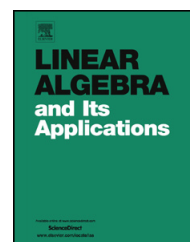




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Classification of finite potent endomorphisms [☆]



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ABSTRACT

The aim of this work is to offer a family of invariants that allows us to classify finite potent endomorphisms on arbitrary vector spaces, generalizing the classification of endomorphisms on finite-dimensional vector spaces. As a particular case we classify nilpotent endomorphisms on infinite-dimensional vector spaces.

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1. Introduction

The classification of mathematical objects is a classical problem: try to determine the structure of a quotient set up to some equivalence. The classification of endomorphisms on finite-dimensional vector spaces, where the group of automorphisms acts by conjugation, is well known.

In this work we generalize this classification to finite potent endomorphisms on arbitrary vector spaces (this notion was introduced by J. Tate as a tool for his elegant definition of Abstract Residues [5]): that is, if φ and ϕ are finite potent endomorphisms on a k -vector space V , we give conditions for the existence of an automorphism $\tau \in \text{Aut}_k(V)$ such that $\phi = \tau\varphi\tau^{-1}$. As a particular case, we classify nilpotent endomorphisms on arbitrary infinite-dimensional vector spaces. As far as we know, this result is not stated explicitly in the literature.

If X_V^{fp} is the subset of $\text{End}_k(V)$ consisting of all the finite potent endomorphisms of V , we also provide an explicit description of the quotient set $X_V^{\text{fp}}/\text{Aut}_k(V)$ for countable dimensional vector spaces.

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The paper is organized as follows. In Section 2 we briefly recall the basic definitions of this work: the definition of finite potent endomorphisms with the decomposition of the vector space given by M. Argerami, F. Szechtman and R. Tifenbach in [1]; Tate's definition of the trace of a finite potent endomorphism [5]; and the definition of the determinant for these objects recently offered by D. Hernández Serrano and the author in [3]. Moreover, in this section we describe the well-known theory of the classification of endomorphisms on finite-dimensional vector spaces.

Section 3 is devoted to giving the main results of this work. Indeed, we offer invariants to classify nilpotent endomorphisms on arbitrary vector spaces (Theorem 3.10). As an example we offer the explicit description of the quotient set obtained from the classification of nilpotent endomorphisms on a countable dimensional vector space (Example 1). Using this classification of nilpotent endomorphisms, we offer invariants to solve the proposed problem: the classification of finite potent endomorphisms on arbitrary vector spaces (Theorem 3.18). Finally, we characterize the quotient set of finite potent endomorphisms for countable dimensional vector spaces (Example 2), and we show that the determinant and the trace of a finite potent operator are invariant under the classification offered.

2. Preliminaries

This section is added for the sake of completeness.

2.1. Basic definitions

Let k be an arbitrary field, and let V be a k -vector space.

Let us now consider an endomorphism φ of V . We say that φ is “finite potent” if $\varphi^n V$ is finite dimensional for some n . This definition was introduced by J. Tate in [5] as a basic tool for his elegant definition of Abstract Residues.

In 2007 M. Argerami, F. Szechtman and R. Tifenbach showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ -invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, W_φ is finite dimensional, and $\varphi|_{W_\varphi} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k , we may view V as $k[x]$ -module via φ , and the explicit definition of the above φ -invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } x^m v = 0 \text{ for some } m\}$.
- $W_\varphi = \{v \in V \text{ such that } p(x)v = 0 \text{ for some } p(x) \in k[x] \text{ relative prime to } x\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \text{Ker } \varphi^m$ and $W_\varphi = \text{Ker } p(\varphi)$.

Hence, this decomposition is unique. In this paper we shall call this decomposition the φ -invariant AST-decomposition of V .

For a finite potent endomorphism φ , a trace $\text{tr}_V(\varphi) \in k$ may be defined as $\text{tr}_V(\varphi) = \text{tr}_W(\varphi|_{W_\varphi})$.

This trace has the following properties:

1. If V is finite dimensional, then $\text{tr}_V(\varphi)$ is the ordinary trace.
2. If W is a subspace of V such that $\varphi W \subset W$ then

$$\text{tr}_V(\varphi) = \text{tr}_W(\varphi) + \text{tr}_{V/W}(\varphi).$$

3. If φ is nilpotent, then $\text{tr}_V(\varphi) = 0$.

For details readers are referred to [5].

With the previous notation, and using this AST-decomposition of V , recently D. Hernández Serrano and the author have offered in [3] a definition of a determinant for finite potent endomorphisms as follows:

$$\det_V^k(1 + \varphi) := \det_{W_\varphi}^k(1 + \varphi|_{W_\varphi}).$$

This definition generalizes the one given by A. Grothendieck for operators of finite rank in [2], and satisfies the following properties:

- If V is finite dimensional, then $\det_V^k(1 + \varphi)$ is the ordinary determinant.
- If W is a subspace of V such that $\varphi W \subset W$, then

$$\det_V^k(1 + \varphi) = \det_W^k(1 + \varphi) \cdot \det_{V/W}^k(1 + \varphi).$$

- If φ is nilpotent, then $\det_V^k(1 + \varphi) = 1$.

2.2. Classification of endomorphisms on finite-dimensional vector spaces

Let E be a finite-dimensional vector space over a field k , and let $T \in \text{End}_k(E)$ be an endomorphism of E . We shall write E_T to denote the vector space E with this $k[x]$ -module structure induced by T .

It is known that two endomorphisms $T, \tilde{T} \in \text{End}_k(E)$ are equivalent, i.e. there exists an automorphism $\tau \in \text{Aut}_k(V)$ such that $T = \tau \tilde{T} \tau^{-1}$ if and only if the $k[x]$ -modules E_T and $E_{\tilde{T}}$ are isomorphic.

Let $a_T(x) = p_1(x)^{n_1} \cdot \dots \cdot p_r(x)^{n_r}$ be the annihilator polynomial of T , where $p_i(x)$ are irreducible polynomials on $k[x]$. The decomposition of k -vector spaces

$$E = \text{Ker } p_1(T)^{n_1} \oplus \dots \oplus \text{Ker } p_r(T)^{n_r},$$

where the subspaces $\text{Ker } p_i(T)^{n_i} \subset E$ are invariant by T , is compatible with the respective $k[x]$ -module structures.

Indeed, the classification of endomorphisms on finite-dimensional vector spaces is reduced to studying the $k[x]$ -module structure of $\text{Ker } p(T)^n$, $p(x)$ being an irreducible polynomial on $k[x]$. This structure is determined by a decomposition of $k[x]$ -modules:

$$\text{Ker } p(T)^n \simeq [k[x]/p(x)^n]^{\nu_n(E, p(T))} \oplus \dots \oplus [k[x]/p(x)]^{\nu_1(E, p(T))},$$

where $\nu_n(E, p(T)) \neq 0$ and

$$\nu_i(E, p(T)) = \dim_K (\text{Ker } p(T)^i / [\text{Ker } p(T)^{i-1} + p(T) \text{Ker } p(T)^{i+1}]),$$

with $K = k[x]/p(x)$.

Again writing the annihilator polynomial of T as

$$a_T(x) = p_1(x)^{n_1} \cdot \dots \cdot p_r(x)^{n_r},$$

the invariant factors $\{\nu_i(E, p_j(T))\}_{1 \leq j \leq r; 1 \leq i \leq n_j}$ determine the $k[x]$ -module structure of E_T and, therefore, they classify the endomorphism T .

Furthermore, if $K_j = k[x]/p_j(x)$, then $p_j(x)$ is a polynomial of degree $d_j = \dim_k K_j$.

The well-known theory of classification of endomorphisms on finite-dimensional vector spaces shows that there exist families of vectors $\{e_h^{ij}\}_{1 \leq h \leq \nu_i(E, p_j(T))}$ with

$$\begin{aligned} e_h^{ij} &\in \text{Ker } p_j(T)^i, \\ e_h^{ij} &\notin \text{Ker } p_j(T)^{i-1} + p_j(T) \text{Ker } p_j(T)^{i+1}, \end{aligned}$$

for all $1 \leq j \leq r$ and $1 \leq i \leq n_j$, such that if we set

$$\langle e_h^{ij} \rangle_T = \bigoplus_{0 \leq s \leq i-1} \langle p_j(T)^s [e_h^{ij}], p_j(T)^s [T(e_h^{ij})], \dots, p_j(T)^s [T^{d_j-1}(e_h^{ij})] \rangle,$$

then

$$E = \bigoplus_{\substack{1 \leq j \leq r \\ 1 \leq i \leq n_j \\ 1 \leq h \leq \nu_i(E, p_j(T))}} \langle e_h^{ij} \rangle_T.$$

Indeed, the family of vectors

$$\bigcup_{\substack{1 \leq j \leq r \\ 1 \leq i \leq n_j}} \{e_h^{ij}\}_{1 \leq h \leq v_i(E, p_j(T))} \tag{2.1}$$

generates a Jordan basis of E for T .

3. Classification of finite potent endomorphisms

Let V be an arbitrary k -vector space, and let $\text{End}_k(V)$ be the k -vector space of endomorphisms of V . Let us consider the subset $X_V^{fp} \subset \text{End}_k(V)$ consisting of all the finite potent endomorphisms of V (note that X_V^{fp} is not a vector subspace of $\text{End}_k(V)$ because, in general, the sum of two finite potent endomorphisms is not finite potent).

We have an action of the group of automorphisms of V , $\text{Aut}_k(V)$, on X_V^{fp} by conjugation:

$$\begin{aligned} \text{Aut}_k(V) \times X_V^{fp} &\longrightarrow X_V^{fp}, \\ (\tau, \varphi) &\longmapsto \tau\varphi\tau^{-1}. \end{aligned}$$

This section is devoted to offering the main result of this work: the characterization of the quotient set $X_V^{fp} / \text{Aut}_k(V)$.

Henceforth, if $f \in \text{End}_k(V)$ and H is a k -subspace of V invariant for f , to simplify we shall again write $f: H \rightarrow H$ and $f: V/H \rightarrow V/H$ to refer to the induced linear operators.

3.1. Classification of nilpotent endomorphisms

Let V again be an arbitrary vector space over a ground field k , and let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis of V . It is known that $\dim(V) = \#\mathcal{B}$ is independent of the basis chosen, $\#\mathcal{B}$ being the cardinal of the set \mathcal{B} .

Let X_V^N be the subset of X_V^{fp} consisting of all nilpotent endomorphisms of V , that is:

$$X_V^N = \{f \in \text{End}_k(V) \text{ such that } f^n = 0 \text{ for a certain } n \in \mathbb{N}\}.$$

It is clear that the group of automorphisms of V , $\text{Aut}_k(V)$, acts on X_V^N by conjugation. We shall characterize the quotient set $X_V^N / \text{Aut}_k(V)$.

To start, we shall construct a Jordan basis of V for a nilpotent endomorphism.

Let us consider $f \in X_V^N$ of order n ($f^n = 0$ and $f^{n-1} \neq 0$). We have a sequence of k -subspaces of V :

$$\{0\} \subsetneq \text{Ker } f \subsetneq \text{Ker } f^2 \subsetneq \dots \subsetneq \text{Ker } f^{n-1} \subsetneq \text{Ker } f^n = V.$$

Let H_n^f be a supplementary subspace of $\text{Ker } f^{n-1}$ on V , i.e.

$$\text{Ker } f^{n-1} \oplus H_n^f = V.$$

Note that the dimension of H_n^f , $\dim H_n^f$, is independent of the choice made because

$$H_n^f \simeq V / \text{Ker } f^{n-1}.$$

Lemma 3.1. *With the previous notation, we have that:*

1. $f(H_n^f) \cap H_n^f = \{0\}$.
2. $f(H_n^f) \cap \text{Ker } f^{n-2} = \{0\}$.

Proof.

1. Since $f(H_n^f) \subset \text{Ker } f^{n-1}$ and $H_n^f \cap \text{Ker } f^{n-1} = \{0\}$, the first assertion is deduced.

2. Let us consider $v \in f(H_n^f) \cap \text{Ker } f^{n-2}$.

If $v = f(h)$, with $h \in H_n^f$, since $v \in \text{Ker } f^{n-2}$, then $f^{n-1}(h) = 0$.

Thus, $h \in \text{Ker } f^{n-1} \cap H_n^f = \{0\}$, and we conclude that $v = 0$. \square

Hence, we can now consider a k -subspace of V , H_{n-1}^f , such that

$$\text{Ker } f^{n-1} = \text{Ker } f^{n-2} \oplus f(H_n^f) \oplus H_{n-1}^f.$$

Since $V = \text{Ker } f^n$ and $f(\text{Ker } f^{n-1}) \subseteq \text{Ker } f^{n-2}$, it is clear that

$$\text{Ker } f^{n-1} = [\text{Ker } f^{n-2} + f(\text{Ker } f^n)] \oplus H_{n-1}^f.$$

Recurrently, we can fix k -vector subspaces of V , $\{H_r^f\}_{1 \leq r \leq n}$, such that

$$\text{Ker } f^i = \text{Ker } f^{i-1} \oplus f^{n-i}(H_n^f) \oplus \dots \oplus f(H_{i+1}^f) \oplus H_i^f, \tag{3.1}$$

for every $1 \leq i < n$.

Similarly to Lemma 3.1, since

$$f^{j-i+1}(H_{j+1}^f) + \dots + f^{n-i}(H_n^f) + \text{Ker } f^{i-1} \in \text{Ker } f^i,$$

we can prove that

Lemma 3.2. *If $1 \leq i < n$, one has that*

$$f^{j-i}(H_j^f) \cap [f^{j-i+1}(H_{j+1}^f) + \dots + f^{n-i}(H_n^f) + \text{Ker } f^{i-1}] = \{0\},$$

for all $i + 1 \leq j < n$.

As above, bearing in mind that

$$\text{Ker } f^{i-1} \oplus f^{n-i}(H_n^f) \oplus \dots \oplus f(H_{i+1}^f) = \text{Ker } f^{i-1} + f(\text{Ker } f^{i+1}),$$

one has that

$$\text{Ker } f^i = [\text{Ker } f^{i-1} + f(\text{Ker } f^{i+1})] \oplus H_i^f.$$

If we now set

$$\begin{aligned} W_n^\varphi &= V / \text{Ker } f^{n-1}, \\ W_{n-1}^\varphi &= \text{Ker } f^{n-1} / [\text{Ker } f^{n-2} + f(\text{Ker } f^n)], \\ &\vdots \\ W_i^\varphi &= \text{Ker } f^i / [\text{Ker } f^{i-1} + f(\text{Ker } f^{i+1})], \\ &\vdots \\ W_2^\varphi &= \text{Ker } f^2 / [\text{Ker } f + f(\text{Ker } f^3)], \\ W_1^\varphi &= \text{Ker } f / f(\text{Ker } f^2), \end{aligned}$$

by construction the dimension of the k -subspace H_i^f is independent of the choices made for all $1 \leq i \leq n$, and

$$\dim H_i^f = \dim W_i^\varphi.$$

Accordingly, we can assign to each $f \in X_V^N$ a family of cardinals $\{\mu_i(V, f)\}_{1 \leq i \leq n}$ where $\mu_i(V, f) = \dim W_i^\varphi$.

Remark 3.3. If $f \in X_V^N$, with $f^n = 0$ and $f^{n-1} \neq 0$, one has that $\mu_n(V, f) \neq 0$.

We should emphasize that there is no relationship of order between the invariants $\{\mu_i(V, f)\}$.

Thus, if V is a k -vector space of countable dimension with a basis $\{e_1, e_2, \dots, e_n, \dots\}$, and we consider $f, g \in X_V^N$ defined by:

$$f(e_i) = \begin{cases} e_3 & \text{if } i = 1, 2, \\ 0 & \text{if } i \geq 3, \end{cases}$$

$$g(e_j) = \begin{cases} 0 & \text{if } j = 1, 2, \\ e_{j+1} & \text{if } j \geq 3 \text{ odd,} \\ 0 & \text{if } j \geq 4 \text{ even,} \end{cases}$$

then $f^2 = g^2 = 0$, and one can choose

- $H_2^f = \langle e_1 \rangle$, $f(H_2^f) = \langle e_3 \rangle$, and $H_1^f = \langle e_1 - e_2, e_4, e_5, \dots \rangle$;
- $\mu_1(V, f) = \aleph_0$, and $\mu_2(V, f) = 1$;
- $H_2^g = \langle e_{2j+1} \rangle_{j \geq 1}$, $g(H_2^g) = \langle e_{2j} \rangle_{j \geq 2}$, and $H_1^g = \langle e_1, e_2 \rangle$;
- $\mu_1(V, g) = 2$, and $\mu_2(V, g) = \aleph_0$;

\aleph_0 being the cardinal of the set of all natural numbers.

Henceforth, for indexing bases, $S_{\mu_i(V, f)}$ will be a set such that $\#S_{\mu_i(V, f)} = \mu_i(V, f)$, with $S_{\mu_i(V, f)} \cap S_{\mu_j(V, f)} = \emptyset$ for $i \neq j$. In particular, if $\mu_i(V, f)$ is a natural number N , then $S_{\mu_i(V, f)} = \{i_1, i_2, \dots, i_N\}$. Note that

$$\# \left[\bigcup_{1 \leq i \leq n} (S_{\mu_i(V, f)} \sqcup \overbrace{\dots}^{(i)} \sqcup S_{\mu_i(V, f)}) \right] = \dim(V).$$

Proposition 3.4. If $1 \leq i \leq n$ and $\{v_{s_i}\}_{s_i \in S_{\mu_i(V, f)}}$ is a basis of H_i^f , then:

1. $\{f^r(v_{s_i})\}_{s_i \in S_{\mu_i(V, f)}}$ is a basis of $f^r(H_i^f)$ for all $1 \leq r < i$.

2.
$$\bigcup_{\substack{s_i \in S_{\mu_i(V, f)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}$$

is a Jordan basis of V for φ .

Proof.

1. We only have to show that $\{f^r(v_{s_i})\}_{s_i \in S_{\mu_i(V, f)}}$ is a system of linearly-independent vectors.

If $J \subset S_{\mu_i(V, f)}$, and $\sum_{j \in J} \lambda_j f^r(e_j) = 0$, since f^r is linear one has that

$$f^r \left[\sum_{j \in J} \lambda_j e_j \right] = 0.$$

Thus, $\sum_{j \in J} \lambda_j e_j \in \text{Ker } f^r \cap H_i^f = \{0\}$ with $r < i$, and we conclude that $\lambda_j = 0$ for every $j \in J$.

2. The existence of bases of V with the required structure is a direct consequence of the decomposition of V obtained recurrently from expressions (3.1). \square

Remark 3.5. Recall from [4] that for every $\phi \in \text{End}(V)$ possessing an annihilating polynomial of an arbitrary infinite-dimensional vector space V there exists a Jordan basis of V associated with ϕ . We should note that the above construction of Jordan bases for nilpotent endomorphisms is compatible with the results of [4]. However, from the proof of the existence of Jordan bases given in [4] a Classification Theorem for these endomorphisms is not obtained, because from the statements of this paper it is not possible to deduce that the dimensions of the vector subspaces that determine a Jordan basis are independent of the choices made.

We shall use the existence of Jordan bases for nilpotent endomorphisms to characterize the quotient set $X_V^N / \text{Aut}_k(V)$.

Note that a Jordan basis of V for a nilpotent endomorphism of order n , φ , is determined by a family of vectors

$$\{v_{s_1}\}_{s_1 \in S_{\mu_1(V,f)}} \cup \dots \cup \{v_{s_n}\}_{s_n \in S_{\mu_n(V,f)}}, \tag{3.2}$$

$\{v_{s_i}\}_{s_i \in S_{\mu_i(V,f)}}$ being a basis of H_i^f .

Let τ be an automorphism of V .

Lemma 3.6. *If $v \in V$, then*

$$f^s(v) = 0 \iff \bar{f}^s(\tau(v)) = 0$$

with $\bar{f} = \tau f \tau^{-1}$.

Proof. One has that

$$\bar{f}^s(\tau(v)) \iff \tau[f^s(v)] = 0 \iff f^s(v) = 0. \quad \square$$

Corollary 3.7. *If $\bar{f} = \tau f \tau^{-1}$, then:*

- $\tau[\text{Ker } f^r] = \text{Ker } \bar{f}^r$ for all $r \geq 1$.
- $\tau[\text{Ker } f^{i-1} + f \text{Ker } f^{i+1}] = \text{Ker } \bar{f}^{i-1} + \bar{f} \text{Ker } \bar{f}^{i+1}$ for all $i \geq 1$.

Proposition 3.8. *If $\bar{f} = \tau f \tau^{-1}$, then $\mu_i(V, f) = \mu_i(V, \bar{f})$ for all $1 \leq i \leq n$.*

Proof. The statement can be deduced bearing in mind that Corollary 3.7 implies that $H_j^{\bar{f}} = \tau(H_j^f)$ is a valid choice for all $1 \leq j \leq n$. \square

Remark 3.9. Let f be a nilpotent endomorphism of order n , and let

$$\{v_{s_1}\}_{s_1 \in S_{\mu_1(V,f)}} \cup \dots \cup \{v_{s_n}\}_{s_n \in S_{\mu_n(V,f)}}$$

be a family of vectors determining a Jordan basis of V for f .

If $\bar{f} = \tau f \tau^{-1}$, one has that

$$\bar{f}^s[\tau(v_{s_i})] = \tau[f^s(v_{s_i})]$$

for all $1 \leq i \leq n$ and $s \geq 1$, and we have that

$$\{\tau(v_{s_1})\}_{s_1 \in S_{\mu_1(V,f)}} \cup \dots \cup \{\tau(v_{s_n})\}_{s_n \in S_{\mu_n(V,f)}}$$

determines a Jordan basis of V for \bar{f} .

Theorem 3.10 (Classification Theorem). *Let $f, g \in X_V^N$ be two nilpotent endomorphisms of order n . Thus, $f \sim g \pmod{\text{Aut}_k(V)}$ if and only if $\mu_i(V, f) = \mu_i(V, g)$ for all $1 \leq i \leq n$.*

Proof. (\implies) If $[f] = [g] \in X_V^N / \text{Aut}_k(V)$, there exists $\tau \in \text{Aut}_k(V)$ such that $g = \tau f \tau^{-1}$, and it follows from Proposition 3.8 that $\mu_i(V, f) = \mu_i(V, g)$ for all $1 \leq i \leq n$.

(\impliedby) If $\mu_i(V, f) = \mu_i(V, g)$ for all $1 \leq i \leq n$, let us consider two families of vectors $\bigcup_{1 \leq i \leq n} \{v_{s_i}\}_{s_i \in S_{\mu_i(V, f)}}$ and $\bigcup_{1 \leq i \leq n} \{w_{s_i}\}_{s_i \in S_{\mu_i(V, g)}}$ determining Jordan bases of V for f and g respectively.

Let $\tau \in \text{Aut}_k(V)$ be the automorphism defined by

$$\tau[f^r(v_{s_i})] = g^r(w_{s_i}),$$

for all $s_i \in S_{\mu_i(V, f)}$, $1 \leq i \leq n$ and $0 \leq r < i$.

By construction, one has that

$$(\tau f \tau^{-1})[g^r(w_{s_i})] = (\tau f)[f^r(v_{s_i})] = \tau[f^{r+1}(v_{s_i})] = g^{r+1}(w_{s_i}).$$

Accordingly, $\tau f \tau^{-1} = g$ and $[f] = [g] \in X_V^N / \text{Aut}_k(V)$. \square

Example 1. Let V be a k -vector space of countable dimension. If $Y = \{0\} \cup \mathbb{N} \cup \{\aleph_0\}$, \aleph_0 being the cardinal of the set of all natural numbers, one has a bijection

$$X_V^N / \text{Aut}_k(V) \longrightarrow \bigcup_{n \in \mathbb{N}} \left[\prod_n' Y \right],$$

$$[f] \longmapsto (\mu_1(V, f), \dots, \mu_n(V, f)),$$

where

$$\prod_n' Y = \{(y_1, \dots, y_n) \text{ with } y_i \in Y, y_n \neq 0, \text{ and } y_j = \aleph_0 \text{ for at least one } j\}.$$

Remark 3.11. If E is a finite-dimensional k -vector space, and $f \in \text{End}_k(E)$ is nilpotent of order n with invariants $\{\mu_i(V, f)\}_{1 \leq i \leq n}$, we should note that $\mu_i(V, f) = \nu_i(V, f)$ for all i (see Subsection 2.2), and, hence, the structure of E as a $k[x]$ -module induced by f is:

$$E_f \simeq (k[x]/x)^{\mu_1(V, f)} \oplus \dots \oplus (k[x]/x^i)^{\mu_i(V, f)} \oplus \dots \oplus (k[x]/x^n)^{\mu_n(V, f)}.$$

3.2. Invariants for finite potent endomorphisms

Let V be an arbitrary k -vector space, and let φ be a finite potent endomorphism of V . Let us consider the AST-decomposition of V induced by φ , that is: $V = U_\varphi \oplus W_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, W_φ is finite dimensional, and $\varphi|_{W_\varphi} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Let $\tau \in \text{Aut}_k(V)$ again be an automorphism of V .

Lemma 3.12. If $\bar{\varphi} = \tau \varphi \tau^{-1}$, and $V = U_{\bar{\varphi}} \oplus W_{\bar{\varphi}}$ is the AST-decomposition of V induced by $\bar{\varphi}$, then $\tau[U_\varphi] = U_{\bar{\varphi}}$ and $\tau[W_\varphi] = W_{\bar{\varphi}}$.

Proof. From the uniqueness of the AST-decomposition, since $\bar{\varphi}$ is nilpotent on $\tau[U_\varphi]$ and it is an isomorphism on $\tau[W_\varphi]$, the claim is deduced. \square

Corollary 3.13. If $\bar{\varphi} = \tau \varphi \tau^{-1}$, then:

- $\tau[\varphi|_{U_\varphi}] \tau^{-1} = \bar{\varphi}|_{U_{\bar{\varphi}}}$.
- $\tau[\varphi|_{W_\varphi}] \tau^{-1} = \bar{\varphi}|_{W_{\bar{\varphi}}}$.

With similar arguments to [Lemma 3.6](#) and [Corollary 3.7](#) one has that

Lemma 3.14. *If $\bar{\varphi} = \tau\varphi\tau^{-1}$, then:*

- $\tau[\text{Ker}(\varphi|_{U_\varphi})^r] = \text{Ker}(\bar{\varphi}|_{U_{\bar{\varphi}}})^r \subset U_{\bar{\varphi}}$ for all $r \geq 1$.
- $\tau[(\varphi|_{U_\varphi}) \text{Ker}(\varphi|_{U_\varphi})^i] = (\bar{\varphi}|_{U_{\bar{\varphi}}}) \text{Ker}(\bar{\varphi}|_{U_{\bar{\varphi}}})^i \subset U_{\bar{\varphi}}$ for all $i \geq 1$.

And, analogously, one can see that

Lemma 3.15. *If $\bar{\varphi} = \tau\varphi\tau^{-1}$, and $p(x) \in k[x]$, then:*

- $\tau[\text{Ker } p(\varphi|_{W_\varphi})^r] = \text{Ker } p(\bar{\varphi}|_{W_{\bar{\varphi}}})^r \subset W_{\bar{\varphi}}$ for all $r \geq 1$.
- $\tau[(\varphi|_{W_\varphi}) \text{Ker } p(\varphi|_{W_\varphi})^i] = p(\bar{\varphi}|_{W_{\bar{\varphi}}}) \text{Ker}(\bar{\varphi}|_{W_{\bar{\varphi}}})^i \subset W_{\bar{\varphi}}$ for all $i \geq 1$.

If $\varphi|_{U_\varphi}$ is a nilpotent endomorphism of order n of V_φ , and

$$a_{\varphi|_{W_\varphi}}(x) = p_1(x)^{n_1} \cdot \dots \cdot p_r(x)^{n_r}$$

is the annihilator polynomial of $\varphi|_{W_\varphi}$, let us consider the invariant factors that classify $\varphi|_{U_\varphi} \in X_{U_\varphi}^N$ and $\varphi|_{W_\varphi} \in \text{End}_k(W_\varphi)$:

- $\{\mu_i(U_\varphi, \varphi|_{U_\varphi})\}_{1 \leq i \leq n}$ (see [Subsection 3.1](#));
- $\{\nu_s(W_\varphi, p_j(\varphi|_{W_\varphi}))\}_{1 \leq j \leq r; 1 \leq s \leq n_j}$ (see [Subsection 2.2](#)).

It is clear that

Lemma 3.16. *Keeping the previous notation, if $\varphi|_{U_\varphi}$ is a nilpotent endomorphism of order n of U_φ , and $a_{\varphi|_{W_\varphi}}(x)$ is the annihilator polynomial of $\varphi|_{W_\varphi}$, then:*

1. $\bar{\varphi}|_{U_{\bar{\varphi}}}$ is also a nilpotent endomorphism of order n of $U_{\bar{\varphi}}$;
2. $a_{\varphi|_{W_\varphi}}(x) = a_{\bar{\varphi}|_{W_{\bar{\varphi}}}}(x)$.

Proposition 3.17. *If φ is a finite potent endomorphism of V , and $\bar{\varphi} = \tau\varphi\tau^{-1}$, one has that:*

- $\{\mu_i(U_\varphi, \varphi|_{U_\varphi})\} = \{\mu_i(U_{\bar{\varphi}}, \bar{\varphi}|_{U_{\bar{\varphi}}})\}$ for all $1 \leq i \leq n$;
- $\{\nu_s(W_\varphi, p_j(\varphi|_{W_\varphi}))\} = \{\nu_s(W_{\bar{\varphi}}, p_j(\bar{\varphi}|_{W_{\bar{\varphi}}}))\}$ for all $1 \leq j \leq r; 1 \leq s \leq n_j$.

Proof. Similarly to [Proposition 3.8](#), the claim is a direct consequence of the statements of [Lemma 3.14](#) and [Lemma 3.15](#) because if ψ is a finite potent endomorphism with

$$a_{\psi|_{W_\psi}}(x) = p_1(x)^{n_1} \cdot \dots \cdot p_r(x)^{n_r},$$

then

$$\{\mu_i(U_\psi, \psi|_{U_\psi})\} = \#(\text{Ker}(\psi|_{U_\psi})^i / [\text{Ker}(\psi|_{U_\psi})^{i-1} + \psi(\text{Ker}(\psi|_{U_\psi})^{i+1})])$$

and

$$\begin{aligned} &\{\nu_s(W_\psi, p_j(\psi|_{W_\psi}))\} \\ &= \dim_{K_j}(\text{Ker } p_j(\psi|_{W_\psi})^s / [\text{Ker } p_j(\psi|_{W_\psi})^{s-1} + p_j(\psi|_{W_\psi}) \text{Ker } p_j(\psi|_{W_\psi})^{s+1}]), \end{aligned}$$

where $K_j = k[x]/p_j(x)$. \square

Theorem 3.18 (Classification Theorem). Let $\varphi, \phi \in X_V^{fp}$ be two finite potent endomorphisms of an arbitrary k -vector space V . Thus, $\varphi \sim \phi \pmod{\text{Aut}_k(V)}$ if and only if

- $\{\mu_i(U_\varphi, \varphi|_{U_\varphi})\} = \{\mu_i(U_\phi, \phi|_{U_\phi})\}$ for all $1 \leq i \leq n$;
- $\{\nu_s(W_\varphi, p_j(\varphi|_{W_\varphi}))\} = \{\nu_s(W_\phi, p_j(\phi|_{W_\phi}))\}$ for all $1 \leq j \leq r; 1 \leq s \leq n_j$.

Proof. (\implies) If $[\varphi] = [\phi] \in X_V^{fp} / \text{Aut}_k(V)$, there exists $\tau \in \text{Aut}_k(V)$ such that $\phi = \tau\varphi\tau^{-1}$, and hence $\varphi|_{U_\varphi}$ and $\phi|_{U_\phi}$ have the same order of nilpotency, and $a_{\varphi|_{W_\varphi}}(x) = a_{\phi|_{W_\phi}}(x)$.

Accordingly, it follows from Proposition 3.17 that:

- $\{\mu_i(U_\varphi, \varphi|_{U_\varphi})\} = \{\mu_i(U_\phi, \phi|_{U_\phi})\}$ for all $1 \leq i \leq n$;
- $\{\nu_s(W_\varphi, p_j(\varphi|_{W_\varphi}))\} = \{\nu_s(W_\phi, p_j(\phi|_{W_\phi}))\}$ for all $1 \leq j \leq r; 1 \leq s \leq n_j$.

(\impliedby) Let us now assume that the invariant factors of φ and ϕ referred to in the statement are equal.

Let us consider families of vectors: $\bigcup_{1 \leq i \leq n} \{u_{s_i}\}_{s_i \in S_{\mu_i(U_\varphi, \varphi|_{U_\varphi})}}$ determining a Jordan basis of U_φ for $\varphi|_{U_\varphi}$; $\bigcup_{1 \leq i \leq n} \{v_{s_i}\}_{s_i \in S_{\mu_i(U_\phi, \phi|_{U_\phi})}}$ determining a Jordan basis of U_ϕ for $\phi|_{U_\phi}$;

$$\bigcup_{1 \leq j \leq r; 1 \leq s \leq n_j} \{w_h^{s_j}\}_{1 \leq h \leq \nu_s(W_\varphi, p_j(\varphi|_{W_\varphi}))}$$

generating a Jordan basis of W_φ for $\varphi|_{W_\varphi}$; and $\bigcup_{1 \leq j \leq r; 1 \leq s \leq n_j} \{e_h^{s_j}\}_{1 \leq h \leq \nu_s(W_\phi, p_j(\phi|_{W_\phi}))}$ generating a Jordan basis of W_ϕ for $\phi|_{W_\phi}$ (see (2.1) and (3.2)).

Thus, we can construct isomorphisms $\tau_1: U_\varphi \longrightarrow U_\phi$ and $\tau_2: W_\varphi \longrightarrow W_\phi$ from the following assignments:

$$\begin{aligned} \tau_1[\varphi^a(u_{s_i})] &= \phi^a(v_{s_i}), \\ \tau_2[p_j(\varphi)^b[\varphi^c(w_h^{s_j})]] &= p_j(\phi)^b[\phi^c(e_h^{s_j})] \end{aligned}$$

for all $1 \leq i \leq n; s_i \in S_{\mu_i(U_\varphi, \varphi|_{U_\varphi})}; 0 \leq a \leq i - 1; 1 \leq j \leq r; 1 \leq s \leq n_j; 1 \leq h \leq \nu_s(W_\phi, p_j(\phi|_{W_\phi}));$
 $0 \leq b \leq s - 1$ and $0 \leq c \leq d_j - 1$.

Moreover, since $V = U_\varphi \oplus W_\varphi$ and $V = U_\phi \oplus W_\phi$, there exists a unique automorphism $\tau \in \text{Aut}_k(V)$ such that $\tau|_{U_\varphi} = \tau_1$ and $\tau|_{W_\varphi} = \tau_2$. Thus, an easy check shows that $\phi = \tau\varphi\tau^{-1}$, and, therefore, $[\varphi] = [\phi] \in X_V^{fp} / \text{Aut}_k(V)$. \square

Remark 3.19. If V and \tilde{V} are two k -vector spaces, and $\psi: V \longrightarrow \tilde{V}$ is an isomorphism, it follows from the above statements that the induced map

$$\begin{aligned} \psi_*: X_V^{fp} / \text{Aut}_k(V) &\longrightarrow X_{\tilde{V}}^{fp} / \text{Aut}_k(\tilde{V}), \\ [\varphi] &\longmapsto [\psi\varphi\psi^{-1}] \end{aligned}$$

is bijective.

Example 2. Let V be a k -vector space of countable dimension. For each finite potent endomorphism $\varphi \in X_V^{fp}$, since W_φ is a finite-dimensional vector space, one has that V and U_φ are isomorphic as k -vector spaces.

For each $n \in \mathbb{N}$, let E_n be a k -vector space with $\dim_k(E_n) = n$. If we consider the quotient sets $X_n = \text{End}_k(E_n) / \text{Aut}_k(E_n)$, Theorem 3.18 shows that

$$X_V^{fp} / \text{Aut}_k(V) = X_V^N / \text{Aut}_k(V) \times \left[\bigcup_{n \in \mathbb{N}} X_n \right],$$

$X_V^N / \text{Aut}_k(V)$ being the set characterized in [Example 1](#).

3.2.1. Trace and determinant of finite potent endomorphisms

Given an arbitrary k -vector space V , for each finite potent endomorphism φ on V , and from the AST-decomposition $V = U_\varphi \oplus W_\varphi$, it is possible to define a trace $\text{tr}_V(\varphi)$ and a determinant $\det_V^k(1 + \varphi)$ (Subsection 2.1).

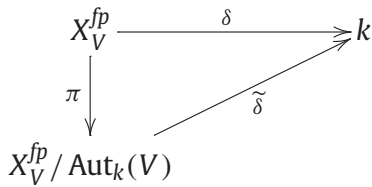
If $\tau \in \text{Aut}_k(V)$, it is known that

$$\det_V^k(1 + \tau\varphi\tau^{-1}) = \det_V^k(1 + \varphi) \quad ([3], \text{Lemma 3.13}).$$

Thus, considering the map

$$\begin{aligned} \delta: X_V^{fp} &\longrightarrow k, \\ \varphi &\longmapsto \det_V^k(1 + \varphi), \end{aligned}$$

the commutative diagram



makes sense with $\tilde{\delta}([\varphi]) = \det_V^k(1 + \varphi)$, π being the quotient map.

On the other hand, if $c_{\varphi|_{W_\varphi}}(x)$ is the characteristic polynomial of the endomorphism $\varphi|_{W_\varphi} \in \text{End}_k(W_\varphi)$, it is clear that

$$c_{\varphi|_{W_\varphi}}(x) = c_{\phi|_{W_\phi}}(x), \tag{3.3}$$

for all endomorphisms $\varphi \sim \phi \pmod{\text{Aut}_k(V)}$.

Since $c_{\varphi|_{W_\varphi}}(x) = x^m - \text{tr}_V(\varphi)x^{m-1} + \dots$, with $m = \dim(W_\varphi)$, from (3.3) we deduce that

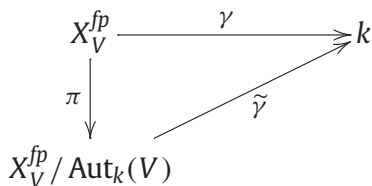
$$\text{tr}_V(\varphi) = \text{tr}_V(\phi),$$

for all $\varphi, \phi \in X_V^{fp}$ such that $[\varphi] = [\phi] \in X_V^{fp} / \text{Aut}_k(V)$.

Then, writing

$$\begin{aligned} \gamma: X_V^{fp} &\longrightarrow k, \\ \varphi &\longmapsto \text{tr}_V(\varphi), \end{aligned}$$

one has the following commutative diagram



with $\tilde{\gamma}([\varphi]) = \text{tr}_V(\varphi)$.

Remark 3.20 (*Final Consideration*). We have shown that the determinant and the trace of a finite potent operator are invariant under the classification offered. It should be noted that this invariance also holds for all the coefficients of the characteristic polynomial $c_{\varphi|W_\varphi}(x)$. Thus, it makes sense to define objects from these coefficients, and to explore their properties, to attempt to obtain new intrinsic reciprocity laws using similar arguments to the proofs of the Residue Theorem [5] or the Reciprocity Law for the Segal–Wilson pairing [3].

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