, the so-called *S-stars*, orbiting Sgr A*, allows testing the orbital precession and the gravitational redshift around a supermassive compact object, while the direct observation of Sgr A* (using revolutionary techniques like Very Long Baseline Interferometry) allows to probe, around the same object, the deflection of light in a strong-field regime that was previously unexplored. Moreover, the potential discovery of pulsars in tight orbits around Sgr A* promises not only to improve our characterization of its environment but also to bring tests of gravity in the strong-field regime to a whole new level of precision.

As such, the Galactic Center represents a unique laboratory to probe gravity, where one is not only able to test the basic predictions of General Relativity but also to falsify it against competing theories that, over time, have been proposed to modify or extend it, to test different paradigms of dark matter, and to place constraints on putative models that have been formulated as alternatives to the standard black hole paradigm in General Relativity. We are now moving towards an era of precision astronomy in the Galactic Center, with new observational facilities that will not only bring the current sensitivity

⁷ LIGO/VIRGO Collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger*, 2016 [23].

⁸ Webster, B. L., Murdin, P., *Cygnus X-1—a Spectroscopic Binary with a Heavy Companion?*, 1972 [24].

⁸ Schramm D. at Arthur M. Sackler Colloquia of the National Academy of Sciences: Physical Cosmology, 1992.

¹⁰ <https://www.nobelprize.org/prizes/physics/2020>

threshold down by orders of magnitude, but will also bring to the discovery of fainter and closer stars to Sgr A*, thus diving even deeper in the strong-field gravitational regime, and, as mentioned before, pulsars. The discovery of such new probes calls for a revisitation of the standard techniques used to model our observables in the Galactic Center. Whilst with the current observational sensitivity, orbital models based on approximations (*e.g.* the post-Newtonian limit) of the general relativistic equations of motion for test particles allow to derive a sufficiently precise reconstruction of the astronomical observables, as we will show, this will no longer be true for such new probes. For this reason, developing an orbital model based on a fully relativistic description of the motion of test particles around a supermassive compact object and devising a self-consistent strategy not only to test General Relativity but also to derive constraints on its alternatives from the comparison with observational data, is of uttermost importance. This is the main motivation and scientific goal of this thesis.

In Chapter 1 we will introduce the reader to the environment of the Galactic Center and to its peculiar role, at the edge of astrophysics and fundamental physics. We will thus present a historical overview of the decades-long observational efforts of the Galactic Center, and of the technological and theoretical advancement achieved to improve our characterization of this environment and of the compact radio source Sgr A* at its heart. Chapter 2, on the other hand, will be devoted to an introduction to more theoretical aspects of General Relativity and, in particular, to relativistic celestial mechanics which allows us to derive a versatile orbital model for test particles in a generic spherically symmetric spacetime, that we will use throughout the rest of the thesis. In Chapter 3, we will present the first scientific application of this methodology to derive constraints on the extended dark matter component in the Galactic Center, focusing in particular on the Fuzzy Dark Matter model, which has become increasingly popular over the last years as an alternative to the standard Cold Dark Matter model. In Chapter 4, we will consider black hole mimicker geometries as alternatives to the classical general relativistic Schwarzschild model to characterize the gravitational field of Sgr A*. In Chapter 5, on the other hand, several distinct models of extended and alternative theories of gravity will be taken into account, showing the feasibility of testing the underlying theory of gravity using the observations of the S-

stars in the Galactic Center. Finally, Chapter 6 will be devoted to the application of our geodesic methodology to characterize the observable times of arrival of pulses of electromagnetic radiation emitted by potential pulsars in the Galactic Center, showing the breakthrough improvement that these objects would lead in recognizing the fundamental nature of Sgr A*, the underlying theory of gravity and the determination of the physical properties of the compact central object.

The Galactic Center

1.

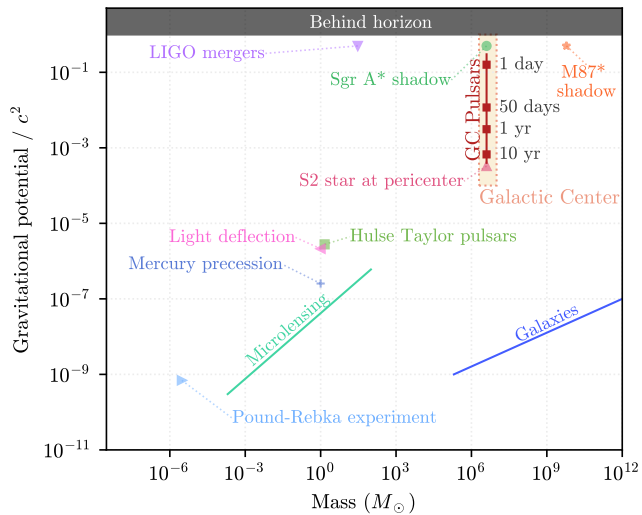
The central few hundred parsecs (pc) of the Milky Way, usually referred to as the Galactic Center, represent a challenging, yet very popular, target for astronomical observations [25]. The tremendous technological efforts invested in the development of observational facilities and techniques required to penetrate the complex environment of absorbing and obscuring features in the Galactic Center (and in the Galactic plane, through which observations are made) are motivated by one clear and ambitious scientific goal: advance our understanding of physical processes at the edge of astrophysics and fundamental physics.

The Galactic Center is a rather populated environment [26]. The remarkable variety of astrophysical phenomena in the Interstellar Medium within this region, embracing different stages of its evolution, from supernova remnants [27] and compact star-forming regions to large populations of radio filaments [28–31] and radio arcs [32–34], has motivated decades-long observational campaigns over the entire electromagnetic spectrum. High-resolution observations of the Galactic Center present the unique opportunity to study the innermost regions of galaxies with unmatched detail and precision, surpassing the capabilities available for other galaxies (the center of the Milky Way is 10^5 times closer to us than the center of any other galaxy [35]). As we will show, this not only enables the testing of the Supermassive Black Hole (SMBH) paradigm within our Galaxy and the study of the influence of Massive Black Holes (MBHs) on stellar and interstellar environments but also provides a unique chance to investigate gravity in a regime that was previously unexplored, as shown in Figure 1.1. The gravitational field in the Galactic Center is dominated by a point-like source of $\sim 4 \times 10^6 M_\odot$ and, as we will show in this chapter, the scales of our currently available probes span from the orbits of stars at a few hundred astronomical units (AU) to observations on the horizon-scale, on the order of 10^{-2} AU. The gravitational potential-mass regime probed with these observations sits in a region unexplored by any other available gravitational source.

Nonetheless, since the Galactic Center is observed from within the galactic plane, observations face challenges posed by

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Figure 1.1. The gravitational potential for different astrophysical historically-relevant and potential (as in the case of pulsars, *i.e.* the red line) probes of gravity, as a function of the mass of the respective gravitational source. The yellow rectangle delimits probes at the Galactic Center, whose gravitational field is dominated by the point-like source Sgr A* whose mass is estimated to be $\sim 4 \times 10^6 M_\odot$. The gravitational potential-mass regime probed with observations at the Galactic Center is unexplored by any other available gravitational source, identifying this environment as a novel laboratory for gravity.



the presence of interstellar dust particles in the plane of the disk (which causes prominent extinction, obscuring the galactic nucleus, especially in visible wavelengths) and by the crowdedness of the region in terms of stellar populations, gaseous environments and transient phenomena. Hence, much of our current understanding of the Galactic Center has been gleaned from observations across various wavelengths, spanning from the infrared and microwave to high-energy phenomena captured by X-ray and γ -ray facilities (see Figure 1.2 for a multi-wavelength depiction of the central few pc of the Galactic Center).

For instance, the Central Molecular Zone is a twisted torus-like structure that can be observed in the far-infrared [36], rich in molecular gas that accounts for $\sim 5 - 10\%$ of the content of the entire Galaxy and of its infrared luminosity (see [37] for a comprehensive review about this region). Its high density ($\sim 10^7 M_\odot \text{ pc}^{-3}$) [38–40] leads to increased gas temperature and pressure with respect to the surrounding environment [41–44], creating one of the most extreme and turbulent [45–47] regions of star formation that can be directly observed. Deeper within the central few pc of the Milky Way, a diverse array of physical phenomena manifest across the entire electromagnetic spectrum. Notably, the Galactic Center hosts various clusters of young and massive stars [26]. Among these clusters are the Arches [48, 49], the Quintuplet [50, 51], and, significantly, the Nuclear Star Cluster (NSC) located

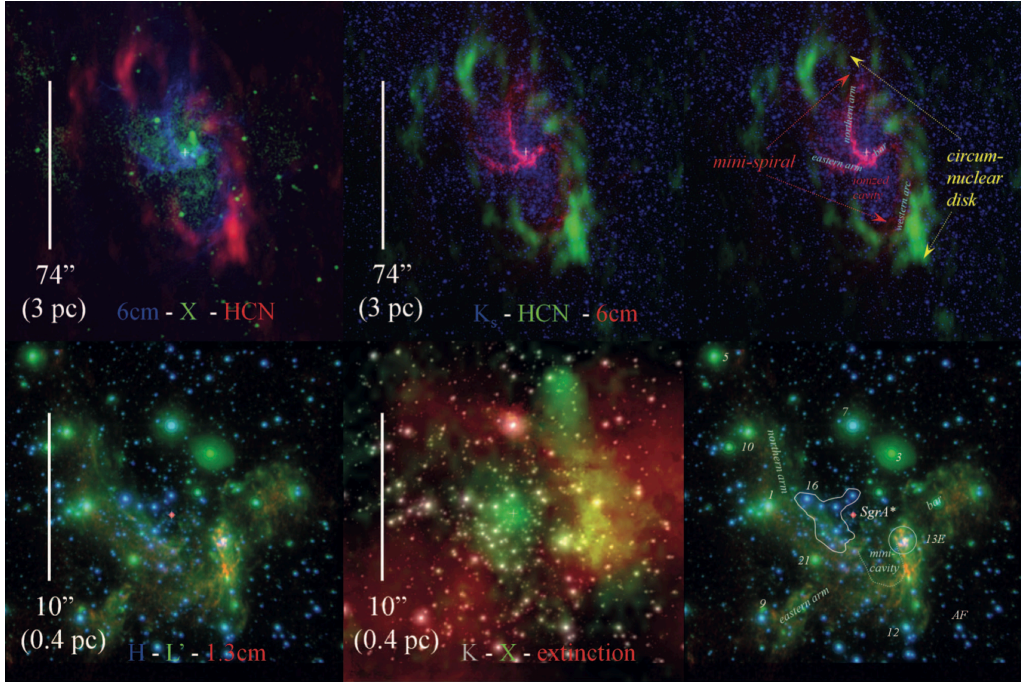


Figure 1.2. A multi-wavelength perspective of the central few pc of the Galactic Center, within a field-of-view of approximately ~ 3 pc for the top row and ~ 0.4 pc for the bottom one. The cross in the image indicates the position of Sgr A* in the sky, while the galactic plane intersects the image at a position angle of 32° in the SW-NE direction (with N denoting up and E denoting left in the images). Each panel illustrates color-composite images from the spectral bands specified in each box. Specifically, the 6 cm and 1.3 cm observations represent the radio continuum from the VLA ([64–66]), the HCN emission line is from the OVRO [67], X-ray emission is observed with the Chandra satellite [62], K denotes the $2.2 \mu\text{m}$ K-band near-infrared observation at the VLT with the ISAAC instrument [68] for the top-middle panel and with the NACO AO camera [69], and L' and H represent the $3.8 \mu\text{m}$ and $1.6 \mu\text{m}$ bands from VLT-NACO [69]. The two right-most panels replicate the top-central and bottom-left panels, respectively, with emphasized main interstellar features. The figure appears in [35].

within the central ~ 300 pc around the compact radio source Sgr A* [52–55]. In addition to these star clusters, both neutral and intensely hot ionized gas have been observed [56–60] in the same region (refer to Figure 1.2). The central pc, where a predominant portion of the gas is ionized, showcases a prominent HII region (Sgr A West) and a luminous X-ray emitting region comprising hot ($T \sim 10^6$ K) gas [61–63].

To overcome the observational difficulties involved in Galactic Center observations, state-of-the-art observational facilities have been employed over the years, requiring the application of pioneering astronomical techniques, decades-long efforts of large scientific collaborations in parallel to better theoretical understanding. In this thesis, we will study the dynamics of those stellar populations that, thanks to the unprecedented observational accuracy reached over the years,

have allowed us to identify the SMBH Sgr A* and to study its physical properties, opening an entirely new window on fundamental tests of gravity in a regime that was yet unexplored.

These stellar populations are not only interesting from a theoretical physics point of view. Their existence, formation history and evolution pose serious problems in our current understanding of stellar formation mechanisms in the presence of a SMBH. This multifaceted domain of interest of the Galactic Center, placed at the edge of observational multi-wavelength astronomy, theoretical astrophysics and more fundamental subjects, is what makes this environment attractive, motivating the numerous studies from where we begin our discussion.

1.1. An astrophysical dilemma: the paradox of youth

Among the three stellar clusters identified in the Galactic Center, the NSC stands out as particularly intriguing. Situated at the heart of the Milky Way, it lies within a few pc of Sgr A*. The NSC's morphology and the surprising presence of a population of very young stars (along with the population of older stars that one would expect in such an environment) in close proximity to a SMBH, raises several unanswered questions about its formation mechanism [26].

Early infrared observations [70, 71] of the NSC initially revealed an unexpectedly high number of bright stars, which were identified as older (> 1 Gyr) red giants or Asymptotic Giant Branch (AGB) stars. Moreover, infrared observations of the proper three-dimensional motions revealed random isotropic motions and minimal bulk rotation (in the same sense as the Galaxy rotation), supporting the notion of a gravitationally relaxed and aged system [72–75].

During the late 1980s and early 1990s, however, pioneering observations by Forrest et al. [76] and Allen and Sanders [77] led to the discovery of an increasingly numerous population of bright early-type hot stars in the NSC [52, 73, 78–86], which were either Wolf-Rayet (WR) stars with ages below 10 million years and zero-age-main-sequence masses ranging from ~ 30 to $100M_{\odot}$ [87–89]. Moreover, from a kinematic point of view

these stars exhibit a coherent rotation pattern [73, 86, 90] with an intricate structure composed of two disks spinning in opposite directions and inclined at two different angles relative to each other [26, 90], revealed by proper-motion observations using the European Southern Observatory¹ (ESO) 3 m New Technology Telescope² (NTT).

The distinctive disk structure of these stars, along with their estimated masses, orientations, eccentricities and inferred young ages, poses a challenge for a comprehensive model, especially given the concurrent existence of a potential SMBH that would typically hinder cluster formation. In typical star-forming environments, in order to trigger star formation in a molecular cloud the accumulation of matter above a certain mass and density threshold to induce Jeans instability is required [91, 92]. However, in an environment characterized by a steep mass profile, such as within the central kpc of the Galaxy, where the compact mass of Sgr A* dominates the gravitational potential well, the strong tidal field induces shear on the collapsing cloud. This could disrupt the cloud completely, impeding star formation unless the cloud is sufficiently massive for its self-gravity to withstand the exerted shear [93]. It is estimated that a cloud situated at a distance of 1 pc from Sgr A* would need to possess a density $\gtrsim 6 \times 10^7 \text{ cm}^{-3}$ to withstand tidal disruption [26]. By comparison, star-forming giant molecular clouds in the Milky Way typically exhibit densities in the range of 1-100 cm^{-3} . This renders the Galactic Center a hostile environment for star formation, which can only happen through some either “in-situ” or “ex-situ” peculiar mechanisms [26].

The “ex-situ” scenario contemplates the migration towards the center of a star cluster formed at greater distances (where the tidal field is weaker). While the migration of individual stars is a slow process, insufficient to explain the observed relaxed and temporally stratified multi-disk population, the presence of dynamical friction can facilitate a faster rate of migration within the lifespan of its early-type stars [94–99]. In order for the infall mechanism to be capable of depositing stars within 6 million years (the age of the oldest early-type stars in the disks), resisting tidal shears adequately to position stars at the required radii and leaving no discernible trail of young stars during migration towards the center [100], the cluster should have a very high central stellar density (around $\sim 10^8 M_{\odot}$

¹ <https://www.eso.org/public/>

² <https://www.eso.org/public/teles-instr/lasilla/ntt/>

pc^{-3} , as discussed in [96]). This scenario, involving the infall of different clusters has been demonstrated to be plausible, through recent simulations [101, 102]. Alternatively, the cluster central concentration could be explained by the formation of an Intermediate Mass Black Hole (IMBH, $M \sim 10^3 - 10^4 M_\odot$) from runaway collisions between massive stars [103, 104]. Such an IMBH might offer the required core density for a stellar cluster to withstand tidal shear during infall, even at galactocentric distances as short as $\sim 0.1 \text{ pc}$ [105]. N-body simulations of this process have successfully reproduced the overall properties of the stellar population post-infall, when at least two distinct clusters are considered. This scenario is considered a plausible formation mechanism for the early-type stellar disks in the Galactic Center, especially following the recent discovery of a potential IMBH in the small star cluster IRS 13E. Estimates suggest that for it to avoid tidal disruption, an IMBH in the range of $10^3 - 10^4 M_\odot$ should reside within it [85, 106–109]. Moreover, in recent years, the count of potential IMBHs in the Galactic Center has increased [110–114].

A second plausible scenario for the formation of the early-type nuclear stellar cluster within the central pc of the Milky Way involves the “in-situ” formation of these stars at or near their current positions. In this mechanism, a self-gravitating gaseous disk, either the remnant of an infalling molecular cloud or the result of the continuous accumulation of gas (or a combination of both processes) [115, 116], can become unstable due to its self-gravity. Fragments exceeding the Roche limit of the disk can then collapse [117], leading to the formation of stars [118–120]. The absence of massive stars at distances greater than 0.5 pc from Sgr A* supports this in-situ scenario [121], as does the existence of several candidate massive clouds within approximately 20 pc from the Galactic Center [122, 123]. Increasingly precise and refined numerical simulations have been successful in reproducing specific characteristics of the early-type cluster, such as mass, orientation, eccentricity distributions, surface density, the presence of a counter-rotating disk, and age stratification in the cluster [124–131].

Despite the numerous studies on the subject, the presence of a young stellar population in the Galactic Center and the lack of a comprehensive model for their formation represents an astrophysical conundrum, that has been termed the “Paradox of Youth” by Ghez et al. in 2003 [132], reflecting the ongoing

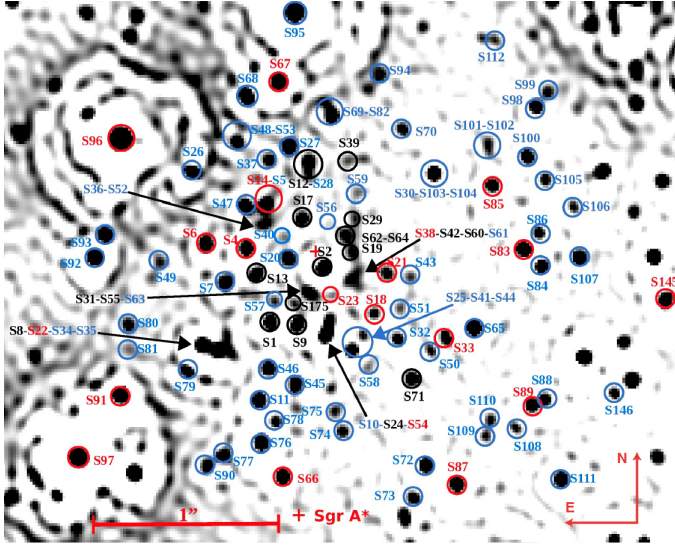


Figure 1.3. VLT-NACO finding chart of the S-star cluster obtained in 2018. In the plot, East direction is left, North is up. Sgr A* is denoted by a red cross. Stars in red and black circles correspond to the two separate disk populations recognized in [133]. Orbital elements for all of the 29 stars are known. Blue circles, on the other hand, correspond to the stars for which only linear fit to their orbits is currently possible. The figure appears in [133].

challenges in understanding this intriguing phenomenon.

1.1.1. The S-stars

Moving even closer to the Galactic Center, a notable drop in the density of the young star population is observed at a distance of approximately 0.04 pc from Sgr A*. This discontinuity signifies the separation between the young stars in the clockwise and counterclockwise disks described earlier and a third prominent component of the NSC. This component consists of a remarkable concentration of very fast-moving dwarf B-type stars, aged between 6 and 400 Myrs [134], observed in the near-infrared within the central arcsecond (arcsec) around Sgr A*, called “S-stars” [135–138]. The naming convention for these stars, initially proposed by Eckart and Genzel in 1997 for the first 11 identified fast-moving components of the cluster [139], is still in use today, although the number of recognized stars has expanded to over 200 components [137, 138]. Despite the possibility of considering the S-stars as a low-density, unbound extension of the young stellar disks, several arguments suggest that the S-stars should be regarded as an entirely distinct population. Understanding the association of this cluster with the young stellar disks, could provide insights into the formation history of both populations.

First, from a spectroscopic standpoint, the great majority of the

S-stars are spectrally classified as types B2-9 [134], in contrast to the giant and supergiant stars (OB/WR) belonging to the young disks. This different spectral classification suggests that the two populations likely formed at different epochs, given the shorter average lifetime of main-sequence O stars compared to B stars. Second, the kinematic properties of the S-stars are highly inconsistent with the coherent planar rotation of both the clockwise and counterclockwise stellar disks. The orbits of the brightest S-stars have been tracked since the 1990s using speckle imaging methods at NTT [90, 140] and later with the Keck telescope [132, 136, 141]. This tracking has provided a set of astrometric positions, enabling the constraint of stellar proper motion from data acquired over multiple years. These studies have revealed that these stars have randomly distributed orbits with no evidence of co-planarity with either the clockwise or counterclockwise disk. Monitoring the orbits of the S-stars has continued (and still continues) because characterizing their motion not only enhances our understanding of this peculiar stellar population but also serves as a probe for the gravitational field of Sgr A*, allowing constraints on its mass and distance with unprecedented accuracy [137, 142], as we will thoroughly explain in Section 1.2. For the first time these observation allowed to constrain the orbital parameters for some of the B stars in the S-stars cluster [134, 137, 142, 143] due to the accelerated motion that some of them appeared to undergo on the sky plane. The S-stars exhibited eccentricities of $e > 0.5$, with some going as high as $e > 0.9$, and angular momentum pointing in random directions in the sky. This suggested that the S-stars, while showing a somewhat excess of strongly non-circular orbits, were consistent with the expected probability distribution of eccentricities and angular momentum direction for a spherical, isotropic, and virialized distribution of particles orbiting a central mass [143]. This excess of high eccentricities suggests that the possible mechanism of formation of these stars somehow selects highly non-circular orbits. Moreover, since the S-stars have been established as a completely separate stellar component of the Galactic Center from other stellar clusters [136, 139], their formation may be explained by a different mechanism. Such a mechanism has to be compatible with existing early-type formation scenarios on one hand and, on the other, must account for the observed properties of the S-star cluster. The formation of the S-stars cluster through an in-situ process is generally considered unlikely, given the suppression of clustering by the tidal field in the vicinity of a

SMBH. While the migration of some cluster members from the stellar disk of the Milky Way might be possible due to phenomena like Lense-Thirring precession [115] or torques exerted by Lin-Shu-type spiral density waves on stars at an inner Lindblad resonance [144], these mechanisms cannot account for the majority of long-period eccentric orbits. Migration from the OB/WR-type stellar disks into the central arcsec-wide region could be a plausible explanation. Simulations have indicated that two interacting stellar disks could result in the creation of a population of captured stars an order of magnitude closer to the central SMBH [145]. Precession and Kozai-Lidov interactions [146, 147] on individual stars could lead to extremely high eccentricities at random orientations. Furthermore, stellar binaries on such eccentric orbits might be disrupted due to close passages near the SMBH, leaving behind a single S-star and ejecting the companion from the vicinity of the SMBH, providing an explanation for the observed hyper-velocity stars in the Milky Way halo [145].

However, this process is still considered unlikely for two main reasons: (i) the circumnuclear disks are predominantly populated by OB/WR giant stars, and only a small number of main-sequence B-type stars have been observed. A possible explanation for this is that a much larger population of B-stars than the observed one could exist in the disks, but it has evaded observation due to the crowded stellar environment, low spatial resolution of observations, and the multi-spectral emission dominated by O-type stars [26]; (ii) it has been demonstrated that spherical stellar bulges suppress Kozai oscillations when the enclosed mass inside the orbit of a test body is of the order of the mass in the stellar disk [148]. This situation appears to be satisfied in the Galactic Center, where the stellar bulge is much larger than the stellar discs.

This leaves with only two possible formation mechanisms, both involving processes that started outside of the central arcsec: the cluster infall scenario, and the binary capture scenario. The first scenario is a direct extension of the cluster infall mechanism for the formation of early-type stars in the circumnuclear disks (Section 1.1). This process can reproduce the observed eccentricities and apparent thermal isotropic distribution of the S-stars within the lifetime of the stars [149]. In the binary capture scenario [150] the S-stars might have reached the Galactic Center in the form of binary systems on orbits with high eccentricities ($e > 0.98$), which is then smoothed out over the first Myr through a relaxation process.

The number of observed S-stars would require a reservoir of binary systems in the form of a “massive perturber” (such as a molecular cloud or a dense star cluster) that might induce the migration of binary pairs toward the SMBH [151]. However, the presence of this perturber is still debated today [133].

1.1.2. The challenge of observing the S-stars

Two international observational groups, with leading scientists Reinhard Genzel and Andrea Ghez, both recipients of the 2020 Nobel Prize in Physics, based at the Max Planck Institute for Extraterrestrial Physics (MPE) and the University of California, Los Angeles (UCLA), respectively, have been meticulously tracking the orbits of S-stars in the Galactic Center for over thirty years [25, 138, 152]. The MPE group conducts observations at ESO facilities in Chile, while the UCLA Galactic Center Group utilizes the W.M. Keck Observatory telescopes at Mauna Kea, Hawaii. Despite the notable blue emission from B-type stars in the S-stars cluster [134], optical observations of the Galactic Center are challenging. Diffuse interstellar dust along the line of sight to the center of the Milky Way causes approximately ~ 30 magnitudes of visible extinction [56, 71, 153, 154]. However, the extinction curve steeply falls off in the near-infrared, reaching only about ~ 3 magnitudes of extinction toward Sgr A* in the K-band ($\lambda = 2.2\mu\text{m}$). This drop-off extends up to a few μm , beyond which obscuration by dust grains increases again. This makes the K-band an optimal spectral window for observing individual stars in the Galactic Center.

Tracking individual stars in the crowded Galactic Center demands exceptional spatial resolution and prolonged measurement times, both in single observations (long exposure times) and across observational campaigns (following stars over their orbital periods). Due to these requirements, space-based observations of the S-stars were ruled out. Instead, efforts were directed toward improving ground-based facilities and analysis techniques to mitigate the blurring effects caused by Earth’s atmosphere during extended observation periods. To overcome atmospheric turbulence and achieve high-resolution observations, the speckle imaging technique with the “shift-and-add” method³ proved highly successful for near-infrared

³ This approach, now commonly known as “image-stacking,” represents a form of speckle imaging frequently employed to generate high-fidelity

observations at the Galactic Center [155, 156]. This technique takes advantage of the fact that atmospheric turbulence causes distortions on timescales shorter than about one second, limiting the angular resolution of long-exposure images. By capturing numerous very short exposures (around 0.1 s) with a sensitive detector, spatially shifting them to align sources, and then adding them together, a significantly sharper and deeper image can be produced, approaching the diffraction limit of the observing telescope. The MPE and UCLA groups initially adopted this technique. The MPE group successfully implemented it between 1991 and 1996 at ESO NTT using the SHARP camera developed specifically for the project [135, 139], achieving a spatial resolution of 0.15 arcsec. However, the breakthrough came in 1998 when the UCLA group employed this technique at the W. M. Keck I 10m telescope [136]. They reached the diffraction limit of the telescope (0.05 arcsec at $2.2 \mu\text{m}$ wavelength in the K band) and were able to resolve spatial scales as small as 2.5 light days. This marked a significant advancement in the study of the Galactic Center.

While speckle imaging enables the achievement of maximum nominal resolution from ground-based telescopes, it has a notable drawback. Limited to short-time exposures, it can only monitor the brightest sources. Additionally, extended surveys are required to extract reliable estimates of stars' proper motion. Addressing these challenges was crucial in developing a technique that could leverage the full potential of observing facilities. This need led to a “new era” in observations at the Galactic Center with the introduction of Adaptive Optics (AO) techniques [158, 159] at the Keck II observatory in 2000 [160] (though Galactic Center observations benefited from this technique only from 2004 onward, with the development of laser guide-star AO in 2004 on the 10 m Keck II telescope [161]). A few years later, AO techniques were also implemented at the ESO Very Large Telescope (VLT)⁴ with the Nasmyth Adaptive Optics System (NAOS) - Near-Infrared Imager and Spectrograph (CONICA), together forming NACO⁵ instrument [162].

images from multiple short exposures exhibiting varied image shifts. By aligning the short exposure images using the brightest speckle (a process involving the computation of the differential shifts among images, which is readily feasible in astronomical contexts due to the presence of stars for alignment), they are then averaged to produce a single output image.

⁴ <https://www.eso.org/public/teles-instr/paranal-observatory/vlt/>

⁵ <https://www.eso.org/public/teles-instr/paranal-observatory/vlt/vlt-instr/naco/>

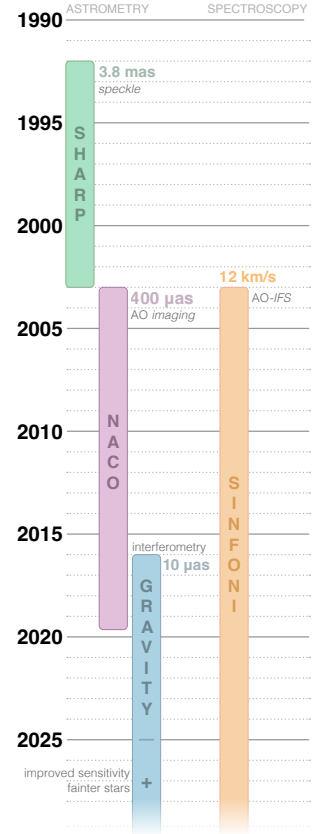


Figure 1.4. Timeline of observational facilities and techniques (with the corresponding uncertainties) for the astrometric and spectroscopic observables used for Galactic Center observations by the MPE group (later Gravity Collaboration) over the last three decades. The + in the post-2025 portion of the GRAVITY instruments stands for the series of improvements of the VLT interferometer, named GRAVITY+ [157], that will enable much greater sensitivities and the possible discovery of fainter and closer stars to Sgr A*.

To consistently measure sky-projected positions for the S-stars relative to Sgr A*, it was necessary to establish an astrometric reference frame. The concept of using a group of red giant stars at the Galactic Center to create a radio-to-infrared reference frame can be traced back to the work of Menten and collaborators in the mid-1990s [163]. These stars were observable in both the radio (via SiO maser emission in their envelopes) and the near-infrared, with their positions and proper motions characterized through radio-interferometric astrometry. However, neither the Keck II telescope⁶ (with a field of view of 10 arcsec) nor the NACO camera at VLT (with a field of view of 14 arcsec) had a large enough field of view to observe the S-stars and the masers simultaneously. Therefore, to link the observed positions of the stars to the radio-to-infrared reference frame, a dither pattern of overlapping telescope pointings had to be adopted, involving secondary astrometric referencing (either using matching coordinate lists or mosaic images). This process introduces systematic astrometric errors due to geometric distortions, field dependence of the point-spread function, and higher-order aberrations of the optics [164–166].

The application of AO techniques to observe the S-stars at the Galactic Center not only enhanced astrometry but also enabled spectroscopy on individual stars. This breakthrough allowed for the spectral classification of the population [134]. More critically, the ability to perform spectroscopy provided line-of-sight velocity measurements in addition to sky-projected properties, thereby offering insights into the third dimension. In 2003, the first spectral lines for single stars in the Galactic Center were measured using the NIRC2⁷ spectrograph at the Keck II telescope, which was coupled with a natural guide star AO system, yielding a resolution of 75 km s^{-1} for the K-band Bracket- γ absorption line of hydrogen at $\lambda \sim 2.16612 \mu\text{m}$ (a prominent spectral feature of B-type stars [132, 134]). Since then, spectroscopic measurements combined with AO systems have significantly improved, with the use of dedicated instruments. Specifically, since 2005, the UCLA group has utilized the OSIRIS⁸ (OH-Suppressing Infra-Red Imaging Spectrograph) integral field spectrograph at the Keck Observatory, which employs laser guide star AO, reducing uncertainties in measured line-of-sight velocities to about 10—

⁶ <https://www.keckobservatory.org>

⁷ <https://www2.keck.hawaii.edu/inst/nirc2/>

⁸ <https://oirlab.ucsd.edu/OSIRIS.html>

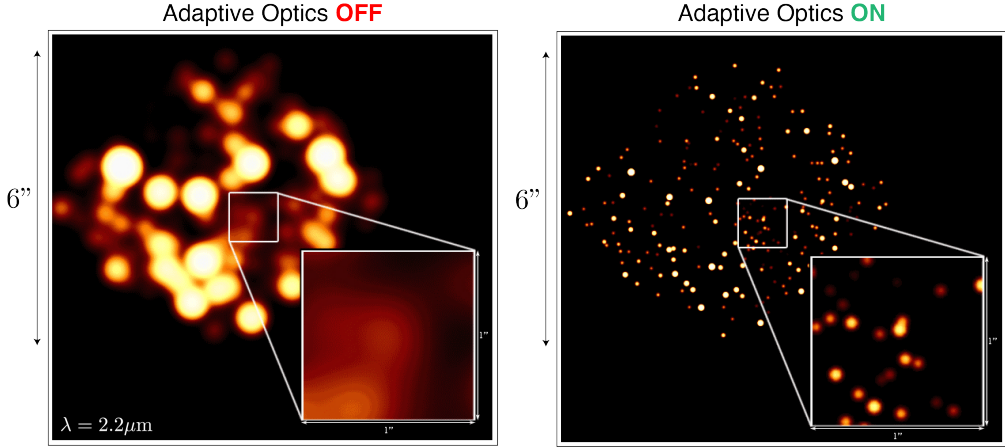


Figure 1.5. Observations of the 6×6 arcsec Galactic Center region with the Keck telescope in the K-band [161] comparing the resolution performances without (left panel) and with (right panel) the application AO techniques. The resolution gain is quite evident. The inset panel reports a zoomed-detail of the central region. The figure is adapted from the UCLA Galactic Center Group [website](#).

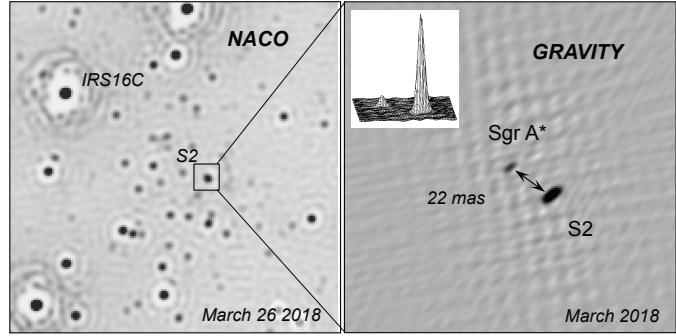
20 km/s [167].

The MPE group also embraced AO-assisted spectrographic techniques to track the line-of-sight motion of the S-stars, starting in 2004 with the commissioning of the SINFONI⁹ (SINgle Faint Object Near-IR Investigation) integral field spectrograph at VLT [168, 169]. Another notable addition to the suite of astronomical facilities used to monitor the orbits of the S-stars is GRAVITY¹⁰, an interferometric astrometry-imager that combines all four Unit Telescopes (UTs) of the VLT as an interferometer [170]. The Gravity Collaboration, with R. Genzel among the principal investigators, presently leverages the exquisite astrometry provided by the GRAVITY interferometer for Galactic Center studies. Interferometric techniques have enabled GRAVITY to achieve an angular resolution of approximately $20 \mu\text{as}$ (with a nominal instrumental goal of $10 \mu\text{as}$), representing a resolution gain of nearly 100 times compared to the initial SHARP speckle images [170]. This improvement is evident in Figure 1.6, which presents a comparison between the NACO field-of-view and reconstructed imagery from GRAVITY data for the star S2. These observations have reached a level of precision where the position of this particular star can be observed

⁹ <https://www.eso.org/public/teles-instr/paranal-observatory/vlt/vlt-instr/sinfoni/>

¹⁰ <https://www.eso.org/public/teles-instr/paranal-observatory/vlt/vlt-instr/gravity/>

Figure 1.6. Observations of the S2 star in the Galactic Center using the NACO AO telescope at VLT (left panel) and the GRAVITY interferometer (right panel), which combines all four UTs at VLT, are shown in the images acquired in March 2018, a few weeks before the pericenter passage of the star around Sgr A*. The right panel demonstrates the ~ 50 -fold improvement in angular resolution achieved with GRAVITY, making a noticeable difference when compared to the NACO observation in the left panel. In the NACO observation, S2 appears slightly elongated due to confusion with Sgr A*, as the two sources are not resolved at such high proximity. In contrast, the GRAVITY observation shows the two sources well separated at approximately 20 mas, making both distinguishable. It's worth noting that the slightly elongated shape of S2 in the GRAVITY image is due to the shape of the interferometric clean beam. This comparison illustrates the significant enhancement in resolution provided by GRAVITY, enabling clearer and more detailed observations. The image is adapted from [171].



to change between consecutive nights [171]. Additionally, interferometric observations of the Galactic Center are not susceptible to the systematic uncertainties of the reference frame. GRAVITY can directly relate the position of the observed star (primarily S2) to that of Sgr A*, as the two sources are well separated. However, GRAVITY observations are subject to other systematic uncertainties arising from aberrations in the optical paths of the individual telescopes [172].

Recent research by Peisßker and collaborators [173, 174] has identified some very close and faint stars (S62, S4711, S4716) using a different image filtering technique to re-analyze NACO observations of the Galactic Center. However, the astrometric precision for these closer stars is still poor and not competitive with the observational data for S2 from GRAVITY. Furthermore, it remains unclear how many faint, low-mass stars exist in the inner few mpc of the Galactic Center, as mass segregation mechanisms and collisional destruction by massive remnants are expected to suppress such a population [175]. While recent deep observations with GRAVITY identified a slowly moving star with a K-band magnitude of 18.9 within 30 mas of Sgr A*, identified as S62, it is at a substantially larger physical distance from Sgr A*. No other faint stars have been detected in current images of the Galactic Center with GRAVITY, possibly due to limitations in the calibration of the instrument [176].

Future upgrades within the GRAVITY+¹¹ project [157, 177], improving calibration and removing point-spread functions from bright stars, are expected to enable deeper observations of the Galactic Center, potentially revealing a new population of closer, fainter stars [176]. Moreover, substantial progress in this direction is anticipated with the development of larger

¹¹ <https://www.mpe.mpg.de/ir/gravityplus>

observing facilities like the Thirty Meter Telescope¹² (TMT) on Hawaii [178] or the 39-meter Extremely Large Telescope¹³ (ELT) at ESO [179].

1.2. Towards fundamental physics: pinpointing the nature of Sagittarius A*

The concept that compact objects such as SMBHs could serve as the energy source for intense radio emissions at the cores of galaxies has its roots in the influential theoretical contributions of Lynden-Bell and Rees [180, 181]. Their proposition suggested that a majority of galactic nuclei, including the Milky Way's center, could harbor such compact objects. Notably, they highlighted Very-Long Baseline Interferometry (VLBI) as a powerful technique for determining the size of any central black hole present in the Galaxy.

The actual discovery of Sgr A* occurred during the nights spanning February 13 to 15, 1974, as reported by B. Balick and R.L. Brown [182]. Their investigation was a continuation of earlier studies that had scrutinized the fine-scale radio characteristics of the Sagittarius A (Sgr A) region. It is worth noting that more than two decades earlier, Australian astronomers Piddington and Minnet [183] had initially pinpointed this region as a plausible location for the center of the Galaxy. In the early 1970s, efforts led by various researchers [184–188] had provided valuable insights, yet none had achieved the necessary resolution to conclusively discern the presence of the compact radio source.

Balick and Brown's pivotal observation, conducted using the Green Bank interferometer and a solitary antenna at the Huntersville West Virginia site, marked the first instance of attaining the requisite resolution and $u - v$ coverage¹⁴ to disentangle the intricate structures within the Sgr A West complex. They noted the atypical sub-structure of the source and its positional alignment with the galactic nucleus, strongly suggesting a physical association with the Galactic Center.

¹² <https://www.tmt.org>

¹³ <https://elt.eso.org>

¹⁴ The portion of the Fourier space available with the baselines covered by a given network of radio telescopes operating as interferometers.

The luminous and compact radio feature was subsequently validated by Ekers and colleagues [189], who combined data from the Westerbork Synthesis Radio Telescope with observations from the Owens Valley Radio Observatory¹⁵ (OVRO) at 6 cm. Notably, the technique of VLBI was first applied in the Galactic Center by Lo and collaborators in their pioneering work [190], providing further confirmation of the compact radio source. It wasn't until 1982 that it was christened "Sagittarius A*", to distinguish it from the more extensive source complex known as Sgr A [191].

In 1981, high-resolution observations using the Very Large Array¹⁶ (VLA) definitively localized Sgr A* at the dynamical center of the gas streamers within the Galactic nucleus [192]. A year later, the same research group made the pioneering estimation of its radio variability, revealing Sgr A*'s characteristics such as rapid variability, high radio luminosity, and apparent compactness. These distinctive features strengthen the evidence in favour of the SMBH hypothesis for Sgr A*.

1.2.1. Constraints from gas dynamics

From a dynamical perspective, the initial evidence supporting the presence of a central "dark" mass concentration in the Galactic Center emerged in the late 1970s and early 1980s [193–195]. Astronomers at the University of California, Berkeley, led by Nobel laureate C.H. Townes, made pioneering observations of the ionized gas in the Galactic Center, specifically targeting the 12.8 μm line of Ne II. These observations revealed radial velocities that increased to several hundred km/s in the central pc of the Milky Way [193]. Virial analysis applied to the velocity and velocity dispersion profiles of the ionized gas suggested the presence of a mass concentration of approximately $4 \times 10^6 M_{\odot}$ within the central 1 pc of the Milky Way.

The same group further characterized the infrared fine-structure line emission of compact clouds of ionized gas in the Sgr A West region [194, 195]. By analyzing the velocity distribution of these clouds, they derived a plausible mass distribution for the region within 1 pc of the center. The most plausible model included a central point-like mass of a few $10^6 M_{\odot}$, along with star clusters having a total mass of $\sim 10^6 M_{\odot}$. For the first

¹⁵ <https://www.ovro.caltech.edu>

¹⁶ <https://public.nrao.edu/telescopes/vla/>

time, the notion that this mass concentration could potentially be a SMBH was brought forward. However, due to the limited number of observed clouds, there was not yet definitive proof of the existence of a central point-like mass.

The evidence for the presence of a dark mass concentration in the Galactic Center has continued to accumulate over the years. In the 1980s, increasingly precise observations of ionized, atomic, and molecular gas within the central 1 pc were conducted [196–204]. These observations appeared to confirm the existence of a mass on the order of a few million solar masses, spatially coinciding with the compact source Sgr A*. However, some astronomers remained skeptical, pointing out that relying solely on gravitational forces might be overly simplistic for modeling observed gas motion. Other non-gravitational forces, such as shocks, magnetic fields, and stellar winds, could significantly modify predictions [57, 205]. Moreover, the expectation that a SMBH would be a luminous X-ray source posed a challenge, as Sgr A* appeared faint in all bands except radio to sub-millimeter [206]. This raised doubts about the actual existence of such a compact object [207–209]. Despite improvements in observations, gas dynamics measurements were subject to numerous variables, and the low spatial resolution made conclusions about the nature of Sgr A* uncertain and ambiguous.

1.2.2. Constraints from the stellar dynamics

Significant progress in the field was achieved with the introduction of stellar dynamics measurements in the Galactic Center. Spectroscopic observations of the $2\ \mu\text{m}$ CO absorption bands in late-type giants and AGB stars, as well as the 18 cm OH maser emission stars [58, 72, 210–212], marked the beginning of stellar velocity dispersion measurements within the range of 10 arcsec to 3 arcmin from Sgr A*. While these observations confirmed the predictions from gas dynamics regarding the presence of a few-million-solar-masses concentration within the Galactic Center, the observed stars were positioned at a relatively large distance (projected distances > 10 arcsec), preventing the constraints of such mass to a point-like source.

The evidence for a SMBH gained strength with the observation of hot “He II-stars”, which are blue supergiants nearing their WR stage, within the central 10 arcsec [52, 76, 77].

A virial analysis of the gravitational field using the radial velocity dispersion of 35 early- and late-type stars within 1 to 12 arcsec from Sgr A* allowed to constrain the derived central dark mass to approximately $\sim 3 \times 10^6 M_{\odot}$ within 0.14 pc of the dynamical center [78].

In 1996, Genzel and collaborators undertook the first comprehensive study of approximately 200 stars, reaching distances as close as 0.1 pc from Sgr A* [73]. Their findings revealed a statistically significant increase in projected stellar velocity dispersion toward the dynamical center, reaching up to 180 km s⁻¹ at a distance of 0.1 pc. Employing a simultaneous fit of observed projected surface densities and velocity dispersion, coupled with Jeans analysis, they derived the mass distribution between 0.1 and 5 pc. This marked the first instance where evidence for a compact central dark mass of $2.5\text{--}3.2 \times 10^6 M_{\odot}$ was obtained at a significance level of approximately 8σ .

1.2.3. Constraints from the S-stars

More stringent constraints on the central mass were achieved with the availability of proper motion data for individual stars in the S-star cluster (see Section 1.1.1) located in the central few arcsec. Near-infrared observations of this stellar cluster, conducted by both the MPE (since 1992, [73, 135, 139]) and UCLA (since 1995 [136]) groups (see Section 1.1.2), enabled the determination of proper motions for these high-velocity stars, reaching speeds up to 1000 km/s. The derived velocities (v) of these stars, obtained from the shifts in their sky-projected positions, successfully revealed the $\sigma(v) \propto r^{-1/2}$ relation, indicative of Keplerian motion around a single massive point source, down to a distance of 0.01 pc from Sgr A*. The relatively small number of recorded stellar proper motions (around 40) and the lack of information regarding the line-of-sight direction imposed a statistical approach, limiting the smallest scale to the average radius of the S-stars cluster, approximately ~ 0.02 pc [35].

The next advance occurred with the capability to track the individual stellar orbits [213–215]. Around the early 2000s, the acceleration of three S-stars (S0-1, S0-2, and S0-4, according to the UCLA nomenclature) was determined [214, 215]. These measurements not only confirmed a central mass of approximately $4 \times 10^6 M_{\odot}$, increasing by a factor of 10 the lower bound on the mass density in the central region of the Galactic Center

but also pinpointed the dark mass within 0.05 ± 0.04 arcsec, compatible with zero, of the nominal position of Sgr A*, as determined by examining the intersection of their acceleration vectors. It was also possible to estimate the orbital period of 16 years for one of these stars, S-02 in the UCLA notation, or S2 in the MPE notation and the remainder of this thesis, the brightest in the cluster with a magnitude $m_K = 14$ in the K -band. This discovery emphasized that continuous monitoring of the region over multiple years could yield orbital phase coverage spanning entire orbits. In 2002, S2 reached its pericenter around Sgr A*, providing the first opportunity to constrain its orbital parameters [132, 216]. S2 revolves Sgr A* on a highly eccentric orbit with $e \sim 0.88$, bringing it as close as 120 AU to Sgr A* at its pericenter (see Table 1.1 for the full set of orbital parameters). Consequently, the $4 \times 10^6 M_\odot$ dark mass must be concentrated in a radius smaller than the pericenter passage distance of S2. The significance of observing stellar orbits at the Galactic Center, especially the bright short-period S2, as precision probes of the gravitational field and for establishing constraints on the compact dark mass associated with Sgr A*, motivated both the MPE and UCLA groups to conduct 30-year-long observational campaigns that still continue today. Over the years, improvements in observing techniques and data analysis methods have increased the number of single orbits tracked and substantially enhanced the characterization of S-star orbital parameters [137, 138, 152, 171, 217–219]. In 2017, the MPE group presented the most updated and comprehensive catalog of orbital parameters, astrometric, and line-of-sight velocity data for the S-stars [138]. This catalog constrained the orbits of 40 stars in the Galactic Center and performed a multi-star fit (Figure 1.7) of the gravitational field, incorporating updated systematic analyses from [165].

The position of Sgr A* in relation to the background quasars has been under continuous monitoring since 1981, initially employing the VLA [220] and later utilizing the Very Long Baseline Array¹⁷ (VLBA) [221, 222]. These observations have consistently revealed that, after correcting for the Solar System motion around the center of the Galaxy, the radio source remains stationary. Specifically, Sgr A* exhibits a small residual motion of -7.2 ± 8.5 km/s along the Galactic Plane, with the high uncertainty primarily arising from imprecise estimates of the Sun’s motion around the Galactic Center (ranging between 220 km/s and 255 km/s). The motion

¹⁷ <https://public.nrao.edu/telescopes/vlba/>

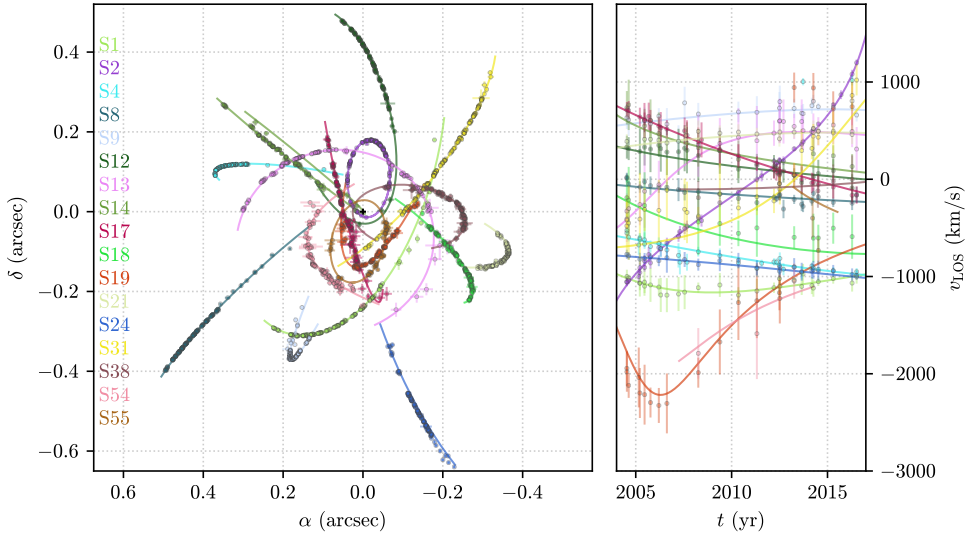


Figure 1.7. Astrometric (left) and line-of-sight velocity (right) data for the 17 stars used for the multi-star fit from [138]. Solid lines represent the best-fitting Keplerian orbits from the multi-star fit, and the color coding is the same for the two plots. The figure has been generated from the electronic tables presented in [138] containing the astrometric and spectroscopic data for the S-stars and the corresponding best-fitting orbital elements.

perpendicular to the Galactic plane is even smaller, at -0.4 ± 0.9 km/s [221]. Consequently, Sgr A* moves at a pace of a few hundred to a few thousand times slower than the S-stars orbiting around it, with orbital speeds estimated to reach up to 10^4 km/s and masses assumed to be in the range of $10 - 15 M_{\odot}$. This observation serves as further evidence that the compact radio source contains a significant fraction of the dynamical mass derived from the dynamics of the S-stars. The equipartition of kinetic energy between a central SMBH and a population of orbiting stars allows to set a lower limit on the mass of Sgr A*, $\gtrsim 4 \times 10^5 M_{\odot}$, solely based on its minimal proper motion [221, 223–225].

Improved astrometry achieved through interferometric observations of the Galactic Center by the Gravity Collaboration using VLT-GRAVITY has led to increasingly precise measurements of orbital parameters, particularly for the S2 star. We report the Keplerian orbital elements (see Section 2.2.3 for more details on their definition) for this star both from the 2017 AO data in [138] and from the post-Newtonian (Section 2.2.4) analysis in [226] with interferometric data. The evident improvement in the tracking of the orbit of the S2 star resulted in more accurate constraints on the properties of the central

Parameter	Gillessen et al. (2017) [138]	Gravity Collaboration (2020) [226]
M ($10^6 M_\odot$)	4.28 ± 0.10	4.261 ± 0.012
D (kpc)	8.32 ± 0.07	8.2467 ± 0.0093
a (mas)	125.5 ± 0.9	125.058 ± 0.041
e	0.8839 ± 0.0019	0.884649 ± 0.000066
i (deg)	134.18 ± 0.40	134.567 ± 0.033
Ω (deg)	226.94 ± 0.60	228.171 ± 0.031
ω (deg)	65.51 ± 0.57	66.263 ± 0.031
T (yr)	16.00 ± 0.02	16.0455 ± 0.0013
t_p (yr)	$2018.33 \pm 0.03^*$	2018.37900 ± 0.00016

compact object, such as its mass (M) and distance (D) to Earth. More importantly, it allowed for the first detection of relativistic effects. These measurements not only demonstrated the relevance of the S-stars (and of the Galactic Center in general) as a powerful gravitational laboratory but also highlighted a remarkable agreement between the observed departures from a purely Keplerian orbit and predictions of General Relativity (we will review thoroughly these predictions in Chapter 2). In the next section (Section 1.2.4) we will analyze in more details the astronomical aspects behind such relativistic effects.

The most recent findings from the UCLA Galactic Center Group regarding the S-stars were reported in 2019. In addition to confirming the detection of relativistic gravitational redshift affecting the line-of-sight velocity of S2, they provided the latest Keck-based estimates of the mass and distance of Sgr A* [152]. The updated values are presented in Table 1.2. As astrometric precision has improved, a more than 5σ tension has emerged between the distance measurements of the two groups, denoted as $\Delta D = 340 \pm 45$ pc. This tension extends beyond D , including the semi-major axis of S2's orbit, its inclination, and its longitude of the ascending node. The Gravity Collaboration investigated the source of this discrepancy in 2021 [172] and concluded that the potential causes could include (i) a potential offset, drift, and distinct orientations of the reference frames used by the two groups, initially observed in [137] by overlapping datasets on the same plot; (ii) differences in the measured line-of-sight velocities of the stars from around 2011 onwards, particularly noticeable for high line-of-sight velocities, partly attributed to the line-of-sight velocity extraction algorithm employed by the UCLA group; and (iii) a potential non-zero offset in the declination direction for Keck telescope measurements, leading to a larger

Table 1.1. Orbital parameters for the S2 star as determined by the Keplerian orbital fit in [138] based on AO VLT data up to 2017 (second column) and by including interferometric data taken with GRAVITY after the 2018 pericenter passage [226] (third column). In both cases the orbital period T is related to the mass M and the semi-major axis a assuming the validity of Kepler's third law.

* The epoch of pericenter passage t_p in the second column has been shifted one orbital period forward (with appropriate error propagation) with respect to what is reported in [138] to make it compatible with the corresponding value in the third column.

UCLA	
M	$3.984 \pm 0.058 \times 10^6 M_\odot$
D	7.971 ± 0.059 kpc

Table 1.2. Most updated values of the Sgr A* mass and distance obtained by UCLA Galactic Center group [152], up to the date of this thesis.

semi-major axis and influencing the entire analysis.

1.2.4. Detection of relativistic effects for the S2 star

When the first measurements of the proper motions of the S-stars orbiting Sgr A* in the Galactic Center were made, it immediately appeared clear that tracking the orbits of such stars could be an efficient way to probe the black hole paradigm and to constrain the distribution of dark matter around Sgr A* [227]. The recorded high eccentricities of these stars combined with the four-million-solar-masses presumed mass for Sgr A*, would imply very close pericenter passages (up to a few hundred gravitational radii of Sgr A*) and subsequent extreme orbital velocities at pericenter ($\sim 10^4$ km/s), that would enable the detection of relativistic effects. This would represent a potential groundwork for interesting tests of General Relativity, the greater the observational accuracy the more stringent such tests.

The first studies where such low-order general-relativistic effects (*i.e.* the first appearing in the post-Newtonian description of relativistic orbital motion, see Section 2.2.4), namely the transversal Doppler shift, the gravitational redshift and the pericenter advance (see Section 2.2.3 for a thorough description of this effects in a general relativistic framework), were taken into account date back to late 1990s - early 2000s [227–231] when the accuracy of observations at the Galactic Center was still poor and such effects could only be theoretically estimated. Due to lack of line-of-sight velocities observations at the time, the effects that were considered were those affecting the sky-projected proper motions of the stars. Subsequent theoretical studies have considered high-order relativistic effects such as frame-dragging and Lense-Thirring effects and those related to the black hole quadrupole moment [232–237], affecting both the astrometry and the spectroscopic observables, and the detection of gravitational lensing by Sgr A* [238–240].

Useful parameters for quantifying the extent of relativistic effects in the orbit of a star orbiting a massive source with mass M with velocity v , are the special-relativistic β -factor, $\beta \equiv v/c$ (being c the speed of light in vacuum) and the general-relativistic parameter $\Gamma \equiv r_g/r$, where $r_g \equiv GM/c^2$ (being G the gravitational constant) represents the gravitational radius

of a massive object, and r is the orbital distance. These parameters serve as a measure of the prominence of deviations from Newtonian dynamics, with higher values corresponding to more pronounced relativistic effects. These parameters vary along the orbit, reaching their maximum at the pericenter, r_p (see Section 2.2.3), when the object delves deeper into the gravitational field of the central massive object and moves at a higher speed and exhibits both special and general relativistic effects.

The S2 star possesses relativistic parameters $\beta \sim 0.026$ (*i.e.* at pericenter it moves at 2.6% of the speed of light) and $\Gamma \sim 7 \times 10^{-4}$. For context, the value of Γ on the surface of a white dwarf is approximately $\sim 10^{-4}$, while on a neutron star, it is on the order of $\sim 10^{-1}$. Several other stars in the S-stars cluster exhibit extreme orbital parameters (detailed in Table 1.3, along with the corresponding relativistic parameters) that would imply the existence of relativistic effects on their orbits. Nevertheless, among all the other objects in the cluster known to date, the S2 star is the easiest to observe due to its intrinsic high brightness (that makes it less prone to confusion with other objects in the cluster) and its almost face-on orientation. Observational efforts over the past decade have therefore been mostly directed at improving the accuracy of observations at the Galactic Center specifically tailored to this particular object.

Spectroscopic observations of the S2 star near its pericenter are expected to reveal an additional ‘kick’ in the reconstructed line-of-sight velocity, denoted as $\Delta v_{\text{LOS}} \simeq \beta^2 c \simeq 200$ km/s, attributable to the combined effects of special and General Relativity, as we will show in Section 2.2.3. The AO-assisted spectrographs utilized by both the UCLA and MPE groups (see Section 1.1.2) deliver line-of-sight velocities with uncertainties on the order of a few 10 km/s, making them capable of detecting this effect on the orbit of S2. This prompted both research groups to conduct intensive monitoring of S2 throughout 2018, the year of its last pericenter passage, to generate datasets suitable for successfully detecting the gravitational redshift signal. As measuring post-Newtonian deviations from a Newtonian orbit relies heavily on precise knowledge of the orbital Keplerian elements, spectroscopic observations in 2018 needed to be complemented with proper motion measurements spanning over 20 years. This approach ensured accurate simultaneous fitting for both the orbital parameters and the redshift effect.

Table 1.3. Main orbital parameters for some of the S-stars, along with the corresponding relativistic parameters and effects. In particular, columns 2 to 6 report the orbital period T , semi-major axis a , eccentricity e and corresponding pericenter and apocenter distances, r_p and r_a respectively. In columns 7 and 8 the relativistic parameters Γ and β (in squared brackets) appear, both evaluated at pericenter. We also report the spatial velocity at pericenter in km/s. Finally, in columns 9 and 10 we report the corresponding special + general relativistic additional redshift contribution at pericenter and the rate of orbital precession per revolution. Orbital data for the stars S62, S4711, S4716 and S4717 result from independent analyses by Peißker and collaborators [173, 174] with near-infrared observations at VLT-NACO. All the other stars have been observed by the MPE/Gravity Collaboration group [138, 177]. The orbital parameters of the S2 star, the best observed and the one on which most of the work of this thesis is based, are highlighted in yellow.

Star	T (yr)	a (AU)	e	r_p (AU)	r_a (AU)	Γ (10^{-4})	v_p (km/s) [β]	Δv_{LOS} (km/s)	$\Delta\phi$ (arcmin)
S4714	12.00	841	0.985	13	1670	33.33	24386 [0.081]	1984	108.82
S62	9.90	746	0.976	18	1475	23.48	20421 [0.068]	1391	77.00
S14	55.30	2361	0.976	56	4666	7.46	11505 [0.038]	442	24.44
S4716	4.00	400	0.756	98	702	4.31	8247 [0.028]	227	15.90
S2	16.05	1031	0.885	119	1944	3.54	7738 [0.026]	200	12.16
S4711	7.60	619	0.768	143	1094	2.93	6825 [0.023]	155	10.74
S12	58.90	2463	0.888	275	4651	1.53	5093 [0.017]	87	5.25
S8	92.90	3337	0.803	657	6018	0.64	3221 [0.011]	35	2.30
S13	49.00	2178	0.425	1252	3104	0.34	2074 [0.007]	14	1.53
S1	166.00	4907	0.556	2179	7635	0.20	1643 [0.005]	9	0.80

UCLA [152]	
ζ_{RS}	0.88 ± 0.16
MPE [171]	
ζ_{RS}	0.901 ± 0.090

Table 1.4. Measurement of the redshift parameter obtained from the 2018 pericenter passage of S2 by the UCLA Galactic Center group [152] and the MPE/Gravity Collaboration [171].

Both the UCLA and MPE groups introduced an additional parameter (Υ in UCLA notation and f in MPE notation, denoted here as ζ_{RS}), to measure the accordance of their observations with a relativistic model. This post-Newtonian parameter uniformly interpolates between an orbit described by Newtonian mechanics ($\zeta_{\text{RS}} = 0$) and an orbit in perfect agreement with General Relativity ($\zeta_{\text{RS}} = 1$). By allowing this parameter to vary freely in the orbital fits, the two groups quantitatively assessed the validity of General Relativity against Newtonian gravity at the 1PN order. During the ~ 2018 pericenter passage, both groups successfully detected a value of ζ_{RS} consistent with unity, confirming the presence of the anticipated 200 km/s ‘kick’ in line-of-sight velocity, yielding the results reported in Table 1.4.

The other measured relativistic effects on the orbit of S2 is the relativistic orbital precession (also known as periastron/pericenter advance or shift), which will be thoroughly described in Section 2.2.2. Differently from the gravitational redshift, the effect is not directly measurable at pericenter, but it is produced (from a dynamical standpoint) there and only becomes noticeable once the body gets farther away from the central object (following a different elliptic orbit than the one on which it arrived at pericenter). When considering the orbital parameters of S2, the orbital precession amounts to

an angle of $\Delta\phi \simeq 12.1$ arcmin for each revolution around Sgr A* (see the last column of Table 1.3 for an estimate of this effect for other S-stars), which corresponds to an observable astrometric shift at apocenter (*i.e.* when the orbital modification is more noticeable) of $\Delta p_a \simeq 1$ mas per orbital period. The amplitude of this effect makes its observations feasible with the current instrumentation [241, 242].

In 2019, the Gravity Collaboration successfully detected [226] the periastron advance of the S2 star, only a few months after the pericenter passage in 2018, thanks to the exquisite astrometric precision delivered by the GRAVITY interferometer¹⁸, revealing a departure from a Keplerian orbit on the order of ~ 0.1 mas (especially in the RA direction, due to the particular inclination of the orbit) following the prediction of General Relativity. In order to establish quantitatively the agreement of the observed data with the relativistic prediction against Newtonian gravity, a new parameter f_{SP} (SP standing for Schwarzschild Precession), which would continuously span between Newtonian ($f_{\text{SP}} = 0$) and relativistic dynamics ($f_{\text{SP}} = 1$), was introduced. Leaving this additional parameter to vary freely in the fit of orbital data presented in [226], along with all other orbital elements, a value compatible with unity was obtained (see Table 1.5), confirming the validity of predictions of General Relativity over Newtonian gravity at more than 5σ [226].

The detection of relativistic effects on the orbits of the S-stars relies heavily on the precision of astrometric and spectroscopic measurements. While, as mentioned, recent efforts from both the MPE and the UCLA groups have focused on improving the orbital tracking of the S2 star, it remains uncertain whether higher-order effects can be detected on its orbit [176] (see Appendix A for an overview of such effects). Higher-order effects decrease more rapidly with distance from the SMBH than lower-order effects. Additionally, stars located at a greater distance from Sgr A* are more affected by Newtonian orbital perturbations due to the stellar population

MPE [226]	
f_{SP}	1.10 ± 0.19

Table 1.5. Measurement of the orbital precession parameter obtained after the 2018 pericenter passage of S2 by the MPE group-Gravity Collaboration [226].

¹⁸ If near-infrared NACO/Keck measurements of the astrometric position of S2 would have not been affected by systematic effects in the construction of the infrared-to-radio references frame it would have been possible to detect the orbital precession of the S2 long before 2019. However, the drift of the orbit on the orbital plane due to the orbital precession of S2 is totally degenerate with a drift of the mass centroid itself on the reference frame and only with the ability of GRAVITY to observe directly the S2-Sgr A* separation it was possible to detect the relativistic precession in a matter of months [226].

itself or a dark mass concentration. Stars or pulsars (see Section 1.3) on closer orbits, due to shorter semi-major axes or higher eccentricities with closer pericenter passages, would offer greater detectability of higher-order orbital effects [175]. Therefore, the next challenge at the Galactic Center involves successfully observing such closer stars that have so far eluded detection. These stars are expected to belong to the fainter end of the luminosity function of the Galactic Center stellar population, making their observation challenging [176]. The prospects for successfully detecting higher-order effects will significantly improve with future observations, for instance, with the 39-meter ELT [179] or TMT [178], enabling the detection of fainter stars on even tighter and more eccentric orbits.

1.2.5. Imaging the event horizon scale structure of Sgr A*

In the pursuit of identifying the fundamental nature of Sgr A* as a SMBH, the observation of the S-stars offers compelling evidence [138, 152, 172–174, 177], and the detection of relativistic effects in perfect agreement with the theoretical predictions of General Relativity for a MBH further narrows down the margin for possible deviations. However, a crucial step forward towards the definitive identification of its nature (or, at least, in undoubtedly rule out several possible alternative descriptions) requires the identification of distinct features unique to SMBHs. The Schwarzschild radius (see Section 2.2.1), characterizes a black hole and determines the radial coordinate of its event horizon [243]. Strong gravitational lensing effects near a black hole, predicted by General Relativity, manifest in observable features, notably a central brightness depression surrounded by a bright emission ring, often termed the “black hole shadow” [244–247]. This shadow, a geometric property reflecting the critical curve on the observer’s image plane, results from photons following paths tangent to the photon sphere [248–250]. For the observation of the shadow, a luminous, optically thin source near the black hole is essential, a condition met by both radiatively efficient and inefficient accretion flows [245, 247].

Despite observational challenges, thanks to breakthrough advancements in VLBI techniques [251–255], Sgr A* has been

observed at progressively shorter wavelengths, with millimeter VLBI being able to capture the presence of internal structure at the horizon scale [256–262]. The Event Horizon Telescope¹⁹ (EHT) collaboration established in 2009 aimed to achieve the angular resolution required to resolve the shadow of Sgr A* [255]. By upgrading and expanding VLBI facilities worldwide, the EHT accomplished a significant improvement in sensitivity [263–269]. In 2022, the EHT provided ground-breaking imaging results of Sgr A*, revealing a compact emission region with intra-hour variability [270–275]. The observed structure, a bright ring with an asymmetric brightness and central depression, aligned with theoretical expectations for a Kerr black hole of four million solar masses at a distance of about 8 kpc.

Parallel to VLBI efforts, Sgr A* was extensively studied at other wavelengths [35, 62, 276–284]. Characterized by a flat radio spectrum and faint X-ray emission, Sgr A* poses challenges to accretion models [62, 285–288]. X-ray and gamma-ray observations revealed a feeble, persistent source emitting thermal bremsstrahlung [270]. The low luminosity and radiative inefficiency raise questions about the accretion properties, with proposed scenarios involving radiatively inefficient accretion or emission from plasma outflows [289–293]. Sgr A* exhibits flares at various wavelengths, with continuous variability in the millimeter band [69, 172, 294–297]. Near-infrared flares have also been tracked astrometrically with the GRAVITY interferometer [172]. Variability of the X-ray and near-infrared emission is detected on timescales of hours, suggesting an origin of the flaring emission from within $\sim 5r_s$ of the SMBH, consistent with emission originating near the Innermost Stable Circular Orbit (ISCO) of the black hole [270]. Multiwavelength observations during the EHT campaign provided a unique opportunity to connect the variability of Sgr A* with changes observed at horizon scales.

1.3. Pulsars at the Galactic Center?

The next major breakthrough in experimental gravitation could come through the discovery of pulsars orbiting SMBHs. Pulsars, renowned for their remarkable intrinsic rotational stability, exhibiting variations on the order of one part in 10^{15}

¹⁹ <https://eventhorizontelescope.org>

per pulse period [298], stand out as exceptional tools for probing gravitational fields [299–301]. Since the 1970s, the exploration of pulsars on binary systems has enabled the detection of various general relativistic effects with increasing sensitivity, including the decay of orbital periods due to the emission of quadrupolar gravitational waves [302–304]. These systems investigate a relatively weak gravitational field in comparison with a SMBH, with component masses on the order of solar masses and compactness $GM/Rc^2 \sim 10^{-5} \div 10^{-7}$ [305], where R represents the binary separation. In contrast, a pulsar in a close orbit (with orbital periods below 1–100 years) around a supermassive compact object, such as Sgr A* at the Galactic Center, would explore an entirely different gravity regime (see Figure 1.1). The analysis of Times Of Arrival (TOAs) of pulses emitted by these objects, known as pulsar timing analysis, holds the potential to surpass all previous tests of General Relativity in the strong field regime [306, 307]. Beyond significantly enhancing the mass determination precision of Sgr A*, pulsar timing analyses in the Galactic Center could provide measurements of the spin magnitude and orientation of the SMBH [305, 307] with a precision of $10^{-4} - 10^{-3}$. Furthermore, these analyses could yield measurements of the SMBH quadrupole moment [306–308], thereby enabling a direct test of the no-hair theorem at the Galactic Center [309, 310].

The significant scientific potential offered by pulsars in the Galactic Center has spurred an increasing number of radio pulsar searches within the central few parsecs of the Milky Way [311–315]. Despite dedicated efforts, these searches have faced challenges in locating the pulsar population of interest. Only five pulsars have been discovered within 15 arcmin of Sgr A* [312, 313], along with a single radio magnetar positioned 2.4 arcsec away from Sgr A* (equivalent to 0.1 pc in projection) [316–318]. To this census, only recently a new addition of a millisecond pulsar (see Chapter 6 for an accurate definition) within 1° from the Galactic Center (thus at an even larger physical distance from Sgr A* than previous discoveries) was made [319]. This scarcity of observed Galactic Center pulsars is attributed to interstellar scattering processes, resulting in the temporal broadening of pulses as they traverse the turbulent and ionized interstellar medium within the Galactic Center [320]. Notably, this temporal broadening exhibits a strong dependence ($\propto \nu^{-4}$) on the observing frequency ν , rendering traditional periodicity search techniques ineffective

at frequencies around $\nu \sim 10^9$ Hz, even for long-period pulsars. Since pulse temporal broadening cannot be corrected instrumentally [321], a potential solution is to conduct pulsar searches at higher observing frequencies. However, the characteristic power-law spectra of pulsars, $\propto \nu^\alpha$ (with $\alpha < 0$ [322]), imply that higher observing frequencies correspond to diminished intrinsic source flux. Consequently and despite the efforts, high-frequency pulsar searches in the Galactic Center, including attempts at 2 and 3 mm [323] and the recent reanalysis for pulsar search of the EHT 2017 data for Sgr A* [324], have not reported the detection of new pulsars. Nevertheless, the abundant population of young and massive stars near Sgr A* suggests the potential presence of pulsars originating from supernovae explosions of stars with masses $> 9M_\odot$. Depending on the specific population model, between 100 and 1000 pulsars could exist within the central parsec of the Galactic Center, with orbital periods < 100 years (including around 100 pulsars with orbital periods < 10 years) and over 10,000 millisecond pulsars [325–328], although for them to be observed they should also point towards Earth.

The detection of at least one pulsar on a tight orbit around Sgr A* stands out as a crucial scientific objective for forthcoming observational facilities, including the Square Kilometre Array²⁰ (SKA) [329], the Five-hundred-meter Aperture Spherical Telescope²¹ (FAST) [330], the next-generation Very Large Array²² (ngVLA) [331], and the next generation Event Horizon Telescope²³ (ngEHT) [270]. These advanced instruments, benefitting from substantial collection areas, not only aim to discover pulsars in proximity to Sgr A* but also aspire to conduct precise timing analyses [332] with an expected accuracy on the pulse TOA measurement of $\sigma_{\text{TOA}} \simeq 100\mu\text{s}$ for normal pulsars at frequencies above 15 GHz [307].

1.4. Conclusions

In this chapter, we have reviewed the historical advancement of observations in the Galactic Center that have not only led to the discovery of the massive compact source Sgr A* at its heart, but also to the decades-long efforts to carefully

²⁰ <https://www.skao.int/en>

²¹ <http://fast.bao.ac.cn>

²² <https://public.nrao.edu/ngvla/>

²³ <https://www.ngeht.org>

measure its physical properties such as its mass and physical size. Thanks to S-stars observations we now know that the gravitational field of the Galactic Center is dominated by a point-like source situated in the correspondence of Sgr A* that behaves like a black hole as described by General Relativity. Even more strikingly, direct imaging of accreting material by the EHT Collaboration revealed a brightness depression that is compatible with the shadow cast by a Kerr black hole. Consequently, the compelling evidence for the presence of a SMBH in the Galactic Center and our current ability not only to track stellar orbits around it but also to measure relativistic effects on their orbits, calls for the development of a fully relativistic methodology to model these observables. This will be the aim of our Chapter 2, in which we will introduce a geodesic orbital model for massive and massless test particles in the Galactic Center and for the reconstruction of relativistic astrometric, spectroscopic and timing observables, that we use in this work to derive constraints on possible alternatives to the standard Schwarzschild scenario.

The General Relativistic motion of test particles

2.

In this chapter, we shall describe the motion of test particles on a curved spacetime. We revise some of the textbook examples by introducing notations and formulae that will come at hand in the following chapters. Furthermore, we introduce the Schwarzschild metric, the paradigmatic spherically symmetric solution that describes the gravitational field of a stationary black hole. This model serves as a basis for developing an understanding on the class of spherically-symmetric spacetimes. The methodology that we will introduce to describe the motion of a test particle in this spacetime will be the starting point for all the subsequent discussion in a more general case. All the trajectories and plots reported in this chapter have been derived with an open-source code, PyGRO (Python integrator for General Relativistic Orbits), that I have developed as part of this thesis project and is currently available on GitHub¹ and as a Python package. Therefore, the examples given not only serve as illustrative tools to study the general-relativistic motion of test particles but also as a test bench for the code itself that has to reproduce all the known results. In the rest of the chapter, we also introduce the post-Newtonian description of the motion of test particles in black hole spacetimes, which allows to develop a great intuition on how General Relativity modifies the orbital dynamics of test particles and allows to easily derive results for the description of relativistic effects. Throughout the chapter (unless is clearly stated) we will make use of geometrized units, in which $G = c = 1$ and thus all quantities can be expressed in units of mass.

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2.1. Basic concepts in General Relativity

In General Relativity the gravitational field produced by massive bodies is described in terms of the curvature of the spacetime. More specifically, General Relativity postulates that spacetime is as a 4-dimensional differentiable manifold $\mathcal{M}$, on which physical events take place, endowed with a

In this section we give a brief and concise overview of the mathematical quantities used in General Relativity. For more details we refer to the classical textbooks on the subject, like [243, 333–335].

¹ <https://github.com/rdellamonica/pygro>

Lorentzian causal structure encoded in an everywhere non-degenerate, smooth, symmetric metric tensor g , with signature $(-, +, +, +)$. The differential structure of this manifold translates in the possibility to define local coordinate systems (or *charts*) in an open neighborhood of any point on the manifold (commonly referred to as *events*). The components of the metric tensor in a local coordinate system $\{x^\mu\}_{\mu=0}^3$ are useful to express the separation between two infinitely nearby events in spacetime,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad | 2.1$$

The metric tensor is a 2-fold covariant tensor field which takes as input a pair of tangent vectors $\vec{v}$ and $\vec{w}$ in the local tangent space to the manifold and returns a real scalar $g(\vec{v}, \vec{w}) \in \mathbb{R}$. This allows to use g to assign an invariant 4-norm to tangent vectors,

$$\|\vec{v}\| = g(\vec{v}, \vec{v}),$$

which, due to the particular signature of a Lorentzian metric, is not positive-definite. This allows for an invariant classification of tangent vectors:

- Time-like vectors: $g(\vec{v}, \vec{v}) < 0$,
- Space-like vectors: $g(\vec{v}, \vec{v}) > 0$,
- Null vectors: $g(\vec{v}, \vec{v}) = 0$.

with dx^μ the infinitesimal coordinate displacements between the two events. This quantity is invariant under arbitrary invertible differentiable coordinate transformations and it is called the *spacetime interval*, or *line element*. The infinitesimal quantity in (2.1) can be integrated along a curve $x(\lambda)$, being λ an affine parameter, on the manifold (connecting two events A and B) to measure its length

$$\ell[x(\lambda)] = \int_{A, x(\lambda)}^B ds = \int_{\lambda_A}^{\lambda_B} \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda, \quad | 2.2$$

hence the name *metric tensor*.

At the conceptual foundation of General Relativity, lies the Equivalence Principle. The classical (Newtonian) formulation of this principle (usually referred to as Weak Equivalence Principle, or WEP) states that, for any given body, the inertial mass and the gravitational mass coincide. As a consequence of the WEP, a physical observer that builds a reference frame tied to a system of particles (that they consider stationary) experiences a cancellation of the local acceleration of gravity when the system (as a whole) undergoes freely falling motion, related to the fact that all bodies *fall* in the same way [336]. The idea that the effects of gravity can always be locally cancelled allows for the more general Strong equivalence principle (SEP): the outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

The SEP implies that freely falling particles follow spacetime trajectories described by

$$\frac{d^2 x^\beta}{ds^2} + \Gamma_{\mu\nu}^\beta \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \quad | 2.3$$

This system of ordinary differential equations has an impor-

tant physical consequence: any change of coordinate system *introduces* a system of (apparent) forces acting on the body. A local gravitational acceleration is thus totally equivalent to an apparent force.

Eq. (2.3) is called the *geodesic equation* and its importance in General Relativity (and in general for all metric theories of gravity) is rather remarkable as its solutions describe the fully relativistic motion of test particles on a given spacetime. The quantities $\Gamma_{\mu\nu}^\beta$ in Eq. (2.3) are called Christoffel symbols and uniquely identify an affine connection on the manifold. In torsion-free manifolds, these symbols can be deduced directly from the metric tensor, defining the Levi-Civita connection

$$\Gamma_{\alpha\beta}^\rho = \frac{1}{2}g^{\rho\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \quad | \quad 2.4$$

where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$. Moreover, it can be proven that solutions of Eq. (2.3) are curves of minimum length, Eq. (2.2), between two events.

Since the geodesic equations in Eq. (2.3) minimize the functional in Eq. (2.2), the quantity

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \quad | \quad 2.5$$

The metric tensor $g_{\mu\nu}$ represents the dynamical variable of General Relativity and it can be completely deduced from the energy-matter content on the manifold. The field equations for $g_{\mu\nu}$ can be derived from the Hilbert-Einstein action [337]

can be taken as a point-particle Lagrangian.

$$\mathcal{S}_{\text{GR}}[g_{\mu\nu}] = \int \sqrt{-g} \left(\frac{1}{16\pi} R + \mathcal{L}_m \right) d^4x, \quad | \quad 2.6$$

where $\mathcal{L}_m$ is the matter Lagrangian and the Ricci scalar, R , is built from the curvature tensor, or *Riemann tensor*,

$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma_{\mu\rho}^\sigma - \partial_\mu \Gamma_{\nu\rho}^\sigma + \Gamma_{\mu\rho}^\gamma \Gamma_{\gamma\nu}^\sigma - \Gamma_{\nu\rho}^\gamma \Gamma_{\gamma\mu}^\sigma,$$

through its non-null contractions

$$\text{the Ricci tensor, } R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma, \quad | \quad 2.7$$

$$\text{and the curvature (or Ricci) scalar, } R = R_{\mu\nu} g^{\mu\nu}. \quad | \quad 2.8$$

Variation of Eq. (2.6) with respect to $g_{\mu\nu}$ leads to the Einstein Field Equations (temporarily restoring the constants G and c)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad | \quad 2.9$$

where $T_{\mu\nu}$, arising from the variation of the matter lagrangian, is the energy-momentum tensor, that describes the energy-matter content of the system. Eq. (2.9) is a system of ten cou-

The SEP implies that only first-order derivatives of the metric tensor can be canceled at a given point in spacetime. Its second derivatives, which appear in the curvature tensor, cannot be generally canceled within a locally inertial frame. Therefore, even if one applies the SEP and builds a local reference frame where the metric is flat and its first derivatives are null, the Riemann tensor at the point would still be non-zero. This is responsible for the appearance of forces of tidal nature that represent the residual, uncancellable, manifestation of gravity.

pled, nonlinear, hyperbolic-elliptic partial differential equations [338] for the ten independent (due to the symmetry) unknown components of the metric tensor field $g_{\mu\nu}(x)$. The left-hand side of equation (2.9) has null divergence (Bianchi identity), reducing the number of independent equations from ten to six, and thus allowing for four gauge-fixing degrees of freedom that correspond to the freedom in the choice of coordinate system. The Bianchi identity implies that the solutions of Eq. (2.9) must satisfy identically

$$\nabla_\mu T^{\mu\nu} = 0,$$

providing equations of motion for the matter fields in the form of a local conservation of energy and momentum.

Due to the high degree of complexity involved in Einstein Field Equations, finding solutions without resorting to numerical methods is in general a very difficult task, and only a handful of known analytical solutions exist [339]. One of the best ways to reduce the complexity of the problem is to impose some physically-motivated symmetries on the metric tensor. In this context, spherically symmetric solutions in vacuum, $T_{\mu\nu} = 0$, hold paramount importance, not only for their implications in astrophysics [334], in gravitational wave astronomy [340] and in cosmology [341–343], but also for the role they had in early experimental confirmations of General Relativity, through the so-called classical tests [21, 22, 344]. We will restrict our study to spherically-symmetric spacetimes, as the current precision of observations at the Galactic Center is not sufficient to probe the rotation of the central object [345].

2.2. Spherically symmetric spacetimes

Spherically symmetric solutions to Einstein's field equation (2.9) in vacuum describe the exterior gravitational field of static and spherically-symmetric bodies. A spacetime is defined to be *spherically symmetric* if its isometry group² contains a subgroup isomorphic to $SO(3)$ and the orbits of this subgroup are 2-spheres. In other terms, isometries within $SO(3)$ can be regarded as rotations and a spherically symmetric

² The group of all the transformations that keep the metric invariant

spacetime is one characterized by a metric tensor that remains invariant under rotations. Because of this rotational symmetry, the metric induced on such 2-spheres has to be completely characterized by their area A . It is thus convenient to introduce a function

$$r = \sqrt{\frac{A}{4\pi}}, \quad | \quad 2.10$$

called *aerial radius*, that uniquely identifies an isometric 2-sphere in spacetime. In flat three-dimensional Euclidean space, r has a non-ambiguous definition in terms of the actual radius of the sphere, namely the geometrical distance between points located on the sphere and its center. This is generally not true in non-flat spacetimes, where the 2-sphere spacetime geometry can differ greatly from Euclidean space and the notion of distance can be in itself not unique. The existence of rotation-invariant 2-spheres, suggests employing the usual angular coordinates (θ, ϕ) to identify points on the sphere, and the metric on each 2-sphere takes the usual form

$$^{(2)}ds^2 = r^2 d\Omega^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad | \quad 2.11$$

Birkhoff's theorem [346] states that any solution to the vacuum field equations characterized by spherical symmetry is necessarily static and asymptotically flat. So that, in the actual derivation of the spacetime metric tensor components in the spherically symmetric case, one can just impose spherical symmetry and automatically find a static solution.

With all the previous discussion in mind, it appears evident that a convenient system of coordinates to employ in spherically symmetric spacetimes is (t, r, θ, ϕ) , usually called the *spherical (or Schwarzschild) coordinate system*. In it, the metric tensor is further simplified, as spherical symmetry imposes that the components of the metric tensor can depend on (θ, ϕ) only through Eq. (2.11), which implies that all off-diagonal terms in g_{ij} must be zero and also gives an explicit expression for $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2 \theta$. In this hypothesis, the problem of finding the metric tensor for a spherically symmetric spacetime thus reduces to only two unknown functions of the sole aerial radius, (corresponding to the g_{tt} and g_{rr} components) and the metric element takes the form:

A spacetime is *stationary* if there exists a one-parameter family of isometries ψ_t , called “time translations”, whose orbits are time-like curves. A spacetime is defined to be *static* if it is also characterized by the existence of space-like hypersurfaces Σ_t that are orthogonal to the orbits of ψ_t . A coordinate system $\{t, x^i\}$, employing the isometry parameter t as a time-like coordinate (referred to as *coordinate time*) has a metric tensor in this coordinate system with all the cross-terms $g_{0i} = 0$.

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2. \quad | \quad 2.12$$

The unknown functions $A(r)$ and $B(r)$ completely fix the geometry of spacetime. It is important to remark here that the line element in Eq. (2.12) only descends from symmetries imposed to a generic spacetime metric, before taking into consideration the equations of General Relativity. For this reason, this line element is rather generic and it might be used for any metric theory of gravity that uses differential geometry to describe the gravitational field of bodies and, in principle, does not require General Relativity. Since our main aim will be to study the geodesic motion in spherically symmetric spacetime solutions that present modifications to the general relativistic pictures, it will be convenient to explicitly write down the geodesic equations for the metric element in Eq. (2.12)

$$\ddot{t} = -\frac{\dot{r}\dot{t}A'(r)}{A(r)}, \quad | \quad 2.13$$

$$\ddot{r} = \frac{r\dot{\theta}^2}{B(r)} - \frac{\dot{r}^2 B'(r)}{2B(r)} - \frac{\dot{t}^2 A'(r)}{2B(r)} + \frac{r\dot{\phi}^2 \sin^2 \theta}{B(r)}, \quad | \quad 2.14$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \dot{\phi}^2 \sin(\theta) \cos(\theta), \quad | \quad 2.15$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}, \quad | \quad 2.16$$

where dots denote derivatives with respect to the geodesic affine parameter and primes denote derivatives with respect to the radial coordinate r .

2.2.1. The Schwarzschild spacetime

When the spherically symmetric spacetime prototype in Eq. (2.12) is plugged into Eq. (2.9), one completely fixes the functional form of the unknown functions in Eq. (2.12) by

$$A(r) = B^{-1}(r) = 1 - \frac{2M}{r}, \quad | \quad 2.17$$

obtaining the *Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad | \quad 2.18$$

Schwarzschild's spacetime is a one-parameter family of solutions that is completely fixed once the mass M of the point-like source of the gravitational field is assigned. In the limit

$r \rightarrow \infty$, the metric reduces to

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad | \quad 2.19$$

which is the metric of flat Minkowski spacetime expressed in spherical polar coordinates. This property of spacetime is called *asymptotic flatness*, implying that far from the source the gravitational field, and hence the curvature, goes gradually to 0. From (2.19) we can infer that the coordinate time t corresponds to the time measured by a clock at infinite distance from origin.

Schwarzschild metric in Eq. (2.18) presents coordinate singularities at $r = 0$ and at $r = 2GM/c^2$, the Schwarzschild radius (temporarily restoring G and c), *i.e.* points where the coefficients of the metric either go to 0 or to infinity. A coordinate singularity does not necessarily imply a genuine spacetime singularity, and it is often possible to remove such singularities by simply changing the coordinate system. However, there are also situations in which the coordinate singularities lead to coordinate independent *curvature singularities*, *i.e.* if the curvature scalar R , or any other scalar formed by contractions of Riemann or Ricci tensors, diverges at some point then those singularities cannot be removed by a coordinate transformation. In the case of the Schwarzschild metric it is easy to show that at $r = 0$ there is an actual curvature singularity since

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12M^2}{r^6}$$

but none of the curvature scalars diverge at the Schwarzschild radius and thus this spherical surface does not represent an actual spacetime singularity. The fact that the line element exhibits anomalous behaviours for $r \leq 2M$ suggests that the Schwarzschild coordinate system does a poor job describing events *inside* this surface in space.

The Schwarzschild radius is usually very small compared to the actual physical size d of a massive body³, and since the vacuum solution is only applicable for $r \geq d \gg 2M$, we can use Schwarzschild coordinates to describe physical phenomena on the exterior of massive spherical bodies. Ultracompact spherical objects, whose physical size is $d < 2M$, expose the

³ For example, the Sun whose mass is $M \sim 10^{30}$ kg, has a Schwarzschild radius of ~ 1 km, much smaller than its physical radius $R \sim 10^6$ km.

surface $r = 2M$ to the outside and the anomalous behavior of the metric coefficients reflects in peculiar characteristics of the motion of particles and light around the body. We call the surface located at the Schwarzschild radius the *event horizon* and the object itself a black hole, since even light cannot escape the gravitational pull from inside the horizon. In particular, inside the event horizon all null and time-like paths can only proceed towards $r = 0$. Once a test particle (regardless of its causal character) has crossed the horizon, moving to smaller values of r is as inevitable as moving forward in time. This is a consequence of the fact that inside the horizon $g_{rr} < 0$, and so the radial coordinate r actually behaves like a time-like coordinate. Subsequently, spacetimes possessing an event horizon are endowed with a non-trivial causal structure in which

- events happening outside the horizon can affect events inside it,
- events happening inside the horizon *cannot* affect events outside it.

The pathological behavior of the Schwarzschild coordinates can be avoided by building different coordinates systems like the Kruskal-Szekeres coordinates [333] that cover the entire maximally extended Schwarzschild solution and have no singularity except for the genuine central one. In this thesis, we will study test particles that move outside of the event horizon, and so using Schwarzschild coordinates, which reduce to standard flat-space spherical coordinates at infinite distance from the central source, is a reasonable choice to retain an immediate geometrical intuition of the particles trajectories.

2.2.2. Geodesic motion in the Schwarzschild spacetime

Freely falling test particles under the action of the gravitational field described by Eq. (2.18) follow the geodesic of the Schwarzschild spacetime obtained as solutions of the system of equations

$$\ddot{t} = \frac{2M\dot{r}\dot{t}}{r^2 \left(\frac{2M}{r} - 1 \right)}, \quad | \quad 2.20$$

$$\ddot{r} = \frac{M\dot{r}^2}{r^2 \left(1 - \frac{2M}{r} \right)} - \frac{Mt^2 \left(1 - \frac{2M}{r} \right)}{r^2} + r\dot{\phi}^2 \left(1 - \frac{2M}{r} \right) \sin^2(\theta) + r\dot{\theta}^2 \left(1 - \frac{2M}{r} \right), \quad | \quad 2.21$$

$$\ddot{\theta} = \dot{\phi}^2 \sin(\theta) \cos(\theta) - \frac{2\dot{r}\dot{\theta}}{r}, \quad | \quad 2.22$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}. \quad | \quad 2.23$$

For the subsequent discussion, it will be convenient to derive the geodesic equations Eq. (2.20)-(2.23) from the test particle Lagrangian, Eq. (2.5), in order to better appreciate how the spacetime symmetries help reduce the complexity of the description of the geodesic motion of a test particle. From Section 2.1 we know that the general relativistic test particle Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \\ &= -\frac{1}{2} \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \frac{1}{2 \left(1 - \frac{2M}{r} \right)} \dot{r}^2 + \\ &\quad + \frac{r^2}{2} \sin^2 \theta \dot{\phi}^2 + \frac{r^2}{2} \dot{\theta}^2, \end{aligned} \quad | \quad 2.24$$

corresponding to the norm of the vector tangent to the geodesic. Since this norm is constant, the Lagrangian itself is a constant of motion. Moreover, since the Lagrangian depends on the metric coefficients, one can apply Noether's theorem and derive conserved quantities by directly invoking the symmetries of spacetime. The Euler-Lagrange equation for the θ component of the geodesic, which leads to the geodesic equation Eq. (2.22)

$$\ddot{\theta} = \dot{\phi}^2 \sin \theta \cos \theta - \frac{2\dot{r}\dot{\theta}}{r}. \quad | \quad 2.25$$

From this equation, if we fix the initial conditions so that $\theta = \pi/2$ and $\dot{\theta} = 0$, corresponding to a geodesic initially on the equatorial plane with velocity parallel to this plane, then $\ddot{\theta} = 0$, identically. Because of the spherical symmetry, we can always perform an isometric rotation and transform any plane into the equatorial plane, thus implying that the test particle motion is always planar. Since we have made no use of the particular

form of the functions $A(r)$ and $B(r)$, this result applies to any spherically symmetric spacetime. Hereafter, let us represent with $\mathcal{L}$ the Lagrangian restricted to the equatorial plane and invoke the other spacetime symmetries. Namely, the metric coefficients are both independent on t and ϕ and we can derive two additional constants of motion

The minus (-) sign in E is conventionally taken because of the signature convention.

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \quad \Rightarrow \quad E \equiv -\frac{\partial \mathcal{L}}{\partial \dot{t}} = \dot{t} \left(1 - \frac{2M}{r} \right) = \text{const.} \quad | \quad 2.26$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad L \equiv \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}} = r^2 \dot{\phi} = \text{const.} \quad | \quad 2.27$$

that we shall call *specific energy* and *specific angular momentum*, respectively. *Specific* stands for “per-unit-mass of the test particle”. We can thus substitute $\dot{t}$ and $\dot{\phi}$ in the Lagrangian, in terms of the constants of motions

$$2\mathcal{L} = -\frac{E^2}{1 - \frac{2M}{r}} + \frac{L^2}{r^2} + \frac{\dot{r}^2}{1 - \frac{2M}{r}}. \quad | \quad 2.28$$

The problem can be particularized for massive test particles that move on time-like geodesics. This fixes the 4-norm of the tangent vector to the test-particle trajectory to $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1$, which fixes the value of the Lagrangian to $\mathcal{L} = -1/2$. Inserting this into (2.28) and solving for $\dot{r}$ yields

$$\dot{r}^2 = E^2 - 1 + \frac{2ML^2}{r^3} - \frac{L^2}{r^2} + \frac{2M}{r}. \quad | \quad 2.29$$

This equation gives the radial evolution of the geodesic parameterized in terms of energy E and angular momentum L . The first two terms can be interpreted as the classical kinetic energy of the test particle (*i.e.* its total mechanical energy subtracted by the particle’s rest energy), so that we can interpret the remaining terms as a one-dimensional effective potential felt by the particle

$$V_{\text{eff}} = -\frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3}. \quad | \quad 2.30$$

The first two terms are the classical contributions given by the attractive Newtonian potential energy and the repulsive centrifugal barrier. The last term $\propto r^{-3}$ is peculiar to General Relativity and contains the orbital modification in relativistic dynamics. In Figure 2.1 we show the effective potential V_{eff} for several different values of L/M . For $r \rightarrow \infty$, $V_{\text{eff}} \rightarrow 0$ as in the Newtonian case, while for $r \rightarrow 0$, $V_{\text{eff}} \rightarrow -\infty$,

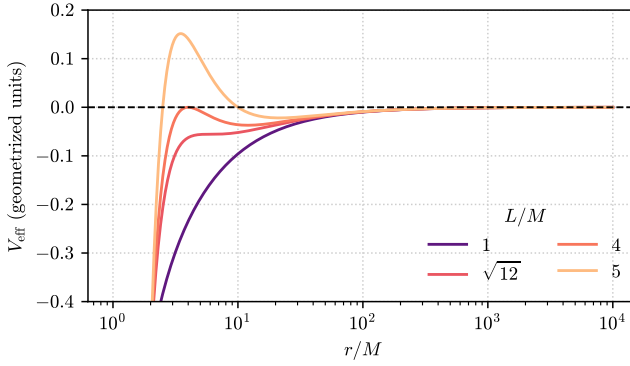


Figure 2.1. The effective potential for different values of the orbital angular momentum L . Note, in particular, the curve corresponding to $L = \sqrt{12}$ for which the effective potential presents a saddle point, thus separating between potentials that do have a stationary point and monotonically increasing potentials.

whereas in the Newtonian case the potential goes to $+\infty$ as r approaches 0. This fact will have a crucial consequence in possible orbital configuration in the Schwarzschild spacetime. The effective potential in Eq. (2.30) allows for a simple classification of possible massive test-particle orbits in terms of the radial turning points, namely points in which the right hand side of Eq. (2.29) and the radial derivative of the effective potential (we might say the *effective force*) vanish. The stationary point of the potential, *i.e.* points where $dV_{\text{eff}}/dr|_{r=\bar{r}} = 0$, are

$$\bar{r} = \frac{L^2 \pm |L|\sqrt{L^2 - 12M^2}}{2M}. \quad 2.31$$

Thus, the effective potential Eq. (2.30) admits real stationary points only above a critical angular momentum

$$|L| > L_c \equiv \sqrt{12}M, \quad 2.32$$

and in this case the root of Eq. (2.31) corresponding to the $-$ sign is always negative. For the orbital classification, it is important to understand whether the effective potential crosses $V_{\text{eff}} = 0$ or not. Let us solve the equation:

$$V_{\text{eff}} = 0 \quad \Rightarrow \quad 2ML^2 - L^2r + 2Mr^2 = 0 \quad 2.33$$

which admits solutions only if $|L| > L_u \equiv 4M (> L_c)$. For $|L| < L_u$ the effective potential is always negative, for $|L| = L_u$ the effective potential touches the $V_{\text{eff}} = 0$ line at its relative maximum (see Figure 2.1) and for $|L| > L_u$ the relative maximum of the effective potential is always positive and one of the roots of Eq. (2.33) is between the maximum and the minimum of the potential. We can thus identify four

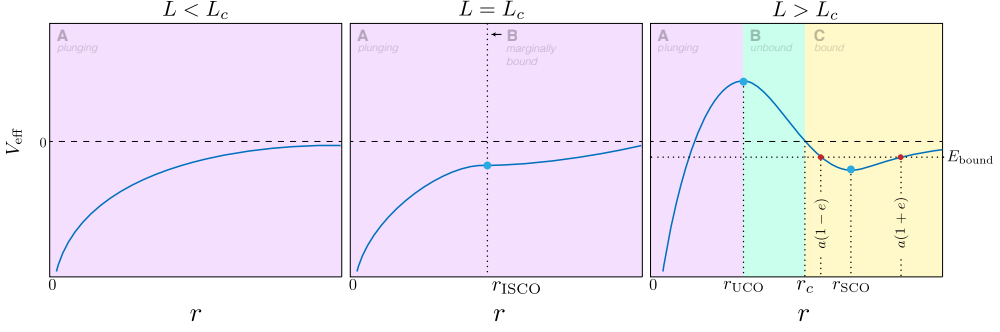


Figure 2.2. Effective potential and corresponding critical points for the three possible configurations $L < L_c$ (left panel), $L = L_c$ (central panel) and $L > L_c$ (right panel). The pink regions correspond to plunging orbital configurations, *i.e.*, while yellow denotes bound orbital configurations (for $L = L_c$ this region reduces to a single point $r = r_{\text{ISCO}} = 6M$ highlighted by a blue dot) and green escaping-to-infinity unbound orbits (a configuration that is only present whenever). In the right panel, the red dots denote the radial turning points ($r_p = a(1 - e)$ and $r_a = a(1 + e)$) for a bound configuration, while the blue dots denote the stationary points of the effective potential, corresponding to the unstable circular orbit (r_{UCO}) and stable circular orbit (r_{SCO}) respectively.

different scenarios:

- $|L| > L_u$. In this case, the effective potential admits two real stationary points and two roots (see right panel of Figure 2.2). More specifically, the root of Eq. (2.31) corresponding to the sign $- (+)$ is a maximum (minimum) for the effective potential, thus implying an unstable (stable) equilibrium. Orbital configurations that realize this condition correspond to bound circular orbits and for this reason they are dubbed as *unstable circular orbit*, at $r = r_{\text{UCO}}$, and *stable circular orbit*, at $r = r_{\text{SCO}}$, respectively. The unstable point r_{UCO} , separates two different classes of orbits. For $r < r_{\text{UCO}}$ we have orbits whose energy and angular momentum are too low to escape the pull of the central object and are thus captured in the event horizon. We call this class of trajectories *plunging orbits* and identify the corresponding region of the parameter space with the letter (A). Identifying by r_c the root of V_{eff} that lies between r_{UCO} and r_{SCO} , we have that for $r_{\text{UCO}} < r \leq r_c$ the effective potential is positive and thus the radial turning point is above the unstable circular orbit, implying unbound orbits that escape to infinity (we identify this region of parameter space with the letter (B)). For $r > r_c$ we found bound orbits that periodically oscillate between the two radial turning points (where the energy E_{bound} matches the effective potential), we identify this region of the parameter space with the letter

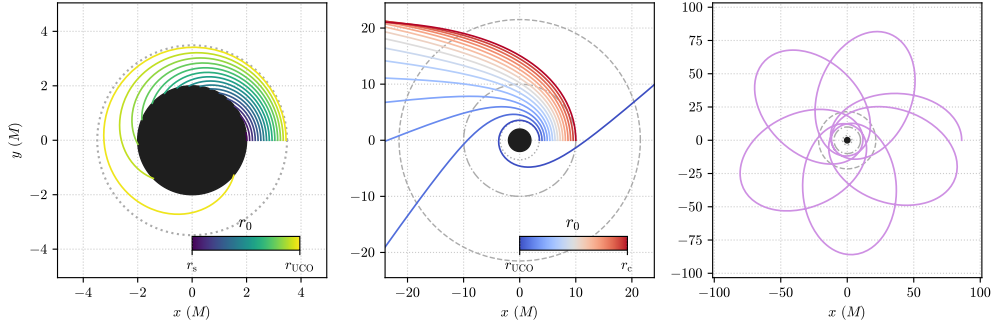
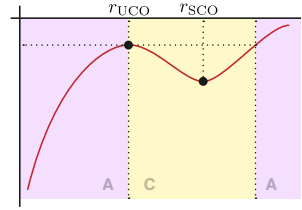


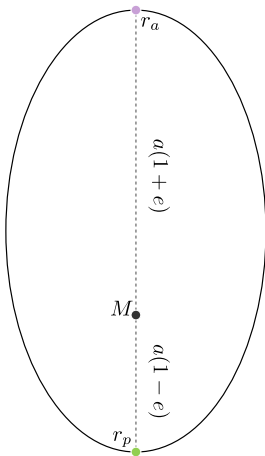
Figure 2.3. Several example time-like orbits in the Schwarzschild spacetime with a fixed value of $L > L_c$ (as in the right panel of Figure 2.2). In the left panel we show orbits with several values of r_0 in region (A), corresponding to orbits plunging into the black hole; in the central panel orbits in region (B), *i.e.* unbound orbits are shown; in the right panel we report one example bound orbit belonging to region (C). All orbits have been numerically integrated with PyGR0.

(C). Example orbits for all the configurations described in this paragraph are reported in Figure 2.3.

- $L_c < |L| \leq L_u$. In this case, there are still a minimum and a maximum for the effective potential, but it admits no root (or only one degenerate root for $|L| = L_u$). There is no unbound orbital configuration and we only have one region of type (C) bounded by two regions of type (A), see the figure in the margin.
- $|L| = L_c$. In this case, there is only one degenerate stationary point for the gravitational potential $r = r_{\text{ISCO}} = 6M$ that corresponds to a saddle point (central panel of Figure 2.2). This orbit is marginally bound, representing the closest marginally stable circular orbit that is possible in the Schwarzschild spacetime, hence the name *innermost stable circular orbits* (ISCO). For this configuration, both orbits with larger and smaller radii than r_{ISCO} belong to the class (A).
- $|L| < L_c$. In this case, the effective potential has neither roots nor stationary points and the only possible orbital configuration is (A).



For the remainder of this thesis, we will discuss configurations of class (C), *i.e.* bound trajectories representing massive test particles in orbit around a central compact object. Specifically, we will particularize the central object to be the supermassive black hole SgrA* in the Galactic Center of the Milky Way and the test particle to be one of the S-stars (most likely S2) or (in the last chapter) a pulsar. Both kinds of orbiting objects



have plausible masses in the range from a few to tens of solar masses and the test-particle approximation is reasonably motivated. As we have discussed, the orbital configuration (C) is characterized by a regular radial motion between two radial turning points corresponding to points in which the orbital energy $E - 1$ matches the effective test-particle potential V_{eff} . If the particle energy is such that $r = r_{\text{SCO}}$ then the orbit is circular and periodic. Orbits like the one depicted in the right panel of Figure 2.3 periodically oscillate between two radii that we can identify as the *pericenter* (identified by the radial coordinate r_p) and *apocenter* (identified with r_a) of the trajectory. We can reformulate the problem in terms of these two parameters. Instead of fixing the specific orbital energy E and angular momentum L of the test particle, one can assign values for r_p and r_e and derive a unique pair $(E, |L|)$. Even more conveniently, one can introduce two orbital parameters, namely the eccentricity e and the semi-major axis a , defined by $r_p = a(1 - e)$ and $r_a = a(1 + e)$ that have a one-to-one mapping to E and $|L|$. In the flat-space limit these parameters perfectly match the concept of semi-major axis and eccentricity classically defined in Newtonian celestial mechanics for Keplerian elliptical orbits. In the General Relativistic case, these parameters depend on the particular choice of coordinates used (for example, in the harmonic gauge of post-Newtonian mechanics, these parameter have a slightly different definition and physical interpretation) and their physical meaning is not uniquely defined in the strong-field regime. As we will show later, even in weak-field a general relativistic orbit is not given by a closed ellipse and thus the definitions themselves of semi-major axis and eccentricity are not perfectly well-defined from a geometrical standpoint. Hence, their purpose is to give a more familiar parametrization of the constants of motion E and L introduced in this section.

Let us now analyze the resulting trajectories in configuration (C). The equation for the relativistic trajectory is already included in Eqs. (2.27) and (2.29). We can express the differential equation for the polar trajectory $r(\phi)$ on the equatorial plane of the Schwarzschild spacetime by considering their ratio

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{E^2 r^4}{L^2} - \frac{r^4}{L^2} \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{r^2}\right). \quad | \quad 2.34$$

When considering the non-relativistic limit $2M/r \rightarrow 0$, this

equation reduces to the usual polar equation for the Keplerian orbits in Newtonian mechanics. The $\propto r^{-3}$ term is what drives the orbital modification in General Relativity. By defining the variable

$$\xi = \frac{L^2}{Mr} \quad \Rightarrow \quad \left(\frac{d\xi}{d\phi} \right)^2 = \left(\frac{Mr^2}{L^2} \right)^2 \left(\frac{d\xi}{d\phi} \right)^2, \quad | \quad 2.35$$

Eq. (2.34) can be recast as

$$\left(\frac{d\xi}{d\phi} \right)^2 = \frac{E^2}{\sigma} - (1 - 2\sigma\xi) \left(\xi^2 + \frac{1}{\sigma} \right), \quad | \quad 2.36$$

where $\sigma \equiv (M/L)^2$. This parameter is $\sigma \sim 0$ in a weak-field non-relativistic regime⁴. We can differentiate Eq. (2.36) with respect to the angle ϕ and bring the equation to the second-order form

$$\xi'' = 1 - \xi + 3\sigma\xi^2, \quad | \quad 2.37$$

which is a differential equation describing *exactly* the polar trajectory of a test particle on the equatorial plane of the Schwarzschild spacetime. This equation does not generally have a closed form analytical solution. However, we can now proceed perturbatively in terms of $\sigma \ll 1$. In order to do so, we can consider a solution of the form

$$\xi(\phi) = \xi_0 + \sigma\xi_1 + \mathcal{O}(\sigma^2) \quad | \quad 2.38$$

and neglect all higher-order terms. Plugging Eq. (2.38) into Eq. (2.37) and keeping only terms up to first order in σ one finds at zero-th order the classical Keplerian equation (supporting the idea that the general relativistic motion, in the weak field regime, can be considered as a small perturbation on the underlying Keplerian motion), while, including the first order correction one finally gets

$$r(\phi) = \frac{a(1 - e^2)}{1 + e \cos \left[\phi \left(1 - \frac{3M}{a(1 - e^2)} \right) \right]}. \quad | \quad 2.39$$

Rather importantly, the radial oscillation between r_p and r_a does not have a period of 2π for the angular coordinate ϕ , namely the *true anomaly* of the related Keplerian orbit. In fact, due to the additional term in the cosine, which is precisely the consequence of the general relativistic term in Eq. (2.30), the

⁴ For example, for the S2 star in the galactic center $\sigma = 1.88 \times 10^{-4}$.

Of course this approximation ceases to be valid in the strong-field regime, where one either has to consider higher order perturbative terms or integrate numerically the geodesic equations.

period of the function at the denominator is larger than 2π . It is thus convenient to introduce a *relativistic anomaly*

$$\chi \equiv \phi \left(1 - \frac{3M}{a(1-e^2)} \right) \quad | \quad 2.40$$

that allows to identify points on the orbit periodically in the interval $\chi \in [0, 2\pi]$. Hence, we can compute what is true anomaly $\Delta\phi$ spanned during one period for the relativistic anomaly

$$\begin{aligned} \chi_1 = 0 & \Rightarrow \phi = 0 \\ \chi_2 = 2\pi & \Rightarrow \phi = 2\pi + \Delta\phi_{\text{GR}}, \end{aligned}$$

thus implying (having restored the physical constants in the last formula for convenience)

$$\Delta\phi_{\text{GR}} \simeq \frac{6\pi GM}{ac^2(1-e^2)}. \quad | \quad 2.41$$

Eq. (2.41) gives the rate of *orbital precession* (or periastron/periapsis advance) of a massive test-particle undergoing free-fall motion in the Schwarzschild spacetime. In the weak-field limit, we can describe general relativistic orbits in the Schwarzschild coordinates just like we would describe Keplerian orbit, except that the Keplerian ellipses constantly precesses about the central body by an amount given by Eq. (2.41). The positive sign of $\Delta\phi$ implies that the orbital precession predicted by General Relativity is *prograde*, *i.e.* the orbit precesses in the same angular orientation of the motion. In Figure 2.4 we show an illustration of this effect. The orbital precession is a characteristic effect of General Relativity and served as an early confirmation of the theory in the Solar System. Indeed, all orbiting bodies experience periastron precession due to General Relativistic effects (along with other classical sources of periastron advance, like multipole moments perturbations by a third body and so on) and the formula Eq. (2.41) allows to account for a periastron advance of $43''/\text{century}$ on the orbit of Mercury that was previously unexplained. Today, this effect has been successfully measured for the orbit of Venus and Earth's orbit [347] and for a range of different objects in the Universe [348–351], including the S2 star in Galactic Center.

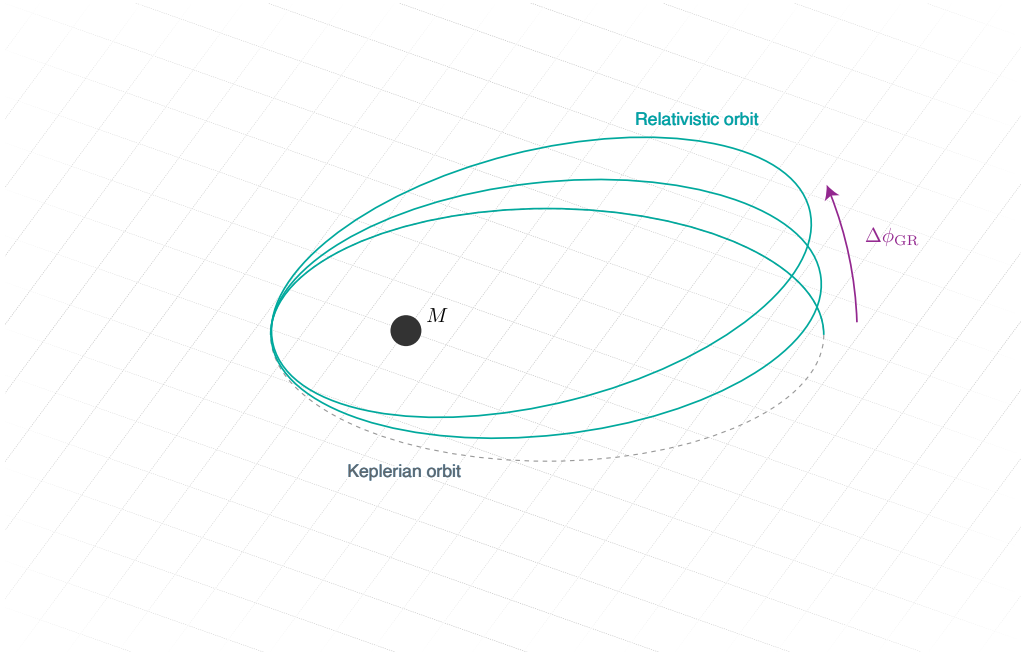


Figure 2.4. Illustration of a precessing orbit as compared to a Keplerian ellipse.

2.2.3. Numerical integration of the geodesic equations

In this section we shall describe the procedure by which we assign initial conditions for massive test particles in terms of orbital elements, we report the numerical schemes that have been implemented in our integration routines and we describe the procedure by which observable quantities are reconstructed starting from the integrated geodesics. We will particularize our calculations for the Schwarzschild spacetime in Eq. (2.18) and the corresponding geodesics, Eqs. (2.20)–(2.23), although the methodology is generally applicable to any spacetime of the form presented in Eq. (2.12).

Assigning initial conditions

The geodesic equations in Eq. (2.16) are a system of four second-order differential equations for the unknown functions $(t(s), r(s), \theta(s), \phi(s))$ that describe the spacetime trajectory of a test particle. The system is well-posed and a solution is uniquely determined once second-order initial conditions are assigned. Namely, one has to specify at a any given proper

time s_0 the initial spacetime position of the test particle ($t(s_0)$, $r(s_0)$, $\theta(s_0)$, $\phi(s_0)$) and spacetime 4-velocity ($\dot{t}(s_0)$, $\dot{r}(s_0)$, $\dot{\theta}(s_0)$, $\dot{\phi}(s_0)$). First, in Section 2.2.2 we have seen that, due to the spherical symmetry, one can always perform appropriate rotations to make the trajectory lie on the equatorial plane for the whole duration of the motion. This reduces the number of free parameters, since $\theta = \pi/2$ and $\dot{\theta} = 0$ are assumed implicitly. Second, since the metric components (and thus the geodesic equations) do not explicitly depend on the time coordinate t , the value $t(s_0)$ can be chosen arbitrarily (and we can fix $t(s_0) = 0$ without loss of generality). Finally, the normalization condition of the 4-velocity for massive test-particles, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$, represents a constraint on the initial data, thus allowing to derive one of the components of the 4-velocity as a function of the others. A complete set of initial data that uniquely identifies a trajectory on the equatorial plane of a spherically symmetric spacetime can thus be $(r(s_0), \phi(s_0), \dot{t}(s_0), \dot{\phi}(s_0))$, with $\dot{r}(s_0)$ fixed by the choice of $\dot{t}(s_0)$ and $\dot{\phi}(s_0)$ through the normalization condition. As explained in Section 2.2.2, instead of fixing the initial conditions directly in terms of the components of the 4-velocity, one can introduce a different parameterization in terms of constants of motion. More specifically, we have seen that, for a closed orbit on the equatorial plane of a Schwarzschild spacetime with mass M belonging to the case (C), for any given specific energy E and specific azimuthal angular momentum L of the test particle the two radial turning points r_a (apocenter) and r_p (pericenter) are identified and that we can parametrize the radial coordinate at these points through the semi-major axis a and an orbital eccentricity e . We can invert this chain of equalities to make it so that unique values for E and L are assigned upon fixing a value for a and e . For the Schwarzschild metric this can be done analytically, by imposing that $r_a = a(1 + e)$ and $r_p = a(1 - e)$ correspond to roots for the radial velocity in Eq. (2.29),

$$E^2 - 1 + \frac{2ML^2}{a(1 \pm e)^3} - \frac{L^2}{(a(1 \pm e))^2} + \frac{2M}{a(1 \pm e)} = 0, \quad | \quad 2.42$$

by which one gets

$$E^2 = \frac{a^2(e+1)(e-1) - 4M^2 + 4Ma}{a(Me^2 + 3M + ae^2 - a)}, \quad | \quad 2.43$$

$$L^2 = \frac{Ma^2(2e^2 - e^4 - 1)}{Me^2 + 3M + ae^2 - a}, \quad | \quad 2.44$$

the signs of E and L being determined by the desired sign of $\dot{t}$ and by the orientation (whether clockwise or not) of the orbital motion, respectively. The specific form of Eq. (2.29) strictly depends on the metric coefficients and the possibility to solve Eq. (2.42) analytically in terms of E and L is not always guaranteed (e.g. considering a different spacetime metric). For this reason, this equation is computed symbolically and then solved numerically for E and L using a Newton-Raphson method in PyGRO. In Figure 2.5 we plot the mapping between pairs of (a, e) and (E, L) obtained numerically for the Schwarzschild metric, assuming that both E and L are positive. The gray area corresponds to orbital configurations that do not belong to the case (C) and hence do not represent precessing-elliptical orbits.

Once the values of E and L are known one can invert equations Eq. (2.26) and (2.27) to obtain $\dot{t}$ and $\dot{\phi}$. For convenience in all our orbital integrations we fix the initial position at the apocenter. We assume that the apocenter time $t_a = t_p - T/2$, where t_p is the time of pericenter passage and T the orbital period, is a known quantity, and

$$\begin{aligned} t(s_0) &= t_a \\ r(s_0) &= a(1 + e) \\ \phi(s_0) &= 0 \\ \dot{r}(s_0) &= 0 \end{aligned}$$

and $\dot{t}(s_0)$ and $\dot{\phi}(s_0)$ are derived with the procedure that we have just explained. This completely fixes the initial conditions on the equatorial plane and allows to integrate the geodesic equations numerically.

Integration scheme

To integrate numerically the geodesic equations we have to recast the system of 4 second-order equations into a system of 8 coupled first-order equations of the form

$$\dot{\vec{y}} = \vec{f}(\vec{y}, s), \quad | \quad 2.45$$

where with $\vec{y}$ we refer to the state vector (which encapsulates all relevant dynamical variables of the system) given by

$$\vec{y} = (t, r, \phi, \dot{t}, \dot{r}, \dot{\phi}) \quad | \quad 2.46$$

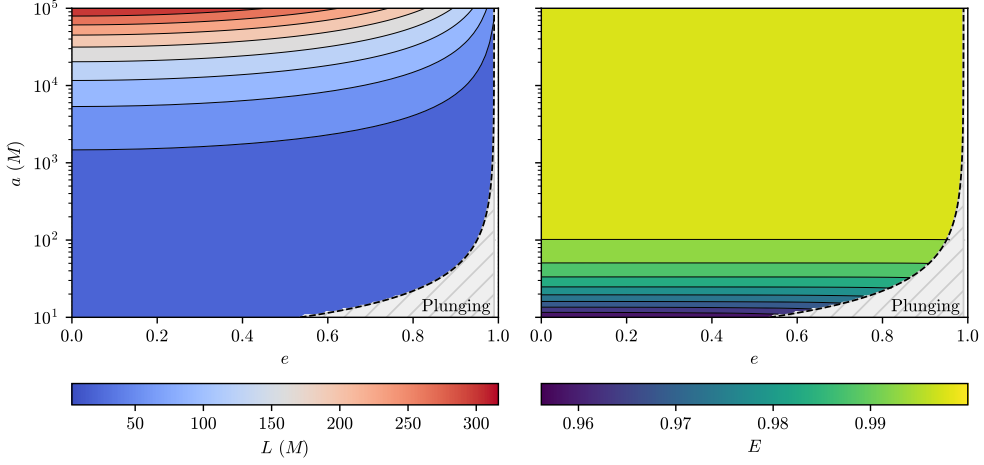


Figure 2.5. Mapping between the on-orbital-plane parameters, the semi-major axis a and eccentricity e , and the two constants of motion E and L whose values are expressed in geometrized units (see the colorbars). The grayed out region, delimited by a black dashed line corresponds to orbital configurations (A), *i.e.* plunging orbits, as opposed to Keplerian mechanics in which, in principle, any pair (a, e) can realize a bounded orbital configuration.

and with $\vec{f}$ to the right-hand side of the system:

$$\begin{aligned}
 \frac{dt}{ds} &= \dot{t}, \\
 \frac{dr}{ds} &= \dot{r}, \\
 \frac{d\phi}{ds} &= \dot{\phi}, \\
 \frac{d\dot{t}}{ds} &= \frac{2M\dot{t}}{r^2 \left(\frac{2M}{r} - 1 \right)}, \\
 \frac{d\dot{r}}{ds} &= \frac{M\dot{r}^2}{r^2 \left(1 - \frac{2M}{r} \right)} - \frac{M\dot{t}^2 \left(1 - \frac{2M}{r} \right)}{r^2} + r\dot{\phi}^2 \left(1 - \frac{2M}{r} \right) \sin^2(\theta) + r\dot{\theta}^2 \left(1 - \frac{2M}{r} \right) \\
 \frac{d\dot{\phi}}{ds} &= -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}.
 \end{aligned}$$

Initial conditions on the state vector are assigned as explained in the previous paragraph,

$$\vec{y}_0 = (t(s_0), r(s_0), \phi(s_0), \dot{t}(s_0), \dot{r}(s_0), \dot{\phi}(s_0)), \quad | \quad 2.47$$

and finally, the system of equations is integrated with a chosen integration method. In PyGRO, several methods of integration have been implemented:

- Euler's method [352];

- Runge-Kutta fourth order explicit method [352];
- Adaptive step-size Runge-Kutta explicit fourth-order method [353];
- Adaptive step-size Runge-Kutta explicit seventh-order method [354];
- Implicit Runge-Kutta fifth-order method of the Radau IIA family [354].

Each of the methods provides with an array of solutions,

$$\vec{y}(s) = (t(s), r(s), \pi/2, \phi(s), \dot{t}(s), \dot{r}(s), 0, \dot{\phi}(s)), \quad | \quad 2.48$$

whose convergence to the exact solution depends on the order of the method used and on the accuracy and precision tolerances (or on the step size for fixed-step methods). The integration is carried out either forward or backward with respect to the affine parameter (s) and it is stopped either when a given final time (s_f) is reached or when some constraint function of the coordinates $K(x^\mu)$ fires a stopping condition for the integration. This is particularly useful to stop the geodesic integration when the geodesic plunges into the event horizon of a black hole. The solution is interpolated with a polynomial whose order matches the order of the integrator, in order to evaluate it at any point in the integration domain.

Reconstructing observable quantities

The numerical integration procedure explained so far leads to the parametric array shown in Eq. (2.48) containing the positions of the test particle on the equatorial plane of the Schwarzschild spacetime. To make it comparable with astronomical observations, it is necessary to reconstruct the observable quantities starting from the integrated spacetime trajectory. As explained in Section 1.1.2 the available data for the S-stars in the Galactic Center consist of sky-projected astrometric positions for an Earth-based observer and spectroscopically reconstructed line of sight velocities. The information on both observables is thus carried by photons emitted by the stars along their orbits and collected by a distant observer. It is clear that, if one wants to approach the problem of reconstructing the astrometric and spectroscopic observables in a fully relativistic way, photon paths represented by

Nevertheless, it is important to remark that the routines implemented in PyGR0 make it capable of adopting the fully relativistic approach if needed. This approach is examined in depth in Appendix A.

null geodesics must be integrated after an appropriate initial-condition-search has been carried out to identify the right null trajectory connecting emitter and observer at each epoch, the so-called *emitter-observer problem*. The advantage of this approach is that one is able to take into account at once all the relativistic effects on the star and the photons. This technique is quite costly in terms of computational resources. On the other hand, the current observational sensitivity in the Galactic Center only allows to detect the leading-order relativistic effects (see Appendix A). For this reason, we can neglect the fully relativistic treatment of light rays and use a semi-classical approach to reconstruct the two observables. Our first assumption is that one can use the aerial coordinate r as representative of the Euclidean concept of distance between the test particle and the central source. This of course implies attributing a physical meaning to a spacetime coordinate that *per se* does not have one. While breaking the general covariance of General Relativity, this assumption allows to work with the integrated trajectory as one would do in Newtonian mechanics, by constructing a cartesian frame of which (r, θ, ϕ) constitute the usual spherical coordinates and in which one can easily build cartesian-like positions

$$x(s) = r(s) \sin(\theta(s)) \cos(\phi(s)) = r(s) \cos(\phi(s)), \quad | \quad 2.49$$

$$y(s) = r(s) \sin(\theta(s)) \sin(\phi(s)) = r(s) \sin(\phi(s)), \quad | \quad 2.50$$

$$z(s) = r(s) \cos(\theta(s)) = 0. \quad | \quad 2.51$$

and velocities

$$v_x(s) = \dot{x}(s) = \dot{r}(s) \cos(\phi(s)) - r(s) \dot{\phi}(s) \sin(\phi(s)), \quad | \quad 2.52$$

$$v_y(s) = \dot{y}(s) = \dot{r}(s) \sin(\phi(s)) + r(s) \dot{\phi}(s) \cos(\phi(s)), \quad | \quad 2.53$$

$$v_z(s) = 0. \quad | \quad 2.54$$

Since the choice of using the radial coordinate as a notion of distance has been made totally arbitrarily and comes from no general prescription, it is important to remark here that any different choice might have been equally appropriate. For example, one could have decided as well to adopt the harmonic radial coordinate used in the post-Newtonian metric (see Section 2.2.4) or the isotropic radial coordinate, as is customary in relativistic celestial mechanics (see Chapter 40 of [333]) as distance and derived a different set of cartesian po-

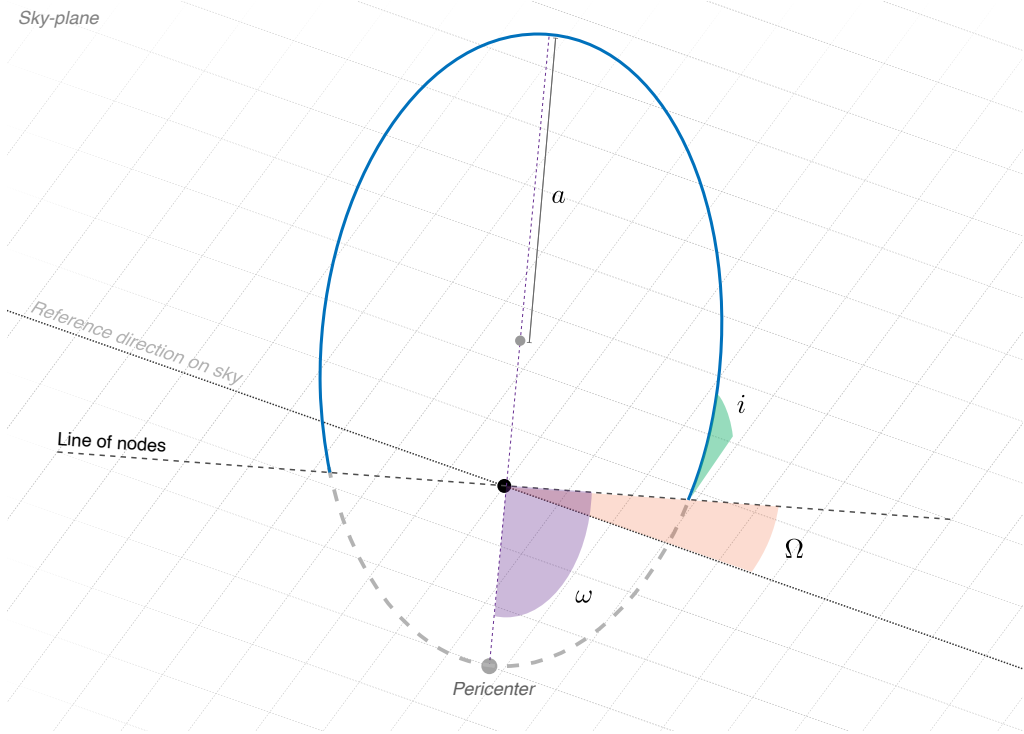


Figure 2.6. Illustration of the angular orbital elements for a Keplerian orbit. The orbital inclination i represent the inclination of the orbital plane with respect to a reference plane (*e.g.* the plane of sky for a distant observer); the longitude of the ascending node Ω corresponds to the angle between a reference direction (usually the E-W direction in sky) and the line of node, namely the line in which the orbital plane intersects the reference plane; the argument of the pericenter ω is the angle, along the orbital plane between the line of nodes and the orbit's pericenter.

sitions and thus a different set of observables. The point, here, is that, as long as one considers sufficiently distant objects, the deviations in the reconstructed observables can be well below the instrumental sensitivity and any choice in the coordinate system will be within the error-bars of the observations. In this regard, the S2 star is sufficiently far away (see Figure A.6 and the related discussion in Appendix A) as not to introduce systematic errors. If one could reduce the instrumental error to arbitrarily low values, the only appropriate procedure would be the fully relativistic integration of photon paths which gives the observables in a truly covariant manner.

Geometrical projection With all this discussion in mind, we can proceed with the semi-classical reconstruction of the astrometric and spectroscopic observables. The first and most obvious step to do is a rotation of the integrated orbit from the equatorial plane into the astrometric reference frame of

a distant observer. In order to do this, one needs to perform a change of reference from the (x, y, z) frame given by Eqs. (2.49)-(2.51) (with the $x-y$ plane corresponding to the orbital plane and the z axis corresponding to the angular momentum direction) to a generic frame (x', y', z') , in which, conventionally, it is assumed that the (x', y') plane corresponds to the plane of the sky and the z' axis corresponds to the line of sight (see Figure 2.6). This brings to the introduction of three new orbital elements (namely three Euler angles) labeled as i , ω and Ω and representing:

The three rotation matrices are defined by:

$$R_1 = \begin{bmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad | \quad 2.55$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{bmatrix}, \quad | \quad 2.56$$

$$R_3 = \begin{bmatrix} \cos(\Omega) & -\sin(\Omega) & 0 \\ \sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad | \quad 2.57$$

- $i = \widehat{zz'}$: the *inclination* of the orbit, which corresponds to the angle between the two directions z and z' and measures how much the orbital plane is inclined with respect to the $x' - y'$ plane in the observer's reference frame. The two planes intersect on a line that is known as *line of nodes*. The orbiting object crosses this line two times over one period. The point where it cuts it from below the plane is known as the *ascending node*; the point where it cuts it from above is known as *descending node*.
- Ω , which is the angle between the x' axis and the ascending node and is called *longitude of the ascending node*.
- The angle ω , on the orbital plane, that is formed between the line of nodes and the semi-major axis of the orbit, specifically its pericenter. It is known as the *argument of the pericenter*.

These angles correspond to three subsequent rotations that bring the orbit into the desired reference frame. In particular one can easily see (Figure 2.7) that, in order to pass from the orbital reference system (x, y, z) to the observer's reference system (x', y', z') , one has to perform the following transformations:

- a rotation R_1 around the z axis by an angle $-\omega$ (the minus sign is due to the fact that by definition ω points *towards* x and not *from* it);
- a rotation R_2 by an angle $-i$ around the new x axis;
- a rotation R_3 around the z' axis by an angle $-\Omega$.

Now, let us consider a generic column vector $\vec{v} = (v_x, v_y, v_z)$ (this might be the vector of positions, Eqs. (2.49)-(2.51), of the test particle or its velocities, Eqs. (2.52)-(2.54)) and, to

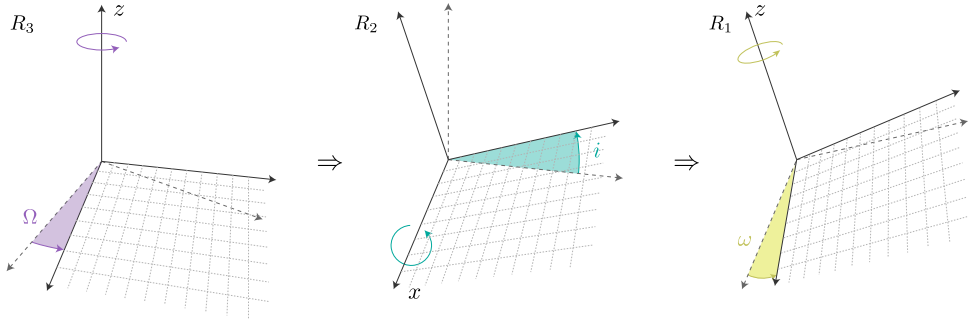


Figure 2.7. The three subsequent rotations needed to transform from the reference frame in which the orbit lies on the equatorial plane, to the one of the distant observer.

express its components in the observer's reference frame $\vec{v}' = (v'_x, v'_y, v'_z)$, apply the three matrices sequentially

$$\vec{v}' = R_3 R_2 R_1 \vec{v}. \quad | \quad 2.58$$

The matrix product of the three matrices can be easily computed and yields

$$R_3 R_2 R_1 = \begin{bmatrix} \mathcal{A} & \mathcal{F} & \mathcal{M} \\ \mathcal{B} & \mathcal{G} & \mathcal{N} \\ \mathcal{C} & \mathcal{H} & \mathcal{P} \end{bmatrix},$$

where we have defined the Thiele-Innes elements [355] as

$$\begin{aligned} \mathcal{A} &= \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i, \\ \mathcal{B} &= \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i, \\ \mathcal{C} &= -\sin \omega \sin i, \\ \mathcal{F} &= -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i, \\ \mathcal{G} &= -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i, \\ \mathcal{H} &= -\cos \omega \sin i, \\ \mathcal{M} &= \sin i \sin \Omega, \\ \mathcal{N} &= -\sin i \cos \Omega, \\ \mathcal{P} &= \cos i, \end{aligned}$$

and thus the transformed components are computed as

$$v'_x = \mathcal{A}v_x + \mathcal{F}v_y + \mathcal{M}v_z, \quad | \quad 2.59$$

$$v'_y = \mathcal{B}v_x + \mathcal{G}v_y + \mathcal{N}v_z, \quad | \quad 2.60$$

$$v'_z = \mathcal{C}v_x + \mathcal{H}v_y + \mathcal{P}v_z. \quad | \quad 2.61$$

Since the trajectory of the star lies in the equatorial plane, we can reconstruct the two sky-plane astrometric observables as

$$x'(s) = \mathcal{A}x(s) + \mathcal{F}y(s) \equiv \alpha(s)D, \quad | \quad 2.62$$

$$y'(s) = \mathcal{B}x(s) + \mathcal{G}y(s) \equiv \delta(s)D, \quad | \quad 2.63$$

where with α and δ we refer to the right ascension and declination as seen by the distant observer and D is the distance of Sgr A* from Earth. The line-of-sight distance and velocities of the test particle, on the other hand, are computed by

$$z'(s) = \mathcal{C}x(s) + \mathcal{H}y(s), \quad | \quad 2.64$$

$$v'_z(s) = -(\mathcal{C}v_x(s) + \mathcal{H}v_y(s)), \quad | \quad 2.65$$

where the minus sign in the last equation is conventionally considered to get a positive line-of-sight velocity when the test particle approaches the observer and a negative sign when receding. In Figure 2.8 we show the orbit of the S2 star computed using the procedure depicted above given the orbital parameters reported in Table 1.1. More specifically, we show the orbit on the equatorial plane and the post-projection trajectory. The colormap in the projected orbit reports the line-of-sight distance. For completeness, we also show astrometric data for S2, as given in [138].

We are implicitly assuming that the Solar System's and Earth's gravitational fields induce a negligible gravitational time dilation for clocks on Earth. This is indeed not true, as has been demonstrated by several Solar System experiments and pulsar timing observations [356]. Nonetheless, the error induced on the astronomical observables of the S-stars is well below the current instrumental sensitivity and we can neglect it, see for example the supplementary materials of [152].

Rømer delay Positions and velocities of the test particles are computed in terms of the affine parameter. For massive test particles, we take for convenience this affine parameter to be the proper time measured by a clock at rest with respect to the test particle. The zero-th component of the integrated spacetime trajectory of the particle, however, $t(s)$, provides with a map between this proper time s and the coordinate time t , namely (in Schwarzschild coordinates) the time measured by a clock at infinity. This quantity is very useful when reconstructing astronomical observables because we can effectively identify it with the time measured by clocks on Earth, very distant from the gravitational field of Sgr A*, that is used to label the observation epochs. We can thus use the coordinate time as a bookkeeper coordinate to label both the moments of emission of the photons, t_{em} , and of arrival at the observer's location, t_{obs} . The relation between

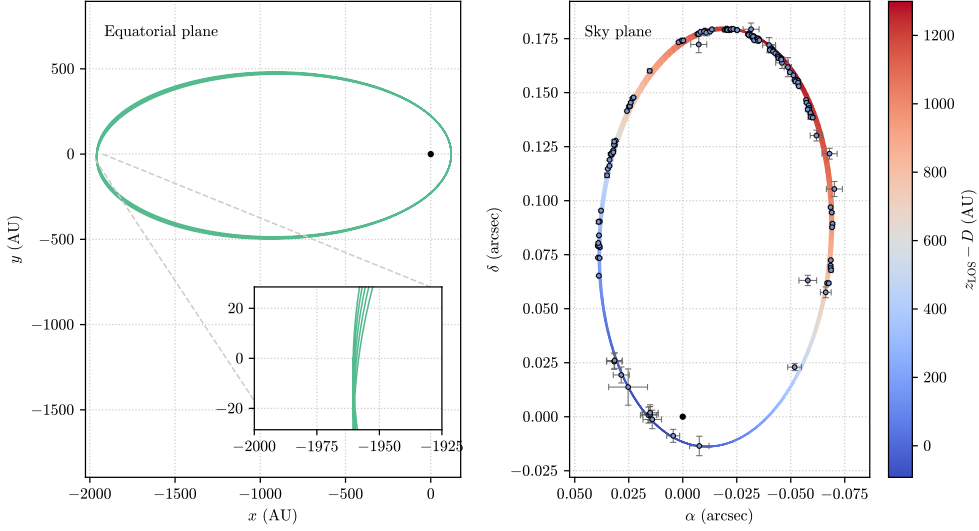


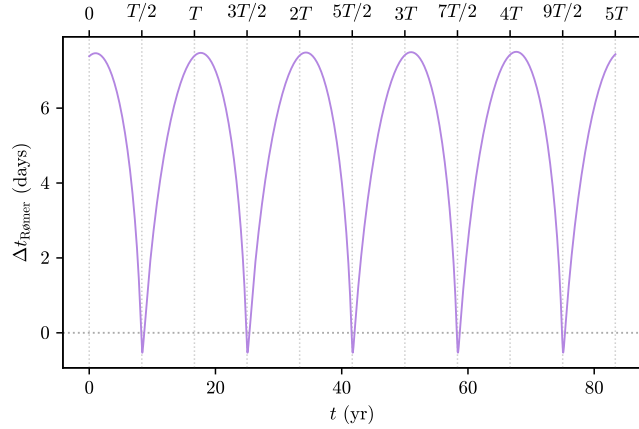
Figure 2.8. Numerically integrated orbit of the S2 star around the SMBH Sgr A* on the equatorial plane of the Schwarzschild spacetime in astronomical units (*left panel*) and after projecting it into an Earth-based observer’s reference frame (*right panel*) over five orbital periods. The inset plot in the left panel reports a zoomed portion of the orbit around the apocenter in which the effect of orbital precession is clearly visible. The colorbar on the right panel reports the line-of-sight distance of the S2 star taking as zero-point the sky-plane intersecting Sgr A*. The superimposed dots correspond to the astrometric observations of the S2 star in [138] and their respective uncertainties.

the two quantities is given by the photon travel time between emission and reception. In Section 2.2.5 we will see how one can compute this propagation time in a fully relativistic way. Nonetheless, for what concerns the S-stars observations the only effect that gives a non-negligible contribution is the classical Rømer delay. This effects simply accounts for the light-ray propagation time assuming straight-line propagation and can be thus expressed as (we restore the speed of light c for a clearer notation)

$$\Delta t_{\text{Rømer}} \equiv t_{\text{em}} - t_{\text{obs}} = \frac{z'(t_{\text{em}})}{c}. \quad | \quad 2.66$$

For orbits with no inclination with respect to the sky-plane z' is a constant and the Rømer delay simply accounts for the distance between the observer and the orbit barycenter. For inclined orbits, on the other hand, the time-varying character of $z'(t_{\text{em}})$ induces a periodic modulation on the time of arrival of the photons that one has to take into account. Solving Eq. (2.66) for t_{obs} gives out a map between observation times and emission times. The zero-th component of the integrated geodesic in Eq. (2.48) relates coordinate times of emission to

Figure 2.9. Rømer delay, *i.e.* difference between time of emission and observations, for the S2 star computed over five orbital periods. At the moment of greatest distance the delay amounts to $\sim +7$ days, while at the closest proximity the delay is negative (*i.e.* the photons arrive earlier as compared to a case with no inclination) and amount so ~ -13 hours.



proper times. Thus, we have an invertible chain of maps that links the proper times with the times of observation and *vice versa*

$$s \leftrightarrow t_{\text{em}}(s) \leftrightarrow t_{\text{obs}}(t_{\text{em}}), \quad | \quad 2.67$$

that we can invert to express the proper time as a function of the observation time

$$s = s(t_{\text{obs}}). \quad | \quad 2.68$$

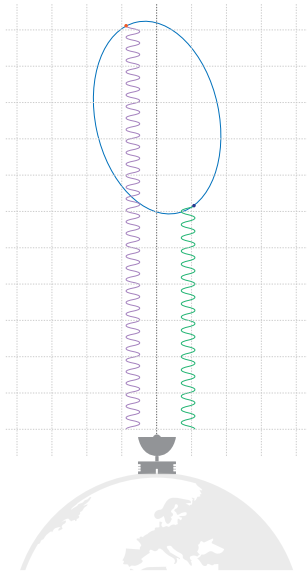
This allows to reparametrize the reconstructed observables in Eqs. (2.63) and (2.65) as function of the observation time

$$\alpha(s) \rightarrow \alpha(t_{\text{obs}}), \quad | \quad 2.69$$

$$\delta(s) \rightarrow \delta(t_{\text{obs}}), \quad | \quad 2.70$$

$$z'(s) \rightarrow z'(t_{\text{obs}}), \quad | \quad 2.71$$

$$v'_z(s) \rightarrow v'_z(t_{\text{obs}}), \quad | \quad 2.72$$



which is a fundamental step for the comparison of synthetic orbits with actual observational data. In Figure 2.9 we report the Rømer delay computed for the S2 star around Sgr A*. Due to its high inclination, observations for S2 are drastically affected by Rømer delay, which amounts to ~ 7 days of delay around the apocenter. The photons emitted at the apocenter will arrive on Earth almost a week later as compared to a case with no inclination, while at the pericenter photons arrive ~ 13 hours earlier.

Redshift As explained in Section 1.1.2, the line-of-sight velocity of the S-stars is reconstructed from spectroscopic observations. Usually, the hydrogen Brackett- γ absorption line is chosen and tracked at different epochs. The apparent redshift ζ is measured and converted into an apparent line-of-sight velocity, by considering that this shift is produced only due to longitudinal Doppler effect (we restore here and for the rest of the paragraph the speed of light c for a clearer notation)

$$\zeta = \frac{\Delta\lambda}{\lambda} - 1 \equiv \frac{v_{\text{LOS}}}{c}. \quad | \quad 2.73$$

For non-relativistic orbits, the line-of-sight velocity reconstructed with this method matches the kinematic velocity of the celestial body along the line-of-sight. While the kinematic line-of-sight velocity v'_z accounts for most of the apparent redshift,

$$\zeta_k(t_{\text{obs}}) = \frac{v'_z(t_{\text{obs}})}{c}, \quad | \quad 2.74$$

the combination of the intense gravitational field produced by the SMBH in the Galactic Center and the subsequent high orbital speed brings the S-stars orbit in the relativistic regime. More specifically, the gravitational time dilation due to the proximity to the supermassive source and the transverse Doppler effect (*i.e.* Lorentz time dilation) due to the high velocity of the S-stars, especially near the pericenter, produce a non-negligible increase in the apparent redshift that must be taken into account to properly reconstruct the spectroscopic observable. The combination of both effects is named Einstein delay. It is possible to estimate the two effects separately using a post-Newtonian approach (see Section 2.2.4). Since we integrate fully relativistic geodesics, however, we can avoid resorting to post-Newtonian formulae and directly estimate the coordinate-to-proper-time shift from the integrated zeroth component of the 4-velocity

$$\zeta_E(t_{\text{obs}}) + 1 = \dot{t}(t_{\text{obs}}). \quad | \quad 2.75$$

We can then combine this redshift component to the kinematical term in Eq. (2.74), giving the total apparent redshift

$$\frac{v_{\text{LOS}}(t_{\text{obs}})}{c} + 1 = (\zeta_k(t_{\text{obs}}) + 1)(\zeta_E(t_{\text{obs}}) + 1). \quad | \quad 2.76$$

In Figure 2.10 we report the relativistic redshift (left panel) and the total apparent line-of-sight velocity (right panel) reconstructed from the integrated geodesic for the S2 star. The

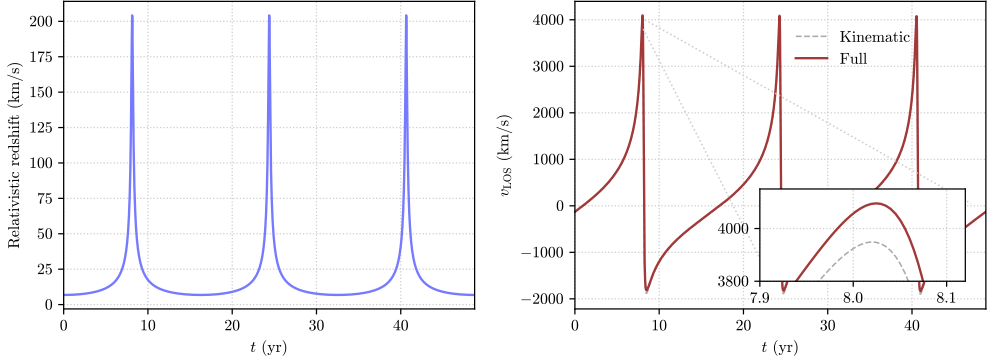


Figure 2.10. *Left panel:* Additional contribution to the apparent line-of-sight velocity due to the relativistic (special Doppler + gravitational) redshift. *Right panel:* total reconstructed line-of-sight velocity from the spectroscopic redshift considering both the kinematical velocity and the relativistic contributions. The inset plot zooms around the pericenter passage to show that the relativistic contributions perceptibly affects the spectroscopic observable well above the current instrumental limit of ~ 10 km/s.

relativistic contribution at the pericenter accounts for an additional ~ 200 km/s kick in the apparent line-of-sight velocity that has been successfully measured by the two independent observational groups as explained in Section 1.1.2 [152, 171].

Orbital precession The geodesic approach also allows to determine the rate of orbital precession directly from the integrated orbit, rather than from a post-Newtonian formula. This is particularly useful for very close objects, for which the assumptions leading to the expression Eq. (2.41) may not hold any more or for spacetimes that differ from the Schwarzschild black hole and for which an analytic expression for the rate of orbital precession is unknown. The periastron advance is naturally taken into account, in a fully relativistic manner, when integrating directly the geodesic equations. To reconstruct the rate of orbital precession from the numerically integrated orbit, one needs to consider the definition of periastron advance. Basically, it is sufficient to determine from the polar equation of the trajectory on the orbital plane, $r(\phi)$, obtained by combining the $r(s)$ and $\phi(s)$ solutions in Eq. (2.48), the radial velocity as a function of angular coordinate, $\dot{r}(\phi)$. The roots of this function (that can be easily obtained numerically with a root finding algorithm, like Newton-Raphson method [357]) correspond to radial turning points on the orbit. Determining the values of ϕ corresponding to two consecutive apocentric or pericentric turning points one can estimate how much the

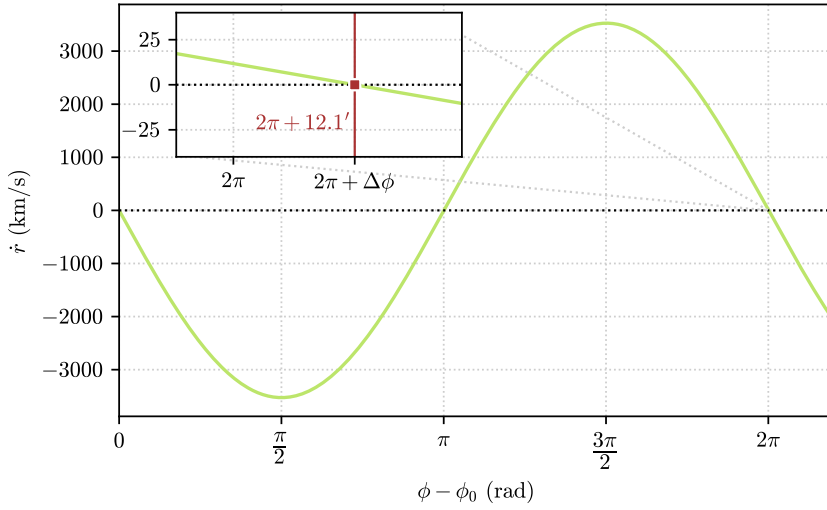
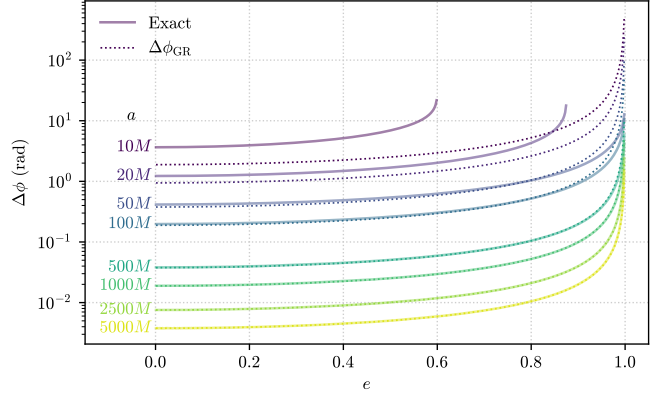


Figure 2.11. Radial velocity profile as a function of ϕ for the S2 star over a bit more than one orbital period. The trajectory starts at apocenter ($\phi = \dot{r} = 0$) and when it reaches apocenter again, after one radial orbital period, the angle spanned by the test particle is $2\pi + \Delta\phi$, with $\Delta\phi = 12.1$ arcmin as shown in the inset zoomed plot. This value matches exactly what one gets applying the post-Newtonian formula Eq. (2.41) for the orbital parameters of S2.

angle spanned in one radial period differs from 2π (which would be the case with no orbital precession). The angle obtained is exactly the orbital precession accumulated over one orbital period, *i.e.* the desired rate of orbital precession.

In Figure 2.11 we show the radial velocity profile as a function of ϕ for the S2 star over slightly more than one orbital period. The orbit starts at the apocenter and when it reaches it again, after one orbital period, the angle spanned is slightly larger (by 12.1 arcmin) than 2π . This value matches exactly what one obtains applying the post-Newtonian formula Eq. (2.41) for the orbital parameters of S2. The discrepancy between the rate of orbital precession reconstructed from the fully relativistic geodesic approach and the post-Newtonian expression is evident at short orbital separations and high eccentricities. In Figure 2.12, we plot the fully relativistic computation and the post-Newtonian prediction for increasingly small values of the semi-major axis a as a function of the eccentricity e . For $a \lesssim 100M$ the discrepancy between the two predictions starts to be noticeable even at low eccentricities, indicating that a cautious use of the post-Newtonian approximations for such highly relativistic objects is required. Moreover, while the post-Newtonian predictions always diverge for $e \rightarrow 1$

Figure 2.12. Rate of orbital precession (per orbital period) as a function of the eccentricity e and for several values of the semi-major axis a (in the range for which bounded orbital configurations are allowed, see Figure 2.5). The solid lines report the values of the orbital precession computed numerically with PyGR0, while the dotted lines correspond to the values of the orbital precession predicted by the first-order low-eccentricity formula in Eq. (2.41). For very close orbits and very-high eccentricities the orbital precession predicted by the analytical formula severely departs from the actual value considering the fully relativistic dynamics.



for every semi-major axis, the fully relativistic prediction takes into account that at small orbital separations not all eccentricities are allowed and that a low angular momentum might lead to an unstable orbital configurations (*e.g.* the plunging case (A) in Figure 2.5) for which a precessing ellipse is not a solution to the geodesic equations.

Systematic effects Near-infrared astrometric observations of the S-stars in the Galactic Center deliver relative positions between stars in the nuclear star cluster, which are then casted in the radio reference frame of the Galactic Center where emission from Sgr A* itself is visible. This radio-to-infrared matching of the positions of the S-stars and of the gravitational source Sgr A* allows for a zero-point offset and drift of the central source in the sky plane. In order to take into account this effect, which, if neglected, will limit the detectability of relativistic effects on the trajectory of the S-stars [226], we consider a linear translation of the massive source on the sky (accelerations on the sky-plane can be neglected [138]), which requires the introduction of four new parameters for the zero-point at the initial epoch for its linear motion, (α_0, δ_0) and $(v_{\alpha,0}, v_{\delta,0})$, respectively, that we add on top of the resulting astrometric observables derived as explained in previous sections.

Since the spectroscopically reconstructed line-of-sight velocity is corrected for the Local Standard of Rest [138], an additional constant offset, $v_{\text{LOS},0}$ in the reconstruction of the line-of-sight velocity must be taken into account and added to the v_{LOS} derived in Eq. (2.76).

Full orbital model

When the whole procedure explained in this chapter is taken into account, a fully relativistic⁵ set of observables for a massive test particle in the Galactic Center (*e.g.* for the S2 star) is completely fixed by the orbital model that we have developed, once the following parameters have been assigned:

$$(M, D, t_p, a, e, i, \omega, \Omega, \alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0}, v_{\text{LOS},0}). \quad | \quad 2.77$$

Here:

- M completely fixes the components of the Schwarzschild metric in Eq. (2.18) and of the related geodesic equations Eqs. (2.20)-(2.23);
- D , the distance to the gravitational source, allows to obtain angular separations from the spatial positions;
- t_p fixes the coordinate time of pericenter passage, setting a zero point for the coordinate time scale to match with the observation epochs;
- a and e select an orbit on the equatorial plane of the Schwarzschild spacetime;
- i, ω and Ω allow to project the equatorial-plane orbit in any given observer's reference frame;
- $\alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0}$ and $v_{\text{LOS},0}$, take into account systematic effects related to the construction of the astrometric and spectroscopic reference frame.

The model returns a set of observables which, for a given set of observation epochs $\{t_{\text{obs}}\}$, is composed by:

- Relative astrometric positions α (right ascension) and δ (declination) between the test particle and the origin of the astrometric reference frame (which corresponds with Sgr A* if $\alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0}$ and $v_{\text{LOS},0}$ are all zero);
- Line-of-sight velocities, v_{LOS} for the test particle corrected for the Local Standard of Rest;
- The rate of orbital precession $\Delta\phi$ of the test particle computed in a fully relativistic manner.

⁵ Keeping in mind the prescription on the light propagation illustrated at the beginning of Section 2.2.3

This orbital model will be used in the rest of this work to produce fully relativistic synthetic orbital data for the S-stars (mainly the S2 star) in the Galactic Center by considering a plethora of spherically symmetric models that differ from the Schwarzschild spacetime and that present one or more additional parameters to be added to Eq. (2.77). A fitting procedure, detailed in Appendix C, of such orbital models on the data presented in Section 1.1.2 will be used to derive experimental bounds on model and orbital parameters.

On several occasions, we will investigate whether future data for the S2 star from the GRAVITY interferometer will help to improve our constraint on such additional parameters in the spacetime metric. In order to perform this forecast analysis we have used the orbital model developed in this chapter to produce a mock catalogue of realistic future observations mirroring the sensitivity and limitations of the VLT instruments GRAVITY, for the astrometry, and SINFONI, for the spectroscopic measurements leading to line-of-sight velocity estimates, over three and a half orbital periods, starting from the last pericenter passage in ~ 2018 . In our baseline model, we assume: (i) that the gravitational field of Sgr A* is described by Schwarzschild spacetime; (ii) the *true* values of the parameters corresponds to their best fitting values from [226] and reported in Table 1.1 (right column); (iii) that the astrometric and spectroscopic observations are taken at exactly the same epochs, following an observational pattern that becomes more frequent at and around the pericenter passage. In particular, following [345] we assume to have:

- one observation per day in the two weeks centered on the pericentre passage;
- one observation every two nights in a month centered on the pericentre passage;
- one observation per week in the two months centered on the pericentre;
- one observation per month in the rest of the year of pericentre passage;
- two observations per year in the rest of the years.

We thus build an array of observation times, $\{t_{\text{obs}}\}$, accordingly and use it to compute the astrometric and spectroscopic observables. Moreover, (iv) we assume that the instruments are always operated in ideal conditions, acquiring data with

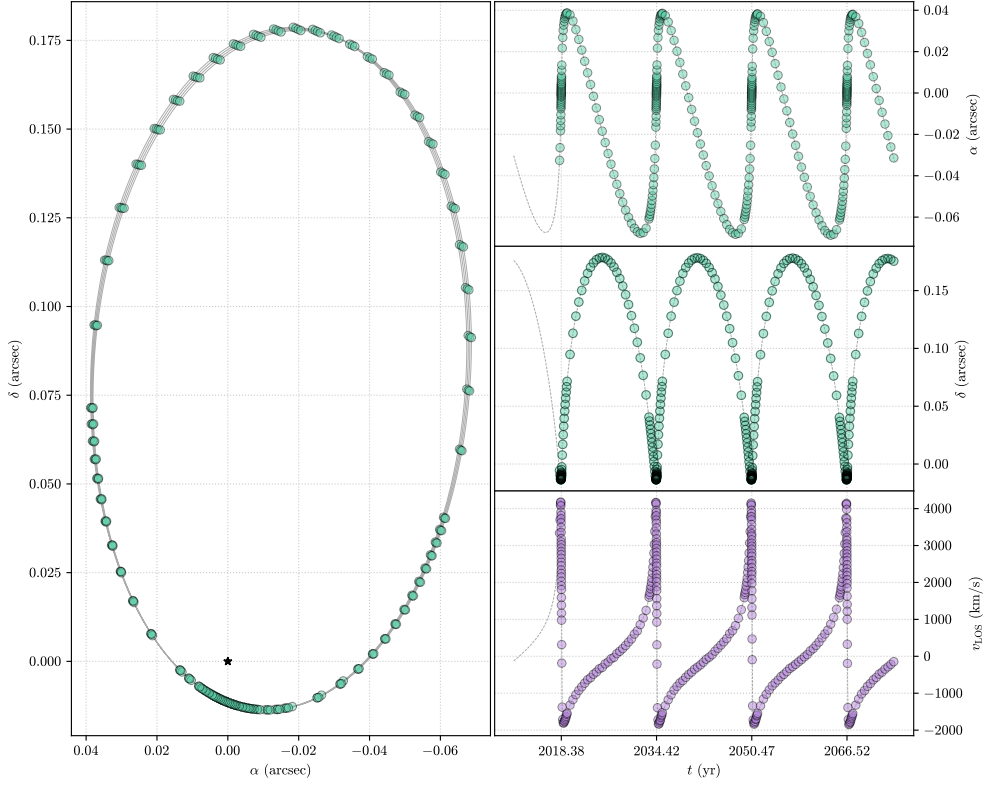


Figure 2.13. Mock catalogue of observations (astrometric positions, left panel and top two panels on the right, and radial velocity, right bottom panel) for the S2 star over three and a half orbital periods starting at the last pericenter passage in ~ 2018 assuming a Schwarzschild baseline model.

an accuracy set to their maximal nominal values $\sigma_A = 10 \mu\text{as}$ for the GRAVITY astrometry and $\sigma_V = 10 \text{ km/s}$ for the SINFONI line-of-sight velocities. Such uncertainties are taken into account by adding to the observables resulting from the orbital model a Gaussian noise with zero-mean and FWHM given by σ_A and σ_V for the astrometric data and line-of-sight velocities, respectively. Finally, (v) we assume that the GRAVITY observations always provide astrometry relative to Sgr A*, meaning that we assume the central object to be fixed in the origin of the celestial reference frame, which fixes to 0 the value of the offset and drift parameters of the reference frame. In Appendix B we report a table containing all such mock observations that are shown in Figure 2.13 for completeness.

2.2.4. Post-Newtonian test-particle orbital mechanics

The post-Newtonian formalism is an approximation scheme used to study general relativistic field equations and their solutions in the weak-field and small-velocity (of the sources of the gravitational field) limit (see Chapter 5 of [340], or Chapter 39 of [333]). More specifically, for a gravitational system characterized by mass M , typical velocity v , and scale r , one can introduce the parameter (restoring momentarily G and c)

$$\epsilon \sim \frac{v^2}{c^2} \sim \frac{GM}{rc^2}. \quad | \quad 2.78$$

Systems for which $\epsilon \sim 1$ are said to be in a gravitational strong-field regime. Conversely, a system for which $\epsilon \sim 0$ is characterized by Newtonian gravity. The dynamics of relativistic system for which $\epsilon \ll 1$ can be described within the post-Newtonian framework by using the parameter ϵ as a perturbative parameter, and by expanding all the relevant quantities and equations at the desired perturbative order (called in this context a *post-Newtonian order*). For our purpose, we will focus on the first-order post-Newtonian (1PN) term that, due to the perturbative nature of the post-Newtonian approximation, represents the dominant perturbation to the test particle dynamics over the classical Newtonian (zero-th order) contribution and, more specifically, we will report here the results for the case of a massive test particle orbiting a massive object. The spatial acceleration of such a test particle in the gravitational field of a massive point-like source with mass M can be thus written as [335],

$$\vec{a} = \vec{a}_N + \vec{a}_{1PN} + \vec{a}_{2PN} + \dots, \quad | \quad 2.79$$

where, the zero-th order Newtonian term $\vec{a}_N$ is the usual Newtonian acceleration

$$\vec{a}_N = -\frac{GM}{r^2} \hat{n}, \quad | \quad 2.80$$

and the 1PN term, proportional to ϵ can be written as

$$\vec{a}_{1PN} = \frac{GM}{c^2 r^2} \left[\left(\frac{4GM}{r} - v^2 \right) \hat{n} + 4(\vec{v} \cdot \hat{n}) \vec{v} \right] \sim \mathcal{O}(\epsilon). \quad | \quad 2.81$$

Here, the vector $\hat{n} \equiv \vec{r}/r$ represent the radial direction of the test-particle with respect to the central source and $\vec{v}$ is its

spatial velocity. Moreover, it is important to keep in mind that the radial coordinate r appearing in the previous expressions is not the standard Schwarzschild aerial radial coordinate appearing in Eq. (2.18), but the harmonic radial coordinate⁶ introduced in the standard derivation of the post-Newtonian formalism in General Relativity. At Newtonian order the two radial coordiantes are completely equivalent, hence the term in Eq. (2.80) is completely coordinate-independent, as one would expect from its non-relativistic nature. The 1PN term, on the other hand, is coordinate dependent and its expression differs if one uses the Schwarzschild radial coordinate instead of the harmonic one [358]. The 1PN term in Eq. (2.81) can be derived in different ways:

- First, one can consider the post-Newtonian expansion of the Schwarzschild metric using harmonic coordinates in Eq. (2.82), stop it at 1PN order and than compute the corresponding geodesic equations. These will automatically be 1PN equations of motion that can be easily recast in the form presented in Eq. (2.81) [335].
- Alternatively, one can compute non-approximated geodesic equations from the full metric in Eq. (2.82) and develop the resulting equations of motion in a post-Newtonian expansion, stopping at 1PN order. The two approaches are completely equivalent [358].
- Finally, one can start from the post-Newtonian treatment of the general-Relativistic two-body problem (which was one of the main goals of the development of the post-Newtonian formalism, for the description of compact binary mergers and pulsar binary systems) and consider the extreme-mass-ratio limit that applies to the case of a test particle around a SMBH [335, 340].

From Eq. (2.81) one can compute the 1PN orbit of a test particle using a perturbative approach that is very similar to the one that we have employed in Section 2.2.2. In particular, the additional acceleration term can be decomposed in a

⁶ The harmonic radial coordinate is defined as

$$r = r_S - M,$$

where r_S is the Schwarzschild radial coordiante. The Schwarzschild metric in this coordinate system thus becomes

$$ds^2 = -\left(\frac{r-M}{r+M}\right) dt^2 + \left(\frac{r+M}{r-M}\right) dr^2 + (r+M)^2 d\Omega^2. \quad | \quad 2.82$$

radial ($\propto \hat{n}$) term and a tangential ($\propto \vec{v}$) one that represents perturbative forces (due to the small ϵ modulation factor). These perturbations are responsible for a variation over time of the orbital elements [335, 359] that one can add on top of the zero-th order Keplerian orbit. The result is a precessing Keplerian ellipse whose recovered rate of orbital precession at 1PN order exactly matches the term in Eq. (2.41).

Similarly, the spectroscopic observable admits a first-order expansion of the form [152, 360, 361]

$$z + 1 = \frac{1 + \frac{1}{c} \hat{k} \cdot \vec{v}_{\text{em}}}{1 + \frac{1}{c} \hat{k} \cdot \vec{v}_{\text{obs}}} \frac{1 - \frac{U_{\text{obs}}}{c^2} - \frac{v_{\text{obs}}^2}{2c^2}}{1 - \frac{U_{\text{em}}}{c^2} - \frac{v_{\text{em}}^2}{2c^2}} + \mathcal{O}\left(\frac{1}{c^3}\right), \quad | \quad 2.83$$

where $\hat{k}$ represents the direction of the line-of-sight, $\vec{v}_{\text{em}}$ and $\vec{v}_{\text{obs}}$ the emitter's and observer's velocities, respectively, and U_{em} and U_{obs} the gravitational potential at the emitter's and observer's locations. The last expression was expanded up to order $\mathcal{O}\left(\frac{1}{c^3}\right)$ in order to obtain up to 1PN terms in the reconstruction of the line-of-sight velocity

$$\begin{aligned} v_{\text{LOS}} \equiv cz \simeq & \hat{k}(v_{\text{em}} - v_{\text{obs}}) + \\ & + \frac{1}{c} \left[(\vec{k} \cdot \vec{v}_{\text{obs}})^2 - (\hat{k} \cdot \vec{v}_{\text{obs}})(\hat{k} \cdot \vec{v}_{\text{em}}) \right] + \\ & + \frac{1}{c^2} \left[U_{\text{em}} - U_{\text{obs}} + \frac{v_{\text{em}}^2}{2} - \frac{v_{\text{obs}}^2}{2} \right] + \\ & + \mathcal{O}\left(\frac{1}{c^2}\right). \end{aligned}$$

The first line in the previous equation is the classical kinematical term related to the longitudinal Doppler shift due to relative velocity between emitter and observer. All other terms are relativistic corrections (as it is clear from the pre-factors in the form of powers of $1/c$). In particular, the terms on the second line are a second-order longitudinal Doppler shift for the observer and a second-order cross-term and they give only marginal contribution to the reconstructed spectroscopic shift (for the S-stars in the Galactic Center they can both be neglected at the current level of sensitivity, as their contribution is below 0.1 km/s [152]). The terms on the third line, on the other hand, contain the relevant relativistic contribution to the reconstructed line-of-sight velocity and are related to the relativistic time dilation (both the general relativistic gravitational time dilation - in the terms containing the gravitational potential, and the special relativistic Lorentz

time dialtion - terms proportional to the Lorentz factor $\beta \equiv v/c$). As a matter of fact, while we can safely neglect both the gravitational time dilation term (U_{obs}/c^2) and the Lorentz time dilation term ($v_{\text{obs}}^2/2c^2$) for an Earth-based observer (as both contribution are well below the current sensitivity limity for spectroscopic observation at the Galactic Center [152]), the corresponding terms on the emitter side give non-negligible contributions. Their effect on the reconstructed line-of-sight velocity amounts to the additional ~ 200 km/s already shown in Figure 2.10 and derived directly from the geodesic integration.

An expression, very similar to Eq. (2.81), exists for the 1PN equations of motion of massless particles [358, 359] From this one can compute approximate photon paths using a 1PN correction for the straight-line classical light ray propagation and also compute the photon propagation time (whose computation we will develop in a fully relativistic way in Section 2.2.5). At 1PN order, one can express the photon travel time from the emitter's position $\vec{r}_e$ to the observer's position $\vec{r}_o$ as a linear sum of different effects [356, 359]:

$$\Delta t_{\text{PN}} = \Delta t_{\text{R}} + \Delta t_{\text{Sh}} + \Delta t_{\text{geo}}. \quad | \quad 2.84$$

Here, Δt_{R} represents the classical Rømer delay related to the photon propagation time across the emitting particle's orbit

$$\Delta t_{\text{R}} = \frac{|\vec{r}_o - \vec{r}_e|}{c}, \quad | \quad 2.85$$

and Δt_{Sh} is the Shapiro time delay related to the time dilation experienced by light rays when grazing the region where the central object curves spacetime substantially, and is given by⁷ [301, 362]:

$$\Delta t_{\text{Sh}} = -\frac{2GM}{c^3} \ln \left(\frac{2|\vec{r}_o|}{|\vec{r}_e| + \vec{r}_e \cdot \hat{k}} \right). \quad | \quad 2.86$$

The last term, dubbed geometrical time delay, is a correction term that takes into account the curved photon path. It is given by [301, 363]

$$\Delta t_{\text{geo}} = \frac{2GM}{c^3} \left(\frac{|\vec{r}_{\pm} - \vec{r}_s|}{R_E} \right)^2. \quad | \quad 2.87$$

⁷ It is worth noticing that both formulas for the Rømer and the Shapiro delay are computed by assuming that the photons propagate on a straight line.

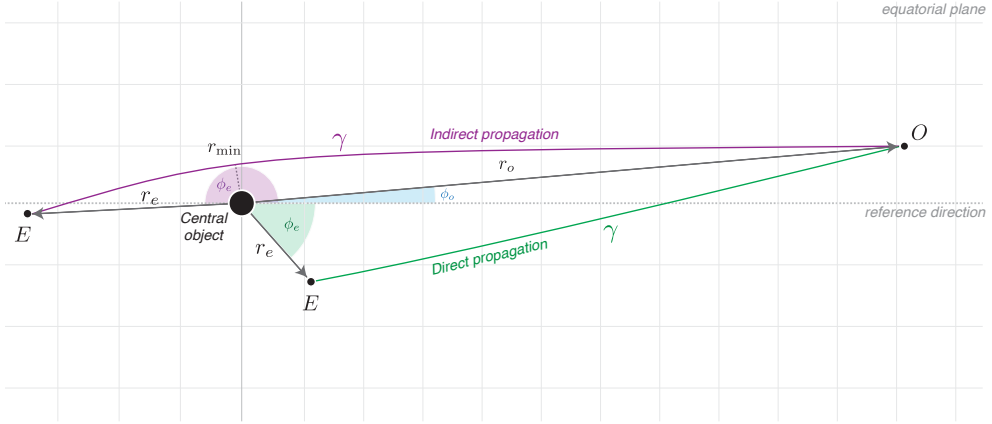


Figure 2.14. Configuration of the emitter-observer problem. The emitter is located at a point E , identified by polar coordinates (r_e, ϕ_e) , while the observer is located at point O , with coordinates (r_o, ϕ_o) . We consider only primary photons received by the observer, those that do not circle the central black hole, for which we only have two possible scenarios: the radial coordinate increases monotonically going from r_e to r_o , a *direct propagation* ($r_e \rightarrow r_o$) from the emitter to the observer, (*green path*); the photon starts with a decreasing radial coordinate (*i.e.* a negative radial velocity) then reaches a minimum distance $r_{\min}$ to the central object and then starts increasing again up to the observer position. We call the latter configuration *indirect propagation* ($r_e \rightarrow r_{\min} \rightarrow r_o$) (*purple path*).

where we have defined the gravitational-lensing Einstein radius

$$R_E = \left(\frac{4GM}{c^2} (\vec{r}_o - \vec{r}_e) \cdot \hat{k} \right) \quad | \quad 2.88$$

and the vector $\vec{r}_s = \vec{r}_e - (\vec{r}_e \cdot \hat{k})\hat{k}$ represents the sky-plane projection of the emitter position (*i.e.* the apparent position that the test particle would have in the observer reference frame in the case where no light bending is present), while

$$\vec{r}_{\pm} = \frac{\vec{r}}{2} \left(1 \pm \sqrt{1 + \frac{4R_E^2}{|\vec{r}_s|^2}} \right). \quad | \quad 2.89$$

is the actual sky-plane lensed position of the test particle as seen by the distant observer. All these expressions will be useful in the next section to quantify the error committed when using a post-Newtonian approximation as compared to the fully-relativistic treatment.

2.2.5. Light-rays propagation

As explained in details in Section 2.2.2, in order to take into account all the relativistic effects experienced by photons when propagating in a curved spacetime, which would allow

to reconstruct fully relativistic observables for test particle orbits, photon paths represented by null geodesics must be integrated after an appropriate initial-condition-search has been carried out to identify the right null trajectory connecting emitter and observer at any epoch. While such an approach gives corrections that lie much below the current instrumental sensitivity for the S-stars in the Galactic Center (allowing us to retain only leading-order photon effects as explained in the previous section), it is of paramount importance to consider all the effects when dealing with the problem of pulsar timing. We approach the problem by developing a numerical methodology for the computation of the photon propagation time in a generic asymptotically-flat spherically-symmetric spacetime described by the line element in Eq. (2.12). As in Section 2.2.3 we can restrict the photon path to the equatorial plane (which may not coincide with the plane on which the orbit of the emitting object lies). One can then define the conserved specific energy, $E = -\partial\mathcal{L}/\partial t$, and angular momentum, $L = \partial\mathcal{L}/\partial\phi$, for a null geodesic, derived from the Lagrangian. For null geodesic, the Lagrangian is identically zero ($g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$). From the conserved quantities, one can define an impact parameter $b \equiv L/E$ which allows to rewrite the equations of motion for a null geodesic as the first-order set of equations:

$$\frac{dt}{ds} = \frac{E}{A(r)}, \quad | \quad 2.90$$

$$\frac{d\phi}{ds} = \frac{L}{r^2}, \quad | \quad 2.91$$

$$\frac{dr}{ds} = \pm \sqrt{\frac{E^2}{r^2} \Upsilon(r)}, \quad | \quad 2.92$$

where we have defined

$$\Upsilon(r) = \left(\frac{r^2 - b^2 A(r)}{A(r)B(r)} \right). \quad | \quad 2.93$$

The ratio between Eq. (2.91) and Eq. (2.90) gives an equation for the polar trajectory of the photon,

$$\frac{d\phi}{dr} = \frac{b}{r\sqrt{\Upsilon(r)}}. \quad | \quad 2.94$$

Similarly, the ratio between Eq. (2.92) and Eq. (2.90) yields a differential equation for the coordinate time, t , as a function

of the radial coordinate, r ,

$$\frac{dt}{dr} = \frac{r}{A(r)\sqrt{\Upsilon(r)}}. \quad | \quad 2.95$$

The latter equation is to be integrated in order to compute the travel time, and thus the propagation delay, for a photon γ originating at the emitter position and reaching the observer. The former, on the other hand, carries information on the geometry of the trajectory with respect to the emitter and observer and will be used to solve for b or, in other words, to solve the *emitter-observer* problem. In particular, given a photon emitted at coordinates $E(r_e, \phi_e)$ at coordinate time t_e , and an observer located at $O(r_o, \phi_o)$ receiving the photon at coordinate time t_o (see Figure 2.14), we are interested in computing the travel time, $\Delta t = t_o - t_e$. Due to the curved photon path, one has to determine the appropriate value for the impact parameter b for the photon γ connecting points E and O . To solve this problem, one needs to integrate Eq. (2.94) over the trajectory of γ and solve for b . This yields

$$\phi_o - \phi_e = \int_{r_e, \gamma}^{r_o} \frac{b}{r\sqrt{\Upsilon(r)}} dr. \quad | \quad 2.96$$

The dependence of the integral in Eq. (2.96) on the photon path γ implies that one has to take into account whether the propagation of γ from E to O is direct or indirect (see Figure 2.14). This, in turn, depends on the specific geometrical configuration of the emitter and the observer with respect to the central object, *i.e.* whether the radial component of the 4-velocity in Eq. (2.92) is positive or negative. Considering only primary photons received by the observer (*i.e.* not considering photons that graze so close to the unstable photon orbit of the central object that their paths bend so strongly, $\Delta\phi > \pi$, that they circle one or more times the central object) only two possible scenarios are possible:

- The radial coordinate increases monotonically going from r_e to r_o (we are assuming, as is usual, that the emitter is located closer to the central object than the observer), corresponding to a positive sign in Eq. (2.92), in which case we say that the propagation is *direct*, thus resulting in

$$\phi_o - \phi_e = \int_{r_e}^{r_o} \frac{b}{r\sqrt{\Upsilon(r)}} dr. \quad | \quad 2.97$$

- The radial coordinate decreases (minus sign in Eq. (2.92)) up to a certain $r_{\min}$ (for which the right-hand side of Eq. (2.92) goes to zero) and then increases monotonically up to r_o . In the latter case, the integral in Eq. (2.96) would read

$$\phi_o - \phi_e = \int_{r_e}^{r_{\min}} \frac{b}{r\sqrt{\Upsilon(r)}} dr + \int_{r_{\min}}^{r_o} \frac{b}{r\sqrt{\Upsilon(r)}} dr, \quad | \quad 2.98$$

and we refer to this condition as *indirect propagation*.

For the Schwarzschild metric, closed form solutions exist in terms of Jacobi elliptic integrals, as has been shown in [362]. However, even if one solves the integrals analytically, solving the resulting equation for b is not generally possible, and one has to resort to numerical methods to find the appropriate value of the impact parameter that solves the emitter-observer problem. Our aim is to develop an algorithm that is able to compute the propagation time delay in any spherically symmetric spacetime, and we cannot rely on the analytical solution of the integrals in Eq. (2.96) that is valid only for the Schwarzschild case. Moreover, in the Schwarzschild case, one can bring the right-hand side of Eq. (2.92) in the form of a third-degree polynomial [362, 364], thus reducing the problem of finding the radius $r_{\min}$ of closest approach for the indirect propagation to that of finding polynomial-roots. For a generic metric, however, the function for which one has to find the root $r_{\min}$ can be indefinitely complicated and may not be provided at all with an analytic solution.

For these reasons, we solve the emitter-observer problem numerically, building an algorithm that, given the coordinates of E and O , does not only give the impact parameter b of the photon connecting the two points but is also able to determine if it is a direct or indirect trajectory. In order to do this, first of all, we consider the following: the radial coordinate $r_{\min}$ does not only represent a root for the right-hand side of Eq. (2.92). Since the radial coordinate decreases monotonically from r_e to $r_{\min}$ and it increases monotonically from $r_{\min}$ to r_o (up to $r_o \rightarrow \infty$), it is also a local minimum for the radial coordinate. This guarantees that $r_{\min}$ is the largest among the roots of the right-hand side of Eq. (2.92) and the inequality $r_{\min} \leq b < r_e < r_o$ will always hold. For a given value of b , we apply a root-finding algorithm, using the non-linear equation solver Minpack implemented in [365], using as an initial guess for $r_{\min}$ the value of b itself, thus guaranteeing

that the algorithm will converge to the root of the function that is the closest to b , and which thus corresponds to the value of $r_{\min}$ that we are seeking. For values of b below a critical impact parameter b_{crit} (which one usually relates to the apparent size of a black hole shadow [364]), the right-hand side of Eq. (2.92) does not have any real root in the range $r > 0$. We check for this condition, and if the algorithm converges to negative roots, we stop looking for indirect photons, as we are guaranteed that the propagation is direct.

Once the value of $r_{\min}$ is known, we can numerically compute the angular integrals in Eqs. (2.97) and (2.98), that, for generic radial coordinates r_1 and r_2 and a given value of b , read as

$$\Phi(r_1, r_2; b) = \int_{r_1}^{r_2} \frac{b}{r \sqrt{\Upsilon(r)}} dr, \quad | \quad 2.99$$

by employing the Quadpack routine implemented in [366]. We carry out this integration for both the direct and indirect photons (provided that the latter exists, *i.e.* that the specific value of b is above its critical value) and compute

$$\Delta\phi_{\text{direct}}(r_e, r_o; b) = \Phi(r_e, r_o; b), \quad | \quad 2.100$$

$$\Delta\phi_{\text{indirect}}(r_e, r_o; b) = \Phi(r_e, r_{\min}; b) + \Phi(r_{\min}, r_o; b). \quad | \quad 2.101$$

We thus have a routine that for fixed values of r_e and r_o and a given value of b provides us with two numerical estimations of the angular distance between emitter and observer. We can apply again the Minpack root-finding routine [365] to look for the value of b that solves the equation

$$\Delta\phi - (\phi_o - \phi_e) = 0. \quad | \quad 2.102$$

At each iteration, $\Delta\phi$ is the value between $\Delta\phi_{\text{direct}}$ and $\Delta\phi_{\text{indirect}}$ that is closer to the target solution $(\phi_o - \phi_e)$. Here, we use as an initial guess $b_0 = r_e \sin(\phi_o - \phi_e)$, which corresponds to the impact parameter in a flat spacetime. The choice between $\Delta\phi_{\text{direct}}$ and $\Delta\phi_{\text{indirect}}$ in the last iteration of the root-solver will determine the direct/indirect nature of the resulting photon path. In Figure 2.15 we show an example of the use of our routine for solving the emitter-observer problem in a Schwarzschild spacetime with mass M . For different values of the emission radius r_e , we fix the position of the observer at a very distant location on the equatorial plane ($r_o = 10^9 M$) and we vary the angle between emitter and observer between

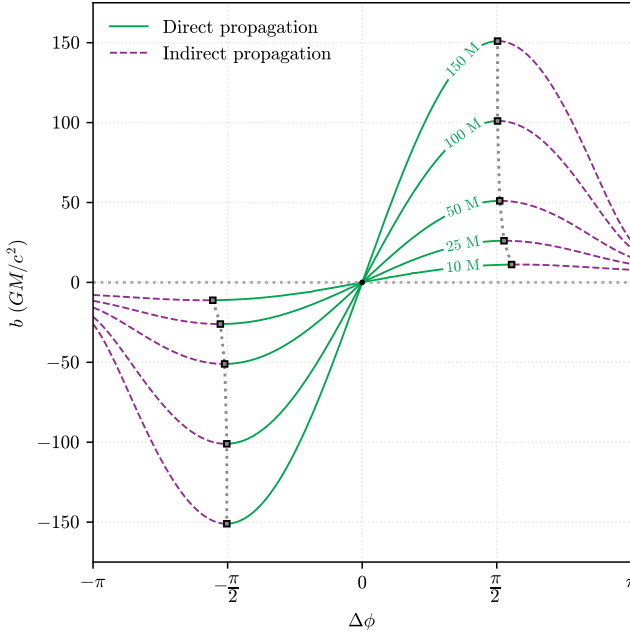


Figure 2.15. Solution of the emitter-observer problem with the root-solver in the Schwarzschild spacetime. For an observer located at $r_o = 10^9 M$, we have considered radial coordinates r_e between $10 M$ and $150 M$ (reported in green on the corresponding lines) and for values of $(\phi_o - \phi_e)$ in the range $[0, 2\pi]$. Green solid lines correspond to photons that propagate directly from E to O , and violet dashed lines to photons on indirect trajectories. Gray squares highlight the separation between direct and indirect propagation. These points tend towards $\Delta\phi = \pi/2$ as the distance of the emitter from the central object increases (meaning that for a sufficiently far emitter, the entire semi-plane containing the observer corresponds to a direct propagation) and deviates from $\pi/2$ as the emitter is closer to the center, due to the strong bending of photon paths.

$-\pi$ and π . We compute the impact parameter for the primary photon as a function of $\Delta\phi$ and we determine the direct/indirect character of the photon. Interestingly, the angular distance corresponding to the separation between direct and indirect propagation (gray dotted line) tends towards $\Delta\phi = \pi/2$ as the distance of the emitter from the central object increases: for a distant emitter, photons propagate directly to the observer when starting in the entire semi-plane containing the observer itself as in the classical case. For an emitter that is closer to the central object, this separation deviates from $\pi/2$ due to the strong lensing effects that bend photon paths.

Once the emitter-observer problem has been solved, we have obtained the impact parameter b of the photon corresponding to the primary image of the source for the observer O . We can now approach the problem of determining the travel time of the photon from E to O . The assumption of asymptotic flatness of the spacetime in Eq. (2.12) is crucial for two reasons. First, it allows us to assume that for a sufficiently far observer the curvature of spacetime produced by the compact central object can be set to zero at the observer location. Of course, in the case of the Earth, one should take into account the presence of the Sun's gravitational field and the motion of Earth around it. However, this contribution can be added later using classical

formulas (e.g. from [359]) since we can consider a weak field approximation for the Sun. As a result, the travel time is simply the integral of Eq. (2.95) over the photon's path γ :

$$\Delta t \equiv t_o - t_e = \int_{r_e, \gamma}^{r_o} \frac{r}{A(r) \sqrt{\Upsilon(r)}} dr, \quad | \quad 2.103$$

Like for the integrals of the angular coordinate, the information on the photon path γ is encoded in the impact parameter b . Considering the solution of the integral between two generic radial coordinates,

$$T(r_1, r_2) = \int_{r_1}^{r_2} \frac{r}{A(r) \sqrt{\Upsilon(r)}} dr, \quad | \quad 2.104$$

we can express the two cases by

$$\Delta t_{\text{direct}} = T(r_e, r_o), \quad | \quad 2.105$$

$$\Delta t_{\text{indirect}} = T(r_e, r_{\min}) + T(r_{\min}, r_o). \quad | \quad 2.106$$

As for the case of the emitter-observer problem, an analytic solution in terms of Jacobi elliptic function exists for the light propagation time in the Schwarzschild metric. In a generic case, one has to solve the problem numerically. However, the integration of Eq. (2.104) presents additional challenges from a computational point of view. The integrand in Eq. (2.99) tends to 0 when $r \rightarrow \infty$, thus the numerical quadrature method implemented in Quadpack is always able to converge within the desired precision tolerance. The integrand in Eq. (2.104), on the other hand, generally tends to a non zero constant value as $r \rightarrow \infty$. From a physical point of view, this is given by the fact that since this integral returns the travel time for a photon, as r_o grows, the travel time has to grow accordingly due to the additional time required to cover the extra distance. This condition, however, can affect the behaviour of the Quadpack routine leading to round-off errors, especially when the range of integration is particularly large (as is the case for a *very distant* observer). Again, the asymptotic-flatness assumption can help solve this issue. Regardless of the specific functional form of $A(r)$ and $B(r)$, in an asymptotically-flat spacetime both of these functions approach 1 as $r \rightarrow \infty$ [243], and subsequently the integrand itself of Eq. (2.104) tends to 1. We

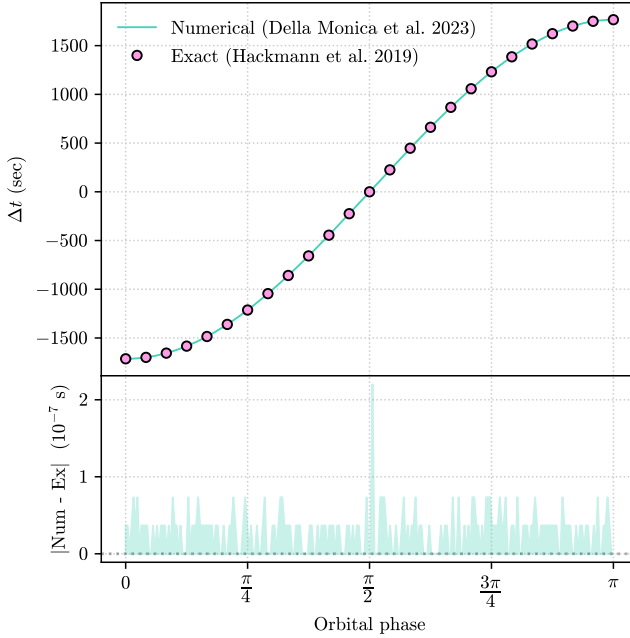


Figure 2.16. *Top panel:* photon propagation time as a function of the orbital phase for a circular orbit of radius $r = 100M$ on the equatorial plane of the Schwarzschild spacetime computed both numerically, using the routine illustrated in this section and analytically, using the exact formula computed in [362]. The two results, that have both been shifted so that the orbital phase $\pi/2$ is taken as a zero-travel-time reference point, perfectly agree with each other. *Bottom panel:* difference in units of 10^{-7} s between our numerical estimation of the photon propagation time and that obtained by applying exact formula by [362]. The results are perfectly compatible within the numerical tolerance of 10^{-6} that has been chosen in this calculation.

can thus express the integral in the following way

$$T(r_1, r_2) = \int_{r_1}^{r_2} \left(\frac{r}{A(r)\sqrt{\Upsilon(r)}} - 1 \right) dr + (r_2 - r_1), \quad | \quad 2.107$$

and, taking advantage that the integrand now tends to 0 for $r \rightarrow \infty$, we can integrate the first term numerically within any desired precision tolerance. Finally, particularizing the calculations for the Schwarzschild spacetime in Eq. (2.18), the photon travel time can be written as the integral

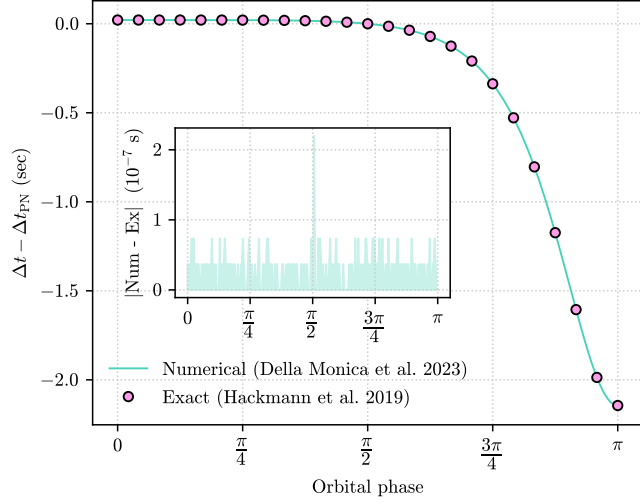
$$\Delta t = \int_{\gamma} \frac{r^2 dr}{b \left(1 - \frac{2M}{r} \right) \sqrt{R(r)}}, \quad | \quad 2.108$$

depending on the photon path γ connecting emitter and observer, on the corresponding impact parameter b and on the fourth-degree polynomial function $R(r) = r^4/b^2 - r^2 + 2Mr$. The integral in Eq. (2.108) has an exact solution given by

$$\Delta t_{\text{ex}} = \frac{GM}{c^3} (T_{\text{ex}}(r_o; b) \pm T_{\text{ex}}(r_e; b)). \quad | \quad 2.109$$

The sign $\pm$ in Eq. (2.109) depends on whether the integration path is direct or indirect and the functions $T_{\text{ex}}(r; b)$ result from

Figure 2.17. Difference in seconds between the fully relativistic delay and the post-Newtonian approximation for both the exact formula (pink dots) and our numerical routine (aqua solid line) as a function of the orbital phase for the same model used in Figure 2.16. The 1PN formula in Eq. (2.84) fails in reproducing the correct result for the photon propagation time by several seconds, for the model that we have considered. The inset reports the residuals between the analytical estimation and the numerical one, as in Figure 2.16.



the integration of Eq. (2.108), given analytically by [362]:

$$T_{\text{ex}}(r; b) = \frac{2}{\sqrt{r_4(r_3 - r_1)}}(T_1 + T_2 + T_3 + T_4) + T_{\infty}, \quad 2.110$$

where we have defined the terms

$$T_1 = \left(\frac{r_3^3}{r - 2} + \frac{1}{2}(r_4 - r_3)(r_3 - r_1 + 4) \right) F(x, k), \quad 2.111$$

$$T_2 = -\frac{1}{2}r_4(r_3 - r_1)E(x, k), \quad 2.112$$

$$T_3 = -2(r_4 - r_3)\Pi\left(x, \frac{k^2}{c_1}, k\right), \quad 2.113$$

$$T_4 = -\frac{8(r_4 - r_3)}{(r_4 - 2)(r_3 - 2)}\Pi(x, c_2, k), \quad 2.114$$

and T_{∞} encodes all the diverging (for $r \rightarrow \infty$) terms as follows

$$T_{\infty} = \frac{b\sqrt{R(r)}}{r - r_3} + 2 \ln \left(\frac{\sqrt{r(r - r_1)} + \sqrt{(r - r_4)(r - r_3)}}{\sqrt{r(r - r_1)} - \sqrt{(r - r_4)(r - r_3)}} \right). \quad 2.115$$

All quantities that appear are expressed in geometrized units; the functions F , E and Π are Jacobian elliptic integrals [367], the quantities r_1 , r_2 , r_3 , r_4 are the roots of the polynomial $R(r)$ of Eq. (2.108), x is an auxiliary variable and k , c_1 and c_2 are all constant terms constructed from the roots r_1 , r_2 , r_3 , r_4 . In Figure 2.16 we show the perfect agreement

between our numerical prediction and the analytical result for the Schwarzschild spacetime for a non-inclined circular orbit of radius $100M$ and an observer at a distance of $10^9 M$. The numerical methods developed in this section correctly reproduce the prediction from the analytical formula in Eq. (2.109) within our numerical tolerance used (of 10^{-6} s)⁸. The timing formula in Eq. (2.109) has been shown to provide a better description of the photon propagation time than the usual post-Newtonian approximations, being able to capture strong field features that the first-order post-Newtonian in Eq. (2.84) effect fail to recover [362]. The difference could be of the order of seconds, as shown in Figure 2.17.

From Eq. (2.109) it is clear that the specific expression derived in [362] is only valid under the assumption of a Schwarzschild spacetime geometry. For models that differ from the Schwarzschild metric, such an expression should be modified accordingly. The methods developed in this section, on the other hand, are completely general and allow a fully relativistic computation of photon paths and propagation times for any spherically symmetric spacetime and any emitter-observer configuration.

2.3. Conclusions

In this chapter we have reviewed the basic concepts of General Relativity, focusing on the geodesic motion of test particles (either massive objects or photons) in spherically symmetric spacetimes. We have introduced notation, equations and a methodology that will be used in the rest of this thesis. We will exploit the full generality of the approach and our open-source code, PyGR0, to study the motion of the S-stars in the Galactic Center in several spherical symmetric spacetimes that arise either from the introduction of an extended dark matter mass component in the Galactic Center (Chapter 3), from considering black hole mimicker spacetimes as alternative to the standard Schwarzschild scenario for Sgr A* (Chapter 4) and considering modifications to the underlying theory of gravity (Chapter 5). Finally, we will make use of the methodology developed in Section 2.2.5 to model the TOAs of pulsar orbiting Sgr A* (Chapter 6).

⁸ We have also tried to reduce the numerical tolerance which reflected in an accordingly lower numerical error in the predicted photon propagation time.

Testing Dark Matter Models

3.

Since the late 1990s, when the first measurements of the proper motions of the S-stars orbiting Sgr A* in the Galactic Center became available (see Section 1.1.2), it was evident that tracking the orbits of such stars could not only probe the black hole paradigm but also constrain possible extended mass distributions surrounding Sgr A* [368], thus serving as a potential tool to study the distribution of dark matter in the Galactic Center [227]. Apart from the general relativistic contribution, the in-plane orbital precession of stars can carry signatures of the presence of an extended gravitational source. A test particle traversing through a continuous mass distribution, assumed to be unobservable otherwise, experiences an additional acceleration imposed by the extended mass component. This extra component could result in an additional variation of the orbital elements, thus including the pericenter advance, even in the Newtonian limit [229, 369]. Such a distribution may include faint stars, stellar remnants, and/or dark matter [242]. Termed *mass precession*, the effect depends on the specific mass profile and the enclosed mass and can be comparable or even dominate the general relativistic precession reported in Eq. (2.41) [229, 369]. Furthermore, for particularly steep density profiles, the effect could yield a retrograde precession (*i.e.* opposite to the sense of orbital revolution of the test particle), while the Schwarzschild precession is always prograde [368] (see Section 2.2.2), making the precession measurements a great observational probe of such models.

The accuracy achieved with current S-star data enables to set upper bounds on the mass within the apocenter of S2. A recent study by the Gravity Collaboration [368] with data of the S2 star up to 2022, has placed an upper limit on the mass of the extended component of $M \lesssim 3 \times 10^3 M_\odot$, corresponding to approximately $\sim 0.1\%$ of the black hole mass. This represents a tenfold improvement from prior studies by the same group [138] (conducted before they had access to the precise astrometry of the GRAVITY interferometer at and after the 2018 pericenter passage), which yielded much weaker constraints on the extended mass, at around $\sim 1\%$ of the mass of the central SMBH. As astrometry becomes

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increasingly precise, it becomes crucial to unambiguously differentiate signatures on the orbits due to the extended mass component from other sources of precession [242]. On secular timescales, this concern is particularly relevant for the relativistic effects of Schwarzschild precession and Lense-Thirring frame-dragging. It is expected that future GRAVITY data will achieve a rms precision of $\sim 50 \mu\text{as}$ collected over an entire orbital period (expected around ~ 2033), and will be able to place a 1σ upper bound on the extended mass component as low as $\sim 1 \times 10^3 M_\odot$.

In this chapter, we explore the possibility that a specific model of dark matter, namely Fuzzy Dark Matter (FDM), is present in the Galactic Center in the form of an extended dark soliton. We test the impact that this model would have on the orbit of the S2 star and the constraints that this object can impose on the mass m_ψ of the FDM particle. Briefly, in Sections 3.1 and 3.2 we describe the ULDM and the FDM models, respectively, the hypotheses and motivations on which they are formulated and the main differences that they have over the Cold Dark Matter (CDM) model. In Section 3.3 we describe the results of a sensitivity analysis, performed at 1PN order to assess the orbital deviations due to the presence of a FDM soliton in the Galactic Center, forecasting the future limit that GRAVITY observations of the S2 star will be able to impose on the FDM particle mass. Finally, in Section 3.4 we derive a constraint on the mass of the FDM particle by applying our geodesic methodology to currently available data, comparing it with other astrophysical and cosmological probes.

3.1. Ultralight Dark Matter

Many aspects of the large-scale structure of the Universe can be described successfully using cosmological models in which $\sim 27\%$ of the mass-energy density consists of CDM [370, 371], composed of non-relativistic particles, that weakly interact with ordinary matter solely through gravitational forces and that, having thus far eluded direct detection, belong to beyond-Standard-Model theoretical scenarios. Despite the success on cosmological scales, predictions of CDM models on scales of $\sim 10 \text{ kpc}$ or less have consistently failed in reproducing the observed properties of low-mass systems, such as the abundance and internal structure of dwarf galaxies, in the local Universe [372–374]. While a possible

explanation for this failure might reside in the difficulty in modeling baryonic physics (star formation, supernova and black-hole feedback, etc.) and tidal forces in these systems [375, 376], an intriguing alternative to CDM is that dark matter is composed of ultralight bosons having astrophysical-sized de Broglie wavelengths [377–379]. It is straightforward to show [379] that, given the bounds on the dark matter density in the Solar System neighborhood [380–382], for $m_\psi \ll 30$ eV the occupancy number of particles in a de Broglie volume of dark matter is so large that the set of particles is best described by classical waves. For this reason, dark matter in this mass range is usually referred to as Wave Dark Matter (commonly abbreviated as ψ DM). Moreover, since the Pauli exclusion principle would preclude the massive occupation of the quantum states required to reach the inferred density of the local dark matter halo [380–382], such an ultralight dark matter particle has to necessarily be bosonic. More specifically, we focus on zero-spin (*i.e.* scalar) particles [383], although a very similar description applies to vector and tensor bosons [384, 385].

From a more fundamental perspective, several candidates for such ultralight scalar bosons exist. First introduced by Peccei and Quinn in 1977 [386] to resolve the strong CP problem, QCD axions are promising candidates in the ultralight dark matter sector. Moreover, in high-energy theories such as string theory, a collection of scalar axion-like particles is predicted, whose broad mass spectrum (that can be as small as 10^{-33} eV) is referred to as the ‘Axiverse’ [387, 388]. These models are interesting because they do not only provide a theoretical foundation for the existence and production mechanism of ultralight particles that are inherently weakly interacting and cold but also because such particles emerge naturally from considerations that are independent of the dark matter problem [378]. Moreover, they naturally predict relic densities of dark matter that match the observed abundance.

As we have anticipated, the dynamics of dark matter composed of ultralight bosons is described in terms of a scalar field, ϕ , minimally coupled to the space-time metric, $g_{\mu\nu}$, for which one neglects self-interaction and coupling to ordinary matter [378]. In the general case, one considers a complex scalar field whose dynamics is regulated by the action

$$S_\phi = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{2} m_\psi \phi^* \phi \right), \quad | \quad 3.1$$

The occupancy number can be computed as

$$N \sim \left(\frac{34 \text{ eV}}{m_\psi} \right)^4 \left(\frac{250 \text{ km/s}}{v} \right)^3,$$

being m_ψ the mass of the dark matter particle and v the velocity dispersion of the galactic halo.

whereas the gravitational sector is regulated by the standard general-relativistic Hilbert-Einstein action, $\mathcal{S}_{\text{GR}}$, given in Eq. (2.6), so that the full system (gravity + scalar field) is governed by the total action $\mathcal{S} = \mathcal{S}_{\text{GR}} + \mathcal{S}_\phi$. Varying this action with respect to $g_{\mu\nu}$ and ϕ one obtains differential equations that govern the fully relativistic non-linear evolution of the system,

$$G_{\mu\nu} = 8\pi \left(\partial_{(\mu} \phi^* \partial_{\nu)} \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi^* \partial_\beta \phi + m_\psi^2 \phi^* \phi \right) \right), \quad | \quad 3.2$$

$$(\square - m_\psi^2) \phi = 0. \quad | \quad 3.3$$

The first equation, corresponding to Eq. (2.9) with a right-hand-side source term due to the energy-momentum tensor of the scalar field, regulates the dynamics of the spacetime metric. The second one, namely the Klein-Gordon equation, governs the dynamics of the massive scalar field and contains its coupling to the gravitational field inside the d'Alembertian operator $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$. Of course, in the case $\phi = 0$ one recovers the standard general relativistic field equations, from which the Schwarzschild and Kerr black hole solutions are derived. However, non-trivial solutions to Eqs. (3.1)-(3.3) have been the subject of intense investigation over the past years. The first interesting results can be derived by an analysis of linearized scalar field perturbations around a Kerr black hole. In Boyler-Lindquist coordinates, one can use the Teukolsky formalism to separate the variables in the Klein-Gordon equation (3.3). When one imposes boundary conditions for the scalar field to be vanishing at spatial infinity and a purely ingoing wave (in a co-rotating frame) at the black hole's horizon, one finds quasi-bound states, which have complex eigenfrequency ω . Their real part is characterized by three wave numbers (n, ℓ, m) that form a discrete spectrum very similar to the energy spectrum of a hydrogen atom [389]. The imaginary part, on the other hand, is generally negative, signaling a field that is decaying, by infalling into the central black hole. There is a critical frequency, however, $\omega_c = m\Omega_H$ (being m the angular momentum quantum number of the given mode), where Ω_H represents the horizon angular velocity, for which the imaginary part of the eigenfrequency goes to zero. This represents the separation between the decaying and the *superradiant* regime [390–396], corresponding to $\omega < \omega_c$. In the latter case, the imaginary part of ω becomes positive, indicating instability of the Kerr black hole in the presence of a massive scalar field. A superradiant mode extracts energy

The superradiant regime occurs when the Compton wavelength of the scalar field is comparable to the gravitational radius of the central black hole.

and angular momentum from the black hole, through the Penrose process [397]. The exponential growth over time of the scalar field implies that the scalar may not necessarily be dark matter. Even a small quantum fluctuation could serve as the initial trigger to develop a massive occupation of the scalar field bound states around the black hole. The resulting structure is a scalar cloud [395, 398, 399], usually referred to as a *gravitational atom* [389], due to the resemblance of the scalar eigenstates to those of a hydrogen atom. Throughout this process, the black hole undergoes a decrease in mass and angular momentum¹, with a significant portion of these losses occurring when the cloud attains considerable size. Simulations suggest that through this process up to $\sim 10\%$ of the black hole's energy can be transferred to the bosonic field [400, 401]. Eventually, there comes a point where the black hole's mass and spin reach values such that the concerned mode is no longer unstable and the superradiant process stops (the formed scalar cloud eventually dissipates through the emission of continuous gravitational waves [400, 402, 403]). In recent years, numerous studies have investigated the detectability of such scalar clouds, especially through their distinctive features on the gravitational wave signal by binary systems embedded within a scalar cloud [404–408], their signatures on test particles orbits [409, 410] and other related effects [411–414]. Furthermore, in 2019, the Gravity Collaboration investigated the orbital impact of a scalar field superradiant structure near Sgr A* on the S2 star [219]. The scalar field cloud, with axial symmetry to the black hole's spin axis, showed effects dependent on the field mass. Numerical simulations indicated significant changes in the S2 star's orbit as it crossed the scalar field density peak. The impact on the argument of the pericenter, ranging between -1.37 and -17.19 arcmin per orbital period, competed with relativistic precession. Moreover, effects on eccentricity, orbital inclination, and ascending node longitude have been predicted, due to the scalar field's asymmetry.

When considering the fully non-linear non-perturbative setup of Eqs. (3.2)–(3.3) a family of solutions describing asymp-

The extraction of rotational energy is a requisite for the generation of stable quasi-bound configuration of the scalar field. For this reason, this kind of structure can only form around rotating black holes.

¹ This happens because, despite the linearity of the Klein-Gordon equation in ϕ , the evolution of the composite black hole-scalar cloud system is nonlinear. As the black hole undergoes changes in mass and spin due to accretion/extraction, the background geometry for the Klein-Gordon equation experiences modifications, influencing the scalar field's evolution. This feedback loop yields non-negligible effects, even though, at any specific moment in time, the geometry is predominantly governed by the black hole rather than the cloud.

totically flat, spinning black holes with scalar hair and a regular horizon can be found [415, 416]. Supported by rotation and without a static limit, such hairy black holes carry a conserved, continuous Noether charge that measures the scalar hair. Moreover, these solutions branch off from the Kerr metric within the superradiant regime and when their horizon area is set to zero, they reduce to spinning boson stars [417, 418], exotic compact objects formed by the gravitational collapse of the boson cloud itself.

3.2. Fuzzy Dark Matter

A concrete realization of ψ DM particles, in the mass range $m_\psi \sim 10^{-21} - 10^{-22}$ eV, is usually termed, for reasons that will briefly be clear, FDM. Its putative existence has been demonstrated to provide a natural mechanism to solve small-scale shortcomings of usual CDM models [370, 377–379, 419], while retaining the same large-scale behavior.

In the context of FDM, one considers a real scalar field ϕ of mass m_ψ described by the action in Eq. (3.1) in the non-relativistic regime, relevant for structure formation on galactic scales, in which the particles fall due to the inherent high density. Differently from the case of superradiantly generated boson clouds, or the scalar hair black hole case, we will consider a self-gravitating system composed of a large collection of bosonic particles, sharing the same quantum state, that form a Bose-Einstein condensate on galactic scales that we thus assume to compose the FDM halo. Under these assumptions, it is convenient to express ϕ as

$$\phi(t, \vec{r}) = \frac{1}{\sqrt{2m_\psi}} [\psi(t, \vec{r}) e^{-im_\psi t} + \psi^*(t, \vec{r}) e^{im_\psi t}], \quad | \quad 3.4$$

where the fast time dependence of ϕ over the scalar field Compton frequency has been factored out, and we have introduced the complex function ψ , corresponding to the wavefunction shared by the system of particles. In the non-relativistic regime, the field equations in Eq. (3.2) reduce to those of purely Newtonian gravity, while the Klein-Gordon equation in Eq. (3.3) reduces to the Gross-Pitaevskii equation [420, 421],

resulting in

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m_\psi} \nabla^2 \psi + m_\psi \Phi \psi, \quad | \quad 3.5$$

$$\nabla^2 \Phi = 4\pi G \rho, \quad | \quad 3.6$$

known as the Schrödinger-Poisson equations. In these equations, the function $\Phi = \Phi(\vec{r}, t)$ represents the gravitational potential generated by the density field associated to the wave-function, $\rho = m_\psi |\psi|^2$. Eqs. (3.5)-(3.6) govern the dynamics of a self-gravitating halo composed of FDM, sustained by an additional internal quantum pressure arising from Heisenberg's uncertainty principle on the de Broglie wavelength scale, thus precluding system formation below this scale [378]. Extensive numerical simulations of FDM halos, regulated by Eqs. (3.5)-(3.6), have thoroughly examined the properties of stable configurations within the system [422, 423]. Importantly, such studies predict the generation of coherent standing waves of dark matter in the centers of gravitationally bound systems. These distinctive dark matter solitonic cores are characterized by surrounding wave interference patterns (hence the term “fuzzy”), with their azimuthal average following a Navarro-Frenk-White (NFW) profile. Additionally, the radial profile of the solitonic core is well approximated by [422, 424]

$$\rho_s(r) = \rho_0 \left[1 + \alpha \left(\frac{r}{r_c} \right)^2 \right]^{-8}. \quad | \quad 3.7$$

Here, $\alpha = \sqrt[8]{2} - 1 \approx 0.091$, ρ_0 corresponds to the central density of the halo

$$\rho_0 = 0.019 \left(\frac{m_\psi}{10^{-22} \text{ eV}} \right)^{-2} \left(\frac{r_c}{\text{kpc}} \right)^{-4} \frac{M_\odot}{\text{pc}^3}, \quad | \quad 3.8$$

r_c is a radial-scale parameter, known as core radius, and corresponds to the radial coordinate at which the density has dropped to $\sim 50\%$ of the central value ρ_0 . The solitonic profile in Eq. (3.7) thus depends on the two parameters m_ψ and r_c . Nonetheless, cosmological simulations of FDM halos have highlighted a scaling relation that relates the core radius to the virial mass of the entire halo via the empirical formula [423]

$$r_c = 1.6 \left(\frac{m_\psi}{10^{-22} \text{ eV}} \right)^{-1} \left(\frac{M_{\text{halo}}}{10^9 M_\odot} \right)^{-1/3} \text{ kpc}. \quad | \quad 3.9$$

For all effective purposes, when taking into account this scaling relation, the solitonic profile in Eq. (3.7) only depends on the FDM mass m_ψ ², provided that the halo mass is known. For the Milky Way, estimates of the total dark matter halo mass have been derived from the Galaxy rotation curve from GAIA³ data [427], that we report in Table 3.1.

GAIA [427]	
$M_{\text{halo}} (M_\odot)$	$1.08^{+0.20}_{-0.14} \times 10^{12}$

Table 3.1. Mass of the Milky Way dark matter halo as inferred from GAIA DR2 rotation curves in [427].

In the left panel of Figure 3.1 we report solitonic profiles $\rho_s(r)$ for several values of the boson mass m_ψ around the range of interest for FDM, $m_\psi \sim 10^{-20}$ eV. The prominent centrally flattened solitonic core is more dense and compact for heavier particles and more diluted and extended for particles towards the lightest end of the considered mass range. When computing the enclosed mass at a given radius (right panel of the same figure),

$$\begin{aligned}
 M_{\text{enclosed}}(r) &\equiv \int_0^r 4\pi r'^2 \rho_s(r') dr' = \\
 &= \frac{\pi r_c^3 \rho_0}{53760\alpha \left(1 + \alpha \left(\frac{r}{r_c}\right)^2\right)^7} \left(\frac{r}{r_c}\right) \left(3465\alpha^6 \left(\frac{r}{r_c}\right)^{12} + 23100\alpha^5 \left(\frac{r}{r_c}\right)^{10} + \right. \\
 &\quad \left. + 65373\alpha^4 \left(\frac{r}{r_c}\right)^8 + 101376\alpha^3 \left(\frac{r}{r_c}\right)^6 + 92323\alpha^2 \left(\frac{r}{r_c}\right)^4 \right. \\
 &\quad \left. + 48580\alpha \left(\frac{r}{r_c}\right)^2 - 3465\right) + \frac{33\pi r_c^3 \rho_0}{512\alpha^{3/2}} \tan^{-1} \left(\sqrt{\alpha} \frac{r}{r_c}\right), \tag{3.10}
 \end{aligned}$$

we can see that the amount of FDM soliton mass enclosed in the S2 orbit (yellow-shaded vertical band) increases for heavier bosons, due to its increased compactness. For values of the mass $m_\psi \sim 10^{-17}$ eV the enclosed mass within the S2 apocenter is around 0.1% of the mass of Sgr A*, corresponding to the currently measured upper limit for the mass of an extended matter component in the Galactic Center

² The core-halo relation in Eq. (3.9) has been first observed in cosmological dark-matter-only simulations of FDM in [422]. The physical meaning of the scaling relation is that the size of the soliton matches the de Broglie wavelength of the halo. Moreover, the relation can be obtained from thermodynamical arguments [425]. The validity of the core-halo relation in Eq. (3.9) has been thoroughly investigated in [426], where dependence on the simulation setup has been reported. More specifically, a sizeable dispersion in the core-halo mass relation is found for higher halo masses, pointing towards heavier solitons, especially in non-cosmological, fully virialized settings [424]. Since a heavier soliton would alter the stellar dynamics more drastically, we take a conservative approach and assume Eq. (3.9) to be valid in our case.

³ https://www.esa.int/Science_Exploration/Space_Science/Gaia

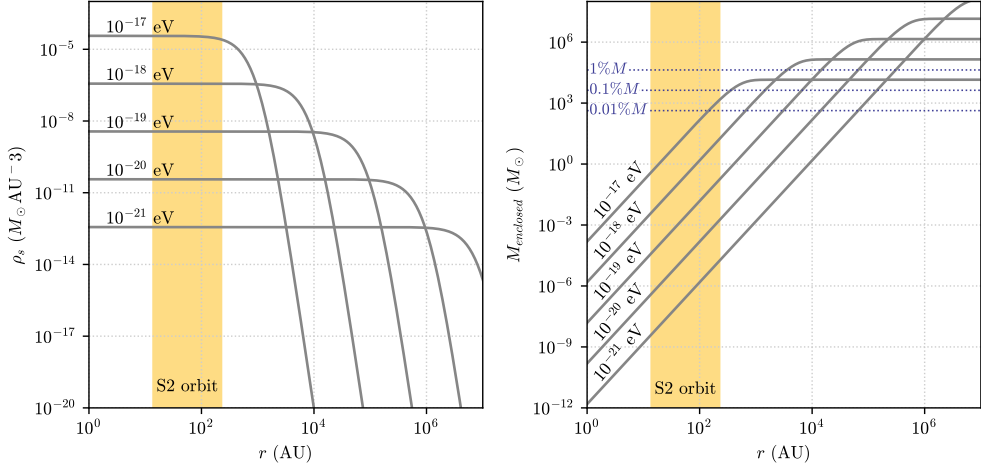


Figure 3.1. *Left panel:* centrally-flattened soliton density profiles, as reported in Eq. (3.7), for different values of the boson mass m_ψ in the range $m_\psi \in [10^{-17}, 10^{-21}]$ eV. *Right panel:* enclosed soliton mass at radius r for the same values of the boson mass from Eq. (3.10). The highlighted vertical stripe corresponds to the orbital separation of the S2 star over one orbital period. The horizontal dotted lines correspond to enclosed masses of the FDM distribution amounting to 0.01%, 0.1% and 1% of the central SMBH mass, respectively.

[368], thus hinting at the possibility of placing an upper bound on the soliton mass using S-stars observations⁴. In the next two sections, we will elaborate on this possibility, by developing an orbital model for the S2 star that takes into account the presence of a FDM soliton in the Galactic Center, as described by Eq. (3.7), along with Sgr A*. It is worth to remark here that the solitonic profile in Eq. (3.7) is derived from the Schrödinger-Poisson equations, Eqs. (3.5)-(3.6), that describe the self-gravitating dynamics of the FDM halo, without the presence of the central SMBH. In this regard, a reasonable question is whether this assumption holds and whether the gravitational interaction of a massive point-like source would drastically alter the final distribution of FDM in its surroundings. Several studies have investigated this possibility. In [429], for example, the impact of the presence of a central point-like source on the relation between the mass and radius of the core has been analyzed with analytical arguments, showing that, as a general result, the main

⁴ This possibility was first investigated in [428] using a keplerian + extended mass – based on the solitonic profile with a modified soliton mass (M_{sol}) core-radius relation due to the presence of Sgr A* – fit on VLT data from [138]. While no statistical evidence is found in favor of the presence of the soliton, an interesting exclusion region is obtained on the $M_{\text{sol}} - m_\psi$ plane, with the tightest constraint on the soliton mass obtained for 2×10^{-20} eV $\lesssim m_\psi \lesssim 8 \times 10^{-19}$ eV.

modification induced by the central mass is a decrease in the radius of the soliton, accompanied by an increase in its mass. A very similar result has been obtained in [425], performing pseudo-spectral simulations of FDM halos in the presence of a point-like supermassive perturber. It has been shown there that, for $m_\psi \gtrsim 10^{-20}$ eV, the overall structure of the halo can indeed be altered by the presence of the SMBH, developing a more compact and massive soliton owing to the point-like gravitational source. On the other hand, in [430] accretion of the FDM halo due to infalling boundary conditions on a central Schwarzschild black hole has been investigated. Outside the Schwarzschild radius and in proximity to the black hole, the scalar dynamics exhibit a stationary solution characterized by a non-vanishing flux. This signifies the infall of FDM toward the horizon. Conversely, at a considerable distance from the center, the dynamics replicate the static soliton behavior of Eq. (3.7). The general picture that comes out of such studies, hence, is that the modifications to the soliton profile in Eq. (3.7), due to the presence of the central SMBH, would lead to a more massive and compact soliton that, in turn, would affect more drastically the orbital dynamics of test particles, thus including those of the S2 star in the Galactic Center. For such reason, and due to the not yet complete characterization of these modifications to the soliton profile, we have decided to take a conservative approach and consider it unaltered by the presence of Sgr A*, when studying the motion of the S2 star.

3.3. Post-newtonian sensitivity analysis

The first analysis that we performed to assess the detectability with orbital data of the S2 star of a FDM solitonic core in the Galactic Center is a post-Newtonian sensitivity analysis [4] using mock GRAVITY data over two orbital periods. In particular, our analysis assumes as ground truth the absence of such a scalar cloud and takes as putative observational data the mock catalogue described in Section 2.2.3 (reported in Appendix B and depicted in Figure 2.13), generated within a Schwarzschild spacetime. On this dataset, we have fitted a 1PN orbital model that takes into account the presence of an FDM soliton as described in Eq. (3.7), which, once taking into account the scaling relation in Eq. (3.9) and fixing the

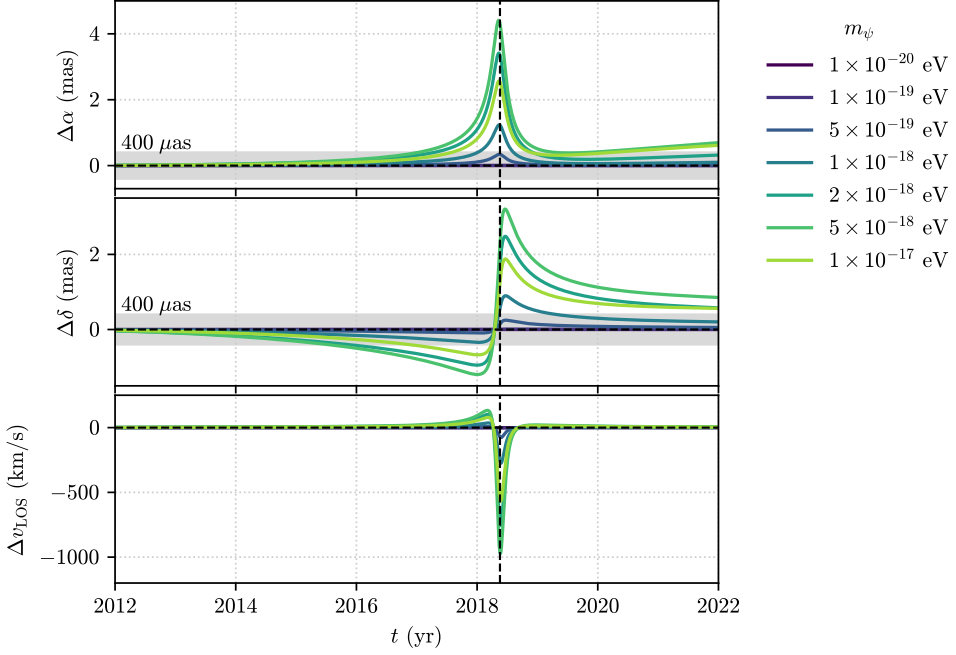


Figure 3.2. Astrometric (right ascension α and declination δ , top two panels) and spectroscopic (line-of-sight velocity, bottom panel) impact of the presence of a FDM soliton as described in Eq. (3.7) in the Galactic Center, computed for different values of the boson mass m_ψ as the difference between the numerically integrated orbits in the FDM case and the Schwarzschild case. The gray shaded region in the top two panels corresponds to the mean astrometric uncertainty for pre-2017 VLT observations for the S2 star reported in [138].

Milky Way halo mass to its value from GAIA reported in Table 3.1, is completely characterized by the boson mass m_ψ . The 1PN orbital model used to describe the motion of the S2 star in the presence of the FDM soliton is built by adding to the Newtonian and 1PN contributions from the central point-like source, reported in Eqs. (2.80) and (2.81) respectively, a contribution arising from the FDM halo:

$$\vec{a} = \vec{a}_N + \vec{a}_{\text{1PN}} + \vec{a}_{\text{FDM}}. \quad | \quad 3.11$$

More specifically, we consider the acceleration due to the FDM soliton being directed radially (under the assumption that the mass centroid of the dark matter distribution coincides with the SMBH) so that we can write $\vec{a}_{\text{FDM}} = a_{\text{FDM}}^r \hat{n}$. On the other hand, this acceleration is produced by the gravitational potential Φ of the solitonic FDM profile

$$-a_{\text{FDM}}^r = \frac{d\Phi}{dr} \quad | \quad 3.12$$

given by the Poisson equation (3.6). When expressed in spherical coordinates this equation reduces to (momentarily restoring Newton's constant G)

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi G \rho_s(r) \quad | \quad 3.13$$

so that we can relate, via direct integration, the radial acceleration to the soliton enclosed mass M_{enclosed} as reported in Eq. (3.10):

$$-a_{\text{FDM}}^r = \frac{d\Phi}{dr} = \frac{G}{r^2} \int 4\pi r'^2 \rho_s(r') dr' = \frac{GM_{\text{enclosed}}(r)}{r^2}. \quad | \quad 3.14$$

We can thus obtain, once appropriate initial conditions have been assigned (see Section 2.2.3), a numerically integrated orbit for the S2 star, described Newton's second law.

From the integrated orbit we reconstruct observable quantities following the procedure described in Section 2.2.3 and we compare the sky-projected orbits and line-of-sight velocities to the ones obtained in the Schwarzschild case, to assess the deviations induced by the FDM soliton. In Figure 3.2, we report the deviations on the two sky-projected directions and on the line-of-sight velocity for different values in the range of interest $m_\psi \leq 10^{-17}$ eV. The gray shaded region in the top two panels, at around $400 \mu\text{as}$ corresponds to the rms astrometric uncertainty for pre-2017 observations at VLT for the S2 star, reported in [138]. The plot shows clearly that pre-pericenter observations carry little to no information about the presence of the FDM soliton, with only a negative shift in declination exceeding the potential instrumental limit for pre-pericenter observations between 2016 and 2017, with boson masses 2×10^{-18} eV $\lesssim m_\psi \lesssim 1 \times 10^{-17}$ eV. The strongest deviation between the FDM case and the Schwarzschild case is produced at and after the pericenter passage in ~ 2018 for both kinds of observables, thus suggesting that the tightest constraints on the boson mass can be produced with observations in this portion of the orbit. This preliminary analysis already points out that fitting only VLT data with the 1PN orbital model for the S2 star would not allow placing sensible constraints on the boson mass m_ψ . For this reason, we have initially focused only on mock observations spanning two orbital periods (so that the data contain at least a close encounter between the test particle and the FDM soliton and the subsequent astrometric impact of the orbital modification over an entire orbital period). We have thus fitted the orbital model described above applying

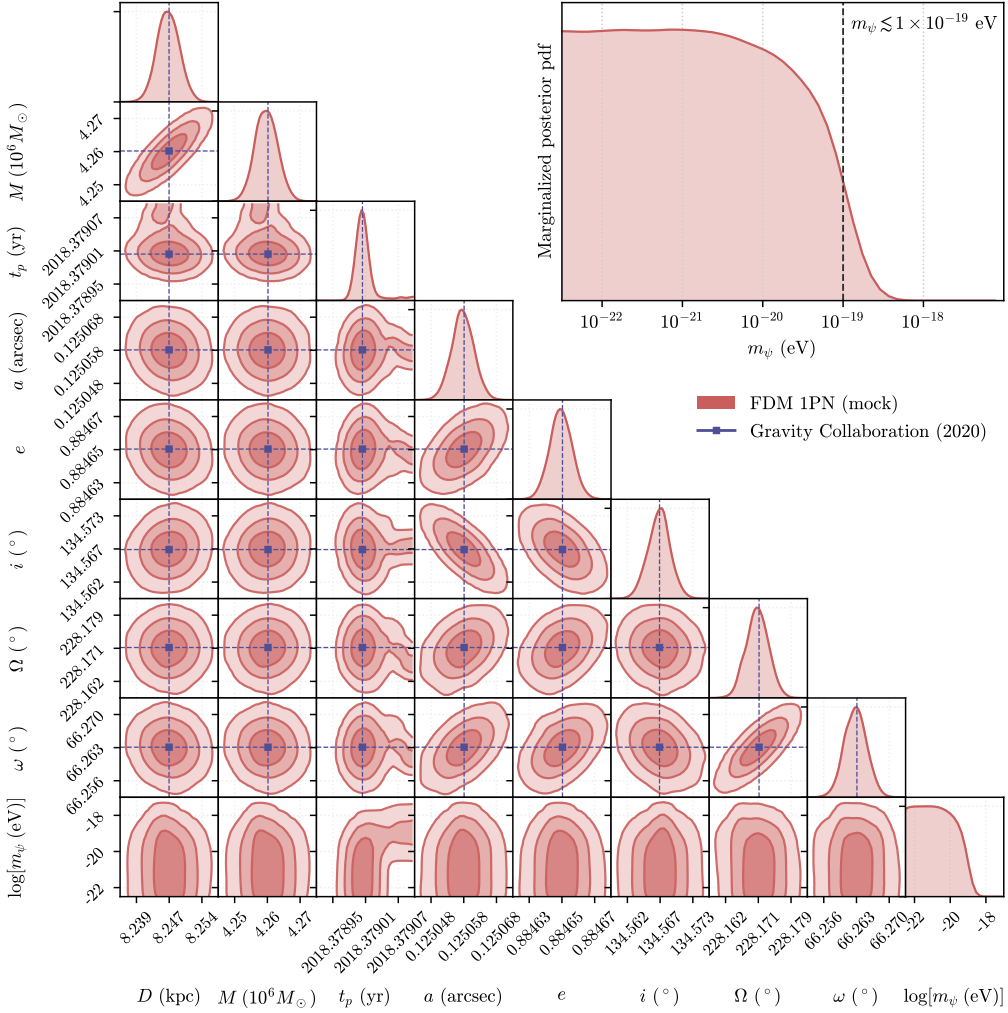


Figure 3.3. Corner plot reporting the full 9-dimensional posterior probability density function from the sensitivity analysis on the 1PN orbital model for the orbit of the S2 star in Galactic Center, in presence of a FDM soliton. The blue dashed lines and markers correspond to the input values used for the generation of the mock catalogue employed in our analysis. The inset plot reports the marginalized posterior probability density function on the parameter m_ψ . The vertical dashed line corresponds to the 95% confidence level upper limit on m_ψ .

the Bayesian analysis technique described in Appendix C. In particular, our orbital model depends on the 8 orbital parameters in Eq. (C.4), plus the parameter m_ψ (that we have sampled in logarithmic scale) that constitutes our parameter of interest. We have thus adopted the likelihood function reported in Eq. (C.5), assigning priors that combine those reported in Table C.1, plus a uniform prior on $\log m_\psi$ reflecting the range of interest for the FDM model that we aim to analyze. Such prior is reported in Table 3.2.

Uniform prior		
	Start	End
$\log[m_\psi(\text{eV})]$	-24	-17

Table 3.2. Log-uniform prior on the parameter m_ψ used for our Bayesian analysis, both in the 1PN sensitivity analysis and in the posterior analysis with actual data in the geodesic approach.

Table 3.3. Credible intervals on all the parameters of the 1PN orbital model resulting from our sensitivity analysis. In particular, all the orbital parameters are bound and their 68% (1σ) credible intervals contain the values employed for the generation of the mock catalogue. The posterior for the parameter $\log m_\psi$, reported in the highlighted bottom row, results in a 95% confidence level upper bound corresponding to $m_\psi \lesssim 1 \times 10^{-19}$ eV.

Parameter (unit)	Posterior
D (kpc)	8.2467 ± 0.0025
M ($10^6 M_\odot$)	4.2608 ± 0.0039
t_p (yr)	$2018.379006^{+0.000008}_{-0.000014}$
a (arcsec)	$0.1250579^{+0.0000033}_{-0.0000032}$
e	$0.8846487^{+0.0000054}_{-0.0000053}$
i ($^\circ$)	$134.5671^{+0.0017}_{-0.0018}$
Ω ($^\circ$)	228.1708 ± 0.0028
ω ($^\circ$)	66.2629 ± 0.0024
$\log[m_\psi(\text{eV})]$	$\lesssim -19.0$ (95% c.l.)

Here, we summarize the results of our 1PN sensitivity analysis. In Table 3.3 we report the 68% (1σ) credible intervals on all the parameters of our orbital model. All parameters are bound within credible intervals that contain the *true* values, *i.e.* those employed for the generation of the mock catalogue (highlighted with dashed blue lines in the plot). Importantly, the posterior for the parameter $\log m_\psi$, highlighted in the last row of Table 3.3, allows to place a 95% confidence level upper bound corresponding to $m_\psi \lesssim 1 \times 10^{-19}$ eV. The corner plot in Figure 3.3 reports in the off-diagonal plots the 68%, 95% and 99.7% contours of the marginal posterior probability distributions of each pair of parameters of our orbital model, while the diagonal plots show the marginalized one-parameter posteriors. The only relevant correlation between the orbital parameters and m_ψ appears to be with the pericenter passage epoch t_p . We interpret this correlation as the effect of the central soliton of modifying the near-pericenter dynamics of the orbit (Figure 3.2), thus resulting in a slightly retarded pericenter epoch for the values of m_ψ that correspond to greatest enclosed mass and steepest density profile (Figure 3.1). For completeness, in the inset plot we also report the marginalized posterior distribution on the parameter m_ψ , which is flat for values of $m_\psi \lesssim 10^{-21}$ eV (thus indicating no sensibility of the S2 orbit in this mass range) and rapidly falls off to zero between 10^{-20} eV and 10^{-18} eV, hence justifying the heuristic priors used for our analysis. Our results suggest that FDM solitons with boson masses below 10^{-19} eV would go undetected in the Galactic Center, even when analyzed via future GRAVITY-like orbital data of the S2 star, spanning two full orbital periods. Heavier bosons, on the other hand, being

outside the confidence region from our sensitivity analysis, would produce detectable signatures in future data.

Finally, we have assessed how our conclusions would be affected by slightly relaxing the constraints imposed by assuming the validity of the scaling relation in (3.9). In particular, we have tweaked the value of the halo mass M_{halo} , by picking extreme values within the confidence interval from the GAIA results [427] reported in Table 3.1 for the Milky Way halo. We have repeated our sensitivity analysis by fixing values for M_{halo} that are 3σ above (below) the reported value. This allowed us to probe whether or not, for a given boson mass m_ψ , a smaller (bigger) soliton radius with respect to the one used in the primary sensitivity analysis would produce drastically different results. The marginalized posterior distribution on m_ψ for the three cases are shown in Figure 3.4 in logarithmic scale, with the corresponding 95% upper limits on the boson mass. While the resulting values for the upper limit on m_ψ are slightly different, namely a bigger value $m_\psi \lesssim 1.9 \times 10^{-19}$ eV is obtained when $M_{\text{halo}} = M_{\text{GAIA}} - 3\sigma$ and a smaller value $m_\psi \lesssim 7.1 \times 10^{-20}$ eV results in the $M_{\text{halo}} = M_{\text{GAIA}} + 3\sigma$ case, the general conclusion doesn't change as, in any case, the sensitivity analysis points out an upper limit on m_ψ around the 10^{-19} eV.

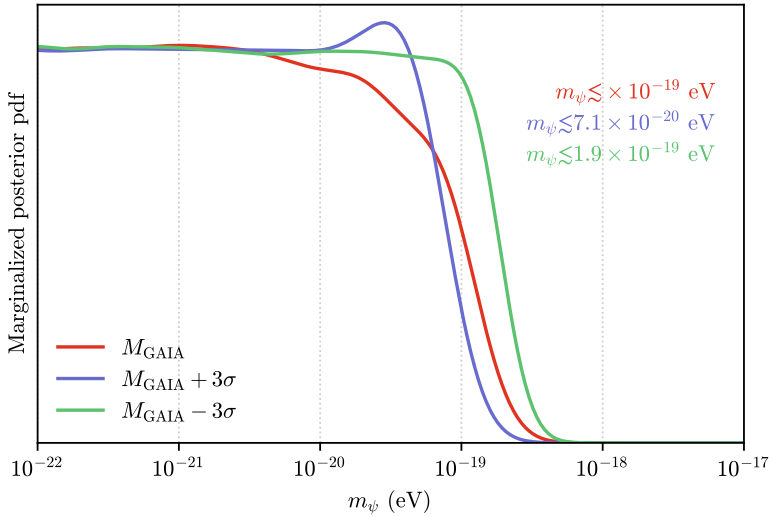


Figure 3.4. Marginalized posterior probability density functions on the parameter m_ψ (in logarithmic scale) resulting from the three sensitivity analyses performed by changing the value of the halo mass M_{halo} in its 3σ confidence region from GAIA data [427].

3.4. Constraint from the geodesic approach

The sensitivity analysis presented in the previous section, motivated us to look further into the potential constraint on the boson mass m_ψ for FDM. As shown in Figure 3.2, most of the deviation for the S2 orbit between the vacuum case and the case of a FDM soliton (and most of the sensitivity to the specific value of m_ψ) is found at pericenter (and in the subsequent motion). Publicly available data, at the time of the writing of this thesis, presented in [138], only cover a period up to ~ 2017 and thus miss the crucial astrometric and spectroscopic information brought by the pericenter passage. Nonetheless, as explained in Section 1.2.4, relativistic effects produced at pericenter have been successfully detected for the S2 star [171, 226] and a possible solution is to include their information by adding their measurements as additional data-points to the orbital data up to 2017. In order to appropriately model relativistic effects, we have decided to go beyond the post-Newtonian description adopted in the sensitivity analysis and to move towards a geodesic description for the orbit of S2, as explained in Chapter 2. In order to do so, we first have to consider a spacetime metric that takes into account the simultaneous presence of the central black hole and the FDM halo. Again, we will make the assumption that the presence of Sgr A* does not alter significantly the structure of the FDM soliton⁵, and look for a way to effectively introduce the FDM halo within a spherically symmetric spacetime. This problem was first analyzed in [431], where a formalism to introduce any spherically symmetric dark matter density distribution into the spherically symmetric spacetime in Eq. (2.12) and obtain an analytical solution for the metric tensor components when simultaneously accounting for the existence of a central black hole was presented. This formalism is based on a methodology first introduced in [432] to produce a general relativistic spacetime metric that only takes into account the presence of a dark matter halo (which, in that work, was applied to the NFW dark matter profile). In particular, on purely geometrical ground, an expression for the tangential velocity of objects

⁵ The description of the fully non-linear backreaction of the scalar cloud to the presence of the SMBH, and *vice versa*, requires the solution of the system of Eqs. (3.2)-(3.3), that we don't want to approach here, so that we can make a connection with the FDM dynamics on galactic scales that is recovered in the non-relativistic regime.

moving in circular stable geodesics, and which depends solely on the metric coefficients, is found. This circular velocity is then matched to the expression for the circular velocity in a given dark matter profile and used as an equation for the metric coefficient, instead. This allows to obtain the spacetime metric for pure dark matter and an expression for the energy-momentum tensor describing such distribution of matter. Finally, the obtained energy-momentum tensors is plugged into Einstein's field equations, Eq. (2.9), and an analytical spherically symmetric black hole solution embedded within a dark matter halo is derived (by requiring that the metric reduces to the Schwarzschild one, Eq. (2.18), in the absence of a dark matter halo). This formalism has been successfully applied to derive the spacetime metric for different dark matter models (also considering modifications to General Relativity) [433–436] and, interestingly, a recent study also applied it for the first time to the case of a FDM halo around a central black hole [437]. In this work, an analytic space-time solution is constructed from the solitonic FDM profile in Eq. (3.7)

$$ds^2 = -F(r) dt^2 + \frac{1}{F(r)} dr^2 + r^2 d\Omega^2, \quad | \quad 3.15$$

where

$$F(r) = \exp \left(\frac{4\pi\rho_0 r_c^3}{91\alpha^8 r} \left(\left(\frac{r_c}{r} \right)^{13} \mathcal{F}(r) - \frac{3003\pi\alpha^{\frac{13}{2}}}{2048} \right) \right) - \frac{2M}{r}, \quad | \quad 3.16$$

being $\mathcal{F}$ an hypergeometric function of the radial coordinate r , given by

$$\mathcal{F}(r) = {}_2F_1 \left(\frac{13}{2}, 7; \frac{15}{2}; -\frac{r_c^2}{r^2\alpha} \right). \quad | \quad 3.17$$

The positive root of the function in Eq. (3.16) identifies the radial coordinate r_H of the event horizon. We derived r_H numerically as a function of the FDM boson mass m_ψ in the range of interest of ultralight axions $m_\psi \in [10^{-23}, 10^{-17}]$ eV, and have found a negligible variation ($< 10^{-6}M$) with respect to the Schwarzschild case, $2M$ (which is not unexpected, since the FDM distribution is diluted and hence does not significantly alter the compactness of the central object). Small deviations of the horizon radius, leading to a comparatively larger size of the black hole shadow for the metric in Eq. (3.16), have been examined in [437], as a function of both the boson mass and of the core radius r_c , assumed as independent

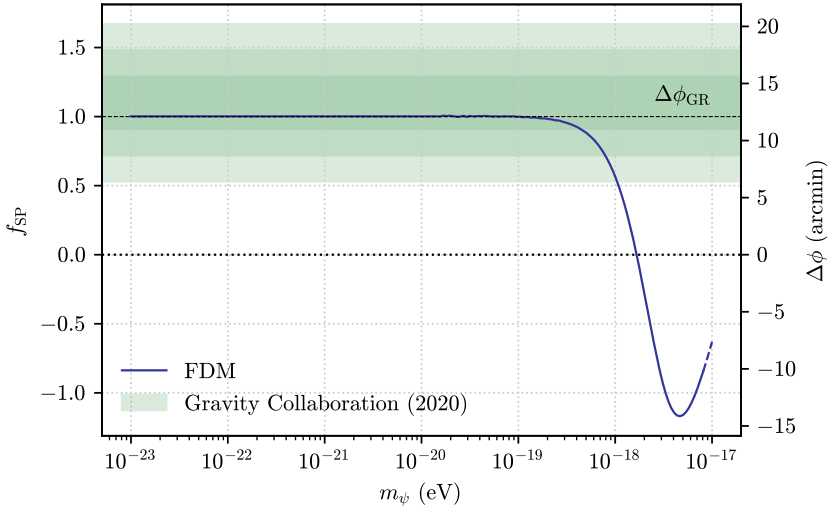


Figure 3.5. Orbital precession for the S2 star in the FDM solitonic spacetime as a function of m_ψ between $\sim 10^{-23}$ eV and $\sim 10^{-17}$ eV. In particular, the right axis reports the orbital precession over one orbital period, while the left axis reports its ratio, f_{SP} to the 1PN general relativistic value in Eq. (2.41). The green shaded horizontal strips report the 1, 2 and 3σ confidence regions on f_{SP} as measured by the Gravity Collaboration in 2020 [226] and reported in Table 1.5.

parameters (*i.e.* without taking into consideration the scaling relation in Eq. (3.9)). Due to the current low-resolution of event horizon imaging [270], however, such horizon-scale tests alone are currently unable to place sensible bound on the boson mass when the scaling relation in Eq. (3.9) is taken into account. Nonetheless, greater effects are expected on bigger scales, due to the larger fraction of enclosed integrated mass of the FDM distribution. For example, stronger effects of weak lensing and a larger Einstein ring due to the presence of the soliton effects are predicted in [437], especially when the impact parameter is comparable to the soliton core.

From Eq. (2.3) applied to the spacetime metric in Eq. (3.16) one derives geodesic equations for the motion of freely falling test particles in this spacetime

$$\ddot{t} = -\frac{\dot{r}\dot{t}F'(r)}{F(r)} \quad | \quad 3.18$$

$$\ddot{r} = -\frac{c^2\dot{t}^2F(r)F'(r)}{2} + r\dot{\phi}^2F(r)\sin^2(\theta) + r\dot{\theta}^2F(r) + \frac{\dot{r}^2F'(r)}{2F(r)} \quad | \quad 3.19$$

$$\ddot{\theta} = \dot{\phi}^2 \sin(\theta) \cos(\theta) - \frac{2\dot{r}\dot{\theta}}{r} \quad | \quad 3.20$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r} \quad | \quad 3.21$$

where, given the derivative of the hypergeometric function in Eq. (3.17)

$$\frac{d {}_2F_1(a, b; c; z)}{dz} = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z), \quad | \quad 3.22$$

the function $F'(r) = dF/dr$ is given by

$$F'(r) = \frac{4\pi r_c^3 \rho_0}{91\alpha^8 r} \left[\frac{182r_c^{15}}{15\alpha r^{16}} {}_2F_1\left(\frac{15}{2}, 8; \frac{17}{2}; -\frac{r_c^2}{\alpha r^2}\right) - \frac{14r_c^{13}}{r^{14}} \mathcal{F}(r) + \frac{3003\pi\alpha^{13/2}}{2048r} \right] \\ \exp\left(\frac{4\pi\rho_0 r_c^3}{91\alpha^8 r} \left(\left(\frac{r_c}{r}\right)^{13} \mathcal{F}(r) - \frac{3003\pi\alpha^{13/2}}{2048}\right)\right) + \frac{2M}{r^2}. \quad | \quad 3.23$$

Integrating these equations numerically, once initial conditions for the S2 star have been appropriately assigned following Section 2.2.3, we have reproduced the same general behaviour already found in the post-Newtonian analysis, with deviations from the vacuum case that match those reported in Figure 3.2. In Figure 3.5, on the other hand, we report the results of our numerically estimated rate of orbital precession for S2 from the geodesic integration as a function of the boson mass m_ψ . The green bands in the plot report the 1σ , 2σ and 3σ bounds on the measured rate of orbital precession for S2 obtained by the precise monitoring of its orbital motion by the Gravity Collaboration [226]. The presence of the FDM solitonic core becomes more prominent at around $m_\psi \sim 10^{-19}$ eV when the orbital precession starts to decrease noticeably from the value of 12.1 arcmin per orbital period expected in a purely Schwarzschild metric. At $m_\psi \sim 10^{-18}$ eV the deviation is so strong that the orbital precession would exceed the 3σ bounds imposed by astrometric observations, and at even greater masses the mass precession caused by the FDM soliton becomes dominant over the prograde Schwarzschild precession, with the overall effect becoming retrograde from $m_\psi \sim 1.5 \times 10^{-18}$ eV on.

This prediction agrees with the result from the 1PN estimation of the orbital deviation presented in the previous section and with the constraints on the possible dark mass enclosed

Table 3.4. Credible intervals on all the parameters of the 14-dimensional geodesic orbital model for S2, resulting from our posterior analysis. In particular, all the orbital (rows 1 to 9) and reference frame (rows 10 to 14) parameters are bound and their 68% (1σ) credible intervals are compatible with their known counterpart in literature [138]. The posterior for the parameter $\log m_\psi$, reported in the highlighted bottom row, results in a 95% confidence level upper bound at $\log[m_\psi(\text{eV})] \lesssim -18.5$ corresponding to $m_\psi \lesssim 3.2 \times 10^{-19}$ eV.

Parameter (unit)	Posterior
D (kpc)	8.03 ± 0.21
M ($10^6 M_\odot$)	4.03 ± 0.22
t_p (yr)	2018.364 ± 0.023
α (arcsec)	0.1260 ± 0.0011
e	0.8843 ± 0.0024
i ($^\circ$)	133.80 ± 0.47
Ω ($^\circ$)	226.75 ± 0.80
ω ($^\circ$)	$65.53^{+0.75}_{-0.74}$
α_0 (mas)	0.33 ± 0.16
δ_0 (mas)	-0.01 ± 0.19
$v_{\alpha,0}$ (mas/yr)	0.066 ± 0.052
$v_{\delta,0}$ (mas/yr)	$0.120^{+0.062}_{-0.063}$
$v_{\text{LOS},0}$ (km/s)	-5.8 ± 4.5
$\log[m_\psi(\text{eV})]$	$\lesssim -18.5$ (95% c.l.)

in the S2 orbit at around 0.1% of the Sgr A* mass [219, 242]. Moreover, it is straightforward to understand why this would be the case by looking at the density and mass profiles in Figure 3.1. Evidently, for boson masses greater than $\sim 10^{-18}$ eV the mass of the solitonic distribution enclosed within the S2 orbit reaches values ranging from 0.01% to 0.1% of the mass of Sgr A*. At the same time, for the same boson masses, the density profiles within the S2 orbit switch from a flat distribution to a steeply ($\rho \propto r^{-16}$) decaying one within the extent to the orbit itself. The combination of significant enclosed mass and steep density gradient within the orbit is responsible for a modification of the orbital dynamics [242] that is consistent with our periastron advance results in the considered range of masses.

We have also estimated how the prediction on the other measured relativistic effect for S2, namely the gravitational redshift, would change due to the presence of the FDM soliton. From our numerically integrated orbits, it resulted that this effect shows very little dependence on the boson mass (with deviations below 0.2% when compared to the general relativistic value, in the considered range of masses). This agrees with the fact that the FDM-induced modifications to the black hole metric do not alter significantly the compactness of the central object, owing to the negligible variation in the event horizon

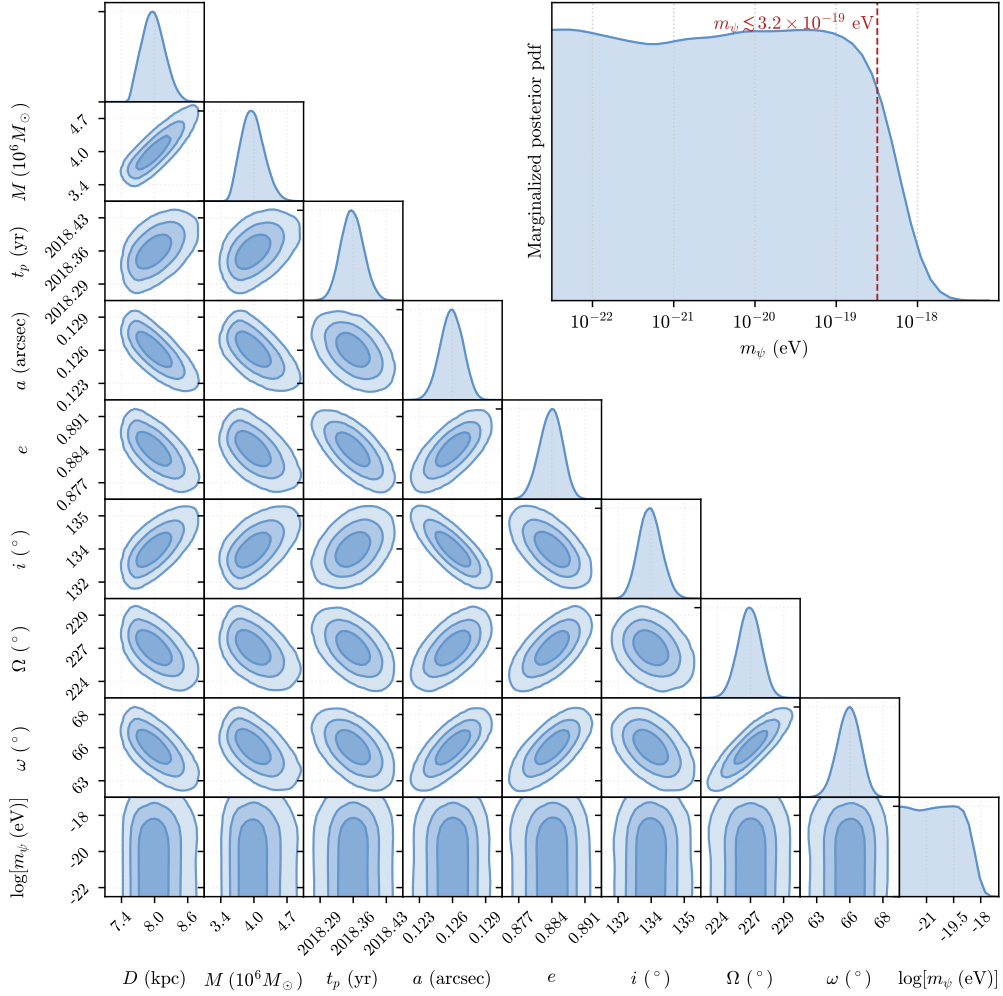


Figure 3.6. Corner plot reporting a 9-dimensional slice of the posterior probability density function with the parameters of interest (we decide to exclude for clearness the 5 astrometric reference frame parameters from the visualization. Their posterior is perfectly compatible with the one reported in Figure C.3 and the reference frame parameters exhibit no significant correlation with m_ψ). The inset top-right-corner plot shows the marginalized posterior probability density function on the parameter m_ψ (in logarithmic scale). The vertical dashed line corresponds to the 95% confidence level upper limit on m_ψ .

radius.

Once we assessed the impact of the FDM soliton on the geodesic motion of the S2 star, we proceeded to the posterior analysis with publicly available data + the information on the orbital precession. This is done following the Bayesian analysis methodology presented in Appendix C and, in particular, by adopting the likelihood reported in Eq. (C.3). In this case, due to the additional parameters introduced to account for

the systematics in the astrometric and spectroscopic reference frame construction, our orbital model depends on a total of 14 parameters, given by the 13 parameters reported in Eq. (C.1), plus the parameter m_ψ (that we have sampled in logarithmic scale) that, again, represents our parameter of interest. Finally, for this analysis, we have adopted the same priors used for the sensitivity analysis in the previous section.

The results of our MCMC analysis are reported in Table 3.4. Median values and 68% confidence level intervals around it are reported for the parameters that result bounded from our analysis. In particular, for both the Keplerian orbital parameters and the reference frame parameters it is possible to derive a confidence interval that is compatible within 1σ with previous results on the same dataset [138] and to the fiducial values reported in Appendix C. On the other hand, we were able to place an upper limit on the boson mass $m_\psi \lesssim 3.22 \times 10^{-19}$ eV at 95% confidence level, which is consistent with both the qualitative bounds derived from the results for the orbital precession reported in Figure 3.6 and with the forecast from the post-Newtonian sensitivity analysis, thus implying that no soliton with a boson mass above the derived upper limit is detected in the Galactic Center. For completeness, in Figure 3.6 we report a corner plot with the full posterior distribution (excluding the reference frame parameters, whose posterior is qualitatively and quantitatively matching the one reported in Figure C.3), highlighting the marginalized posterior distribution for m_ψ in the inset plot.

3.5. Conclusions

The rich non-linear dynamics of FDM self-gravitating halos [422] composed of ultralight bosons, bring distinctive and testable predictions that have fueled interest in this model. Several tests have been conducted to derive observational bounds on the boson mass m_ψ . Cosmological analyses allow for a mass range of $10^{-24} \lesssim m_\psi \lesssim 10^{-17}$ [438, 439], while analyses on the superradiance instability [442] are compatible with intervals $m_\psi \lesssim 2.9 \times 10^{-22}$ eV and $m_\psi \gtrsim 4.6 \times 10^{-22}$ eV. Other astrophysical tests have allowed lower limits on the boson mass, the most stringent of which come from dwarf spheroidal galaxies [440], analysis of the Lyman- α forest [441], the dynamics of ultra-faint dwarf galaxies [443] and the study of the abundance and mass function of galactic subhalos

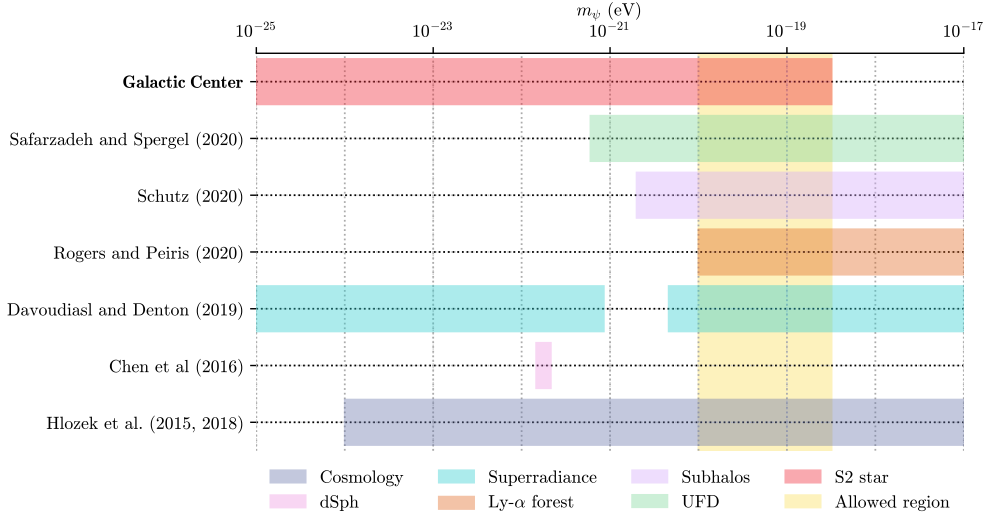


Figure 3.7. Comparison of the allowed range for the mass, m_ψ , of FDM particles from our analysis (Galactic Center top entry) with other constraints in literature coming from analysis at different scales. In dark blue, we report the constraints from cosmological observations (cosmic microwave background and large-scale structures) from [438, 439]. In pink, we report the constraints from analysis in dSph [440]. Orange boxes report constraints from the analysis of the Lyman- α forest from [441]. In aqua, we report the allowed range for m_ψ resulting from the analysis of SMBH superradiance in [442]. In green, we give the constraints from ultra-faint dwarf galaxies taken from [443]. Finally, the lilac box reports the results of the analysis on subhalos from [444]. The vertical yellow strip corresponds to the intersection of the allowed intervals for m_ψ from most of the astrophysical and cosmological probes, except for the ones coming from dSph.

[444]. In this chapter, we have shown how observations of the rapidly moving stars in our Galactic Center, and in particular of the S2 star, can represent a new probe for the existence of a FDM solitonic core and can provide new constraints on the boson mass. First, by means of a post-Newtonian sensitivity analysis, we have assessed the potential detectability of such an FDM solitonic core in the Galactic Center by pointing out the orbital deviations that would arise in this scenario. Next, we have derived actual constraints on the boson mass m_ψ using the publicly available data for S2, once an appropriate matrix, taking into account the presence of the extended dark matter component along with the central massive source, has been fixed. Our analysis has set an upper limit $m_\psi \lesssim 3.22 \times 10^{-19}$ eV at 95% confidence level, implying that no FDM solitonic core generated by bosons with greater masses is detected in the Galactic Center. In Figure 3.7 we show our constraint on m_ψ alongside those from other astrophysical and cosmological data. The constraints from the Galactic Center are complementary to the other aforementioned astrophysical

tests, and, when combining most of the bounds on m_ψ (except for the ones coming from dwarf spheroidal galaxies (dSph)⁶) the allowed region of the parameter space is reduced to an unprecedented level. In particular, combining our result with the 95% constraint $m_\psi \gtrsim 2 \times 10^{-20}$ eV from the analysis of the Lyman- α forest [441], results in an allowed mass range at 95% confidence level of $(2.0 \lesssim m_\psi \lesssim 32.2) \times 10^{-20}$ eV, barely one order of magnitude.

⁶ For which different analyses have highlighted a tension between the predicted boson mass and that derived from analyses at larger scales [440, 445].

Black Hole Mimickers

4.

The evidence accumulated over the years supporting the presence of a large mass confined within a few gravitational radii in Sgr A* is compelling. Analyses of gas dynamics measurements, stellar orbital motion, and the successful observation of horizon-scale emission (see Chapter 1), consistent with accreting plasma and a brightness depression in Sgr A*, all indicate a mass of approximately 4×10^6 solar masses confined within an angular diameter of approximately $50 \mu\text{as}$. However, whether this object precisely conforms to a black hole as described by General Relativity remains unclear [275], and its intrinsic nature, a black hole or an alternative object, is still an open question. By “alternative”, we refer to other classes of exotic compact objects that, while remaining within the framework of General Relativity (i.e., without considering extensions or modifications to the underlying theory of gravity, which will be discussed separately in Chapter 5), could mimic black hole behaviour. A *black hole mimicker* (see [446] for an extensive review of such objects) is, by definition, a potential alternative to a black hole that would observationally appear almost identical, but would either lack a horizon [447–452] or present properties in the strong-field near-horizon region that would differ substantially from a black hole, while its behaviour at large distances would be indistinguishable from one.

Regular black holes are among the most captivating mimickers. In contrast to classical black holes, which exhibit a singularity at their core [453, 454], these objects maintain complete regularity throughout their spacetime. This point is particularly significant, given the expectation that a quantum theory of gravity is needed to resolve the classical singularity problem. Although some attempts have been made to capture the main properties of the fundamental theory [455–457], a clear understanding of its dynamics and the mechanism leading to the formation of these regular spacetimes is still lacking. Consequently, primarily bottom-up approaches have been pursued, first proposed by Bardeen in 1968 [458] prompting intense research in the field [459–467], in which general relativistic solutions are typically modified to test possible deviations in a phenomenological manner. An intriguing feature

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The original scientific results presented in this chapter have been published in [3, 7].

of non-singular black hole models is the presence of a new length scale (hair), which can be hierarchically larger than the Planck scale [461].

Another class of interesting mimickers is constituted by wormholes. The concept of ‘traversable wormhole’ was first introduced by Morris and Thorne [450] and refers to a horizon-free geometry featuring a central throat connecting two asymptotically Minkowski regions of spacetime and satisfying the ‘flare-out’ condition, implying that the surface area of the throat is positive. Various intriguing wormhole spacetimes have been developed and explored in [451, 468–478].

Other forms of exotic compact objects endowed with surfaces (either thermalizing or reflecting), exotic stars like gravastars [479, 480] and recent nested extensions [481], boson and Proca stars (see [446] for an extensive review of such objects and [482, 483] for studies of their properties), or naked singularities exist and have gained increasing popularity in the last decades in the context of black hole mimickers.

Very stringent tests on the fundamental nature of Sgr A* have been unlocked by the comparison between EHT observed images of Sgr A* and numerical simulations. While the observed brightness distribution is consistent with the presence of an event horizon, there is no conclusive evidence ruling out all other possible alternatives [275]. The mere observation of a black hole shadow seems to completely exclude models like non-spinning Joshi-Malafarina-Narayan-2 [484] naked singularities¹, which are anticipated not to have a shadow at all [485]. Similarly, also both stable and unstable boson star models [446, 486] for Sgr A* seem to be ruled out by EHT observations [275] as they would either lack a central brightness depression in their optical appearance or its size would not match the observed light ring of Sgr A*².

In this chapter, we will study how the fundamental nature of Sgr A* can be probed through the orbital motion of the S-stars. As already mentioned, the orbital motion of the S-stars coupled with the successful detection of relativistic effects in astrometric and spectroscopic observables has proven to be

¹ This model is an exact solution of the Tolman-Oppenheimer-Volkoff equations of General Relativity [333], formed as the non-empty end state of the gravitational collapse of a non-thermal perfect fluid from regular Cauchy data.

² Nevertheless, to conclusively rule out boson stars as alternatives to the black hole model for Sgr A*, a more comprehensive study of their spin, compactness, and astrophysical properties is needed [487].

a powerful tool for testing the validity of General Relativity. A natural question arises regarding whether the motion of such stars could provide clues about the intrinsic nature of the central object, be it a black hole or a mimicker.

In particular, in Section 4.1 we will consider a non-singular blackhole spacetime representing a strong deformation of the Schwarzschild solution that presents an additional hair, whose magnitude can be hierarchically larger than the Planck scale and whose role is to smear the classical singularity. This solution is chosen since it introduces corrections to the Schwarzschild geometry that are the largest deformations still compatible with Schwarzschild asymptotics. As we will show, this maximizes the effects of such departure on the geodesic motion of test particles and allows stringent tests on the specific value of the hair, using currently available data for the S2 star. In Section 4.2, we will investigate a slightly more sophisticated geometry, namely the Black Bounce model, that represents a family of solutions that smoothly interpolates between a Schwarzschild black hole, regular black hole solutions and wormhole geometries. We will place observational constraints on its extra parameter, α_{BB} , using the orbital data of S2 and predict the accuracy down to which future GRAVITY observations of S2 and of putative closer stars to Sgr A* will measure this parameter.

4.1. Non-singular black hole with super-Planckian hair

The first black hole mimicker geometry that we consider is a particular case of the Fan-Wang regular black-hole metric [488], belonging to the general class of models explored in [461]. In our analysis, we specifically consider a static and spherically symmetric spacetime expressed in Schwarzschild coordinates as in Eq. (2.12), with

$$A(r) = B^{-1}(r) = 1 - \frac{2m(r)}{r}, \quad | \quad 4.1$$

$$m(r) = 4\pi \int_0^r d\tilde{r} \tilde{r}^2 \rho(\tilde{r}), \quad | \quad 4.2$$

and $m(r)$ represents the Misner-Sharp (MS) mass determined by a density profile $\rho(r)$. Various profiles of the MS mass in Eq. (4.2) lead to a general class of solutions within General

Relativity sourced by an anisotropic fluid, where the equation of state is given by $p_r = -\rho$, with p_r representing the radial component of the fluid pressure [461] (see also [489, 490] for details regarding the significance of anisotropic objects in General Relativity). As discussed in Section 2.2.1, Birkhoff's theorem dictates that the only vacuum solution of Einstein's field equation in Eq. (2.9) in spherical symmetry is the Schwarzschild solution, wherein $m(r) = M$, with M being the Arnowitt-Deser-Misner mass of the object. However, as previously highlighted, this solution is plagued by a curvature singularity at $r = 0$. To circumvent this issue, a commonly employed approach is to impose a de Sitter (dS)-like behavior of the metric in the core. This translates into the requirement $A(r) = 1 - \frac{\alpha r^2}{\ell^2} + \mathcal{O}\left(\frac{r^3}{\ell^3}\right)$ for $r \ll M$, where α is a positive dimensionless constant determining the inner dS length. Here, an additional length scale ℓ is introduced, responsible for the smearing of the classical singularity and interpretable as an extra hair of the black hole. Furthermore, asymptotic flatness at infinity demands $A(r) = 1 - \frac{2M}{r} + \frac{\beta \ell^2}{r^2} + \mathcal{O}\left(\frac{\ell^3}{r^3}\right)$ for $r \gg GM$, where β is, once again, a dimensionless constant.

Within this general class of solutions, we opt for a function of the form [491]

$$A(r) = 1 - \frac{2Mr^2}{(r + \ell)^3}, \quad | \quad 4.3$$

where $\alpha = \frac{2M}{\ell}$ and $\beta = \frac{6M}{\ell}$. This choice represents an exact solution of Einstein's equations sourced by an anisotropic fluid, with the Schwarzschild spacetime retrieved in the limit $\ell \rightarrow 0$. The key motivation behind selecting the metric in Eq. (4.3) is its $\mathcal{O}(\ell^2/r^2)$ corrections to the Schwarzschild geometry³, offering the most powerful deformations still compatible with Schwarzschild asymptotically. This makes the model particularly prone to be tested through the orbital motion of the S2 star in the Galactic Center, particularly when the hair ℓ is super-Planckian, following the idea that, in case this spacetime results not to be testable with current data, no other models belonging to the same class could.

The metric also draws a compelling astrophysical analogy. The density associated with the Misner-Sharp mass in (4.3) is given by $4\pi\rho(r) = \frac{3M\ell}{(r+\ell)^4}$, exhibiting a large r behavior

³ Differently from past studies that have focused on orders such as $\mathcal{O}(\ell^3/r^3)$ (e.g., the Bardeen model [458]), $\mathcal{O}(\ell^4/r^4)$ (e.g., the Hayward black hole [460]), or models with exponentially-suppressed corrections [492].

proportional to $\sim r^{-4}$. This resembles density profiles of dark matter in elliptical and spherical galaxies, such as the Hernquist [493] or Jaffe [494] profiles, albeit with distinct characteristic length scales involved.

The spacetime described by Eq. (4.1) manifests distinct configurations based on the parameter ℓ selection. Specifically, for $0 \leq \ell < \ell_c$, where $\ell_c = 8M/27 \approx 0.296M$, it characterizes black holes with two horizons: an external event horizon r_+ and an internal Cauchy horizon r_- . These two horizons merge into a single (degenerate) horizon, representing an extreme black hole, as ℓ reaches ℓ_c . Beyond ℓ_c , the horizon vanishes, revealing a regular naked dS core. Additionally, an analysis of the thermodynamic properties indicates that the system's stability, determined by the positivity of the specific heat (at constant ℓ), is only achievable within the range $0.245M \leq \ell \leq \ell_c$. This confines the existence of stable black holes within a narrow interval in this model.

We focus on the geodesic motion of test particles in the non-singular black hole spacetime by applying Eqs. (2.3) to the metric in Eq. (4.1), obtaining

$$\ddot{t} = -\frac{2\dot{r}\dot{t}Mr(2\ell - r)}{(\ell + r)(2Mr^2 - (\ell + r)^3)} \quad | \quad 4.4$$

$$\begin{aligned} \ddot{r} = & -\frac{(2Mr^2 - (\ell + r)^3)}{2(\ell + r)^3} \left[2r\dot{\phi}^2 \sin^2(\theta) + 2r\dot{\theta}^2 + \right. \\ & + \dot{r}^2 \left(\frac{(\ell + r)^3(3(\ell + r)^2 - 4Mr)}{(2Mr^2 - (\ell + r)^3)^2} + \frac{3(\ell + r)^2}{2Mr^2 - (\ell + r)^3} \right) + \\ & \left. + \dot{t}^2 \left(\frac{4Mr - 3(\ell + r)^2}{(\ell + r)^3} + \frac{3((\ell + 3)^3 - 2Mr^2)}{(\ell + r)^4} \right) \right], \end{aligned} \quad | \quad 4.5$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \dot{\phi}^2 \sin(\theta) \cos(\theta), \quad | \quad 4.6$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}. \quad | \quad 4.7$$

We apply the procedure to fix initial conditions for time-like geodesics described in Section 2.2.3, and obtain orbits for the S2 star in the spacetime generated by the non-singular black hole in Eq. (4.1). This allows us to assess the orbital deviation from a purely Schwarzschild orbit on the S2 star astrometric and spectroscopic observables arising from the modification to the metric. The results of this analysis are

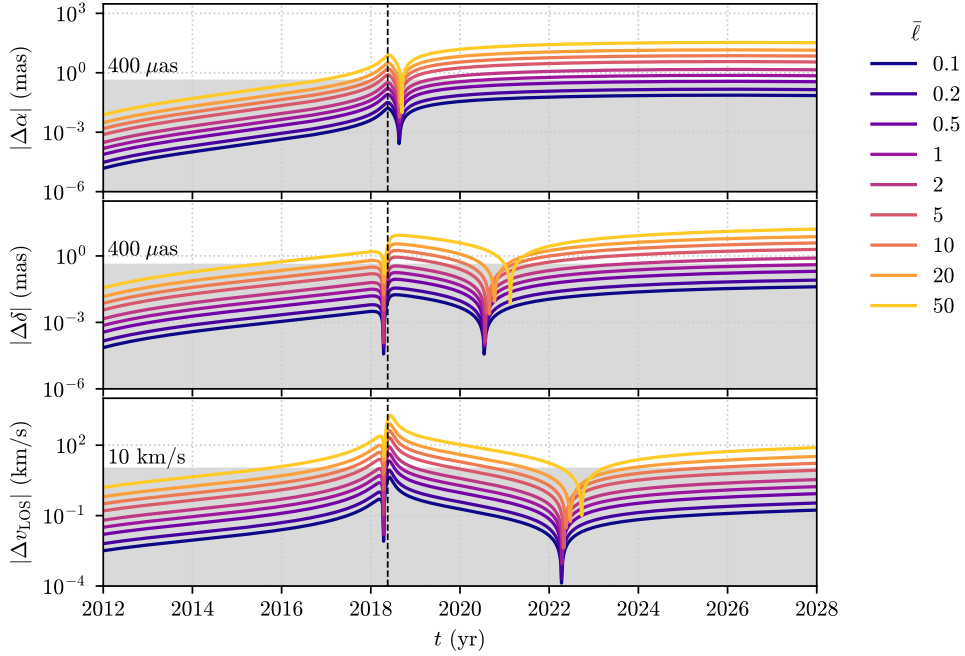


Figure 4.1. Astrometric (right ascension α and declination δ , top two panels) and spectroscopic (line-of-sight velocity, bottom panel) impact of the non-singular black hole metric in Eq. (4.1) on the orbit of the S2 star in the Galactic Center, computed for different values of the parameter $\bar{\ell} \in [0.1, 50]$ over an entire orbital period around the 2018 pericenter passage (highlighted by a vertical dashed line). The shaded regions correspond to the astrometric (top two panels) and spectroscopic (bottom panel) sensitivity thresholds at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and 10 km/s for the SINFONI interferometer.

reported in Figure 4.1 where the strength of the deviations can be appreciated as a function of the hair length ℓ . Here, and in what follows, we define the dimensionless black hole hair $\bar{\ell} = \ell/M$ for an easier parametrization of the deviations from the Schwarzschild geometry. In particular, we consider the range $\bar{\ell} \in [0.1, 50]$ and show that at and after the 2018 pericenter passage, a sensible astrometric deviation from the Schwarzschild trajectory is predicted, whereas pre-2017 data are only sensible to strong modifications of the geometry corresponding to $\bar{\ell} \sim 50$.

This motivates us to look more attentively at the near-pericenter dynamics of massive test particles and, more specifically, at the rate of relativistic precession induced by the non-singular black hole. The functional form of the metric coefficients in Eq. (4.1) is simple enough that one can look for an analytical formula for the rate of periastron advance, by following the same steps that led to the determination of the general relativis-

tic precession in the Schwarzschild case, reported in Eq. (2.41). In particular, when taking into account the geodesic equations for the non-singular black hole, Eqs. (4.4)-(4.7) particularized for the motion of a massive test particle on the equatorial plane one of the geometry, one obtains an equivalent of Eq. (2.37) with an additional term reflecting the impact of the modification to the geometry to the motion of test particles. This reads

$$\xi_1'' + \xi_1 = 3\xi_0(\xi_0 - 2\bar{\ell}). \quad | \quad 4.8$$

One can thus compute the relativistic rate of orbital precession in the non-singular black hole case and find

$$\Delta\phi = \Delta\phi_{\text{GR}}(1 - \bar{\ell}). \quad | \quad 4.9$$

While the usual Schwarzschild result is obtained for $\bar{\ell} \rightarrow 0$, we see that $\Delta\phi$ decreases linearly with the additional hair and it becomes retrograde for $\bar{\ell} > 1$. This result suggests that the measurement of the rate of orbital precession is a strong falsifier for the metric in Eq. (4.1).

In Figure 4.2 we compute the rate of orbital precession of the S2 star using both the analytical formula in Eq. (4.9) and the usual numerical procedure described in Section 2.2.3. The plot shows perfect agreement in the range $\bar{\ell} \lesssim 10$ between the two estimates of the orbital precession, correctly predicting the linear decrement of f_{sp} with ℓ , the cancellation of the orbital precession ($f_{\text{sp}} = 0$) for $\bar{\ell} = 1$ and the retrograde precession for $\bar{\ell} > 1$. For $\bar{\ell} \gtrsim 10$, however, the first-order formula in Eq. (4.9) is not able to capture non-linear effects in the orbital dynamic that the numerical integration of the fully non-linear geodesic equations in Eqs. (4.4)-(4.7) correctly reproduces. Interestingly, the figure also shows that just considering the measurement of the orbital precession for the S2 star by the Gravity Collaboration [226], reported in Table 1.5 and in the plot as a green horizontal band, one can derive qualitative bounds on the hair ℓ at around 0.5, a value for which the predicted rate of orbital precession starts to exceed the 3σ credible interval on f_{sp} .

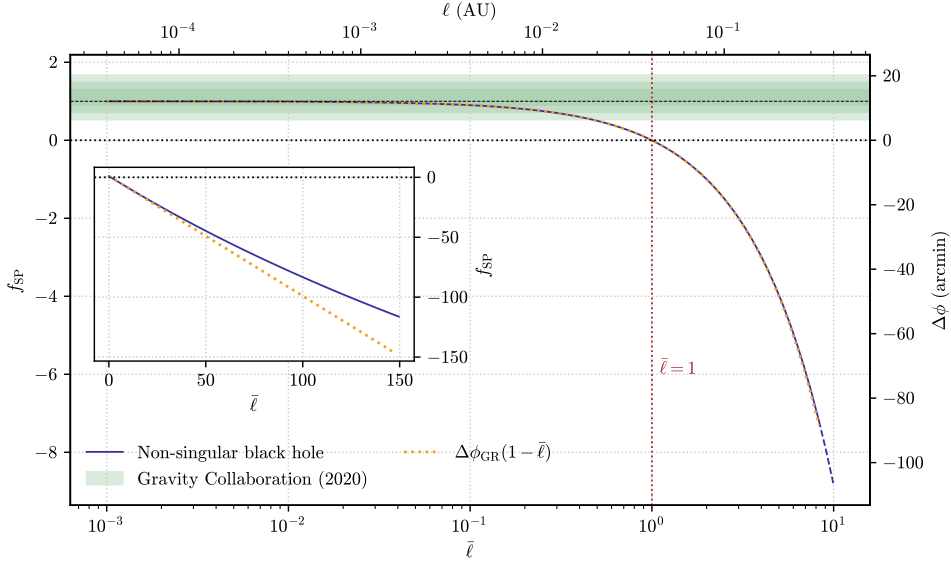


Figure 4.2. Orbital precession for the S2 star in the non-singular black hole spacetime in Eq. (4.1) as a function of the hair $\bar{\ell}$ between $\sim 10^{-3}$ and $\sim 10^1$ (the top axis reports the actual physical length of ℓ in astronomical units). In particular, the right axis reports the orbital precession over one orbital period, while the left axis reports its ratio, f_{SP} to the 1PN general relativistic value in Eq. (2.41). The green shaded horizontal strips report the 1, 2 and 3σ confidence regions on f_{SP} as measured by the Gravity Collaboration in 2020 [226] and reported in Table 1.5. The inset plot extends the estimation of the orbital precession up to $\bar{\ell} \sim 150$ to exhibit the progressive departure of the first-order analytical formula for the orbital precession reported in Eq. (4.9) from the actual numerically computed value. The discrepancy starts to be important for $\bar{\ell} > 10$, while the two predictions perfectly agree with each other for a smaller hair.

Uniform prior		
	Start	End
$\bar{\ell}$	0	5

Table 4.1. Uniform prior on the parameter $\bar{\ell}$ used for our Bayesian analysis.

Assessing the astrometric implications of the non-singular black hole spacetime in Eq. (4.1) on the geodesic trajectory of the S2 star and its rate of orbital precession provides a preliminary insight into potential constraints on the hair $\bar{\ell}$. To establish actual constraints, we conducted a posterior analysis using publicly available data. Specifically, spectroscopic and astrometric data from [138], covering the pre-2017 orbit. This dataset was supplemented with information on the orbital precession measured from [226]. The Bayesian analysis methodology, as outlined in Appendix C, was performed adopting the likelihood function detailed in Eq. (C.3).

The orbital model for S2 involves a total of 14 parameters, encompassing the 13 parameters of Eq. (C.1) (comprising orbital parameters and parameters addressing systematics in the astrometric and spectroscopic reference frame), together with the hair ℓ that, along with central object mass M fixes a unique geometry for the spacetime. The priors utilized for

Parameter (unit)	Posterior
D (kpc)	8.24 ± 0.22
M ($10^6 M_\odot$)	4.23 ± 0.25
t_p (yr)	2018.379 ± 0.024
a (arcsec)	$0.1249^{+0.0011}_{-0.0010}$
e	0.8828 ± 0.0024
i ($^\circ$)	$134.42^{+0.48}_{-0.49}$
Ω ($^\circ$)	$226.75^{+0.83}_{-0.82}$
ω ($^\circ$)	$65.23^{+0.78}_{-0.77}$
α_0 (mas)	0.26 ± 0.16
δ_0 (mas)	$-0.04^{+0.19}_{-0.20}$
$v_{\alpha,0}$ (mas/yr)	$0.071^{+0.053}_{-0.052}$
$v_{\delta,0}$ (mas/yr)	0.092 ± 0.062
$v_{\text{LOS},0}$ (km/s)	-3.4 ± 4.5
ℓ (AU)	$\lesssim 0.019$ (95% c.l.)
$\bar{\ell}$	$\lesssim 0.47$ (95% c.l.)

Table 4.2. Credible intervals on all the parameters of the 14-dimensional geodesic orbital model for S2, resulting from our posterior analysis in the non-singular black hole spacetime in Eq. (4.1). In particular, all the orbital (rows 1 to 8) and reference frame (rows 9 to 13) parameters are bound and their 68% (1σ) credible intervals are compatible with their known counterpart in literature [138]. The posterior for ℓ , reported in the highlighted bottom rows both in physical units and in dimensionless units, results in a 95% confidence level upper bound on this parameter.

this analysis incorporate a combination of those presented in Table C.1, along with heuristic priors—motivated by the aforementioned analysis of the astrometric impact—on the parameter ℓ , as detailed in Table 4.1.

The outcomes of our MCMC analysis are detailed in Table 4.2. Median values along with 68% confidence level intervals are provided for all parameters. Specifically, Keplerian orbital parameters and the reference frame parameters exhibit bounded posterior intervals, consistent within 1σ with previous results from the same dataset [138] and the fiducial values outlined in Appendix C. The full posterior distribution, along with the marginalized posterior for the parameter ℓ , are reported as a corner plot in Figure 4.3. From our analysis, we can establish an upper limit on the hair, $\ell \lesssim 0.019$ AU at the 95% confidence level, corresponding to $\bar{\ell} \lesssim 0.47$. Interestingly, while this result largely restricts the allowed range for the non-singular black hole hair, it cannot completely rule out horizonless solutions and still allows for the existence of thermodynamically stable regular black holes, *i.e.* models with $0.245 \leq \bar{\ell} \leq 0.296$. Our results show that regular black holes with super-Planckian hair are not excluded by the S2 star observational data. The actual proof of the existence of

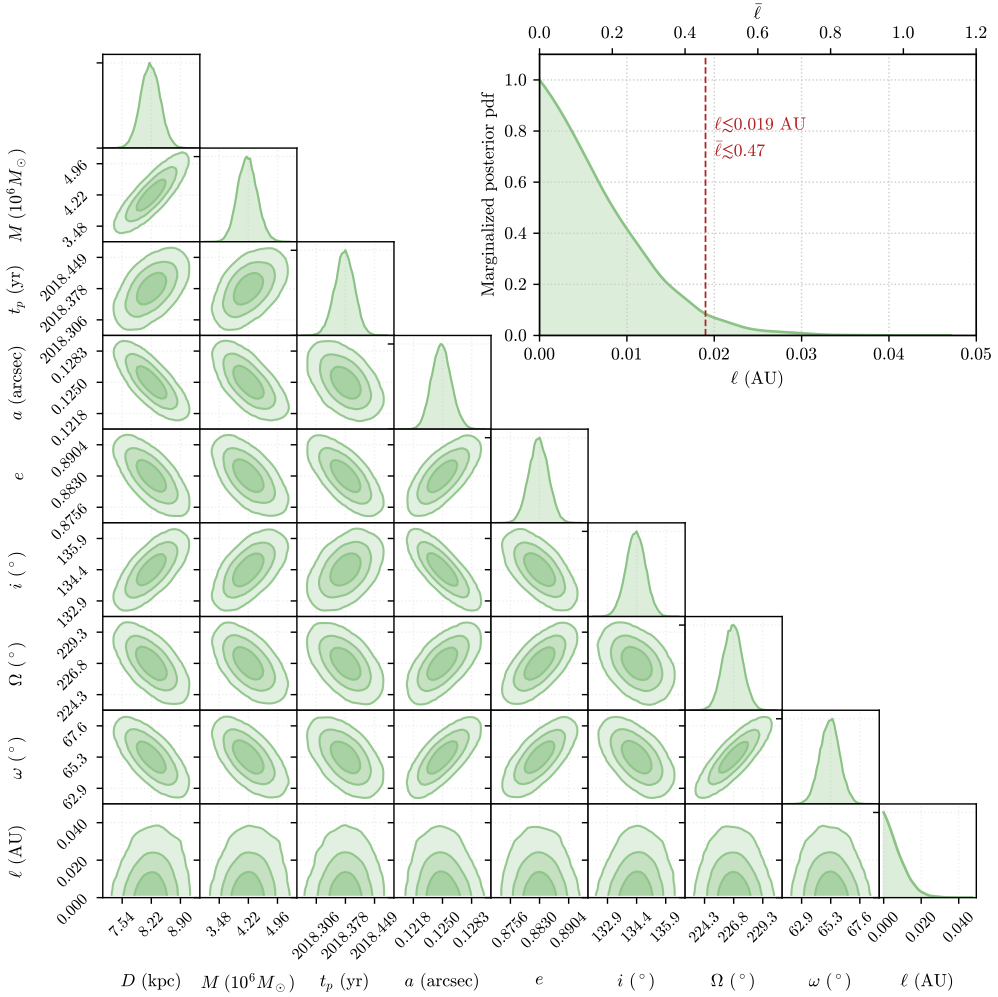


Figure 4.3. Corner plot reporting a 9-dimensional slice of the posterior probability density function with the parameters of interest (we decide to exclude for clearness the 5 astrometric reference frame parameters from the visualization. Their posterior is perfectly compatible with the one reported in Figure C.3 and the reference frame parameters exhibit no significant correlation with ℓ) for the S2 star in the non-singular black hole spacetime. The inset top-right-corner plot shows the marginalized posterior probability density function on the parameter ℓ , both in physical units and in its dimensionless expression. The vertical dashed red line reports the 95% confidence level upper limit on ℓ .

our nonsingular black holes requires the measurement of ℓ -dependent deviations from General Relativity, which could be detected by observations at the light-ring scale, e.g., with the black hole shadow (see for example the results of [14]) or gravitational wave experiments.

4.2. Black Bounce

In the context of candidate spacetimes for black hole mimicker geometries the Black Bounce metric refers to the family of solutions to the field equation of General Relativity, first introduced by A. Simpson and M. Visser in 2019 [495], and described by the line element

$$ds^2 = - \left(1 - \frac{2M}{\sqrt{\xi^2 + \alpha_{\text{BB}}^2}} \right) dt^2 + \left(1 - \frac{2M}{\sqrt{\xi^2 + \alpha_{\text{BB}}^2}} \right)^{-1} d\xi^2 + (\xi^2 + \alpha_{\text{BB}}^2)(d\theta^2 + \sin^2 \theta d\phi^2). \quad | \quad 4.10$$

This static and spherically symmetric one-parameter family of solutions is tuned by the parameter α_{BB} and employs a coordinate system, (t, ξ, θ, ϕ) , that while sharing the same definition for the coordinate time, t , and the angular coordinates, θ and ϕ , with the usual Schwarzschild coordinate system illustrated in Section 2.2.1, presents important differences regarding the radial coordinate ξ . This coordinate is defined in the domain

$$\xi \in (-\infty, +\infty), \quad | \quad 4.11$$

and to better appreciate the difference that it presents with the usual radial coordinate in the Schwarzschild spacetime, let us consider the 2-dimensional line element $ds_{(2)}^2 = (\xi^2 + \alpha_{\text{BB}}^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ on hypersurfaces with $\{t = \text{const.}; \xi = \text{const.}\}$. This hypersurface represents a 2-sphere whose radius, $r^2(\xi) = \xi^2 + \alpha_{\text{BB}}^2$, is a function of the radial coordinate ξ . When taking the surface area, $A = \int ds_{(2)}^2 = 4\pi r^2$, of the sphere, one can immediately regard the function $r(\xi)$ as the standard definition of the aerial radius, as in the standard Schwarzschild spacetime. While setting $\alpha_{\text{BB}} = 0$ reduces Eq. (4.10) precisely to the Schwarzschild metric in Eq. (2.18) and ξ to r , the interesting region of the parameter space that we consider is for $\alpha_{\text{BB}} > 0$. Here, α_{BB} represents the so-called *throat* of the geometry, $r_{\text{th}}^2 = \alpha_{\text{BB}}^2$, which constitutes an absolute minimum for the aerial radius obtained for $\xi = 0^4$. This feature allows for a smooth interpolation between a Schwarzschild black hole and a two-way traversable wormhole in the Morris-Thorne sense [450]. This can be seen by looking at the causal structure of the geometry, *i.e.* by explicitly computing the roots of the g_{00} term (that in the standard black hole geometries identify the radial locations of the event horizon(s) of the geometry).

⁴ This minimum is usually called *bounce*, hence the name of this model.

In the Black Bounce case, one has

$$g_{00} = 1 - \frac{2M}{\sqrt{\xi^2 + \alpha_{\text{BB}}^2}} = 0 \quad \Rightarrow \quad \xi_H = \pm \sqrt{4M^2 - \alpha_{\text{BB}}^2}, \quad |4.12$$

thus implying that an horizon (and thus a black hole behavior of the solution) can exist if and only if the condition $\alpha_{\text{BB}} \leq 2M$ is satisfied. In this case, for each value of α_{BB} in this range, the geometry has two horizons, one on each side of the throat $\xi = 0$. The case $\alpha_{\text{BB}} = 2M$, hence, represents a limiting scenario that divides the family of the Black Bounce solutions into intrinsically different geometries. In particular:

- As mentioned, for $\alpha_{\text{BB}} = 0$, $\xi_H = \pm 2M$ and the geometry reduces to the usual Schwarzschild black hole with mass M and with a genuine curvature singularity at $\xi = r = 0$;
- For $0 < \alpha_{\text{BB}} < 2M$, the metric possesses both an horizon and a non-null throat (corresponding to a space-like hypersurface), but $r^2(\xi_H) = 4M^2 > r_{\text{th}}^2$ and hence, the bounce is “hidden” behind the horizon. This solution corresponds to a regular black hole geometry [460, 496, 497]. As a matter of fact, in [495] both the non-null components of the Riemann tensor and several curvature invariants have been computed, showing that all these terms present in their denominator powers of $(\xi^2 + \alpha_{\text{BB}}^2)$ and are thus only maximized but remain non-diverging in the $\xi \rightarrow 0$ limit⁵, thus the geometry is truly regular for $\xi = 0$. The Carter-Penrose diagram for this class of solutions is depicted in Figure 4.4;
- For $\alpha_{\text{BB}} = 2M$, the horizon and the throat coincide, meaning that the geometry represents a one-way traversable wormhole, in which the throat $\xi = 0$ is a null hypersurface and thus corresponds at the same time with an event horizon [495]. Particles can thus cross the throat from one side (even though the null character of the throat implies that it will take an infinite amount of time to cross it from the point of view of any stationary exterior observer) and emerge at a past null horizon on the other side (a future reincarnation of the universe

⁵ In [495] it is also highlighted that, since the coordinate basis employed in Eq. (4.10) is orthogonal, the (mixed) non-zero components of the Riemann tensor are the same with respect to this basis as to any orthonormal tetrad. This guarantees that the appearance or lack of any singularities is not simply a coordinate artifact but a general property of the geometry.

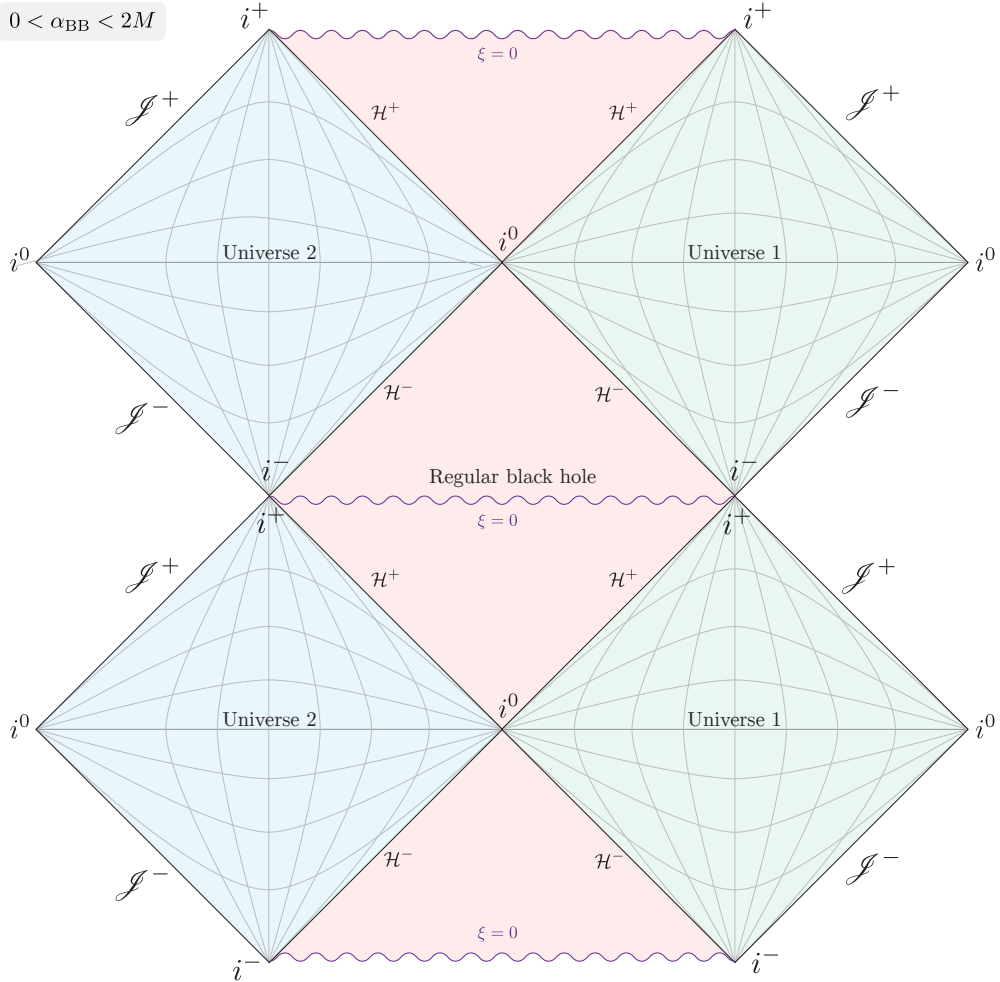
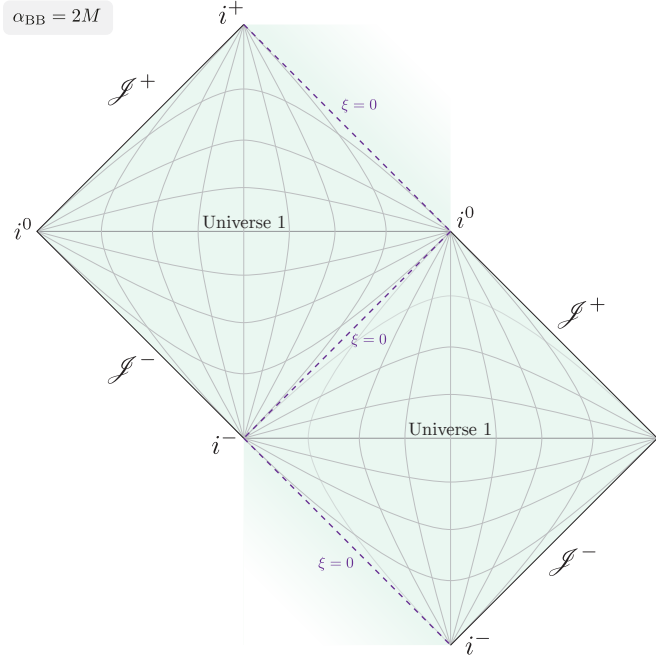


Figure 4.4. Carter-Penrose diagram for the maximal extension of the regular black hole case, $0 < \alpha_{\text{BB}} < 2M$. The throat $\xi = 0$ is a space-like hypersurface with aerial radius $r_{\text{th}} = \alpha_{\text{BB}}$, which lies behind event horizons located on each side of the throat at $\xi_H = \pm \sqrt{4M^2 - \alpha_{\text{BB}}^2}$, corresponding to locations in the two separate universes corresponding to the regions $\xi > 0$ (Universe 1, our universe) and $\xi < 0$ (Universe 2). The event horizons are crossable null hypersurfaces that give access the regular black hole region. After that, an in-going particle is forced to go through the throat (exactly as it would be forced to go towards the singularity in a Schwarzschild black hole) and to emerge in a future incarnation of either Universe 1 or 2, crossing the corresponding past horizon. Infinitely many future (and past) copies of the universe exist, all connected at the throats $\xi = 0$.

Figure 4.5. Carter-Penrose diagram for the maximally extended spacetime in the case when $\alpha_{\text{BB}} = 2M$. In this case the geometry represents a one-way wormhole geometry, in which the throat $\xi = 0$ is a null hypersurface and thus corresponds at the same time with the event horizon. This impedes ingoing particles to cross from Universe 1 ($\xi > 0$) to Universe 2 ($\xi < 0$) and makes it so that a particle that crosses the horizon at $\xi = 0$ bounces into a copy (also identified as a future incarnation) of the Universe from which it was originating.



from which it originated). The Carter-Penrose diagram for this class of solutions is depicted in Figure 4.5;

- Finally, for $\alpha_{\text{BB}} > 2M$, the geometry is free of horizons and the bounce, corresponding to a time-like hypersurface, is exposed on both sides, meaning that the geometry represents a two-ways traversable wormhole [450, 451, 468, 469] that connects two separate regions of spacetime. A Universe 1, labeled by $\xi > 0$ coordinates and a Universe 2, for $\xi < 0$. Particle trajectories originating on each side of the throat can cross (possibly multiple times) the throat in either direction. The Carter-Penrose diagram for this class of solutions is depicted in Figure 4.6.

Alongside the formulation of the spacetime metric for the Black Bounce model, in [495] the energy-momentum tensor sourcing this spacetime is computed. Namely, from the geometry in Eq. (4.10) the curvature tensor and scalar are computed and plugged into the left-hand-side of the general relativistic Einstein field equations, Eq. (2.9). This yields the right-hand-side energy-momentum tensor for the source of this spacetime. From this computation, it is straightforward to evaluate the energy conditions. In particular, in [495] it is shown that

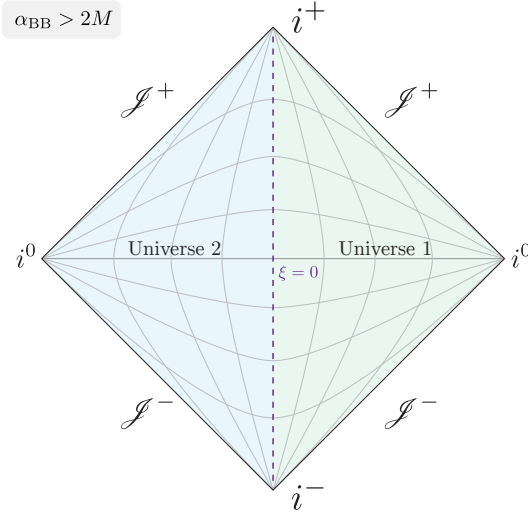


Figure 4.6. Carter-Penrose diagram for the case with $\alpha_{\text{BB}} > 2M$, corresponding to traditional traversable wormhole in the Morris-Thorne sense. No horizon is present in this geometry and the throat $\xi = 0$ is time-like, and can thus be crossed from both sides and in both directions. The throat thus represents a connection between two separate regions of spacetime: Universe 1 (for $\xi > 0$) and Universe 2 (for $\xi < 0$).

the Null Energy Condition (NEC) [243, 334, 498] is violated both outside and inside any possible horizon the geometry might present (for any solution with $\alpha_{\text{BB}} > 0$), pointing that only exotic forms of matter can source the Black Bounce spacetime. Demonstrating the NEC violation is sufficient to conclude that also the weak, strong, and dominant energy conditions will be violated, a condition that is shared by many traversable wormhole spacetimes [471, 499, 500].

Classical energy conditions all start from the assumption that reasonable forms of matter sources to the gravitational field should have energy densities, as measured by any observer in spacetime, that are nonnegative. In particular, the NEC implies that, for any arbitrary future-directed null vector k^μ , the energy-momentum tensor satisfies the relation

$$T_{\mu\nu} k^\mu k^\nu \geq 0.$$

In the case of a perfect fluid, this implies that

$$\rho + p_i \geq 0.$$

4.2.1. Geodesic motion in the Black Bounce spacetime

Freely-falling test particles in the Black Bounce spacetime follow geodesic trajectories derived by applying Eq. (2.3) to the metric in Eq. (4.10). Such equations read:

$$\ddot{t} = - \frac{2M\xi\dot{\xi}}{\Xi(\xi)(\alpha_{\text{BB}}^2 + \xi^2)^{3/2}}, \quad | \quad 4.13$$

$$\ddot{\xi} = \frac{M\xi(\dot{\xi}^2 - \dot{t}^2\Xi^2(\xi))}{\Xi(\xi)(\alpha_{\text{BB}}^2 + \xi^2)^{3/2}} - \Xi(\xi)(\xi\dot{\phi}^2 \sin^2 \theta + \xi\dot{\theta}^2) \quad | \quad 4.14$$

$$\ddot{\theta} = - \frac{2\xi\dot{\theta}}{\alpha_{\text{BB}}^2 + \xi^2} + \frac{\dot{\phi}^2 \sin(2\theta)}{2}, \quad | \quad 4.15$$

$$\ddot{\phi} = - \frac{2\xi\dot{\phi}\dot{\xi}}{\alpha_{\text{BB}}^2 + \xi^2} - 2\dot{\phi}\dot{\theta} \cot(\theta). \quad | \quad 4.16$$

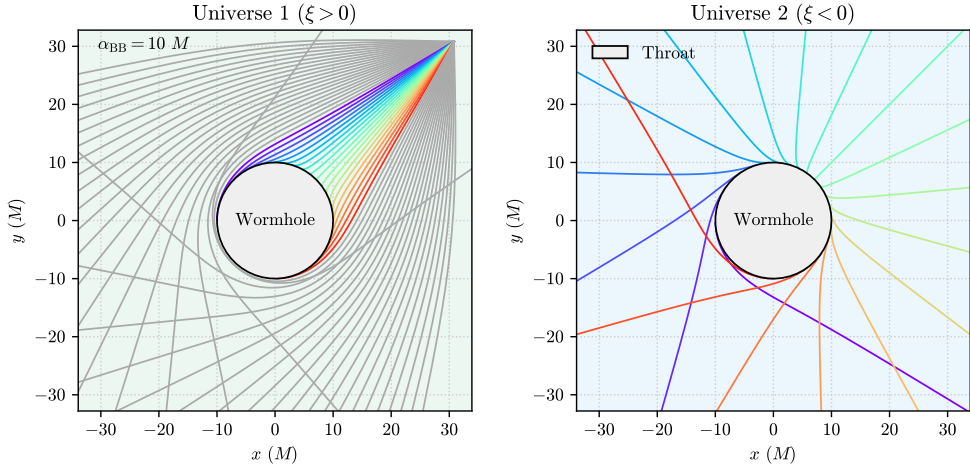


Figure 4.7. Example null geodesic trajectories (*i.e.* photon paths) for the case $\alpha_{\text{BB}} = 10M$, corresponding to a two-way traversable wormhole of throat $r_{\text{th}} = 10M$ (reported as a gray circle in the origin of the axes). The left panel depicts the region of spacetime corresponding to the equatorial plane of the Black Bounce metric in the Universe 1, *i.e.* those points in spacetime with $\xi > 0$. Conversely, the right panel shows the equatorial plane *on the other side* of the throat, corresponding to coordinates $\xi < 0$, *i.e.* Universe 2. The pseudo-cartesian coordinates xy used in this visualization are built by using the aerial radius $r(\xi)$ as the usual radius in spherical coordinates. A series of null trajectories originate at $\xi = 30\sqrt{2}M$ and $\phi = 45^\circ$ on the equatorial plane ($\theta = 0$) with different impact parameters and integrated numerically. The resulting trajectories can be divided into two families: *unbound* geodesics (with impact parameter that is large enough that the minimum radius is greater than the throat radius r_{th}) that propagate to infinity in Universe 1; *plunging* geodesics that fall into the wormhole and emerge in Universe 2, in which they propagate to infinity.

where we have defined

$$\Xi(\xi) \equiv 1 - \frac{2M}{\sqrt{\alpha_{\text{BB}}^2 + \xi^2}}. \quad | \quad 4.17$$

The standard procedure to fix initial conditions, described in Section 2.2.3, is applied to both null and time-like geodesics to derive example solutions in the $\alpha_{\text{BB}} > 2M$ (wormhole) case. Indeed, in the domain $\alpha_{\text{BB}} \leq 2M$, the phenomenology of falling particles mostly matches the Schwarzschild case, illustrated in Section 2.2.2, and consists of either plunging, bound and unbound orbits. Of course, the resulting geodesic trajectory will depend on the specific value of α_{BB} in the range $[0, 2M]$, but the general classification of the geodesics will not be altered. The case is completely different for the wormhole scenario, in which, plunging orbits are not ‘captured’ by an event horizon. Indeed, in this scenario, the throat, which is a time-like hypersurface (see Figure 4.6), can be crossed from both sides and in both directions and this always takes a finite amount of coordinate time. This means that null trajectories

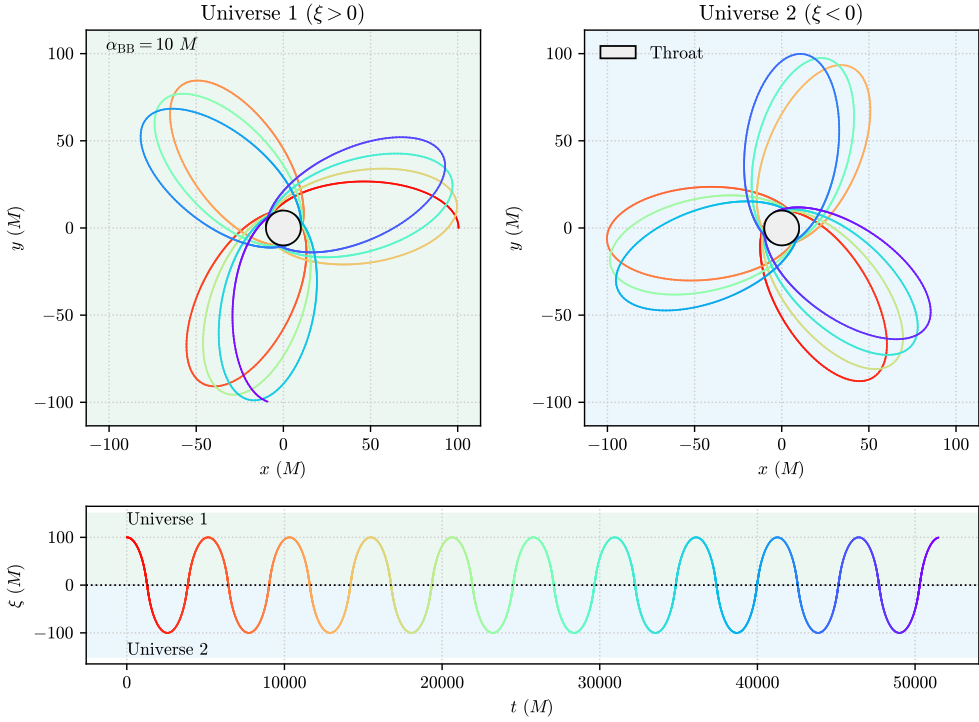


Figure 4.8. Example time-like geodesic trajectory for the same two-way traversable wormhole with $\alpha_{\text{BB}} = 10M$ studied in Figure 4.7. The time-like geodesic evolves on the equatorial plane of the Black Bounce spacetime starting at $\xi = 100M$ with an initially-tangent velocity of $\sim 3\%$ of the speed of light in vacuum. The same orbital configuration would correspond to a plunging geodesic in the Schwarzschild case (Section 2.2.2), *i.e.* a trajectory that ends at the horizon for an observer at any point in Universe 1 and takes an infinite coordinate time to cross it. In the Black Bounce wormhole, on the other hand, the throat is not a null hypersurface, but a time-like one (Figure 4.6) and thus the particle is free to cross it in both directions and this always takes a finite amount of coordinate time (the bottom plot reports the radial coordinate ξ as a function of coordinate time t , which is also used as a color code on the trajectory evolution). The resulting orbit, thus oscillates periodically between Universe 1 and Universe 2 (*i.e.* the radial coordinate is periodically alternating between positive and negative values), while precessing on the orbital plane. This means that the particle takes a full orbit in each of the Universes before plunging into the throat and continuing its motion in the other Universe, repeatedly.

(example geodesics for this case are shown in Figure 4.7), depending on their impact parameter can either be unbound, *i.e.* be deflected by the central source but remain on the same side of the throat (implying that their coordinate ξ always has the same sign), or plunge into the throat and emerge on the other Universe where they propagate to infinity (in this case the ξ coordinate along the geodesic changes sign at the throat crossing). Bound configurations for time-like geodesics exhibit peculiar features connected to the wormhole nature of the central object. As we show for an example configuration in Figure 4.8, plunging time-like geodesics in the wormhole case

correspond to a bound trajectory that oscillates periodically between Universe 1 and Universe 2. In other words, the radial coordinate is periodically alternating between positive and negative values, while the orbit experiences orbital precession on its plane. As a result, the particle takes a full orbit in each of the Universes before plunging into the throat and continuing its motion in the other Universe, repeatedly.

4.2.2. Constraining the nature of Sgr A* with S2

In this section, we use publicly available data for the S2 star to probe the nature of the central object. In particular, by fitting the orbit derived from the geodesic equations in Eqs. (4.13)-(4.16) to the astrometric data, our goal is to constrain the parameter α_{BB} in the Black Bounce metric that allows us to discern between a black hole and a wormhole nature of the gravitational source.

Preliminarily, we assess the orbital deviation from a purely Schwarzschild orbit on the S2 star astrometric and spectroscopic observables arising from the modification of the geodesic equation in the Black Bounce metric. In particular, from the integrated orbit we reconstruct observable quantities following the procedure described in Section 2.2.3 and we compare the sky-projected orbits and line-of-sight velocities to the ones obtained in the $\alpha_{\text{BB}} = 0$ Schwarzschild case. We do this for values of α_{BB} in the range $0.5M - 100M$, with a particular emphasis on the $\alpha_{\text{BB}} = 2M$ case that constitutes the separation between a black hole and wormhole nature for the central object. In Figure 4.9 we report the deviations on the two sky-projected directions and on the line-of-sight velocity for the considered values of α_{BB} . Horizontal dashed lines report to the mean astrometric uncertainty, at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and the nominal astrometric precision of the GRAVITY interferometer [170], at $10 \mu\text{as}$, respectively. As it appears from the plots, the maximum astrometric departure between the Schwarzschild black hole ($\alpha_{\text{BB}} = 0$) and the one-way traversable wormhole ($\alpha_{\text{BB}} = 2M$) cases (that is reached in the orbital apocenter), which amounts to $\sim 0.1 \mu\text{as}$ for the first orbital period after the 2018 pericenter passage and to $\sim 0.5 \mu\text{as}$ for the second, is always well below the sensitivity thresholds of both instruments. The same goes

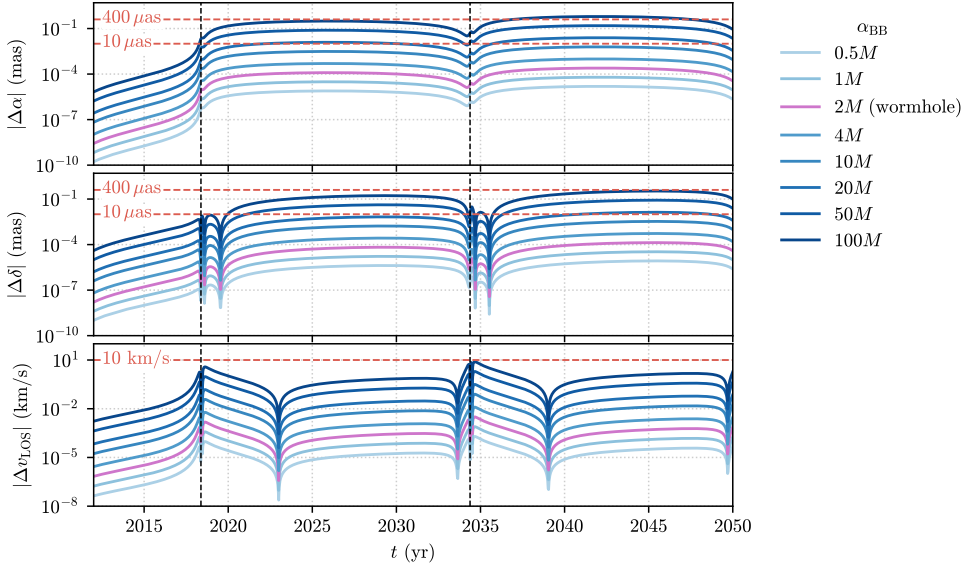


Figure 4.9. Astrometric (right ascension α and declination δ , top two panels) and spectroscopic (line-of-sight velocity, bottom panel) impact of the Black Bounce metric on the orbit of the S2 star in the Galactic Center, computed for different values of the parameter $\alpha_{\text{BB}} \in [0.5, 100]M$ over two entire orbits after the 2018 pericenter passage (this and the next pericenter passage are highlighted by vertical dashed lines). Horizontal dashed lines correspond to the mean astrometric uncertainty, at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and the nominal astrometric precision of the GRAVITY interferometer [170], at $10 \mu\text{as}$, respectively. In the bottom plot, the 10 km/s sensitivity threshold for the SINFONI interferometer is reported. The pink highlighted lines for $\alpha_{\text{BB}} = 2M$ correspond to the case of a one-way traversable wormhole that represents the separation between the family of black hole and wormhole geometries in the family.

for the spectroscopic observable whose maximum departure from the Schwarzschild case is produced around the pericenter passages (due to the additional gravitational redshift) and amounts to only a few m/s in the reconstructed line-of-sight velocity for the one-way traversable wormhole case. Nevertheless, the plots show that orbital deviations that can surpass the nominal instrumental thresholds are predicted for two-way traversable wormholes with larger throats. This analysis suggests that, while the S2 orbit seems to be insensible to the fundamental nature of Sgr A* (either black hole or wormhole), an upper limit on the radius of a possible throat in the geometry could arise from the analysis of observational data for S2 alone.

To further substantiate this result, we have also investigated the impact of the Black Bounce geometry on the relativistic orbital precession. In particular, fixing all the orbital parameters for S2 to the ones resulting from the Gravity Collaboration in

Table 1.1 and by only changing the parameter α_{BB} in the range $[10^{-2}, 10^3]M$, we have computed the rate of orbital precession $\Delta\phi$ numerically for the Black Bounce geodesics, applying the same procedure illustrated in Section 2.2.3. In Figure 4.10 we compare the predicted rate of orbital precession for the Black Bounce case and the reference value $f_{\text{SP}} = 1$ obtained in the Schwarzschild spacetime. We obtain values that are perfectly compatible with unity (and are well within the observational limits imposed by the Gravity Collaboration on f_{SP} [226]) for all values of $\alpha_{\text{BB}} \lesssim 10M$, meaning that, as anticipated from the analysis of the astrometric observables in Figure 3.2, not even the orbital precession can discern between a Schwarzschild black hole and a wormhole geometry for the S2 star. However, the rate of orbital precession starts to be sensible to values of α_{BB} between $10M$ and $100M$, for which a greater rate of periastron advance is predicted ($f_{\text{SP}} > 1$), due to the increased size of the throat.

Uniform prior		
	Start	End
$\alpha_{\text{BB}} (M)$	0	100

Table 4.3. Uniform prior on the parameter α_{BB} used for our Bayesian analysis.

The analysis of the astrometric impact of the Black Bounce spacetime on the geodesic motion of the S2 star gives a rough idea of what possible constraints on α_{BB} can be derived using the orbital data of S2. To obtain actual constraints, we proceeded to the posterior analysis with publicly available data (*i.e.* spectroscopic and astrometric data from [138] covering the pre-2017 portion of the orbit) complementing them with the information on the orbital precession. We have thus followed the Bayesian analysis methodology presented in Appendix C, adopting the likelihood reported in Eq. (C.3). Our orbital model depends on a total of 14 parameters, given by the 13 parameters reported in Eq. (C.1) (consisting of the orbital parameter and the parameters introduced to account for the systematics in the astrometric and spectroscopic reference frame), plus the parameter α_{BB} that represents our parameter of interest. The priors used for this analysis are given by a combination of the priors illustrated in Table C.1 with heuristic priors (motivated by the analysis on the astrometric impact shown above) on the parameter α_{BB} , reported in Table 4.3.

The results of our MCMC analysis are reported in the second column of Table 4.4. Median values and 68% confidence level intervals around it are reported for all the parameters. In particular, Keplerian orbital parameters and the reference frame parameters are bounded and their posterior intervals are compatible within 1σ with previous results on the same dataset [138] and to the fiducial values reported in Appendix C. The

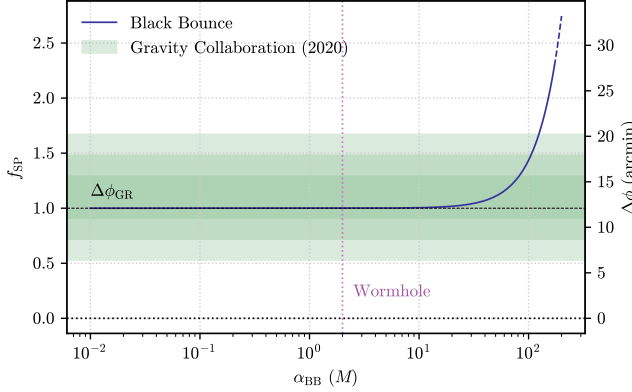
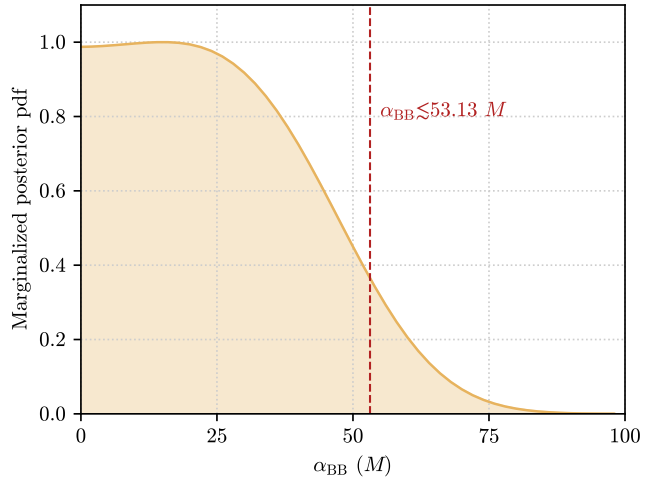


Figure 4.10. Orbital precession for the S2 star in the Black Bounce spacetime as a function of α_{BB} between $\sim 10^{-2}M$ and $\sim 10^{-3}M$. In particular, the right axis reports the orbital precession over one orbital period, while the left axis reports its ratio, f_{SP} to the 1PN general relativistic value in Eq. (2.41). The green shaded horizontal strips report the 1, 2 and 3 σ confidence regions on f_{SP} as measured by the Gravity Collaboration in 2020 [226] and reported in Table 1.5.

marginalized posterior distribution on α_{BB} , on the other hand, that we report for the sake of completeness in Figure 4.11, allowed us to place an upper limit on the Black Bounce throat $\alpha_{\text{BB}} \lesssim 53.15M$ at 95% confidence level. As expected, using currently publicly available data for S2, we are not able to distinguish between a black hole and a wormhole (both natures fall well within our credible interval for the parameter α_{BB}). Nevertheless, it is remarkable that we are able to exclude two-way traversable wormholes whose throat is characterized by a radius $\alpha_{\text{BB}} \gtrsim 53M$, at 2 σ confidence.

As we have shown, currently available data for the S2 star are incapable of distinguishing between the black hole nature and the wormhole one for Sgr A*. This is unsurprising given the relatively poor astrometric accuracy of such observations, notably when compared to the precision provided by the GRAVITY interferometer. The current observational accuracy is evidently insufficient to discern the subtle changes in the geodesic motion of the star resulting from the distinct fundamental natures of Sgr A*. We have already shown in Figure 4.9 that an instrument with the GRAVITY accuracy would be sensible to much smaller values of α_{BB} with respect to NACO observations. Moreover, a better astrometric accuracy would also improve the accuracy of the measurement of the orbital precession, thus reducing even further the allowed range α_{BB} in Figure 4.10. We can thus anticipate that the upper limit placed on α_{BB} in the previous paragraph could significantly improve with a more precise dataset. Moreover, observations of an entire orbit of S2 from the GRAVITY instrument would not only yield much smaller astrometric error bars but would also eliminate the need for introducing the offset and drift

Figure 4.11. Marginalized posterior probability density function on the parameter α_{BB} resulting from our MCMC analysis on publicly available astrometric and spectroscopic data from [138], complemented with the measurement of the orbital precession for S2 from [226]. The vertical dashed line corresponds to the 95% confidence level upper limit on α_{BB} .



reference frame parameters required for the NACO observations. Consequently, we would have superior data quality and a reduced dimensionality of the parameter space for constraints. It is hence a natural question to ask whether such an improvement would be able to unambiguously identify the nature of the central object. In order to assess whether this is the case or not, we have performed an additional posterior analysis, by using the mock catalogue of future observations of S2 illustrated in Section 2.2.3, mimicking the GRAVITY accuracy over two entire orbits. We have employed the same priors used in the previous analysis but adopted the likelihood in Eq. (C.5) on the dataset reported in Appendix B. Since the mock catalogue has been generated by assuming a Schwarzschild black hole nature for Sgr A*, our *true* reference value for this parameter is $\alpha_{\text{BB}} = 0$. Thus, from the posterior distribution of the parameter α_{BB} resulting from our analysis we can establish the ability of the upcoming observations with the GRAVITY/SINFONI instruments to rule out the wormhole nature for Sgr A*. In Figure 4.12 we report the full posterior distribution of the 9-dimensional parameter space of this analysis, highlighting its 1σ , 2σ and 3σ contours and the posterior of each parameter. The inset plot reports the marginalized posterior distribution for the parameter α_{BB} . The vertical dashed blue line, marks its inferred 95%-confidence upper limit, while the red one denotes the one-way traversable wormhole, separating the two classes of natures for Sgr A*. As it appears from our analysis, reaching an astrometric precision as that nominally provided by the GRAVITY instrument, over

two entire orbits of the star, is not yet sufficient to distinguish between the black hole nature and a wormhole scenario for the central source in the Galactic Center, being able to only provide with an upper limit $\alpha_{\text{BB}} \lesssim 8.13M$ and thus excluding wormholes with a larger throat at 95% confidence. Significantly, this would represent an improvement of a factor ~ 10 compared to the results with currently available data.

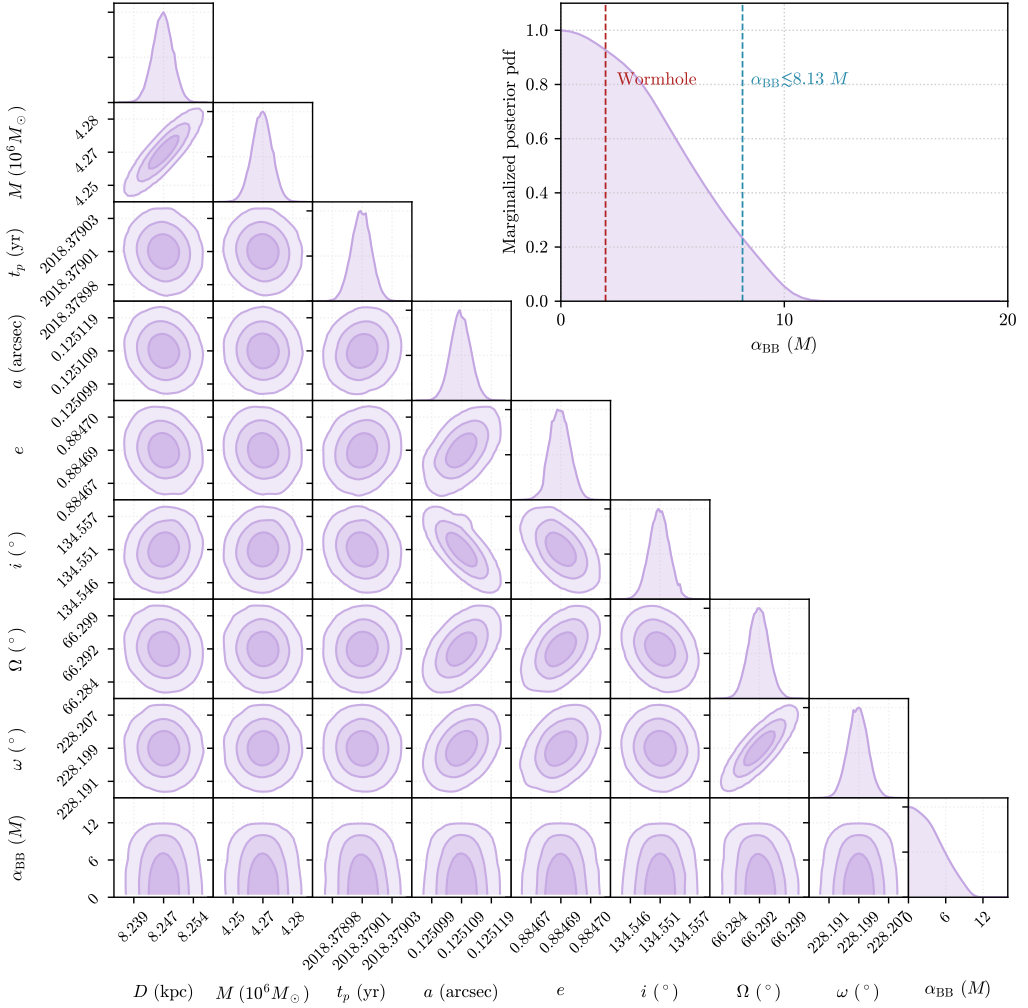


Figure 4.12. Full posterior distribution of the 9-dimensional parameter space of the Black Bounce orbital model for S2 from the MCMC analysis on mock observations. The inset plot zooms on the marginalized posterior distribution for α_{BB} . In particular, the blue dashed line corresponds to the 95% upper limit on α_{BB} , while the red dashed line reports the separation between the black hole and wormhole classes of solutions.

Table 4.4. Results from the MCMC analyses for the Black Bounce model from orbital data of the S2 star in the Galactic Center (second column) and from the forecast analysis using future mock observations of the same object through the GRAVITY interferometer over two orbital periods (third column). In particular, we report the 68% credible intervals around the median for all the parameters of the model, except for α_{BB} (highlighted in the last row), for which we report a 95% confidence upper limit. From the comparison between the two columns, the improvement yielded by the increased accuracy of GRAVITY observations is evident on all the parameters, and especially on α_{BB} .

Parameter (unit)	Posterior (data + precession)	Posterior (mock)
D (kpc)	8.11 ± 0.22	8.2469 ± 0.0025
M ($10^6 M_{\odot}$)	4.10 ± 0.24	$4.2663^{+0.0039}_{-0.0038}$
t_p (yr)	2018.363 ± 0.024	$2018.3790063^{+0.0000088}_{-0.0000089}$
a (arcsec)	0.1254 ± 0.0011	$0.1251092^{+0.0000033}_{-0.0000032}$
e	0.8832 ± 0.0024	0.8846881 ± 0.0000052
i ($^{\circ}$)	134.07 ± 0.50	134.5512 ± 0.0018
Ω ($^{\circ}$)	$227.06^{+0.79}_{-0.80}$	228.1991 ± 0.0027
ω ($^{\circ}$)	65.44 ± 0.75	66.2916 ± 0.0024
α_0 (mas)	-0.30 ± 0.16	-
δ_0 (mas)	0.09 ± 0.19	-
$v_{\alpha,0}$ (mas/yr)	$0.076^{+0.052}_{-0.053}$	-
$v_{\delta,0}$ (mas/yr)	0.132 ± 0.064	-
$v_{\text{LOS},0}$ (km/s)	-2.8 ± 4.5	-
α_{BB} (M)	$\lesssim 53.15$ (95% c.l.)	$\lesssim 8.13$ (95% c.l.)

4.2.3. Constraints from putative closer stars

In Section 4.2.2, our analysis exclusively focused on the S2 star. This choice is justified as, up to the present date, S2 stands out as the most extensively studied star in our Galactic Center, both in terms of observational accuracy and the detectability of relativistic effects on its orbit. Nevertheless, our analysis has undoubtedly demonstrated that neither the publicly available data nor future observations for S2 utilizing the GRAVITY instrument, which represents the most precise tool available today for astrometry, can effectively constrain the parameter space of a the Black Bounce geometry, described in Eq. (4.10). Consequently, distinguishing between the black hole nature and that of a wormhole remains challenging.

However, as mentioned in Chapter 1, there is a prospect that stars orbiting closer to Sgr A* than S2 may provide more accurate insights into the gravitational field generated by the central object. Such stars have been hypothesized [173, 174, 176, 501], and their orbits could potentially offer a higher

degree of sensitivity to relativistic effects. In this section, we explore several toy models, selecting orbital parameters to produce increasingly closer pericenter passages. This deliberate choice results in orbits exhibiting a more pronounced deviation when comparing black hole and wormhole geometries. Specifically, we considered three orbital periods of nine, five, and three years, respectively. For each of these periods, we fixed the orbital eccentricity at either 0.98 or 0.99. The semi-major axis (a) is determined by Kepler's third law through the relation: $a = \sqrt[3]{GMT^2/4\pi^2}$, and the aerial radius of the star at pericenter can be estimated as $r_p = a(1 - e)$ (that we then converted to the radial coordinate ξ). For convenience, we set the time of the next pericenter passage for all stars to 2025.00, without loss of generality. This choice is permissible because the metric coefficients in Eq. (4.10) do not explicitly depend on the coordinate time t , making the problem symmetric with respect to time translations. Thus, regardless of the specific value of t_p , the same conclusions would be drawn.

The three angular orbital parameters (i, Ω, ω) are set to zero, ensuring that the orbit remains parallel to the sky-plane. This alignment facilitates the prominence of the astrometric shift of the star resulting from its periastron advance, especially in the right ascension direction. Essentially, the orbits always start at pericenter with the semi-major axis positioned vertically with respect to the α - δ reference system. Consequently, as the orbit's periastron advances, the noticeable shift primarily occurs in the α direction, allowing us to focus on that specific direction. Additionally, since the orbit presents no inclination with respect to the sky-plane of a distant observer, there is no need to account for the Rømer delay. While this geometrical configuration is rather peculiar and putative closer stars in the Galactic Center might have any inclination with respect to the line-of-sight, we stick to this simple case to maximize the astrometric deviations. The full set of orbital parameters chosen for the considered toy models is reported in Table 4.5

The numerically integrated, sky-projected orbits for these toy models are illustrated in Figure 4.13. The figure displays both the orbit for the black hole model (depicted as a blue line) and the one for a wormhole with $\alpha_{\text{BB}} = 2M$ (represented as pink dots). Additionally, Keplerian orbits are shown as grey lines. While the figure highlights the significant effect of periastron advance, indicated by the non-closed appearance of the orbits (in contrast to their Keplerian counterparts), at this scale, no

Table 4.5. The orbital parameters for toy model stars orbiting Sgr A*. In particular, for each star we report the orbital period and the semi-major axis (related through the third Kepler’s law by considering the value of the central mass fixed), the aerial radius at pericenter and the eccentricity. For all the toy models the three angular orbital parameters (i , Ω , ω) have been set to zero and, additionally, the mass and distance of Sgr A* are fixed to the ones from [226].

Model	T (yr)	a (AU)	r_p (AU)	e
Toy star 1	9	701	14.0	0.98
Toy star 2	5	474	9.5	0.98
Toy star 3	3	337	6.7	0.98
Toy star 4	9	701	7.0	0.99
Toy star 5	5	474	4.7	0.99
Toy star 6	3	337	3.4	0.99

observable departure is evident between the black hole and wormhole solutions.

To enhance the visibility of this departure, Figure 4.14 focuses exclusively on the right ascension direction. It illustrates deviations in this direction of the geodesic in the regular black hole case (for $\alpha_{\text{BB}} = M$), in the one-way traversable wormhole case ($\alpha_{\text{BB}} = 2M$) and in the two traversable wormhole case ($\alpha_{\text{BB}} = 3M$) from the geodesic in the Schwarzschild black hole case ($\alpha_{\text{BB}} = 0$). The shaded region corresponds to the nominal astrometric sensitivity threshold of the GRAVITY interferometer. The figure clearly shows that the orbital motion of all our toy stars, over multiple orbital periods in some cases, eventually gains enough astrometric shift to be potentially detected through observations. This suggests that such objects could serve as probes to differentiate between the wormhole and black hole nature of Sgr A*. As anticipated, the most significant departure is observed at apocenter, and depending on the orbital parameters, this departure surpasses the detectability threshold either after one orbit (for Toy stars 3, 4, 5, and 6) or after multiple orbits (Toy stars 1 and 2).

Assuming observations cover only one entire orbital period (from one apocenter to the next one), we focused on a specific toy model, namely Toy star 5, and constructed various mock catalogues. These catalogues explored different observational strategies to assess how the wormhole nature could be ruled out (assuming, as done in the forecast analysis for the S2 star, that the true value is $\alpha_{\text{BB}} = 0$). Given that the most significant astrometric discrepancy between the orbits of the star around a black hole and a wormhole occurs at apocenter, we propose that observations should be particularly dense around this region to capture the most substantial deviation. This assumption is made under the condition that no extended

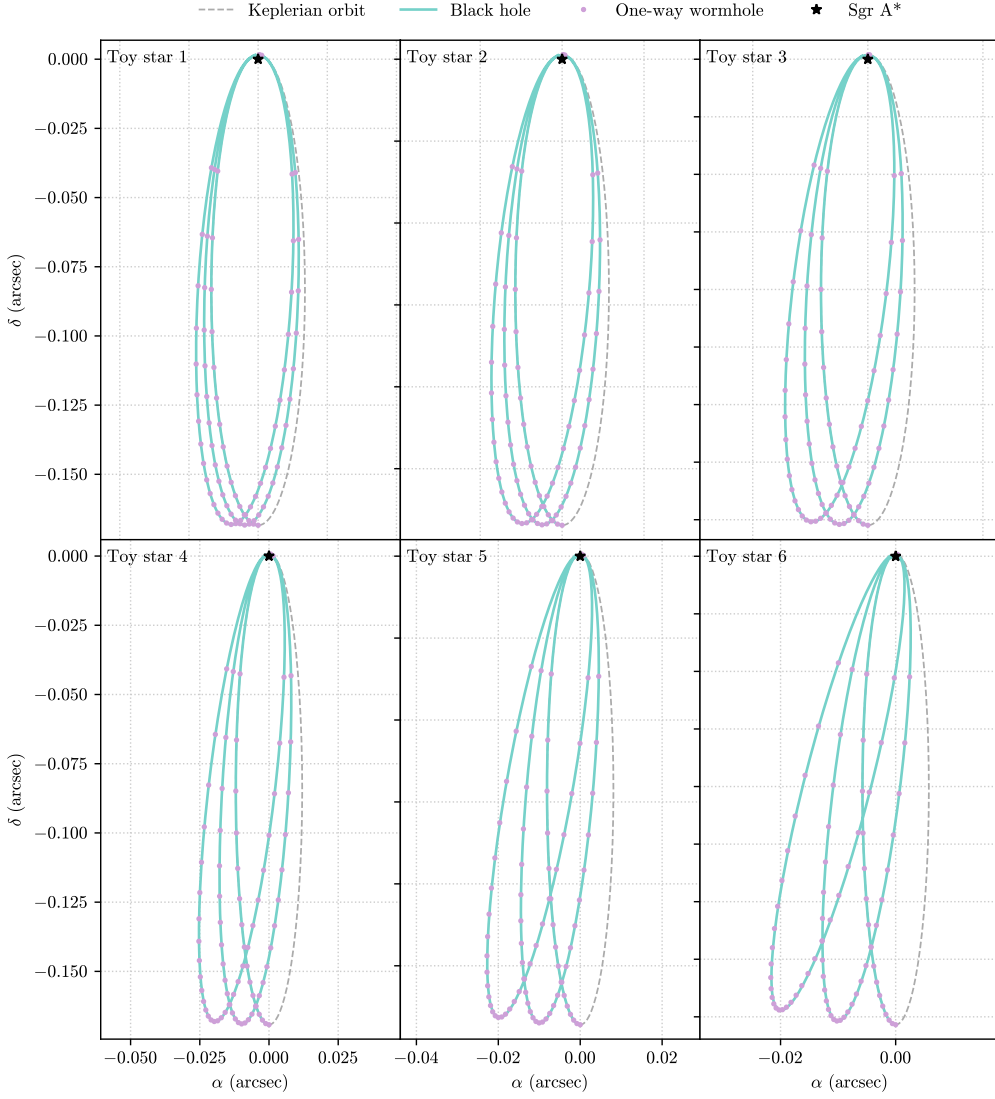


Figure 4.13. Orbital motion of the toy models listed in Table 4.5. The gray lines report the corresponding Keplerian orbit. Geodesic trajectories are reported for a Schwarzschild black hole ($\alpha_{\text{BB}} = 0$, turquoise line) and for a one-way traversable wormhole ($\alpha_{\text{BB}} = 2M$, pink dots).

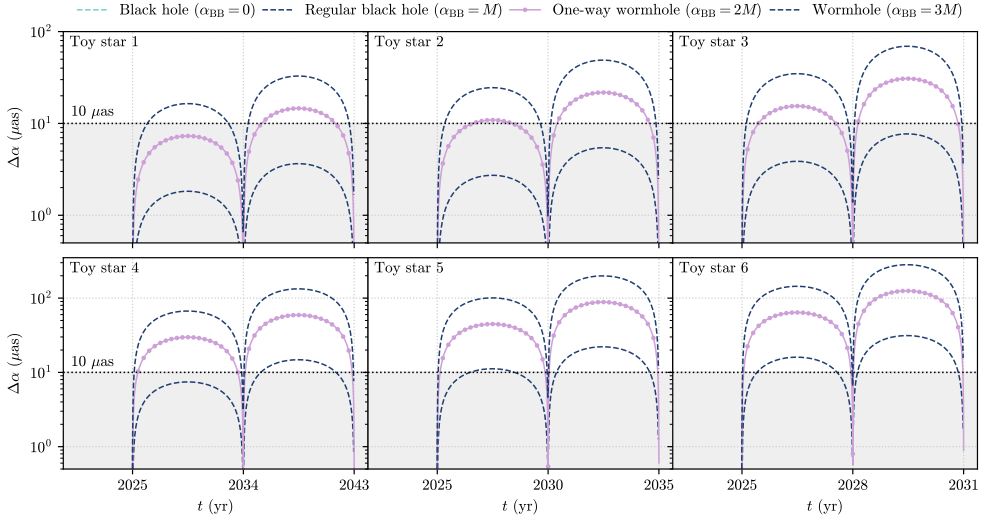


Figure 4.14. A focus on the departure between the geodesic orbit around a Schwarzschild black hole (which we take as a reference at level 0) and one way traversable wormhole (the pink line) in the right ascension direction. The gray shaded area depicts the nominal astrometric accuracy of the GRAVITY interferometer, showing that observations with GRAVITY can actually be able to detect departures between the two geometries for the toy model orbits that we have considered. that is more noticeable as the orbital parameters gets more *extreme*. We also report the geodesic orbit (dashed and dotted blue lines, respectively) for a regular black hole with $\alpha_{\text{BB}} = M$ and a two-way traversable wormhole for $\alpha_{\text{BB}} = 3M$.

mass distribution is present around Sgr A*, whose mass precession effect on the periastron advance would dominate over the general relativistic orbital precession in the apocenter region [242] (see also Chapter 3). Furthermore, observations at apocenter should be complemented with observations at pericenter and throughout the rest of the orbit to properly constrain the orbital period of the star, its eccentricity, and its time of pericenter passage. Different observational strategies are characterized by a varying number of data points (and hence nights of observations) in different regions of the orbit. In particular, we divided the orbit of the toy star into three regions.

These are reported in Figure 4.15 and correspond to:

- Region $\mathcal{A}$ is centered on the star's pericenter and is defined by the points along the orbit whose eccentric relativistic anomaly $\mathcal{E}$ is within 60° of the argument of pericenter. We have already introduced the relativistic anomaly, χ in Eq. (2.40), defined in such a way that the radial coordinate ξ has a maximum (apocenter) for $\chi = \pi$ and a minimum (pericenter) for $\chi = 0$. This means

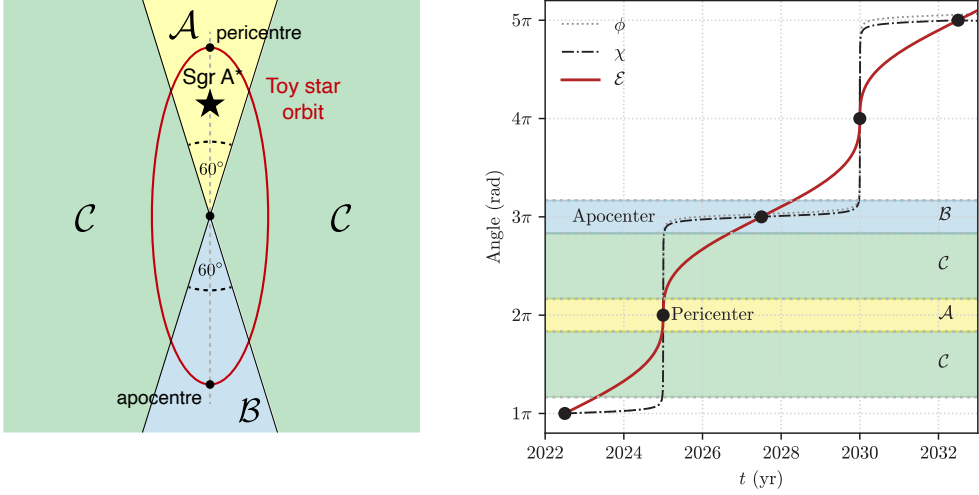


Figure 4.15. *Left panel:* We have divided the orbit into three regions: region $\mathcal{A}$ (in yellow) is centered on the pericenter of the orbit and has an (eccentric anomaly) amplitude of 60° ; region $\mathcal{B}$ (light blue) has the same amplitude of $\mathcal{A}$ but is centered on the star’s apocenter; region $\mathcal{C}$ (in green) covers the rest of the orbit. The red line must be imagined as an ellipse following the star during the periastron advance and the angles representing the relativistic eccentric anomaly $\mathcal{E}$. *Right panel:* the actual values of the different angles along the integrated orbit. The red line represents the relativistic eccentric anomaly $\mathcal{E}$, the black dash-dotted line the relativistic anomaly χ , while the gray dotted line is the true anomaly ϕ of the star (which gradually piles up a shift with respect to χ , due to the orbital precession).

that the angle χ describes positions on the geodesic as the ellipse precesses. From this, we can compute the *eccentric relativistic anomaly* $\mathcal{E}$ by changing the origin of the angles from the central mass (located in one of the ellipse’s foci) to the center of the ellipse. This is necessary if we want to select angles symmetrically with respect to a fixed point on the star’s relativistic path. The relation between the anomaly and the eccentric anomaly is given by:

$$\tan \frac{\chi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{\mathcal{E}}{2}. \quad | \quad 4.18$$

In Figure 4.15, we provide a visual illustration of the different regions and the angular coordinates used.

- Region $\mathcal{B}$ is constructed with the same idea as Region $\mathcal{A}$ but is centered on the star’s apocenter ($\chi = \pi$).
- Region $\mathcal{C}$ covers the rest of the orbit.

We have constructed six mock catalogues of observations (following the same procedure used for the S2 star) by changing

Table 4.6. We report the number of observations that are performed in the regions of the toy star’s orbit that we have defined in Figure 4.15 for the six different mock catalogues that we have built.

Mock catalogue	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
M1	5	0	6
M2	10	0	20
M3	10	10	10
M4	20	20	20
M5	35	20	20
M6	45	20	20

the number of observation nights in each orbital region. The chosen observation counts aim to evaluate how the boundaries on the parameter α_{BB} improve or worsen depending on the observational strategy and on possible limitation to the number of available observations (*e.g.* selection effects, confusion, etc.). These are outlined in Table 4.6. In particular, the M6 dataset represents the best-case scenario, featuring the same number of observations per region as the mock catalogue built for S2 in section 2.2.3. The other mock catalogues have fewer observations in each region, reflecting the greater observational challenge posed by a close and fast-moving star like Toy Model 5 compared to S2.

Mock catalogues M5 and M4 have progressively fewer points at pericenter while maintaining the same number of observations in regions $\mathcal{B}$ and $\mathcal{C}$. In M3, we have halved the number of observations in each region compared to M4, resulting in a total of 30 observations over an entire orbit. Considering the maximum astrometric deviation between a black hole and a wormhole model occurs around apocenter (as shown in Figure 4.14), M2 maintains a total of 30 observations, but shifts the observations from region $\mathcal{B}$ to region $\mathcal{C}$. In the most drastic scenario, M1, we have reduced the total number of observations by approximately 1/3, leaving only 5 data points around pericenter and 6 in the rest of the orbit, with no observations in region $\mathcal{B}$.

For each of these mock catalogues, we performed an MCMC analysis, similar to what was done in section 4.2.2. We allowed all nine parameters of our orbital model to vary, assigning uniform large priors centered on the true values used to build the catalogues (as listed in Table 4.7). The credible intervals on all the parameter resulting from our posterior analyses are reported in the same table. Additionally, Figure 4.16 illustrates the marginalized posterior for the parameter α_{BB}

Table 4.7. A summary of the priors and best-fit values from our MCMC analysis on the different mock catalogues that we have built for the Toy star 5. More specifically, column 2 reports the true value of the fit parameters, i.e. the one used to build the mock catalogue; columns 3 and 4 report the upper and lower limits of the uniform distributions that we used as priors; column 5 to 7 report the best fitting values resulting from the MCMC posterior analysis with their 1σ credible intervals and 5σ upper confidence limit for the parameter α_{BB} , also reported in Figure 4.16.

Parameter	Uniform prior			M1	M2	M3
	True	Start	End			
D (kpc)	8.246	7	9	8.249 ± 0.023	8.246 ± 0.012	8.248 ± 0.011
T (yr)	5	4	6	5.0007 ± 0.0062	5.0000 ± 0.0023	5.00004 ± 0.00049
$t_p - 2025$ (yr)	0	2024	2026	0.0000008 ± 0.0000076	0.0000004 ± 0.0000056	0.0000004 ± 0.0000057
a (AU)	474	450	500	474.1 ± 1.4	474.12 ± 0.72	474.06 ± 0.66
e	0.99	0.98	1	0.990002 ± 0.000035	0.990001 ± 0.000019	0.990002 ± 0.000017
i ($^\circ$)	0	-10	10	0.06 ± 0.33	0.02 ± 0.21	0.01 ± 0.20
ω ($^\circ$)	0	-10	10	0.10 ± 0.56	0.04 ± 0.45	0.06 ± 0.59
Ω ($^\circ$)	0	-10	10	0.08 ± 0.58	0.08 ± 0.46	0.08 ± 0.61
$\alpha_{\text{BB}} (M)$	0	0	20	$\lesssim 3.22 (5\sigma)$	$\lesssim 2.52 (5\sigma)$	$\lesssim 2.22 (5\sigma)$
				M4	M5	M6
D (kpc)	8.246	7	9	8.2467 ± 0.0053	8.2469 ± 0.0049	8.2466 ± 0.0049
T (yr)	5	4	6	5.00004 ± 0.00034	5.00000 ± 0.00032	5.00005 ± 0.00032
$t_p - 2025$ (yr)	0	2024	2026	0.0000007 ± 0.0000039	0.0000003 ± 0.0000030	0.0000001 ± 0.0000027
a (AU)	474	450	500	474.03 ± 0.31	474.01 ± 0.29	474.01 ± 0.28
e	0.99	0.98	1	0.9900001 ± 0.0000064	0.9900003 ± 0.0000061	0.9900008 ± 0.9900060
i ($^\circ$)	0	-10	10	0.0011 ± 0.0095	0.0006 ± 0.0048	0.0007 ± 0.0038
ω ($^\circ$)	0	-10	10	0.08 ± 0.41	0.06 ± 0.35	0.01 ± 0.35
Ω ($^\circ$)	0	-10	10	0.06 ± 0.42	0.04 ± 0.36	0.05 ± 0.35
$\alpha_{\text{BB}} (M)$	0	0	20	$\lesssim 1.92 (5\sigma)$	$\lesssim 1.46 (5\sigma)$	$\lesssim 1.28 (5\sigma)$

and the upper 1σ , 3σ , and 5σ bounds obtained for the different observational strategies. As expected, the constraining power on the parameter α_{BB} of our datasets increases, leading to a significant improvement in the upper bounds on α_{BB} from the worst to the best-case scenario. In particular, the data in M1 can only exclude the wormhole nature of the central object at 1σ significance, with the limit $\alpha_{\text{BB}} = 2M$ well within both the 3σ and 5σ credible intervals. Both M2 and M3 were able to exclude the wormhole nature at 3σ significance, but the 5σ upper limit on α_{BB} is above the $\alpha_{\text{BB}} = 2M$ limit and slightly improves between M2 and M3 due to the additional information from observations recorded at apocenter. M4 is the first dataset to exclude at more than 5σ the wormhole nature of Sgr A*, with M5 and M6 increasingly improving this result.

Our analysis, thus, suggests that in order to distinguish at

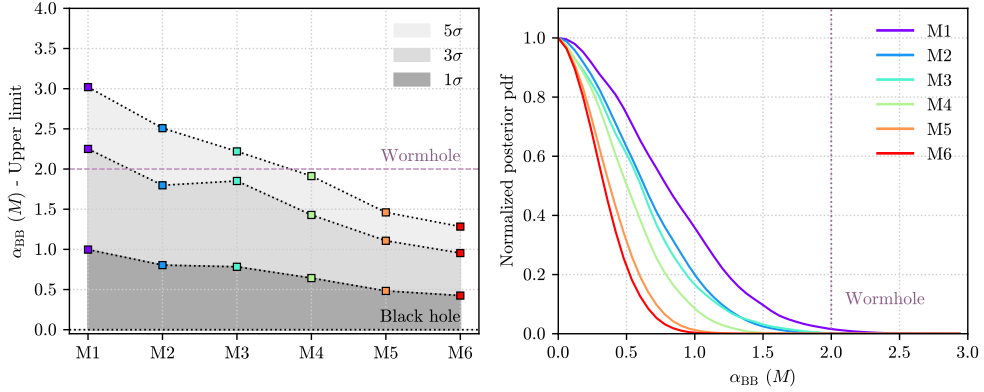


Figure 4.16. In the left panel we report the 1 σ , 3 σ and 5 σ upper bounds on the parameter α_{BB} resulting from the posterior distributions (that we report on the right panel all normalized to have maximum value to unity), as a function of the observational strategies in Table 4.6 for the Toy star 5 in Table 4.5.

a 5 σ significance level the black hole nature to that of a wormhole for Sgr A*, a uniform astrometric coverage of at least 20 observations per orbital region is required for a 5-year-period highly-eccentric ($e = 0.99$) star in the Galactic Center, maintaining a nominal astrometric accuracy of 10 μas over the entire observational campaign.

4.3. Conclusions

In this chapter we have analyzed the viability of probing black hole mimicker geometries as an alternative model to describe Sgr A*, relying solely on the orbital motion of the S-stars and on the detection of the relativistic effects for these objects. We have explored two interesting proposals in the context of regular black holes.

First, we have analyzed a hairy black hole endowed with a super-Planckian hair, ℓ , that smears the central singularity, which in this geometry is replaced by a dS core. While many different possible spacetime metrics can be formulated that respect this requirement, we have focused on a model that results in the strongest deviations from the Schwarzschild black hole in the near-horizon region, while retaining the same asymptotic behavior (see Eq. (4.1)). Interestingly, this model has an astrophysical analogy since it resembles known dark matter profiles. We have studied the orbital motion of a massive test particle in this spacetime, particularizing it to

the precession of the S2 star, showing how strongly it can constrain the value of the hair ℓ . We have then performed a MCMC analysis using publicly available orbital data for S2 and the measurement of the orbital precession after the last pericenter passage in 2018 to get a posterior probability distribution on ℓ . As anticipated, the parameter ℓ is tightly constrained and our posterior analysis sets an upper limit $\ell \lesssim 0.019$ AU ($\bar{\ell} \lesssim 0.47$) at 95% confidence level. While the results is remarkable, as it shows that the orbital dynamics of S2 are sensible to a scale length of the black hole hair that is as small as 0.001% of S2 semi-major axis, it doesn't allow to completely exclude horizonless objects in the studied family of non-singular black holes.

Similar conclusions were derived from our analysis of the Black Bounce model. This geometry is particularly interesting as the only additional free parameter, α_{BB} , in the geometry allows to smoothly interpolate between very different geometries. Depending on the value of α_{BB} , the line element in Eq. (4.10) either describes a Schwarzschild black hole, a regular black hole, a one-way traversable wormhole or a two-way traversable wormhole in the Morris-Thorne sense. Being able to establish observational constraints on α_{BB} , would thus allow to constrain the fundamental nature of Sgr A*. Our analysis, however, has demonstrated that neither the currently available data for S2 nor future forecasted data would be able to conclusively rule out one of the possible geometries in the Black Bounce family. We have then analyzed what possible constraints on the model could come from the observations of putative stars closer to the Galactic Center. Testing several toy stars and devising several mock catalogues of potential observations, corresponding to different observational strategies for these stars, we found that, in order to distinguish at 5σ the black hole from wormhole for Sgr A*, a uniform astrometric coverage of at least 20 observations per orbital region is required for a 5-year-period highly-eccentric ($e = 0.99$) star in the Galactic Center, maintaining a nominal astrometric accuracy of $10 \mu\text{as}$ over the entire observational campaign.

Extended theories of gravity

5.

Extended Theories of Gravity (ETGs) try to address the limitations of General Relativity across both infrared and ultraviolet scales [502]. The motivation behind their formulation is their potential to naturally account for cosmological and astrophysical observations across various scales without requiring the introduction of exotic forms of energy and matter. General Relativity encounters challenges in accurately predicting the rotation curves of nearby galaxies [503], explaining velocity dispersion [504], and describing the strong and weak gravitational lensing observed in galaxy clusters [505–507]. On cosmological scales, General Relativity struggles to elucidate the formation of Large Scale Structures in the observable Universe, fails to accommodate the current accelerated expansion, and faces limitations in describing the early stages of the Universe within the framework of the Big Bang theory. To overcome such shortcomings, the inclusion in Einstein’s Field Equations of a cosmological constant Λ , representing a form of negative-pressure or another type of energy known as dark energy [508–510]. The existence of non-baryonic dark matter, as we have seen in Chapter 3, is postulated in order to address the discrepancies observed on galactic and galaxy cluster scales. The assumption that General Relativity is the correct description of gravity and the postulated existence of dark matter and dark energy constitute the Λ CDM model, considered nowadays the concordance cosmological model for the description of the Universe. However, dark matter has thus far eluded direct detection, while dark energy seems even more difficult to describe at a fundamental level.

ETGs offer a phenomenological framework that naturally accounts for these diverse observations, providing a cohesive explanation for the shortcomings of General Relativity across various scales. It is imperative that any extensions to General Relativity maintain its successful predictions in the weak field limit, particularly within the Solar System where stringent tests have validated its accuracy [301]. There are numerous methods for constructing a self-consistent theory of gravity that extends beyond General Relativity. In Figure 5.1, we provide a concise (though not exhaustive) overview of the plethora of possible extensions to General Relativity (for more

5.1	Scalar-Tensor-Vector Gravity	149
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The original scientific results presented in this chapter have been published in [1, 2, 5, 6, 8, 10, 12], and are presented here in chronological order of publication.

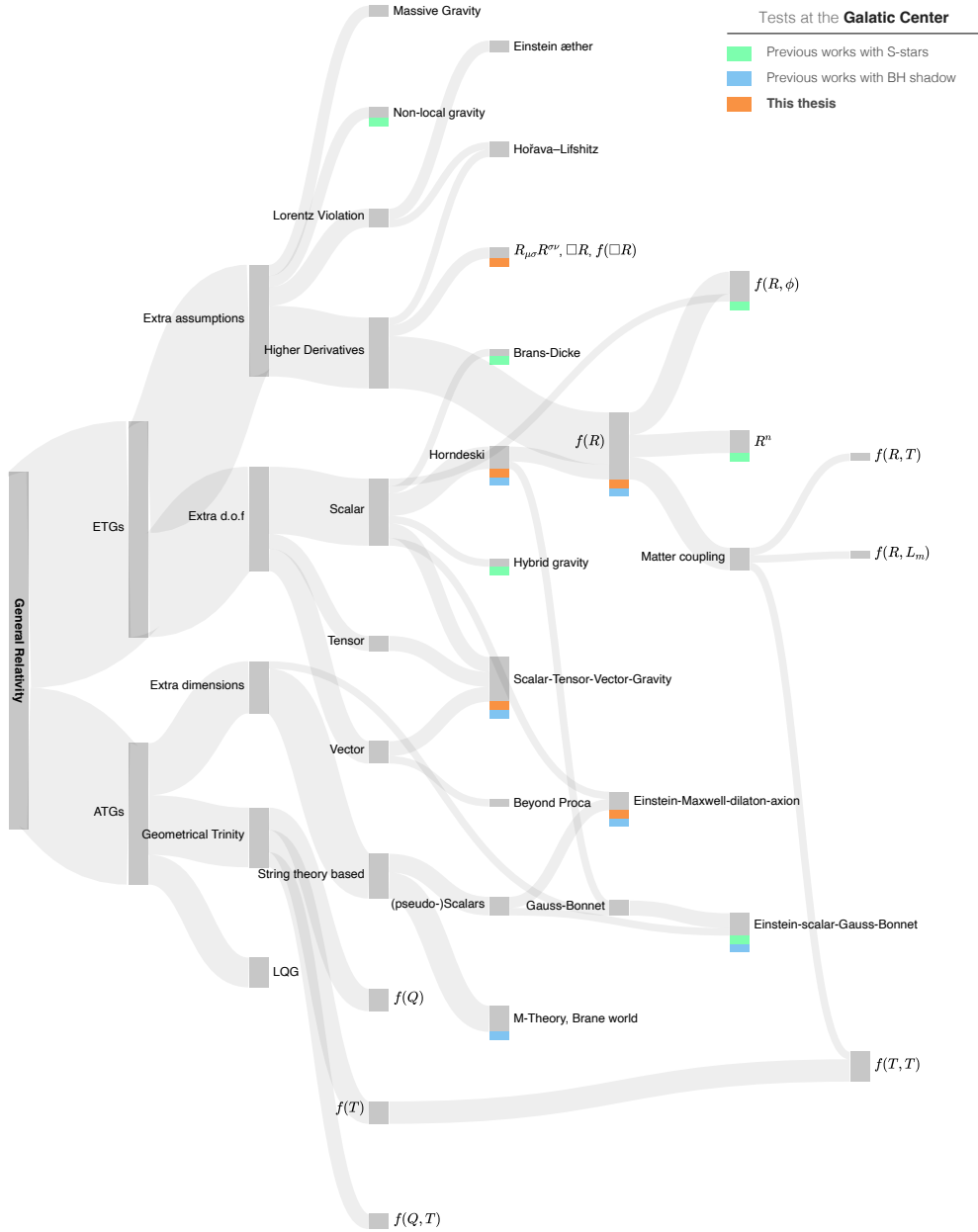


Figure 5.1. Diagram of families of modifications/extensions/alternatives to General Relativity organized by the assumptions from which they arise. Theories for which tests at the Galactic Center, either using the S-stars orbits or the shadow of Sgr A*, are present in literature have been highlighted with a green or a pink square, respectively. We highlight results presented in this thesis with orange squares.

comprehensive summaries, see [502, 511, 512]). One alternative is to change the foundational hypotheses of General Relativity, either by introducing additional assumptions or by violating existing ones. For example, massive gravity theories introduce a massive tensorial degree of freedom [513], while other theories assume that the gravitational interaction is considered as non-local [514]. Violation of Lorentz invariance [515] gives rise to theories where a preferred reference frame exists, such as Einstein-aether theory [516]. Hořava-Lifshitz gravity [517], introduces field equations with higher spatial derivatives [518]. More generally, theories can incorporate higher-curvature terms in the action, leading to higher-order field equations, as in $f(R)$ gravity or in theories containing terms such as $R_{\mu\nu}R^{\mu\nu}$, $\square R$, etc. [502]. Theories can also be postulated to include extra scalar, tensor or vector degrees of freedom in the action. In 1974, G.W. Horndeski derived the most general scalar-tensor theory resulting in second-order field equations [519], which includes a wide range of ETGs, sub-classes of $f(R)$ and Gauss-Bonnet gravity [520]. More recently, Scalar-Tensor-Vector Gravity (STVG) has been formulated to account for all possible additional degrees of freedom [521]. Such modifications to the General Relativity are introduced in the form of extra degrees of freedom (additional fields, higher-order terms in the action, etc.) without altering its fundamental geometrical formulation (metric + connection + curvature). However, Alternative Theories of Gravity (ATGs) based on distinct geometrical formulations have been proposed. While classical General Relativity describes gravity in terms of curvature on a torsion-free ($T = 0$, where T is the torsion tensor) metric ($Q = 0$, where Q is the non-metricity tensor) manifold, the same theory can be equivalently described in terms of either torsion ($T \neq 0$) or non-metricity ($Q \neq 0$) on a flat ($R = 0$) manifold. While these alternatives provide fully equivalent descriptions of gravity in General Relativity, forming the so called *Geometrical Trinity of Gravity* [522], ATGs have been developed to extend the theory from these equivalent descriptions. Examples include teleparallel $f(T)$ gravity, which modifies the teleparallel equivalent of the General Relativity action using the idea behind $f(R)$ modifications [523], $f(Q)$ theories that modify the non-metricity equivalent [524]¹, and $f(Q, T)$ theories where both entities are modified [526]. These theories, along with other ATGs such as string theory Brane-

¹ Which have been lately demonstrated to possess pathologies on cosmological scales [525], and thus are ruled out.

World gravity [527], exhibit different phenomenology, which has garnered considerable interest in cosmological scenarios [524, 528–532]. ATGs have been applied only in a limited number of cases to non-cosmological settings, with only a few tests existing in the Galactic Center [533, 534], demonstrating that observable features for compact object geometries arising in these models cannot currently be distinguished from those of a classical Schwarzschild black holes.

In this chapter, we primarily focus on the much broader literature available for ETGs. In particular, we have chosen several different well-motivated extensions of General Relativity and considered their spherically symmetric black hole solutions, that present one or more additional parameters modulating the deviation from General Relativity. For each of these models we have computed the geodesic equations and applied our methods, detailed in Chapter 2, to compute fully relativistic orbits in the Galactic Center (we focus primarily on the S2 star) and sets of astrometric and spectroscopic observables for this objects. We place observational constraints on the extra parameters of each model, through the application of our Bayesian procedure to either publicly available data for the S-stars or mock data, as explained in Appendix C. We provide a brief overview of each model that we have considered, the hypotheses behind their formulation, the expressions for the metric tensor for spherically symmetric solutions arising in them and the corresponding geodesic equations. We then focus on the deviations that each model induces on the astrometric observables for the S-stars in the Galactic Center and on the reconstructed relativistic effects. Finally, we present the results of our posterior analyses and the constraints derived on the extra parameters for each model. In particular, in Sections 5.1, 5.2 and 5.3 we study STVG, $f(R)$ and Horndeski gravities, respectively. All these theories fall into the definition of ETGs that we have given above and have been extensively studied in literature both theoretically and in different astrophysical and cosmological settings. In Section 5.4 we study Einstein-Maxwell-Dilaton-Axion (EMDA), a class of low-energy effective field theories originating from string theory and supergravity and that thus fall into the category of ATGs. Finally, in Section 5.5 we analyze a generic quadratic metric-affine extension of General Relativity endowed with an electric field, which produces a regular black hole solution in spherical symmetry.

5.1. Scalar-Tensor-Vector Gravity

As mentioned, considering extra scalar, tensorial, and vectorial degrees of freedom in the general relativistic action is a viable way of formulating meaningful extensions to General Relativity. This is the case, for example, of STVG². As its name suggests, this ETG, first presented in [521], considers the introduction of extra degrees of freedom in the description of the gravitational interaction in the form of scalar fields and a massive vector field ϕ^μ (along with metric tensor field $g_{\mu\nu}$). In particular, while the gravitational constant³ G and the mass μ of the vector field are elevated to the role of scalar fields, a vector field ϕ^μ , that couples with massive test particles (of mass m), is introduced. Considering all such ingredients, the generally covariant action can be written as [521]

$$\mathcal{S} = \mathcal{S}_{\text{GR}} + \mathcal{S}_{\text{m}} + \mathcal{S}_{\phi} + \mathcal{S}_S. \quad | \quad 5.1$$

The first two terms are the classical Hilbert-Einstein action of General Relativity, reported in Eq. (2.6), and the action for ordinary matter. The two additional terms, $\mathcal{S}_{\phi}$ and $\mathcal{S}_S$, encode the novel features of STVG, namely

$$\mathcal{S}_{\phi} = - \int d^4x \sqrt{-g} \left(\frac{1}{4} B^{\sigma\nu} B_{\sigma\nu} - \frac{1}{2} \mu^2 \phi^\nu \phi_\nu + V(\phi) \right), \quad | \quad 5.2$$

$$\begin{aligned} \mathcal{S}_S = \int d^4x \sqrt{-g} \frac{\omega_M}{G^3} & \left(\frac{1}{2} g^{\sigma\nu} \nabla_\sigma G \nabla_\nu G - V(G) \right) + \\ & + \int d^4x \frac{1}{\mu^2 G} \left(\frac{1}{2} g^{\rho\nu} \nabla_\rho \mu \nabla_\nu \mu - V(\mu) \right). \end{aligned} \quad | \quad 5.3$$

Here, g stands for the determinant of the metric tensor $g_{\mu\nu}$, ω_M is a constant, the vector field defines the Maxwell tensor $B_{\sigma\nu} := \nabla_\sigma \phi_\nu - \nabla_\nu \phi_\sigma$, and $V(\phi)$, $V(G)$ and $V(\mu)$ are scalar potentials arising from the self-interaction associated with the vector field and the scalar fields, respectively. From the minimization of the action Eq. (5.1), the field equations in

² It is worth mentioning that STVG has also been referred to as Modified Gravity (MOG) in literature, on several occasions.

³ Only within the scope of this Section, we will express the Newtonian value of the gravitational constant with G_N , while with the letter G we refer to its scalar field counterpart in STVG.

vacuum (*i.e.* $T_{\mu\nu}^m = 0$) for STVG can be derived [535]

$$G_{\mu\nu} = -\frac{\omega_M}{\chi^2} \left(\nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \chi \nabla_\sigma \chi \right) + \quad | \quad 5.4$$

$$-\frac{1}{\chi} (\nabla_\mu \chi \nabla_\nu \chi - g_{\mu\nu} \square \chi) + \frac{8\pi}{\chi} T_{\mu\nu}^\phi, \quad | \quad 5.5$$

where the scalar field $\chi = 1/G$, and $T_{\mu\nu}^\phi$ is the gravitational ϕ -field energy momentum tensor given by

$$T_{\mu\nu}^\phi = - \left(B_\mu^\sigma B_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} B^{\sigma\rho} B_{\sigma\rho} \right). \quad | \quad 5.6$$

One of the key features of STVG is that the presence of the vector field ϕ^μ , and its coupling to massive test particles, introduces a modification of the usual geodesics equations in Eq. (2.3)

$$\left(\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} \right) = \frac{q}{m} B^\mu{}_\nu \frac{dx^\nu}{d\lambda}. \quad | \quad 5.7$$

In particular, q acts as a coupling constant between the point-like particles of mass m and the vector field ϕ^μ , resulting in an extra Lorentz-type force, usually called fifth force, which depends on the velocity of the particle. This constant is postulated to be positive, implying a repulsive fifth force, in order to describe physically stable stars, galaxies, galaxy clusters, and agreement with solar system observational data [536–542] and its value is assumed to be proportional to the mass of the particle itself, $q = \kappa m$, so that the WEP can be recovered.

In [541], an explicit black hole solution in STVG was derived. Among the hypotheses that are involved in such derivation, G can be regarded as a constant ($\partial_\nu G = 0$) with a larger value than its Newtonian counterpart (which makes the constant ω_M in Eq. (5.3) be incorporated in the definition of G itself), which we refer to as G_N , encoded in an additional dimensionless parameter α_{STVG} ,

$$G = G_N (1 + \alpha_{\text{STVG}}). \quad | \quad 5.8$$

Moreover, it is postulated that the mass μ of the vector field can be neglected on the scales of compact objects⁴. From

⁴ More specifically, it is assumed that the vector field mass μ produces Yukawa-like modifications of the Newtonian gravitational potential, whose effects manifest on kpc scales from the source of the gravitational field, *i.e.* on galactic scales [538, 543] that can thus be ignored on compact object

these assumptions, a spherically symmetric spacetime metric is derived,

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2d\Omega^2, \quad | \quad 5.9$$

where

$$\Delta = r^2 - 2Mr + \alpha_{\text{STVG}}M((1 + \alpha_{\text{STVG}})M - 2r). \quad | \quad 5.10$$

This spacetime is formally equivalent to that of a Reissner-Nordström charged black hole [544] where in this case, the charge is the fifth-force charge of the central object and reduces to the usual Schwarzschild metric when $\alpha_{\text{STVG}} = 0$. Moreover, by matching the weak field dynamics in STVG to that of Newtonian dynamics [539], one can derive the proportionality constant κ in the geodesic equations, given by

$$\kappa = \sqrt{\alpha_{\text{STVG}}G_N}, \quad | \quad 5.11$$

so that, when α_{STVG} vanishes, both the metric and the geodesic equations in STVG reduce to their general relativistic counterparts.

Under the same spherical symmetry assumptions, the dynamical equations for the vector field ϕ^μ read

$$\nabla_\mu B^{\mu\nu} = 0, \quad | \quad 5.12$$

$$\nabla_\sigma B_{\mu\nu} + \nabla_\mu B_{\nu\sigma} + \nabla_\nu B_{\sigma\mu} = 0, \quad | \quad 5.13$$

where ∇_μ is the covariant derivative operator related to the metric tensor and $B_{\mu\nu} = \nabla_\mu\phi_\nu - \nabla_\nu\phi_\mu$, can be solved, resulting in [545]

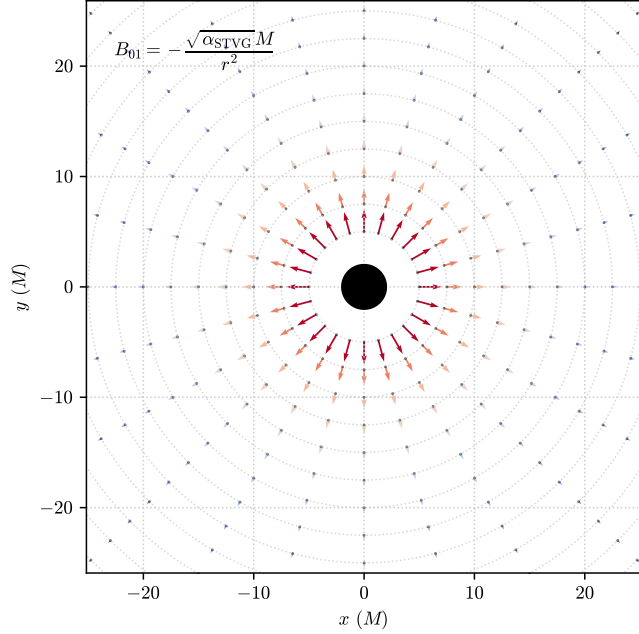
$$\phi_\mu = \left(-\frac{\sqrt{\alpha_{\text{STVG}}G_N}M}{r}, 0, 0, 0 \right). \quad | \quad 5.14$$

The vector field ϕ_μ behaves like the vector potential for a purely electrical radial field generated by a point source located at the black hole position, whose charge is $Q = \sqrt{\alpha_{\text{STVG}}G_N}M$ and whose action on massive test particles is a *repulsive* force counteracting the enhanced gravitational constant G . This configuration is shown in Figure 5.2.

Due to the modified geodesic equations in Eq. (5.7), which

scales.

Figure 5.2. The only non-null component of the Maxwell tensor for the STVG massive vector field ϕ_μ in the spherically symmetric case considered in Eq. (5.9) is the radial component of the mixed terms (01 term). This corresponds to a radial outward-pointing vector field with a $\propto r^{-2}$ decay, as illustrated in the figure, at regular radii from the event horizon of the central black hole.



when particularized for the metric in Eq. (5.9) take the form (by setting $G_N = c = 1$ for simplicity)

$$\ddot{t} = \frac{M\dot{r}(2M\alpha_{\text{STVG}}^2\dot{t} + 2M\alpha_{\text{STVG}}\dot{t} - 2\alpha_{\text{STVG}}r\dot{t} + \alpha_{\text{STVG}}r - 2r\dot{t})}{r(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)}, \quad | \quad 5.15$$

$$\begin{aligned} \ddot{r} = & \frac{M\alpha_{\text{STVG}}r\dot{t}(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)^2}{r^5(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)} + \\ & + \frac{r^4\dot{r}^2(-M^2\alpha_{\text{STVG}}^2 - M^2\alpha_{\text{STVG}} + 2M\alpha_{\text{STVG}}r + 2Mr - r^2 - r(M\alpha_{\text{STVG}} + M - r))}{r^5(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)} + \\ & + \frac{r^4(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2)(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)^2}{r^5(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)} + \\ & + \frac{\dot{t}^2}{r^5}(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} - 2M\alpha_{\text{STVG}}r - 2Mr + r^2)(M^2\alpha_{\text{STVG}}^2 + M^2\alpha_{\text{STVG}} + \\ & - 2M\alpha_{\text{STVG}}r - 2Mr + r^2 + r(M\alpha_{\text{STVG}} + M - r)), \end{aligned} \quad | \quad 5.16$$

$$\ddot{\theta} = \frac{\dot{\phi}^2\sin(2\theta)}{2} - \frac{2\dot{r}\dot{\theta}}{r}, \quad | \quad 5.17$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta}\cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}, \quad | \quad 5.18$$

in STVG the dynamics of freely falling test particles is not only affected by modifications in the spacetime geometry

introduced by the STVG black hole metric in Eq. (5.9), but also by the presence of the radial repulsive vector field. In particular, due to this additional interaction the constants of motion, of which we have made large use in the definition itself of the initial condition for the geodesic motion (see Section 2.2.3), have to be modified accordingly. While the Lagrangian function per unit mass of the test particle can be defined in the usual way from the metric elements of Eq. (5.9),

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \begin{cases} 0 & (\text{mass-less particle}) \\ -1 & (\text{massive test particle}) \end{cases}, \quad | \quad 5.19$$

one has to define generalized momenta, π_μ , to incorporate the presence of the vector field. Due to the particular form of ϕ_μ in Eq. (5.14), only the time component of the 4-momentum is affected by the vector field [333]:

$$\pi_t \equiv \frac{\partial \mathcal{L}}{\partial \dot{t}} - \frac{q}{m} \phi_t, \quad | \quad 5.20$$

$$\pi_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{r}}, \quad | \quad 5.21$$

$$\pi_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}}, \quad | \quad 5.22$$

$$\pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad | \quad 5.23$$

Due to the symmetry of the Lagrangian with respect to the coordinates t and ϕ , the two quantities

$$E \equiv \pi_t = \frac{\Delta}{r^2} \dot{t} + \frac{MG_N \alpha_{\text{STVG}}}{r}, \quad | \quad 5.24$$

$$L \equiv -\pi_\phi = r^2 \sin^2 \theta \dot{\phi}, \quad | \quad 5.25$$

are constants of motion and can be regarded as the specific energy and specific angular momentum of the particle at infinity. Moreover, from

$$\pi_\theta = -r^2 \dot{\theta}, \quad | \quad 5.26$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = -r^2 \dot{\theta} \sin \theta \cos \theta, \quad | \quad 5.27$$

one can see that upon setting initial conditions so that the test particle initially lies on the equatorial plane $\theta = \pi/2$ with $\dot{\theta} = 0$, the entire motion will take place on this plane and

thus $\theta(\tau) = \pi/2$ identically. So the planarity of the motion is preserved in STVG, in spite of the presence of the additional vector field. We can now use equations Eq. (5.24) and Eq. (5.25) to express $\dot{\phi}$ and $\dot{t}$ in terms of E and L and replacing them in the equation Eq. (5.19) for a test particle we can solve for $\dot{r}^2/\dot{\phi}^2 = (dr/d\phi)^2$:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{L^2} \left(E - \frac{M\alpha_{\text{STVG}}}{r}\right)^2 - \Delta(r) \left(1 + \frac{r^2}{L^2}\right), \quad | \quad 5.28$$

where G_N has been set to unity for simplicity. Eq. (5.28) is the equivalent of Eq. (2.34) for STVG. We can thus repeat the calculations for the rate of orbital precession for a massive test particle and derive an expression for how this quantity is modified in STVG. In particular, one can make the variable change $\xi = L^2/Mr$, so that Eq. (5.28) becomes

$$\begin{aligned} \left(\frac{L^2}{M\xi^2}\right)^2 \left(\frac{d\xi}{d\phi}\right)^2 &= \\ &= \frac{L^6}{M^4\xi^4} \left(E - \frac{M^2\alpha_{\text{STVG}}}{L^2}\xi\right) - \Delta(r(\xi)) \left(1 + \frac{L^2}{M^2\xi^2}\right). \end{aligned} \quad | \quad 5.29$$

Defining the parameter $\sigma = (M/L)^2$, the latter equation can be rewritten as

$$\begin{aligned} \left(\frac{d\xi}{d\phi}\right)^2 &= \frac{1}{\sigma} (E - \alpha_{\text{STVG}}\sigma\xi)^2 + \\ &- (1 - 2\sigma(1 + \alpha_{\text{STVG}})\xi + \sigma^2\alpha_{\text{STVG}}(1 + \alpha_{\text{STVG}})\xi^2) \left(\xi^2 + \frac{1}{\sigma}\right) \end{aligned} \quad | \quad 5.30$$

where the definition of Δ given in Eq. (5.10) has been used. The latter equation can be differentiated with respect to ξ resulting in

$$\xi'' = 1 + \alpha_{\text{STVG}}(1 - E) - \xi(1 + \sigma\alpha_{\text{STVG}}) + 3\sigma(1 + \alpha_{\text{STVG}})\xi^2 - 2\alpha_{\text{STVG}}(1 + \alpha_{\text{STVG}})\xi^3. \quad | \quad 5.31$$

Where primes stand for derivatives with respect to the angle ϕ . Differently from the classical Schwarzschild case of General Relativity reported in Chapter 2 (in particular, Eq. (2.37)), the equation that we have obtained presents a non-homogeneous term, $\alpha_{\text{STVG}}\delta$, where $\delta \equiv 1 - E$, that directly descends from the different definition of energy in STVG due to the presence of the vector field. This non-homogeneous term, however, can be neglected. In fact, if we restore all the dimensional

constants and particularize the equation for the orbit of S2 (e.g. at pericenter), we obtain

$$\delta = 1 - \frac{\Delta(r)}{r^2} \dot{t} - \frac{G_N M \alpha_{\text{STVG}}}{c^2 r} \sim \begin{cases} 2.10 \times 10^{-5} & (\text{for } \alpha_{\text{STVG}} = 0) \\ \sim 2.13 \sim 10^{-5} & (\text{for } \alpha_{\text{STVG}} = 2) \end{cases} \quad | \quad 5.32$$

Neglecting the contribution of δ in Eq. (5.31) and applying a perturbative expansion of ξ using σ as a perturbation parameter (this is possible because, generally, $\sigma \ll 1$, e.g. for the S2 star $\sigma = 1.88 \times 10^{-4}$), we get at zeroth and first, respectively

$$\xi_0'' + \xi_0 = 1, \quad | \quad 5.33$$

and

$$\xi_1'' + \xi_1 = -\xi_0(\alpha_{\text{STVG}} - 3(1 + \alpha_{\text{STVG}})\xi_0). \quad | \quad 5.34$$

We have thus brought the equation for the polar motion of a freely-falling massive test particle exactly in the same form used for the derivation of the rate of orbital precession in General Relativity (which we recover in the limit $\alpha_{\text{STVG}} \rightarrow 0$). Proceeding as already illustrated in Section 2.2.3, one finally obtains the rate of orbital precession in STVG, given by

$$\Delta\phi_{\text{STVG}} = \Delta\phi_{\text{GR}} \left(1 + \frac{5}{6} \alpha_{\text{STVG}} \right). \quad | \quad 5.35$$

The precession scales linearly with the parameter α_{STVG} and reduces to the usual general relativistic expression in Equation (2.41) when $\alpha_{\text{STVG}} = 0$.

Particularizing the geodesic equation for the S2 star in the Galactic Center, in Figure 5.3 we report the deviations on the two sky-projected directions and the line-of-sight velocity for values of α_{STVG} in the range $[0.01, 5]$ and compare them to the $\alpha_{\text{STVG}} = 0$ case. The horizontal shaded region reports the mean astrometric uncertainty, at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and a dashed horizontal line corresponds to the nominal astrometric precision of the GRAVITY interferometer [170], at $10 \mu\text{as}$, respectively. As it can be seen, the astrometric deviation induced by the STVG metric in Eq. (5.9) plus the additional interaction with the vector field, strongly modify the orbital dynamics for the S2 star, making the deviations exceed the instrumental limit for both the GRAVITY interferometer and pre-2017 VLT observations in the considered range. This means that even without introducing the information on the relativistic effect of orbital precession we can exploit the orbit

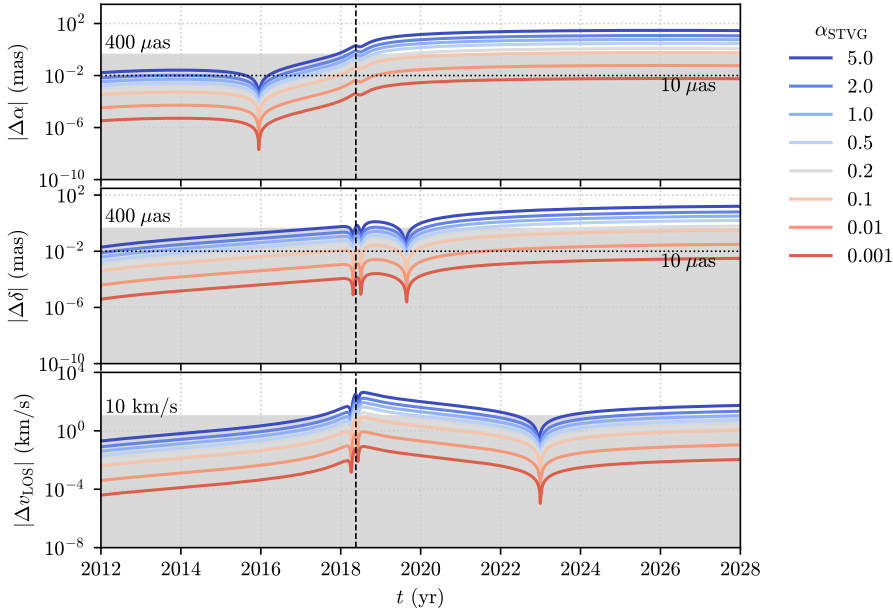


Figure 5.3. Astrometric (right ascension α and declination δ , top two panels) and spectroscopic (line-of-sight velocity, bottom panel) impact of STVG on the orbit of the S2 star in the Galactic Center, as a function of the extra parameter $\alpha_{\text{STVG}} \in [0.001, 5]$ before and after the 2018 pericenter passage. Horizontal dashed lines correspond to the mean astrometric uncertainty, at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and the nominal astrometric precision of the GRAVITY interferometer [170], at $10 \mu\text{as}$, respectively. In the bottom plot, the 10 km/s sensitivity threshold for the SINFONI interferometer is reported.

of S2 to impose constraints on the parameter α_{STVG} of STVG using NACO-only observations. Of course, such constraints improve by a lot when considering the much finer astrometry by GRAVITY.

To quantify this, we report in Figure 5.4 the rate of orbital precession of the S2 star in STVG as compared to the measured value by the Gravity Collaboration, appearing in the plot as a green horizontal band, and reported in Table 1.5. Here, we report both the results using the analytical formula in Eq. (5.35) and those obtained using the usual numerical procedure described in Section 2.2.3. On one hand, the plot shows perfect agreement between the two predictions for $\alpha_{\text{STVG}} \lesssim 10$, correctly predicting the linear increase of f_{SP} with α_{STVG} . For $\alpha_{\text{STVG}} \gtrsim 10$, the first-order formula in Eq. (5.35) is not able to capture non-linear effects in the orbital dynamic that the numerical integration of the fully non-linear geodesic equations in Eqs. (5.15)-(5.18) correctly reproduces. On the other hand, the figure also shows that considering the

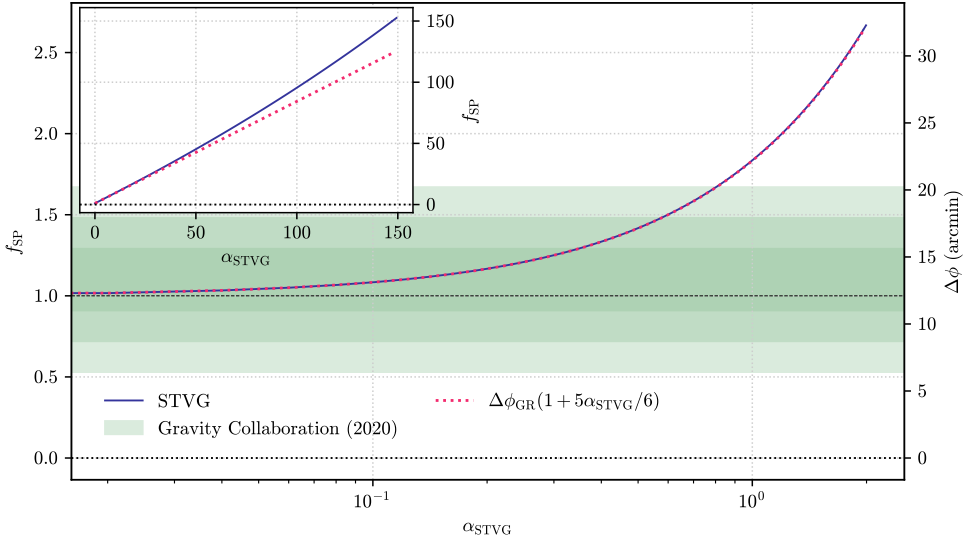


Figure 5.4. Rate of orbital precession for the S2 star in STVG computed numerically (blue solid line) and using the first order analytical expression in Eq. (5.35) (purple dotted line), in the range $[10^{-2}, 150]$ (the inset plot zooms out onto the entire range). The linear formula for the rate of orbital precession correctly reproduces the numerical results below $\alpha_{\text{STVG}} \approx 10$. For greater values of the STVG parameter, the analytical formula is not able to capture non-linear effects that our numerical procedure correctly reports. The green shaded bands report the 1σ , 2σ and 3σ bounds on the measured rate of orbital precession for S2 by the Gravity Collaboration [226], reported in Table 1.5. For $\alpha_{\text{STVG}} \approx 0.4$ the precession rate exceeds the 1σ bound for the measured value of f_{SP} , while at $\alpha_{\text{STVG}} \approx 0.9$ the strong deviations bring the predicted rate outside the 3σ bound.

measurement of the orbital precession for the S2 star by the Gravity Collaboration one can derive stringent bounds on the parameter α_{STVG} below 0.5.

Motivated by the analysis of the astrometric deviations induced by the STVG metric and the modification to the rate of orbital precession, we have proceeded to perform a series of MCMC analyses devoted to derive more quantitative bounds on the parameter α_{STVG} of STVG, applying the methodology described in Appendix C. More specifically, we have performed three different analyses⁵:

⁵ Note that, since this was chronologically the first work within this thesis project, we tested several different approaches and statistical strategies (*i.e.* different sets of priors and likelihoods) that later drove us towards the more consolidated approach that we have then used throughout the thesis and detailed in Appendix C. In particular, we have concluded that the use of Gaussian priors on all the orbital parameters, used in the second and third analyses reported here, is too conservative. Firstly, such bounds had been derived using exactly the same dataset that we used and so those priors would not be independent of the particular dataset at hand. Secondly, from a more theoretical point of view, the analysis from which those bounds have been derived was performed by fitting a Keplerian orbital model to the data. Since our aim was to study possible deviations from the standard Schwarzschild black hole model for Sgr A*, using those Gaussian priors on

Log-uniform prior		
	Start	End
α_{STVG}	0	2

Table 5.1. Log-uniform prior on the parameter α_{STVG} used for our Bayesian analysis on STVG.

- In the first one, we don't include the information on the orbital precession and we only fit the STVG orbital model for S2 to publicly available NACO + SINFONI data up to 2017. Here, we adopt Gaussian priors on all the orbital parameters with central value coinciding with the best-fitting values resulting from the post-Newtonian analysis in [226] and amplitudes that are 10 times larger than those from the same analysis. For the parameter α_{STVG} , that we sampled logarithmically, we assumed heuristic (motivated by our qualitative analysis) log-uniform priors as detailed in Table 5.1. The likelihood used in this case is given in Eq. (C.2).
- In the second analysis, we assume exactly the same priors as done in the previous case, but we consider the addition of the measured rate of orbital precession into our likelihood, now given by Eq. (C.3).
- In the third case, we relax our priors not pushing the orbital parameters towards the values emerging from the post-Newtonian analysis and adopting the uniform priors that have later been used throughout this thesis, reported in Table C.1. In this case we do include the information on the orbital precession and, for this reason, we adopt the same likelihood used in the second case, given in Eq. (C.3).

In Table 5.2 we summarize the results of the three analyses that we performed. In particular, we report the 68% credible intervals on all the parameters of our orbital model. All parameters are bound within credible intervals that contain the values reported in the literature for the same parameters. Remarkably, in all three cases the posteriors for the parameter α_{STVG} , highlighted in the last row, allow to place an upper limit on α_{STVG} . More specifically, when considering Gaussian priors we obtain a 95% (99.7%) confidence upper bound on $\alpha_{\text{STVG}} \lesssim 0.496$ (1.528) when not including the measured rate of orbital precession, and $\alpha_{\text{STVG}} \lesssim 0.275$ (0.662) when the additional datapoint is included in the analysis, implying an improvement by about a factor 2 carried by the measurement of the relativistic effect. In the third case, *i.e.* when relaxing the priors allowing all the parameters to vary freely, the posteriors

all the parameters would mean deliberately pushing our analysis towards the general relativistic limit. For this reason, in all subsequent analyses, we adopted more relaxed flat priors on all the orbital parameters (except the reference frame parameters, whose bounds come from independent analyses in [165]) as delineated in Appendix C.

Table 5.2. Results from the MCMC analyses for the STVG model from orbital data of the S2 star in the Galactic Center assuming Gaussian priors from [226] without (second column) and including (third column) information for the orbital precession and using the agnostic approach, *i.e.* adopting flat priors on all the orbital parameters. In particular, we report the 68% credible intervals around the median for all the parameters of the model, except for α_{STVG} (highlighted in the last row), for which we report a 95% confidence upper limit (and, in parenthesis the corresponding 99.7% confidence level upper limit). From the comparison between the three results columns, the improvement yielded by the relativistic information included in the measurement of the rate of orbital precession by the Gravity Collaboration appears evident.

Parameter (unit)	Posterior		
	Gaussian	Gaussian + precession	Flat + precession
M ($10^6 M_\odot$)	$4.251^{+0.071}_{-0.072}$	$4.265^{+0.081}_{-0.080}$	$3.96^{+0.28}_{-0.27}$
D (kpc)	$8.197^{+0.052}_{-0.051}$	8.215 ± 0.055	7.86 ± 0.30
t_p (yr)	$2018.3791^{+0.0015}_{-0.0016}$	2018.3790 ± 0.0016	2018.3780 ± 0.0017
a (arcsec)	0.12577 ± 0.00028	0.12560 ± 0.00029	0.1280 ± 0.0020
e	0.88435 ± 0.00057	$0.88442^{+0.00059}_{-0.00060}$	$0.8873^{+0.0039}_{-0.0038}$
i ($^\circ$)	134.60 ± 0.17	134.70 ± 0.19	133.36 ± 0.75
Ω ($^\circ$)	227.65 ± 0.22	227.83 ± 0.23	$227.25^{+0.95}_{-0.94}$
ω ($^\circ$)	$66.17^{+0.18}_{-0.17}$	$66.26^{+0.18}_{-0.19}$	$66.25^{+0.98}_{-0.97}$
α_0 (mas)	0.96 ± 0.18	0.94 ± 0.24	0.92 ± 0.26
δ_0 (mas)	-0.66 ± 0.28	-0.55 ± 0.32	$-1.31^{+0.73}_{-0.74}$
$v_{\alpha,0}$ (mas/yr)	0.162 ± 0.031	0.156 ± 0.037	0.119 ± 0.063
$v_{\delta,0}$ (mas/yr)	-0.009 ± 0.037	-0.022 ± 0.042	0.141 ± 0.066
$v_{\text{LOS},0}$ (km/s)	-9.8 ± 5.7	$-8.2^{+7.2}_{-7.3}$	$-10.0^{+9.4}_{-9.5}$
α_{STVG}	$\lesssim 0.496$ (1.528)	$\lesssim 0.275$ (0.662)	$\lesssim 0.286$ (0.708)

and all the orbital parameters are much more smeared as compared to the previous case (as evident from the larger bounds that result in this case), but this does not drastically worsen the upper limit on the STVG parameter, which is set to $\alpha_{\text{STVG}} \lesssim 0.286$ (0.708) in this case. These results match the expectations from the qualitative analysis on the astrometric deviation and the impact of orbital precession performed above. The full posteriors for the orbital and STVG parameters in the three cases are shown in Figure 5.5. Comparison between the three posteriors shows the much larger confidence regions resulting from the uniform priors on the orbital parameters while the marginalized posteriors on α_{STVG} , reported in logarithmic scale in the inset plot, results to be independent of the priors used (in all cases we obtain a flat posterior for $\alpha_{\text{STVG}} \lesssim 0.1$ that rapidly falls to zero for greater values of α_{STVG}) while being crucially affected by the measurement of the orbital precession as resulting from the comparison between the first and the other two cases (that are almost equivalent between each other).

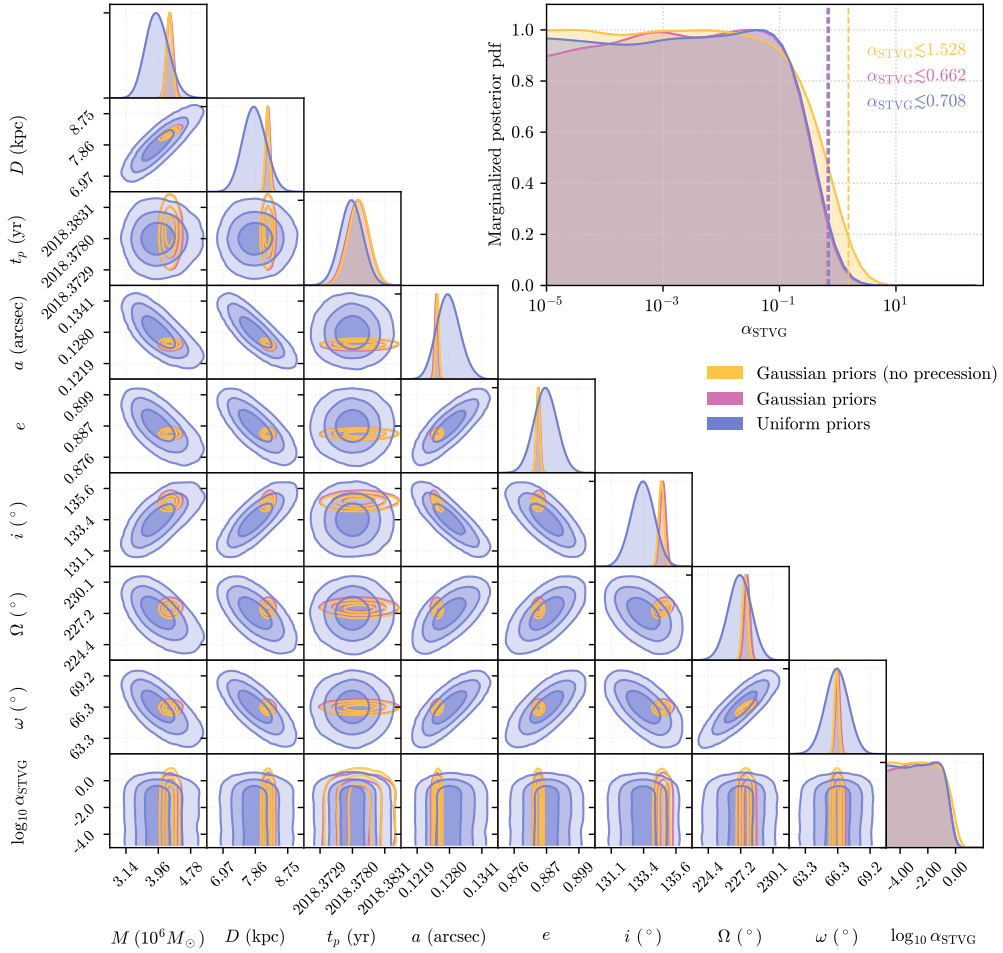


Figure 5.5. Corner plot reporting 9-dimensional slices of the posterior probability density functions of the orbital models for the S2 star in the STVG black hole spacetime in the three cases considered including the orbital parameters of S2 and the additional parameter α_{STVG} . The inset top-right-corner plot shows the marginalized posterior probability density functions on the parameter α_{STVG} , with the vertical dashed lines reporting the 99.7% confidence level upper limit on α_{STVG} in the three cases.

Our analysis resulted in the first constraint on the dimensionless parameter α_{STVG} of STVG on the scale of masses of Sgr A* and on astrophysical scales as small as those of the S2 star orbit in the Galactic Center. The results that we have obtained represent a complementary probe of this ETG to the ones already found in literature on galactic [539, 543, 546, 547], extra-galactic [536, 537, 540, 548, 549] and cosmological [550] scales.

5.2. $f(R)$ theories

Among various attempts to extend General Relativity, $f(R)$ gravity has historically played a significant role [551]. This ETG is based on a direct extension of the Lagrangian in the Einstein-Hilbert action of General Relativity, $\mathcal{S}_{\text{GR}}$ appearing in Eq. (2.6), which is extended to depend on a generic function $f(R)$ of the Ricci scalar, an idea that was first brought forward by Buchdal in 1970 [552],

$$\mathcal{S} = \int \sqrt{-g} f(R) d^4x. \quad | \quad 5.36$$

This modification introduces a non-linear dependence of the Lagrangian on R , resulting in field equations of the fourth order⁶, whereas General Relativity is characterized by second-order equations (recovered when $f(R) = R$). Consequently, these modified field equations exhibit a much richer phenomenology compared to general relativistic solutions [502], including the propagation of an additional scalar degree of freedom in the metric formalism [556]. $f(R)$ gravity has shown potential to address limitations of General Relativity across various astrophysical [559–569] and cosmological scenarios [570–577]. Additionally, the propagation of gravitational waves in $f(R)$ gravity significantly differs from that in General Relativity due to the extra degree of freedom in the theory [578, 579], representing a potential powerful way to test the predictions of the theory.

Here we will focus on spherically symmetric solutions for the spacetime metric, in the form of Eq. (2.12), obtained in the weak-field limit of the theory. In particular, the exact field

⁶ It is a common practice to employ a conformal transformation to transition from the Jordan frame to the Einstein frame (see [553, 554] for a rigorous definition of the two frames in scalar-tensor theories and the correspondence between the two frames and the duality between the metric and Palatini formalism of General Relativity and its extensions). In the Einstein frame, the field equations are reduced from fourth-order partial differential equations to second-order ones, and a scalar field emerges from the additional degrees of freedom. This transformation simplifies calculations significantly, but it requires the introduction of a mechanism to screen the scalar field in high-density environments, such as short distances [553, 555, 556]. While mathematically equivalent, the physical equivalence of the two frames is currently under debate [557, 558]. The alternative is to retain the formulation in the Jordan frame and accept having to deal with fourth-order field equations. This approach avoids the need to introduce *ad-hoc* screening mechanisms because of the scale dependence of the theory. For this reason, here we make the choice of sticking with the Jordan frame throughout the section.

equations obtained by varying the action in Eq. (5.36) with respect to the metric are

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = 8\pi T_{\mu\nu} \quad | \quad 5.37$$

and their trace

$$3\square f'(R) + f'(R)R - 2f(R) = 8\pi T. \quad | \quad 5.38$$

In general, whenever one wants to find solutions to the field equation in Eq. (5.37), a given functional form for the function f has to be fixed. This is the case of some specific families of $f(R)$ theories that have been proposed over the years [570–573, 580]. A different approach, that retains the generality of the $f(R)$ formulation, at the price of introducing several additional parameters in the theory, is considering classes of $f(R)$ theories that are analytic functions of the Ricci scalar and that can thus be expanded in a Taylor series of the form

$$f(R) = \sum_n \frac{f^{(n)}(R_0)}{n!} (R - R_0)^n = f_0 + f'_0 R + f''_0 R^2 + f'''_0 R^3 + \dots, \quad | \quad 5.39$$

where R_0 is the Ricci scalar for a given known background solution (which we have fixed to a Ricci-flat, $R = 0$, background case in the second expression) and the Taylor coefficients, $f_0^{(n)} = df/dR|_{R=R_0}$, tune the particular $f(R)$ theory considered. Analytic $f(R)$ models like the one in Eq. (5.39), are characterized by well-posed Cauchy problems only in the case $f''_0 < 0$, which guarantees the existence of stable cosmological solutions for perfect fluids [555]. Moreover, from the weak field limit at the zero-th perturbative order one can easily see that $f_0 = 0$. Considering a test solution of the form given in Eq. (2.12) and a post-Newtonian expansion up to the second order of the field equations in Eq. (5.37) under the analytic assumption for the $f(R)$ function, Eq. (5.39), one obtains [581–583] the expression for the spacetime around a point-like mass M in spherical coordinates, given by the line element

$$ds^2 = (1 + \Phi(r))dt^2 + (1 - \Psi(r))dr^2 - r^2 d\Omega^2, \quad | \quad 5.40$$

where $\Phi(r)$ and $\Psi(r)$ are two post-Newtonian gravitational potentials whose definitions can be found in [583]. In the same reference, it is shown that for small-scale systems and for small deviations from General Relativity ($|f'_0| \ll 1$), hypotheses that we both assume to be valid on the scales of the

S-stars in Galactic Center, one can approximate $\Psi(r) \sim \Phi(r)$, with the latter assuming the expression (temporarily restoring the constants G and c)

$$\Phi(r) = -\frac{2GM}{(1+\zeta)c^2r}(1+\zeta e^{-r/\lambda}). \quad | \quad 5.41$$

The modification to the standard weak-field spherically-symmetric metric incorporates a Yukawa interaction with a scale length λ , modulated by the parameter ζ . These parameters are related to the coefficients of the Taylor expansion of the function $f(R)$ in Eq. (5.39):

$$\zeta = f'_0 - 1; \quad \lambda = \sqrt{-\frac{6f''_0}{f'_0}}. \quad | \quad 5.42$$

Freely-falling test particles in the spacetime considered above undergo geodesic motion, described by the equations

$$\ddot{t} = \frac{2M\dot{r}\dot{t}(\zeta(\zeta r + \lambda(\zeta + 1) + r) + \lambda(\zeta + 1)e^{\frac{r}{\lambda}})}{\lambda r(\zeta + 1)(2M(\zeta + e^{\frac{r}{\lambda}}) - r(\zeta + 1)e^{\frac{r}{\lambda}})}, \quad | \quad 5.43$$

$$\begin{aligned} \ddot{r} = & \frac{M(-\dot{t}^2 + \dot{r}^2)(\zeta(\zeta r + \lambda(\zeta + 1) + r) + \lambda(\zeta + 1)e^{\frac{r}{\lambda}})}{\lambda r(\zeta + 1)(2M\zeta + 2Me^{\frac{r}{\lambda}} + r(\zeta + 1)e^{\frac{r}{\lambda}})} + \\ & + \frac{\lambda r^3(\zeta + 1)^2(\dot{\phi}^2 \sin^2(\theta) + \dot{\theta}^2)e^{\frac{r}{\lambda}}}{\lambda r(\zeta + 1)(2M\zeta + 2Me^{\frac{r}{\lambda}} + r(\zeta + 1)e^{\frac{r}{\lambda}})}, \end{aligned} \quad | \quad 5.44$$

$$\ddot{\theta} = \frac{\dot{\phi}^2 \sin(2\theta)}{2} - \frac{2\dot{r}\dot{\theta}}{r}, \quad | \quad 5.45$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}. \quad | \quad 5.46$$

When numerically solving the geodesic equations governing the motion of the S2 star within the $f(R)$ spacetime, described by Eqs. (5.43)-(5.46), and applying the methodology outlined in Section 2.2.3 to fix initial conditions and reconstruct astronomic observables from the integrated orbits, we find deviations from the Schwarzschild case in the two sky-projected observables and the line-of-sight velocity, as illustrated in Figure 5.6. These deviations are depicted for several different values of λ ranging from 100 AU to 10^5 AU. The horizontal lines in the figure represent the mean astrometric uncertainty, approximately $400 \mu\text{as}$, for pre-2017 VLT observations of the S2 star [138], along with the nominal astrometric precision of the GRAVITY interferometer [170],

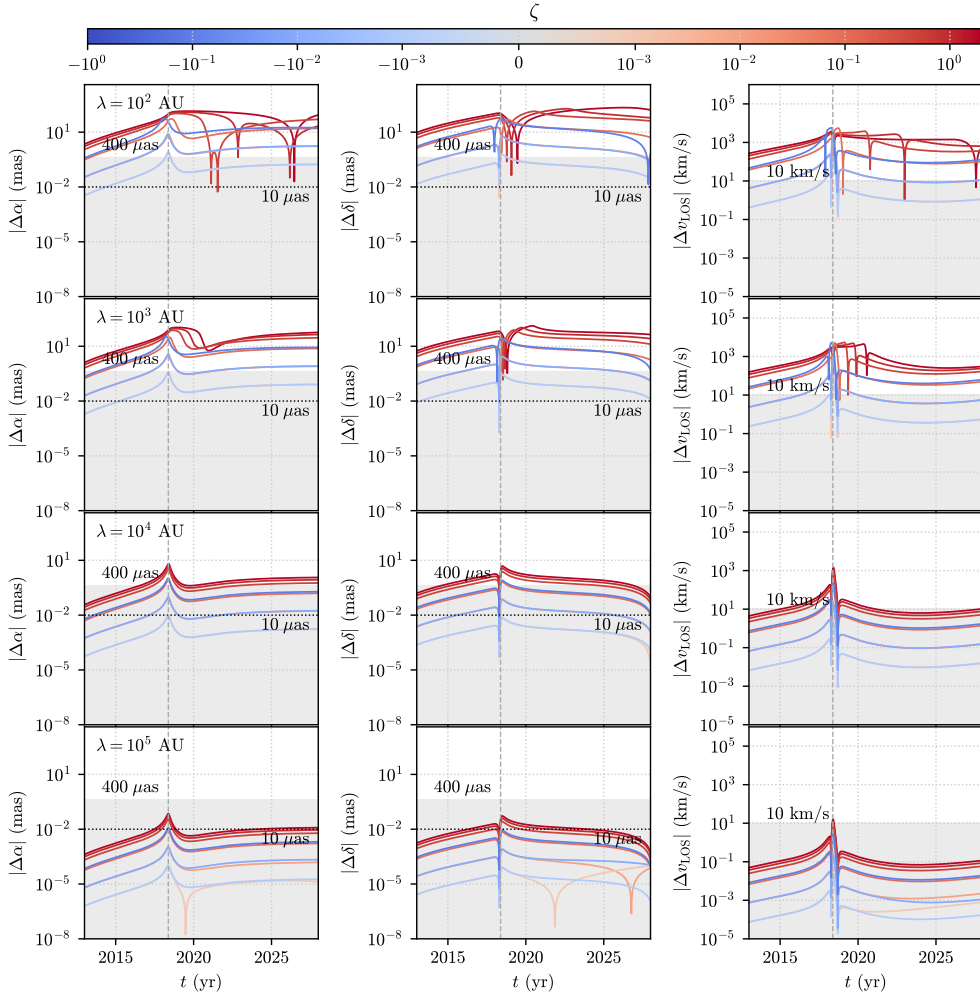


Figure 5.6. Deviation of the astrometric (right ascension α and declination δ , left-most and central panels) and spectroscopic (line-of-sight velocity v_{LOS} , right-most panels) observables for the orbit of the S2 star in the Galactic Center from the general relativistic case for the $f(R)$ spherically symmetric weak field solution in Eq. (5.41). From top to bottom, we consider increasingly high values of the scale length λ , in the range $[100, 100\,000]$ AU and for each of these cases we consider 7 values of the parameter ζ between -0.1 and 1 as detailed in the upper colorbar. As expected, the deviation is generally stronger for the smallest values of λ that we have considered, for which the astrometric deviations exceed both the NACO (shaded area) and GRAVITY (dotted horizontal line) sensitivity thresholds. Moreover, the deviations are smaller for values of ζ closer to 0 (which recovers the general relativistic behaviour). For $\lambda = 10^5$ AU, on the other hand, none of the considered values of ζ make the deviations for S2 detectable for NACO observations, while slightly exceeding the GRAVITY threshold for some highly different than zero values of ζ . The intermediate values considered for λ smoothly interpolate between the two cases.

standing at $10 \mu\text{as}$. The astrometric perturbation induced by the $f(R)$ metric results in slight-to-severe, depending on the particular combinations of parameters considered, alterations to the orbital dynamics of the S2 star. These deviations can exceed both instrumental thresholds also before the 2018 pericenter passage. Consequently, this qualitative analysis suggests that utilizing pre-pericenter NACO observations of the S2 star could already provide stringent constraints on the parameter space of $f(R)$, especially bringing to the exclusion of smaller values for the scale length λ . Moreover, GRAVITY data are shown to be sensitive to much smaller values of ζ , for the different values of λ considered and this suggests that future post-pericenter observations of the S2 star with this instrument will lead to much tighter constraints to $f(R)$ gravity.

To further substantiate these qualitative conclusions, we have focused on the relativistic observables for the S2 star. In particular, in the weak-field limit, it is possible to derive a first-order analytic expression for the rate of orbital precession of a test particle [583], that is given by (temporarily restoring G and c)

$$\Delta\phi_{f(R)} = \frac{\Delta\phi_{\text{GR}}}{(\zeta + 1)} \left(1 + \frac{2\zeta G^2 M^2}{3a^2 c^4 (1 - e^2)^2} - \frac{2\pi\zeta G^2 M^2}{ac^4 (1 - e^2)\lambda} - \frac{3\zeta GM}{ac^2 (1 - e^2)} + \right. \\ \left. - \frac{\zeta G^2 M^2}{6c^4 (\zeta + 1)\lambda^2} + \frac{\zeta GM}{3\lambda c^2} \right), \quad | \quad 5.47$$

where $\Delta\phi_{\text{GR}}$ represents the general relativistic value of the rate of orbital precession given by Eq. (2.41). Note that for $\zeta = 0$ the expression reduces to the general relativistic one, as expected. A qualitative analysis of the impact of Yukawa-like modifications to gravitational potential (like the one introduced in the weak field limit of $f(R)$ gravity) on the periastron shift of S2 can be found in the same paper [583]. Here we extend the analysis performed there, by computing the rate of orbital precession using our numerical methodology illustrated in Section 2.2.3. The results of our calculations are shown in Figure 5.7 for values of $\zeta \in [-1, 4]$ and $\lambda \in [10, 50\,000]$ AU (that we have sampled logarithmically). We explore the dependence of the ratio f_{SP} of the rate of orbital precession to its general relativistic counterpart over a large region of the parameter space of $f(R)$ gravity. The resulting value of f_{SP} can be substantially different from the prediction of General Relativity ($f_{\text{SP}} = 1$) and, as a general trend, it

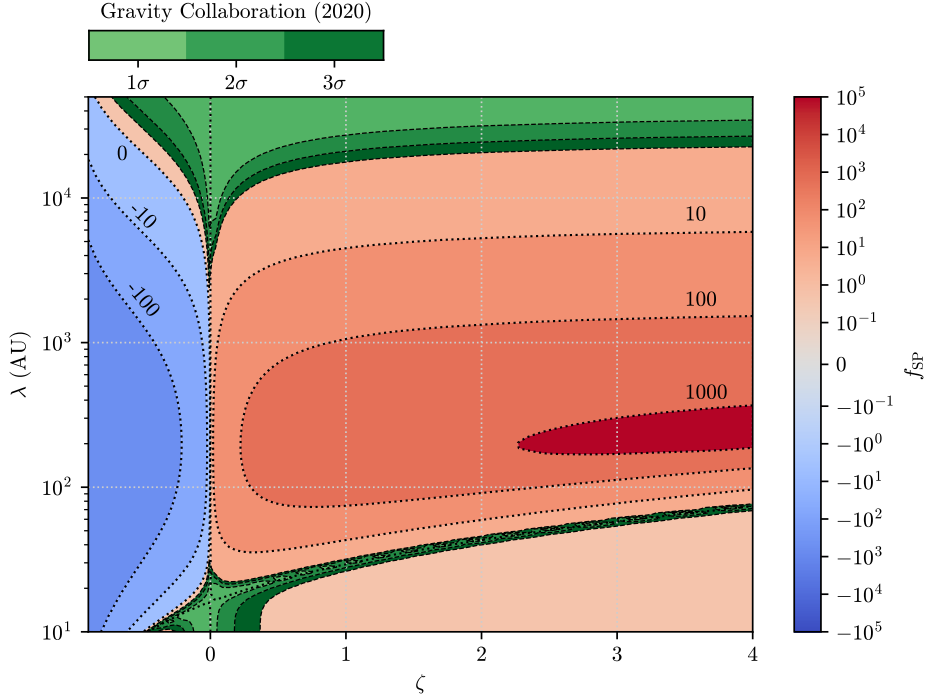


Figure 5.7. Rate of orbital precession for the S2 star in the $f(R)$ spherically symmetric spacetime described by Eq. (5.41). In particular, we report for $\zeta \in [-1, 4]$ and $\lambda \in [10, 10^5]$ AU the ratio f_{SP} of the numerically computed rate of orbital precession for that specific combination of parameters to the relativistic rate of orbital precession in the Schwarzschild case, whose value is given by Eq. (2.41). As displayed in the colorbar on the right side, the colormap used assigns red shades to positive (prograde) rates of precessions that are 10, 100 or 1000 times larger than their general relativistic counterparts (as reported with the contour labels) and blue shades to negative (retrograde) rates of orbital precession. The green shades, on the other hands, correspond to values of the rate of orbital precession for S2 that fall within the 1σ , 2σ and 3σ confidence regions of the experimental measurement of this relativistic observable by the Gravity Collaboration in 2020 [226]. This is perfectly centered on the $\zeta = 0$ value, that corresponds to the general relativistic limit of the theory. The figure already hints at the possible constraints that the measurement of the orbital precession enables on the parameter space of $f(R)$ gravity.

tends to be positive and greater than unity for positive values of ζ and smaller than unity for negative values of ζ , even becoming negative (implying a retrograde precession) for some combinations of parameters. The strongest deviations from General Relativity are found for λ between ~ 100 AU and ~ 1000 AU, corresponding to the physical size of the S2 orbit (*i.e.* between its pericenter distance of ~ 120 AU and its apocentral distance of ~ 2000 AU). In this region, the prograde precession can exceed a value that is 1000 times larger than the general relativistic rate of precession for values of $\zeta \gtrsim 2.3$, while in the retrograde case the periastron advance can exceed values of $f_{\text{SP}} \sim -100$. Both cases imply extremely strong perturbations of the orbital dynamics of

S2 that exceed by much the experimental bounds on f_{SP} by post-2018 observations of the S2 star with the GRAVITY interferometer and reported in Table 1.5. The regions of the (ζ, λ) parameter space for which the predicted rate of orbital precession for S2 falls within the 1σ , 2σ and 3σ intervals of such measured value are reported with increasingly dark green contours on the same plot. This analysis manifestly shows how considering the experimental value of the rate of orbital precession for S2 strongly limits the allowed region for the parameter space of $f(R)$ gravity.

Prompted by the qualitative analysis of the astrometric deviations and the estimation of the orbital precession of the S2 star, we conducted two Bayesian analyses aimed at deriving actual observational constraints on the parameter space of the $f(R)$ metric presented in Eq. (5.41), following the procedure detailed in Appendix C. In the first analysis we only took into account publicly-available orbital positions and velocities from [138]. In the second one we incorporated the precession measurement from [226]. Consequently, in the analysis involving only pre-2017 NACO observations we adopted the likelihood reported in Eq. (C.2), while we adopted the likelihood in Eq. (C.3) in the second case. In both cases we have considered the priors detailed in Table C.1 for the orbital parameters, complemented with heuristically-fixed uniform priors on the parameters (ζ, λ) as reported in Table 5.3. In particular, given the shape of the green contours in Figure 5.7 and to make our results comparable with other constraints in literature for the parameters of $f(R)$ (that we detail at the end of this section) we have decided to assign a prior on $\lambda > 100$ AU to target the upper allowed region for ζ and λ .

In Figure 5.8 and Table 5.4 we illustrate the results of our posterior analysis for the $f(R)$ parameter space. In particular, in the corner plot we report 68%, 95%, and 99.7% confidence regions for each pair of orbital parameters plus the extra parameters (ζ, λ) coming from the Taylor expansion of the $f(R)$ function. The two analyses reveal nearly overlapping contours for the Keplerian orbital elements of S2, centered on identical values, and both consistent with past studies in the literature [138]. Interestingly, the marginalized posteriors on the parameters (ζ, λ) , reported in more detail in Figures 5.9 and 5.10 for the two cases, respectively, are notably more constrained for the second case, benefiting from the additional information provided by the orbital precession. Indeed, in the analysis without considering the orbital precession, neither ζ

Uniform prior		
	Start	End
ζ	-0.9	2
λ (AU)	100	50 000

Table 5.3. Uniform prior on the parameters ζ and λ used for our Bayesian analysis.

Table 5.4. Resulting 68% confidence intervals for all the orbital parameters from our MCMC analyses on the $f(R)$ spherically symmetric weak field limit. In particular, in the left column we report the results of the posterior analysis only including pre-2017 astrometric and spectroscopic observations of S2 from [138]. The right column, on the other hand, reports the results of the analysis including the measured rate of orbital precession of S2 in [226] reported in Table 1.5. All orbital parameters are bound and consistent within 1σ between each other (from the two analyses) and with their best fitting values from past studies in the literature [138]. The posterior on the parameters ζ and λ are highlighted for the two cases in the bottom two rows.

Parameter (unit)	Posterior	
	Data	Data + precession
M ($10^6 M_\odot$)	4.24 ± 0.18	$4.28^{+0.24}_{-0.25}$
D (kpc)	8.21 ± 0.17	8.26 ± 0.22
t_p (yr)	2018.3789 ± 0.0017	2018.378 ± 0.0017
a (arcsec)	$0.12542 \pm 0.0.00087$	0.1251 ± 0.00110
e	0.8817 ± 0.0017	0.8815 ± 0.002
i ($^\circ$)	$134.15^{+0.38}_{-0.37}$	134.25 ± 0.48
Ω ($^\circ$)	226.29 ± 0.63	$226.17^{+0.81}_{-0.8}$
ω ($^\circ$)	65.1 ± 0.6	64.92 ± 0.78
α_0 (mas)	-0.54 ± 0.13	-0.43 ± 0.16
δ_0 (mas)	0.25 ± 0.19	0.17 ± 0.19
$v_{\alpha,0}$ (mas/yr)	0.125 ± 0.043	0.102 ± 0.053
$v_{\delta,0}$ (mas/yr)	$0.131^{+0.047}_{-0.046}$	$0.117^{+0.058}_{-0.057}$
$v_{\text{LOS},0}$ (km/s)	$-15.8^{+6.9}_{-6.8}$	$-16.0^{+9.1}_{-9.0}$
ζ	$\gtrsim -0.07$	$-0.01^{+0.61}_{-0.14}$
λ (AU)	$\gtrsim 9540$ (68% c.l.)	$\gtrsim 6300$ (68% c.l.)

nor λ could be constrained with an upper limit. However, in the second analysis we successfully constrained the parameter ζ , emphasizing the enhanced constraining power of relativistic effects like the orbital precession. Consequently, our analysis has provided the first constraint on the strength of the Yukawa-like potential of $f(R)$ at the Galactic Centre: $\zeta = -0.01^{+0.61}_{-0.14}$. On the other hand, examining the two-dimensional contours in Figures 5.9 and 5.10, we only derive in both cases a lower bound on the scale length of the Yukawa-like potential given by $\lambda \gtrsim 9540$ AU in the first case, and $\lambda \gtrsim 6300$ AU in the second, at 1σ confidence. This outcome aligns with expectations, as this parameter tends to be better constrained on larger astrophysical scales [370], reaffirming findings in previous studies [569, 583]. Additionally, the 95% confidence contours from our analysis align fully with the exclusion regions on the fifth force determined for the S2 star in [584] for a similar Yukawa-like potential within the region of the parameter space we have examined ($\lambda > 100$ AU).

Finally, we conducted a thorough comparison of our results with existing constraints on $f(R)$ gravity obtained from analyses at both astrophysical and cosmological scales:

- Various constraints exist at Solar System scales for different $f(R)$ Lagrangians, such as those presented in [585] and [586]. However, direct comparisons with our

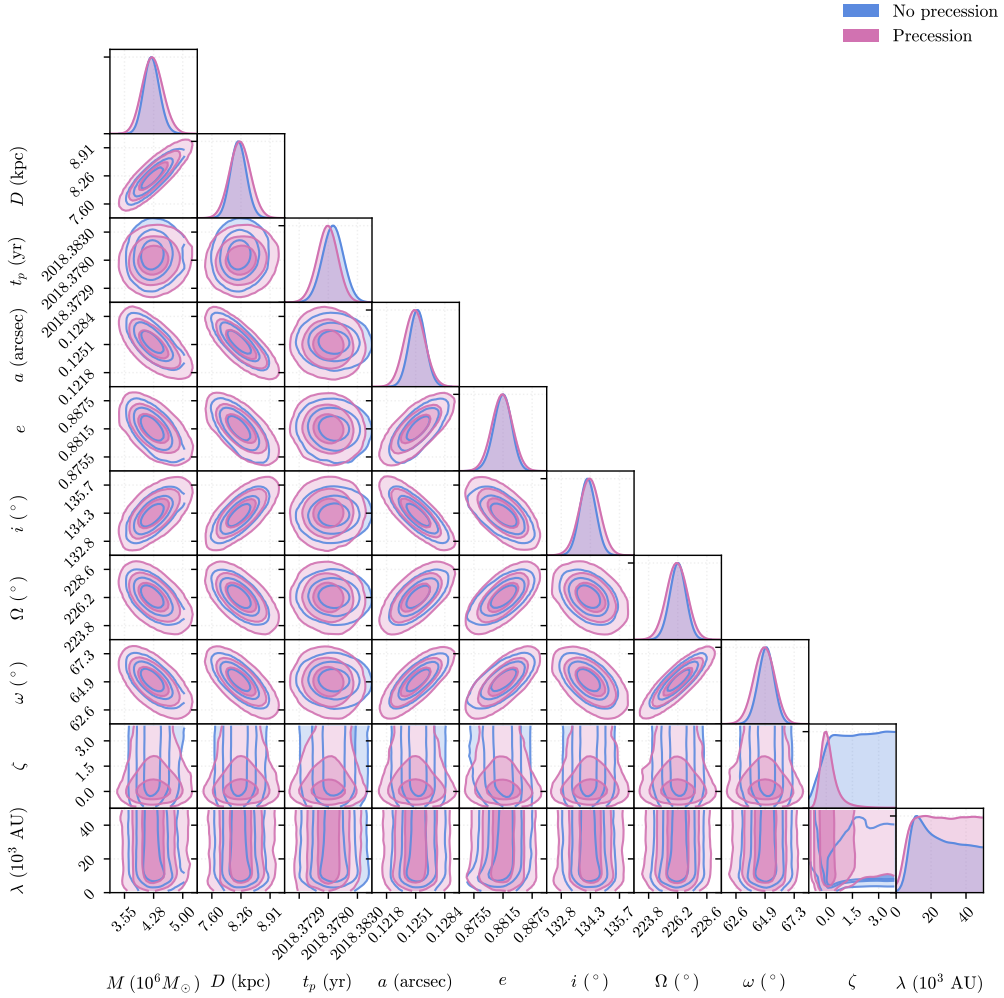


Figure 5.8. Posterior distributions for the $f(R)$ orbital model for the S2 star in the two cases that we have analyzed, as specified in the legend. In particular, we report the 68%, 95% and 99.7% confidence regions of the two-dimensional marginalized posterior distribution for each pair of orbital parameters and for the parameter of interest in out weak-field limit of $f(R)$, ζ and λ , along with the marginalized posterior distributions for each parameter. The two posteriors are almost exactly overlapping and are centered on the same values, except for the two $f(R)$ extra parameters, for which the posteriors are substantially different, due to the additional constraining power brought by the orbital precession information in the second analysis, which leads to a much narrower confidence region for such parameter.

results are challenging due to differences in methodologies and scales. Notably, [587] favored strongly positive values of ζ based on the orbit of the S2 star, although it remains unclear whether all observational and relativistic effects were considered, and orbits were predicted using Newton's Law rather than integrating geodesic equations. In contrast, our analysis comprehensively

Figure 5.9. Two-dimensional slice of the full posterior for the (ζ, λ) parameters in the spherically symmetric solution for the weak field of $f(R)$ gravity. Here we report the results from the MCMC analysis only taking into account the pre-2017 astrometric observations at the Galactic Center, without the information on the orbital precession and thus resulting only in lower bounds on both ζ and λ , as reported in left column of Table 5.4 and here as dashed pink vertical lines.

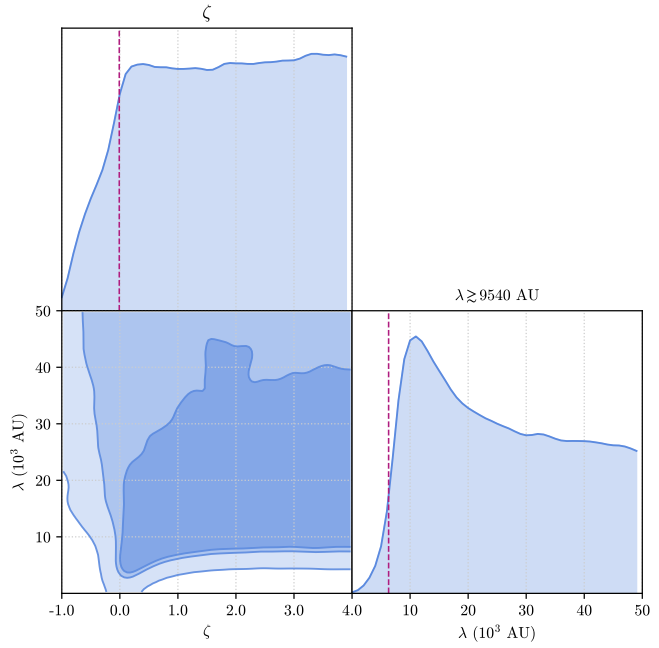
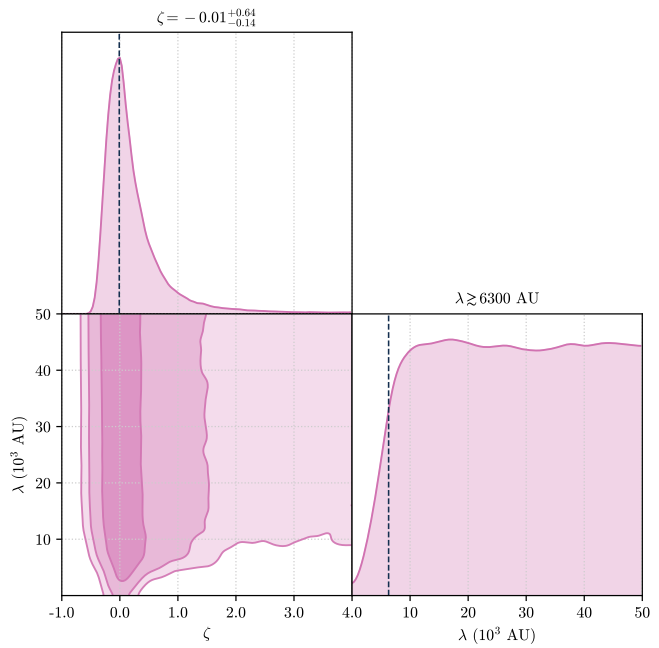


Figure 5.10. As in Figure 5.9 but for the second analysis, including the measurement of the relativistic rate of orbital precession for S2, along with the orbital data. This allowed us to improve the constraints imposed on the two parameters, as reported in the right column of Table 5.4.



accounts for observational and relativistic effects, and we compute orbits using geodesic equations. Moreover, the lower limit on the scale length that we have obtained represents a significant improvement compared to similar analyses in [588].

- Our results differ from those obtained for elliptical galaxies [589], which indicated a significant departure from General Relativity, constraining $\zeta \sim -0.8$ and $\lambda \geq 10$ kpc. Their errors, at the level of 10%, preclude reconciliation with General Relativity, whereas we do not detect any departure from General Relativity.
- Our results achieve comparable precision to those obtained using galaxy clusters [567]. However, they contrast with [590], where authors found a value of f'_0 incompatible with unity and a value of f''_0 weakly compatible with zero (corresponding to the general relativistic limit). Comparisons with other constraints at the scale of galaxy clusters are more challenging due to differences in the $f(R)$ Lagrangian used, such as the Hu-Sawicki model or R^n .

It is important to note that there are additional constraints based on PPN parameters and pulsar timing, which are significantly more accurate (by at least 5-6 orders of magnitude) in the parameter ζ compared to our results [591]. Nevertheless, our analysis complements these efforts and our qualitative analysis of the astrometric deviations has shown the potential to achieve comparable accuracy in the near future.

5.3. Horndeski gravity

The Horndeski theory of gravity, a well-known and widely studied class of ETGs, stands out as one of the most comprehensive frameworks in modified gravity [519, 592]. It represents a broad spectrum of gravitational theories coupled to a scalar field, ϕ , that yields second-order field equations for both the metric tensor $g_{\mu\nu}$ and ϕ .

Extensively explored in both cosmological and astrophysical contexts, Horndeski theory has garnered significant attention, especially in studies related to black holes [593, 594] and

neutron stars [595, 596]. The gravitational action in Horndeski theory is encapsulated in the expression

$$\mathcal{S} = \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i[g_{\mu\nu}, \phi] + \mathcal{S}_m[g_{\mu\nu}, \chi_m], \quad | \quad 5.48$$

where $\mathcal{S}_m$ represents the matter action and χ_m encompasses all matter fields. The gravitational sector of the action comprises various terms, each defined by functions G_2 , G_3 , G_4 , and G_5 of the scalar field ϕ and its kinetic term $X \equiv -1/2 \nabla^\mu \phi \nabla_\mu \phi$, as illustrated below:

$$\begin{aligned} \mathcal{L}_2 &= G_2(\phi, X), \quad \mathcal{L}_3 = -G_3(\phi, X) \square \phi, \\ \mathcal{L}_4 &= G_4(\phi, X) R + G_{4X}(\phi, X) [(\square \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2], \\ \mathcal{L}_5 &= G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} G_{5X}(\phi, X) \left[(\square \phi)^3 - 3(\square \phi)(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right]. \end{aligned} \quad | \quad 5.49$$

It's important to note that within the broader class of Horndeski theories, not only those with explicit appearances of scalar fields in the gravitational action and field equations are included, but also theories like $f(R)$ gravity or $f(\mathcal{G})$ Gauss–Bonnet gravity, which inherits higher-order field equations but can be equivalently represented by a scalar field [592]. Moreover, a spherically symmetric expression for the spacetime in the Newtonian and Post-Newtonian expansion of the theory was found in 2015 [597], revealing an effective general Yukawa-like potential. It is in this Newtonian limit that we will develop our orbital model for the S-stars in the Galactic Center. In fact, as we will show, even at Newtonian order, Horndeski gravity brings five extra parameters to spherically symmetric spacetime solutions, with 10 additional parameters appearing at 1PN order. These parameters would add on top of the 13 parameters that we need to fix a unique orbit for S-stars in the Galactic Center, as outlined in Section 2.2.3. In order not to introduce an unmanageable amount of extra parameters, we hence decided to only consider the Newtonian order, with the intention to move to the 1PN order only in case we would obtain extremely good results in the first case.

The Newtonian limit of Horndeski gravity theory can be obtained by perturbatively expanding the field equations around a Minkowski background $\eta_{\mu\nu}$, whose expression in spherical symmetry is given in Eq. (2.19), and a constant cosmological

background value Φ of the scalar field [597]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \phi = \Phi + \psi, \quad X = -\frac{1}{2}\nabla^\mu\psi\nabla_\mu\psi. \quad | \quad 5.50$$

A perturbative solution can be looked for, requiring that the background is homogeneous, isotropic, and stationary, and further assuming that the four free functions G_2 , G_3 , G_4 , and G_5 can be expanded in Taylor series as

$$G_i(\phi, X) = \sum_{m,n=0}^{\infty} G_{i(m,n)} \psi^m X^n, \quad | \quad 5.51$$

where the coefficients $G_{i(m,n)}$ are given by

$$G_{i(m,n)} = \frac{1}{m!n!} \left. \frac{\partial^{m+n}}{\partial \phi^m \partial X^n} G_i(\phi, X) \right|_{\phi=\Phi, X=0}, \quad | \quad 5.52$$

with each term $G_{i(m,n)} \psi^m X^n$ being of order $\mathcal{O}(\psi^{m+2n})$. Following the multi-messenger event GW170817/GRB170817 A, only theories for which $G_{4,X} = 0$ and $G_5 = 0$ remain viable [598]. Additionally, it's important to note that at the Newtonian order, $G_{4,X}$ and G_5 do not contribute (see Eq. (28) in [597]).

For a mass point source, the solution at Newtonian order assuming spherical coordinates (being r the usual aerial radius) can be written as [597]

$$h_{00}^{(2)}(r) = \frac{M}{4\pi r} \left[\frac{1}{2G_{4(0,0)}} + \frac{1}{2} \left(\frac{G_{4(1,0)}}{G_{4(0,0)}} \right)^2 \Upsilon^{-1} e^{-m_\psi r} \right], \quad | \quad 5.53$$

where $G_{4(0,0)}$ plays the role of the inverse of the gravitational constant G ,

$$m_\psi = \sqrt{\frac{-2G_{2(2,0)}}{\Upsilon}}, \quad | \quad 5.54$$

and, for the sake of simplicity, we have introduced the following definition

$$\Upsilon \equiv G_{2(0,1)} - 2G_{3(1,0)} + 3\frac{G_{4(1,0)}^2}{G_{4(0,0)}}. \quad | \quad 5.55$$

Following the seminal work of [592], and the more recent results of [597], Eq. (5.54) requires $\Upsilon > 0$, which implies $G_{2(0,0)} < 0$ and reduces the allowed parameter space of the theory.

In a general fashion, Horndeski theory of gravity has a unique limit that recovers General Relativity. In particular, by looking at the gravitational action in Eqs. (5.48) and (5.49), it is clear that General Relativity ($\mathcal{L} = R/16\pi$) is recovered if and only if all the functions G_2 , G_3 , and G_5 are identically zero (and hence their Taylor expansion coefficients are all zero as well), and $G_4 = (16\pi)^{-1}$. This is the only physical limit to General Relativity as it involves the cancellation of the additional scalar field ϕ introduced to describe the gravitational interaction. On the other hand, at a metric level, the Newtonian approximation of General Relativity for the metric of spacetime around a massive point source is given by:

$$h_{\mu\nu}^{(2)}(r) = \text{diag} \left(\frac{2M}{r}, 0, 0, 0 \right). \quad | \quad 5.56$$

If we compare Eq. (5.53) with Eq. (5.56), we see that we can recover the general relativistic expression by putting $G_{4(0,0)} = 1/16\pi$ and performing one of the following limits:

- **A:** $G_{4(1,0)} \rightarrow 0$, which recovers Eq. (5.56) for any value of the other Taylor coefficients;
- **B:** $G_{2(2,0)} \rightarrow \pm\infty$, which brings $m_\psi \rightarrow \infty$ and hence to the cancellation of the Yukawa-like term in Eq. (5.53);
- **C:** Any combination of the coefficients $G_{2(0,1)}$, $G_{3(1,0)}$, and $G_{4(1,0)}$ which brings $\Upsilon \rightarrow 0$ and hence $m_\psi \rightarrow \infty$, driving the entire Yukawa-like term to an indeterminate form of the kind $e^{-\infty}/0$ whose limit is 0;
- **D:** The simultaneous limit $G_{2(0,1)}, G_{2(2,0)}, G_{4(1,0)}, G_{3(1,0)} \rightarrow 0$, which, again, recovers the expression in Eq. (5.56) for the metric coefficient.

It is evident that the presence of multiple general relativistic limits, among which only the case **D** corresponds to the physical limit of General Relativity of the theory itself, is a spurious effect due to having performed the Newtonian approximation. Care must be taken, hence, in fixing the values of the coefficients in Eq. (5.53) as only one region of the parameter space corresponds to being close to the general relativistic limit, even though at a metric (and thus geodesic) level multiple regions result in an orbit that is compatible with General Relativity. The above considerations involve notable degenerations in the parameter space, as different combinations of these parameters return the same physical

Table 5.5. The set of parameters chosen to build our toy models. The second column reports the value of the scale length of the Horndeski modification to the gravitational potential at Newtonian order. Columns 3 to 7, on the other hand, report the particular set of parameters resulting in the chosen value of $1/m_\psi$. We have temporarily restored the values of the constants G and c .

Model	$1/m_\psi$ (AU)	$G_{4(0,0)}$ ($M_\odot \text{AU}/\text{s}^2$)	$G_{4(1,0)}$ ($M_\odot \text{AU}/\text{s}^2$)	$G_{3(1,0)}$ ($M_\odot \text{AU}/\text{s}^2$)	$G_{2(0,1)}$ ($M_\odot \text{AU}/\text{s}^2$)	$G_{2(2,0)}$ ($M_\odot/\text{AU s}^2$)
Toy 1	0.0	$c^4/16\pi G$	0	0	0	0
Toy 2	66.0	$c^4/16\pi G$	10	0	50	-0.01
Toy 3	208.6	$c^4/16\pi G$	10	0	50	-0.001
Toy 4	433.4	$c^4/16\pi G$	100	0	50	-0.01
Toy 5	1370.5	$c^4/16\pi G$	100	0	50	-0.001

limit, which cannot be broken without being able to impose *a priori* limits on these parameters.

From the line element in Eq. (5.50), one can compute the geodesic equations that describe the motion of freely falling test particles in the Newtonian limit that we have considered

$$\ddot{t} = - \frac{M \dot{r} \left[\frac{1}{r} \left(G_{4(0,0)} \Upsilon + G_{4(1,0)}^2 e^{-m_\psi r} \right) + G_{4(1,0)}^2 m_\psi e^{-m_\psi r} \right]}{8\pi G_{4(0,0)} \Upsilon r - G_{4(0,0)} \Upsilon M - G_{4(1,0)}^2 M e^{-m_\psi r}}, \quad | \quad 5.57$$

$$\ddot{r} = r \dot{\phi}^2 \sin^2 \theta + r \dot{\theta}^2 + \frac{i^2 M}{16\pi G_{4(0,0)} r} \left[-\frac{1}{r} - \frac{G_{4(1,0)}^2 e^{-m_\psi r}}{G_{4(0,0)} \Upsilon} \left(m_\psi + \frac{1}{r} \right) \right], \quad | \quad 5.58$$

$$\ddot{\theta} = \dot{\phi}^2 \sin \theta \cos \theta - \frac{2\dot{r}\dot{\theta}}{r}, \quad | \quad 5.59$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos \theta}{\sin \theta} - \frac{2\dot{\phi}\dot{r}}{r}, \quad | \quad 5.60$$

where, as usual, a dot represents a derivative with respect to the affine parameter. As a preliminary analysis to study the motion of test particles in the Newtonian expansion of Horndeski gravity for a spherically symmetric spacetime, we have selected five toy models for the S2 star with the same orbital parameters (the ones in Table 1.1) and with varying values for the Horndeski Taylor coefficients. The choice of such parameters has been made in order to assess the deformation of the orbit as a function of $1/m_\psi$ (which has dimensions of a length) and $G_{4(1,0)}$ which, together, modulate the strength and scale length of the modification to the Newtonian potential in Eq. (5.50). In particular, we have fixed the general relativistic value of $G_{4(0,0)} = c^4/16\pi G$

(having temporarily restored G and c in the expression), while we have chosen values for the parameters $G_{4(1,0)}$, $G_{3(1,0)}$, $G_{2(0,1)}$, $G_{2(2,0)}$ that would give a value of $1/m_\psi$ that is a given fraction of the typical scale length of the S2 orbit (e.g. its semi-major axis, $a \sim 1000$ AU). The chosen values are reported in Table 5.5 and the corresponding orbits from the numerical integration are shown in Figure 5.11. In particular, the model Toy 1 represents a star moving under the influence of an unperturbed Newtonian potential, due to the fact that all parameters have been set to zero. Models Toy 2 and Toy 4 have $G_{2(2,0)} = -0.01 M_\odot/\text{AU}^2$, while for Toy 3 and Toy 5 a value of $G_{2(2,0)} = -0.001 M_\odot/\text{AU}^2$ has been chosen. On the other hand, while Toy 2 and Toy 3 have a value $G_{4(1,0)} = 10 M_\odot \text{AU}/\text{s}^2$, a value of $G_{4(1,0)} = 100 M_\odot \text{AU}/\text{s}^2$ has been fixed for Toy 4 and Toy 5. This choice results in an increasingly high scale length $1/m_\psi$ for the different models, associated with a strength of the Yukawa-like potential in Eq. (5.50), modulated by $G_{4(1,0)}$, that is increasingly higher. As can be seen from the integrated orbits in Figure 5.11, the main effect related to the variation of the length $1/m_\psi$ is a modification of the orbit and, in particular, its pericentre shift on the orbital plane. It is important, however, to keep in mind that the estimation of this effect is biased by the fact that we are considering an expansion at Newtonian order of the Horndeski metric. This means that even considering the limit **D** to General Relativity of the model and computing the corresponding orbit, would not reproduce a general relativistic orbit that precesses by a rate given by Eq. (2.41), which arises at 1PN level, but the effect is underestimated ($f_{\text{SP}} \sim 0$ instead of $f_{\text{SP}} = 1$). For this reason we cannot confidently estimate the relativistic rate of orbital precession from the numerically integrated orbits at Newtonian order and we cannot introduce in our subsequent posterior analysis the measured rate of periastron advance for S2 by the Gravity Collaboration [226], that could lead to biases in our results.

In order to derive observational constraints on the additional parameters introduced in the spherically symmetric solution for the Newtonian expansion of Horndeski gravity in Eq. (5.53), we have applied the Bayesian inference methodology outlined in Appendix C to publicly available orbital data for the S2 star in the Galactic Center. As mentioned above, since we stop at Newtonian order, we cannot include the orbital precession information in our analysis. For this reason, we adopt the likelihood function reported in Eq. (C.2).

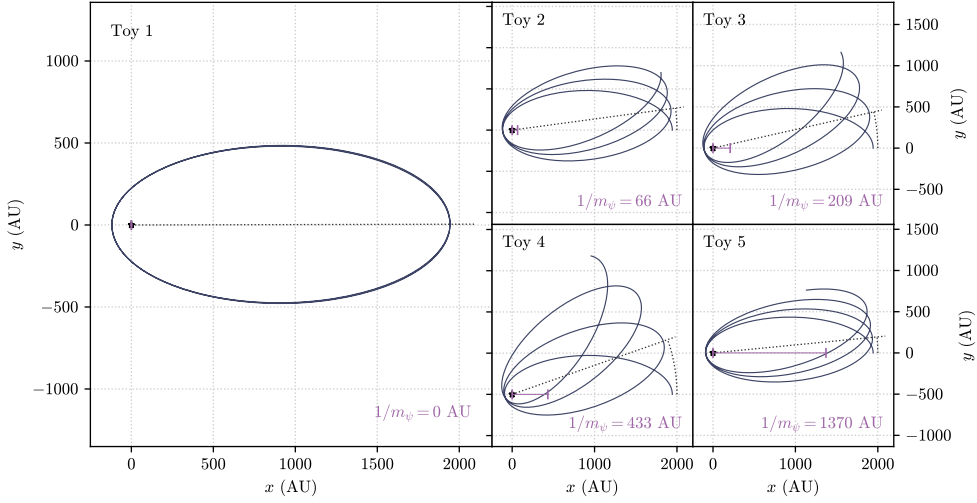


Figure 5.11. The integrated orbits for the toy models in Table 5.5. The pink segment illustrates the varying scale length $1/m_\psi$. The plots report how modification of the orbit are predicted as the coefficient of the Newtonian expansion of Horndeski gravity are varied. The most noticeable effect that we can visually see on the plots is the pericentre shift on the orbital plane of the star. The quantity $1/m_\psi$ behaves like a scale and it gets more prominent for smaller values of $1/m_\psi$.

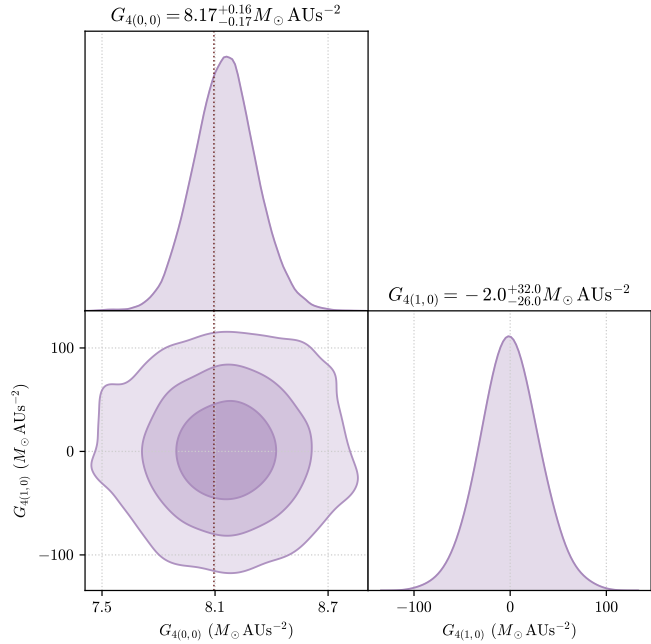
Our orbital model for the S2 in Horndeski gravity leads to an 18-dimensional parameter space composed of the 13 orbital parameters required to uniquely identify an orbit for S2 and the reference frame parameters for the radio-to-infrared positioning of the astrometric frame, reported in Eq. (C.1), plus the additional 5 parameters in the Newtonian limit of Horndeski gravity, $G_{4(0,0)}$, $G_{4(1,0)}$, $G_{2(0,1)}$, $G_{3(1,0)}$ and $G_{2(2,0)}$. Consequently, the priors that we adopted are a combination of the 13-dimensional priors considered throughout this thesis, given in Table C.1 for the S2 parameters, and a set of very large uniform priors on the Taylor coefficients, heuristically motivated by our preliminary analysis using toy models and respecting the constraints on the signs of the parameter $G_{2(2,0)}$ and the combination Υ in Eq. (5.55). These are reported in Table 5.6.

Uniform prior		
	Start	End
$G_{4(0,0)}$	$0.95(c^4/16\pi G)$	$1.05(c^4/16\pi G)$
$G_{4(1,0)}$	-100	100
$G_{3(1,0)}$	-100 000	100 000
$G_{2(0,1)}$	-100 000	100 000
$G_{2(2,0)}$	-10	0

Table 5.6. The uniform priors on the parameter space of the Newtonian limit of Horndeski gravity. For aesthetical reasons we do not explicitly report the units of measure of the considered parameters. However the numbers on the right columns are intended in the correct units reported in Table 5.7.

The results of our posterior analysis for our orbital model for the S2 star in Horndeski gravity are summarized in Table 5.7. In particular, we report the 68% confidence intervals for all the parameters that result to be bounded from our analysis. Specifically, all the orbital elements and the reference frame parameters for the S2 star are bounded and recover, within 3σ (at most), the values reported in Table 1.1 coming from

Figure 5.12. Two-dimensional slice of the posterior distribution on the Taylor coefficients of Horndeski gravity corresponding to the marginalized posterior for the parameters $G_{4(0,0)}$ and $G_{4(1,0)}$. The vertical red line in the left panels depicts the expected value in General Relativity of the Taylor coefficient $G_{4(0,0)}$.



previous studies [138]. Only the semi-major axis a and the inclination angle i represent an exception, being correlated with the parameter $G_{4(0,0)}$, and are only marginally compatible with their counterpart obtained in a Newtonian framework. This is anyway rather expected since $G_{4(0,0)}$ modulates the Newtonian gravitational constant G hence affecting the posterior distribution for the semi-major axis and of the inclination angle.

Regarding the additional Horndeski parameters, we place constraints on $G_{4(0,0)}$ and $G_{4(1,0)}$, while the other parameter remains unconstrained (although we have been able to place a constraint on a combination of two of the three remaining parameters). Clearly, introducing 10 additional parameters, as prescribed by an expansion up to the 1PN order, would inevitably lead to the impossibility of placing sensible constraints on such parameters and would, additionally, worsen the bounds that we have placed at Newtonian order (due to the much higher degree of degeneracy introduced and the poor constraining power of currently available data). In detail, Figure 5.12 is an inset of the corner plot of the full posterior reporting the 68, 95, and 99.7% confidence intervals of $G_{4(0,0)}$ and $G_{4(1,0)}$ parameters, and their corresponding posterior distributions. As reported, we find $G_{4(0,0)} = 8.17^{+0.16}_{-0.17} M_{\odot} AU/s^2$

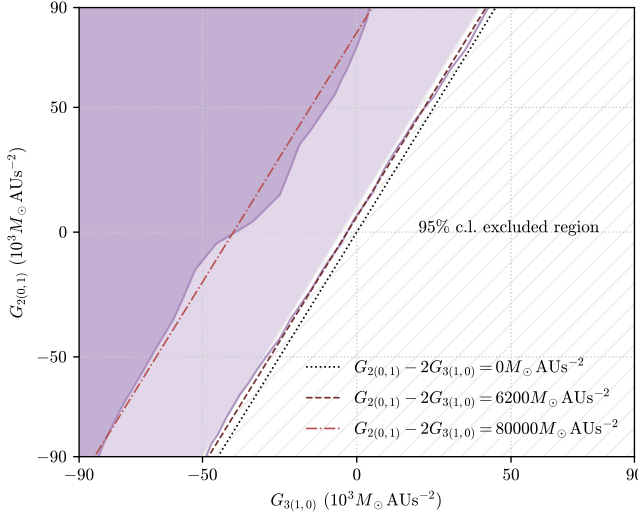
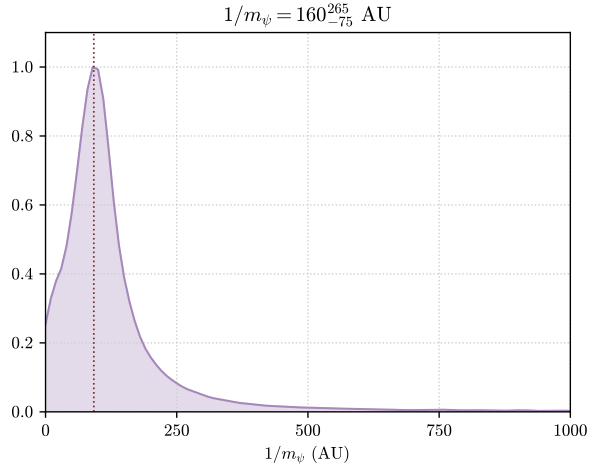


Figure 5.13. 68% and 95% confidence level contours of posteriors of the $G_{2(0,1)} - G_{3(1,0)}$ slice of the parameter space from our MCMC analysis. The dark red dashed line corresponds to the line $G_{2(0,1)} - 2G_{3(1,0)} = 6200 M_\odot \text{AU s}^{-2}$ and coincides with the margin of the 95% confidence level exclusion region (hatched portion of the plot), while the red dash-dotted line coincides with the 68% lower limit on this combination of parameters at $G_{2(0,1)} - 2G_{3(1,0)} = 80000 M_\odot \text{AU s}^{-2}$.

and $G_{4(1,0)} = -2^{+32}_{-26} M_\odot \text{AU/s}^2$. The parameter $G_{4(0,0)}$ is fully compatible with its expected value in General Relativity (which is depicted as a vertical red line in the figure), and the parameter $G_{4(1,0)}$ is compatible with zero at 1σ reducing the gravitational potential to the Newtonian one: $h_{00} = \frac{M}{8\pi G_{4(0,0)} r}$. The result was rather expected but still surprising because the orbital motion data of S2 star, which we are using to test the Newtonian limit of Horndeski theory of gravity, cannot distinguish General Relativity from Newton's theory either [138]. Previous tests of the weak field limit of Horndeski gravity strongly constrained $G_{4(0,0)}$, but they consider a subclass of the whole theory by setting $G_{2(2,0)} = 0$ which means $m_\psi = 0$ [599]. In this case, the Yukawa-term disappears reducing itself to an additional constant in Eq. (5.53). On the other hand, none was able to place constraints on $G_{4(1,0)}$ in the weak field limit. Additionally to those new bounds, looking at the contours of the $G_{2(0,1)} - G_{3(1,0)}$ slice of the whole parameter space, we can see that even though we are not able to place a definitive constraint on either one of the two parameters, we are able to exclude a region of the parameter space that results in a lower limit on the combination $G_{2(0,1)} - 2G_{3(1,0)} > 6200 M_\odot \text{AU/s}^2$ as shown in Figure 5.13. Finally, we were able to bound the scale length m_ψ^{-1} , whose posterior was built from the posteriors of the parameters appearing in the definition of m_ψ in Eq. (5.54).

As shown in Figure 5.14, we found $m_\psi^{-1} = 160^{+265}_{-75} \text{AU}$.

Figure 5.14. Normalized posterior distribution on the inverse of m_ψ in AU derived from the posteriors of the four parameters in Eq. (5.54). With a dashed vertical line we report the maximum likelihood indicator of the posterior distribution, also reported in Table 5.7.



Noticeably, this represents the best constraint up to date on m_ψ^{-1} since other analyses only place a lower limit $m_\psi^{-1} \geq 9$ AU using binary systems composed by a pulsar plus a white dwarf [600]. Finally, let us point out that, even though other constraints exist in literature for a phenomenological Yukawa-like modification to the Newtonian potential using the S2 star at the Galactic Center (see e.g. [584]), due to the specific dependence of Eq. (5.53) on the Taylor parameter of the Horndeski functions, there is no way to consistently map constraints on the former potential into bounds on the latter. Indeed, phenomenological Yukawa-like modifications are parametrized by a scale length (λ) and a modulation factor (α), they are treated (both conceptually and statistically) as independent parameters. The equivalent scale length $1/m_\psi$ in our metric (5.53), on the other hand, is not independent of the coefficient $1/2\Upsilon \left(G_{4(1,0)}/G_{4(0,0)} \right)^2$ of the exponential term in the metric, due to the fact that m_ψ itself depends on all the four free parameters of the Newtonian limit studied here. This, along with the intrinsic non-linearity of the relation between the Horndeski parameters and those in the Yukawa-like potential, makes it conceptually impossible to map pre-existing bounds on α and λ into bounds on $G_{4(0,0)}$, $G_{4(1,0)}$, $G_{3(1,0)}$ and $G_{2(0,1)}$. As previously highlighted, our investigation provides a complementary result to previous studies and, despite not identifying any deviations from General Relativity, illustrates how observations of the S-Stars in the Galactic Center can effectively constrain the parameter space of ETGs.

Parameter	Unit	Posterior
D	kpc	$8.53^{+0.13}_{-0.12}$
M	$10^6 M_\odot$	4.55 ± 0.19
$t_p - 2018$	yr	$0.33^{+0.11}_{-0.08}$
a	arcsec	$0.12323^{+0.00052}_{-0.00048}$
e		$0.8821^{+0.0027}_{-0.0027}$
i	($^\circ$)	$135.12^{+0.18}_{-0.17}$
Ω	($^\circ$)	$226.09^{+0.54}_{-0.54}$
ω	($^\circ$)	$64.25^{+0.54}_{-0.44}$
α_0	mas	$-0.4^{+1.6}_{-2.4}$
δ_0	mas	$0.2^{+1.8}_{-2.1}$
$v_{\alpha,0}$	mas/yr	$-0.02^{+0.12}_{-0.08}$
$v_{\delta,0}$	mas/yr	$0.00^{+0.09}_{-0.11}$
$v_{\text{LOS},0}$	km/s	$7.5^{+3.9}_{-4.0}$
$G_{4(0,0)}$	$M_\odot \text{ AU s}^{-2}$	$8.17^{+0.16}_{-0.17}$
$G_{4(1,0)}$	$M_\odot \text{ AU s}^{-2}$	$-2.0^{+32.0}_{-26.0}$
$G_{2(0,1)} - 2G_{3(1,0)}$	$M_\odot \text{ AU s}^{-2}$	$\gtrsim 6200$
$G_{2(2,0)}$	$M_\odot \text{ AU}^{-1} \text{ s}^2$	Unbounded
$1/m_\psi$	AU	160^{+265}_{-75}

Table 5.7. Results of our posterior analysis on all the parameters of our orbital model for the S2 star in the Newtonian expansion of Horndeski's gravity. In particular, we report the 68% confidence interval for all the orbital and reference frame parameters; the 68% confidence interval for $G_{4(0,0)}$ (which is bounded and perfectly compatible with the Newtonian value $c^4/16\pi G$) and for $G_{4(1,0)}$ (which is centered on 0). The parameter $G_{2(2,0)}$ results unbounded from our analysis, while we are only able to put a constraint on the combination $G_{2(0,1)} - 2G_{3(1,0)}$ which appears as an exclusion region on the corresponding contour plot in Figure 5.13. Finally, we report the 68% confidence interval for the scale length $1/m_\psi$ derived from the individual parameter distributions and applying the definition in Eq. (5.54).

5.4. Einstein-Maxwell-dilaton-axion

The EMDA is an ATG that originates from the low-energy Lagrangian of superstring theory [601]. Specifically, it arises from the compactification of ten-dimensional heterotic string theory on a six-dimensional torus. This operation yields pure $(N = 4, d = 4)$ -supergravity coupled to $(N = 4)$ -super-Yang-Mills theory in the low-energy limit [602, 603]. The action $\mathcal{S}$ associated with EMDA gravity encompasses couplings between the metric $g_{\mu\nu}$, an anti-symmetric electromagnetic tensor field $F_{\mu\nu}$, a $U(1)$ gauge field A_μ , the dilaton field χ , and an axion pseudo-scalar field ξ . It can be expressed as [604, 605]

$$\mathcal{S} = \mathcal{S}_{\text{GR}} + \frac{1}{16\pi} \int \sqrt{-g} d^4x \left(-2\partial_\nu \chi \partial^\nu \chi - \frac{1}{2} e^{4\chi} \partial_\nu \xi \partial^\nu \xi + e^{-2\chi} F_{\rho\sigma} F^{\rho\sigma} + \xi F_{\rho\sigma} \tilde{F}^{\rho\sigma} \right), \quad | \quad 5.61$$

where g is the determinant of the spacetime metric, and $\mathcal{S}_{\text{GR}}$ is the usual general relativistic expression of the covariant action in Eq. (2.6). This action represents a low-energy effective limit of the heterotic string action truncated at first order in the inverse string tension, to include only terms involving up to two derivatives. Hence, while it can formally be regarded as an ETG that includes additional vectorial and tensorial degrees of freedom, these are motivated by deeper and more fundamental considerations. Variation of the action with respect to the metric yields the vacuum field equations

$$G_{\mu\nu} = T_{\mu\nu}(F, \chi, \xi). \quad | \quad 5.62$$

The right-hand side energy-momentum tensor $T_{\mu\nu}$ encapsulates the effective energy-momentum content of the extra fields in the theory, given by

$$\begin{aligned} T_{\mu\nu}(F, \chi, \xi) = & e^{2\chi}(4F_{\mu\rho}F^{\rho\nu} - g_{\mu\nu}F^2) - g_{\mu\nu}(2\partial_\gamma\chi\partial^\gamma\chi + \frac{1}{2}e^{4\chi}\partial_\gamma\xi\partial^\gamma\xi) + \\ & + \partial_\mu\chi\partial_\nu\chi + e^{4\chi}\partial_\mu\xi\partial_\nu\xi. \end{aligned} \quad | \quad 5.63$$

The field equations (5.62) have been shown to admit static black hole solutions in the context of the EMDA theory [601, 606–608], generally characterized by one or more charges associated with the extra fields. The cancellation of such charges reduces these solutions to the standard Schwarzschild black hole.

The existence of a rotating, stationary, and axisymmetric solution for EMDA has been explored in [604]. In the charge-neutral case, the solution reverts to the general relativistic Kerr solution, while in the non-null charge case, a distinct solution is derived, known as the Kerr-Sen black hole. This solution exhibits peculiarities that differentiate it from its charged counterpart in General Relativity [609–612]. When considering $G = c = 1$ and utilizing Boyer-Lindquist coordinates, this solution takes the form

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2M\tilde{r}}{\tilde{\Sigma}}\right) dt^2 + \frac{\tilde{\Sigma}}{\Delta}(d\tilde{r}^2 + \Delta d\theta^2) - \frac{4aM\tilde{r}}{\tilde{\Sigma}} \sin^2\theta dtd\phi + \\ & + \sin^2\theta d\phi^2 \left(\tilde{r}(\tilde{r} + r_2) + a^2 + \frac{2M\tilde{r}a^2 \sin^2\theta}{\tilde{\Sigma}}\right), \end{aligned} \quad | \quad 5.64$$

where

$$\tilde{\Sigma} = \tilde{r}(\tilde{r} + r_2) + a^2 \cos^2 \theta, \quad | \quad 5.65$$

$$\Delta = \tilde{r}(\tilde{r} + r_2) - 2M\tilde{r} + a^2. \quad | \quad 5.66$$

In addition to the mass M of the central object and the black hole spin parameter a , the EMDA rotating geometry introduces the extra parameter $r_2 = \frac{q^2}{M} e^{2\chi_0}$, known as the dilaton parameter. Its value encodes information about both the asymptotic value of the dilatonic field χ_0 and the electric charge q of the black hole, originating from the coupling of the photon with the axion pseudo-scalar. When $q = 0$, the Kerr metric is recovered. The geometry in (5.64) harbors a horizon as long as r_2 satisfies the condition

$$0 \leq r_2 \leq 2M, \quad | \quad 5.67$$

thus delineating the theoretical bounds for a black hole solution in EMDA. As mentioned before in this thesis, for the purpose of studying S-stars orbits in the Galactic Center we can disregard the black hole's rotation and hence consider the non-rotating limit $a \rightarrow 0$, where both the spin and the axionic field vanish. A pure dilaton black hole metric is therefore given by [610]

$$ds^2 = - \left(\frac{\tilde{r} - 2\mu}{\tilde{r} + 2b_{\text{EMDA}}} \right) dt^2 + \left(\frac{\tilde{r} + 2b_{\text{EMDA}}}{\tilde{r} - 2\mu} \right) dr^2 + (\tilde{r}^2 + 2b_{\text{EMDA}}\tilde{r}) d\Omega^2, \quad | \quad 5.68$$

where the aerial radial coordinate r and mass M are defined as:

$$r^2 = \tilde{r}^2 + 2b_{\text{EMDA}}\tilde{r}, \quad M = \mu + b_{\text{EMDA}}. \quad | \quad 5.69$$

Here, a new dilaton parameter $b_{\text{EMDA}} \equiv r_2/2$ is introduced, constrained theoretically to $0 \leq b_{\text{EMDA}} \leq M$ for a black hole geometry, with the limit $b_{\text{EMDA}} \rightarrow 0$ recovering the Schwarzschild case in Eq. (2.18).

Applying the geodesic equations, Eq. (2.3), to the EMDA spacetime metric in Eq. (5.68) we get the equations of motion of freely falling test particles in this theory, given by

$$\ddot{t} = - \frac{\dot{r}t\Theta'(r)}{\Theta(r)}, \quad | \quad 5.70$$

$$\ddot{r} = \frac{\dot{r}^2\Theta'(r) + (-\dot{t}^2\Theta'(r) + 2r\dot{\phi}^2 \sin^2(\theta) + 2r\dot{\theta}^2)\Theta^2(r)}{2\Theta(r)}, \quad | \quad 5.71$$

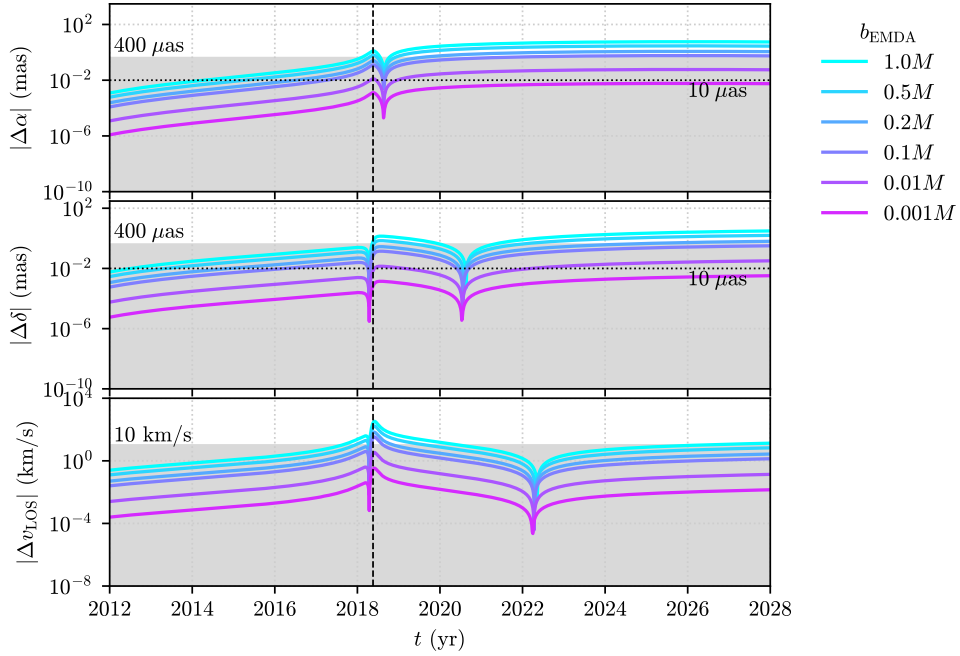


Figure 5.15. Astrometric (right ascension α and declination δ , top two panels) and spectroscopic (line-of-sight velocity, bottom panel) impact of a EMDA black hole on the orbit of the S2 star in the Galactic Center, for several values of the extra parameter b_{EMDA} in its theoretically allowed region $[0.001M, M]$ before and after the 2018 pericenter passage. Horizontal lines correspond to the mean astrometric uncertainty, at around $400 \mu\text{as}$, for pre-2017 VLT observations for the S2 star reported in [138] and the nominal astrometric precision of the GRAVITY interferometer [170], at $10 \mu\text{as}$, respectively. In the bottom plot, the 10 km/s sensitivity threshold for the SINFONI interferometer is reported.

$$\ddot{\theta} = \frac{\dot{\phi}^2 \sin(2\theta)}{2} - \frac{2\dot{r}\dot{\theta}}{r}, \quad | \quad 5.72$$

$$\ddot{\phi} = -\frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{r}}{r}, \quad | \quad 5.73$$

being,

$$\Theta(r) = \frac{b_{\text{EMDA}} - 2M + \sqrt{b_{\text{EMDA}}^2 + r^2}}{b_{\text{EMDA}} + \sqrt{b_{\text{EMDA}}^2 + r^2}}. \quad | \quad 5.74$$

When solving numerically the geodesic equation governing the motion of the S2 star for the EMDA spacetime, Eqs. (5.70)-(5.73), we obtain the deviations in the two sky-projected observables and the line-of-sight velocity that are depicted in Figure 5.15 for values of b_{EMDA} ranging from $0.001M$ to $1M$ (*i.e.* within the theoretically allowed interval for this param-

ter), juxtaposed with the $b_{\text{EMDA}} = 0$ case. The horizontal stripe denotes the mean astrometric uncertainty, approximately $400 \mu\text{as}$, for pre-2017 VLT observations of the S2 star [138], while the horizontal dotted line corresponds to the nominal astrometric precision of the GRAVITY interferometer [170], standing at $10 \mu\text{as}$. The astrometric perturbation induced by the EMDA metric slightly alters the orbital dynamics of the S2 star, with deviations that only surpass the instrumental threshold for GRAVITY interferometer after the 2018 pericenter passage. This suggests that using pre-pericenter NACO observations alone would not be able to sensibly constrain the dilaton parameter b_{EMDA} in its theoretical range, while GRAVITY data should be sensible to much smaller values for b_{EMDA} on the order of $10^{-2}M$.

Additionally, we derive numerically the rate of orbital precession for S2 in the EMDA spacetime as a function of the dilaton parameter b_{EMDA} . The results of our calculations are reported in Figure 5.16 for values of $b_{\text{EMDA}} \in [10^{-2}, 10]$ (thus exceeding the theoretically allowed values of b_{EMDA} for a Sen black hole) and compared to the measured rate of orbital precession from the Gravity Collaboration, reported in Table 1.5. As it appears, in the allowed range for b_{EMDA} , the orbital precession for S2 never exceeds the 3σ bounds on f_{SP} , with a maximum deviation of $f_{\text{SP}} \sim 0.65$ predicted for $b_{\text{EMDA}} = M$. Nevertheless, evident departures at more than 2σ from the purely general relativistic case appear in the considered range for b_{EMDA} which suggests that the orbital precession can play a crucial role in constraining the value of b_{EMDA} .

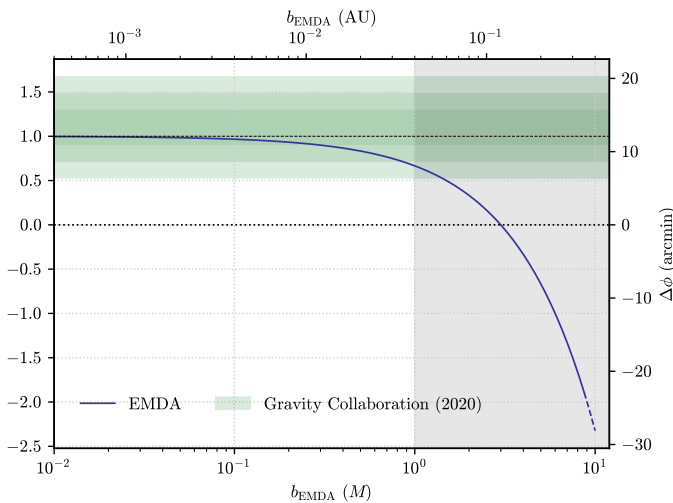


Figure 5.16. Rate of orbital precession for the S2 star in EMDA as a function of the dilaton parameter b_{EMDA} (blue solid line) as computed through our numerical methodology. The green horizontal shaded bands correspond to the 1σ , 2σ and 3σ bounds placed by the analysis in [226], while the gray shaded region is the theoretically excluded region for a black hole solution in EMDA. Within the theoretically allowed region for b_{EMDA} , we predict only slight deviations in the relativistic observable for the S2 star that fall nonetheless within the 2σ experimental bounds.

The analysis of the orbital precession, along with the previous study of the astrometric deviations, motivated us to carry out three MCMC analyses in which we expected to obtain increasing constraining power on the parameter b_{EMDA} . In particular, we first consider the orbital data for S2 only including the pre-pericenter publicly available NACO observations. In this case, we adopt the likelihood presented in Eq. (C.2) and the priors reported in Table C.1, complemented with a heuristically-fixed uniform prior on b_{EMDA} given in Table 5.8⁷. In our second analysis, we consider the same sets of data and the same priors, but we add the measurement of the orbital precession by the Gravity Collaboration [226], thus adopting the likelihood function detailed in Eq. (C.3). Finally, we forecast the accuracy that upcoming observations using the refined astrometry of the GRAVITY interferometer will achieve on the inference of the EMDA parameter b_{EMDA} , employing as dataset our mock catalogue of observations for S2 (reported in Appendix B and detailed in Section 2.2.3) over one and a half orbital period (thus intrinsically containing the information on the orbital precession) based on the Schwarzschild spacetime. In this last case, the likelihood adopted is the one given in Eq. (C.5).

Uniform prior		
	Start	End
$b_{\text{EMDA}} (M)$	0	50

Table 5.8. Uniform prior on the parameter b_{EMDA} used for our Bayesian analysis in EMDA.

The outcomes of our three MCMC analyses are presented in Figures 5.17, and summarized in Table 5.9. Specifically, contour plots representing the 68%, 95% and 99.7% confidence regions for each pair of parameters (we restrict to show only the shared parameters between the three analyses, namely the orbital parameters and the dilaton parameter b_{EMDA}), alongside marginalized posterior histograms for individual parameters in the orbital model, are showcased. The first two analyses (using real data without and with the information on the orbital precession, respectively) exhibit nearly overlapping contours for the Keplerian orbital elements of S2, centered on identical values and both consistent with other studies in literature [138]. Notably, the marginalized posterior on the parameter b_{EMDA} is substantially more constrained for the second case, leveraging the additional information from the orbital precession. The reduced uncertainties on the astrometric observables in our mock catalog result in

⁷ Note that this prior is much larger than the theoretically allowed region for the parameter b_{EMDA} for an EMDA black hole. We do this because, from our preliminary analyses, we do not expect to be able to constrain the value of b_{EMDA} within the theoretical limit, using only pre-2017 data for S2. In any case, we will later exclude the region of the posterior outside the theoretically allowed range for b_{EMDA} .

Table 5.9. Median values and 68% confidence interval for all the orbital parameters of the S2 star as resulting from our three MCMC analyses in the EMDA spacetime. The two bottom rows highlight the 95% confidence level upper limit on our parameter of interest, b_{EMDA} , reported both in AU and in units of gravitational radii (M , being $G = c = 1$) of the central object.

Parameter (unit)	Data	Data + Precession	Mock
D (kpc)	8.21 ± 0.17	8.23 ± 0.23	$8.2468^{+0.0029}_{-0.0030}$
M ($10^6 M_\odot$)	4.22 ± 0.18	4.23 ± 0.25	$4.2610^{+0.0045}_{-0.0046}$
t_p -2018.38 (yr)	-0.003 ± 0.020	$-0.001^{+0.023}_{-0.024}$	$-0.0010165^{+0.0000097}_{-0.0000095}$
a (mas)	$125.13^{+0.84}_{-0.85}$	125.0 ± 1.1	125.0592 ± 0.0036
e	0.8835 ± 0.0020	0.8828 ± 0.0025	0.8846640 ± 0.0000063
i ($^\circ$)	$134.37^{+0.40}_{-0.41}$	134.40 ± 0.50	134.5682 ± 0.0020
Ω ($^\circ$)	226.84 ± 0.58	$226.75^{+0.83}_{-0.84}$	228.1761 ± 0.0033
ω ($^\circ$)	65.35 ± 0.56	$65.23^{+0.78}_{-0.79}$	$66.2678^{+0.0031}_{-0.0030}$
α_0 (mas)	0.38 ± 0.14	0.26 ± 0.16	
δ_0 (mas)	-0.14 ± 0.19	-0.05 ± 0.19	
$v_{\alpha,0}$ (mas/yr)	0.063 ± 0.046	$0.072^{+0.053}_{-0.052}$	
$v_{\delta,0}$ (mas/yr)	0.091 ± 0.053	$0.094^{+0.062}_{-0.063}$	
$v_{\text{LOS},0}$ (km/s)	-5.1 ± 4.1	-3.5 ± 4.5	
b_{EMDA} (AU)	$\lesssim 0.48$	$\lesssim 0.058$	$\lesssim 0.0013$
b_{EMDA} (M)	$\lesssim 12.0$	$\lesssim 1.4$	$\lesssim 0.033$

significantly narrower confidence regions for all parameters, including the dilaton parameter b_{EMDA} , which becomes much more tightly constrained, as can be seen comparing the third column of Table 5.9 to the first two. All analyses yield an upper limit for the parameter b_{EMDA} , with the general relativistic limit $b_{\text{EMDA}} = 0$ consistently within the 1σ confidence region. However, owing to the different datasets employed, the upper limit values on b_{EMDA} from the various analyses differ substantially. To facilitate comparison of these distinct constraints on the dilaton parameter, we also display the log-scale marginalized likelihoods on b_{EMDA} from the three analyses, with respective 95% confidence level upper limits denoted by vertical dashed lines. Furthermore, Figure 5.18 depicts the upper limits imposed at 68%, 95%, and 99.7% confidence levels for the parameter b_{EMDA} from our three analyses, ordered by increasing constraining power on this parameter and contrasted with the theoretical bound $b_{\text{EMDA}} < M$ compatible with a black hole spacetime. Notably, the analysis solely relying on public astrometric data yields the weakest constraints on the parameter b_{EMDA} , with $b_{\text{EMDA}} \lesssim 12M$ at 95% confidence, exceeding the theoretical bound.

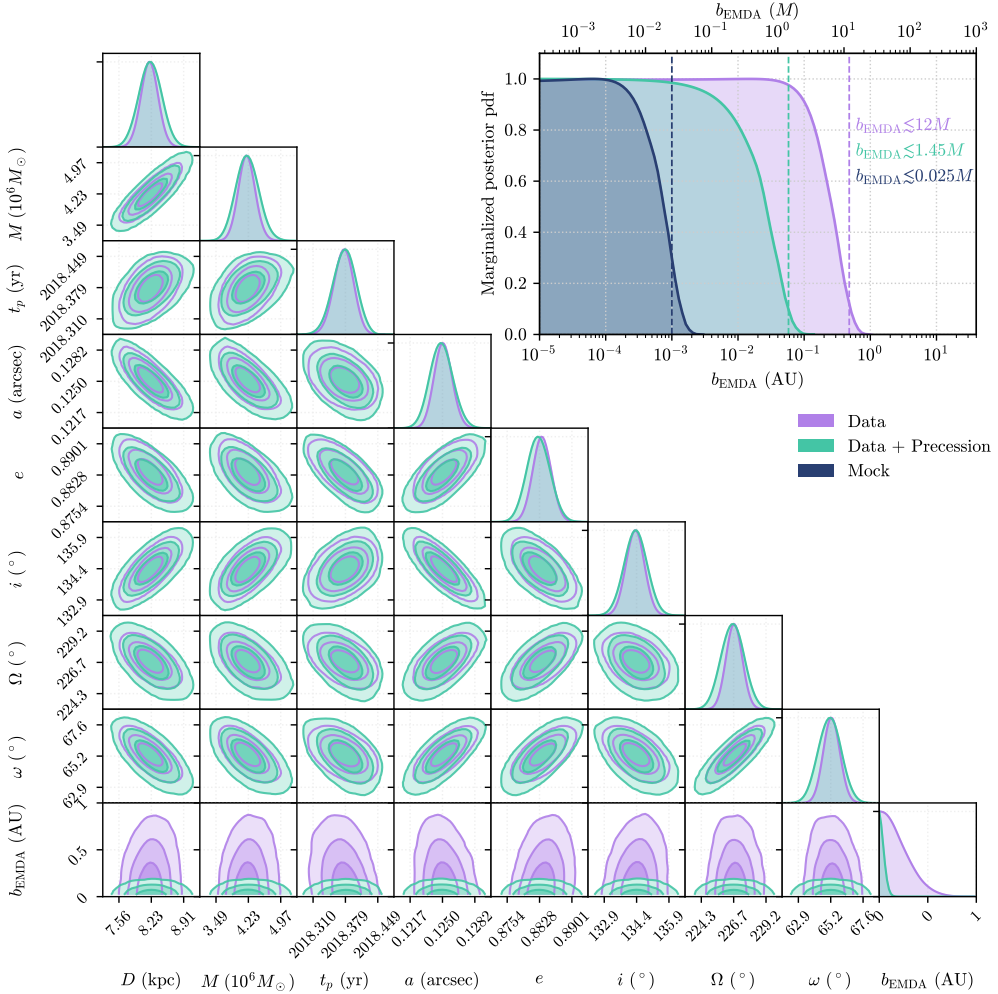


Figure 5.17. Posterior distributions for the EMDA orbital model for the S2 star in the three analyses that we have performed, as specified in the legend. In particular, we depict the 68%, 95% and 99.7% confidence regions of the two-dimensional marginalized posterior distribution for each pair of orbital parameters and for the parameter of interest in EMDA, b_{EMDA} , along with the marginalized posterior distributions for each parameter. The inset plot reports the logarithmic-scale marginalized likelihoods for the EMDA parameter b_{EMDA} in the three cases analyzed both in AU (bottom axis) and in units of gravitational radii (top axis). Dashed vertical lines correspond to the respective 95% confidence upper limits reported in Table 5.9. In the corner plot, we exclude the contours for the third analysis (using mock data) that are too small, as compared to the ones from the first two analyses, to be shown on the same plot.

Incorporating information on the orbital precession allows us to reduce the upper limit on b_{EMDA} by almost one order of magnitude, resulting in $b_{\text{EMDA}} \lesssim 1.4M$ at 95% confidence. Although this limit still surpasses the theoretical bound $b_{\text{EMDA}} < M$, at a confidence level of 68%, the same analysis yields an upper limit $b_{\text{EMDA}} \lesssim 0.8M$, thereby constraining the theoretically allowed parameter space for b_{EMDA} . Lastly, the

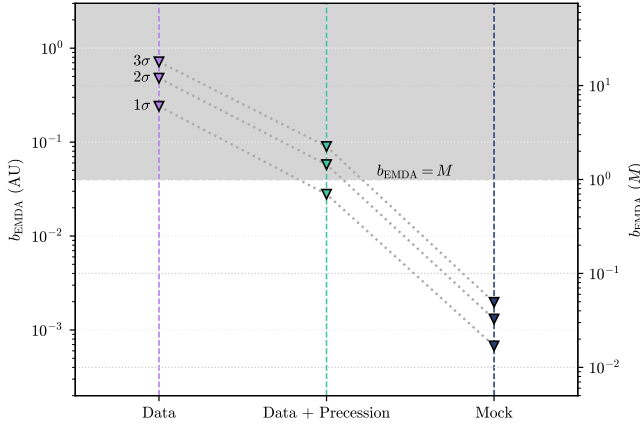


Figure 5.18. Upper limits at 68% (1σ), 95% (2σ) and 99.7% (3σ) confidence level for the parameter b_{EMDA} from our three analyses, sorted in order of increasing constraining power on this parameter. The shaded area corresponds to the theoretically excluded region $b_{\text{EMDA}} > M$.

most stringent constraints on the EMDA dilaton parameter are obtained via our sensitivity analysis employing mock data for S2. In this scenario, we predict that observations over one and a half orbits for S2, measured at a nominal accuracy of $10 \mu\text{s}$, could lower the upper limit on b_{EMDA} by nearly two orders of magnitude, yielding an upper bound at 95% confidence of $b_{\text{EMDA}} \lesssim 0.033M$, thus significantly narrowing the EMDA parameter space. This confirms the qualitative bounds derived from the analysis of the astrometric deviations.

Previous investigations in the literature have aimed to establish experimental constraints on b_{EMDA} through astrophysical observations. For instance, in [613], utilizing the optical continuum spectrum of quasars, a preferred value of $r_2 \sim 0.2$ (equivalent to $b_{\text{EMDA}} \sim 0.1$) is identified, while the range $r_2 \gtrsim 1.6$ ($b_{\text{EMDA}} \sim 0.8$) is shown to be disfavored. Conversely, in [614], the jet power and radiative efficiency of microquasars are utilized to argue that the parameter space region $r_2 \gtrsim 1$ ($b_{\text{EMDA}} \gtrsim 0.5$) is generally disfavored, with $r_2 = b_{\text{EMDA}} = 0$, namely the standard general relativistic Kerr solution, consistently favored over the EMDA Kerr-Sen black hole. Lastly, a recent investigation in [615] has established an observational constraint on the dilaton parameter by analyzing the measured shadow diameter for M87* and Sgr A*. While the shadow of M87* exhibits a preference for the general relativistic scenario, the shadow of Sgr A* surprisingly leans towards the Kerr-Sen scenario, with an allowed interval $0.1 \lesssim r_2 \lesssim 0.4$ ($0.05 \lesssim b_{\text{EMDA}} \lesssim 0.2$), attributed to the smaller shadow cast by black holes with larger dilaton parameters. Our analysis offers a complementary probe to such previous

studies and, while not detecting any deviation from General Relativity, demonstrates how observations of the S-Stars in the Galactic Center (especially with the exquisite astrometry provided by the GRAVITY interferometer) can strongly constrain the value of the dilaton parameter b_{EMDA} of EMDA.

5.5. Quadratic metric-affine gravity

As mentioned in Chapter 2, in the conventional metric formulation of General Relativity, the metric stands as the primary character, with the affine connection deemed as a secondary object and set as the Levi-Civita one, corresponding, in General Relativity, to the Christoffel symbols of the metric (Eq. (2.4)). Conversely, in the metric-affine formulation (also referred to as *Palatini formalism*), both metric and connection are regarded as independent entities, to be determined through variations of the theory's action with respect to each of them [616]. For a broad class of such metric-affine theories, that only include contractions with the symmetric part of the Ricci tensor (referred to as Ricci-based gravities [617]), the resulting field equations are of second order, devoid of ghost-like propagating degrees of freedom (for further elaboration on this matter, refer to [618]). In this section, we focus on a quadratic extension of General Relativity coupled to a sourceless electric field, that was first studied in depth in [619], described by the action

$$\mathcal{S} = \mathcal{S}_{\text{GR}} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} [\ell^2 (aR^2 + R_{\mu\nu}R^{\mu\nu})] + \left. \begin{aligned} & - \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu}F^{\mu\nu}. \end{aligned} \right|_{5.75}$$

Here, $\mathcal{S}_{\text{GR}}$ is the usual Hilbert-Einstein action of General Relativity expressed in Eq. (2.6), a represents a dimensionless constant, g denotes the determinant of the spacetime metric $g_{\mu\nu}$, and ℓ (with dimensions of length) regulates the higher-curvature corrections within the Lagrangian. The quantities $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and $R_{\mu\nu}(\Gamma)$ respectively denote the Ricci scalar and the Ricci tensor associated with the affine connection $\Gamma \equiv \Gamma_{\mu\nu}^\lambda$, which remains independent of the spacetime metric $g_{\mu\nu}$. Regarding the matter fields, the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is introduced, linked to the 4-vector potential A_μ .

The choice of the action (5.75) is justified on the grounds of results from the theory of quantized fields in curved space-times [620], which requires the inclusion of higher-order curvature corrections suppressed by a (length-squared) scale, conventionally associated with Planck-scale effects. However, for our analysis, we treat ℓ as a free parameter, subject to observational constraints.

We focus on this specific extension of General Relativity owing to the fact that it has been extensively studied in the literature [619, 621–624] including a recent determination of the optical appearance of its black hole solutions [625] that exhibits a rich phenomenology with the appearance of additional light rings when compared to the general relativistic case, hinting at a possible detectability of the observational features of such object. In fact, exact analytical solutions for the spacetime metric associated with such a theory have first been derived in [622]⁸. Employing ingoing Eddington-Finkelstein coordinates and adopting geometrized units, $G = c = 1$, the corresponding line element can be expressed as

$$ds^2 = -A(x)dv^2 + \frac{2}{\sigma_+}dvdx + r^2(x)d\Omega^2, \quad | \quad 5.76$$

with the following definitions:

$$A(x) = \frac{1}{\sigma_+} \left[1 - \frac{2M}{r} \frac{(1 + \delta_1 G(r))}{\sigma_-^{1/2}} \right], \quad | \quad 5.77$$

$$\delta_1 = \frac{1}{4M} \sqrt{\frac{r_q^3}{\ell}}, \quad | \quad 5.78$$

$$\sigma_{\pm} = 1 \pm \frac{r_c^4}{r^4(x)}, \quad | \quad 5.79$$

$$r^2(x) = \frac{x^2 + \sqrt{x^4 + 4r_c^4}}{2}. \quad | \quad 5.80$$

Here, M denotes as usual the mass of the central object as observed from an asymptotic observer, $r_c = \sqrt{\ell r_q}$ is a characteristic length given by the geometric mean of the higher-curvature scale length ℓ of the theory and the charge-related length $r_q^2 = 2Gq^2$. On the other hand, the function $G(z)$ that characterizes the metric function, where $z = r/r_c$

⁸ It's worth noting that, as elucidated in [617], owing to a fortuitous coincidence these solutions also emerge within another metric-affine theory of gravity, the so-called Eddington-inspired Born-Infeld theory [626]

serves as a dimensionless radial coordinate, is given by

$$G(z) = -\frac{1}{\delta_c} + \frac{1}{2}\sqrt{z^4 - 1} \left[f_{3/4}(z) + f_{7/4}(z) \right], \quad | \quad 5.81$$

where $\delta_c \approx 0.572069$ is a constant resulting from matching the large and small z expansions, and the function $f_\lambda(z)$ involves the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$:

$$f_\lambda(z) = {}_2F_1 \left[\frac{1}{2}, \lambda; \frac{3}{2}; 1 - z^4 \right] \quad | \quad 5.82$$

The line element in Eq. (5.76) serves as a generalization of the Reissner-Nordström (RN) geometry of General Relativity. Notably, for large values of z (corresponding to the limit $r \gg r_c$), the function $G(z)$ in Eq. (5.81) tends towards $G(z) \sim -1/z$. Concurrently, in the same limit, $\sigma_\pm \rightarrow 1$, causing the two radial coordinates to converge, $r^2(x) \rightarrow x^2$. Consequently, in this limit, the metric function simplifies to:

$$A(x) \approx 1 - \frac{2M}{r} + \frac{r_q^2}{2r^2}, \quad | \quad 5.83$$

which corresponds to the usual RN geometry of General Relativity expressed in our variables.

By redefining the time coordinate in Eq. (5.76) according to $dv = dt + dx/(A\sigma_+)$, an alternative expression for the line element emerges, bearing a closer resemblance to the classical RN solution:

$$ds^2 = -A(x)dt^2 + \frac{1}{B(x)}dx^2 + r^2(x)d\Omega^2, \quad | \quad 5.84$$

where $B(x) = A(x)\sigma_+^2$. In order to preserve the simplicity of the representation of $r^2(x)$, introduced in Eq. (5.80), we refrain from absorbing the factor σ_+ into a redefinition of the coordinate x . This radial coordinate, x , spans the entire real line, *i.e.* $x \in]-\infty, +\infty[$, causing the quantity $r(x)$ to have a minimum r_c for $x = 0$. This circumstance, much resembling the case of the Black Bounce metric analyzed in Section 4.2, implies that the surface area of hypersurfaces of constant t and x (with the geometry of a 2-sphere of radius $r(x)$) reaches a minimum at $x = 0$. Similarly, this bounce in the (radial coordinate of the) geometry, signifies a non-trivial topological structure in the form of a wormhole [627], with $r = r_c$ serving as its throat. Additionally, as $dx^2 = \sigma_+^2 dr^2 / \sigma_-$, at $x = 0$ (*i.e.* at $r = r_c$), we find $dr/dx = 0$. Consequently, a coordinate

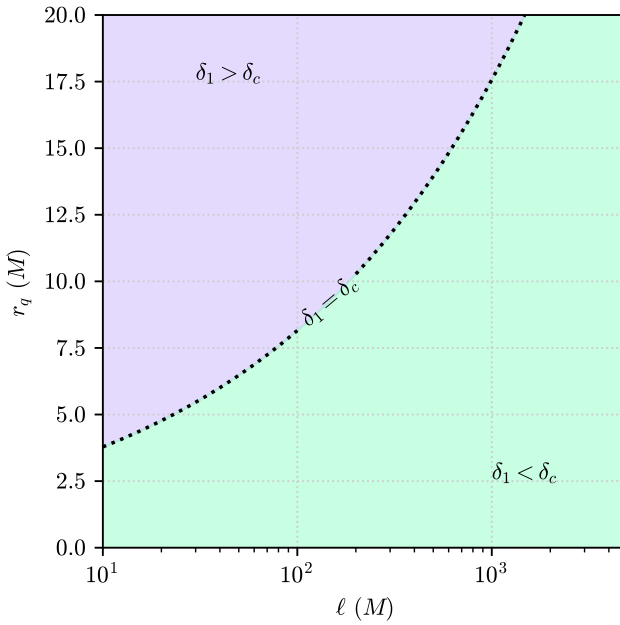


Figure 5.19. The (ℓ, r_q) parameters separated by the extremal case (dotted-line) $\delta_1 = \delta_c$.

transformation from x to $r(x)$ becomes undefined at the throat. Hence, if one aims to employ r as the radial coordinate, it becomes necessary to restrict intervals in which $r(x)$ is a monotonic function [628]. This requires dividing the domain of r into two regions, $x > 0$, and $x < 0$; nevertheless, the transition from one region to the other is seamless, allowing for the completion of all geodesic trajectories across $x = 0$ [624]. Moreover, it has been demonstrated in [629] that the presence of potential curvature divergences does not lead to catastrophic processes for extended (time-like) observers. These features allow to consistently interpret these solutions as regular black holes and naked geometries, depending on the interplay of parameters. However, here we find it more convenient to utilize $x > 0$ as the radial coordinate, as we are solely concerned with the geometry of one side of the throat.

The causal structure of the geometry described by Eq. (5.84) can be characterized in terms of the values of the free parameters. Close to the central object, *i.e.* for $r \rightarrow r_c$ (or equivalently $x \rightarrow 0$), significant deviations from the RN solution emerge. If we consider a power series expansion of $A(r(x))$ around $x = 0$, we obtain

$$\lim_{r \rightarrow r_c} A(x) \approx \frac{N_q}{4N_c} \frac{(\delta_1 - \delta_c)}{\delta_1 \delta_c} \sqrt{\frac{r_c}{r - r_c}} + \frac{N_c - N_q}{2N_c} + \mathcal{O}(\sqrt{r - r_c}), \quad | \quad 5.85$$

where $N_q = q/e$ represents the number of charges (with e being the electron charge), and $N_c \equiv \sqrt{2/\alpha} \approx 16.55$ (where α is the fine structure constant). It can then be shown that the causal structure of the spacetime in Eq. (5.84) depends on the value of the charge-to-mass ratio, δ_1 , relative to the critical value δ_c [619, 622, 623]. Specifically:

- For $\delta_1 < \delta_c$, an event horizon always exists on both sides of the throat, irrespective of the value of N_q . Due to the presence of the horizons, this can be considered a black hole solution, closely resembling the Schwarzschild solution, from the perspective of observers situated on either side of the throat, outside the respective horizon.
- For $\delta_1 > \delta_c$, however, the spacetime's causal structure is more complex and exhibits either two (non-degenerate) horizons, a degenerate one, or no horizons at all, on each side of the throat, depending on the value of N_q (refer to [619] for an exhaustive analysis of the horizons and causal structure in this scenario). Regardless of the configuration, though, the geometry of spacetime near the center undergoes significant alterations compared to its general relativistic counterpart [619].
- In the extremal case $\delta_1 = \delta_c$, three sub-cases arise depending on the value of N_q . Specifically, if $N_q > N_c$, two horizons symmetrically positioned with respect to the throat emerge; if $N_q = N_c$, the two horizons coincide at the throat, $r = r_c$ (or $x = 0$); finally, if $N_q < N_c$, the horizons vanish, resulting in a sort of black hole remnant, possibly representing the endpoint of Hawking evaporation or arising from large density fluctuations in the early Universe [630]. Importantly, for all these configurations, all the curvature scalars remain finite [629].

When one particularizes these conditions on δ_1 to an object with mass matching that of Sgr A*, one arrives at the limit $N_q = N_c$ for an exceedingly low value of $r_q = 3.2315 \times 10^{-35}$ m, especially when compared with the scales of the astrophysical objects relevant for our purpose, *i.e.* the S-stars in the Galactic Center. Consequently, we will disregard the

cases $N_q \leq N_c$. For completeness, Figure 5.19 illustrates where the demarcation between the $\delta_1 > \delta_c$ and $\delta_1 < \delta_c$ cases lies on the (ℓ, r_q) plane.

Despite the fundamental paradigm shift in the metric affine formalism, that considers the connection as independent of the metric, an important result in literature [631] is that freely-falling test particles in these theories are determined by the field equations alone and turn out to be the geodesics associated with the metric compatible connection. This means that we can keep using the same geodesic equations defined in Eq. (2.3) with the usual definition of the Christoffel symbols, Eq. (2.4), built upon the metric in Eq. (5.84). This yields the following set of equations

$$\ddot{t} = - \frac{\dot{t}\dot{x}A'(x)}{A(x)}, \quad | \quad 5.86$$

$$\begin{aligned} \ddot{x} = & \dot{\phi}^2 B(x)r(x) \sin^2(\theta)r'(x) - \frac{\dot{t}^2 B(x)A'(x)}{2} + \\ & + \dot{\theta}^2 B(x)r(x)r'(x) + \frac{\dot{x}^2 B'(x)}{2B(x)}, \end{aligned} \quad | \quad 5.87$$

$$\ddot{\theta} = \dot{\phi}^2 \sin(\theta) \cos(\theta) - \frac{2\dot{\theta}\dot{x}r'(x)}{r(x)}, \quad | \quad 5.88$$

$$\ddot{\phi} = - \frac{2\dot{\phi}\dot{\theta} \cos(\theta)}{\sin(\theta)} - \frac{2\dot{\phi}\dot{x}r'(x)}{r(x)}. \quad | \quad 5.89$$

We solve the geodesic equations Eqs. (5.86)-(5.89) numerically, by applying the procedure to fix initial conditions for time-like geodesics delineated in Section 2.2.3, and obtain orbits for the S2 star in the spacetime described Eq. (5.84). This allows us to assess the orbital deviation from a purely Schwarzschild orbit on the S2 star astrometric and spectroscopic observables arising from the modification to the metric. In particular, we calculate the deviations in the two sky-projected observables and the line-of-sight velocity, as illustrated in Figure 5.20 for several different values of ℓ ranging from $0M$ (corresponding to a purely RN black hole) to $7000M$, and for different values of the charge-related scale length r_q between $0M$ and $7M$. Interestingly, the smallest astrometric deviation from the general relativistic case is obtained for $\ell = 1400M \approx 60 \text{ AU}$ which approximately corresponds to the semi-distance of the S2 at apocenter from the central black hole. This feature, which as we will see is also found in the estimation of the relativistic observables (*i.e.* the orbital

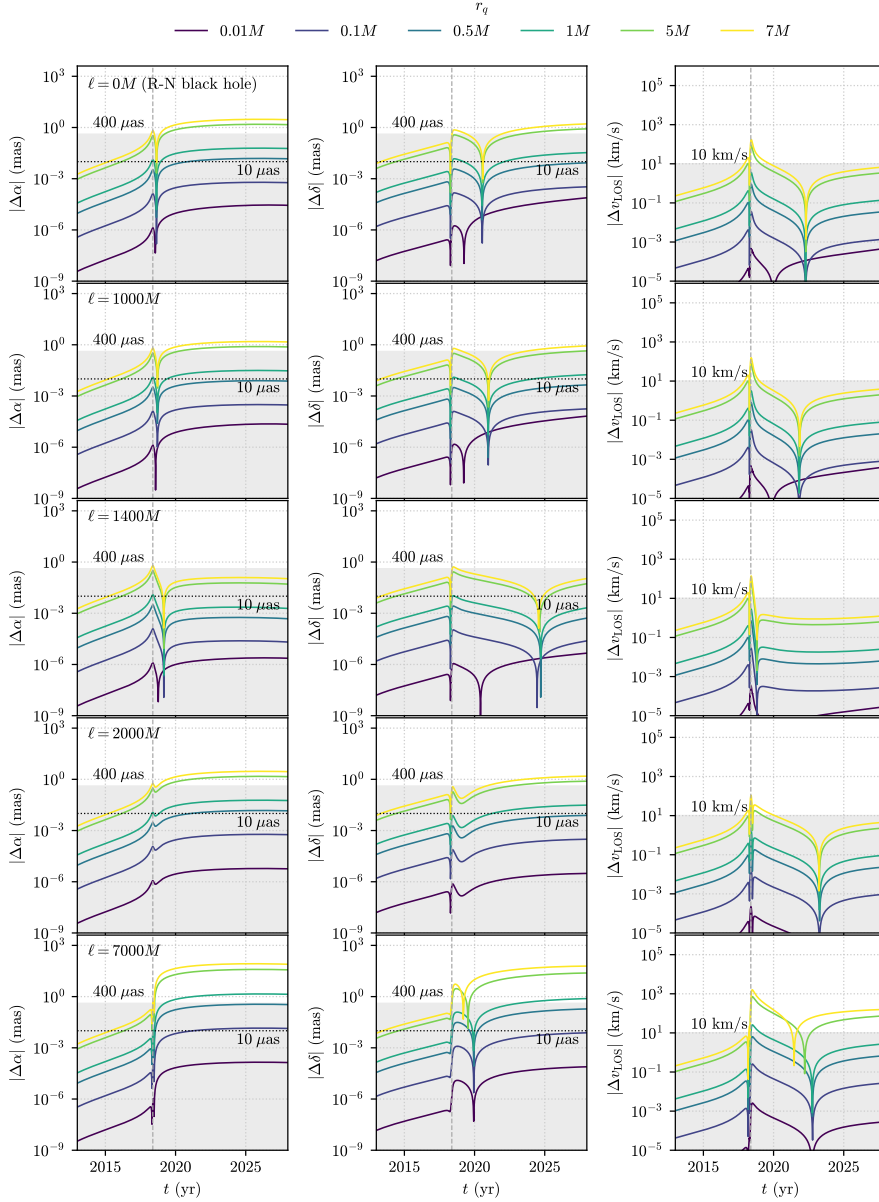


Figure 5.20. Deviation of the astrometric (right ascension α and declination δ , left-most and central panels) and spectroscopic (line-of-sight velocity v_{LOS} , right-most panels) observables for the orbit of the S2 star in the Galactic Center from the general relativistic case for the metric-affine black hole considered in this section. From top to bottom we consider increasingly high values of the metric-affine length ℓ , starting from $\ell = 0$ (i.e. a purely RN black hole) to $\ell = 7000M$ and for each of these cases we consider 6 values of the parameter r_q between $0.01M$ and $7M$ as detailed in the legend. The minimal astrometric deviation is obtained for $\ell \approx 1400M$, incidentally coinciding with the semi-separation between Sgr A* and S2 at pericenter passage.

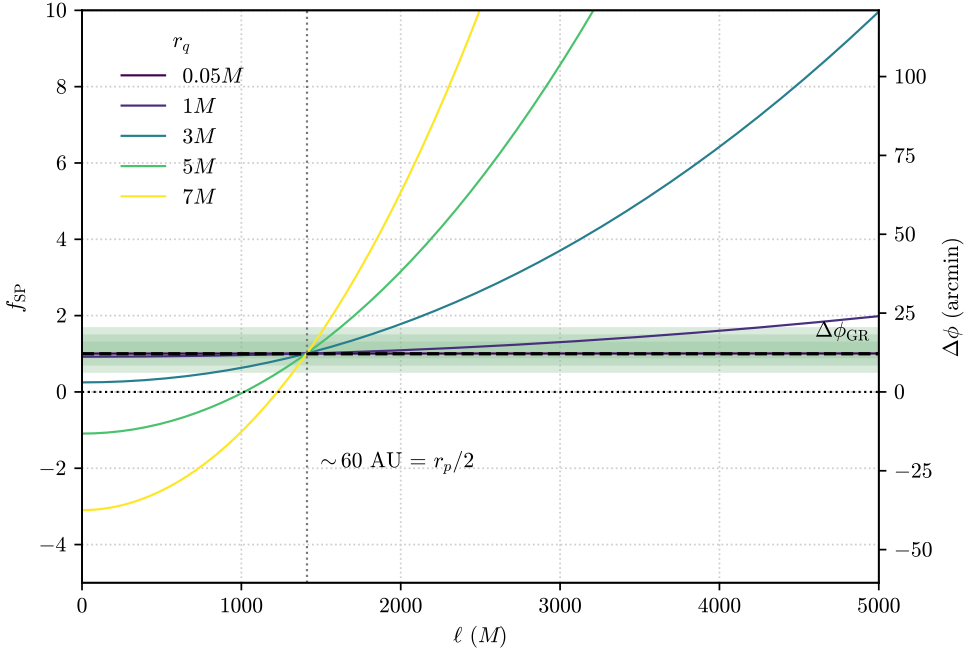


Figure 5.21. Rate of orbital precession for the S2 star in the charged black hole spacetime for the quadratic metric-affine extension of General Relativity that we have considered. In particular, we compute it as a function of the of the higher curvature parameter parameter ℓ for several different values of the charge length, as detailed in the legend. The green horizontal shaded bands correspond to the 1σ , 2σ and 3σ bounds placed by the analysis in [226]. We exhibit here the strong deviations predicted in the relativistic observable for S2 in this extension of General Relativity. Remarkably, when the parameter ℓ matches the semi-separation between S2 and Sgr A* at pericenter, the predicted rate of orbital precession equals the General Relativity prediction ($f_{\text{SP}} = 1$), regardless of the value of r_q , thus implying a compensation of effects in the orbital dynamics, even for underlying geometries that depart greatly from that of a Schwarzschild black hole.

precession) will strongly affect the shape of the constraints obtained on the parameters ℓ and r_q of this model. As done before, with horizontal lines, we report the mean astrometric uncertainty, at approximately $400 \mu\text{as}$, for pre-2017 VLT observations of the S2 star [138], and the nominal astrometric precision of the GRAVITY interferometer [170], set at $10 \mu\text{as}$. For some combinations of the metric-affine parameters, the astrometric deviation exceeds both the NACO and the GRAVITY sensitivity thresholds hinting at the possibility of placing constraints on the spacetime metric in Eq. (5.84) even with current observations.

Additionally, we compute from our numerically integrated orbits the rate of orbital precession f_{SP} , which we display in Figure 5.21 as a function of ℓ and for several values of r_q , as considered before. Strong modifications to the spacetime

metric have a significant impact on the relativistic periastron advance, which, depending on the parameters combination, can exceed by much (f_{SP} can reach ~ 10 in the considered range of the parameters) the value predicted by General Relativity, in Eq. (2.41), or even become negative (e.g. $f_{\text{SP}} \sim -4$ for $\ell = 0$ and $r_q = 7M$), implying retrograde precession. Furthermore, when ℓ approaches values around $1400M \sim 60 \text{ AU}$ (incidentally corresponding to the semi-distance of S2 from Sgr A* at its pericenter passage. r_p), the orbital precession in metric-affine gravity consistently aligns with the value in General Relativity, regardless of the value of r_q . This effect, already noted in the analysis of the astrometric deviations, implies a compensation of effects in the orbital dynamics, even for underlying geometries that deviate significantly from that of a Schwarzschild black hole. As usual, the horizontal green stripes in the plot represent the confidence intervals for the experimentally derived rate of orbital precession by the Gravity Collaboration after the last pericenter passage of S2 in 2018 [226]. The significant deviations highlighted by our analysis qualitatively indicate that the orbital data for S2 should be capable of constraining these two additional parameters despite potential degeneracies with the orbital parameters.

Assessing the astrometric implications of the non-singular black hole spacetime in Eq. (5.84) on the geodesic trajectory of the S2 star and its rate of orbital precession provides a preliminary insight into potential constraints on the parameters ℓ and r_q . To establish actual constraints, we conducted a Bayesian analysis using publicly available data. Specifically, spectroscopic and astrometric data from [138], covering the pre-2017 orbit. This dataset was supplemented with information on the orbital precession measured from [226]. For this reason, our analysis, as outlined in Appendix C, was performed adopting the likelihood function detailed in Eq. (C.3).

Uniform prior		
	Start	End
$\ell (M)$	0	5000
$r_q (M)$	0	7

Table 5.10. Uniform prior on the parameters ℓ and r_q used for our Bayesian analysis.

In this case, the orbital model for S2 involves a total of 15 parameters, including the 13 parameters of Eq. (C.1), together with the additional parameters r_q and ℓ characterizing the metric-affine quadratic extension to General Relativity considered here. The priors utilized for this analysis are composed by a combination of those presented in Table C.1, along with heuristic priors on r_q and ℓ , motivated by the aforementioned analysis of the astrometric impact. These are reported in Table 5.10.

Parameter (unit)	Posterior
D (kpc)	8.07 ± 0.20
M ($10^6 M_\odot$)	$4.08^{+0.22}_{-0.21}$
t_p (yr)	2018.363 ± 0.023
a (arcsec)	0.1260 ± 0.0010
e	0.8840 ± 0.0023
i ($^\circ$)	133.93 ± 0.48
Ω ($^\circ$)	227.01 ± 0.77
ω ($^\circ$)	$65.67^{+0.72}_{-0.71}$
α_0 (mas)	0.32 ± 0.16
δ_0 (mas)	-0.07 ± 0.19
$v_{\alpha,0}$ (mas/yr)	0.084 ± 0.052
$v_{\delta,0}$ (mas/yr)	0.126 ± 0.063
$v_{\text{LOS},0}$ (km/s)	-2.0 ± 4.4
ℓ (M)	See Figure 5.23
r_q (M)	See Figure 5.23

Table 5.11. Resulting 68% confidence intervals for all the orbital parameters from our MCMC analysis on the quadratic metric-affine gravity. All orbital parameters are bound and consistent within 1σ with their best fitting values from other analyses in the literature [138]. The posterior on the parameters ℓ and r_q does not allow to derive easily expressable bounds on these two parameters, and for this, we refer to Figure 5.23.

In Table 5.11, we present the median-centered 68% confidence intervals for the orbital and reference frame parameters obtained from the sampled posterior distribution, while in Figure 5.22 we report a corner plot displaying all the marginalized posterior distributions for the orbital parameters and the extra parameters r_q and ℓ in the metric-affine extension considered. All parameters are well constrained, and the confidence intervals align with those from their previously found values. Figure 5.23 focuses on the confidence regions for our parameters of interest for the metric-affine spacetime solution in Eq. (5.84). In particular, the contour plot showcases the allowed regions (at 1, 2, and 3σ confidence) for the parameters r_q and ℓ derived from our analysis, while the marginalized posterior distributions for the two parameters are shown on the diagonal of the corner plot.

Our orbital model for S2 tends to favor small values of r_q , typically below $2.7M$, for values of ℓ below the semi-radius of the pericenter passage of S2, around 60 AU (indicated as a dotted vertical line in our contour plot). For $\ell = 60$ AU, the orbit of S2 appears to be insensitive to the particular value of r_q , as previously highlighted in our qualitative analysis of the astrometric deviations and the orbital precession as a function

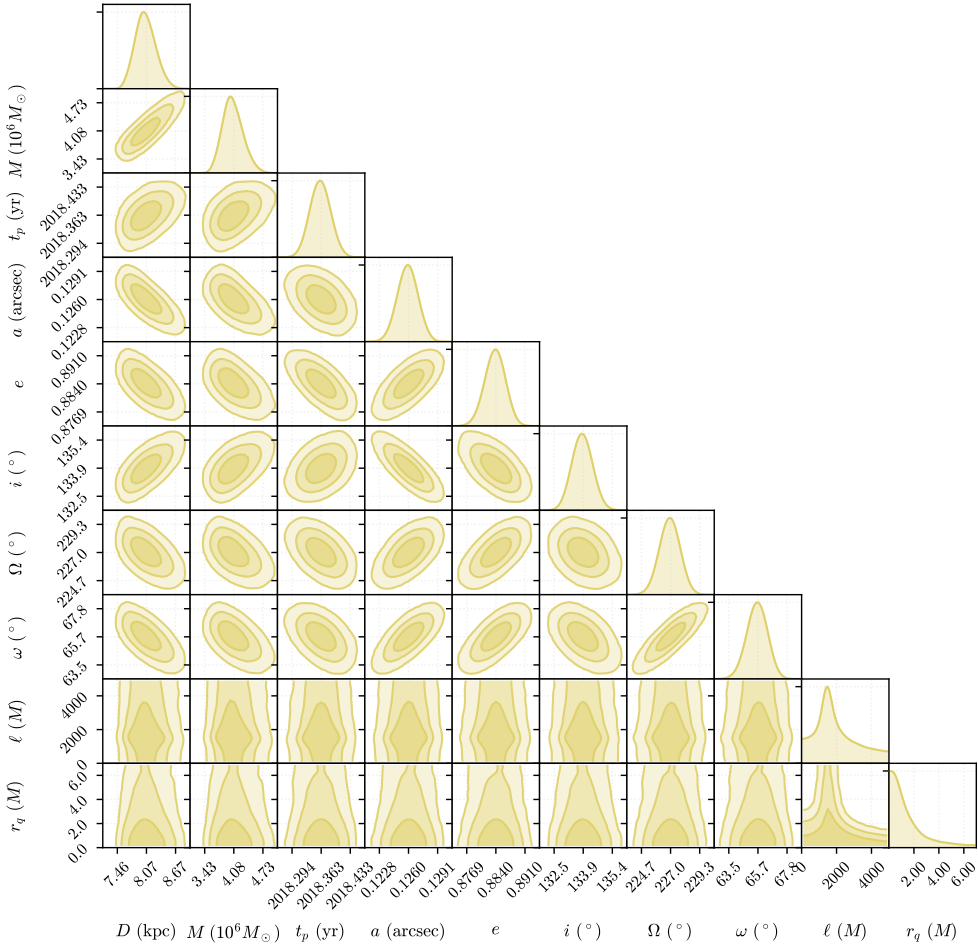


Figure 5.22. Posterior distributions on the orbital model for the S2 star in the quadratic metric-affine extension to General Relativity that we have considered. In particular, we depict the 68%, 95% and 99.7% confidence regions of the two-dimensional marginalized posterior distribution for each pair of orbital parameters and for the parameters of interest in this theory, r_q and ℓ (both expressed in units of gravitational radii M).

of ℓ and r_q (Figures 5.20 and 5.21). For values of $\ell > 60$ AU, deviations in the orbital dynamics of S2 become pronounced even for very small values of r_q , resulting in a narrowed credible interval for this parameter, with $r_q < M$ at $\ell \sim 5000M$. Remarkably, our analysis constrains r_q and ℓ within the $\delta_1 < \delta_c$ region (also shown in the plot), implying that the orbit for S2 favors a central object resembling a Schwarzschild black hole when described by the spacetime metric in Eq. (5.84). This exclusion, within the range of our priors, indicates severe modifications to the underlying geometries predicted

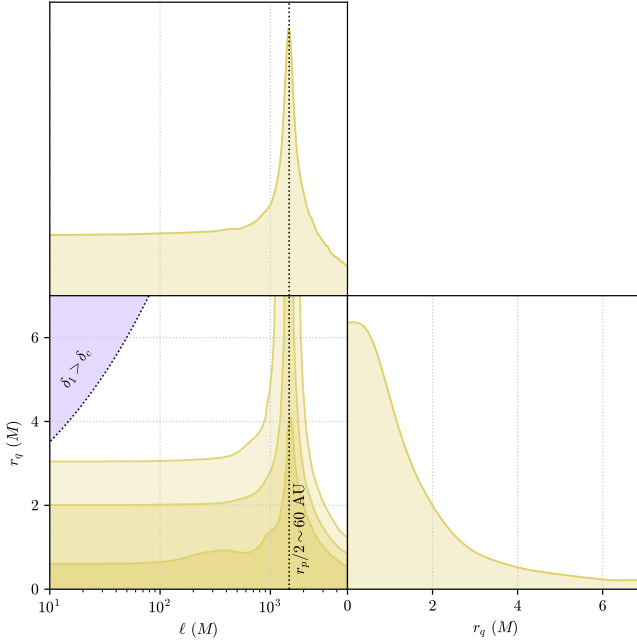


Figure 5.23. Posterior probability distribution for the parameters of interest of our analysis, namely r_q and ℓ . In particular, the top and right panels show the marginalized posterior distributions for the two parameters, while the bottom-left panel shows the 2d-slice of the full posterior distribution corresponding to the r_q - ℓ plane. The 1σ , 2σ and 3σ regions are highlighted with increasingly darker shades. Moreover, the vertical dashed red line corresponds to a value of $\ell \sim r_p/2$. The shaded violet region corresponds to the class of models with $\delta_1 > \delta_c$. Remarkably, our analysis bounds the parameter space in the region $\delta_1 < \delta_c$, showing that the orbit of S2 is sensible to such strong modifications of the spacetime metric.

for cases $\delta_1 \geq \delta_c$, that confrontation with actual orbital data seems to exclude.

5.6. Conclusions

We have explored the ability of observations at the Galactic Center to constrain the underlying theory of gravity. Several different extensions and alternative formulations to General Relativity exist, each based on a different assumption. We have shown how the methodology developed in Chapter 2 to describe the general relativistic geodesic motion of massive test particles can be applied to obtain astrometric, spectroscopic and relativistic observables for the S2 star in the Galactic Center in a variety of different theoretical scenarios. We have analyzed several different classes of ETGs, starting with STVG, $f(R)$ gravity, Horndeski gravity and a quadratic extension to General Relativity, and a popular example of ATG, namely EMDA, for which an analytic black hole solution exists. In all cases, we have performed a posterior analysis on the orbital model for the S2 star built upon the geodesic equations for spherically symmetric solutions in each

of the models considered. We have taken into account publicly available data for S2, comparing the results obtained without and with the introduction of the measured rate of orbital precession for S2, demonstrating the great constraining power brought on by this relativistic observable for modifications at a metric level of the gravitational field.

None of the modifications that we have considered to General Relativity, all involving additional complexity in the theoretical formulation and thus a richer phenomenology, have been found to explain *better* the orbital data for the S-stars. In fact, all the posterior analyses performed have restricted the allowed parameter space to only small deviations from General Relativity. Quite notably, in all cases, although we have always adopted an agnostic approach, by assigning uniform priors to each theory extra parameters, we have always obtained posteriors that are centered on the region of the parameter space that corresponds to the general relativistic limit of the theories that we have studied. This is remarkable and, once again, demonstrates the great predictive power of General Relativity that, to date, gives the simplest and best description of the observational data.

Pulsars at the Galactic Center: the next breakthrough

6.

A hypothetical pulsar orbiting a SMBH and the timing analysis of its radio pulses would provide an ideal probe to measure the mass, the spin, and the quadrupole moment of a black hole, and consequently to test the no-hair theorem [305–307]. The potential impact of these observations on our current understanding of gravity would be unparalleled, opening up to the possibility not only to test General Relativity in the strong-field regime at an unprecedented level of accuracy but also to falsify it against alternative theories of gravity with astounding precision [308].

As mentioned in Section 1.3, the significant scientific potential offered by pulsars in the Galactic Center has spurred an increasing number of radio pulsar searches within the central few parsecs of the Milky Way [311–315, 324]. Despite dedicated efforts, these searches have faced challenges in locating the pulsar population of interest. Only a handful of such objects have been discovered in the region [313, 319], none of which is close enough to Sgr A* to enable such tests.

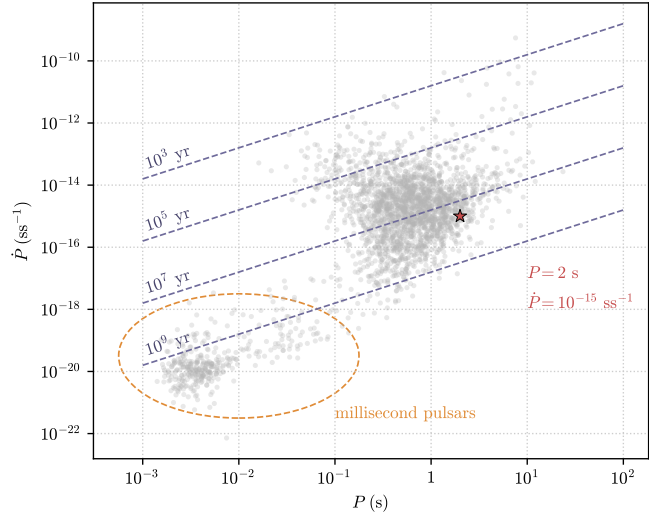
The detection of at least one pulsar on a tight orbit around Sgr A* represents a crucial scientific objective for forthcoming observational facilities like SKA [329]. Benefitting from substantial collection areas, SKA has been demonstrated to potentially be able to conduct precise timing analyses [332] with an expected accuracy on the pulse TOA measurement of $\sigma_{\text{TOA}} \simeq 100 \mu\text{s}$ for normal pulsars (see Figure 6.1 where we report a diagram of all known pulsars and their intrinsic period and spin down rate) at frequencies above 15 GHz [307]. This estimate comes from considering all possible limiting effects on the precision of TOA measurements of young pulsars in the Galactic Center environment, including the signal-to-noise ratio of the measured pulses, the intrinsic pulse phase jitter and the changes in pulse shape caused by interstellar scintillation.

In this chapter, we will assume that pulsars at the Galactic Center actually exist and can be successfully detected and timed within the accuracy goal of $100 \mu\text{s}$ per TOA, and we will apply the methodology developed in Section 2.2.5 to derive credible TOAs estimate for some realistic toy models. In

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Figure 6.1. The period P versus the period derivative $\dot{P}$ diagram for all known pulsars, data taken from the ATNF Pulsar Catalogue version 2.1.1 [632] (<https://www.atnf.csiro.au/research/pulsar/psrcat/>).

Dashed lines correspond to constant characteristic age assuming that pulsars are spinning down solely because of magnetic dipole radiation. Two distinct populations appear: normal pulsars with periods between ~ 0.1 s and 10 s, and millisecond pulsars with periods between 0.001 s and 0.01 s (highlighted in the orange circle). The red star corresponds to a pulsar with period $P = 2$ s and rate of change of period of $\dot{P} = 10^{-15}$, falling within the class of normal pulsars, that we will use as a reference model throughout this chapter.



particular, in Section 6.1 we first estimate how a modification of the underlying theory of gravity or the fundamental nature of the central object would impact the timing residuals from a Schwarzschild scenario, for three different well-motivated alternative models. Then, in Section 6.2, we perform parameter estimation from pulsar TOAs, developing a fully relativistic timing model and fitting technique built upon our photon propagation time calculations. We finally apply our methodology on a TOAs mock catalogue for one of the toy models introduced and show the capability of the algorithm to recover the input parameters with outstanding accuracy.

6.1. Photon propagation time for pulsars in alternatives to the Schwarzschild spacetime

The original scientific results presented in this section have been published in [11] and awarded a public prize issued by the Spanish Society for Gravity and General Relativity (SEGRe) in July 2023.

In this section, we consider the photon propagation time for light rays emitted by orbiting pulsars in several spherically symmetric spacetimes that represent alternatives to the classical Schwarzschild black hole, that we have already analyzed through the orbital motion of the S-stars in previous chapters of this thesis. Specifically, we focus on the Black Bounce (Section 4.2) black hole mimicker model, which explores alternative central object natures within General Relativity;

Model	a (AU)	a (mas)	e	T
Toy 0	40	4.8	0	1.43 days
Toy 1	175.4	21.1	0.800	1.1615 yr
Toy 2	43.8	5.28	0.800	52.9 days
Toy 3	5	0.60	0.786	2.0 days

Table 6.1. The orbital parameters (semi-major axis in both physical units and angular dimension assuming a distance of $D = 8$ kpc for the Galactic Center, eccentricity and orbital period) of the toy models used for our analysis.

the black hole solution in STVG (discussed in Section 5.1), an ETG incorporating additional scalar and vector degrees of freedom; and the Sen black hole solution in EMDA (detailed in Section 5.4), which emerges from a string theory framework as an ATG.

As we have thoroughly analyzed in the corresponding sections for each of the alternative models considered, these spherically symmetric spacetimes are all characterized by the addition of one extra parameter encoding the deviations from General Relativity. In particular, in the Black Bounce model the spacetime metric is tuned by the throat radius α_{BB} (which has thus dimensions of a length) whose value determines the fundamental nature of the central object (either a regular black hole or a two-way traversable wormhole, with $\alpha_{\text{BB}} = 2M$ separating the two families of geometries). Similarly, STVG presents an increased gravitational constant regulated by a dimensionless parameter α_{STVG} , which also modulates the strength of the additional vectorial degree of freedom in the theory. Finally, EMDA, which encompasses an extra dilatonic field, presents a modified black hole spacetime metric modulated by a dilaton parameter b_{EMDA} (with dimensions of a length) whose allowed range of values is $0 < b_{\text{EMDA}} < M$.

The aim of this section is to quantify the deviations that are produced in the TOAs of photons emitted by a pulsar orbiting each of the alternative models considered as a function of the theory parameters. This would give us an idea of the possible range of constraints that future pulsar timing analysis at the Galactic Center would be able to provide on such alternative models. To do so, we particularized the methodology developed in full generality in Section 2.2.5 for a spherically symmetric spacetime, to the specific cases of Black Bounce, STVG, and EMDA solutions.

Initially, we focused on the same toy object considered at the end of Section 2.2.5 representing a circular orbit with a constant distance from the central object of $100M$ on an edge-

on configuration (we will refer to this object as Toy 0 and report its properties for completeness in Table 6.1 along with other toy objects on which we will focus later in this section), calculating the difference between the resulting propagation delay and that of the Schwarzschild scenario. Throughout this section we fix, without loss of generality, the distance of the distant observer at $D = 8$ kpc from the central object, thus mimicking an Earth-based observatory for a putative pulsar orbiting the SMBH in the Galactic Center. The outcomes of our computations are illustrated in Fig. 6.2 for all three models, covering various values of the additional parameters within their respective ranges. Specifically, for each model, we subtracted the propagation delay in the Schwarzschild scenario (achieved by setting all theory parameters to zero) from that obtained with four distinct values of the theory parameter, uniformly sampled across the range of interest. To maintain consistency, we subtracted a constant value, ensuring that the difference between the two values at $\varphi = \pi/2$ would be 0. Across all cases, a higher value of the theory parameter correlated with a greater deviation from the Schwarzschild scenario. Furthermore, the most significant difference between the computed delay and the Schwarzschild scenario emerged at $\varphi = \pi$. Due to the edge-on configuration of the orbit, this case corresponds to the situation in which the pulsar is farthest from the observer lying right *behind* the central object from the point of view of the distant observer, thus requiring the photon to traverse the longest possible path within the gravitational field of the SMBH, and thereby resulting in the highest deviation. Our results underscore that even in cases such as Toy 0's perfectly circular orbit, devoid of relativistic effects on the orbit itself, the propagation delay still provides substantial insights into the spacetime metric. Modifications to the spacetime metric for the range of parameters considered lead to differences of up to several seconds in the photon arrival time. For instance, in the Black Bounce model, such differences amount to approximately 1 second for $\alpha_{\text{BB}} = 2M$ (red line, corresponding to the wormhole spacetime) and can extend to around 3 seconds for $\alpha_{\text{BB}} = 4M$, signifying an unambiguously measurable effect linked to the fundamental nature of the central object.

While the results obtained for Toy 0 demonstrate the potential to detect spacetime modifications at a metric level through photon propagation delays from pulsars, this system may not be the most suitable tool for such investigations. Firstly, Toy 0

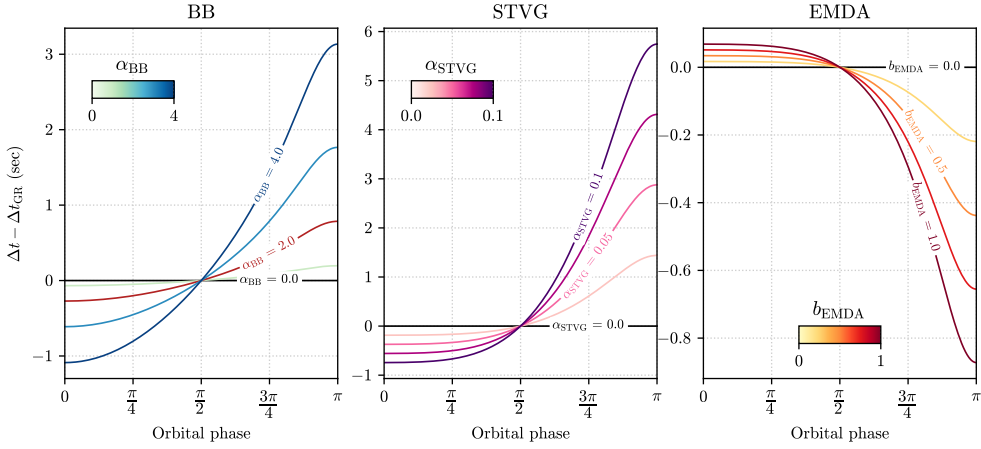


Figure 6.2. Differences in seconds from the Schwarzschild case of the propagation delay in the Black Bounce (left panel; the red line corresponding to $\alpha_{\text{BB}} = 2M$ marks the separation between black hole and wormhole solutions in the model), STVG (central panel), and EMDA (right panel) models, respectively, computed numerically with our code for a central object of $M = 4 \times 10^6 M_{\odot}$ and for the orbiting object Toy 0 in Table 6.1. The 0-level black solid lines correspond to the general relativistic Schwarzschild limit of each model. Colored solid lines correspond to increasingly higher deviations from the Schwarzschild limit obtained for different values of the theory parameter in their range of interest, as reported in the colorbars and on the plotted labels.

lacks realistic orbital characteristics expected from objects in the Galactic Center. Known stellar populations in the Galactic Center, which could serve as progenitors for a pulsar population [332], exhibit notably high eccentricities [137], which are anticipated to persist in the pulsars therein. Consequently, a pulsar on a highly eccentric orbit would experience a broader array of relativistic effects compared to Toy 0. In fact, on a circular orbit, the object maintains a constant distance from the central object, leading to a situation where relativistic orbital effects, such as periastron advance (Eq. 2.41) and Einstein delay, result in a constant shift between the pulsar's proper time and the coordinate time. This constant shift is entirely degenerate with the intrinsic period of the pulses. Conversely, for an object on a highly eccentric orbit, orbital precession significantly contributes, with the amplitude of deviations from a Keplerian orbit growing over time due to its secular nature [301]. Moreover, relativistic time dilation effects in highly eccentric orbits vary along the orbit due to the varying distance from the central object, resulting in a phase-dependent modulation of the propagation delay. These effects, directly related to the metric coefficients, can yield significant signatures of the underlying spacetime geometry on timing profiles.

To explore this possibility, we considered three different orbital models for pulsars in the Galactic Center, initially introduced in [583], with their orbital parameters detailed in Table 6.1. These models aim to investigate increasingly strong gravitational field regimes by reaching closer pericenter distances, providing deeper insights into the gravitational field of the central object. For these models, we considered three orbital inclinations: $i = 0^\circ$ (edge-on configuration), 30° , and 60° , to examine how different geometrical configurations between the emitter and observer affect the resulting time delay. We numerically integrated the geodesic equation for the trajectory of the pulsar, following the procedure delineated in Section 2.2.3, accounting for all relativistic effects on the pulsar orbit and its pulse emission times.

Then, for each position on the pulsar's orbit, we applied the methodology developed in Section 2.2.5 to compute the corresponding photon propagation delay. Given the prograde periastron advance, the time T for the pulsar to complete a full revolution around the central SMBH (*i.e.*, spanning an angle 2π on the orbital plane) is less than the time T_{pre} for the pulsar to travel between two consecutive radial turning points (*i.e.*, two consecutive pericenter or apocenter passages). Therefore, we report our results in terms of the relativistic orbital phase

$$\varphi_{\text{rel}} = \frac{2\pi}{T_{\text{pre}}}(t - t_p), \quad | \quad 6.1$$

for the orbits of the eccentric toy models.

The results of our computations are depicted in Figures 6.3, 6.4, and 6.5, for the toy models listed in Table 6.1, and for the three different orbital inclinations considered, respectively. As done above, we take the propagation time computed for the Schwarzschild spacetime (obtained by setting the theory parameter to zero for all models) as a reference to which we compare the results obtained in each alternative considered. The profiles showcased in Figures 6.3, 6.4, and 6.5 thus represent the amplitudes of the timing residuals that one would obtain when fitting TOAs data for a pulsar that orbits one of the alternative scenarios considered here with a timing model built upon a Schwarzschild black hole (see Section 6.2). For each value of the theory parameter within the range of interest, we perform the following steps: integrate the geodesic equations for the pulsar using the specific parameter value, apply our methodology to derive the phase-dependent

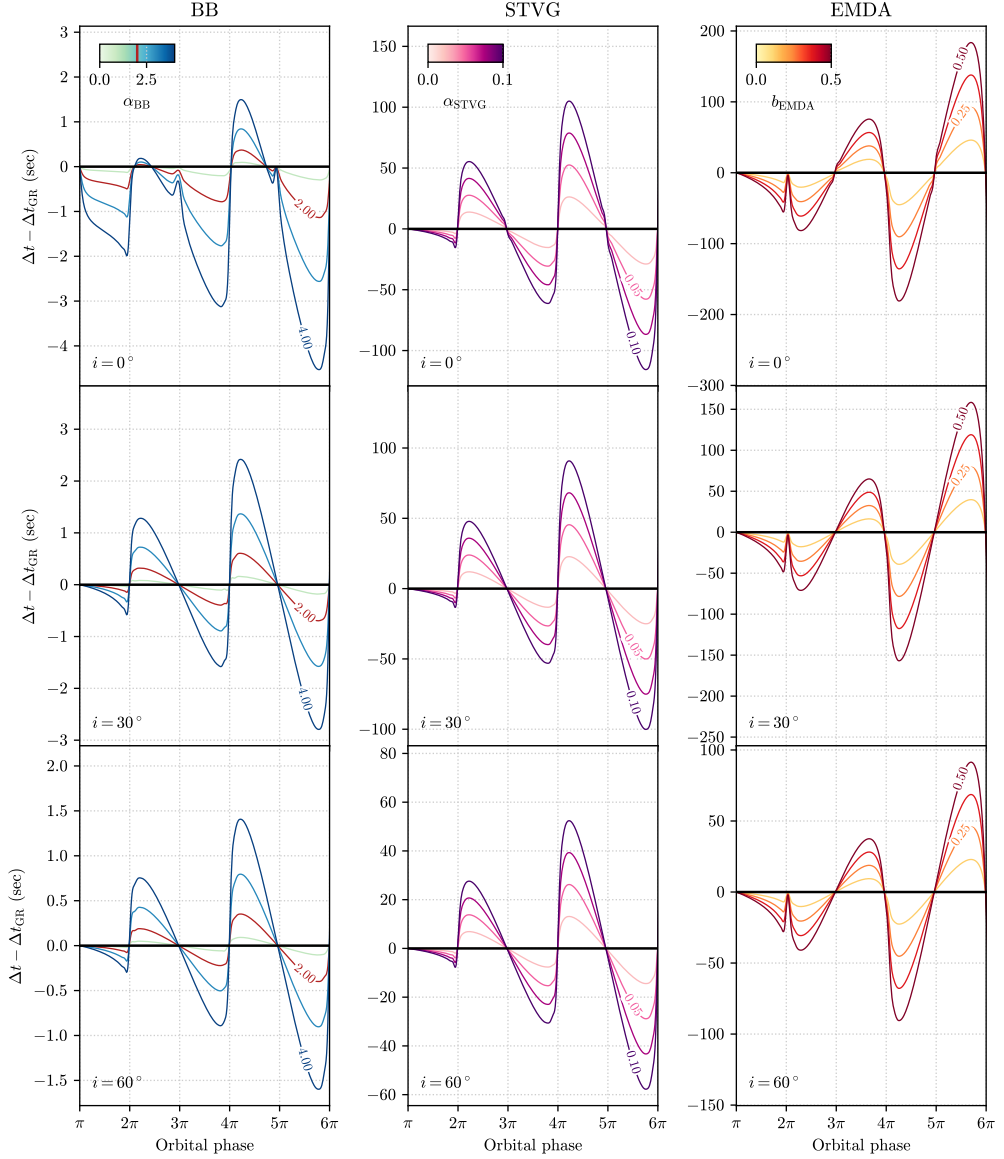


Figure 6.3. Differences in seconds from the Schwarzschild black hole of the propagation delay in the Black Bounce (left panels), STVG (center panels) and EMDA (right panels) models, respectively, numerically computed with our code for a central object of $M = 4 \times 10^6 M_\odot$ and for the orbiting object Toy 1, for three inclinations of $i = 0^\circ$ (top panels), $i = 30^\circ$ (center panels) and $i = 60^\circ$ (bottom panels) and over two and a half orbital periods (starting from apocenter $\varphi_{\text{rel}} = \pi$ to $\varphi_{\text{rel}} = 6\pi$). Coloured solid lines correspond to increasingly higher deviations from the Schwarzschild limit obtained for different values of the theory parameter in their range of interest, as reported in the colorbars and on the plotted labels.

photon propagation time, and then subtract the corresponding Schwarzschild propagation time. Consequently, the resulting

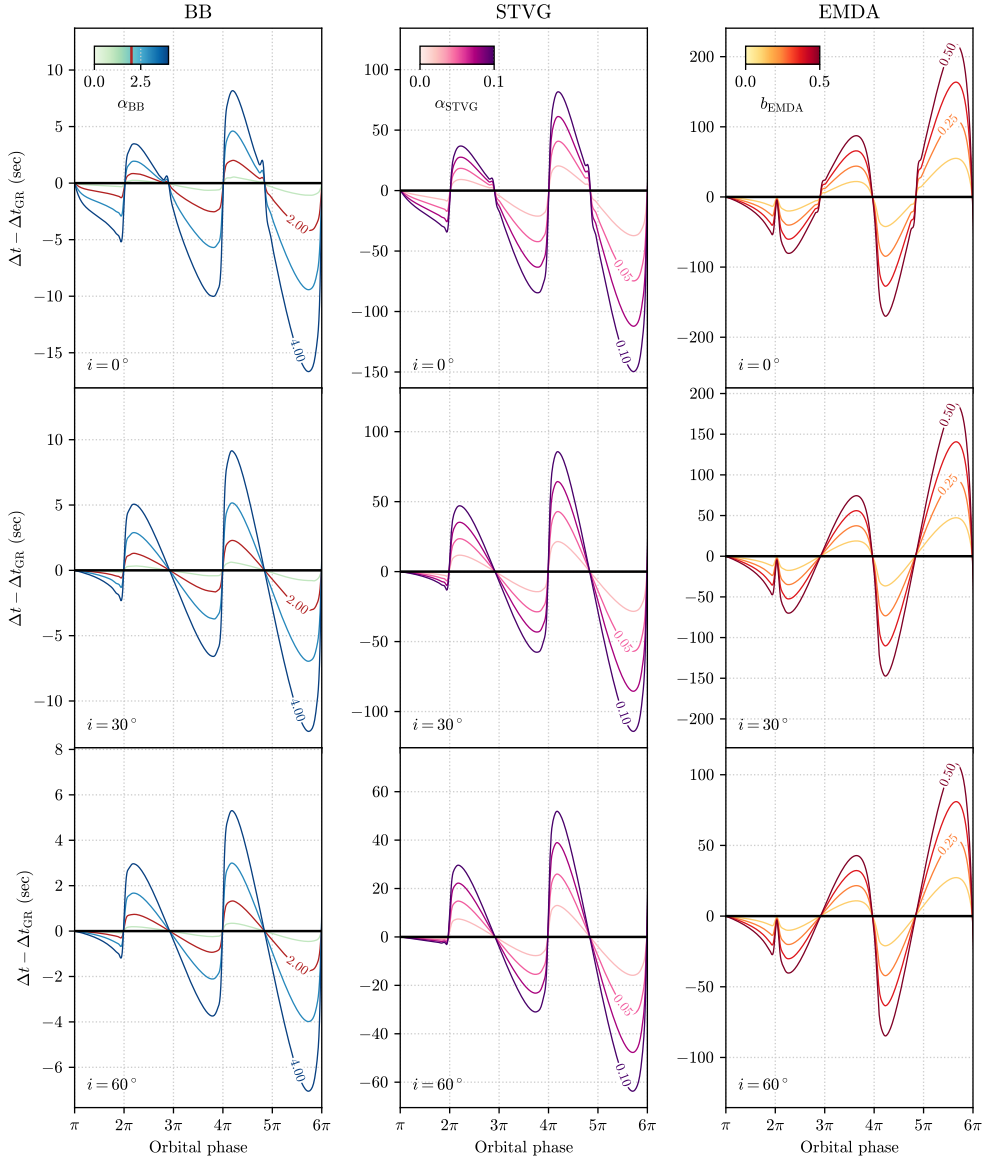


Figure 6.4. As in Figure 6.3 but for the model Toy 2.

profiles encode information about both the modification to the pulsar's trajectory and the photon paths relative to the theory parameter. Our orbital integration starts at apocenter (*i.e.*, at $\varphi_{\text{rel}} = \pi$), and we adopt this as our reference point for computing the delays, ensuring synchronization among all models at the initial time. Consequently, all profiles start from zero deviation when $\varphi_{\text{rel}} = \pi$.

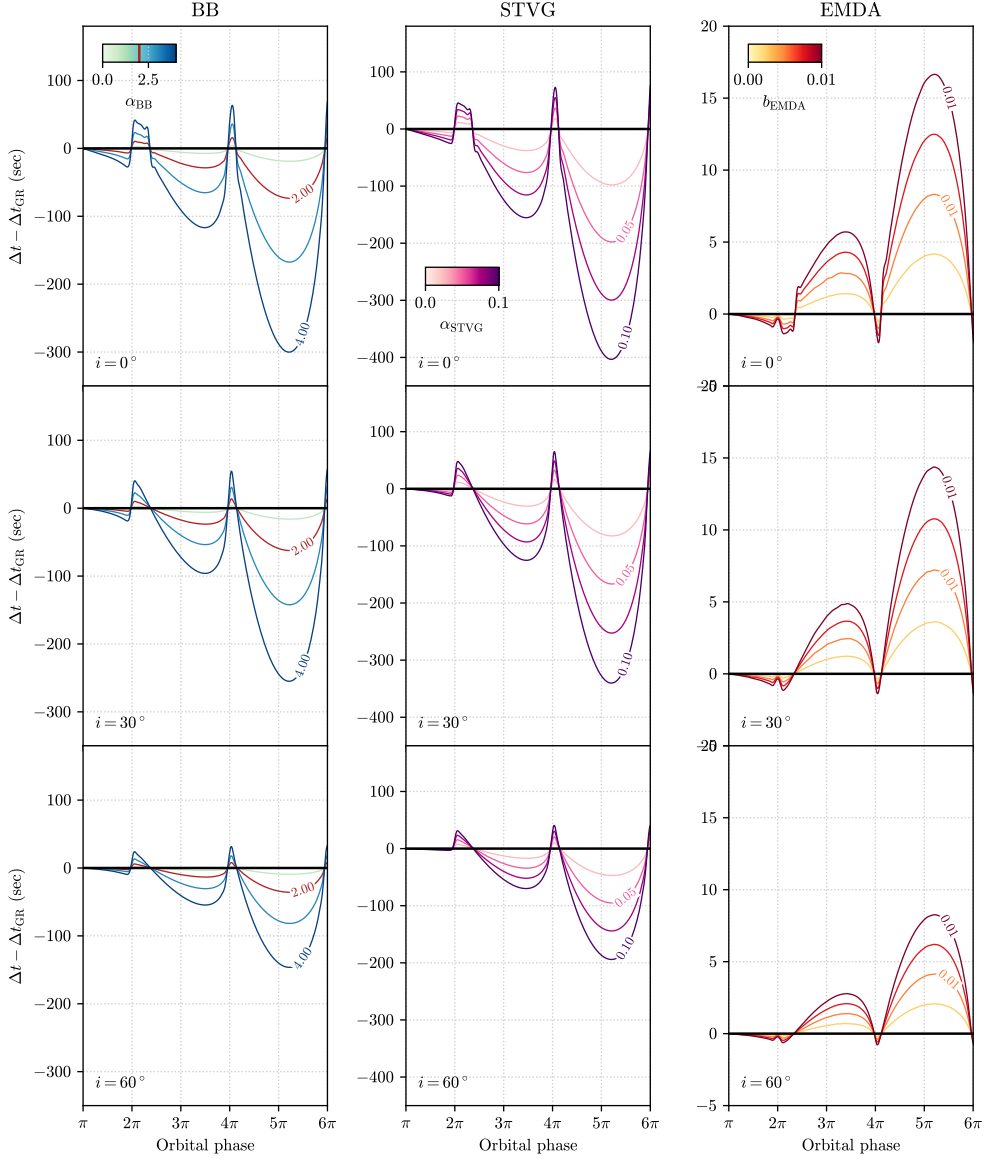


Figure 6.5. As in Figure 6.3 but for the model Toy 3.

A significant observation from our profiles is that in the case of edge-on orbits ($i = 0^\circ$), we consistently observe a sudden shift in propagation time when the pulsar reaches superior conjunction, corresponding to values of φ_{rel} around odd multiples of π . In this configuration, where the pulsar aligns directly behind the central object from the observer's perspective, we encounter a strong curvature of photon paths.

This curvature, coupled with the pronounced impact of the Shapiro delay for such photons, leads to an abrupt change in propagation time. Notably, this effect is absent for inclined orbits, where superior conjunction is never realized. In general, across various orbital models and spacetime geometries, inclined orbits consistently exhibit a smaller impact on photon propagation delay compared to the edge-on case.

Moreover, the deviation from the Schwarzschild case shown in our profiles demonstrates a secular increase, indicating that with each successive orbital period, the departure amplifies. This phenomenon is directly related to the increased rate of orbital precession resulting from alterations in the underlying spacetime geometry, as exemplified by Eq. (5.35) and in the Figure 4.10, 5.35 and 5.16 respectively, for the three models considered. Consequently, the spatial position along the orbit from which the photon originates experiences a progressive divergence. Furthermore, these effects manifest with an amplitude that depends on the theory parameter's value and the object's orbital characteristics. The deviations in photon propagation time, quantified in seconds, can range across several orders of magnitude—from a few to hundreds of seconds in the most extreme scenarios—when applied to the Milky Way's SMBH. Notably, the same object orbiting distinct spacetime geometries generates distinct propagation delay profiles, not only in terms of amplitude but also in functional dependence on orbital phase. These distinctive features, we claim, offer a promising avenue for discerning the underlying theory of gravity.

6.2. Pulsar timing in the Schwarzschild spacetime

The original scientific content presented in this section is part of a currently ongoing project and thus represents a work-in-progress perspective that will be published in the near future.

In this section, we delve into the problem of pulsar timing for a pulsar orbiting a SMBH. In general, by pulsar timing, we refer to a technique that involves measuring the TOAs of pulses emitted by a pulsar far away from the observatory at which they are recorded, monitoring them on a timescale of years, and fitting these TOAs to a model, namely a *timing model* to establish a connection between the measured TOA and the time of emission at the pulsar, allowing for the computation of a *pulse phase of emission* using a model that accounts for the inherent variations in the pulse period. This fitting

technique allows for a very precise estimation of the pulsar intrinsic parameters, like its period and spin-down rate, and of the orbital parameters (thus including the masses) for pulsars in binary systems. Most of the known pulsars belong to non-relativistic binary systems (*i.e.* characterized by a low mass ratio between the pulsar and its companion), for which a post-Newtonian treatment of both the orbital motion and photon propagation, as described in Section 2.2.4, allows for the development of a timing model whose precision is below the experimental uncertainty on the TOAs. For this reason, all the pulsar timing codes available nowadays, for which we provide an overview in Section 6.2.1, adopt this approach. Conversely, when the pulsar is in a strong-field gravitational regime (*i.e.* for example in high mass ratio binary systems with the pulsar orbiting a much heavier body), like pulsars in tight orbits around Sgr A* as those considered in Table 6.1, the 1PN description of both the orbital motion and the photon propagation time fails at providing a complete characterization of all the relativistic effects involved in an accurate determination of the TOAs (as shown in Figure 2.17). This limitation, as we will show in Section 6.2.2 spoils the ability to derive phase-connected residuals for the timing model, undermining the fitting capabilities of usual timing codes, in this scenarios. For this reason, in the same section, we propose a novel approach to implement the fully relativistic calculations of the photon travel time developed in Section 2.2.5 into a more robust timing model for pulsars orbiting SMBHs, showing how a misestimation of the pulsar parameters can lead to the appearance of phase-dependent residuals and performing a proof-of-concept fitting analysis to forecast the accuracy that can be achieved with future observational facilities like SKA.

6.2.1. Overview of current timing techniques

At the core of any code designed for pulsar timing lies the mathematical framework that specifies the TOA model for pulses received at a given observatory. With the rising sensitivity of radio observational facilities and the increasing diversity of pulsar populations discovered, the demand for precision in timing models has grown accordingly. For many years, the preferred choice for pulsar timing was the code TEMPO¹ that was born from the contributions and collaboration

¹ <https://tempo.sourceforge.net>

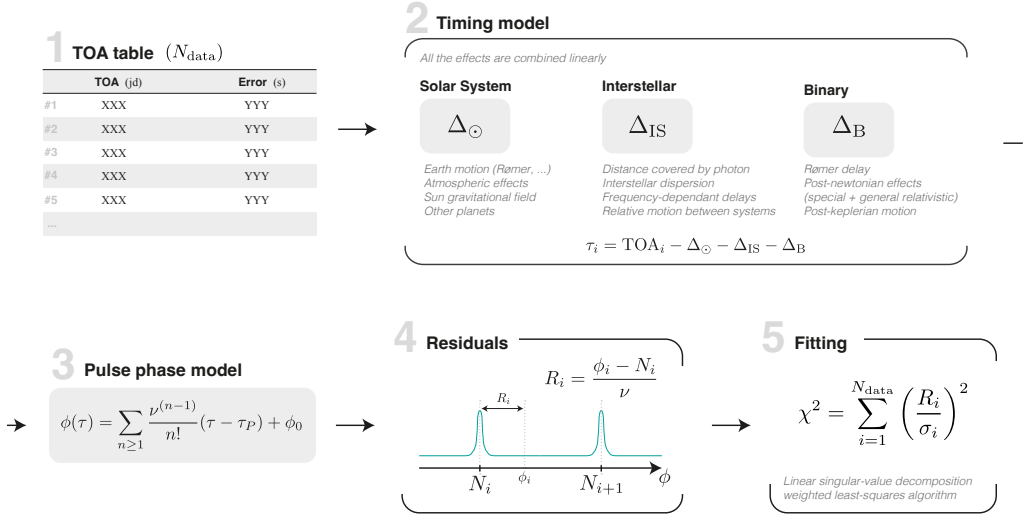


Figure 6.6. Schematic illustration of the timing technique behind popular codes like TEMPO2. The observed TOAs are converted, by means of a timing model encompassing all the different classical and relativistic delays, into barycentric proper emission time in the reference frame of the pulsar, τ . These can be connected to the emitted pulse sequence by modeling the pulse phase as a Taylor series involving the pulse period (or frequency) its variation over time and, possibly, higher derivatives. The difference between the retrieved pulse phase and nearest integer N_i constitutes the timing residual R_i (that can be expressed in units of time from the nearest pulse through division by the frequency or multiplication by the period). A χ^2 statistic is used to quantify the agreement of the overall timing model to the observed TOAs. The best-fit timing model is given by the set of parameters that minimizes the residuals, canceling any significant phase-dependent systematic in their shape.

of many authors and maintained and distributed by Princeton University and the Australia Telescope National Facility² (ATNF). This code provides a timing accuracy at the level of 100 ns for normal binary pulsar systems (*i.e.* with the binary companion being either a white dwarf or a neutron star, possibly another pulsar as well). This level of precision sufficed for numerous significant discoveries, such as the indirect detection of gravitational wave emission by pulsar binaries and other relativistic effects [633, 634], the first detection of extrasolar planetary systems [635] and the discovery of millisecond and submillisecond populations of pulsars [636, 637]. However, with the need for improved accuracy required for the potential direct detection of gravitational radiation by Pulsar Timing Array collaborations [638–641], the bar is set significantly higher in terms of both data precision and volume. This led to the development of an updated version of TEMPO, namely TEMPO2 [356, 642], which, thanks to improved modeling of all the systematic effects that contribute to the

² <https://www.atnf.csiro.au>

resulting TOA, is able to attain a precision of order ~ 1 ns for normal binary pulsar systems. Recently, a modern timing code, named PINT [643], was proposed, based on the same principles of TEMPO and TEMPO2 but built from scratch in Python, using an object-oriented modular approach. This well-tested and thoroughly validated code provides a fresh alternative to older timing software, allowing for a cross-check of the results obtained across the different implementations.

In essence, the process (schematically depicted in Figure 6.6) involves measuring the TOAs of pulses from a specific pulsar at a radio observatory over an extended period. To ensure that these TOAs accurately reflect arrival times in an inertial reference frame, each measured arrival time is transformed into the reference frame of the pulsar. This transformation, namely the timing model, involves initially computing arrival times at the Solar System barycenter and then incorporating additional terms to model the pulsar's orbital motion and interaction of the electromagnetic radiation with the interstellar medium.

We call N_{data} the number of recorded TOAs for a specific pulsar.

If notable systematic deviations, usually referred to as *residuals*, emerge when comparing the reconstructed pulse phase from the measured TOAs to the closest integer number (we will specify this further below), it suggests that the model does not fully capture the true pulsar parameters. A positive residual indicates that the pulse arrives later than predicted, while a negative residual signals an underestimation of the photon travel time. Such discrepancies can arise from various factors, including unaccounted-for binary companions or parameters, irregularities in the pulsar's spin-down, or inaccurate estimation of astrometric or rotational parameters. The full set of intrinsic, astrometric and binary parameters from which the timing model depends comprises more than 60 entries (albeit not all independent of each other) and is reported in Table A.1 of [356].

In general, a timing model for a non-relativistic binary pulsar system is based on a high-level timing formula of the form

$$\tau_i = \text{TOA}_i - \Delta_{\odot} - \Delta_{\text{IS}} - \Delta_{\text{B}}. \quad | \quad 6.2$$

Here, TOA_i is the time of arrival at the observatory of the i -th pulse in the dataset, while τ_i is the proper time in the pulsar's inertial reference frame. The various terms on the right-hand side of Eq. (6.2) contain all the different contributions to the overall photon travel time. In particular, $\Delta_{\odot}$ corresponds to the

conversion to the Solar System barycenter frame, accounting for vacuum propagation delays influenced by the Earth's orbital motion and its spin, as well as the Earth's precession and nutation, considering that the observatory is situated on Earth. Additionally, it encompasses any extra delays arising from the signal's passage through the Earth's atmosphere and the Solar system. This component links the observed TOAs to the arrival time in the barycentric coordinate time at the Solar System barycenter. Next, Δ_{IS} incorporates the transformation to the binary barycenter frame, along with vacuum propagation delays attributed to the system's secular motion, and any additional propagation delays due to the signal traversing the interstellar medium. This term connects the barycentric coordinate time at the Solar System barycenter to the arrival time in the binary barycenter coordinate time. Finally, Δ_{B} includes the conversion to the pulsar frame, incorporating vacuum delays due to the binary orbital motion and any excess delays due to the signal's passage through the gravitational field of the companion. This component links the binary barycentric coordinate TOA at the binary barycenter to the pulsar's proper time of emission, τ_i . The term Δ_{B} contains the crucial relativistic contribution to the total photon propagation time produced in the binary pulsar system. In particular, the photon propagation is described in terms of a linear sum of post-Newtonian delays, as explained in Section 2.2.4. These comprise the geometric Rømer delay, responsible for the additional vacuum light travel time due to the Euclidean displacement of the pulsar, a pseudo-delay reflecting the aberration of the radio beam caused by binary motion, the Einstein delay encompassing both gravitational redshift and special relativistic time dilation in the pulsar frame, and the Shapiro delay arising from gravitational time dilation near the companion. Similarly, the binary motion is obtained from a post-Keplerian description of a binary orbit, based on the post-Newtonian approximation of the two-body problem in General Relativity, derived in [306, 359, 634, 644–648].

From the proper time computed by applying the high-level timing formula in Eq. (6.2) one then computes the pulse phase corresponding to that τ . In particular, if the pulsar period P is perfectly constant (or, equivalently its angular frequency ν), all the proper times reconstructed applying the timing model should be equally spaced and thus, by dividing the difference of each proper time to a given initial epoch τ_P by the period P ,

one would get integer steps starting from an initial phase ϕ_0 . In general, however, pulsars experience a rotational spin-down due to the magnetic-dipole braking [648]. For this reason, one can allow for temporal derivatives of the pulsar frequency or period (for all practical purposes one can consider only the first derivative, namely $\dot{\nu} = -\frac{\dot{P}}{P^2}$) which modifies the relation between the pulse phase ϕ and the proper time of the pulses through a Taylor expansion

$$\phi(\tau) = \sum_{n \geq 1} \frac{\nu^{(n-1)}}{n!} (\tau - \tau_P) + \phi_0 \approx \phi_0 + \frac{\tau - \tau_P}{P} - \frac{\dot{P}}{2P^2} (\tau - \tau_P)^2, \quad | \quad 6.3$$

where in the latter expression we have considered a constant rate of change of the rotational period.

As mentioned above, if the parameters used for the timing model exactly match their actual physical counterparts, the computed $\phi_i - \phi_0$ will be an integer number N_i corresponding to the position of the given pulse in the pulse sequence (with the pulse at ϕ_0 corresponding to the 1st received). In practice, the exact parameters are not known *a priori*, and measurement noise is present so that the deviations of these pulse phases from the nearest integer are finite. To quantify these deviations, one defines the timing residual of the i -th pulse as

$$R_i = \frac{\phi_i - N_i}{\nu} = (\phi_i - N_i)P \quad | \quad 6.4$$

corresponding to the time since (or to) the nearest actual pulse emitted by the pulsar (see panel 4 in Figure 6.6). Thus, given a set of N_{data} recorded TOAs for a specific pulsar, one can build a χ -squared statistic to quantify the agreement of the timing model (and of the set of parameters used) to the observational data

$$\chi^2 = \sum_{i=1}^{N_{\text{data}}} \left(\frac{R_i}{\sigma_i} \right)^2, \quad | \quad 6.5$$

where the σ_i represent the observational uncertainties on the TOAs. Applying an iterative least-squares method one can look for the set of parameters that minimizes the χ^2 in Eq. (6.5), producing the smallest residuals and canceling any significant phase-dependent systematic in their shape. For this set of parameters, and assuming there are no shortcomings in the structure of the timing model, the resulting residual deviations are due only to measurement noise.

6.2.2. A new approach

The technique that we have illustrated in the previous section is nowadays widely used and tested and, as mentioned, has led to the achievement of great scientific results for normal binary pulsar systems. A crucial point to keep in mind, however, is that it heavily relies on two assumptions:

- i) in order for the timing model to work appropriately, the gravitational regime has to be sufficiently weak that one can consider the combination of effects leading to the overall propagation time to be a linear composition of effects, as the one considered in Eq. (6.2) (and in all its constituent terms [356]);
- ii) the deviation introduced by truncating at first post-Newtonian order the description of the motion of both the binary system and photons are below the accuracy goal (~ 1 ns) and, most importantly, are much smaller than the pulsar intrinsic period P .

The latter point is crucial, as this is fundamental in order to obtain a sensible interpretation of the timing residuals. If one cannot rely on the timing model to accurately identify the correct pulse within the pulse sequence emitted by the pulsar, the whole procedure loses its predictive capability. This limitation of current timing models is clearly stated in [356] and regarded as the principal source of error in the formulation of the timing delay due to the presence of the binary companion. Already for double-neutron-star binaries, these deviations are large compared to the 1 ns accuracy goal. For more extreme cases, like the relativistic pulsar binaries that we are interested in (*i.e.* a stellar mass pulsar around a SMBH) these deviations are huge and, even for a mildly relativistic pulsar toy model like the Toy 0 in Table 6.1, we have demonstrated (Section 2.2.5) that the deviations between the 1PN formulation of the photon propagation time and the geodesic approach can be as large as 2 s. Moreover, in that case, we were considering only the 1PN approximation on the photon propagation while relying on a fully geodesic description of the pulsar orbit. The situation would only get worse if we were to consider a 1PN approximation of the orbital motion of the pulsar, as well. This, along with the impossibility of linearly separating the different effects in the fully geodesic description of the photon propagation time pushed us to develop a fully relativistic approach to the problem of pulsar timing, which, as we will show, does

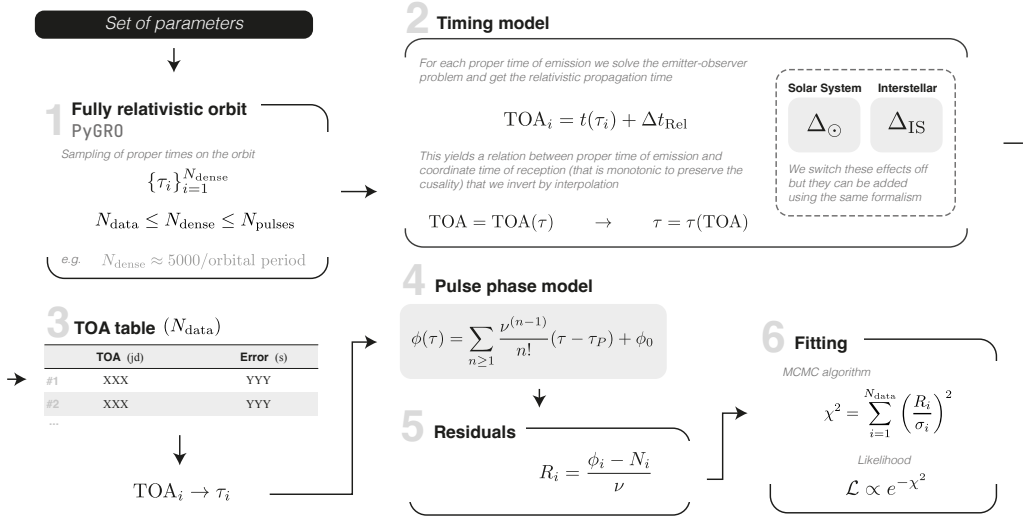


Figure 6.7. Schematic illustration of the timing technique that we have developed in this chapter. For a given set of intrinsic, astrometric and orbital parameters for a pulsar around a SMBH we compute a fully relativistic orbit by applying the procedure illustrated in Section 2.2.3. The orbit is then sampled in N_{dense} equally spaced (in terms of proper time τ) points, $\{\tau_i\}_{i=1}^{N_{\text{dense}}}$. For each of these points, the emitter-observer problem is solved and an estimate of the relativistic propagation time for a distant observer is computed. This gives out a relation between the TOA at the distant observer location and the proper time of the emission that we invert by spline interpolation. The resulting function $\tau(\text{TOA})$ is thus applied to the observed TOAs to get barycentric proper emission time in the reference frame of the pulsar. Following the same procedure used in standard timing techniques we compute the residuals R_i and the χ^2 statistic that is used to define a likelihood $\mathcal{L}$ adopted to derive Bayesian constraints on the model parameters.

not only require incorporating our technique developed for solving the emitter-observer problem and to get a relativistic computation of the photon travel time in the estimation of the relationship between TOAs and proper times of emission, but also implies a paradigm shift in the way this relation is inverted to generate a phase-connected estimation of the timing residuals.

The whole process is schematically outlined in Figure 6.7. From the numerically integrated orbit we sample a given number N_{dense} of equally spaced (in terms of proper time τ) points, $\{\tau_i\}_{i=1}^{N_{\text{dense}}}$, which constitutes a subset of the whole set of all the N_{pulses} pulses emitted over the considered timespan. For each of the sampled points, the emitter-observer problem is solved and an estimate of the relativistic propagation time, Δt_{Rel} , for a distant observer is computed following Eq. (2.106) and the methodology outlined in Section 2.2.5. We thus have at this point an estimation for the time of arrival at the distant observer for all the photons emitted at proper times $\{\tau_i\}_{i=1}^{N_{\text{dense}}}$,

For a pulsar characterized by a pulse period $P = 2$ s (see the red star in Figure 6.1) and an orbital model like Toy 1 in Table 6.1, we would have $N_{\text{pulses}} = 1.84 \times 10^7$ pulses emitted over every orbital period. Similarly, this number would be $N_{\text{pulses}} = 2.29 \times 10^6$ for Toy 2 and $N_{\text{pulses}} = 8.64 \times 10^4$ for Toy 3, assuming the same pulse period.

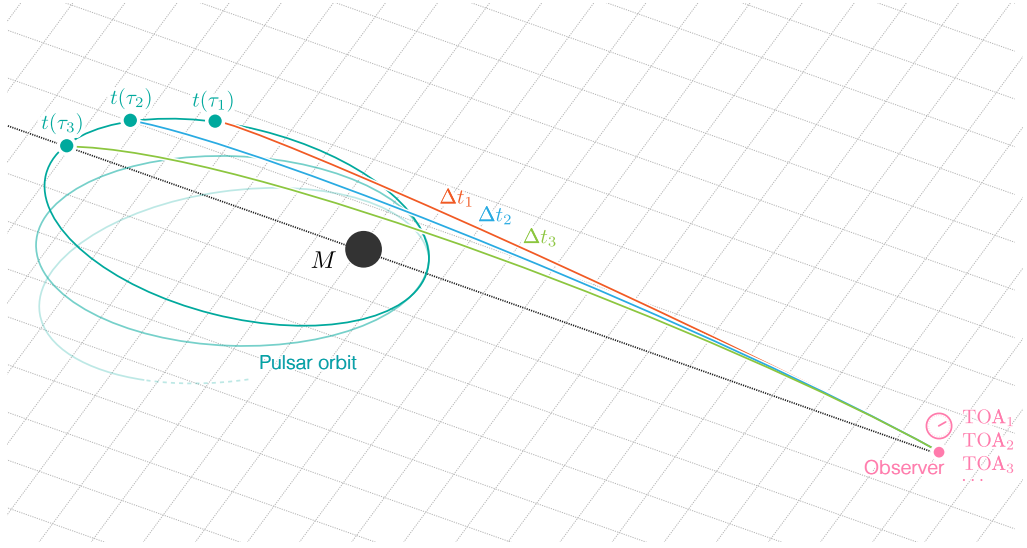


Figure 6.8. Illustration of photon propagation for the pulses emitted by a pulsar around a central black hole following a relativistic photon path up to a distant observer. There is a monotonic (in order to preserve the causality relation between events) relation between proper times of emission, τ_i , corresponding to a coordinate time of emission $t(\tau_i)$, and the coordinate times of arrival at the distant observer, TOA_i .

given by

$$\text{TOA}_i = t(\tau_i) + \Delta t_{\text{Rel},i},$$

6.6

where $t(\tau_i)$ is the coordinate time of emission that is computed directly from the integrated 0-th geodesics equation. We can thus reconstruct the function $\text{TOA}(\tau)$ between proper times of emission in the pulsar's reference frame and the TOAs for a distant observer. This relation, since the principle of causality is preserved in General Relativity is monotonically increasing and it contains contributions from all the different kinds of delays, either classic or relativistic. For the IPN timing model, this function is given right away from Eq. (6.2). In our case, since we sample the relation at N_{dense} points and for each of them we carry out the computations numerically, we do not have an analytic expression to invert, rather a series of samples taken from the function that we can thus invert by interpolation (we opt for a spline interpolator whose order of interpolation is coherent with the order of convergence of the Runge-Kutta algorithm used for the geodesic integration). The number of points, N_{dense} , at which we sample the $\text{TOA}(\tau)$ relation has to sit in the interval

$$N_{\text{data}} \leq N_{\text{dense}} \leq N_{\text{pulses}},$$

where, as mentioned above, N_{data} is the number of measurements in our TOA dataset, and N_{pulses} is the total number of pulses emitted by the pulsar over the considered timespan. Clearly, in an ideal case (*i.e.* with infinite resources and available computational time), the choice that results in the most accurate reconstruction of the $\text{TOA}(\tau)$ relation would be $N_{\text{dense}} = N_{\text{pulses}}$, *i.e.* a 1 : 1 correspondence between emitted pulses, integrated photon paths, and computed TOAs at the observer's location. The cheapest option in terms of computational resources, on the other hand, would be $N_{\text{dense}} = N_{\text{data}}$, but this would result in an accurate reconstruction of the $\text{TOA}(\tau)$ function only in the case that the N_{dense} computed photon TOAs correspond exactly to the pulses, in the pulses sequence, that are actually measured at the observatory. In principle, this information is not known *a priori* and one cannot know to what position in the pulses sequence a given measured TOA corresponds. For this reason, a trade-off is necessary to reconstruct in a trustworthy way the relation between emitted proper times and recorded TOAs with a reasonable amount of computational resources. We have found that, given $N_{\text{data}} = 120$ (see Section 6.2.3), a number $N_{\text{dense}} = 5000$, *i.e.* roughly 50 times N_{data} , provides a good reconstruction of the photon travel time profile. However, it is important to stress that this parameter is fully tunable by the user on a case-by-case basis and that, even in a real-world scenario (*i.e.* when future data will be available), the parameter N_{dense} can be tuned by checking that increasing it would result in an improvement on the reconstruction of the $\text{TOA}(\tau)$ function that is below the instrumental sensitivity or the available measurement uncertainty. Moreover, it is worth mentioning that the term $\Delta t_{\text{Rel},i}$ only takes into account the relativistic contribution to the photon travel time (thus also including the full length of the curved path undertaken by the photon) and no contribution from the interstellar medium, Δ_{IS} , nor any gravitational interaction of the photon with the Solar System gravitational field, $\Delta_{\odot}$, along with the clock and atmospheric corrections included in TEMP02 are considered. This, however, does not change our conclusions, and, as done in TEMP02, all these effects can indeed be added linearly because their contribution truly belongs to a weak-field linear regime even for relativistic binaries (fundamentally, the gravitational field of the binary system does not play a role in these terms) and all the non-linearities are included in the $\Delta t_{\text{Rel},i}$ term. The timing model that we have thus developed allows us to translate TOAs in the observer reference frame into proper

times in the pulsar's reference frame. From this moment on we can proceed exactly as done in TEMP02, by computing the pulse phase (considering the first-order evolution of the pulsar period in Eq. (6.3)) and the residuals R_i for each pulse, as given in Eq. (6.4). For the model fitting (that we carry on explicitly as a proof-of-concept of our methodology in Section 6.2.3), as done throughout this thesis we adopt a Bayesian approach based on an MCMC algorithm (resembling the one outlined in Appendix C) based on a likelihood function given by

$$\mathcal{L} = e^{-\chi^2/2} \quad \Rightarrow \quad \log \mathcal{L} = -\frac{1}{2} \sum_{i=1}^{N_{\text{data}}} \left(\frac{R_i}{\sigma_i} \right)^2, \quad | \quad 6.7$$

where the χ^2 statistic is computed exactly as in the previous case in Eq. (6.5).

In order to appreciate the shape and amplitude of the timing residuals arising from an over- (under-) estimation of the orbital and intrinsic parameters of a pulsar around a SMBH we first perform a qualitative analysis of the TOA deviations. More specifically, we have considered the same toy models used in the previous sections, whose orbital parameters are outlined in Table 6.1, and computed the time of arrival for pulses emitted by assuming an intrinsic period of $P = 2$ s and a period spin-down of $\dot{P} = 10^{-15} \text{ ss}^{-1}$ (this object is reported as a red star in Figure 6.1 within the known pulsar population). Moreover, we assume that the central object is exactly a spherically symmetric black hole described by the Schwarzschild metric in Eq. (2.18)³ for a mass $M = 4.261 \times 10^6 M_\odot$ (as measured in [226] and reported in Table 1.1) for an observer at a fixed inclination $i = \pi/3$ (and all the other orbital inclination parameters to zero, *i.e.* $\Omega = \omega = 0$) and distance $D = 8$ kpc. In fact, differently from the S-stars case, for which astrometric observations allow obtaining a direct estimate of the distance D , in the case of pulsar's TOAs, assuming that $D \gg 2M$ a change in D would only contribute to a constant shift of all photon travel times that would get absorbed in the overall phase shift ϕ_0 included in the pulse phase model in Eq. (6.3). We thus fix this parameter, without loss of generality. Taking this all into consideration, a set of TOAs is uniquely

³ This clearly represents a limitation of our current approach, due to the fact that the methodology developed in Section 2.2.5 only allows us to study spacetimes with spherical symmetry, thus not including rotation. Nevertheless, this suffices for the proof of concept that we want to develop in this chapter and we plan to extend the methodology to be able to include the rotation of the central object in the future.

fixed once the following set of parameters is given:

$$\vec{\theta} = (P, \dot{P}, \tau_P, \phi_0, M, t_p, a, e, i, \omega, \Omega). \quad | \quad 6.8$$

The previous considerations completely fix $P, \dot{P}, M, a, e, i, \Omega$ and ω . Without loss of generality, we fix the temporal zero-point parameters so that the pulsar emits its first detected pulse when it is at the pericenter (so that the epoch of t_p corresponds to the epoch of reference of the pulsar's period, τ_P) and that the first pulse for an Earth-based observer arrives exactly on the year 2025.0. This gives us a set of baseline time of arrivals that we take as a reference for comparison once we start to change the values of the parameters.

In particular, we first consider variations of the intrinsic pulsar period P and its derivative $\dot{P}$, both parameters drastically alter the time of arrivals by inducing linearly drifting residuals in the case of a variation of P and quadratically drifting ones for variations of its derivative. These are shown in the top panels of Figure 6.9 for the model Toy 2 over around 9 orbital periods, as an example. In this case, changing the intrinsic pulsar period by only $\pm 10^{-9}\%$ brings the timing residuals to exceed the sensitivity threshold in only two complete orbits (amounting to about three months of observations). Similarly, the quadratic departure induced by a variation of $\dot{P}$ of order 0.1% produces residuals that exceed the sensitivity threshold after ~ 4.5 full orbits (corresponding to slightly over 7.5 months). Since the shape and amplitude of the residuals induced by the change of the intrinsic parameters does not alter the orbit of the pulsar (nor, hence, the photon travel time), this effect does not depend on the particular orbital characteristic of the object considered and even though Toy 1 and Toy 3 take a different number of orbits for their residuals to exceed the experimental detectability threshold, the amount of time required for this to happen is consistent across the three toy models.

The picture drastically changes when variations of the mass M of the central object (third panel of Figure 6.9) or of the orbital elements (panels 4 and 5 of Figure 6.9 for the semi-major axis a and eccentricity e , respectively) are taken into consideration. These parameters affect the orbital dynamic itself of the pulsar, implying the incurrence of secular effects that on each orbit accumulate and grow in amplitude, leaving a detectable imprint on the shape and amplitude of the residuals even for microscopic changes of the parameters. A summary of the

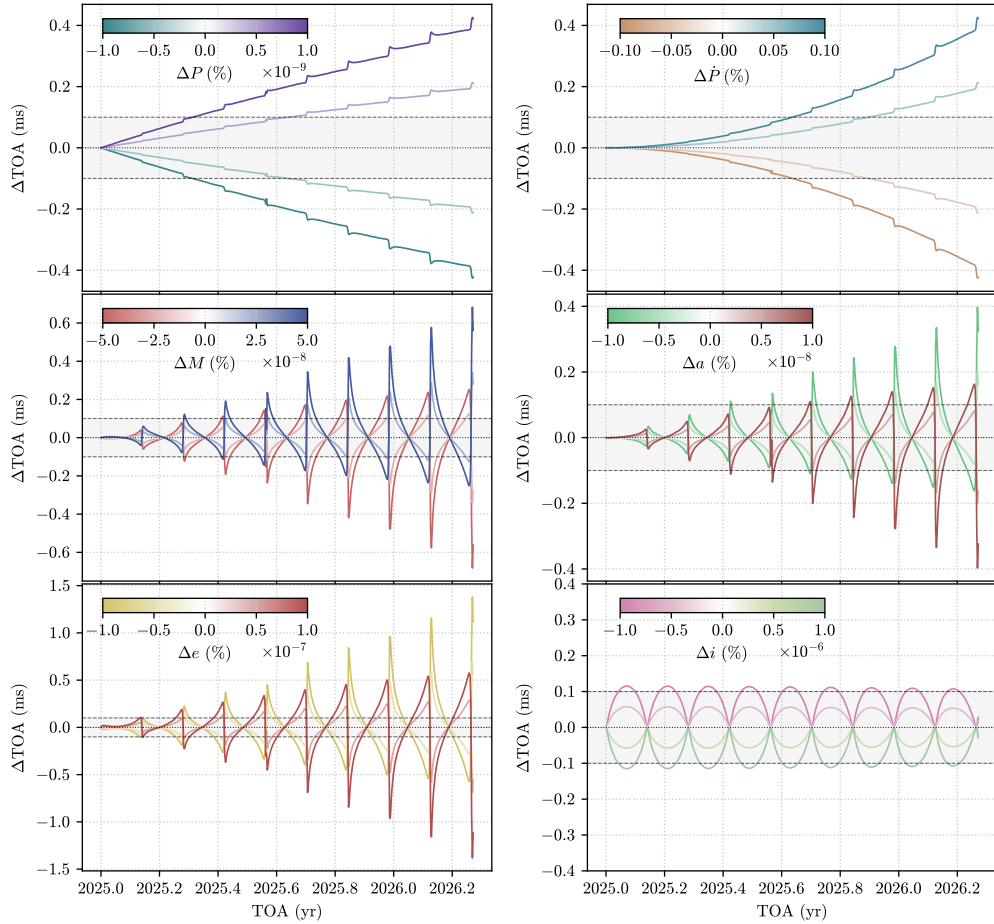


Figure 6.9. Timing residuals over 9 orbits obtained for the model Toy 2 in Table 6.1 by slightly altering the value of each of the intrinsic or orbital parameters of the pulsar by the percentage amount reported in the colorbar of each panel (the residuals are obtained by changing only one parameter at a time and leaving all the others fixed). The horizontal shaded regions correspond to the nominal timing accuracy expected for SKA observations in the Galactic Center at $100 \mu\text{s}$.

number of orbits and the observational time required to reach a given percentage precision (*i.e.* for the timing residuals to exceed the timing sensitivity threshold of $\sim 100 \mu\text{s}$ expected to be reached by SKA in the Galactic Center) on the orbital parameters that we have varied can be found in Table 6.2 for all the toy models reported in Table 6.1.

Finally, we have considered the impact of a change in the orbital inclination i on the residuals. In this case, the amplitude is fixed due to the fact that a misestimation of the angular orbital parameters does not induce (at first order) a secular variation

Table 6.2. The time (and number of complete orbits) required for the TOA residuals to exceed the nominal timing accuracy of $100 \mu\text{s}$ for SKA observations of pulsars in the Galactic Center. In particular, for each toy model considered, we report the time required to obtain the precision reported in the second column (in percentage units) on each of the orbital and intrinsic parameters of the pulsar.

Parameter	Precision	Toy 1		Toy 2		Toy 3	
		Orbits	Time	Orbits	Time	Orbits	Time
P	$5 \times 10^{-10}\%$	0.5	6 months	4	6 months	90	6 months
	$1 \times 10^{-9}\%$	0.25	3 months	2	3 months	45	3 months
$\dot{P}$	$5 \times 10^{-2}\%$	0.75	10 months	6	10 months	110	10 months
	$1 \times 10^{-1}\%$	0.55	7.5 months	4.5	7.5 months	60	7.5 months
M	$2.5 \times 10^{-8}\%$	2	2.5 years	3	5 months	17	1 month
	$5 \times 10^{-8}\%$	1	1 year	2	3.5 months	12	25 days
a	$5 \times 10^{-9}\%$	4	3.5 years	10	1.5 years	30	60 days
	$1 \times 10^{-8}\%$	3	2.5 years	3	5 months	16	32 days
e	$5 \times 10^{-8}\%$	4	4.5 years	3	5 months	2	4 days
	$1 \times 10^{-7}\%$	2	2 years	2	3.5 months	1	2 days
i	Case by case	0.5	6 months	0.5	25 days	0.5	1 day
		$(7 \times 10^{-7}\%)$		$(1 \times 10^{-6}\%)$		$(1 \times 10^{-5}\%)$	

of the orbits and thus, if the deviation of the inclination is sufficiently large (*e.g.* $\sim 10^{-6}\%$ for the model Toy 2) the residuals can exceed the sensitivity threshold already at the first orbit. In the last row of Table 6.2 we report the precision reached on the orbital inclination i for each of the pulsar toy models in Table 6.1 after half orbit.

6.2.3. Bayesian fitting of the residuals

The qualitative analysis that we have performed in the previous sections shows promising results in terms of the constraining power of timing observations for Galactic Center pulsars. Motivated by this, we have generated a mock catalogue of observations for the Toy 1 model in Table 6.1, encompassing $N_{\text{pulse}} = 120$ TOAs recorded over the course of two full orbital periods (~ 2.3 yr) and recorded at an Earth-based observatory working with the nominal SKA timing accuracy of $\approx 100 \mu\text{s}$. We thus built a fully relativistic orbit spanning the two periods, we select 120 of the $\sim 3.6 \times 10^7$ pulses emitted over this period and we integrate the propagation time for the distant observer. This results in a list of TOAs, to which we add a Gaussian noise amounting to $\sigma = 100 \mu\text{s}$ (we don't consider here for the sake of simplicity more sophisticated forms of noise for the TOAs). The resulting mock catalogue is reported in Table D.1.

Table 6.3. The true values for each free parameter in our proof-of-concept MCMC analysis for the fully relativistic Schwarzschild pulsar timing model and the set of uniform priors used for the analysis. In particular, for each parameter, we consider uniform prior probability functions centered on the true values and with amplitudes given by the percentage units reported in the last two columns. This priors are heuristically based on our qualitative analysis.

Parameter (unit)	True	Uniform priors	
		Start	End
M ($10^6 M_\odot$)	4.261	$-10^{-5}\%$	$+10^{-5}\%$
a (AU)	175.4	$-10^{-7}\%$	$+10^{-7}\%$
e	0.8	$-5 \times 10^{-7}\%$	$+5 \times 10^{-7}\%$
P (s)	2	$-10^{-9}\%$	$+10^{-9}\%$
$\dot{P}$ (10^{-15} s)	1	$-10^{-1}\%$	$+10^{-1}\%$

We have then performed a proof-of-concept MCMC analysis to fit our timing model to the mock catalogue of TOAs for the model Toy 1 that we have generated above. In particular, we consider a selection of five parameters, given by a combination of intrinsic pulsar parameters and orbital parameters,

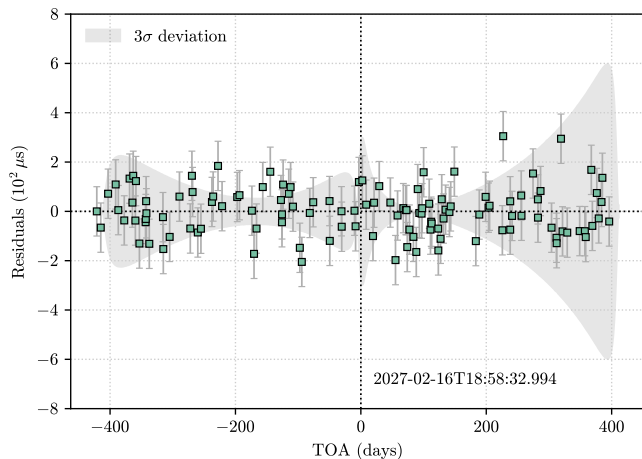
$$(M, a, e, P, \dot{P}),$$

6.9

with the rest of the parameters in full set, Eq. (6.7), fixed to their *true* values. In our analysis, we adopt the likelihood given in Eq. (6.7), with $\sigma_i = 100\mu\text{s}$ for all the TOAs and computing the R_i using our fully relativistic timing model. We adopt uniform priors on all the free parameters, that we report in Table 6.3 along with their true values (*i.e.* the ones used for the generation of the mock catalogue), that are heuristically based on the qualitative analysis reported in Table 6.2.

The results of our five-parameters posterior analysis on the mock data over two orbital periods for the pulsar Toy 1

Figure 6.10. Post-fit residuals obtained in correspondence of the maximum-likelihood best fitting values of the five free parameters in our analysis. The post-fit residuals show no phase-dependent trend and are completely dominated by the instrumental noise (error-bars), thus showing the capability of our approach to correctly fit a timing model to TOAs data. The shaded area corresponds the residuals that would be obtained for models at $\pm 3\sigma$ deviation from the best-fitting ones.



Parameter (unit)	Posterior	Precision (%)
$M (10^6 M_\odot)$	4.2610000011(41)	9×10^{-8}
a (AU)	175.400000006(55)	3×10^{-8}
e	0.80000000045(46)	5×10^{-8}
P (s)	2.0000000000009(41)	2×10^{-10}
$\dot{P} (10^{-15} \text{ s})$	0.99998(12)	1×10^{-2}

Table 6.4. Best fitting values and, in parenthesis, 1σ uncertainties resulting from our MCMC analysis on the five-parameters timing model for the pulsar model Toy 1. We report, in the third column the percentage precision reached on the parameters (given by the ratio of the parameter uncertainty to its best-fitting value) over the two full orbits analyzed.

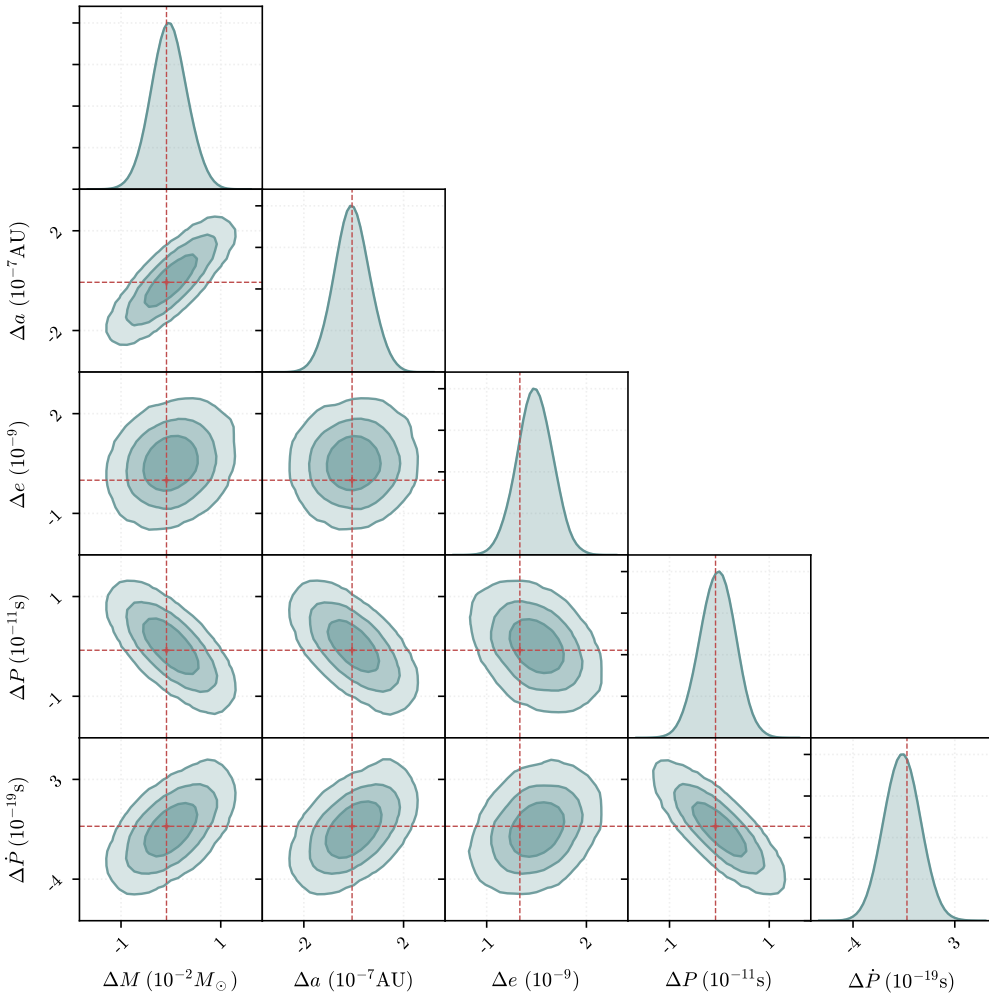


Figure 6.11. Full five-dimensional posterior distribution from our MCMC analysis on the mock catalogue for the pulsar model Toy 1. The red lines report the *true* values of the parameters, used as input for the generation of the mock catalogue, that are correctly recovered within 1σ by our methodology, thus showing that not only we reach astoundingly precise, but also accurate results.

model are reported in Figures 6.10 and 6.11 and in Table 6.4. In particular, in the first figure, we report the post-fit residuals obtained in correspondence of the best-fitting (maximum likelihood) values for each of the parameters considered. These values along with the corresponding 1σ uncertainty (reported in parenthesis) are detailed in Table 6.4. There, we also report the percentage precision achieved on the parameter, computed as the ratio between the uncertainty and the best-fitting value. Finally, Figure 6.11 reports the full corner plot of the posterior distribution on the parameters. The red dashed lines and dots correspond to the true values for each parameter, showing the perfect capability of our approach to recover the input model used for the generation of the mock catalogue. Altogether, our results show that not only incredible accuracy on the estimation of the intrinsic and orbital parameters of a pulsar (on par with current pulsar timing results in normal binary systems) can be achieved, but also that a pulsar around the Galactic Center SMBH would allow an unprecedented accuracy in the estimation of its mass⁴. Moreover, our analysis shows that our approach not only leads to such extreme precision in the parameter estimation but also that the results obtained are accurate (capable of recovering baseline input parameters) with no significant bias emerging from the parameter estimation (as demonstrated in Figure 6.11).

6.3. Conclusions

In this chapter, we have considered the problem of modeling the timing observables for pulsars around a SMBH like Sgr A* in the Galactic Center.

We have shown how the methodology that we have developed in Section 2.2.5 can be applied to estimate the observational impact on the timing residuals of a change in the fundamental nature of the central object or in the underlying theory of gravity. In particular, in Section 6.1 we have introduced a series of toy models and have shown that the timing residuals produced by alternatives to the Schwarzschild metric can surpass the potential observational sensitivity for future SKA timing observations in the Galactic Center. In perspective, our study qualitatively suggests that in all the cases considered the possible constraints on the extra parameters, encoding

⁴ And its spin, as shown in [307], thus enabling strong field tests of the no-hair theorem.

deviations from General Relativity, can surpass by orders of magnitude the constraints arising from other probes in the Galactic Center, like the S-stars. This further demonstrates the paramount importance of pulsars orbiting SMBH as strong-field probes of gravity, adding new results on top of an already broad literature on the subject.

In Sections 6.2.2 and 6.2.3 we have extended our methodology to develop a timing model able to generate phase-connected residuals and perform fitting of TOAs data to the Schwarzschild metric. We estimated quantitatively the phase-dependent residuals arising from a misestimation of one of the pulsar's parameters. Such estimates gave us a clear idea of the possible precision that would be achieved through an actual fitting analysis of our timing model on TOAs data. To analyze further this possibility, we generated a mock catalogue of timing observations for one of the toy models considered previously, representing a pulsar on a highly-eccentric and relativistic orbit around Sgr A*, and applied a Bayesian inference methodology to constrain a subset of parameters of the full problem. Our analysis shows that not only the input parameters used to generate the mock catalogue are perfectly recovered by the posterior analysis, but also that the precision reached in the estimation of the orbital parameters and the mass of Sgr A* can surpass by several orders of magnitude the one achieved thus far with the S-stars observations.

Our work-in-progress study highlights the great constraining power enabled by pulsar timing observations and the importance of properly accounting for all relativistic effects by adopting a geodesic timing model. In order to give a more realistic description, our next major step will be to extend our methodology to the Kerr black hole, or to its mimickers and extensions.

Conclusions and future prospects

In 2010, Genzel, Eisenhauer and Gillessen concluded their review article with the statement: “[*In the next decade*] the Galactic Center may then become a test bed for probing general relativity in the strong-field limit”¹. As we have illustrated throughout this thesis, fourteen years later this prediction has been widely confirmed. Technological advancements in near-infrared astronomy of the Galactic Center over the last decades have enabled to confidently measure the post-Newtonian effects of general relativistic gravitational redshift and Schwarzschild precession on the orbits of the S-stars. The groundbreaking event-horizon scale observation of strongly-lensed emission features in Sgr A* by the EHT, made possible by advancements in theoretical, computational, and technological fields, has validated the predictions of a black hole shadow and provided another tool for testing gravity in the strong-field regime. The Galactic Center has stood out as a powerful laboratory in which the three classical tests of General Relativity could be verified under much more testing conditions.

The remarkable discoveries made at the Galactic Center do not represent the endpoint of this line of research. Rather, they motivate further theoretical and observational efforts to reach new frontiers in the understanding of this unique environment. Over the next decade, new observational facilities will allow us to peer through the Galactic Center with new eyes. Ongoing and future upgrades of the GRAVITY instrument (the GRAVITY+ project) will enable deeper observations of the Galactic Center, potentially leading to the identification of a new population of closer, fainter stars, expected to yield more accurate measurements of the SMBH properties like mass, spin, and quadrupole moment. Further progress is anticipated with the advent of greater observing facilities such as the TMT in Hawaii or the 39-meter ELT at ESO. Additionally, radio pulsars in relativistic orbits around Sgr A* are expected to be detected within a few years with next-generation millimeter observatories like the SKA. Such pulsars, often regarded as the *holy grail* for experimental gravitation, are expected to provide the most stringent tests of gravity theories and black hole physics to date.

Due to current and future challenges in the modeling of

¹ Genzel, R., Eisenhauer, F. and Gillessen, S., *The Galactic Center massive black hole and nuclear star cluster*, 2010 [35]

observable signatures for these objects, the scientific goal for this thesis was, on one hand, the development of a numerical methodology to provide a fully relativistic orbital model for the motion of test particles around a supermassive compact object and, on the other, the implementation of Bayesian inference techniques, based on MCMC algorithms, to devise a self-consistent strategy to both test General Relativity and derive constraints on its alternatives through comparison with observational data at the Galactic Center.

This resulted in the development of an open-source code, `PyGR0`. It combines symbolic and numerical calculations, providing a user-friendly interface to integrate the geodesic equations for any given spacetime metric. We have focused on spacetime geometries characterized by spherical symmetry, which drastically simplifies the equations of motion and the setup of initial conditions for both massless and massive test particles. This allowed us to build an orbital model for the S-stars around Sgr A* in a generic spherically symmetric spacetime and a fully relativistic way to compute the photon paths and propagation times therein, useful for the modeling of pulsar TOAs at an Earth-based observatory.

We have applied our novel methodology in a variety of different scenarios:

- First, we have considered the impact that an extended component of dark matter in the Galactic Center in the form of a solitonic core, composed of ultralight boson particles described within the model of FDM, would have on S-stars orbits. The signature of the presence of such a soliton on the astrometric observables for the S2 star has been thoroughly analyzed and some novel constraints on the boson mass have been derived, which, when combined with complementary astrophysical and cosmological constraints in literature, reduce the allowed region for this parameter to only one order of magnitude.
- Subsequently, we have analyzed the viability of probing black hole mimicker geometries as alternatives to the classical general relativistic black hole model for Sgr A*, relying solely on the orbital motion of the S-stars and on the detection of the relativistic effects for these objects. We have shown that, while currently available data for the known S-stars in the Galactic Center are not able to unambiguously identify the fundamental nature

of Sgr A*, this picture might drastically change with the discovery of stars closer to the central object. Thanks to the forecast analysis that we have conducted, we have not only identified the possible orbital properties that such objects should present but also suggested the observational strategy that future observational campaigns should adopt, in order to maximize the constraining power of their observations.

- Next, we have considered several well-motivated extended or alternative theories to General Relativity. In particular, we have analyzed the STVG, $f(R)$ gravity, Horndeski gravity, EMDA and a quadratic extension to General Relativity, for all of which an analytic black hole solution exists. We have derived novel constraints on the extra parameters encoding the departures from General Relativity, on scales on which they were previously untested. Although we have always adopted an agnostic approach, by assigning uniform priors on such extra parameters, we have obtained posteriors that are centered on the region of the parameter space that corresponds to the general relativistic limit of the theories that we have studied. These remarkable results once again demonstrate the great predictive power of General Relativity.
- Finally, we have considered the case of pulsars on tight orbits around Sgr A*. We have shown how the fully relativistic methodology introduced can be applied to the realistic modeling of TOAs, whereas the usual post-Newtonian technique fails to capture the full non-linearity of the problem. Performing a forecast analysis, we have shown that the level of accuracy and precision in the characterization of the physical properties of Sgr A* through pulsar timing can surpass by several orders of magnitude the one achieved through S-stars observations, even after only a few years of observations. This is reflected in much tighter constraints that such observations would enable on the underlying theory of gravity and the fundamental nature of Sgr A*.

Despite the work done so far and the results achieved with this thesis, some delicate aspects will need to be explored. For example, with the forecasted improvements in the astrometric accuracy, the effects related to the rotation of Sgr A* will cease to be negligible. It is precisely from the possibility of de-

tecting orbital signatures related to the spin and the quadrupole moments of the central object, that the most stringent tests of gravity (*e.g.* the no-hair theorem) are expected to arise. While the numerical methodology developed in this thesis for the geodesic integration is completely general and can be easily applied to rotating spacetime metrics, the same is not true for the procedure to assign initial conditions for massive and massless particles. Changing the symmetries that the spacetime obeys also changes the definitions of the conserved quantities and results in an additional layer of complexity due to the presence of a preferred direction in spacetime (the rotation axis of the central object) that breaks most of the simplifications that we have exploited throughout this thesis. This crucially affects both the time-like orbits of stars or pulsars in the Galactic Center and the null paths followed by photons in their travel toward Earth. While introducing these effects does represent a substantial challenge in terms of theoretical modeling, the valuable insight that one would obtain in return would represent a major leap forward toward a more thorough characterization of Sgr A* and its gravitational field, especially for the case of orbiting pulsars. Due to the peculiar role of the Galactic Center, at the edge of theoretical physics and astrophysics, such advancements on the fundamental side might lead to breakthrough discoveries on the role of SMBHs on stellar formation, of their cosmological co-evolution with their galactic hosts and in our understanding of galactic cores in general.

To conclude, looking in retrospect at the remarkable progress made in the last decades in the astronomical facilities, in the quality of the resulting observations and the theoretical modeling of the Galactic Center, we can anticipate even more exciting advancement in the years to come, both from new objects that will be found orbiting Sgr A* and from the direct observation of Sgr A* itself. In this context, the results obtained within this thesis and the methodologies developed will represent a valuable tool for the greater benefit of the research community in this field.

Appendices

Higher order geodesic effects

A.

The numerical integration scheme explained in Section 2.2.3 and the subsequent reconstruction of astrometric and spectroscopic observables is based on some important assumptions. To reproduce sky-projected astrometric positions for an Earth-based observer and spectroscopically reconstructed line of sight velocities from the integrated trajectories, we have not considered relativistic effects on the photon itself. If one wants to approach the problem of reconstructing the astrometric and spectroscopic observables in a fully relativistic way, photon paths represented by null geodesics should be integrated after an appropriate initial-condition-search has been carried out to identify the null trajectory connecting emitter and observer at each epoch (the so-called *emitter-observer problem*). The advantage of this approach is that one is able to take into account at once all the relativistic effects that affect both the massive test particle trajectory and the photons, including all the appropriate time delays (Figure A.1). In Section 2.2.5 we have described our numerical methodology to address this problem. There, we have also highlighted the improvement over the post-Newtonian approach in the determination of a relativistic propagation time for photons and of an accurate reconstruction of the curved photon path. While this approach is completely general and can easily be extended to models beyond the classical Schwarzschild black hole, it comes with an additional computational cost. In this Appendix we apply the full procedure to the S2 star and show that all the relativistic photon effects, which we collectively refer to as *higher order geodesic effects*, introduce deviations on the observable quantities that are well below the current observational sensitivity thresholds. For this reason, with current instrumentations for the observations in the Galactic Center, neglecting the fully relativistic propagation of light rays and using a semi-classical approach to reconstruct the two observables does not introduce any sensible bias to the analysis of the gravitational field of Sgr A*.

As explained in Section 2.2.5, photons in any spherically symmetric spacetime travel along a plane that contains the emitter, the observer and the central object. This is shown in Figure A.1. On this plane, one can consider the straight-line propagation of photons going from the emitter (located at $\vec{r}_e$)

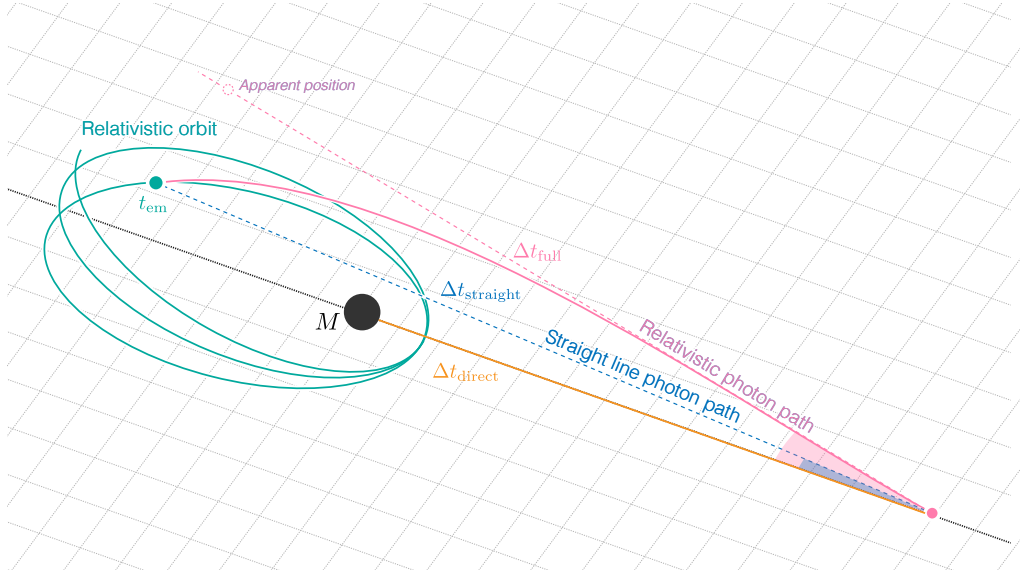


Figure A.1. Illustration of the procedure to produce synthetic relativistic observables from geodesic orbits. For each position on the particle relativistic trajectory, the null photon path (pink line, corresponding to primary photons received by a distant observer) has to be determined. This is then integrated up to the distant observer, yielding the apparent sky-position (determined by the angles formed by the incoming photon with respect to some reference directions) of the emitter and the photon travel time. If one does not take into account the relativistic light propagation (dashed blue line), one would obtain a different photon path thus implying a different reconstructed astrometric position (the angles formed in case of straight line propagation are always smaller) and a different (shorter) propagation time.

to the observer (located at $\vec{r}_o$) and compute the straight-line propagation time as (temporarily restoring the speed of light c)

$$\Delta t_{\text{straight}} = \frac{|\vec{r}_o - \vec{r}_e|}{c}.$$

A.1

This coincides with the classical geometrical propagation time of Newtonian mechanics, affected over the orbit by Rømer delay, already found in Eq. (2.66). If one does not take into account the modulation along the orbit of the light travel time and assumes that all the photons only have to cover the distance of the observer from the central object (in other words, if one neglects the Rømer delay), one can estimate the travel time simply by

$$\Delta t_{\text{direct}} = \frac{|\vec{r}_o|}{c}.$$

A.2

Moreover, in the case of a very distant observer ($|\vec{r}_o| \gg |\vec{r}_e|$), one can neglect all the geometrical distortion effects related to perspective and perform the geometrical projection of the test particle trajectory by simply applying the rotation matrices

(eventually leading to the Thiele-Innes elements) that result in the expressions in Eqs. (2.49)-(2.51) and Eq. (2.63). We refer to this set of astrometric observables as $(\alpha_{\text{geom}}, \delta_{\text{geom}})$.

Applying the procedure shown in Section 2.2.5 for each position of the test particle on its orbit, one is able to determine a plane on which the photon propagates (identified by a position angle of the plane with respect to the chosen reference direction), an impact parameter b of the photon, uniquely identifying a photon on such plane and a relativistic propagation time (reported in Eq. (2.103)), Δt_{full} . From the impact parameter and the position angle of the plane on which the photon propagates, one can construct relativistic astrometric observables, $(\alpha_{\text{lensed}}, \delta_{\text{lensed}})$, (see Figure A.2). The two astrometric observables are shown in Figure A.3 over three and a half orbital periods. The maximum astrometric deviation is found at around pericenter passage (see the zoomed-in inset plot) and it amounts to $\sim 20 \mu\text{as}$.

The relativistic contribution to the photon propagation time can be thus computed as

$$\Delta t_{\text{Rel}} = \Delta t_{\text{full}} - \Delta t_{\text{straight}}.$$

This time difference quantifies the error introduced in the computation of the photon travel time when considering the Rømer delay alone and no additional effect on the photon propagation. For the S2 star, this time discrepancy is maximized around the time of pericenter passage. More specifically, the maximum Δt_{Rel} is produced shortly after the pericenter passage, when the combination of greater Shapiro contribution and greater geometrical bending of the photon paths from the point of emission to the observer, given by the specific orbital inclination of S2. This effect amounts to a modulation of ~ 80 s over one entire orbit, as we show in Figure A.4.

In order to compare the different observables reconstructed by applying the full relativistic orbital model or the different simplifying assumptions over the photon propagation time or photon path, we have considered four distinct orbital models for the S2 star. In particular:

- **A:** this model is the simplest one, the observables are reconstructed by a geometrical orthogonal projection of the geodesic orbital trajectory and the photon travel time is assumed to be the same for all photons, *i.e.* we

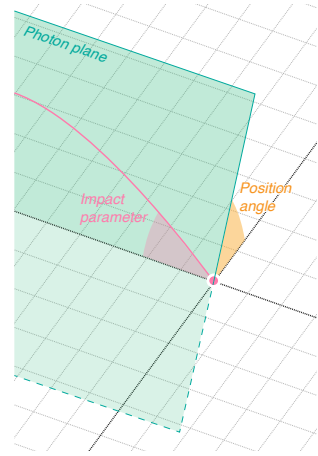


Figure A.2. Position and impact parameter allow to uniquely identify a propagation direction for incoming photons, *i.e.* a pair of astrometric observables $(\alpha_{\text{lensed}}, \delta_{\text{lensed}})$.

A.3

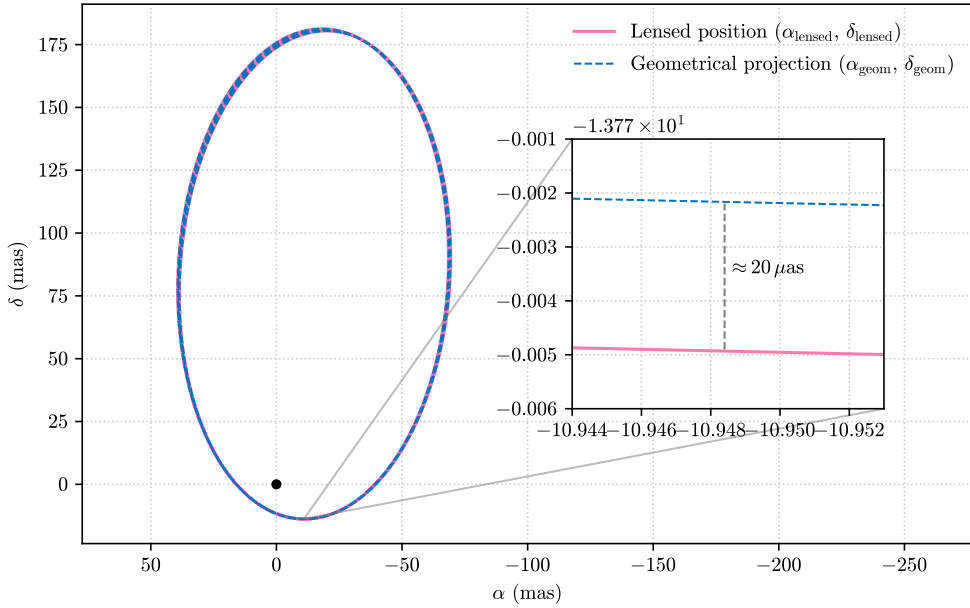


Figure A.3. Comparison between the astrometric observables reconstructed using the simple geometrical projection $(\alpha_{\text{geom}}, \delta_{\text{geom}})$ (dashed blue line) and the lensed position $(\alpha_{\text{lensed}}, \delta_{\text{lensed}})$ (solid pink line) reconstructed using the relativistic photon paths. The inset plot zooms around the pericenter passage, where the maximum astrometric deviation (at around $\sim 20 \mu\text{as}$) is predicted between the two observables.

assume Δt_{direct} in Eq. (A.2). Hence, we parametrize the observables as:

$$(\alpha, \delta)_A = (\alpha_{\text{geom}}(t_A), \delta_{\text{geom}}(t_A)), \quad \text{A.4}$$

with

$$t_A = t_{\text{em}} + \Delta t_{\text{direct}}, \quad \text{A.5}$$

being t_{em} the coordinate time on the geodesic of the massive test particle.

- **B:** in this model we consider the same astrometric observables used in model **A**, but we add the Rømer delay. More specifically, we use the $\Delta t_{\text{straight}}$ to parametrize the astrometric observables, which are given by:

$$(\alpha, \delta)_B = (\alpha_{\text{geom}}(t_B), \delta_{\text{geom}}(t_B)), \quad \text{A.6}$$

with

$$t_B = t_{\text{em}} + \Delta t_{\text{straight}}, \quad \text{A.7}$$

being t_{em} the coordinate time on the geodesic of the

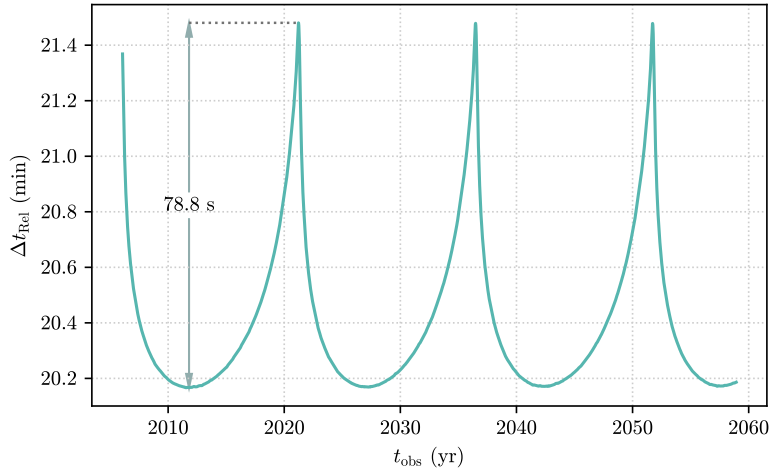


Figure A.4. Estimation of the relativistic contribution to the photon propagation time estimated as in Eq. (A.3) for the S2 star in the Galactic Center. The contribution is never 0, because due to the compactness of the central source, photons take time to emerge from the gravitational potential well of Sgr A*, as compared to the straight-line Newtonian propagation of photons. This effect, thus shifts all incoming photons by an additional ~ 20 min if compared to the non-relativistic propagation. Of course, this effect is not directly detectable with Earth-based observations, because the distance of Earth from Sgr A* is not known a priori and thus this additional shift can be always regarded as an increased distance from the Galactic Center. What indeed is detectable, is the modulation over time of the relativistic contribution, which, amounts to ~ 1.31 mins (~ 80 s) over an orbit and is maximized around pericenter, when the photons are emitted more deeply within the gravitational field of Sgr A* and thus they are more affected by the bending of their path and by the Shapiro delay experienced in emerging from the potential well.

massive test particle.

- **C:** this model takes into account the relativistic photon propagation, by reconstructing lensed astrometric positions for the particle, but it does not consider the actual relativistic photon travel time. Instead, we use the straight-line light travel time. In other words, model **C** returns lensed positions for S2, taking into account the Rømer delay but no relativistic effects on the photon travel time. The observables are thus parametrized by:

$$(\alpha, \delta)_C = (\alpha_{\text{lensed}}(t_C), \delta_{\text{lensed}}(t_C)), \quad \text{A.8}$$

with

$$t_C = t_{\text{em}} + \Delta t_{\text{straight}}, \quad \text{A.9}$$

being t_{em} the coordinate time on the geodesic of the massive test particle.

- **D:** this is our fully relativistic model. Not only the lensed positions are now reconstructed, but photon travel times

are computed following the relativistic procedure described in Section 2.2.5. So we now have fully relativistic observables for S2 parametrized as:

$$(\alpha, \delta)_D = (\alpha_{\text{lensed}}(t_D), \delta_{\text{lensed}}(t_D)), \quad | \text{A.10}$$

with

$$t_D = t_{\text{em}} + \Delta t_{\text{full}}, \quad | \text{A.11}$$

being t_{em} the coordinate time on the geodesic of the massive test particle.

We compare the first three models with the fully relativistic model **D** by computing the astrometric deviation $\Delta \equiv \sqrt{\Delta\alpha^2 + \Delta\delta^2}$ between each of the models and the fully relativistic one. The results of this comparison over three and a half orbital periods are reported in Figure A.5, along with the average accuracy of NACO adaptive optics astrometry for pre-2017 observations at $400 \mu\text{as}$, the currently achieved astrometric accuracy using the GRAVITY interferometer at $60 \mu\text{as}$ [226, 368] and its nominal sensitivity at $10 \mu\text{as}$, respectively [170]. As it appears, neglecting the classical Roemer delay (model **A**) produces an astrometric shift of $\sim 500 \mu\text{as}$ that surpasses the sensitivity threshold of both instruments. On the other hand, including Rømer delay but neglecting lensing effects (model **B**) introduces an astrometric shift whose maximum is at $\sim 20 \mu\text{as}$ that, while potentially detectable using GRAVITY data of the S2 star at the nominal uncertainty of $10 \mu\text{as}$ has currently gone undetected due to the higher astrometric uncertainty that has thus far been achieved with this instrument. Finally, including the gravitational lensing but neglecting effects related to a longer photon propagation time in a fully relativistic setting, introduces an astrometric shift that at its maximum is $\sim 8 \mu\text{as}$ is below the nominal GRAVITY threshold. All the results described here are perfectly compatible with the sensitivity analysis found in [345] for a Kerr black hole and suggest that with current data, a model based on the geometric orbital projection of the geodesic trajectory and taking into account only the classical Rømer delay. For this reason, model **B** was used throughout this thesis.

Finally, we analyze the impact of the choice of a given coordinate system on the resulting astrometric observables, when using the geometrical orthogonal projection. As remarked in Section 2.2.3, using the aerial radius coordinate

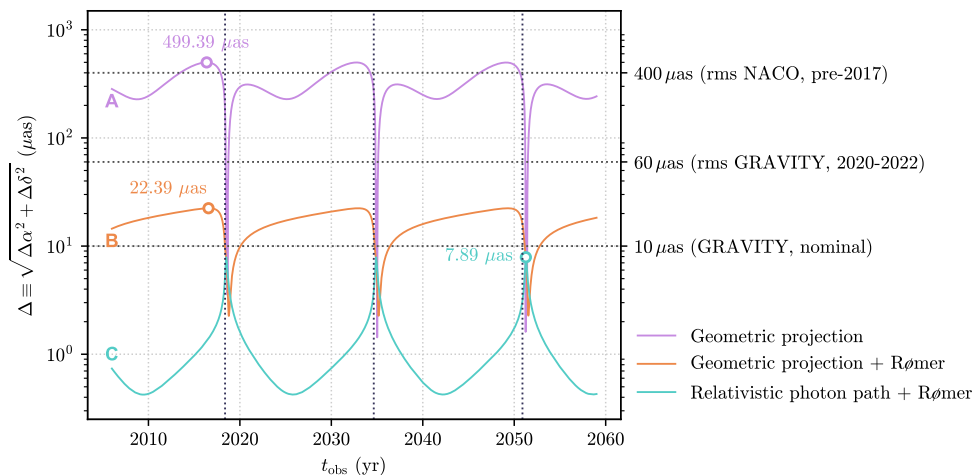


Figure A.5. The astrometric deviations $\Delta = \sqrt{\Delta\alpha^2 + \Delta\delta^2}$, where $\Delta\alpha$ and $\Delta\delta$ are the deviations in the right ascension and declination, respectively of the considered models from the fully relativistic one (*i.e.*, model **D**). Vertical dotted lines report the times of pericenter passage over the three orbits. In particular, as reported in the legend, model **A** (pink line) is built from the geometric projection and without considering the Rømer delay. It is the one with the highest astrometric deviation (reaching $\sim 500\mu\text{as}$), produced shortly before pericenter passage when the star is faster and thus the astrometric departure implied by a mis-reconstruction of the time of light propagation is maximized. Conversely, model **B** (orange line), takes into account the Rømer delay but still assumes geometric projection, so that no relativistic effect on the photon is considered. Similarly to the previous case, the highest astrometric deviation (reaching $\sim 22\mu\text{as}$) is reached shortly before pericenter passage. Finally, model **C** takes into account the relativistic curved photon path, but does not consider the propagation time in the relativistic setting. This produces an astrometric deviation that is maximized at pericenter where it reaches $\sim 8\mu\text{as}$. Horizontal dashed lines report the average accuracy of NACO adaptive optics astrometry for pre-2017 observations at $400\mu\text{as}$, the currently achieved astrometric accuracy using the GRAVITY interferometer at $60\mu\text{as}$ and its nominal sensitivity at $10\mu\text{as}$, respectively.

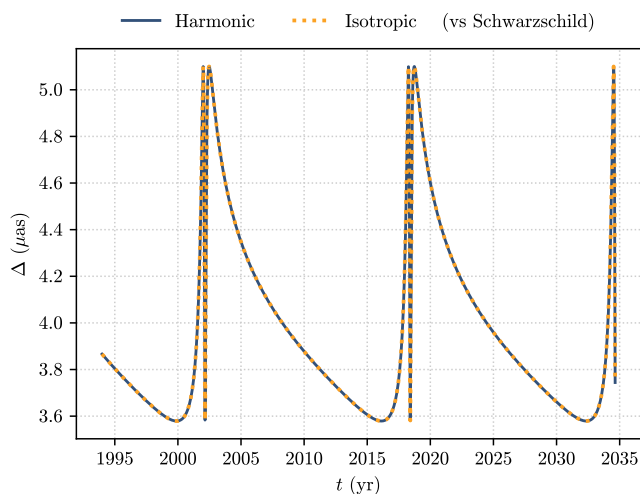


Figure A.6. Deviation of the astrometric observables reconstructed using our model **B**, resulting from a choice of the notion of distance from the central object equivalent to the harmonic radial coordinate (blue solid line) and the isotropic coordinate (orange dotted line), respectively, from the Schwarzschild radial coordinate case. The two predictions are almost equivalent to each other and their deviation from the Schwarzschild case is at most amounting to $\sim 5\mu\text{as}$ and thus undetectable with the current instrumentations.

as a notion of distance to compute the geometric projection is a totally arbitrary choice that comes from no general prescription. In general, any different choice might have been equally appropriate. For example, one might have as well decided to adopt the harmonic radial coordinate used in the post-Newtonian metric (see Eq. (2.82)) or the isotropic radial coordinate (as is customary in relativistic celestial mechanics, see Chapter 40 of [333]) as distance and derived a different set of cartesian positions and thus a different set of observables. Here, we compare what deviation would a distinct choice of the radial coordinate imply on the astrometric observables. To do that, we transform the Schwarzschild radial coordinate for the integrated orbit of S2 into its harmonic and isotropic counterparts and build, using our model **B**, new sets of astrometric observables. In Figure A.6 we report the resulting astrometric deviation Δ between these astrometric positions for S2 and the ones obtained using the standard Schwarzschild aerial coordinate. The slight difference in the radial coordinate definitions produces tiny departures on the astrometric observables that are always $\lesssim 5 \mu\text{as}$ and result completely equivalent. If one could reduce the instrumental noise to arbitrarily low values, the only appropriate procedure is the fully relativistic integration of photon paths, our model **D**, which returns the observables in a truly covariant manner and for which the results are not affected by the specific choice taken.

The mock catalogue for S2

B.

	Date (yyyy-mm-dd)	t (yr)	α (as)	δ (as)	v_{LOS} (km/s)
	First period				
1	2018-30-01	2018.08214	−0.0326	−0.00549	3328
2	2018-01-03	2018.16427	−0.02645	−0.00951	3727
3	2018-31-03	2018.24641	−0.01821	−0.01272	4114
4	2018-07-04	2018.26557	−0.01587	−0.01322	4144
5	2018-14-04	2018.28474	−0.01331	−0.01354	4159
6	2018-21-04	2018.3039	−0.01056	−0.01364	4090
7	2018-28-04	2018.32307	−0.00754	−0.01345	3906
8	2018-30-04	2018.32854	−0.00666	−0.01332	3812
9	2018-02-05	2018.33402	−0.00576	−0.01318	3724
10	2018-04-05	2018.33949	−0.00484	−0.013	3615
11	2018-06-05	2018.34497	−0.00391	−0.01276	3502
12	2018-08-05	2018.35044	−0.00296	−0.01254	3333
13	2018-10-05	2018.35592	−0.00203	−0.01225	3197
14	2018-11-05	2018.35866	−0.00155	−0.01211	3117
15	2018-12-05	2018.3614	−0.00106	−0.01194	3032
16	2018-13-05	2018.36413	−0.00061	−0.01177	2915
17	2018-14-05	2018.36687	−0.00012	−0.01158	2843
18	2018-15-05	2018.36961	0.00038	−0.01142	2761
19	2018-16-05	2018.37235	0.00086	−0.01121	2644
20	2018-17-05	2018.37509	0.00134	−0.01101	2550
21	2018-18-05	2018.37782	0.0018	−0.01078	2441
22	2018-19-05	2018.38056	0.00228	−0.01058	2352
23	2018-20-05	2018.3833	0.00274	−0.01036	2244
24	2018-21-05	2018.38604	0.00323	−0.01011	2142
25	2018-22-05	2018.38877	0.00368	−0.00986	2022
26	2018-23-05	2018.39151	0.00414	−0.00961	1904
27	2018-24-05	2018.39425	0.00461	−0.00935	1814
28	2018-26-05	2018.39973	0.00551	−0.0088	1605
29	2018-28-05	2018.4052	0.00641	−0.00824	1391
30	2018-30-05	2018.41068	0.0073	−0.00764	1170
31	2018-01-06	2018.41615	0.00816	−0.00706	946

	Date (yyyy-mm-dd)	t (yr)	α (as)	δ (as)	v_{LOS} (km/s)
32	2018-08-06	2018.43532	0.01098	-0.0048	310
33	2018-15-06	2018.45448	0.01354	-0.00244	-196
34	2018-15-07	2018.53662	0.02168	0.00773	-1369
35	2018-14-08	2018.61875	0.02684	0.01701	-1710
36	2018-13-09	2018.70089	0.03033	0.02528	-1801
37	2018-13-10	2018.78303	0.03282	0.03271	-1812
38	2018-12-11	2018.86516	0.03464	0.03949	-1806
39	2018-12-12	2018.9473	0.03597	0.04574	-1732
40	2019-11-01	2019.02943	0.03698	0.05153	-1713
41	2019-10-02	2019.11157	0.0377	0.05697	-1683
42	2019-12-03	2019.1937	0.03819	0.06207	-1622
43	2019-11-04	2019.27584	0.03852	0.06689	-1582
44	2019-11-05	2019.35797	0.03872	0.07145	-1531
45	2019-07-11	2019.85079	0.03776	0.09477	-1340
46	2020-05-05	2020.3436	0.03476	0.11308	-1170
47	2020-01-11	2020.83641	0.03067	0.12794	-1027
48	2021-01-05	2021.32923	0.02591	0.14013	-922
49	2021-28-10	2021.82204	0.02077	0.15016	-811
50	2022-25-04	2022.31485	0.01532	0.15835	-729
51	2022-22-10	2022.80767	0.00975	0.16494	-624
52	2023-20-04	2023.30048	0.00411	0.17013	-570
53	2023-17-10	2023.79329	-0.00157	0.174	-507
54	2024-14-04	2024.28611	-0.00726	0.17666	-398
55	2024-11-10	2024.77892	-0.0129	0.17821	-363
56	2025-10-04	2025.27173	-0.01846	0.17866	-278
57	2025-07-10	2025.76454	-0.02394	0.17809	-220
58	2026-04-04	2026.25736	-0.02924	0.17651	-163
59	2026-01-10	2026.75017	-0.03441	0.17393	-90
60	2027-30-03	2027.24298	-0.03939	0.1704	-20
61	2027-26-09	2027.7358	-0.04414	0.16591	54
62	2028-24-03	2028.22861	-0.04862	0.16043	122
63	2028-20-09	2028.72142	-0.0528	0.15395	197
64	2029-20-03	2029.21424	-0.05665	0.14647	293
65	2029-16-09	2029.70705	-0.06008	0.13792	387
66	2030-14-03	2030.19986	-0.06305	0.12827	477
67	2030-10-09	2030.69268	-0.06542	0.11745	560

	Date	t	α	δ	v_{LOS}
	(yyyy-mm-dd)	(yr)	(as)	(as)	(km/s)
68	2031-09-03	2031.18549	−0.06708	0.10534	697
69	2031-05-09	2031.6783	−0.06785	0.0918	875
70	2032-03-03	2032.17112	−0.06747	0.07674	1019
71	2032-30-08	2032.66393	−0.06541	0.05979	1248
72	2033-27-02	2033.15674	−0.06089	0.04064	1598
73	2033-29-03	2033.23888	−0.05982	0.03717	1664
74	2033-28-04	2033.32101	−0.05857	0.03364	1734
75	2033-28-05	2033.40315	−0.0572	0.03003	1832
76	2033-27-06	2033.48528	−0.05565	0.02631	1888
77	2033-27-07	2033.56742	−0.05393	0.02252	2014
78	2033-26-08	2033.64956	−0.05199	0.01866	2121
79	2033-25-09	2033.73169	−0.04978	0.01466	2250
80	2033-25-10	2033.81383	−0.04721	0.01058	2388
81	2033-24-11	2033.89596	−0.04428	0.00643	2587
82	2033-24-12	2033.9781	−0.04083	0.00217	2792
Second period					
83	2034-22-01	2034.06023	−0.03672	−0.00209	3064
84	2034-21-02	2034.14237	−0.03166	−0.00631	3373
85	2034-23-03	2034.2245	−0.0252	−0.01023	3787
86	2034-22-04	2034.30664	−0.0165	−0.01314	4133
87	2034-29-04	2034.3258	−0.01401	−0.01351	4160
88	2034-06-05	2034.34497	−0.0113	−0.01367	4113
89	2034-13-05	2034.36413	−0.00835	−0.01354	3941
90	2034-15-05	2034.36961	−0.0075	−0.01347	3873
91	2034-17-05	2034.37509	−0.00661	−0.01333	3796
92	2034-19-05	2034.38056	−0.00569	−0.01318	3700
93	2034-21-05	2034.38604	−0.00478	−0.013	3596
94	2034-23-05	2034.39151	−0.00384	−0.0128	3482
95	2034-25-05	2034.39699	−0.00289	−0.01253	3355
96	2034-27-05	2034.40246	−0.00195	−0.01224	3189
97	2034-28-05	2034.4052	−0.00146	−0.01208	3092
98	2034-29-05	2034.40794	−0.00101	−0.01193	3007
99	2034-30-05	2034.41068	−0.00051	−0.01178	2919
100	2034-31-05	2034.41342	−3e − 05	−0.01158	2852
101	2034-01-06	2034.41615	0.00044	−0.0114	2728
102	2034-02-06	2034.41889	0.0009	−0.0112	2639

	Date (yyyy-mm-dd)	t (yr)	α (as)	δ (as)	v_{LOS} (km/s)
103	2034-03-06	2034.42163	0.00139	−0.01097	2516
104	2034-04-06	2034.42437	0.00187	−0.01075	2427
105	2034-05-06	2034.4271	0.00234	−0.01055	2335
106	2034-06-06	2034.42984	0.0028	−0.01029	2225
107	2034-07-06	2034.43258	0.0033	−0.01005	2115
108	2034-08-06	2034.43532	0.00375	−0.00983	2009
109	2034-09-06	2034.43806	0.00421	−0.00955	1889
110	2034-10-06	2034.44079	0.00469	−0.0093	1788
111	2034-12-06	2034.44627	0.00558	−0.00876	1593
112	2034-14-06	2034.45175	0.00648	−0.00819	1362
113	2034-16-06	2034.45722	0.00735	−0.00759	1145
114	2034-18-06	2034.4627	0.00821	−0.007	933
115	2034-25-06	2034.48186	0.01104	−0.00475	294
116	2034-02-07	2034.50103	0.01357	−0.00237	−216
117	2034-01-08	2034.58316	0.0217	0.00781	−1387
118	2034-31-08	2034.6653	0.02681	0.0171	−1739
119	2034-30-09	2034.74743	0.03026	0.02535	−1829
120	2034-30-10	2034.82957	0.03273	0.03278	−1811
121	2034-29-11	2034.9117	0.03453	0.03959	−1772
122	2034-29-12	2034.99384	0.03587	0.04582	−1741
123	2035-28-01	2035.07598	0.03683	0.05162	−1711
124	2035-27-02	2035.15811	0.03752	0.05703	−1671
125	2035-29-03	2035.24025	0.038	0.06213	−1632
126	2035-28-04	2035.32238	0.03833	0.06695	−1584
127	2035-28-05	2035.40452	0.03847	0.07152	−1518
128	2035-24-11	2035.89733	0.03744	0.09479	−1315
129	2036-22-05	2036.39014	0.03437	0.11307	−1147
130	2036-18-11	2036.88296	0.03021	0.12789	−1040
131	2037-18-05	2037.37577	0.02539	0.14004	−926
132	2037-14-11	2037.86858	0.02019	0.15001	−832
133	2038-12-05	2038.3614	0.01475	0.15822	−733
134	2038-08-11	2038.85421	0.00913	0.1648	−648
135	2039-07-05	2039.34702	0.00347	0.16993	−544
136	2039-03-11	2039.83984	−0.00225	0.17377	−499
137	2040-01-05	2040.33265	−0.00797	0.17641	−415
138	2040-28-10	2040.82546	−0.01362	0.17794	−358

	Date	t	α	δ	v_{LOS}
	(yyyy-mm-dd)	(yr)	(as)	(as)	(km/s)
139	2041-27-04	2041.31828	-0.01919	0.17837	-292
140	2041-24-10	2041.81109	-0.02465	0.17776	-210
141	2042-21-04	2042.3039	-0.02999	0.17618	-155
142	2042-18-10	2042.79671	-0.03513	0.17359	-91
143	2043-16-04	2043.28953	-0.0401	0.17006	-22
144	2043-13-10	2043.78234	-0.04484	0.16554	59
145	2044-10-04	2044.27515	-0.04932	0.16004	127
146	2044-07-10	2044.76797	-0.05349	0.15355	210
147	2045-06-04	2045.26078	-0.05728	0.14608	303
148	2045-03-10	2045.75359	-0.06073	0.13753	368
149	2046-31-03	2046.24641	-0.06364	0.12786	476
150	2046-27-09	2046.73922	-0.06597	0.11706	574
151	2047-26-03	2047.23203	-0.06757	0.10497	710
152	2047-22-09	2047.72485	-0.06831	0.09145	855
153	2048-20-03	2048.21766	-0.06782	0.07639	1041
154	2048-16-09	2048.71047	-0.06573	0.05947	1263
155	2049-16-03	2049.20329	-0.06113	0.04033	1579
156	2049-15-04	2049.28542	-0.06	0.03689	1671
157	2049-15-05	2049.36756	-0.05878	0.03338	1739
158	2049-14-06	2049.44969	-0.05738	0.02978	1807
159	2049-14-07	2049.53183	-0.05581	0.02608	1898
160	2049-13-08	2049.61396	-0.05406	0.0223	2010
161	2049-12-09	2049.6961	-0.05211	0.01843	2121
162	2049-12-10	2049.77823	-0.04986	0.01447	2255
163	2049-11-11	2049.86037	-0.0473	0.0104	2414
164	2049-11-12	2049.94251	-0.04436	0.00626	2599
	Third period				
165	2050-09-01	2050.02464	-0.04089	0.00204	2806
166	2050-08-02	2050.10678	-0.03676	-0.00225	3036
167	2050-10-03	2050.18891	-0.03167	-0.00644	3388
168	2050-09-04	2050.27105	-0.0252	-0.01031	3793
169	2050-09-05	2050.35318	-0.01645	-0.0132	4115
170	2050-16-05	2050.37235	-0.01395	-0.01353	4146
171	2050-23-05	2050.39151	-0.01125	-0.01368	4092
172	2050-30-05	2050.41068	-0.00832	-0.01355	3939
173	2050-01-06	2050.41615	-0.00744	-0.01347	3871

	Date (yyyy-mm-dd)	t (yr)	α (as)	δ (as)	v_{LOS} (km/s)
174	2050-03-06	2050.42163	-0.00655	-0.01334	3783
175	2050-05-06	2050.4271	-0.00563	-0.01318	3696
176	2050-07-06	2050.43258	-0.0047	-0.013	3576
177	2050-09-06	2050.43806	-0.00376	-0.01275	3445
178	2050-11-06	2050.44353	-0.00283	-0.01252	3307
179	2050-12-06	2050.44627	-0.00236	-0.01237	3239
180	2050-13-06	2050.44901	-0.0019	-0.01222	3149
181	2050-14-06	2050.45175	-0.0014	-0.01208	3076
182	2050-15-06	2050.45448	-0.00094	-0.0119	2992
183	2050-16-06	2050.45722	-0.00045	-0.01173	2871
184	2050-17-06	2050.45996	$2e - 05$	-0.01156	2807
185	2050-18-06	2050.4627	0.00051	-0.01138	2703
186	2050-19-06	2050.46543	0.00097	-0.01116	2605
187	2050-20-06	2050.46817	0.00145	-0.01095	2494
188	2050-21-06	2050.47091	0.00194	-0.01073	2399
189	2050-22-06	2050.47365	0.00239	-0.0105	2304
190	2050-23-06	2050.47639	0.00287	-0.01028	2180
191	2050-24-06	2050.47912	0.00333	-0.01003	2083
192	2050-25-06	2050.48186	0.00381	-0.00979	1979
193	2050-27-06	2050.48734	0.00473	-0.00928	1748
194	2050-29-06	2050.49281	0.00564	-0.00871	1561
195	2050-01-07	2050.49829	0.00653	-0.00815	1330
196	2050-03-07	2050.50376	0.00741	-0.00756	1142
197	2050-10-07	2050.52293	0.0103	-0.00534	444
198	2050-17-07	2050.54209	0.01292	-0.003	-110
199	2050-16-08	2050.62423	0.02129	0.00724	-1359
200	2050-15-09	2050.70637	0.02653	0.0166	-1737
201	2050-15-10	2050.7885	0.03003	0.02493	-1800
202	2050-14-11	2050.87064	0.0325	0.03239	-1825
203	2050-14-12	2050.95277	0.0343	0.03921	-1809
204	2051-13-01	2051.03491	0.03566	0.0455	-1765
205	2051-12-02	2051.11704	0.0366	0.05132	-1726
206	2051-14-03	2051.19918	0.03732	0.05674	-1666
207	2051-13-04	2051.28131	0.0378	0.06187	-1637
208	2051-13-05	2051.36345	0.03809	0.0667	-1580
209	2051-12-06	2051.44559	0.03824	0.07127	-1543

	Date	t	α	δ	v_{LOS}
	(yyyy-mm-dd)	(yr)	(as)	(as)	(km/s)
210	2051-09-12	2051.9384	0.03716	0.09458	-1349
211	2052-06-06	2052.43121	0.03401	0.11288	-1170
212	2052-03-12	2052.92402	0.02977	0.12769	-1030
213	2053-02-06	2053.41684	0.02491	0.13984	-930
214	2053-29-11	2053.90965	0.01969	0.14982	-809
215	2054-27-05	2054.40246	0.01419	0.15797	-742
216	2054-23-11	2054.89528	0.00855	0.16454	-639
217	2055-22-05	2055.38809	0.00286	0.16969	-561
218	2055-18-11	2055.8809	-0.00291	0.17349	-479
219	2056-16-05	2056.37372	-0.0086	0.17614	-426
220	2056-12-11	2056.86653	-0.01427	0.17766	-342
221	2057-12-05	2057.35934	-0.01984	0.17806	-281
222	2057-08-11	2057.85216	-0.02533	0.17746	-223
223	2058-06-05	2058.34497	-0.03065	0.17588	-158
224	2058-02-11	2058.83778	-0.03581	0.17327	-88
225	2059-01-05	2059.3306	-0.04076	0.16973	-22
226	2059-28-10	2059.82341	-0.04549	0.16523	61
227	2060-25-04	2060.31622	-0.04995	0.15973	135
228	2060-22-10	2060.80903	-0.0541	0.15325	221
229	2061-21-04	2061.30185	-0.0579	0.14577	291
230	2061-18-10	2061.79466	-0.06129	0.13722	381
231	2062-15-04	2062.28747	-0.0642	0.12759	498
232	2062-12-10	2062.78029	-0.06648	0.11681	585
233	2063-10-04	2063.2731	-0.06807	0.10471	723
234	2063-07-10	2063.76591	-0.06874	0.09124	851
235	2064-04-04	2064.25873	-0.06823	0.0762	1033
236	2064-01-10	2064.75154	-0.06609	0.05935	1273
237	2065-31-03	2065.24435	-0.06143	0.0403	1586
238	2065-30-04	2065.32649	-0.06034	0.03684	1652
239	2065-30-05	2065.40862	-0.05907	0.03336	1736
240	2065-29-06	2065.49076	-0.05767	0.02977	1808
241	2065-29-07	2065.5729	-0.05611	0.02608	1895
242	2065-28-08	2065.65503	-0.05437	0.02231	1996
243	2065-27-09	2065.73717	-0.05238	0.01847	2120
244	2065-27-10	2065.8193	-0.05014	0.01451	2245
245	2065-26-11	2065.90144	-0.04758	0.01048	2402

	Date	t	α	δ	v_{LOS}
	(yyyy-mm-dd)	(yr)	(as)	(as)	(km/s)
246	2065-26-12	2065.98357	-0.04463	0.00634	2580
Fourth period					
247	2066-24-01	2066.06571	-0.04119	0.00213	2784
248	2066-23-02	2066.14784	-0.03709	-0.00209	3040
249	2066-25-03	2066.22998	-0.03204	-0.00626	3349
250	2066-24-04	2066.31211	-0.02567	-0.01015	3755
251	2066-24-05	2066.39425	-0.0171	-0.0131	4097
252	2066-31-05	2066.41342	-0.01463	-0.01349	4145
253	2066-07-06	2066.43258	-0.01198	-0.0137	4116
254	2066-14-06	2066.45175	-0.0091	-0.01366	3983
255	2066-21-06	2066.47091	-0.00602	-0.01326	3706
256	2066-23-06	2066.47639	-0.00512	-0.01311	3632
257	2066-25-06	2066.48186	-0.00417	-0.01288	3502
258	2066-27-06	2066.48734	-0.00323	-0.01264	3375
259	2066-29-06	2066.49281	-0.0023	-0.01239	3208
260	2066-30-06	2066.49555	-0.0018	-0.01222	3119
261	2066-01-07	2066.49829	-0.00134	-0.01205	3066
262	2066-02-07	2066.50103	-0.00086	-0.01189	2961
263	2066-03-07	2066.50376	-0.00039	-0.01172	2880
264	2066-04-07	2066.5065	0.00011	-0.01153	2792
265	2066-05-07	2066.50924	0.00058	-0.01133	2684
266	2066-06-07	2066.51198	0.00104	-0.01112	2600
267	2066-07-07	2066.51472	0.00153	-0.01091	2491
268	2066-08-07	2066.51745	0.00198	-0.01072	2388
269	2066-09-07	2066.52019	0.00247	-0.01048	2298
270	2066-10-07	2066.52293	0.00293	-0.01021	2175
271	2066-11-07	2066.52567	0.00342	-0.00999	2068
272	2066-12-07	2066.52841	0.00387	-0.00974	1946
273	2066-13-07	2066.53114	0.00433	-0.00948	1863
274	2066-15-07	2066.53662	0.00526	-0.00895	1624
275	2066-17-07	2066.54209	0.00619	-0.00839	1414
276	2066-19-07	2066.54757	0.00703	-0.0078	1213
277	2066-21-07	2066.55305	0.00791	-0.00719	993
278	2066-28-07	2066.57221	0.01075	-0.00497	338
279	2066-04-08	2066.59138	0.01331	-0.00259	-188
280	2066-03-09	2066.67351	0.0215	0.00766	-1397

	Date	t	α	δ	v_{LOS}
	(yyyy-mm-dd)	(yr)	(as)	(as)	(km/s)
281	2066-03-10	2066.75565	0.02664	0.01697	-1733
282	2066-02-11	2066.83778	0.03007	0.02529	-1837
283	2066-02-12	2066.91992	0.03249	0.03274	-1823
284	2067-01-01	2067.00205	0.03425	0.03952	-1811
285	2067-31-01	2067.08419	0.03554	0.04579	-1765
286	2067-02-03	2067.16632	0.0365	0.05158	-1724
287	2067-01-04	2067.24846	0.03715	0.05699	-1692
288	2067-01-05	2067.3306	0.03761	0.06209	-1620
289	2067-31-05	2067.41273	0.03788	0.06689	-1611
290	2067-30-06	2067.49487	0.03803	0.07147	-1562
291	2067-27-12	2067.98768	0.03681	0.09472	-1337
292	2068-24-06	2068.48049	0.03358	0.11295	-1171
293	2068-21-12	2068.97331	0.0293	0.1277	-1029
294	2069-20-06	2069.46612	0.02439	0.13979	-920
295	2069-17-12	2069.95893	0.0191	0.14975	-818
296	2070-14-06	2070.45175	0.01358	0.15786	-719
297	2070-11-12	2070.94456	0.00788	0.16439	-670
298	2071-09-06	2071.43737	0.00216	0.16949	-580
299	2071-06-12	2071.93018	-0.00362	0.17328	-497
300	2072-03-06	2072.423	-0.00935	0.1759	-418
301	2072-30-11	2072.91581	-0.01501	0.17736	-378
302	2073-30-05	2073.40862	-0.0206	0.17779	-293
303	2073-26-11	2073.90144	-0.02607	0.17715	-208
304	2074-24-05	2074.39425	-0.03142	0.17551	-154

Table B.1. Mock catalogue of observations for the S2 star over three and a half orbital periods (the dashed horizontal lines correspond to each full period since the first epoch in early 2018). For each epoch (reported both as civil date and decimal year) we report a synthetic recording for the astrometric position (α , δ) and the spectroscopic line-of-sight velocity (v_{LOS}) obtained assuming observational accuracies of $10 \mu\text{as}$ and 10 km/s , respectively, mimicking the nominal astrometric and spectroscopic precision of the GRAVITY interferometer and of the SINFONI spectrographer (see Chapter 1).

Bayesian model fitting



Throughout this thesis several different orbital models based on modifications of the Schwarzschild metric, have been analyzed in light of current and future data for the S2 star in the Galactic Center (or, in some instances, using synthetic data for closer stars). The models considered introduce one or more additional parameters that, along with the mass M of the central object, uniquely fix the gravitational field experienced by these freely-falling test particles. Our aim was to constrain the value of any additional parameters of the model, in order to derive experimental limits on the deviations from the classical Schwarzschild black hole solution of General Relativity. In order to do so, we applied a Bayesian model fitting algorithm to constrain the parameter space of our models. Let us briefly detail the Markov Chain Monte Carlo (MCMC) algorithm used, the parameter spaces explored, the likelihood and the prior probability distribution used. As a proof of concept, we discuss the results of applying our methodology to the Schwarzschild case, recovering known values for the orbital parameters of the S2 star in literature, as a validation of our analysis. Finally, we also show the ability to work with forecasted data for S2.

Firstly, the MCMC algorithm employed is the ensemble sampler with affine invariance proposed by Goodman and Weare [649]. This is a family of MCMC methods whose performance is unaffected by affine transformations of the parameter space. These methods are particularly useful for sampling badly scaled distributions and they have been shown to be significantly faster than standard MCMC methods on highly skewed distributions. More specifically, we have decided to adopt the Python implementation of the Goodman and Ware algorithm found in the package `emcee` [650]. This well-tested code provides a user friendly, easily customizable and parallelizable implementation of the MCMC algorithm and it features built-in methods to compute the autocorrelation time (or function calls per independent sample) of the random walkers, which constitutes a widely used diagnostic for convergence of the algorithm.

As shown in Section 2.2.3 for the baseline Schwarzschild model, a unique set of astrometric and spectroscopic observables, $\{(\alpha_i(\theta), \delta_i(\theta), v_{\text{LOS},j}(\theta))\}_{i,j}$, is fixed once a 13-dimensional

vector $\vec{\theta}$ in the parameter space of the problem has been fixed

$$\vec{\theta} = \underbrace{(M, D, t_p, a, e, i, \omega, \Omega)}_{\text{orbital model}}, \underbrace{(\alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0}, v_{\text{LOS},0})}_{\text{systematics}}. \quad \text{C.1}$$

Reference values for these parameters, resulting from a fit of a non-relativistic Keplerian orbit to astrometric and spectroscopic data in [138], can be found in Table 1.1. More specifically, the orbital data for S2 given in the same reference and publicly available consist of:

- 145 sky-projected astrometric position of the S2 star, as observed in the near-infrared K-band. These data provide right ascensions (α) and declination (δ) for S2 relative to the ‘Galactic Center infrared reference system’ [165] and, hence relative to the point on the sky where Sgr A* is mapped from radio observations of the Galactic Center (except for an offset and drift of the central source with respect to the origin of the reference frame that is included into our orbital model through the $(\alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0})$ parameters in Eq. (C.1)). As explained in Chapter 1, these data cover the time intervals: from 1992.224 to 2002 with the observations made at the ESO New Technology Telescope, by means of the speckle camera SHARP, which provided with astrometric accuracy of order 3.8 mas; from 2002 with observations performed at the VLT using the adaptive optics assisted infrared NACO instrument, which provided with an astrometric accuracy of $\approx 400\mu\text{as}$.
- Measurement of the line-of-sight velocity v_{LOS} of the S2 on a total of 44 epochs. These data come from the measurement of the Brackett- γ line shift. In particular, up to 2003 these measurements have been done with NIRC2, a near-infrared Adaptive Optic imager and spectrometer located at the Keck observatory; later recordings of the radial velocity were made using SINFONI, mounted at the VLT. The data are corrected for the Local Standard of Rest (LSR), leaving a possible additional offset of the reference frame in the parameter $v_{\text{LOS},0}$.

All these data are plotted in their corresponding time spans in Figure C.1.

The agreement between the model computed for a given set

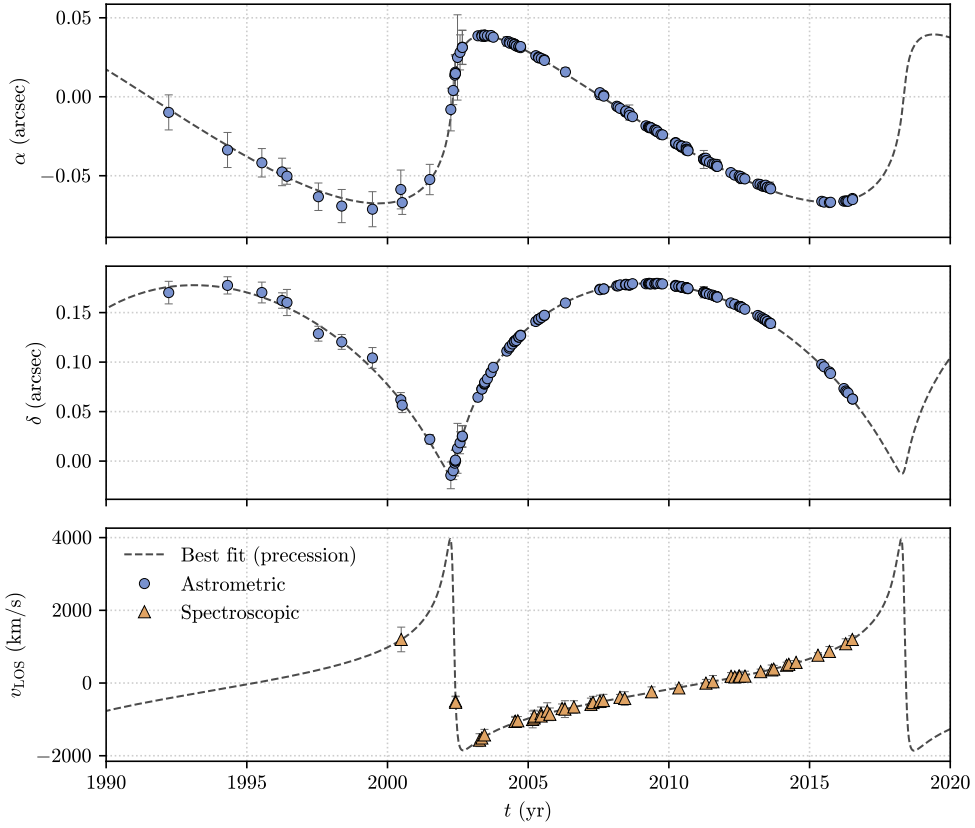


Figure C.1. Blue dots represent astrometric observables from ~ 1992 to ~ 2017 and corresponding uncertainties, as reported in [138]. Orange triangles report spectroscopic line-of-sight velocities and uncertainties starting from 2003 [138]. The dashed line is the best-fitting orbital model computed with the the maximum-likelihood set of parameters derived from the MCMC analysis taking into account the orbital precession.

of parameters, $\{(\alpha_i(\vec{\theta}), \delta_i(\vec{\theta}), v_{\text{LOS},j}(\vec{\theta}))\}_{i,j}$, and the aforementioned data is assessed through the log-likelihood function

$$\log \mathcal{L} = -\frac{1}{2} \left[\sum_i^{145} \left(\frac{\alpha_i(\vec{\theta}) - \alpha_i^{\text{obs}}}{\sigma_{\alpha,i}} \right)^2 + \sum_i^{145} \left(\frac{\delta_i(\vec{\theta}) - \delta_i^{\text{obs}}}{\sigma_{\delta,i}} \right)^2 + \sum_i^{44} \left(\frac{v_{\text{LOS},i}(\vec{\theta}) - v_{\text{LOS},i}^{\text{obs}}}{\sigma_{v_{\text{LOS},i}}} \right)^2 \right], \quad | \text{C.2}$$

where quantities with an *obs* superscript refer to observational data and the σ s are the observational uncertainties of each given datapoint. The log-likelihood in Eq. (C.2) assumes a normal distribution of the observables and that all the measurements are independent (hence the linear sum).

Prior knowledge on the orbital parameters for the S2 star is encoded in a prior probability log-distribution $\log \Pi(\vec{\theta})$ that,

Table C.1. Set of priors for the MCMC analyses used throughout this thesis. In particular, for the orbital parameters, the mass and the distance of Sgr A* we adopt uniform prior distributions centered on the best-fitting values in [138] and with an amplitude that is ~ 10 times larger than the inferred uncertainty on these parameters from [138], thus containing the best-fitting values from this analysis. For the reference frame parameters, on the other hand, we adopt Gaussian priors based on the results of the independent analysis in [165].

Parameter (unit)	Uniform priors	
	Start	End
M ($10^6 M_\odot$)	2.50	5.70
D (kpc)	6.63	9.63
t_p (yr)	2018.03	2018.63
a (arcsec)	0.1165	0.1345
e	0.8649	0.9029
i ($^\circ$)	130.18	138.18
Ω ($^\circ$)	220.94	232.94
ω ($^\circ$)	59.81	71.21
	Gaussian priors	
	μ	σ
α_0 (mas)	0.0	0.2
δ_0 (mas)	0.0	0.2
$v_{\alpha,0}$ (mas/yr)	0.0	0.1
$v_{\delta,0}$ (mas/yr)	0.0	0.1
$v_{\text{LOS},0}$ (km/s)	0	5

for a given set of parameters $\vec{\theta}$, is summed to the log-likelihood to obtain a point estimation of the posterior probability of the parameter set. Estimates of the orbital parameters for S2, along with the mass of Sgr A* and its distance from Earth, are known from the Keplerian analyses on the dataset that we consider here [138]. To study possible deviations from General Relativity using exactly the same dataset, a more agnostic approach is required. To avoid forcing the MCMC chains to explore the exact same region of parameter space resulting from the Keplerian case, we assume large uniform priors centered on best-fitting values resulting from [138] with an amplitude that is ~ 10 times larger than the inferred uncertainty from the same analysis. For the reference frame parameters, we have information coming from independent datasets and analysis, which allows us to place Gaussian priors on these parameters. In particular, from [165] we can place limits on the sky-projected position of Sgr A* in the radio-to-infrared reference frame, and for the line-of-sight velocity offset after correcting spectroscopic measurements to the LSR the motion of Sgr A*. The full set of priors used for our analysis is given in Table C.1.

Parameter (unit)	Posterior (data)	Posterior (data + precession)
D (kpc)	$8.01^{+0.22}_{-0.21}$	8.00 ± 0.16
M ($10^6 M_{\odot}$)	$4.01^{+0.23}_{-0.22}$	4.00 ± 0.17
t_p (yr)	2018.362 ± 0.024	2018.366 ± 0.018
a (arcsec)	0.1261 ± 0.0011	0.12619 ± 0.00085
e	$0.8844^{+0.0024}_{-0.0025}$	0.8843 ± 0.0019
i ($^{\circ}$)	133.77 ± 0.50	133.70 ± 0.38
Ω ($^{\circ}$)	226.80 ± 0.82	226.70 ± 0.60
ω ($^{\circ}$)	$65.58^{+0.77}_{-0.78}$	$65.58^{+0.57}_{-0.56}$
α_0 (mas)	0.33 ± 0.16	0.48 ± 0.13
δ_0 (mas)	-0.02 ± 0.19	-0.05 ± 0.19
$v_{\alpha,0}$ (mas/yr)	$0.068^{+0.054}_{-0.053}$	0.079 ± 0.041
$v_{\delta,0}$ (mas/yr)	$0.123^{+0.063}_{-0.064}$	$0.139^{+0.050}_{-0.049}$
$v_{\text{LOS},0}$ (km/s)	-6.0 ± 4.4	-9.4 ± 4.2

Table C.2. Results from the MCMC analyses for the Schwarzschild orbital model from orbital data of the S2 star in the Galactic Center (second column) and from the forecast analysis using future mock observations of the same object through the GRAVITY interferometer over two orbital periods (third column). In particular, we report the 68% confidence intervals around the median for all the parameters of the model.

Once a set of priors and a likelihood function have been assigned, we can finally proceed to extract samples from the posterior distribution by letting the MCMC algorithm run and computing at each iteration the mean autocorrelation time, τ of the walkers. We adopt a convergence criterion based on the estimate of this quantity as proposed in [649, 650]. The algorithm is said to have converged when the length ℓ of the chains is $\ell > 100\tau$ and when the estimates of τ have not varied (on average) more than 1 percent over the last iteration. Once both criteria have been met, we stop the sampling of the posterior distribution and we cut the first $\sim 2\tau$ samples that we consider belonging to the burn-in phase of the sampling. The resulting array can be thus considered as a collection of samples extracted directly from the posterior distribution of the parameter space, on which we run our statistics to compute maximum likelihood estimators, medians and quantiles, in order to derive confidence intervals for each parameter in the model. In particular, in the first column of Table C.2 we report the 68% confidence intervals derived from our MCMC analysis on an orbital model based exclusively on the Schwarzschild spacetime. On the other hand, 68% (1σ), 95% (2σ) and 99.7% (3σ) confidence regions for every pair of orbital parameters and every pair of reference frame parameters are reported in the corner plots in Figures C.2 and C.3. We also report the best-fitting values from the analysis in [138], which always fall within the 1σ median-centered confidence intervals of the

marginalized posterior distributions for each parameter. We are thus able to recover with our relativistic model and our posterior analysis the already known values from the literature on the orbital parameters of S2.

The pre-2017 data do not include information on the relativistic orbital precession of the S2 star, which has indeed been measured using refined astrometry with the GRAVITY interferometer [226], as thoroughly illustrated in Chapter 1. On the other hand, up to the date of this thesis, the Gravity Collaboration has not made available such improved astrometric data for S2 yet, and thus our analysis could only be limited to the pre-pericenter portion. Including the information on the relativistic effects is crucial if we want to perform metric tests of the spacetime in the Galactic Center. In order to do that, we can include an additional data point carrying the information on the measurement of the orbital precession rate f_{sp} , reported in [226]. Particular care must be taken in this step. As a matter of fact, the analysis performed in [226], from which the estimate of f_{sp} is derived, does not rely solely on the improved astrometry from the GRAVITY interferometer, but also includes the two and a half decade worth of astrometric and spectroscopic data measured up to that point. Consequently, the analysis in [226] takes into account the same dataset that we take into account, plus the additional information from the GRAVITY interferometer. For this reason, adding an extra term to the likelihood in Eq. (C.2) containing the measurement of the precession, as if it was independent of all the other terms, is not the most correct and conservative choice and might lead to an unrealistic over-constraining of the parameters of the model. Conversely, the most robust way of adding the relativistic information is to have access to the actual data from the GRAVITY interferometer and only fit for the positions and velocities of the star. Since such data is not publicly available, though, an alternative route must be taken. A possible way to introduce the measurement of the relativistic rate of orbital precession f_{sp} into our fitting procedure, is to assume that the likelihood for f_{sp} contains already all the astrometric terms and hence dividing all the terms in the sum of the log-likelihood by a factor two would avoid double counting datapoints. This procedure is equivalent to a multiplication by a factor $\sqrt{2}$ of all the observational uncertainties in the likelihood, smearing its peak. Then, the log-likelihood adopted when adding the information on the orbital precession is given by

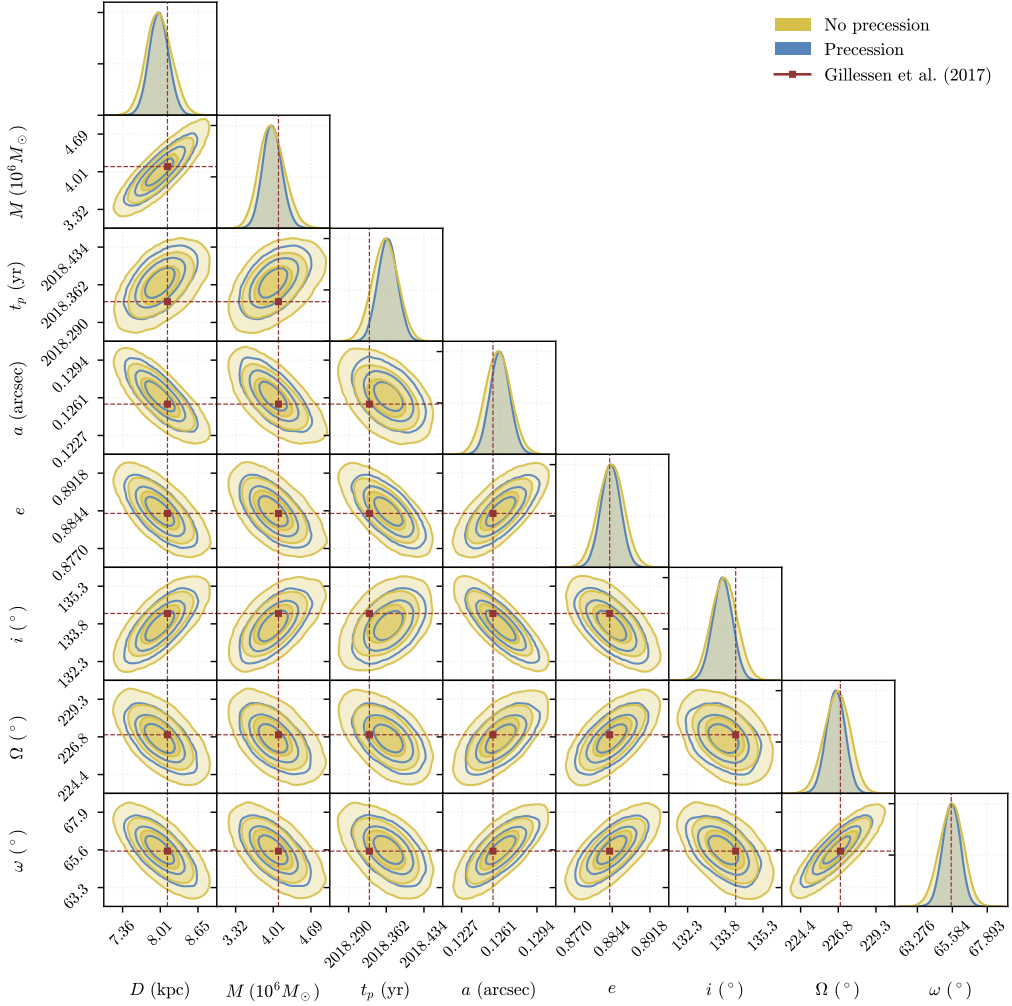
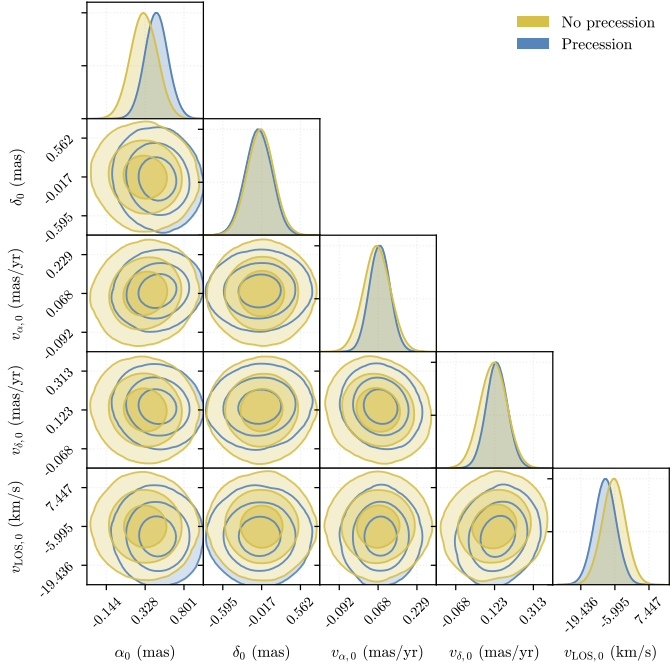


Figure C.2. Yellow contours in the off-diagonal plots report 68% (1σ), 95% (2σ) and 99.7% (3σ) confidence regions for every pair of orbital parameters, (M , D , t_p , a , e , i , ω , Ω) in the Schwarzschild case using only publicly available data from [138] with no relativistic information. The on-diagonal plots report the marginalized posterior probability distribution on each parameter. The superimposed blue distributions correspond to the results of the analysis done by adding the information on the measurement of the orbital precession for S2 in [226]. Red dots and the corresponding dashed lines report the best fitting values from the Keplerian analysis reported in [138]. These values always fall within the confidence regions of each parameter, making the results of our analysis compatible with that in the Keplerian case.

Figure C.3. As in Figure C.2, but for the reference frame parameters, $(\alpha_0, \delta_0, v_{\alpha,0}, v_{\delta,0}, v_{\text{LOS},0})$.



$$\log \mathcal{L} = -\frac{1}{2} \left[\sum_i^{145} \left(\frac{\alpha_i(\vec{\theta}) - \alpha_i^{obs}}{\sqrt{2}\sigma_{\alpha,i}} \right)^2 + \sum_i^{145} \left(\frac{\delta_i(\vec{\theta}) - \delta_i^{obs}}{\sqrt{2}\sigma_{\delta,i}} \right)^2 + \sum_i^{45} \left(\frac{v_{\text{LOS},i}(\vec{\theta}) - v_{\text{LOS},i}^{obs}}{\sqrt{2}\sigma_{v_{\text{LOS},i}}} \right)^2 + \left(\frac{f_{\text{SP}}(\vec{\theta}) - f_{\text{SP}}^{obs}}{\sqrt{2}\sigma_{f_{\text{SP}}}} \right)^2 \right], \quad \text{C.3}$$

where f_{SP}^{obs} and $\sigma_{f_{\text{SP}}}$ represent the experimental value of the rate of orbital precession and its uncertainty from [226], reported in Table 1.5, while f_{SP} is the rate of orbital precession theoretically predicted for each model considered.

Using the likelihood in Eq. (C.3) and adopting the same priors as in the previous case, we repeat the MCMC analysis and from the resulting samples of the posterior probability distribution we derive confidence intervals for each parameter, given in the second column of Table C.2, and the blue contours shown in Figures C.2 and C.3. Our results show that, in the Schwarzschild case, the additional information on the orbital precession, which to a certain extent encodes the improved astrometry of the GRAVITY interferometer, while leading to inferred values of the model parameters that are perfectly compatible with those of the analysis in the previous case,

results in slightly narrower confidence regions for all the orbital parameters (albeit not as narrow as the ones derived from the Gravity collaboration, reported in Table 1.1, using the full astrometric information from GRAVITY). The addition of the measurement of the relativistic precession has proven to be crucial once one considers modifications to the Schwarzschild geometry, as done throughout this thesis, and provides strong constraining power on the extra parameters introduced in each model. The orbit for the S2 star reproduced from the best-fitting (maximum likelihood) parameters from the MCMC analysis including the orbital precession is shown in Figure C.1 and perfectly reproduces the observational data.

In this work we have also used forecasted data for S2, using a mock catalogue of observations for S2 as illustrated in Section 2.2.3 and reported in Appendix B. Since we assume that the entire dataset is composed of GRAVITY observations, there is no need to match the astrometric radio-to-infrared reference frame, by introducing the extra reference frame parameters. This reduces the dimensionality of the parameter space to

$$\vec{\theta} = (M, D, t_p, a, e, i, \omega, \Omega). \quad | \text{ C.4}$$

Fixing a duration for the forecasted dataset, fixes the number N of astrometric and spectroscopic measurements to take into account. In this case the likelihood is computed as:

$$\log \mathcal{L} = -\frac{1}{2} \sum_i^N \left[\left(\frac{\alpha_i(\vec{\theta}) - \alpha_i^{mock}}{\sigma_A} \right)^2 + \left(\frac{\delta_i(\vec{\theta}) - \delta_i^{mock}}{\sigma_A} \right)^2 + \left(\frac{v_{\text{LOS},i}(\vec{\theta}) - v_{\text{LOS},i}^{mock}}{\sigma_V} \right)^2 \right], \quad | \text{ C.5}$$

where the *mock* superscript refers to the data reported in Appendix B, and the uncertainties $\sigma_A = 10 \mu\text{as}$ and $\sigma_V = 10 \text{ km/s}$ are the nominal astrometric and spectroscopic uncertainties from the GRAVITY interferometer and the SINFONI spectrographer, used in the generation of the mock catalogue itself, as discussed in Section 2.2.3. Repeating the MCMC analysis using the mock catalogue over two and a half orbital periods, adopting the likelihood in Eq. (C.5) and the same priors used in the previous analyses, we were able to recover the fiducial values for the orbital parameters, *i.e.* the values used to construct the mock catalogue given in the third column of Table 1.1. This is shown in the corner plot reported in Figure C.4.

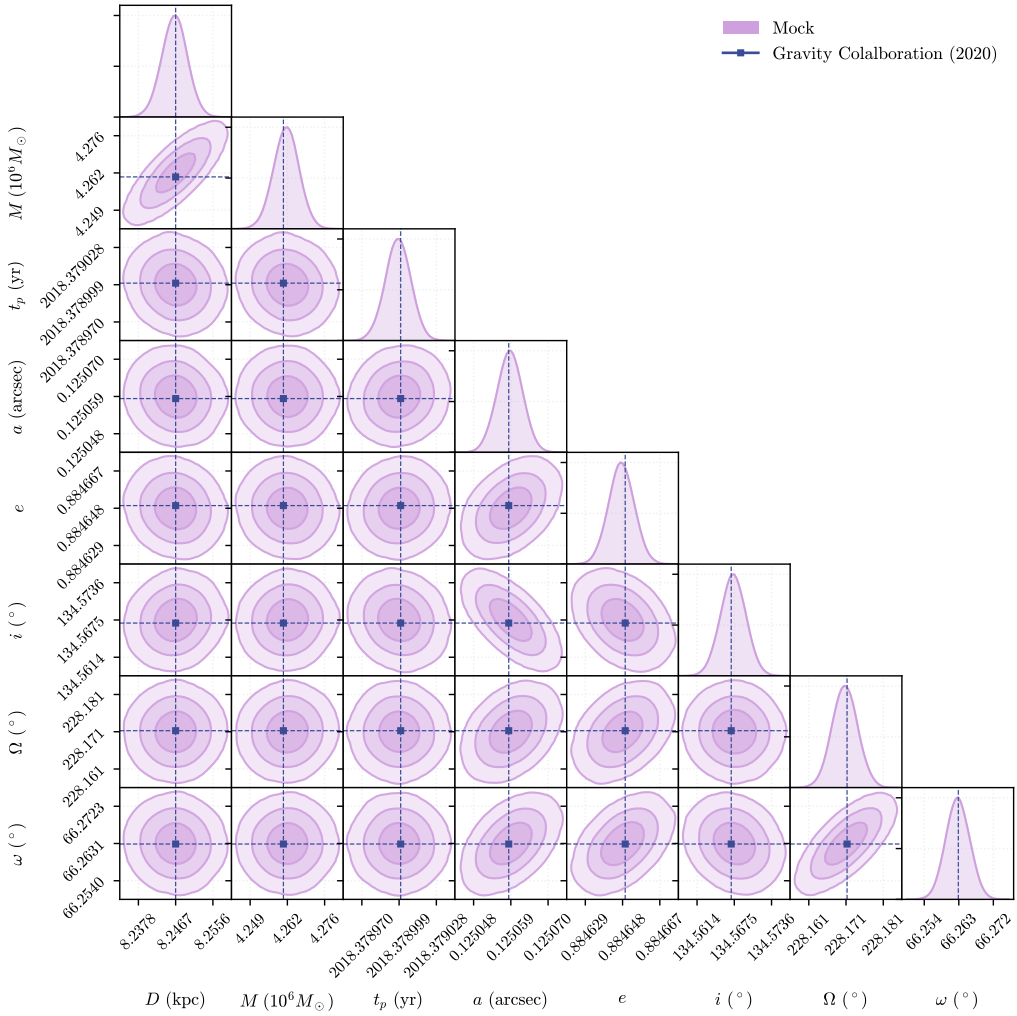


Figure C.4. As in Figure C.2, but for the full parameter space from the analysis on mock data for S2. The blue dots and dashed lines report the fiducial values used for the generation of the mock catalogue, which are perfectly recovered by our MCMC analysis.

The Galactic Center pulsar mock catalogue

D.

	TOA (MJD)	UTC date
1	60676.00000000000	2025-01-01T00:00:00.0000
2	60682.35127949146	2025-01-07T08:25:50.5481
3	60694.03230664622	2025-01-19T00:46:31.2942
4	60706.10730122727	2025-01-31T02:34:30.8260
5	60710.32969563003	2025-02-04T07:54:45.7024
6	60719.79899224274	2025-02-13T19:10:32.9298
7	60728.29749032517	2025-02-22T07:08:23.1641
8	60732.80880507547	2025-02-26T19:24:40.7585
9	60734.11539725900	2025-02-28T02:46:10.3232
10	60737.52653072171	2025-03-03T12:38:12.2544
11	60738.44241136809	2025-03-04T10:37:04.3422
12	60743.97456607073	2025-03-09T23:23:22.5085
13	60753.84406549550	2025-03-19T20:15:27.2588
14	60754.16309198515	2025-03-20T03:54:51.1475
15	60754.67169014076	2025-03-20T16:07:14.0282
16	60755.10813997121	2025-03-21T02:35:43.2935
17	60759.74883054751	2025-03-25T17:58:18.9593
18	60781.53712991840	2025-04-16T12:53:28.0249
19	60782.09994472325	2025-04-17T02:23:55.2241
20	60792.23119280286	2025-04-27T05:32:55.0582
21	60807.85659105321	2025-05-12T20:33:29.4670
22	60825.11904943929	2025-05-30T02:51:25.8716
23	60827.72769018236	2025-06-01T17:27:52.4318
24	60828.78887194460	2025-06-02T18:55:58.5360
25	60837.10134502347	2025-06-11T02:25:56.2100
26	60842.10062299319	2025-06-16T02:24:53.8266
27	60859.66045794300	2025-07-03T15:51:03.5663
28	60861.59838838926	2025-07-05T14:21:40.7568
29	60869.36080931694	2025-07-13T08:39:33.9250
30	60875.82041075145	2025-07-19T19:41:23.4889
31	60899.96570067576	2025-08-12T23:10:36.5384
32	60903.57738690899	2025-08-16T13:51:26.2289
33	60923.41520792897	2025-09-05T09:57:53.9651

	TOA (MJD)	UTC date
34	60926.79430701191	2025-09-08T19:03:48.1258
35	60930.73102858109	2025-09-12T17:32:40.8694
36	60940.74372547596	2025-09-22T17:50:57.8811
37	60952.69965278281	2025-10-04T16:47:30.0004
38	60969.74698597711	2025-10-21T17:55:39.5884
39	60971.37594664071	2025-10-23T09:01:21.7898
40	60971.44174390693	2025-10-23T10:36:06.6736
41	60973.16822822204	2025-10-25T04:02:14.9184
42	60982.17107098910	2025-11-03T04:06:20.5335
43	60985.11812012306	2025-11-06T02:50:05.5786
44	60988.91641925461	2025-11-09T21:59:38.6236
45	60999.82821968786	2025-11-20T19:52:38.1810
46	61003.00466374378	2025-11-24T00:06:42.9475
47	61015.73104400238	2025-12-06T17:32:42.2018
48	61021.20148624310	2025-12-12T04:50:08.4114
49	61047.21106535733	2026-01-07T05:03:56.0469
50	61047.59766165679	2026-01-07T14:20:37.9671
51	61066.28559770816	2026-01-26T06:51:15.6420
52	61066.66356244727	2026-01-26T15:55:31.7954
53	61087.04504217175	2026-02-16T01:04:51.6436
54	61088.74837236643	2026-02-17T17:57:39.3725
55	61094.01464949124	2026-02-23T00:21:05.7160
56	61099.47656439780	2026-02-28T11:26:15.1640
57	61105.63543607363	2026-03-06T15:15:01.6768
58	61116.52457589732	2026-03-17T12:35:23.3575
59	61118.24724270179	2026-03-19T05:56:01.7694
60	61126.50241288969	2026-03-27T12:03:28.4737
61	61144.20025661958	2026-04-14T04:48:22.1719
62	61152.42201420384	2026-04-22T10:07:42.0272
63	61155.70502055788	2026-04-25T16:55:13.7762
64	61164.89100054197	2026-05-04T21:23:02.4468
65	61168.80002242937	2026-05-08T19:12:01.9379
66	61169.99181530149	2026-05-09T23:48:12.8420
67	61171.17015758601	2026-05-11T04:05:01.6154
68	61174.59236143645	2026-05-14T14:13:00.0281
69	61181.02674632766	2026-05-21T00:38:30.8827
70	61185.57725682788	2026-05-25T13:51:14.9899

	TOA (MJD)	UTC date
71	61187.88764879417	2026-05-27T21:18:12.8558
72	61190.40630548547	2026-05-30T09:45:04.7939
73	61194.05155077138	2026-06-03T01:14:13.9866
74	61194.94396769446	2026-06-03T22:39:18.8088
75	61197.29285424121	2026-06-06T07:01:42.6064
76	61206.19376033575	2026-06-15T04:39:00.8930
77	61208.20941830308	2026-06-17T05:01:33.7414
78	61209.34553500591	2026-06-18T08:17:34.2245
79	61210.12726284403	2026-06-19T03:03:15.5097
80	61220.58913691738	2026-06-29T14:08:21.4297
81	61220.74217881296	2026-06-29T17:48:44.2494
82	61224.29480116608	2026-07-03T07:04:30.8207
83	61226.15436374121	2026-07-05T03:42:17.0272
84	61229.04688246305	2026-07-08T01:07:30.6448
85	61232.28338971235	2026-07-11T06:48:04.8711
86	61238.43496470108	2026-07-17T10:26:20.9502
87	61240.49527736750	2026-07-19T11:53:11.9646
88	61246.27428466020	2026-07-25T06:34:58.1946
89	61280.54581650582	2026-08-28T13:05:58.5461
90	61286.07285839980	2026-09-03T01:44:54.9657
91	61296.11956183551	2026-09-13T02:52:10.1426
92	61300.82776218936	2026-09-17T19:51:58.6532
93	61302.44748061661	2026-09-19T10:44:22.3253
94	61322.86427094943	2026-10-09T20:44:33.0100
95	61324.17463500631	2026-10-11T04:11:28.4645
96	61335.34810981645	2026-10-22T08:21:16.6881
97	61335.39306820710	2026-10-22T09:26:01.0931
98	61338.39549008887	2026-10-25T09:29:30.3437
99	61352.94503190175	2026-11-08T22:40:50.7563
100	61353.34852281897	2026-11-09T08:21:52.3716
101	61371.76416226220	2026-11-27T18:20:23.6195
102	61379.52276915406	2026-12-05T12:32:47.2549
103	61379.64021138525	2026-12-05T15:21:54.2637
104	61383.75412623314	2026-12-09T18:05:56.5065
105	61401.29478623215	2026-12-27T07:04:29.5305
106	61409.22019883151	2027-01-04T05:17:05.1790
107	61409.44082237555	2027-01-04T10:34:47.0532

	TOA (MJD)	UTC date
108	61416.38466053278	2027-01-11T09:13:54.6700
109	61418.81393800017	2027-01-13T19:32:04.2432
110	61426.31856973115	2027-01-21T07:38:44.4248
111	61446.16203654155	2027-02-10T03:53:19.9572
112	61454.43812413832	2027-02-18T10:30:53.9256
113	61455.81060934788	2027-02-19T19:27:16.6477
114	61464.71256513619	2027-02-28T17:06:05.6278
115	61466.39727821902	2027-03-02T09:32:04.8381
116	61473.49647426256	2027-03-09T11:54:55.3763
117	61476.54149642808	2027-03-12T12:59:45.2914
118	61480.98778408739	2027-03-16T23:42:24.5452
119	61482.39014870145	2027-03-18T09:21:48.8478
120	61493.24933406177	2027-03-29T05:59:02.4629

Table D.1. Mock catalogue of TOAs for a pulsar around Sgr A* with orbital parameters corresponding to those of the object Toy 1 in Table 6.1 at 120 epochs over two full orbital periods (2 years and 4 months approximately). The timing accuracy considered is the predicted accuracy for SKA timing at 20 GHz of $\approx 100 \mu\text{s}$.

List of acronyms

Abbreviation	Definition
AGB	Asymptotic Giant Branch
AO	Adaptive Optics
arcsec	arcsecond(s)
ATG	Alternative Theory of Gravity
ATNF	Australia Telescope National Facility
AU	Astronomical Unit
BB	Black Bounce
CDM	Cold Dark Matter
CP	Charge-conjugation Parity
DM	Dark Matter
dS	de Sitter
dSph	dwarf spheroidal galaxy
EHT	Event Horizon Telescope
ELT	Extremely Large Telescope
EMDA	Einstein Maxwell Dilaton Axion
ESO	European Southern Observatory
ETG	Extended Theory of Gravity
FAST	Five-hundred meter Aperture Spherical Telescope
FDM	Fuzzy Dark Matter
FWHM	Full-Width Half Maximum
GMT	Greenwich Mean Time
IMBH	Intermediate Mass Black Hole
IR	Infra-red
ISCO	Innermost Stable Circular Orbit
LOS	Line Of Sight
LSR	Local Standard of Rest
MBH	Massive Black Hole
MCMC	Markov Chain Monte Carlo
MJD	Modified Julian Day
MOG	MOdified Gravity

Abbreviation	Definition
MPE	Max Planck Institute for extraterrestrial Physics
MS	Misner-Sharp
NACO	NAOS + CONICA
NAOS	Nasmyth Adaptive Optics System
NEC	Null Energy Condition
NFW	Navarro-Frenk-White
NSC	Nuclear Star (Stellar) Cluster
NTT	New Technology Telescope
OSIRIS	OH-Suppressing Infra-Red Imaging Spectrograph
OVRO	Owens Valley Radio Observatory
pc	parsec(s)
PN	Post Newtonian
PPN	Parametrized Post Newtonian
QCD	Quantum Chromo Dynamics
RA	Right Ascension
RMS/rms	Root mean square
RN	Reissner-Nordström
SCO	Stable Circular Orbit
SEP	Strong Equivalence Principle
SINFONI	SINGLE Faint Object Near-IR Investigation
SKA	Square Kilometer Array
SMBH	Supermassive Black Hole
SP	Schwarzschild Precession
STVG	Scalar Tensor Vector Gravity
TMT	Thirty Meter Telescope
TOA	Time Of Arrival
UCLA	University of California, Los Angeles
UCO	Unstable Circular Orbit
ULDM	Ultralight Dark Matter
UTC	Coordinated Universal Time
VLA	Very Large Array
VLBA	Very Long Baseline Array
VLBI	Very Long Baseline Interferometry
VLT	Very Large Telescope
WEP	Weak Equivalence Principle
WR	Wolf-Rayet

Bibliography

Shown below are the references in the order by which they have been cited in this thesis and the corresponding pages where the reference appear.

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Tesis doctoral

El centro de la Galaxia como laboratorio de materia oscura y de teorías de gravedad modificada

RESUMEN EN CASTELLANO

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Publicaciones

Esta tesis doctoral se basa en el trabajo realizado en los tres últimos años en el Departamento de Física Fundamental de la Universidad de Salamanca. El trabajo dió lugar a 14 publicaciones en revistas de alto impacto de las cuales las 12 que detallamos a continuación están directamente relacionadas con el tema de la tesis:

- [1]: Ivan De Martino, Riccardo della Monica, and Mariafelicia De Laurentis. “f (R) gravity after the detection of the orbital precession of the S2 star around the Galactic Center massive black hole”. In: *Phys. Rev. D* 104.10, L101502 (2021). arXiv: [2106.06821 \[gr-qc\]](#).
- [2]: Riccardo Della Monica, Ivan de Martino, and Mariafelicia de Laurentis. “Orbital precession of the S2 star in Scalar-Tensor-Vector Gravity”. In: *MNRAS* 510.4 (2022). arXiv: [2105.12687 \[gr-qc\]](#).
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- [12]: Ivan De Martino, Riccardo Della Monica, and Diego Rubiera-García. “Constraints on metric-affine gravity black holes from the stellar motion at the Galactic Center”. In: *Phys. Rev. D* 109.2, 024016 (2024). arXiv: [2311.08031 \[gr-qc\]](#).

Otras 2 publicaciones, aunque pertenecen a la misma macroárea, están enfocadas en un tema distinto y por lo tanto no se incluyen en el contenido de la misma:

- [13]: Riccardo Della Monica, Ivan de Martino, and Tom Broadhurst. “Explaining the oblate morphology of dwarf spheroidals with Wave Dark Matter perturbations”. In: *arXiv e-prints*, arXiv:2308.14664 (2023). arXiv: [2308.14664 \[astro-ph.GA\]](#),
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Finalmente, otras 2 publicaciones se encuentran en preparación o en proceso de revisión por pares:

- [15]: Riccardo Della Monica and Richard Brito. “Environmental effects of ultralight axions on compact binaries mergers”. In: *en preparación* (2024),
- [16]: Riccardo Della Monica, Ivan de Martino, and Mariafelicia de Laurentis. “Testing fifth force effects in the Algol three-body system”. In: *artículo enviado* (2024).

Premios

- Una comunicación oral basada en el artículo [11] y presentada en la conferencia EREP2023 (Encuentros Relativistas Españoles y Portugueses) en Bilbao ha sido premiada como mejor contribución, premio otorgado por la Sociedad Española de Gravitación y Relatividad General (SEGRE).

Visitas Académicas

Durante los estudios de doctorado se realizaron dos visitas académicas:

- **Institución:** Università degli Studi di Napoli - “Federico II”
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Tema: “Problema de 3 cuerpos en un potencial tipo Yukawa”
- **Institución:** Center for Astrophysics and Gravitation (CENTRA), Instituto Superior Técnico, Lisboa
Supervisor: Richard Brito
Fechas: 15-09-2023 / 15-12-2023
Tema: “Efectos ambientales de axiones ultraligeros en la fusión de binarias compactas”

Declaración

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y

Yo, **Prof. Dr. Fernando Atrio Barandela**, Profesor Catedrático del área de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca, como director del trabajo de investigación

CERTIFICAMOS

Que el trabajo de investigación desarrollado en la presente tesis doctoral por el investigador **D. Riccardo Della Monica** para optar al Título de Doctor por la Universidad de Salamanca con la Mención de Doctorado Internacional cumple con los requisitos formales, académicos y de calidad requeridos, autorizando por tanto su presentación.

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30/05/2024 con un certificado

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Barandela el 30/05/2024.

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Resumen

La teoría de la Relatividad General, formulada hace poco más de un siglo por Albert Einstein, implicó un cambio en nuestra comprensión de la gravedad y en la actualidad ha sido validada con numerosas observaciones. El Centro Galáctico se destaca como un laboratorio único en la frontera entre la física fundamental y astrofísica en la que verificar la teoría einsteniana de la gravedad. Numerosas evidencias observacionales indican que la fuente Sagitario A* que se encuentra en esta región es un agujero negro supermasivo de cuatro millones de masas solares. Las observaciones de alta resolución de este objeto y las estrellas en su vecindad no solo abren un nuevo escenario en que validar las predicciones de la teoría de Einstein en el límite de campo fuerte, sino también podrían falsar dicha teoría frente a alguna de sus alternativas, probar diferentes paradigmas de la naturaleza de la materia oscura o imponer restricciones sobre las características físicas de la propia fuente Sagitario A*.

En esta tesis exploraremos algunas de estas posibilidades. Primero, desarrollamos una metodología numérica con la modelización apropiada para predecir observables relativistas para estrellas y pulsares en el Centro Galáctico. Seguidamente utilizamos datos actuales y catálogos simulados de fuentes y técnicas de inferencia bayesiana para predecir qué tipo de restricciones podrán aportar futuras observaciones. Consideraremos distintos escenarios, incluyendo componentes extendidos de materia oscura alrededor de Sagitario A*, modelos de imitadores de agujeros negros y modificaciones a la teoría subyacente de la gravedad. Más allá de la importancia de las cotas derivadas para estos modelos, que presentan resultados complementarios a los obtenidos en diferentes escalas astrofísicas o cosmológicas en la literatura, nuestro trabajo demuestra la poderosa capacidad de las observaciones del Centro Galáctico para validar la teoría de la Relatividad General y el concepto de agujero negro. Las metodologías que hemos desarrollado representarán una herramienta valiosa para las futuras investigaciones en este campo.

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Introducción

En 1915 Albert Einstein formuló la Teoría de la Relatividad General. En esta teoría la gravedad deja de ser considerada como una fuerza entre masas, idea que estableció Newton en 1687 con la formulación de la ley de la gravitación universal, y pasó a ser descrita como un efecto emergente de las propiedades geométricas del espacio-tiempo que nos rodea. Espacio y tiempo perdieron su carácter absoluto y dejaron de ser considerados como un mero escenario donde ocurren los eventos físicos, con una estructura geométrica fija y eterna para ser reemplazados por una entidad dinámica. La geometría no está fijada *a priori*, sino que evoluciona continuamente y es determinada por la materia y la energía que contiene. Usando las palabras de John Archibald Wheeler: “*El espacio-tiempo le dice a la materia cómo moverse, la materia le dice al espacio-tiempo cómo curvarse*”¹.

¹Wheeler, J. A., *Geons, black holes, and quantum foam: A life in physics*, 1998 [17]

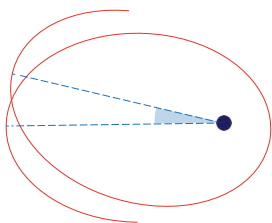
Muchos investigadores describen la Relatividad General como “*la más hermosa de todas las teorías físicas existentes*”² debido a su elegante formulación matemática, intrínsecamente construida con conceptos de geometría diferencial, y a su coherencia y consistencia internas. Tanto es así, que el propio Einstein consideraba el problema de la confirmación experimental de la Relatividad General como de importancia secundaria y respondía en tono de broma a su asistente Ilse Rosenthal-Schneider, cuando le preguntaba qué haría si la teoría resultara ser incorrecta, que “[...] *lamentaría por el buen Señor. La teoría es correcta*”³.

² Landau, L. D. y Lifshitz, E. M., *The Classical Theory of Fields*, 1975 [18]

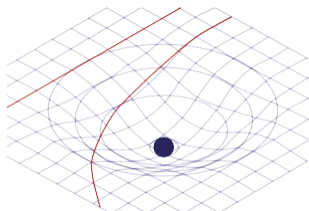
³Rosenthal-Schneider, I., *Reality and Scientific Truth: Discussions with Einstein*, 1980 [19]

Efectivamente, aparte de su elegancia matemática, la Relatividad General tiene una fenomenología muy rica en comparación con la gravedad newtoniana clásica. Poco después de su publicación se idearon una serie de pruebas experimentales para verificar la validez de la teoría en el Sistema Solar. Estos experimentos se conocen como las *pruebas clásicas de la Relatividad General*. Cada una de estas pruebas clásicas se basa en una predicción distintiva de la teoría de la gravedad de Einstein que la diferencia de manera inequívoca del modelo newtoniano:

- Las partículas test en caída libre en un campo gravitatorio experimentan un movimiento geodésico, mientras que en la gravedad newtoniana siguen las leyes de Kepler.

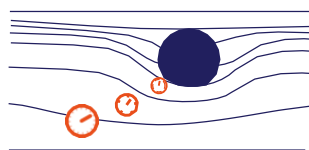


Precesión del perihelio de órbitas elípticas.



Efecto lente producido por la masa central sobre la radiación.

⁴ Dyson, F. W., Eddington, A. S. y Davidson, C., *A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919* [20]



Dilatación gravitacional del tiempo.

⁵ Pound, R. V. y Rebka, G. A., *Variation with Temperature of the Energy of Recoil-Free Gamma Rays from Solids*, 1959 [21]

Las órbitas relativistas no son elipses cerradas, sino elipses que precesan alrededor del centro de masa del sistema. Sorprendentemente, esta predicción coincidió cuantitativamente y con perfecto acuerdo con la tasa anómala de la *precesión del perihelio de la órbita de Mercurio*, reconocida por Urbain Le Verrier en 1859 como un problema en la mecánica celeste y que aún no tenía explicación en el momento de la formulación de la Relatividad General.

- Como consecuencia del principio de equivalencia, las partículas sin masa (como los fotones) también experimentan movimiento geodésico y, por lo tanto, en lugar de propagarse en línea recta se curvan, acomodándose a la curvatura del espacio-tiempo, fenómeno conocido como *efecto lente gravitatoria*. Los fotones emitidos por estrellas distantes y cercanos a la superficie del Sol serían desviados en una cantidad detectable, lo que motivó varias expediciones para observar la luz de estrellas fijas durante un eclipse solar total. La famosa expedición de Sir Arthur Stanley Eddington en 1919⁴ proporcionó la primera confirmación experimental de este fenómeno, añadiendo una pieza importante al rompecabezas.
- La curvatura del espacio-tiempo en presencia de un cuerpo masivo implica que el paso del tiempo no es universal, ni siquiera para observadores estacionarios. Los relojes más cercanos a una fuente gravitacional experimentarían un *retraso gravitacional* y cualquier luz que emitan experimentaría un *corrimiento gravitacional al rojo* cuando sea recogida por un observador más distante. Este efecto fue confirmado experimentalmente por primera vez con el experimento de Pound-Rebka⁵, llevado a cabo en 1959 dentro del campo gravitatorio de la Tierra. En la actualidad este efecto se comprueba a diario y se corrige para permitir el correcto funcionamiento del Sistema de Posicionamiento Global (GPS).

El éxito de las pruebas clásicas proporcionó una confirmación temprana para la teoría y ha sido de suma importancia para la imposición de la Relatividad General como uno de los pilares de la física moderna. La comprensión de que, a pesar de las complejas matemáticas involucradas en la teoría y el profundo cambio de paradigma conceptual que implicaba, la Relatividad General ofrece una descripción más completa

de la interacción gravitacional que la gravedad newtoniana, avivó el campo, motivando numerosos estudios teóricos en la primera mitad del siglo pasado.

Con el avance tecnológico en los años más recientes y un gran esfuerzo científico, a menudo involucrando grandes colaboraciones, tales predicciones teóricas se han demostrado sistemáticamente correctas, tanto cualitativa como cuantitativamente. Desde la detección de ondas gravitacionales⁶, el descubrimiento de los aparentemente paradójicos agujeros negros que ahora son una realidad física⁷, hasta el logro de la llamada “Edad de Oro de la Cosmología”⁸, el papel central de la Relatividad General se ha consolidado aún más.

En los últimos años, gracias a una serie de fructíferas campañas observacionales, el Centro Galáctico ha ganado popularidad, situándose en la intersección de la física fundamental y la astrofísica. El descubrimiento de que una fuente compacta supermasiva Sagitario A* (en adelante Sgr A*) se encuentra en este peculiar entorno y los instrumentos de alta resolución angular que han permitido observar esa región ha llevado a una serie de descubrimientos científicos en las últimas tres décadas que culminaron en el premio Nobel de física de 2020 otorgado a Roger Penrose, Reinhard Genzel y Andrea Ghez⁹. Estas observaciones del Centro Galáctico han permitido nuevos tests observacionales de la Relatividad General en un entorno nuevo y completamente diferente. El descubrimiento de una población de estrellas en rápido movimiento, las llamadas *estrellas S*, orbitando Sgr A*, permite probar la precesión orbital y el corrimiento gravitacional al rojo alrededor de un objeto compacto supermasivo, mientras que la observación directa de Sgr A* (usando técnicas como la Interferometría de Muy Larga Base) permite verificar la deflexión de la luz en un régimen de campo fuerte previamente inexplorado. El potencial descubrimiento de púlsares en órbitas cercanas a Sgr A* promete no solo mejorar nuestra caracterización de su entorno, sino también llevar las pruebas de la gravedad en el régimen de campo fuerte a un nivel completamente nuevo de precisión.

El Centro Galáctico no representa solamente un laboratorio único para validar la Relatividad General, sino también para falsarla frente a teorías competidoras que, con el tiempo, se han ido proponiendo para modificarla o complementarla. Se podrán testar diferentes concepciones sobre la naturaleza de la materia oscura, restringir modelos alternativos al de agujero

⁶ LIGO/VIRGO Collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger*, 2016 [22].

⁷ Webster, B. L., Murdin, P., *Cygnus X-1—a Spectroscopic Binary with a Heavy Companion?*, 1972 [23].

⁸ Schramm D. en Arthur M. Sackler Colloquia of the National Academy of Sciences: Physical Cosmology, 1992.

⁹ <https://www.nobelprize.org/prizes/physics/2020>

negro en la Relatividad de Einstein, etc. Nos acercamos a una astronomía de gran precisión. Nuevas instalaciones observacionales aumentarán la sensibilidad actual en órdenes de magnitud que conllevará el descubrimiento de estrellas más débiles y cercanas a Sgr A*, y púlsares que permitirán tests en un régimen de campo gravitatorio aún más intenso. El descubrimiento de nuevas sondas exige una revisión de las técnicas estándar utilizadas para modelizar nuestros observables en el Centro Galáctico. Mientras que con la sensibilidad observacional actual, los modelos orbitales basados en aproximaciones (*e.g.* el límite post-newtoniano) de las ecuaciones de movimiento relativistas generales para partículas de prueba permiten derivar una reconstrucción suficientemente precisa de los observables astronómicos, como mostraremos, esto ya no será cierto en el futuro. Por esta razón, desarrollar un modelo orbital basado en una descripción completamente relativista del movimiento de partículas de prueba alrededor de un objeto compacto supermasivo y diseñar una estrategia autoconsistente no solo para probar la Relatividad General sino también para obtener restricciones sobre teorías alternativas a partir de la comparación con datos observacionales, es de suma importancia. Esta es la principal motivación y objetivo científico de esta tesis.

Resumen de la tesis

Esta tesis doctoral se estructura en tres partes, con un total de seis capítulos, que detallamos a continuación.

1 Introducción

Los dos primeros capítulos proporcionan una introducción al contexto histórico de las observaciones del Centro Galáctico, centrándose en las escalas observables más pequeñas observadas, e introducen toda la metodología, notaciones y formalismo para la descripción relativista de observables en el Centro Galáctico que se utilizarán en el resto del trabajo.

1.1 Capítulo 1. El Centro Galactico

Este capítulo se basa en el artículo de revisión bibliográfica [9] y en él se detalla el avance histórico de las observaciones en el Centro Galáctico que ha llevado al descubrimiento de Sgr A* y los esfuerzos de décadas para medir cuidadosamente propiedades físicas tales como masa y tamaño físico. Gracias a las observaciones de las estrellas S, sabemos que el campo gravitacional del Centro Galáctico está dominado por una fuente puntual que se comporta como un agujero negro supermasivo (SMBH) de cuatro millones de masas solares, y obedece a las leyes formuladas por la Relatividad General. Aún más sorprendentemente, la imagen directa del material en acreción por la colaboración Event Horizon Telescope (EHT) reveló una depresión de brillo que es compatible con la sombra proyectada por un agujero negro de Kerr. En consecuencia, la evidente presencia de un SMBH en el Centro Galáctico y nuestra capacidad actual para describir órbitas estelares a su alrededor y medir los efectos relativistas que experimentan sus órbitas, exige el desarrollo de nuevos métodos basados en un modelo orbital geodésico que sea completamente relativista. Los conceptos asociados a nuestra metodología se detallan en el Capítulo 2.

1.2 Capítulo 2. Descripción relativista del movimiento de partículas de prueba

Revisamos los conceptos básicos de Relatividad General centrándonos en el movimiento geodésico de partículas de prueba, ya sean objetos masivos o fotones, en espacios-tiempo con simetría esférica. Dado que las estrellas S en el Centro Galáctico (de masas $\lesssim 10M_{\odot}$) orbitan alrededor de un SMBH de cuatro millones de masas solares, la razón entre las masas es tan pequeña que se estas estrellas se pueden considerar partículas test cuyo movimiento orbital es una caída libre en el espacio-tiempo generado por el SMBH. Para seguir su evolución, desarrollamos un código abierto en entorno Python, PyGRO, que permite la integración numérica de las ecuaciones de las geodésicas en cualquier espacio-tiempo esféricamente simétrico, con un método autoconsistente de asignación de condiciones iniciales basado en cantidades conservadas a lo largo de la trayectoria. Esto permite clasificar las órbitas mediante un conjunto restringido de parámetros y asignar la posición y velocidad iniciales de una partícula de prueba. Estos parámetros se relacionan inmediatamente con los parámetros keplerianos de la mecánica celeste clásica, lo que facilita su interpretación física y la comparación con resultados anteriores en la literatura.

En este capítulo se presentan también parte de los resultados publicados en [11] sobre los códigos de resolución numérica del problema emisor-observador para un fotón que se propaga en un espacio-tiempo esféricamente simétrico y la consiguiente integración del tiempo de propagación, utilizando un enfoque totalmente relativista. Esto no sólo permite reconstruir los observables astronómicos de las estrellas S con un enfoque totalmente relativista, sino que además constituirá una pieza fundamental de la metodología para describir los tiempos de llegada de los fotones emitidos por púlsares en el Centro Galáctico.

Todos los ejemplos, gráficos y cálculos de este capítulo se desarrollan sobre la métrica de Schwarzschild, la solución paradigmática de simetría esférica en Relatividad General. Sin embargo, el tratamiento desarrollado pueden aplicarse a cualquier otro espacio-tiempo esféricamente simétrico con toda generalidad.

2 Las estrellas S

En la segunda parte de la tesis aprovechamos la generalidad de nuestro enfoque para estudiar el movimiento de las estrellas S en el Centro Galáctico en varios ejemplos de espacio-tiempo con simetría esférica, bien sea con la introducción de un componente de masa de materia oscura extendida en el Centro Galáctico, considerando espacio-tiempos que imitan agujeros negros, como alternativa al escenario estándar de Schwarzschild para Sgr A* y, finalmente, considerando modificaciones a la teoría subyacente de la gravedad. Todos los resultados obtenidos en este capítulo se resumen en la Tabla 1.

2.1 Capítulo 3. Pruebas de modelos de materia oscura

Esta sección está basada en las siguientes publicaciones: [4, 10]. En ellas hemos mostrado cómo las observaciones de las estrellas que se mueven rápidamente en el centro de la Galaxia, en particular de la estrella S2, pueden representar una nueva sonda para la determinar presencia de un núcleo solitónico compuesto por Fuzzy Dark Matter ψ^1 (FDM) en esta región y pueden proporcionar nuevas restricciones sobre la masa del bosón que compone dicho núcleo. La dinámica de los halos autogravitantes de FDM [24] compuestos de bosones ultraligeros es altamente no-lineal y las predicciones de su comportamiento son específicas de su naturaleza bosónica. Ello ha motivado un gran interés en este modelo y los tests efectuados han permitido derivar límites observacionales sobre la masa del bosón m_ψ . Nuestro trabajo ha consistido en, primero, evaluar la potencial detectabilidad de la existencia de dicho núcleo solitónico en el Centro Galáctico, señalando las desviaciones orbitales que surgirían en este escenario. Segundo, hemos derivado restricciones reales sobre la masa del bosón m_ψ aplicando un análisis estadístico de Montecarlo basado en cadenas de Markov (MCMC) a los datos públicos para S2, una vez fijada una métrica del espacio-tiempo apropiada, teniendo en cuenta la presencia de la componente extendida de materia oscura junto con la fuente masiva central. En conjunto, nuestro análisis ha resultado en un límite superior para la masa del bosón, $m_\psi \lesssim 3.22 \times 10^{-19}$ eV con un nivel

¹ Literalmente, materia oscura difusa.

de confianza del 95%, lo que implica que no se detecta ningún núcleo solitónico de FDM generado por bosones con masas mayores en el Centro Galáctico. Este resultado concuerda con otros estudios publicados recientemente centrados en la búsqueda de una nube escalar o vectorial en el Centro Galáctico utilizando también los datos orbitales de S2.

Nuestras cotas sobre la masa del bosón, m_ψ , en el Centro Galáctico son complementarias a las obtenidas en otros contextos astrofísicos y cosmológicos y, al combinar todos los límites sobre m_ψ , excepto los que provienen de galaxias esferoidales enanas²), hemos conseguido reducir de manera significativa la región permitida del espacio de parámetros. Al combinar nuestro resultado con la restricción del 95% $m_\psi \gtrsim 2 \times 10^{-20}$ eV del análisis del bosque de Lyman- α [27], se obtiene un rango de masa permitido con un nivel de confianza del 95% de $(2.0 \lesssim m_\psi \lesssim 32.2) \times 10^{-20}$ eV.

2.2 Capítulo 4. Pruebas de imitadores de agujeros negros

Se analiza la viabilidad de validar y/o restringir geometrías que imitan agujeros negros y que representan alternativas al agujero negro descrito en la teoría de la Relatividad General. Focalizamos este estudio en el movimiento orbital de las estrellas S alrededor de Sgr A* centrándonos en la detección de los efectos relativistas. Más específicamente, hemos explorado dos propuestas en el contexto de los agujeros negros regulares y que han sido publicadas en [7] y [3].

El primer modelo analizado es el de un agujero negro con cabello (hairy black hole), introducido por primera vez en [28], dotado de un cabello super-Planckiano, ℓ , que difumina la singularidad central, la cual en esta geometría es reemplazada por un core de de Sitter. Aunque se pueden formular muchas métricas del espacio-tiempo diferentes que respeten este requisito, nos hemos centrado en un modelo que resulta en las desviaciones más fuertes del agujero negro de Schwarzschild en la región cercana al horizonte, manteniendo el mismo comportamiento asintótico. Curiosamente, este modelo también tiene una analogía astrofísica ya que se asemeja a los perfiles de materia oscura conocidos. Hemos estudiado el movimiento orbital de la estrella S2 en este espacio-tiempo. Nos hemos

² Para las cuales diferentes análisis han destacado una tensión entre la masa del bosón predicha y la derivada de análisis a escalas mayores [25, 26].

centrado en la precesión orbital relativista, mostrando cómo la comparación entre la predicción teórica y la tasa observada del avance del periastron de S2 restringe fuertemente el valor del cabello ℓ . Luego procedimos a realizar un análisis de Monte Carlo basado en cadenas de Markov utilizando datos orbitales públicos de S2 y la medición de la precesión orbital después del último paso por el pericentro en 2018 para obtener una distribución de probabilidad posterior en ℓ . Nuestro análisis nos permitió establecer un límite superior $\ell \lesssim 0.019$ AU ($\ell \lesssim 0.47$) con un nivel de confianza del 95%. Si bien el resultado es notable, ya que muestra que la dinámica orbital de S2 es sensible a una longitud de escala del cabello del agujero negro tan pequeña como el 0.001% del semieje mayor de S2, no permite excluir completamente objetos sin horizonte en la familia estudiada de agujeros negros no singulares.

Conclusiones similares se obtuvieron de nuestro análisis del modelo Black Bounce. Esta geometría, introducida en [29], es particularmente interesante ya que el único parámetro libre adicional, α_{BB} , permite interpolar entre geometrías muy diferentes. Dependiendo del valor de α_{BB} , el elemento de línea describe un agujero negro de Schwarzschild, un agujero negro regular, un agujero de gusano unidireccionalmente transitable o un agujero de gusano bidireccionalmente transitable en el sentido de Morris-Thorne. Establecer restricciones observacionales sobre α_{BB} permitiría así restringir la naturaleza fundamental de Sgr A*. Sin embargo, nuestro análisis ha demostrado que ni los datos actualmente disponibles para S2 (ni siquiera teniendo en cuenta la detección de efectos relativistas) ni futuros datos previstos con la precisión del interferómetro GRAVITY serían capaces de descartar de manera concluyente una de las posibles geometrías en la familia Black Bounce.

Finalmente, procedimos a analizar qué posibles restricciones sobre el modelo de Black Bounce podrían surgir de las observaciones de estrellas hipotéticamente más cercanas en el Centro Galáctico. Probando varias estrellas ficticias y diseñando varios catálogos simulados de datos futuros, correspondientes a diferentes estrategias observacionales para estas estrellas, encontramos que, para distinguir a un nivel de confianza de 5σ entre un agujero negro y uno de gusano se requiere información astrométrica uniforme de al menos 20 observaciones por región orbital durante un período de 5 años para una estrella de alta excentricidad ($e = 0.99$) en el Centro Galáctico, manteniendo una precisión astrométrica nominal de $10 \mu\text{as}$ durante toda la campaña de observación.

2.3 Capítulo 5. Pruebas de gravedad modificada

En este capítulo hemos explorado la capacidad de las observaciones de las estrellas S, para restringir la teoría subyacente de la gravedad. Como hemos mencionado, existen varias extensiones diferentes (ETGs) y formulaciones alternativas (ATGs) a la Relatividad General, cada una basada en una forma distinta de incorporar grados de libertad adicionales al sector gravitacional o en una formulación geométrica diferente de la teoría a un nivel fundamental. Hemos mostrado cómo nuestro tratamiento, desarrollado en el Capítulo 2 y que nos permite describir el movimiento geodésico relativista de partículas test masivas puede aplicarse, de manera muy general, para obtener observables astrométricos, espectroscópicos y relativistas para la estrella S2 en el Centro Galáctico en una variedad de escenarios teóricos diferentes. En particular, hemos analizado varias clases diferentes de ETG, comenzando con la Scalar-Tensor-Vector Gravity (STVG), gravedad $f(R)$, gravedad de Horndeski y una extensión cuadrática a la Relatividad General, la teoría de Einstein-Maxwell-dilaton-axion (EMDA), que es un ejemplo muy popular de ATG y para la cuál existe una solución de agujero negro analítica. En todos los casos, hemos realizado un análisis posterior sobre el modelo orbital para la estrella S2 basado en las ecuaciones geodésicas para soluciones esféricamente simétricas en cada uno de los modelos considerados. En la mayoría de las ocasiones, hemos tenido en cuenta los datos públicos disponibles para S2, comparando los resultados obtenidos con y sin introducción de la tasa medida de precesión orbital de S2, demostrando el gran poder estadístico de este observable para imponer restricciones a modificaciones métricas del campo gravitacional.

Ninguna de las alternativas a la Relatividad General que hemos considerado, y que involucran tanto una mayor complejidad de la formulación teórica como una fenomenología más rica, han demostrado explicar *mejor* los datos orbitales para las estrellas S. De hecho, nuestros análisis estadísticos han mostrado que estas teorías no pueden ser muy diferentes a la Relatividad General una vez que se tienen en cuenta la tasa medida de precesión orbital. De manera bastante notable, en todos los casos, aunque siempre hemos adoptado un enfoque agnóstico, asignando *prioris* uniformes a dichos parámetros adicionales, siempre hemos obtenido *posterioris* que se cen-

Table 1 Resultados de todos los análisis MCMC realizados en los distintos modelos considerados en esta tesis sobre datos orbitales de las estrellas S en el Centro Galáctico.

Modelo	Parámetro(s)	Resultados	Referencia(s)
Materia oscura			
FDM	m_ψ	$\lesssim 3.22 \times 10^{-19}$ eV	[4, 10]
Imitadores de agujeros negros			
Agujero negro regular con cabello super-Planckiano	ℓ	$\lesssim 0.019$ AU	[7]
EMDA	α_{BB}	$\lesssim 53.13M$ (S2, datos actuales)	[3]
		$\lesssim 8.13M$ (S2, datos futuros)	
		$\lesssim 1.28M$ (estrella más cercana)	
Teorías de gravedad modificada			
STVG	α_{STVG}	$\lesssim 0.708$	[2, 6]
$f(R)$	ζ	$-0.01^{+0.61}$	[1]
	λ	$\gtrsim 6300$ AU	
Horndeski	$G_{4(0,0)}$	$8.17^{+0.16}_{-0.17} M_\odot \text{ AU s}^{-2}$	[5]
	$G_{4(1,0)}$	$-2.0^{+32.0}_{-26.0} M_\odot \text{ AU s}^{-2}$	
	$G_{2(0,1)} - 2G_{3(1,0)}$	$\gtrsim 6200 M_\odot \text{ AU s}^{-2}$	
	$G_{2(2,0)}$	No restringido	
	$1/m_\psi$	$160^{+265}_{-75} \text{ AU}$	
EMDA	b_{EMDA}	$\lesssim 12M$ (S2, no precesión)	[8]
		$\lesssim 1.45M$ (S2, con precesión)	
		$\lesssim 0.025M$ (S2, datos futuros)	
Extensión cuadrática métrico-afín	ℓ, r_q	Ver Figura 5.22 en la tesis	[12]

tran en la región del espacio de parámetros que corresponde al límite de Relatividad General. Esto es notable y, una vez más, demuestra el gran poder predictivo de la Relatividad General que, hasta la fecha, sigue superando con éxito todas las pruebas experimentales a las que se enfrenta.

3 Pulsares en el Centro Galáctico

En la tercera y última parte de esta tesis se presenta una aplicación del formalismo desarrollado a la caracterización de observables astronómicos para púlsares en el Centro Galáctico. Parte de los resultados presentados han sido publicados en [11], mientras que la última parte es un trabajo en curso y ofrece una perspectiva de futuro a las técnicas que hemos elaborado en esta tesis.

3.1 Capítulo 6. Púlsares en el Centro Galáctico: el próximo avance

El próximo gran avance de la gravitación experimental será cuando se descubran púlsares orbitando alrededor de un SMBH. Estos objetos astronómicos son conocidos por su notable estabilidad rotacional intrínseca, que exhiben variaciones del orden de una parte en 10^{15} por período de pulso, lo que los convierte en herramientas excepcionales para medir efectos gravitatorios en el límite de campo fuerte. Un púlsar en una órbita cercana (con períodos orbitales inferiores a 1-100 años) alrededor de un objeto compacto supermasivo como Sgr A* exploraría un régimen gravitatorio completamente diferente de lo que otras pruebas de gravedad actualmente conocidas prueban. El análisis de los Tiempos de Llegada (TOAs) de los pulsos emitidos por estos objetos, conocido como análisis de tiempo de púlsar, superaría todas las pruebas existentes de la Relatividad General y permitiría también compararla con teorías alternativas de la gravedad con una precisión desconocida. El descubrimiento y detección de al menos un púlsar en una órbita cercana a Sgr A* es un objetivo científico crucial para futuras observatorios como el Square Kilometer Array (SKA). Aprovechando áreas de recolección sustanciales, se ha demostrado que SKA podría realizar análisis de tiempo [30] con una precisión esperada en la medición del TOA del pulso de $\sigma_{\text{TOA}} \approx 100 \mu\text{s}$ para púlsares normales a frecuencias superiores a 15 GHz [31].

En particular, hemos aplicado nuestro formalismo al cálculo numérico de los tiempos de propagación relativistas de fotones emitidos en un espacio-tiempo esféricamente simétrico genérico para estimar el impacto en los residuos de timing de un cambio en la naturaleza fundamental del objeto central o en la teoría subyacente de la gravedad. Hemos introducido una serie de modelos de prueba, explorando un campo gravitacional cada vez más fuerte, y hemos mostrado cómo mejorarían las cotas actuales a las desviaciones de la teoría einsteniana de la gravedad. Nuestro estudio sugiere que las posibles restricciones que permitirían las futuras observaciones pueden superar órdenes de magnitud las restricciones actuales. Esto demuestra aún más la importancia capital de descubrir púlsares en órbita alrededor de SMBH como sondas de gravedad en campos fuertes, agregando nuevos resultados a una literatura ya amplia sobre el tema.

Hemos generalizado nuestro tratamiento no solo para calcular

el tiempo de propagación de fotones en general, sino también para desarrollar un modelo de timing para generar residuos conectados por fase y realizar ajustes de datos TOAs a un modelo de Schwarzschild. Esto nos permitió estimar cuantitativamente los residuos dependientes de la fase que surgen de una estimación incorrecta de uno de los parámetros del púlsar. Tales estimaciones, por otro lado, nos dieron una idea cualitativa de la posible precisión que se lograría a través de un análisis de ajuste real de nuestro modelo de timing en datos TOAs. Esto nos motivó a analizar más a fondo esta posibilidad, generando un catálogo simulado de observaciones de timing para uno de los modelos de prueba considerados en la sección anterior y aplicando una metodología de inferencia bayesiana, basada en un modelo de verosimilitud construida sobre los residuos de timing calculados a través de nuestro datos de timing, para restringir un subconjunto de parámetros del problema completo. Nuestros resultados muestran que no solo los parámetros de entrada (utilizados para la generación del catálogo simulado) son recuperados perfectamente por el análisis posterior, sino también que la precisión alcanzada en la estimación de los parámetros orbitales (y la masa de Sgr A*) puede superar por varios órdenes de magnitud la alcanzada a través de observaciones de estrellas S, incluso después de solo unos pocos años de observaciones.

Nuestro estudio en curso destaca el gran poder de restricción que poseerían las observaciones de timing de púlsares y la importancia de tener en cuenta adecuadamente todos los efectos relativistas adoptando un cálculo completamente geodésico. El próximo gran paso a implementar, para proporcionar una descripción más realista del objeto central, es extender la metodología que hemos desarrollado a una solución de agujero negro con espín, como el agujero negro de Kerr en la Relatividad General o sus imitadores y extensiones en teorías modificadas de la gravedad.

Conclusiones y perspectivas futuras

En 2010, el futuro premio Nobel R. Genzel concluyó su artículo de revisión sobre el Centro Galáctico, coescrito con F. Eisenhauer y S. Gillessen, con la afirmación: “[*En la próxima década*] el Centro Galáctico podría convertirse en un banco de pruebas para sondear la Relatividad General en el límite de campo fuerte”. Como hemos ilustrado a lo largo de esta tesis, hoy, catorce años después, esta predicción ha sido ampliamente confirmada. Los avances tecnológicos en la astronomía de infrarrojo cercano del Centro Galáctico en las últimas décadas nos han permitido medir con confianza los efectos post-newtonianos del corrimiento al rojo gravitacional relativista general y la precesión de Schwarzschild en las órbitas de las estrellas S. Del mismo modo, la observación innovadora de Sgr A* por el Event Horizon Telescope, posible gracias a avances en campos teóricos, computacionales y tecnológicos, ha validado las predicciones de una sombra de agujero negro y ha proporcionado otra herramienta para validar las teorías del campo gravitatorio en el régimen de campo fuerte. Sin duda, el Centro Galáctico se ha destacado como un poderoso laboratorio en el cual las tres pruebas clásicas de la Relatividad General se han podido verificar en un régimen mucho más exigente.

Los notables descubrimientos realizados hasta ahora en el Centro Galáctico no representan el punto final de esta línea de investigación. Más bien, motivan nuevos esfuerzos teóricos y observacionales para alcanzar nuevas fronteras en la comprensión de este entorno único. Durante la próxima década, nuevas instalaciones de observación nos permitirán examinar el Centro Galáctico con nuevos ojos. Las mejoras en curso y futuras del instrumento GRAVITY (el proyecto GRAVITY+) permitirán observaciones más profundas del Centro Galáctico, potencialmente llevando a la identificación de una nueva población de estrellas más cercanas y débiles, que se espera que produzcan mediciones más precisas de las propiedades del SMBH, como masa, espín y momento cuadrupolar. Se anticipa un mayor progreso con la llegada de instalaciones de observación más grandes, como el TMT en Hawái o el ELT de 39 metros en el ESO. La posible detección de púlsares en órbitas relativistas alrededor de Sgr A* dentro de unos pocos años con radiotelescopios operando en el rango

de ondas milimétricas de próxima generación como el SKA conseguirán resultados aún más espectaculares. Se espera que estos púlsares, a menudo considerados como el santo grial para la gravitación experimental, proporcionen las pruebas más estrictas hasta la fecha de las teorías de la gravedad y la física de agujeros negros.

Por esta razón, debido a los desafíos actuales y futuros en la modelización de características observables para estos objetos, el objetivo científico de esta tesis fue, por un lado, el desarrollo de un formalismo que proporcionara un modelo orbital completamente relativista para el movimiento de partículas de prueba alrededor de un objeto compacto supermasivo; y por otro, la implementación de técnicas de inferencia bayesiana, basadas en algoritmos MCMC, para idear una estrategia autoconsistente que permita tanto validar/falsar la Relatividad General como derivar restricciones sobre sus alternativas.

Esto resultó en el desarrollo de un código de fuente abierta, PyGRO, que, combinando cálculos simbólicos y numéricos, proporciona una interfaz fácil de usar para integrar las ecuaciones geodésicas para cualquier métrica espacio-temporal dada. A lo largo de esta tesis, nos hemos centrado en geometrías caracterizadas por simetría esférica, lo que, a su vez, simplifica drásticamente las ecuaciones de movimiento y la configuración de las condiciones iniciales tanto para partículas de prueba masivas como fotones. Esto nos permitió construir un modelo orbital para las estrellas S que orbitan alrededor de Sgr A* en un espacio-tiempo esféricamente simétrico y una forma completamente relativista de calcular las trayectorias de fotones y los tiempos de propagación en él, útil para la modelización de TOAs de púlsares en un observatorio terrestre.

En resumen, hemos aplicado nuestro novedoso formalismo a una variedad de escenarios diferentes:

- Hemos considerado el impacto que una componente extendida de materia oscura en el Centro Galáctico en forma de un núcleo solitónico, compuesto de partículas de bosones ultraligeros descritos dentro del modelo de FDM, tendría en las órbitas de las estrellas S. La presencia de tal solitón en magnitudes astrométricas observables de la estrella S2 ha sido analizada a fondo y se han obtenido nuevas restricciones sobre la masa del bosón, que, cuando se combinan con otras restricciones astrofísicas y cosmológicas existentes en la literatura,

reducen la región permitida para este parámetro a solo un orden de magnitud.

- Hemos analizado la viabilidad de validar/falsar geometrías que imitan agujeros negros como alternativas al modelo clásico de agujero negro relativista de Sgr A*, basándonos únicamente en el movimiento orbital de las estrellas S y en la detección de los efectos relativistas para este objeto. Hemos demostrado que, aunque los datos disponibles actualmente para las estrellas S conocidas en el Centro Galáctico no son capaces de identificar inequívocamente la naturaleza fundamental de Sgr A*, este panorama podría cambiar drásticamente con el descubrimiento de estrellas en órbitas mucho más relativistas. Gracias al análisis de pronóstico que hemos realizado, no solo hemos identificado las posibles propiedades orbitales que tales objetos deberían presentar, sino también la estrategia observacional que las futuras campañas de observación deberían adoptar, con el fin de maximizar el poder de restricción de sus observaciones.
- Hemos considerado varias teorías extendidas o alternativas bien motivadas a la Relatividad General. En particular, hemos analizado la STVG, la gravedad $f(R)$, la gravedad de Horndeski, EMDA y una extensión cuadrática de la Relatividad General, para todas las cuales existe una solución de agujero negro analítica. En todos los casos, hemos derivado nuevas restricciones sobre los parámetros adicionales que codifican las desviaciones de la Relatividad General, en escalas en las que no habían sido probadas anteriormente. Aunque siempre hemos adoptado un enfoque agnóstico, asignando priores uniformes a dichos parámetros adicionales, siempre hemos obtenido posterioris que se centran en la región del espacio de parámetros que corresponde al límite relativista general de las teorías que hemos estudiado. Estos notables resultados demuestran una vez más el gran poder predictivo de la Relatividad General.
- Finalmente, hemos considerado el caso de los púlsares en órbitas cerradas alrededor de Sgr A*. Hemos demostrado cómo la metodología completamente relativista que hemos introducido puede aplicarse a la modelización realista de TOAs, mientras que la técnica post-newtoniana habitual no logra capturar toda la no linealidad del

problema. Realizando un análisis de pronóstico, hemos mostrado que el nivel de precisión en la caracterización de las propiedades físicas de Sgr A* a través de la cronometraje de púlsares puede superar por varios órdenes de magnitud al logrado mediante observaciones de estrellas S, incluso después de solo unos pocos años de observaciones. Esto también se refleja en restricciones mucho más estrictas que tales observaciones permitirían sobre la teoría subyacente de la gravedad y la naturaleza fundamental de Sgr A*.

A pesar del trabajo realizado hasta ahora y los resultados alcanzados con esta tesis, aún deben explorarse algunos aspectos delicados. Por ejemplo, con las mejoras previstas en la precisión astrométrica, especialmente para aquellos objetos hipotéticos en órbitas mucho más cerradas que las actualmente observadas, los efectos relacionados con la rotación de Sgr A* dejarán de ser despreciables. Por el contrario, es precisamente de la posibilidad de detectar firmas orbitales relacionadas con el espín y los momentos cuadrupolares del objeto central, de donde se esperan las pruebas más estrictas de la gravedad (*e.g.* el teorema de la calvicie). Aunque el código numérico desarrollado en esta tesis para la integración de curvas geodésicas es completamente general y puede aplicarse fácilmente a métricas espacio-tiempo con rotación, no ocurre lo mismo con el procedimiento que hemos desarrollado para asignar condiciones iniciales para partículas masivas y sin masa. De hecho, cambiar las simetrías que obedece el espacio-tiempo también cambia las definiciones de las cantidades conservadas y resulta en una capa adicional de complejidad debido a la presencia de una dirección privilegiada, el eje de rotación del objeto central, que rompe la mayoría de las simplificaciones que hemos explotado en este trabajo. Esto afecta crucialmente tanto a las órbitas temporales de estrellas o púlsares en el Centro Galáctico como a las geodésicas nulas que siguen los fotones en su trayectoria hacia el observador. Aunque introducir estos efectos representa un desafío sustancial en términos de modelización teórica, la valiosa información que se obtendría a cambio representaría un gran avance hacia una caracterización más completa de Sgr A* y su campo gravitacional, especialmente para el caso de púlsares en órbita. Además, debido al peculiar papel del Centro Galáctico, en la frontera de la física teórica y la astrofísica, tales avances en el aspecto fundamental podrían conducir a descubrimientos revolucionarios sobre el papel de los SMBH en la forma-

ción estelar, su coevolución a escala cosmológica con sus anfitriones galácticos y en nuestra comprensión de los núcleos galácticos en general.

Para concluir, mirando retrospectivamente al notable progreso realizado en las últimas décadas con la instrumentación astronómica que se ha ido desarrollando, en la calidad de las observaciones resultantes y la modelización teórica del Centro Galáctico, podemos anticipar avances aún más emocionantes en los próximos años, tanto por los nuevos objetos que se encontrarán orbitando Sgr A* como por la observación directa de Sgr A* mismo. En este contexto, los resultados alcanzados en esta tesis y las metodologías desarrolladas representarán una herramienta valiosa para el mayor beneficio de la comunidad investigadora en este campo.

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