Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Pure and Applied Algebra 205 (2006) 94–116
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

# 3-Cocycles, symbols and reciprocity laws on curves☆

Fernando Pablos Romo

*Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4,37008 Salamanca, España*

Received 27 February 2005; received in revised form 28 April 2005
Available online 29 August 2005
Communicated by A.S. Merkurjev

**Abstract**

We introduce a new approach for the study of two-dimensional symbols, $\mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* \to G$, where $\mathscr{F}$ is a discrete valuation field and $G$ is a commutative group. From central extensions of groups we obtain a three-cocycle $\{\cdot,\cdot,\cdot\}^{\varphi}_{v_{\mathscr{F}},v_{\mathscr{K}}}$ and the symbol is a differentiated element of the cohomology class $[\{\cdot,\cdot,\cdot\}^{\varphi}_{v_{\mathscr{F}},v_{\mathscr{K}}}] \in H^3(\mathscr{F}^*, G)$. Our construction generalizes well-known two-dimensional symbols, such as the Parshin symbol on a surface, and we offer a proof and a conjecture for reciprocity laws on curves related to these symbols.


*MSC:* 19F15; 14H05; 19C09

## 1. Introduction

The aim of this work is to contribute to a better understanding of two-dimensional Steinberg symbols. Given a field $\mathscr{F}$ and a commutative group $G$, a two-dimensional

☆ This work is partially supported by the DGESYC research contract no. BFM2003-00078 and Castilla y León regional government contract SA071/04.

*E-mail address:* fpablos@usal.es.

0022-4049/$ - see front matter 
doi:10.1016/j.jpaa.2005.06.004

Steinberg symbol is a map $\psi: \mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* \to G$ such that:

- $\psi$ is multiplicative in each argument:

$$\psi(f_1 \cdot f_2, g, h) = \psi(f_1, g, h) \cdot \psi(f_2, g, h),$$
$$\psi(f, g_1 \cdot g_2, h) = \psi(f, g_1, h) \cdot \psi(f, g_2, h),$$
$$\psi(f, g, h_1 \cdot h_2) = \psi(f, g, h_1) \cdot \psi(f, g, h_2)$$

  for all $f_i, g_i, h_i \in \mathscr{F}^*$.
- $\psi(f, 1-f, g) = \psi(f, g, 1-f) = \psi(g, f, 1-f) = 1$ for all $f \neq 1$.

Parshin [11], and Brylinsky and McLaughlin [2] have studied these symbols on surfaces and have offered reciprocity laws for them. Explicitly, in 1985 Parshin introduced a symbol associated with a sequence $p \in C \subset S$, where $C$ is a curve on an algebraic surface $S$, and $p$ is a closed point of $C$. If $f$, $g$ and $h$ are three functions on $S$, the expression of the symbol is $\langle f, g, h\rangle_{(p,C)} =$

$$(-1)^{\alpha_{(p,C)}} \left( \frac{f^{v_C(g)\cdot \overline{v_p}(h) - v_C(h)\cdot \overline{v_p}(g)}}{g^{v_C(f)\cdot \overline{v_p}(h) - v_C(h)\cdot \overline{v_p}(f)}} \cdot h^{v_C(f)\cdot \overline{v_p}(g) - v_C(g)\cdot \overline{v_p}(f)} \right)_{|C} (p),$$

where

$$\alpha_{(p,C)} = v_C(f)\cdot v_C(g)\cdot \overline{v_p}(h) + v_C(f)\cdot v_C(h)\cdot \overline{v_p}(g) + v_C(g)\cdot v_C(h)\cdot \overline{v_p}(f) + v_C(f)\cdot \overline{v_p}(g)\cdot \overline{v_p}(h) + v_C(g)\cdot \overline{v_p}(f)\cdot \overline{v_p}(h) + v_C(h)\cdot \overline{v_p}(f)\cdot \overline{v_p}(g),$$

$v_C$ being the discrete valuation induced by $C$ (a codimension one subvariety of $S$), and $\overline{v_p}$ being a discrete valuation induced by the closed point $p$ and a function $z$ on $S$, such that $v_C(z) = 1$.

We point out that this explicit expression is not completely due to Parshin: i.e., the higher-dimensional tame symbol was defined by Parshin up to the sign, and the full definition, including the sign, was given by Fesenko in his thesis in 1986 and published in 1988 [4] (for the English translation, see [5]).

This symbol is independent of the choice of $z$, and it satisfies the reciprocity laws:

(1) $\prod_p \langle f, g, h\rangle_{(p,C)} = 1$,
(2) $\prod_C \langle f, g, h\rangle_{(p,C)} = 1$,

where $C$ is a complete, irreducible and non-singular curve in (1), and the second reciprocity law is the product over all irreducible curves containing a fixed point $p \in S$. Moreover, in 1996 Brylinski and McLaughlin [2] interpreted the expression of this symbol as the holonomy of a gerbe around a torus and provided a new proof of the above reciprocity laws. Recently, the author [10] offered a new interpretation of that symbol as iterated tame symbols in order to deduce its first reciprocity law from the finiteness of the cohomology groups $H^0(C, \mathscr{O}_C)$ and $H^1(C, \mathscr{O}_C)$. Here we offer a definition of Steinberg symbols on discrete valuation fields that generalizes the Parshin symbol on a surface. From “tame central extensions” associated with two discrete valuation fields, $\mathscr{F}$ and $\mathscr{K}$ (related with each other), we obtain a 3-cocycle $\{\cdot,\cdot,\cdot\}^{\varphi}_{v_{\mathscr{F}}, v_{\mathscr{K}}}$ that coincides, except for the sign, with the expected expression of the symbol. The main result of this work (Theorem 2.15 and Corollary 2.16) shows that the symbol offered is a differentiated element of the cohomology class $[\{\cdot,\cdot,\cdot\}^{\varphi}_{v_{\mathscr{F}},v_{\mathscr{K}}}] \in H^3(\mathscr{F}^*, G)$. Indeed, this symbol is the only Steinberg symbol in the cohomology class that satisfies a usual property in the theory of arithmetical symbols. Several examples of geometric two-dimensional Steinberg symbols are provided, and we prove, with the same method as Tate's proof of the Residue Theorem [13], a reciprocity law that generalizes the first Parshin reciprocity law. Finally, we formulate a conjecture in order to generalize the second Parshin reciprocity law.

## 2. 3-Cocyles and symbols

### 2.1. Central extensions and 3-cocycles

Given a central extension of groups

$$1 \to \Pi \longrightarrow \widetilde{H} \xrightarrow{\phi} H \to 1,$$

if $h_1$ and $h_2$ are two commuting elements of $H$, and $\widetilde{h_1}, \widetilde{h_2} \in \widetilde{H}$ are elements such that $\phi(\widetilde{h_1}) = h_1$ and $\phi(\widetilde{h_2}) = h_2$, then one has a commutator pairing:

$$\{h_1, h_2\}_{\widetilde{H}} = \widetilde{h_1} \cdot \widetilde{h_2} \cdot \widetilde{h_1}^{-1} \cdot \widetilde{h_2}^{-1} \in \Pi.$$

Let us now consider three commutative groups $H$, $\Pi$ and $G$.

If we have two central extensions of groups,

$$1 \to \Pi \to \widetilde{H} \to H \to 1, \tag{2.1}$$

$$1 \to G \to \widetilde{\Pi} \to \Pi \to 1, \tag{2.2}$$

by fixing a morphism of groups $\tau: H \to \mathbb{Z}$ and an element $z \in H$ we can define the map

$$\{\cdot,\cdot,\cdot\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}: H \times H \times H \longrightarrow G$$

by the way:

$$\{f_{i_0}, f_{i_1}, f_{i_2}\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \prod_{j\in\mathbb{Z}/3} [\{\{f_{i_j}, z\}_{\widetilde{H}}, \{f_{i_{j+1}}, z\}_{\widetilde{H}}\}_{\widetilde{\Pi}}]^{\tau(f_{i_{j+2}})},$$

where $\{\cdot,\cdot\}_{\widetilde{H}}$ is the commutator of the central extension (2.1); where $\{\cdot,\cdot\}_{\widetilde{\Pi}}$ is the commutator of the central extension (2.2), and where $[\dots]^{\tau(f)}$ means a power with exponent $\tau(f)$, which makes sense because $\tau(f)$ is an integer number.

Bearing in mind that the commutator of a central extension is a bimultiplicative map, one has that $\{\cdot,\cdot,\cdot\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}$ is multiplicative in each argument:

$$\{f_1 \cdot f_2, g, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{f_1, g, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}\{f_2, g, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}},$$
$$\{f, g_1 \cdot g_2, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{f, g_1, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}\{f, g_2, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}},$$
$$\{f, g, h_1 \cdot h_2\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{f, g, h_1\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}\{f, g, h_2\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}},$$

with $f_i, g_i, h_i \in H$. Moreover, since the commutator of a central extension is skew-symmetric, the map $\{\cdot,\cdot,\cdot\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}$ satisfies the property

$$\{f, f, g\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{f, g, f\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{g, f, f\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = 1$$

for all $f, g \in H$. Furthermore, from the definition of $\{\cdot,\cdot,\cdot\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}$, one also has that

$$\{f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = [\{f_1, f_2, f_3\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}]^{\operatorname{sign}\sigma}$$

for any permutation $\sigma$.

**Lemma 2.1.** *The map* $\{\cdot,\cdot,\cdot\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}$ *is a* 3*-cocycle.*

**Proof.** It is clear that

$$\{g, h, t\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} \cdot \{f, g\cdot h, t\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} \cdot \{f, g, h\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} = \{f\cdot g, h, t\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}} \cdot \{f, g, h\cdot t\}^{\tau,z}_{\widetilde{H},\widetilde{\Pi}}$$

which is the definition of a 3-cocycle [3]. □

*2.2. 3-Cocycles on discrete valuation fields*

Let $\mathscr{F}$ be a discrete valuation field and let $\mathscr{K}(v_{\mathscr{F}})$ be its residue class field. The valuation ring associated with $v_{\mathscr{F}}$ is denoted by $\mathscr{O}_{v_{\mathscr{F}}}$, and $\mathfrak{m}_{v_{\mathscr{F}}}$ is its maximal ideal. Let us consider a field $\mathscr{K}$, such that $\mathscr{K}(v_{\mathscr{F}})$ is a finite separable extension of $\mathscr{K}$, $\deg(v_{\mathscr{F}}) = \dim_{\mathscr{K}} \mathscr{K}(v_{\mathscr{F}})$, and $N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}$ is the norm of the extension of fields $\mathscr{K} \hookrightarrow \mathscr{K}(v_{\mathscr{F}})$.

**Definition 2.2.** We shall use the term “tame central extension” to refer to a central extension of groups

$$1 \to \mathscr{K}^* \to \widetilde{\mathscr{F}}^* \to \mathscr{F}^* \to 1,$$

such that its commutator is

$$\{f, g\}_{\widetilde{\mathscr{F}}^*} = N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left[\frac{f^{v_{\mathscr{F}}(g)}}{g^{v_{\mathscr{F}}(f)}} (\operatorname{mod} \mathfrak{m}_{v_{\mathscr{F}}})\right] \in \mathscr{K}^* \quad \text{for } f, g \in \mathscr{F}^*.$$

**Remark 2.3.** Let $C$ be an irreducible and non-singular curve over a perfect field $k$ and let $x \in C$ be a closed point on it that defines a discrete valuation $v_x$ on $\Sigma_C$ (the function field of $C$). We use $k(x)$ to denote the residue class field of $x$, which is a finite separable extension of $k$, with $\deg(x) = \dim_k k(x)$. Let us consider $\hat{\mathscr{O}}_x = A_x$ (the completion of the local ring $\mathscr{O}_x$) and $(\hat{\mathscr{O}}_x)_0 = K_x$ (the field of fractions of $\hat{\mathscr{O}}_x$, which coincides with the completion of $\Sigma_C$ with respect to the valuation ring $\mathscr{O}_x$). By using commensurable subspaces, it follows from the results of [8] that there exists a central extension of groups

$$1 \to k^* \to \widetilde{\Sigma_C^*} \to \Sigma_C^* \to 1,$$

whose commutator is

$$\{f, g\}_{A_x}^{K_x} = N_{k(x)/k}\left(\frac{f^{v_x(g)}}{g^{v_x(f)}}(x)\right) \in k^*.$$

Thus, this exact sequence of groups is a tame central extension, and we should note that it was defined by Arbarello et al. [1] for an algebraically closed ground field. In this case, the commutator $\{f, g\}_{A_x}^{K_x}$ coincides, except for the sign, with the tame symbol

$$(\cdot,\cdot)_{v_x} : \Sigma_C^* \times \Sigma_C^* \to k(x)^*,$$

defined by Milnor [7]. Moreover, we recall that the main results of the theory of commensurable infinite subspaces are related to the integer number

$$\operatorname{ind}_{A_x}^{K_x}(f) = \dim_k(A_x/A_x \cap fA_x) - \dim_k(fA_x/A_x \cap fA_x) = \deg(x)v_x(f),$$

where $f \in \Sigma_C^*$, and $A_x$, $fA_x$ are subspaces of $K_x$.

**Proposition 2.4.** *Let $\mathscr{F}$ be an arbitrary field with a discrete valuation $v_{\mathscr{F}}$, whose residue class field is $\mathscr{K}(v_{\mathscr{F}})$. For each field $\mathscr{K}$ such that $\mathscr{K}(v_{\mathscr{F}})$ is a finite separable extension of $\mathscr{K}$ there exists a tame central extension associated with $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$.*

**Proof.** This follows from the statements in [10] that there exists a central extension of groups

$$1 \to \mathscr{K}(v_{\mathscr{F}})^* \to \widetilde{\mathscr{F}}_v^* \to \mathscr{F}^* \to 1 \tag{2.3}$$

such that its commutator is

$$\{f, g\}_{\widetilde{\mathscr{F}}_v^*} = \{f, g\}_{\mathfrak{m}_{v_{\mathscr{F}}}}^{\mathcal{O}_{v_{\mathscr{F}}}} = \frac{f^{v_{\mathscr{F}}(g)}}{g^{v_{\mathscr{F}}(f)}} (\operatorname{mod} \mathfrak{m}_{v_{\mathscr{F}}}) \in \mathscr{K}(v_{\mathscr{F}})^* \quad \text{for } f, g \in \mathscr{F}^*.$$

Hence, from 2.3 the morphism of groups $N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}} : \mathscr{K}(v_{\mathscr{F}})^* \to \mathscr{K}^*$ determines a tame central extension

$$1 \to \mathscr{K}^* \to \widetilde{\mathscr{F}}^* \to \mathscr{F}^* \to 1,$$

because its commutator is

$$\{f, g\}_{\widetilde{\mathscr{F}}^*} = \{f, g\}_{\mathfrak{m}_{v_{\mathscr{F}}}, N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}}^{\mathcal{O}_{v_{\mathscr{F}}}} = N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left[\frac{f^{v_{\mathscr{F}}(g)}}{g^{v_{\mathscr{F}}(f)}} (\operatorname{mod} \mathfrak{m}_{v_{\mathscr{F}}})\right] \in \mathscr{K}^*$$

for all $f, g \in \mathscr{F}^*$. □

If we now consider two tame central extensions of groups,

$$1 \to \mathscr{K}^* \to \widetilde{\mathscr{F}}^* \to \mathscr{F}^* \to 1, \tag{2.4}$$

$$1 \to k^* \to \widetilde{\mathscr{K}}^* \to \mathscr{K}^* \to 1, \tag{2.5}$$

fixing an element $z \in \mathscr{F}^*$ such that $v_{\mathscr{F}}(z) = 1$, and the group morphism

$$\begin{aligned} -\mathrm{ind}^{\mathscr{K}}_{\mathscr{K}(v_{\mathscr{F}})} \colon \mathscr{F}^* &\longrightarrow \mathbb{Z} \\ f &\longmapsto -v_{\mathscr{F}}(f)\cdot \deg(v_{\mathscr{F}})' \end{aligned}$$

by using the method of the previous section, we obtain an induced 3-cocycle

$$\{\cdot,\cdot,\cdot\}^{-\mathrm{ind}^{\mathscr{K}}_{\mathscr{K}(v_{\mathscr{F}})},z}_{\widetilde{\mathscr{F}}^*,\widetilde{\mathscr{K}}^*} \colon \mathscr{F}^*\times\mathscr{F}^*\times\mathscr{F}^* \longrightarrow k^*.$$

To simplify, we denote this 3-cocycle by $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$. Moreover, if $f \in \mathscr{F}^*$, we write

$$f(v_{\mathscr{F}}, v_{\mathscr{K}}) = N_{k(v_{\mathscr{K}})/k}[N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(f(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{F}}}))(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{K}}})] \in k^*,$$

which is a well-defined map when $v_{\mathscr{F}}(f) = 0$ and

$$v_{\mathscr{K}}[N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(f(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{F}}}))] = 0.$$

**Proposition 2.5.** *If $f, g, h \in \mathscr{F}^*$, and $v^z_{\mathscr{K}}(f) = v_{\mathscr{K}}(\{f,z\}_{\widetilde{\mathscr{F}}^*})$, the value of the 3-cocycle $\{f,g,h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ is*

$$\left(\frac{f^{[v_{\mathscr{F}}(g)\cdot v^z_{\mathscr{K}}(h) - v_{\mathscr{F}}(h)\cdot v^z_{\mathscr{K}}(g)]}}{g^{[v_{\mathscr{F}}(f)\cdot v^z_{\mathscr{K}}(h) - v_{\mathscr{F}}(h)\cdot v^z_{\mathscr{K}}(f)]}} \cdot h^{[v_{\mathscr{F}}(f)\cdot v^z_{\mathscr{K}}(g) - v_{\mathscr{F}}(g)\cdot v^z_{\mathscr{K}}(f)]}\right)^{\deg(v_{\mathscr{F}})}(v_{\mathscr{F}}, v_{\mathscr{K}}) \in k^*.$$

**Proof.** Since $\{f,z\}_{\widetilde{\mathscr{F}}^*} = N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}[\frac{f}{z^{v_{\mathscr{F}}(f)}}(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{F}}})]$, one has that

$$\begin{aligned} &\{\{f,z\}_{\widetilde{\mathscr{F}}^*}, \{g,z\}_{\widetilde{\mathscr{F}}^*}\}_{\widetilde{\mathscr{K}}^*} \\ &= N_{k(v_{\mathscr{K}})/k}\left(\frac{N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left[\frac{f}{z^{v_{\mathscr{F}}(f)}}(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{F}}})\right]^{v^z_{\mathscr{K}}(g)}}{N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left[\frac{g}{z^{v_{\mathscr{F}}(g)}}(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{F}}})\right]^{v^z_{\mathscr{K}}(f)}}(\mathrm{mod}\,\mathfrak{m}_{v_{\mathscr{K}}})\right) \\ &= \left(\frac{f^{v^z_{\mathscr{K}}(g)}}{g^{v^z_{\mathscr{K}}(f)}} \cdot z^{v_{\mathscr{F}}(g)\cdot v^z_{\mathscr{K}}(f) - v_{\mathscr{F}}(f)\cdot v^z_{\mathscr{K}}(g)}\right)(v_{\mathscr{F}}, v_{\mathscr{K}}), \end{aligned}$$

which is a well-defined element of $k^*$. Bearing in mind the definition of the 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$, a computation shows that the statement holds. □

Note that $v^z_{\mathscr{K}}$ is a discrete valuation field on $\mathscr{F}$ when $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$.

**Corollary 2.6.** *The 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ is independent of the choice of $z$.*

**Proof.** If we replace $z$ with another $z' \in \mathscr{F}^*$ such that $v_{\mathscr{F}}(z') = 1$, we have induced a transformation $v^z_{\mathscr{K}} \longmapsto v^{z'}_{\mathscr{K}} + \lambda v_{\mathscr{F}}$, for some integer $\lambda$. Thus, bearing in mind the explicit expression computed in the previous Proposition, one has that $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ is invariant under transformations $v^z_{\mathscr{K}} \longmapsto v^{z'}_{\mathscr{K}} + \lambda v_{\mathscr{F}}$, and hence it is independent of the choice of $z$. □

**Lemma 2.7.** *Fixing an element $g \in \mathscr{F}^*$, let us consider the morphism of groups*

$$\begin{aligned} v^g_{\mathscr{K}}\colon \mathscr{F}^* &\to \mathbb{Z} \\ f &\mapsto v_{\mathscr{K}}(\{f, g\}_{\widetilde{\mathscr{F}}^*}). \end{aligned}$$

*One has that*:

(1) $v^g_{\mathscr{K}}(f) = v_{\mathscr{F}}(g) \cdot v^z_{\mathscr{K}}(f) - v_{\mathscr{F}}(f) \cdot v^z_{\mathscr{K}}(g)$ *for all* $f, g \in \mathscr{F}^*$.
(2) *If* $f, h \in \mathscr{F}^*$ *and* $f \cdot h^{-1} \equiv -1 \,(\mathrm{mod}\, \mathfrak{m}_{v_{\mathscr{F}}})$, *then* $v^g_{\mathscr{K}}(f) = v^g_{\mathscr{K}}(h)$ *for each* $g \in \mathscr{F}^*$.

**Proof.** If $z \in \mathscr{F}^*$ with $v_{\mathscr{F}}(z) = 1$, and $f, g \in \mathscr{F}^*$, the first part of the statement can be deduced from the computation:

$$\begin{aligned} v^g_{\mathscr{K}}(f) &= v_{\mathscr{K}}\left(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left(\frac{f^{v_{\mathscr{F}}(g)}}{g^{v_{\mathscr{F}}(f)}}\,(\mathrm{mod}\, \mathfrak{m}_{v_{\mathscr{F}}})\right)\right) \\ &= v_{\mathscr{K}}\left(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left(\frac{f}{z^{v_{\mathscr{F}}(f)}}\,(\mathrm{mod}\, \mathfrak{m}_{v_{\mathscr{F}}})\right)^{v_{\mathscr{F}}(g)}\right) \\ &\quad - v_{\mathscr{K}}\left(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left(\frac{g}{z^{v_{\mathscr{F}}(g)}}\,(\mathrm{mod}\, \mathfrak{m}_{v_{\mathscr{F}}})\right)^{v_{\mathscr{F}}(f)}\right) \\ &= v_{\mathscr{F}}(g) \cdot v^z_{\mathscr{K}}(f) - v_{\mathscr{F}}(f) \cdot v^z_{\mathscr{K}}(g). \end{aligned}$$

Moreover, if $f, h \in \mathscr{F}^*$ and $f \cdot h^{-1} \equiv -1 \,(\mathrm{mod}\, \mathfrak{m}_{v_{\mathscr{F}}})$, then $v^g_{\mathscr{K}}(f \cdot h^{-1}) = 0$, and thus $v^g_{\mathscr{K}}(f) = v^g_{\mathscr{K}}(h)$ for all $g \in \mathscr{F}^*$. □

**Remark 2.8.** It follows from Lemma 2.7 that another explicit expression of the 3-cocycle $\{\cdot, \cdot, \cdot\}^z_{v_{\mathscr{F}}, v_{\mathscr{K}}}$ is

$$\begin{aligned} \{f, g, h\}^z_{v_{\mathscr{F}}, v_{\mathscr{K}}} &= (f^{v_{\mathscr{K}}(\{h,g\}_{\widetilde{\mathscr{F}}^*})} \cdot g^{v_{\mathscr{K}}(\{f,h\}_{\widetilde{\mathscr{F}}^*})} \cdot h^{v_{\mathscr{K}}(\{g,f\}_{\widetilde{\mathscr{F}}^*})})^{\deg(v_{\mathscr{F}})}(v_{\mathscr{F}}, v_{\mathscr{K}}) \\ &= (f^{v^g_{\mathscr{K}}(h)} \cdot g^{v^h_{\mathscr{K}}(f)} \cdot h^{v^f_{\mathscr{K}}(g)})^{\deg(v_{\mathscr{F}})}(v_{\mathscr{F}}, v_{\mathscr{K}}) \in k^*, \end{aligned}$$

which is clearly independent of the choice of $z$.

**Remark 2.9.** With the hypothesis of Proposition 2.5, for each morphism of commutative groups $\varphi\colon k^* \to G$, the tame central extension of groups associated with $(\mathscr{K}, v_{\mathscr{K}}, k)$ determines another central extension

$$1 \to G \to \widetilde{\mathscr{K}}^*_{\varphi} \to \mathscr{K}^* \to 1 \tag{2.6}$$

such that the 3-cocycle, induced by the tame central extension associated with $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$ and the central extension 2.6, is

$$\{f, g, h\}^z_{v_{\mathscr{F}}, v_{\mathscr{K}}, \varphi} = \varphi(\{f, g, h\}^z_{v_{\mathscr{F}}, v_{\mathscr{K}}}),$$

which is an element of $G$ and is also independent of the choice of $z$.

### 2.3. Steinberg symbols

This subsection is devoted to constructing symbols from the 3-cocycles $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ and $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ referred to previously. With the above notations, let us now consider a map

$$\psi\colon \mathscr{F}^*\times\mathscr{F}^*\times\mathscr{F}^*\to G.$$

We say that $\psi$ satisfies the Steinberg relations when:

(1) $\psi$ is multiplicative in each argument:

$$\psi(f_1\cdot f_2,g,h)=\psi(f_1,g,h)\cdot\psi(f_2,g,h),$$
$$\psi(f,g_1\cdot g_2,h)=\psi(f,g_1,h)\cdot\psi(f,g_2,h),$$
$$\psi(f,g,h_1\cdot h_2)=\psi(f,g,h_1)\cdot\psi(f,g,h_2)$$

for all $f_i,g_i,h_i\in\mathscr{F}^*$.

(2) $\psi(f,1-f,g)=\psi(f,g,1-f)=\psi(g,f,1-f)=1$ for all $f\neq 1$.

**Definition 2.10.** A map $\psi\colon\mathscr{F}^*\times\mathscr{F}^*\times\mathscr{F}^*\to G$ is called a "Steinberg symbol" when it satisfies the Steinberg relations.

**Remark 2.11.** It follows from its definition that a Steinberg symbol also satisfies the relations:

- $\psi(f_{\sigma(1)},f_{\sigma(2)},f_{\sigma(3)})=\psi(f_1,f_2,f_3)^{\operatorname{sign}\sigma}$ for any permutation $\sigma$.
- $\psi(f,g,g)=\psi(f,g,-1)$ and $\psi(f,g,-g)=1$.

The 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ satisfies the first property of the list of Steinberg relations. However, $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ is not a Steinberg symbol because it follows from Proposition 2.5 that:

$$\{f,g,-g\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}=(-1)^{\deg(v_{\mathscr{F}})\deg(v_{\mathscr{K}})[v_{\mathscr{F}}(f)v^z_{\mathscr{K}}(g)+v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)]}.$$

We shall now give a cohomological definition of a Steinberg symbol from the 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$. According to the definition of the 3rd-cohomology group, $H^3(\mathscr{F}^*,k^*)=Z^3(\mathscr{F}^*,k^*)/B^3(\mathscr{F}^*,k^*)$ ([3, p. 53]), one has that $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ determines a cohomology class

$$[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}]\in H^3(\mathscr{F}^*,k^*).$$

Moreover, this symbol is independent of the choice of $z$. We should recall that $\bar{c}\in Z^3(\mathscr{F}^*,k^*)$ is a 3-coboundary—i.e., $\bar{c}\in B^3(\mathscr{F}^*,k^*)$—if there exists a function on two variables $\phi$ on $\mathscr{F}^*$ to $k^*$ such that

$$\bar{c}(x,y,z)=(\delta\phi)(x,y,z)=\phi(y,z)\phi(x\cdot y,z)^{-1}\phi(x,y\cdot z)\phi(x,y)^{-1}$$

for all $x,y,z\in\mathscr{F}^*$.

**Lemma 2.12.** *There exists a unique* 3*-cocycle* $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ *in the cohomology class* $[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}] \in H^3(\mathscr{F}^*, k^*)$ *satisfying the conditions*:

(1) $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ *is multiplicative in each argument.*
(2) $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = [(f_1, f_2, f_3)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}]^{\operatorname{sign}\sigma}$ *for any* $\sigma \in S_3$.
(3) $(f, g, -g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = 1$ *for all* $f, g \in \mathscr{F}^*$.
(4) $(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ if $v_{\mathscr{F}}(f) = v^z_{\mathscr{K}}(f) = 0$.

**Proof.** Since $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} \in [\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}]$, one has that

$$(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = c(f, g, h) \cdot \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}},$$

$c$ being a 3-coboundary. It follows from the properties of $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ and the above hypothesis that $c$ is multiplicative in each argument and that

$$c(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}) = c(f_1, f_2, f_3)^{\operatorname{sign}\sigma}$$

for any permutation $\sigma$. Let us now consider the morphism of groups:

$$v_{\mathscr{F}} \times v^z_{\mathscr{K}} : \mathscr{F}^* \longrightarrow \mathbb{Z} \times \mathbb{Z}.$$

Bearing in mind that $c(f, g, h) = 1$ when $v_{\mathscr{F}}(f) = v^z_{\mathscr{K}}(f) = 0$, and that the same property holds when $v_{\mathscr{F}}(g) = v^z_{\mathscr{K}}(g) = 0$ and when $v_{\mathscr{F}}(h) = v^z_{\mathscr{K}}(h) = 0$, one has a commutative diagram of morphisms of groups:

$$\begin{array}{ccc} \mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* & & \\ {\scriptstyle \prod_{i=1}^3 (v_{\mathscr{F}} \times v^z_{\mathcal{K}})} \downarrow & \searrow^{c} & \\ \prod_{i=1}^3 (\mathbb{Z} \times \mathbb{Z}) & \xrightarrow{\widetilde{c}} & k^*, \end{array}$$

where $\widetilde{c}$ is a 3-coboundary satisfying the properties:

- $\widetilde{c}(x_1 + x_2, y, h) = \widetilde{c}(x_1, y, h) \cdot \widetilde{c}(x_2, y, h)$;
- $\widetilde{c}(x, y_1 + y_2, h) = \widetilde{c}(x, y_1, h) \cdot \widetilde{c}(x, y_2, h)$;
- $\widetilde{c}(x, y, h_1 + h_2) = \widetilde{c}(x, y, h_1) \cdot \widetilde{c}(x, y, h_2)$;
- $\widetilde{c}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \widetilde{c}(x_1, y_2, z_3)^{\operatorname{sign}\sigma}$ for any permutation $\sigma$

for all $x_i, y_j, h_k \in \mathbb{Z} \times \mathbb{Z}$. To simplify, we put $a = \deg(v_{\mathscr{F}}) \cdot \deg(v_{\mathscr{K}})$. Moreover, it follows from the expression

$$\{f, g, -g\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = (-1)^{a[v_{\mathscr{F}}(f)v^z_{\mathscr{K}}(g) + v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)]}$$

that $\widetilde{c}(x, y, y) = (-1)^{a[x_1y_2 + y_1x_2]}$ for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{Z} \times \mathbb{Z}$. Thus,

$$\widetilde{c}(x, (1, 0), (1, 0)) = (-1)^{a \cdot x_2},$$
$$\widetilde{c}(x, (0, 1), (0, 1)) = (-1)^{a \cdot x_1}$$

for all $x = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z}$. Furthermore, $\widetilde{c}(x, (1, 1), (1, 1)) = (-1)^{a(x_1 + x_2)}$, and hence

$$\widetilde{c}(x, (0, 1), (1, 0)) = \widetilde{c}(x, (1, 0), (0, 1))^{-1}.$$

One also has that

$$\widetilde{c}(x,(1,0),(1,0)) = (-1)^{a\cdot x_2} = \widetilde{c}((1,0),(1,0),x_1(1,0)+x_2(0,1)) = \widetilde{c}((1,0),(1,0),(0,1))^{x_2} \quad \text{for all } x \in \mathbb{Z}\times\mathbb{Z}.$$

Therefore, $\widetilde{c}((1,0),(1,0),(0,1)) = (-1)^a$. Moreover, carrying out a similar computation for $\widetilde{c}(x,(0,1),(0,1))$, one sees that $\widetilde{c}((0,1),(0,1),(1,0)) = (-1)^a$. Hence, $\widetilde{c}(x,(1,0),(0,1)) = \widetilde{c}(x,(0,1),(1,0)) = (-1)^{a(x_1+x_2)}$, and the only $\widetilde{c} \in Z^3(\mathbb{Z}\times\mathbb{Z},k^*)$ that satisfies the required properties is

$$\widetilde{c}(x,y,z) = (-1)^{a(x_1y_1z_2+x_1y_2z_1+x_1y_2z_2+x_2y_1z_1+x_2y_1z_2+x_1y_2z_1)}.$$

This 3-cocycle is a 3-coboundary because $\widetilde{c} = \delta(\phi)$, $\phi$ being the function on two variables on $\mathbb{Z}\times\mathbb{Z}$ to $k^*$ defined by

$$\phi(x,y) = (-1)^{a\frac{(x_2+y_1)(x_2y_1+2x_1y_2)}{2}} \text{for each } x=(x_1,x_2),\ y=(y_1,y_2) \in \mathbb{Z}\times\mathbb{Z}.$$

Thus, the only 3-cocycle in the cohomology class

$$[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}] \in H^3(\mathscr{F}^*,k^*)$$

satisfying the conditions of the lemma is

$$(f,g,h)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = c(f,g,h)\cdot\{f,g,h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$$

for all $f,g,h \in \mathscr{F}^*$, $c$ being the 3-coboundary $c(f,g,h) = (-1)^{\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,g,h)}$ with

$$\begin{aligned}\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,g,h) = {} & \deg(v_{\mathscr{F}})\deg(v_{\mathscr{K}})[v_{\mathscr{F}}(f)v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(h) + v_{\mathscr{F}}(f)v^z_{\mathscr{K}}(g)v_{\mathscr{F}}(h) \\ & + v_{\mathscr{F}}(f)v^z_{\mathscr{K}}(g)v^z_{\mathscr{K}}(h) + v^z_{\mathscr{K}}(f)v_{\mathscr{F}}(g)v_{\mathscr{F}}(h) \\ & + v^z_{\mathscr{K}}(f)v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(h) + v^z_{\mathscr{K}}(f)v^z_{\mathscr{K}}(g)v_{\mathscr{F}}(h)].\end{aligned}$$

Since the expression of $c$ is invariant under transformations

$$v^z_{\mathscr{K}} \longmapsto v^{z'}_{\mathscr{K}} + \lambda v_{\mathscr{F}},$$

the defined 3-cocycle is independent of the choice of $z$. □

**Proposition 2.13.** *If* $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$, *one has that* $(f,1-f,g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = 1$ *for all* $f,g \in \mathscr{F}^*$, *with* $f \neq 1$.

**Proof.** Let us consider two elements $f,g \in \mathscr{F}^*$, with $f \neq 1$. Bearing in mind that

$$\{f,1-f\}_{\widetilde{\mathscr{F}}^*} = (-1)^{v_{\mathscr{F}}(f)\cdot v_{\mathscr{F}}(1-f)} \in \mathscr{K}^*,$$

one has that $v_{\mathscr{K}}(\{f,1-f\}_{\widetilde{\mathscr{F}}^*}) = v_{\mathscr{F}}(1-f)\cdot v^z_{\mathscr{K}}(f) - v_{\mathscr{F}}(f)\cdot v^z_{\mathscr{K}}(1-f) = 0$, and it follows from the above lemma and the explicit expression of the 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ referred to in Remark 2.8 that

$$(f,1-f,g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = (-1)^{\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,1-f,g)}\cdot\frac{f^{v^g_{\mathscr{K}}(1-f)}}{(1-f)^{v^g_{\mathscr{K}}(f)}}(v_{\mathscr{F}},v_{\mathscr{K}}) \in k^*,$$

$\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f, 1-f, g)$ being the integer number

$$\deg(v_{\mathscr{K}})[v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)v^z_{\mathscr{K}}(1-f)+v^z_{\mathscr{K}}(g)v_{\mathscr{F}}(f)v_{\mathscr{F}}(1-f)].$$

As in the proof of Milnor [7] related to the tame symbol, the proof of this proposition will be divided into several cases. If $v_{\mathscr{F}}(f)<0$, then $f^{-1}\in \mathfrak{m}_{v_{\mathscr{F}}}$, and hence

$$\frac{1-f}{f}=-1+f^{-1}\equiv -1 (\operatorname{mod} \mathfrak{m}_{v_{\mathscr{F}}}).$$

Therefore, $v_{\mathscr{F}}(f)=v_{\mathscr{F}}(1-f)$. It follows from Lemma 2.7 that $v^g_{\mathscr{K}}(f)=v^g_{\mathscr{K}}(1-f)$ for all $g\in\mathscr{F}^*$, and one has that

$$\begin{aligned}\{f, 1-f, g\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}} &= (-1)^{\deg(v_{\mathscr{K}})v^g_{\mathscr{K}}(f)}\\ &= (-1)^{\deg(v_{\mathscr{K}})[v^z_{\mathscr{K}}(g)v_{\mathscr{F}}(f)+v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)]}.\end{aligned}$$

Thus, $(f, 1-f, g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}=1$ in this case.

Let us now assume that $v_{\mathscr{F}}(f)>0$. Then $1-f\equiv 1 (\operatorname{mod} \mathfrak{m}_{v_{\mathscr{F}}})$, $v_{\mathscr{F}}(1-f)=0$, and $v^g_{\mathscr{K}}(1-f)=0$ for all $g\in\mathscr{F}^*$. Hence,

$$(f, 1-f, g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}=\frac{f^{v^g_{\mathscr{K}}(1-f)}}{(1-f)^{v^g_{\mathscr{K}}(f)}}(v_{\mathscr{F}}, v_{\mathscr{K}})=1,$$

with the assumptions made. Moreover, the case $v_{\mathscr{F}}(1-f)>0$ is similar.

Finally, when $v_{\mathscr{F}}(f)=v_{\mathscr{F}}(1-f)=0$, the explicit expression of the symbol is

$$(f, 1-f, g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}=(-1)^{\deg(v_{\mathscr{K}})v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)v^z_{\mathscr{K}}(1-f)}\cdot\{f, 1-f, g\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}},$$

with

$$\begin{aligned}\{f, 1-f, g\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}} &= \left[\frac{f^{v^z_{\mathscr{K}}(1-f)}}{(1-f)^{v^z_{\mathscr{K}}(f)}}(v_{\mathscr{F}}, v_{\mathscr{K}})\right]^{v_{\mathscr{F}}(g)}\\ &= N_{k(v_{\mathscr{K}})/k}\left(\left[\frac{\{f,z\}_{\widetilde{\mathscr{F}}^*}^{v_{\mathscr{K}}(1-\{f,z\}_{\widetilde{\mathscr{F}}^*})}}{(1-\{f,z\}_{\widetilde{\mathscr{F}}^*})^{v_{\mathscr{K}}(\{f,z\}_{\widetilde{\mathscr{F}}^*})}}\right](\operatorname{mod} \mathfrak{m}_{v_{\mathscr{K}}})\right)^{v_{\mathscr{F}}(g)}\\ &= N_{k(v_{\mathscr{K}})/k}\left(\left[(-1)^{v^z_{\mathscr{K}}(f)v^z_{\mathscr{K}}(1-f)}\right]\right)^{v_{\mathscr{F}}(g)}\\ &= (-1)^{\deg(v_{\mathscr{K}})v_{\mathscr{F}}(g)v^z_{\mathscr{K}}(f)v^z_{\mathscr{K}}(1-f)}.\end{aligned}$$

Accordingly, one also sees that $(f, 1-f, g)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}=1$ in this latter case, and the claim is proved. □

**Remark 2.14.** When $\mathscr{F}$ is a two-dimensional local field, one can directly prove that the 3-cocycle $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ coincides with the two-dimensional tame symbol [4], which will therefore imply its Steinberg property.

**Theorem 2.15.** *Let us assume that* $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$. *Then,* $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ *is the unique Steinberg symbol in the cohomology class* $[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}}] \in H^3(\mathscr{F}^*, k^*)$ *satisfying the condition*:

$$(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}}} = \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}}} \quad \text{if } v_{\mathscr{F}}(f) = v^z_{\mathscr{K}}(f) = 0.$$

**Proof.** The statement follows immediately from the results proved in Lemma 2.12 and Proposition 2.13. □

**Corollary 2.16.** *With the notations of Remark* 2.9, *if* $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$, *one has that there exists a unique Steinberg symbol* $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ *in the cohomology class*

$$[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}] \in H^3(\mathscr{F}^*, G)$$

*satisfying the condition*

$$(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} = \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} \quad \text{if } v_{\mathscr{F}}(f) = v^z_{\mathscr{K}}(f) = 0.$$

**Proof.** A direct consequence of the previous theorem and the definition of the 3-cocycle $\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ is that

$$(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} = c_\varphi(f, g, h) \cdot \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} \quad \text{for all } f, g, h \in \mathscr{F}^*,$$

where $c_\varphi(f, g, h) = (h_{-1})^{\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,g,h)}$, with $h_{-1} = \varphi(-1)$. □

**Remark 2.17.** According to [12], if $A$ is an additive commutative group, $F_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic number field and $U_1 := 1 + \mathfrak{p}$, a symbol $c$: $F^*_{\mathfrak{p}} \times F^*_{\mathfrak{p}} \to A$ is called "tame" if $c(U_1, F^*_{\mathfrak{p}}) = 0$. With the notations of the previous corollary, the condition

$$(f, g, h)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} = \{f, g, h\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} \quad \text{if } v_{\mathscr{F}}(f) = v^z_{\mathscr{K}}(f) = 0$$

implies that $(U_1^{v_{\mathscr{F}}} \cap U_1^{v^z_{\mathscr{K}}}, \mathscr{F}^*, \mathscr{F}^*)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} = 1$, with $U_1^{v_{\mathscr{F}}} := 1 + \mathfrak{m}_{v_{\mathscr{F}}}$ and $U_1^{v^z_{\mathscr{K}}} := 1 + \mathfrak{m}_{v^z_{\mathscr{K}}}$. Thus, the required condition is not strange in the theory of symbols and $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ is somehow a "two-dimensional tame symbol".

Let $B$ be a system of representatives of $\mathscr{K}(v_{\mathscr{F}})$ in $\mathscr{O}_{v_{\mathscr{F}}}$, where $\mathscr{O}_{v_{\mathscr{F}}}$ is the valuation ring associated with the discrete valuation $v_{\mathscr{F}}$; $\mathfrak{m}_{v_{\mathscr{F}}}$ is the unique maximal ideal, and $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{O}_{v_{\mathscr{F}}}/\mathfrak{m}_{v_{\mathscr{F}}}$ is the residue class field. Let us assume that $0 \in B$. If $t \in \mathscr{O}_{v_{\mathscr{F}}}$ is a parameter such that $\mathfrak{m}_{v_{\mathscr{F}}} = (t)$, and $\hat{\mathscr{O}}_{v_{\mathscr{F}}}$ is the $\mathfrak{m}_{v_{\mathscr{F}}}$-adic completion of $\mathscr{O}_{v_{\mathscr{F}}}$, each element $\hat{a}$ of $\hat{\mathscr{O}}_{v_{\mathscr{F}}}$ can be written in the form

$$\hat{a} = \sum_{i \geqslant 0} b_i t^i \text{with } b_i \in B.$$

In general $B$ is not a subring of $\mathscr{O}_{v_{\mathscr{F}}}$, and the multiplication of two elements of $B$ is a element of $\hat{\mathscr{O}}_{v_{\mathscr{F}}}$ that must be expanded in a power series of $t$. Moreover, if $f \in (\hat{\mathscr{O}}_{v_{\mathscr{F}}})^*_0$ then one has that

$$f = t^{\beta} b_0 \left(1 + \sum_{i \geqslant 1} b_i t^i\right),$$

with $b_i \in B$, $\beta \in \mathbb{Z}$ and $b_0 \neq 0$. Thus, with the above notations, one has that

$$U_1^{v_{\mathscr{F}}} = \{f \in (\hat{\mathcal{O}}_{v_{\mathscr{F}}})_0^* \quad \text{such that } f = 1 + \sum_{i \geqslant 1} b_i t^i\}.$$

It is clear that $U_1^{v_{\mathscr{F}}}$ is a multiplicative group, and that $U_1^{v_{\mathscr{F}}} \cap \mathscr{F}^*$ is an open subset of $\mathscr{F}^*$ with the structure of the topological group induced by the valuation $v_{\mathscr{F}}$. Therefore, if we consider $\mathscr{F}^*$ as a topological group with the $v_{\mathscr{F}}$-topology and we consider $G$ as a topological group with the discrete topology, the symbol

$$(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} : \mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* \to G$$

is a continuous map because

$$\begin{aligned}[(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}]^{-1}(g) &= \bigcup_{\substack{\lambda,\mu,\delta \in B;\ \alpha,\beta,\gamma \in \mathbb{Z} \\ \varphi[(-1)^{s(\alpha,\beta,\gamma,\lambda,\mu,\delta)} r(\alpha,\beta,\gamma,\lambda,\mu,\delta)] = g}} [(\lambda \cdot t^{\alpha} \cdot U_1^{v_{\mathscr{F}}} \times \mu \cdot t^{\beta} \cdot U_1^{v_{\mathscr{F}}} \times \delta \cdot t^{\gamma} \cdot U_1^{v_{\mathscr{F}}}) \\ &\qquad \cap (\mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^*)],\end{aligned}$$

with

$$\begin{aligned} r(\alpha,\beta,\gamma,\lambda,\mu,\delta) &= N_{k(v_{\mathscr{K}})/k}[N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\lambda}^{v_{\mathscr{K}}(\frac{\bar{\delta}^{\beta}}{\bar{\mu}^{\gamma}})} \cdot \bar{\mu}^{v_{\mathscr{K}}(\frac{\bar{\lambda}^{\gamma}}{\bar{\delta}^{\alpha}})} \cdot \bar{\delta}^{v_{\mathscr{K}}(\frac{\bar{\mu}^{\alpha}}{\bar{\lambda}^{\beta}})}) \pmod{\mathfrak{m}_{v_{\mathscr{K}}}}]^{\deg(v_{\mathscr{F}})} \in k^*,\end{aligned}$$

$\bar{\lambda}, \bar{\mu}, \bar{\delta} \in \mathscr{K}(v_{\mathscr{F}})^*$ being the respective classes of $\lambda$, $\mu$ and $\delta$ in $\mathcal{O}_{v_{\mathscr{F}}}/\mathfrak{m}_{v_{\mathscr{F}}}$, and with

$$\begin{aligned} s(\alpha,\beta,\gamma,\lambda,\mu,\delta) = \deg(v_{\mathscr{F}}) \deg(v_{\mathscr{K}})[&\alpha v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\mu})) v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\delta})) \\ &+ \beta v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\lambda})) v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\delta})) \\ &+ \gamma v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\lambda})) v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\mu})) \\ &+ \alpha\beta v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\delta})) + \beta\gamma v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\lambda})) \\ &+ \alpha\gamma v_{\mathscr{K}}(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}(\bar{\mu}))].\end{aligned}$$

Moreover, when $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$, since $v_{\mathscr{F}}(t) = 1$ and

$$U_1^{v^t_{\mathscr{K}}} = \left\{ f \in \mathscr{F}^* \text{such that } f = 1 + \sum_{i \geqslant h_0} b_i t^i \text{with } v_{\mathscr{K}}(\bar{b}_{h_0}) \geqslant 1 \right\},$$

one analogously sees that the symbol

$$(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi} : \mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* \to G$$

is a continuous map when $\mathscr{F}^*$ is a topological group with the $v^t_{\mathscr{K}}$-topology and that $G$ is again a topological group with the discrete topology. Furthermore, if $z \in \mathscr{F}^*$ with $v_{\mathscr{F}}(z)=1$, it follows from the equality $v^z_{\mathscr{K}} = v^t_{\mathscr{K}} + \lambda v_{\mathscr{F}}$ that our symbol is also a continuous map when $\mathscr{F}^*$ is a topological group with the $v^z_{\mathscr{K}}$-topology for a general $z \in \mathscr{F}^*$ satisfying the required condition.

**Remark 2.18.** Assuming again that $\mathscr{K}(v_{\mathscr{F}}) = \mathscr{K}$, we can consider in $\mathscr{F}^*$ two structures of a topological groups:

- the structure of the topological group induced by the valuations $v_{\mathscr{F}}$ and $v^z_{\mathscr{K}}$ (the smallest topology that contains the $v_{\mathscr{F}}$-topology and the $v^z_{\mathscr{K}}$-topology);
- the structure determined by the product of the discrete topology on two copies of $\mathbb{Z}$ (corresponding to the choice of two local parameters of the discrete valuation fields $\mathscr{F}$ and $\mathscr{K}$), of the discrete topology on the group of multiplicative representatives of $\mathscr{F}$, and of the trivial topology on the groups of principal units of $\mathscr{F}$.

Let us consider $G$ as a topological group with the discrete topology. Similar to previous works by the author [8–10], an interesting issue in this theory is to determine which is the best topology for studying these symbols and also to offer an answer for the question:

How many continuous Steinberg symbols are in the cohomology class

$$[\{\cdot,\cdot,\cdot\}^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}] \in H^3(\mathscr{F}^*, G)?$$

In this case we are not sufficiently confident to conjecture that $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ is the only continuous Steinberg symbol in the cohomology class referred to.

Since $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}}}$ and $(\cdot,\cdot,\cdot)^z_{v_{\mathscr{F}},v_{\mathscr{K}},\varphi}$ are independent of the choice of $z$, we shall henceforth denote these symbols by $(\cdot,\cdot,\cdot)_{v_{\mathscr{F}},v_{\mathscr{K}}}$ and $(\cdot,\cdot,\cdot)^{\varphi}_{v_{\mathscr{F}},v_{\mathscr{K}}}$.

**Example 1.** Let $\mathscr{F}$ be a two-dimensional local field: that is, a complete discrete valuation field whose residue field $\mathscr{K}$ is a local field. If $k$ is the residue field of $\mathscr{K}$, the respective tame central extensions induce a symbol

$$(\cdot,\cdot,\cdot)_{v_{\mathscr{F}},v_{\mathscr{K}}} : \mathscr{F}^* \times \mathscr{F}^* \times \mathscr{F}^* \longrightarrow k^*.$$

In particular, if we consider $\mathscr{F} = k((u))((s))$, and $v_{\mathscr{F}}$ is the valuation induced by the parameter $u$, $\mathscr{K} = k((s))$, $v_{\mathscr{K}}$ is the valuation induced by the parameter $s$, $z = u$, $v^u_{\mathscr{K}}(f) = v_{\mathscr{K}}(\frac{f}{u^{v_{\mathscr{F}}(f)}})$, and $f, g, z \in k((u))((s))^*$, we have that the value of the symbol $(f,g,h)_{v_{\mathscr{F}},v_{\mathscr{K}}}$ is

$$(-1)^{\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,g,h)} \left( \frac{f(u,s)^{v_{\mathscr{F}}(g)\cdot v^u_{\mathscr{K}}(h) - v_{\mathscr{F}}(h)\cdot v^u_{\mathscr{K}}(g)}}{g(u,s)^{v_{\mathscr{F}}(f)\cdot v^u_{\mathscr{K}}(h) - v_{\mathscr{F}}(h)\cdot v^u_{\mathscr{K}}(f)}} h(u,s)^{v_{\mathscr{F}}(f)\cdot v^u_{\mathscr{K}}(g) - v_{\mathscr{F}}(g)\cdot v^u_{\mathscr{K}}(f)} \right)_{|u=0\ s=0},$$

with

$$\begin{aligned}\alpha_{v_{\mathscr{F}},v_{\mathscr{K}}}(f,g,h) = {} & \deg(v_{\mathscr{F}})\deg(v_{\mathscr{K}})[v_{\mathscr{F}}(f)v_{\mathscr{F}}(g)v^u_{\mathscr{K}}(h) + v_{\mathscr{F}}(f)v^u_{\mathscr{K}}(g)v_{\mathscr{F}}(h) \\ & + v_{\mathscr{F}}(f)v^u_{\mathscr{K}}(g)v^u_{\mathscr{K}}(h) + v^u_{\mathscr{K}}(f)v_{\mathscr{F}}(g)v_{\mathscr{F}}(h) \\ & + v^u_{\mathscr{K}}(f)v_{\mathscr{F}}(g)v^u_{\mathscr{K}}(h) + v^u_{\mathscr{K}}(f)v^u_{\mathscr{K}}(g)v_{\mathscr{F}}(h)].\end{aligned}$$

**Example 2.** Let $C$ be an irreducible and non-singular algebraic curve on a smooth, proper, geometrically irreducible surface $S$ over an algebraically closed field $k$. If $\Sigma_S$ is the function field of $S$, the curve $C$ defines a discrete valuation $v_C: \Sigma_S^* \to \mathbb{Z}$, whose residue class field is $\Sigma_C$ (the function field of $C$). Moreover, since $C$ is non-singular, each closed point $x \in C$ defines another discrete valuation $v_x: \Sigma_C^* \to \mathbb{Z}$, whose residue class field is $k$. Hence, for $\mathscr{F} = \Sigma_S$, $v_{\mathscr{F}} = v_C$, $\mathscr{K} = \Sigma_C$, $v_x^z(f) = v_x(\frac{f}{z^{v_C(f)}})$ ($z$ being an element of $\Sigma_S$ such that $v_C(z) = 1$), and $f, g, z \in \Sigma_S^*$, one has that

$$(f, g, h)_{v_C, v_x} = (-1)^{\alpha_{v_C, v_x}(f,g,h)} \cdot \{f, g, h\}_{v_C, v_x},$$

where $\alpha_{v_C, v_x}(f, g, h)$ is the integer number referred to in Example 1 (by replacing $v_{\mathscr{F}}$ with $v_C$, and $v^u_{\mathscr{K}}$ with $v_x^z$), and

$$\{f, g, h\}_{v_C, v_x} = \left( \frac{f^{v_C(g) \cdot v_x^z(h) - v_C(h) \cdot v_x^z(g)}}{g^{v_C(f) \cdot v_x^z(h) - v_C(h) \cdot v_x^z(f)}} \cdot h^{v_C(f) \cdot v_x^z(g) - v_C(g) \cdot v_x^z(f)} \right)_{|C} (x) \in k^*.$$

The symbol $(\cdot, \cdot, \cdot)_{v_C, v_x}$ is the Parshin symbol associated with the sequence of varieties $x \in C \subset S$ ([11]).

**Example 3.** With the hypothesis of Example 2, if the ground field $k$ is perfect (instead of algebraically closed); $k(x)$ is the residue class field of a closed point $x \in C$, $\deg(x) = \dim_k k(x)$, and $N_{k(x)/k}$ is the norm of the finite extension $k \hookrightarrow k(x)$, our method gives a symbol

$$(f, g, h)_{v_C, v_x} = (-1)^{\beta_{v_C, v_x}(f,g,h)} \cdot \{f, g, h\}_{v_C, v_x} \in k^*,$$

where

$$\{f, g, h\}_{v_C, v_x} = N_{k(x)/k} \left[ \left( \frac{f^{v_C(g) \cdot v_x^z(h) - v_C(h) \cdot v_x^z(g)}}{g^{v_C(f) \cdot v_x^z(h) - v_C(h) \cdot v_x^z(f)}} \cdot h^{v_C(f) \cdot v_x^z(g) - v_C(g) \cdot v_x^z(f)} \right)_{|C} (x) \right]$$

and

$$\begin{aligned} \beta_{v_C, v_x}(f, g, h) = \; & \deg(x)[v_C(f) \cdot v_C(g) \cdot v_x^z(h) + v_C(f) \cdot v_C(h) \cdot v_x^z(g) \\ & + v_C(g) \cdot v_C(h) \cdot v_x^z(f) + v_C(f) \cdot v_x^z(g) \cdot v_x^z(h) \\ & + v_C(g) \cdot v_x^z(f) \cdot v_x^z(h) + v_C(h) \cdot v_x^z(f) \cdot v_x^z(g)] \end{aligned}$$

for all $f, g, h \in \Sigma_S^*$. Moreover, if $k$ is a finite field that contains the $m$th roots of unity, with $\#k = q$, one has the morphism of groups

$$\begin{aligned} \phi_m: k^* &\longrightarrow \mu_m \\ \lambda &\longmapsto \lambda^{\frac{q-1}{m}}, \end{aligned}$$

which induces a symbol

$$(f, g, h)^{\phi_m}_{v_C, v_x} = (-1)^{\delta^{\phi_m}_{v_C, v_x}(f,g,h)} \cdot \{f, g, h\}^{\phi_m}_{v_C, v_x} \in \mu_m,$$

$\{f, g, h\}_{v_C, v_x}^{\phi_m}$ being equal to

$$N_{k(x)/k}\left[\left(\frac{f^{v_C(g)\cdot v_x^z(h)-v_C(h)\cdot v_x^z(g)}}{g^{v_C(f)\cdot v_x^z(h)-v_C(h)\cdot v_x^z(f)}}\cdot h^{v_C(f)\cdot v_x^z(g)-v_C(g)\cdot v_x^z(f)}\right)_{|C}(x)\right]^{\frac{q-1}{m}}$$

and $\delta_{v_C, v_x}^{\phi_m}(f, g, h) = \frac{q-1}{m}\beta_{v_C, v_x}(f, g, h)$ for all $f, g, h \in \Sigma_S^*$.

**Example 4.** Let us now consider a perfect field $\mathscr{K}$, together with a discrete valuation $v_{\mathscr{K}}$, whose residue class field is denoted by $k(v_{\mathscr{K}})$. If $C$ is an irreducible and non-singular curve over $\mathscr{K}$; $\mathscr{K}(x)$ is the residue class field of a closed point $x \in C$, and $N_{\mathscr{K}(x)/\mathscr{K}}$ is the norm of the finite extension $\mathscr{K} \hookrightarrow \mathscr{K}(x)$, setting $\deg(x) = \dim_{\mathscr{K}} \mathscr{K}(x)$, $\mathscr{F} = \Sigma_C$, $v_{\mathscr{F}} = v_x$, and $f, g, h \in \Sigma_C^*$, we obtain the symbol

$$(f, g, h)_{v_x, v_{\mathscr{K}}} = (-1)^{\beta_{v_x, v_{\mathscr{K}}}(f,g,h)} \cdot \{f, g, h\}_{v_x, v_{\mathscr{K}}} \in k(v_{\mathscr{K}})^*,$$

where $\{f, g, h\}_{v_x, v_{\mathscr{K}}}$ is equal to

$$\left[N_{\mathscr{K}(x)/\mathscr{K}}\left(\left(\frac{f^{[v_x(g)\cdot v_{\mathscr{K}}^z(h)-v_x(h)\cdot v_{\mathscr{K}}^z(g)]}}{g^{[v_x(f)\cdot v_{\mathscr{K}}^z(h)-v_x(h)\cdot v_{\mathscr{K}}^z(f)]}}\cdot h^{[v_x(f)\cdot v_{\mathscr{K}}^z(g)-v_x(g)\cdot v_{\mathscr{K}}^z(f)]}\right)(x)\right)^{\deg(x)}\right] (\mathrm{mod}\ \mathfrak{m}_{v_{\mathscr{K}}}),$$

where $z \in \Sigma_C^*$ with $v_x(z) = 1$, and $\beta_{v_x, v_{\mathscr{K}}}(f, g, h)$ is the integer number:

$$\begin{aligned}\beta_{v_x, v_{\mathscr{K}}}(f, g, h) = {} & \deg(x)[v_x(f)\cdot v_x(g)\cdot v_{\mathscr{K}}^z(h) + v_x(f)\cdot v_x(h)\cdot v_{\mathscr{K}}^z(g) \\ & + v_x(g)\cdot v_x(h)\cdot v_{\mathscr{K}}^z(f) + v_x(f)\cdot v_{\mathscr{K}}^z(g)\cdot v_{\mathscr{K}}^z(h) \\ & + v_x(g)\cdot v_{\mathscr{K}}^z(f)\cdot v_{\mathscr{K}}^z(h) + v_x(h)\cdot v_{\mathscr{K}}^z(f)\cdot v_{\mathscr{K}}^z(g)].\end{aligned}$$

Thus, when $C$ is a curve over $\mathbb{Q}$, $\mathbb{Q}(x)$ is the residue class field of a closed point $x \in C$, $deg(x) = \dim_{\mathbb{Q}} \mathbb{Q}(x)$; $N_{\mathbb{Q}(x)/\mathbb{Q}}$ is the norm of the finite extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(x)$, and $v_p$ is the $p$-adic valuation on $\mathbb{Q}$, with $p$ a prime number ($p \neq 2$), there exists a symbol $(\cdot, \cdot, \cdot)_{v_x, v_p}$ whose explicit expression is

$$(f, g, h)_{v_x, v_p} = (-1)^{\beta_{v_x, v_p}(f,g,h)} \cdot \{f, g, h\}_{v_x, v_p} \in (\mathbb{Z}/p)^*,$$

where $\{f, g, h\}_{v_x, v_p}$ is equal to

$$N_{\mathbb{Q}(x)/\mathbb{Q}}\left[\left(\left(\frac{f^{v_x(g)\cdot v_p^z(h)-v_x(h)\cdot v_p^z(g)}}{g^{v_x(f)\cdot v_p^z(h)-v_x(h)\cdot v_p^z(f)}}\cdot h^{v_x(f)\cdot v_p^z(g)-v_x(g)\cdot v_p^z(f)}\right)(x)\right)^{\deg(x)}\right] (\mathrm{mod}\ p)$$

for all $f, g, h \in \Sigma_C^*$. Furthermore, if $\mu_2$ consists of the 2nd roots of unity, from the morphism of groups

$$\begin{aligned}\phi_p : (\mathbb{Z}/p)^* &\longrightarrow \mu_2 \\ \lambda &\longmapsto \lambda^{\frac{p-1}{2}},\end{aligned}$$

we can obtain an induced symbol

$$\{\cdot,\cdot,\cdot\}^{\phi_p}_{v_x,v_p}\colon \Sigma_C^* \times \Sigma_C^* \times \Sigma_C^* \longrightarrow \mu_2,$$

with an explicit expression analogous to the symbol $(\cdot,\cdot,\cdot)^{\phi_m}_{v_C,v_x}$ referred to in Example 3.

**Example 5.** Let $X$ be an irreducible and non-singular curve over a perfect field $k$, such that $\Sigma_X$ is also a perfect field. By considering an irreducible and non-singular curve $\widetilde{Y}$ over $\Sigma_X$ and two closed points $\widetilde{y} \in \widetilde{Y}$, $x \in X$, and putting $\deg(\widetilde{y}) = \dim_{\Sigma_X} \Sigma_X(\widetilde{y})$, $\deg(x) = \dim_k k(x)$, $\mathscr{F} = \Sigma_{\widetilde{Y}}$, $v_{\mathscr{F}} = v_{\widetilde{y}}$, $\mathscr{K} = \Sigma_X$, and $v_{\mathscr{K}} = v_x$, it follows from the method offered that there exists a symbol

$$(\cdot,\cdot,\cdot)_{v_{\widetilde{y}},v_x} \in Z^3(\Sigma^*_{\widetilde{Y}}, k^*)$$

whose explicit expression is

$$(f,g,h)_{v_{\widetilde{y}},v_x} = (-1)^{\varepsilon_{v_{\widetilde{y}},v_x}(f,g,h)} \cdot \{f,g,h\}_{v_{\widetilde{y}},v_z} \in k^*,$$

where $\{f,g,h\}_{v_{\widetilde{y}},v_x}$ is equal to

$$N_{k(x)/k}\left[ N_{\Sigma_X(\widetilde{y})/\Sigma_X}\left( \frac{f^{[v_{\widetilde{y}}(g)\cdot v^z_x(h) - v_{\widetilde{y}}(h)\cdot v^z_x(g)]}}{g^{[v_{\widetilde{y}}(f)\cdot v^z_x(h) - v_{\widetilde{y}}(h)\cdot v^z_x(f)]}} \cdot h^{[v_{\widetilde{y}}(f)\cdot v^z_x(g) - v_{\widetilde{y}}(g)\cdot v^z_x(f)]}(\widetilde{y})\right)^{\deg(\widetilde{y})}(x)\right]$$

and

$$\begin{aligned}\varepsilon_{v_{\widetilde{y}},v_x}(f,g,h) = {} & \deg(\widetilde{y})\deg(x)[v_{\widetilde{y}}(f)\cdot v_{\widetilde{y}}(g)\cdot v^z_x(h) + v_{\widetilde{y}}(f)\cdot v_{\widetilde{y}}(h)\cdot v^z_x(g) \\ & + v_{\widetilde{y}}(g)\cdot v_{\widetilde{y}}(h)\cdot v^z_x(f) + v_{\widetilde{y}}(f)\cdot v^z_x(g)\cdot v^z_x(h) \\ & + v_{\widetilde{y}}(g)\cdot v^z_x(f)\cdot v^z_x(h) + v_{\widetilde{y}}(h)\cdot v^z_x(f)\cdot v^z_x(g)],\end{aligned}$$

for all $f,g,h \in \Sigma^*_{\widetilde{Y}}$.

## 3. Reciprocity laws

This final section is devoted to providing reciprocity laws for the symbols defined above by using a method similar to Tate's proof of the Residue Theorem [13].

### 3.1. Reciprocity law for $\mathscr{K} = \Sigma_C$

Let us now consider an irreducible and non-singular curve $C$ over a perfect field $k$ and a closed point $x \in C$. If $\mathscr{K} = \Sigma_C$, $v_{\mathscr{K}} = v_x$, $k(x)$ is the residue class field of a closed point $x \in C$, $deg(x) = \dim_k k(x)$, and $N_{k(x)/k}$ is the norm of the finite extension $k \hookrightarrow k(x)$, for a discrete valuation field $\mathscr{F}$, such that there exists a tame central extension associated with $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$, and for a morphism of commutative groups $\varphi\colon k^* \longrightarrow G$, we recall from

Section 2 that the 3-cocycle $\{\cdot,\cdot,\cdot\}^{\varphi}_{v_{\mathscr{F}},v_x}$ is

$$\{f_{i_0}, f_{i_1}, f_{i_2}\}^{\varphi}_{v_{\mathscr{F}},v_x} = \varphi\left(\prod_{j\in\mathbb{Z}/3}[\{\{f_{i_j},z\}_{\widetilde{\mathscr{F}}^*},\{f_{i_{j+1}},z\}_{\widetilde{\mathscr{F}}^*}\}^{K_x}_{A_x}]^{-v_{\mathscr{F}}(f_{i_{j+2}})\cdot\deg(v_{\mathscr{F}})}\right) \in G$$

and the explicit expression of the induced symbol is

$$(f,g,h)^{\varphi}_{v_{\mathscr{F}},v_x} = (h_{-1})^{\gamma_{v_{\mathscr{F}},v_x}(f,g,h)}\cdot\{f,g,h\}^{\varphi}_{v_{\mathscr{F}},v_x} \in G,$$

where $h_{-1}=\varphi(-1)$, $\{f,g,h\}^{\varphi}_{v_{\mathscr{F}},v_x}$ is equal to

$$\varphi\left(N_{k(x)/k}\left[\left(N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}\left(\frac{f^{\deg(v_{\mathscr{F}})[v_{\mathscr{F}}(g)\cdot v^z_x(h)-v_{\mathscr{F}}(h)\cdot v^z_x(g)]}}{g^{\deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f)\cdot v^z_x(h)-v_{\mathscr{F}}(h)\cdot v^z_x(f)]}}\cdot h^{\deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f)\cdot v^z_x(g)-v_{\mathscr{F}}(g)\cdot v^z_x(f)]}(\operatorname{mod}\mathfrak{m}_{v_{\mathscr{F}}})\right)\right)(x)\right]\right),$$

$N_{\mathscr{K}(v_{\mathscr{F}})/\mathscr{K}}$ being the norm of the finite extension $\mathscr{K}\hookrightarrow\mathscr{K}(v_{\mathscr{F}})$, and

$$\begin{aligned}\gamma_{v_{\mathscr{F}},v_x}(f,g,h) = {} & \deg(x)\deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f)\cdot v_{\mathscr{F}}(g)\cdot v^z_x(h) \\ & + v_{\mathscr{F}}(f)\cdot v_{\mathscr{F}}(h)\cdot v^z_x(g) + v_{\mathscr{F}}(g)\cdot v_{\mathscr{F}}(h)\cdot v^z_x(f) \\ & + v_{\mathscr{F}}(f)\cdot v^z_x(g)\cdot v^z_x(h) + v_{\mathscr{F}}(g)\cdot v^z_x(f)\cdot v^z_x(h) \\ & + v_{\mathscr{F}}(h)\cdot v^z_x(f)\cdot v^z_x(g)] \quad \text{for all } f,g,h\in\mathscr{F}^*.\end{aligned}$$

We shall recall from [8,9] the properties of the commutator $\{\cdot,\cdot\}^{K_x}_{A_x,\varphi}$ of the central extension of groups

$$1\to G\to \widetilde{\Sigma^*_C}^{K_x}_{A_x,\varphi}\to\Sigma^*_C\to 1$$

induced by $\varphi$ on the tame central extension associated with the discrete valuation $v_x$ on $\Sigma^*_C$. To simplify we denote this commutator by $\{\cdot,\cdot\}^{\varphi}_{A_x}$, and it has the following properties:

(1) Given $f,g\in\Sigma^*_C$, if we consider the central extension of groups associated with two closed points $x,y\in C$:

$$1\to G\longrightarrow(\widetilde{\Sigma^*_C})^{\varphi}_{A_x\oplus A_y}\longrightarrow\Sigma^*_C\to 1, \tag{3.1}$$

which is determined by commensurable subspaces to $A_x\oplus A_y$ in $K_x\oplus K_y$, we have that

$$\{f,g\}^{\varphi}_{A_x\oplus A_y} = (h_{-1})^{\deg(x)\deg(y)[v_x(f)v_y(g)+v_x(g)v_y(f)]}\cdot\{f,g\}^{\varphi}_{A_x}\cdot\{f,g\}^{\varphi}_{A_y}$$

with $h_{-1}=\varphi(-1)$.

(2) If $C$ is a complete curve and $X=\{x_1,\ldots,x_k\}$ is a finite subset of closed points of $C$ such that it contains all zeros and poles of $f,g\in\Sigma^*_C$, there exists a central extension of groups

$$1\to G\longrightarrow(\widetilde{\Sigma^*_C})^{\varphi}_{A_{x_1}\oplus\cdots\oplus A_{x_k}}\longrightarrow\Sigma^*_C\to 1, \tag{3.2}$$

whose commutator satisfies the condition

$$\{f, g\}^{\varphi}_{A_{x_1}\oplus\cdots\oplus A_{x_k}} = \prod_{i=1}^{k} (h_{-1})^{\deg(x_i)v_{x_i}(f)v_{x_i}(g)} \cdot \{f, g\}^{\varphi}_{A_{x_i}}.$$

(3) If $C$ is a complete curve, by using the theory of adeles, and with a similar method to Tate's proof of the Residue Theorem, one has that

$$\prod_{x\in C} \{f, g\}^{\varphi}_{A_x} = (h_{-1})^{\sum_{x\in C} \deg(x)v_x(f)v_x(g)}.$$

Let us now assume that $\mathscr{F}$ is an arbitrary discrete valuation field and that there exists a tame central extension associated with $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$, such that $\mathscr{K} = \Sigma_C$ with $C$ an irreducible and non-singular curve over a perfect field $k$. Let us also fix an element $z \in \mathscr{F}^*$, with $v_{\mathscr{F}}(z) = 1$.

**Lemma 3.1.** *For each pair of closed points $x, y \in C$, one has a 3-cocycle*

$$\{\cdot, \cdot, \cdot\}^{\varphi}_{v_{\mathscr{F}}, (\widetilde{\Sigma^*_C})^{\varphi}_{A_x\oplus A_y}} \in Z^3(\mathscr{F}^*, G),$$

*induced by the central extension* (3.1) *and the tame central extension associated with $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$, such that*

$$\{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, (\widetilde{\Sigma^*_C})^{\varphi}_{A_x\oplus A_y}} = (h_{-1})^{v_{v_{\mathscr{F}},\{x,y\}}(f,g,h)} \cdot \{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, v_x} \cdot \{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, v_y},$$

*with*

$$\begin{aligned} v_{v_{\mathscr{F}},\{x,y\}}(f, g, h) = {} & \deg(x)\deg(y)\deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f)(v^z_x(g)v^z_y(h) + v^z_x(h)v^z_y(g)) \\ & + v_{\mathscr{F}}(g)(v^z_x(f)v^z_y(h) + v^z_x(h)v^z_y(f)) + v_{\mathscr{F}}(h)(v^z_x(g)v^z_y(f) \\ & + v^z_x(f)v^z_y(g))] \end{aligned}$$

*for all $f, g, h \in \mathscr{F}^*$.*

**Proof.** Since

$$\begin{aligned} & \{f_{i_0}, f_{i_1}, f_{i_2}\}^{\varphi}_{v_{\mathscr{F}}, (\widetilde{\Sigma^*_C})^{\varphi}_{A_x\oplus A_y}} \\ & = \varphi\left(\prod_{j\in\mathbb{Z}/3} [\{\{f_{i_j}, z\}_{\widetilde{\mathscr{F}}^*}, \{f_{i_{j+1}}, z\}_{\widetilde{\mathscr{F}}^*}\}^{\varphi}_{A_x\oplus A_y}]^{-v_{\mathscr{F}}(f_{i_{j+2}})\cdot\deg(v_{\mathscr{F}})}\right), \end{aligned}$$

the statement is a direct consequence of the properties of the commutator $\{\cdot, \cdot\}^{\varphi}_{A_x}$. □

**Proposition 3.2.** *Given $f, g, h \in \mathscr{F}^*$, if $C$ is a complete curve and $X = \{x_1, \ldots x_k\}$ is a finite subset of closed points of $C$ such that it contains all zeros and poles of*

$\{f, z\}_{\widetilde{\mathscr{F}}^*}$, $\{g, z\}_{\widetilde{\mathscr{F}}^*}$ *and* $\{h, z\}_{\widetilde{\mathscr{F}}^*}$, *the* 3-*cocycle*

$$\{\cdot, \cdot, \cdot\}^{\varphi}_{v_{\mathscr{F}}, (\widetilde{\Sigma^*_C})^{\varphi}_{A_{x_1} \oplus \cdots \oplus A_{x_k}}} \in Z^3(\mathscr{F}^*, G),$$

*induced by the central extension* (3.2) *and the tame central extension associated with* $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$, *satisfies the condition that*

$$\{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, (\widetilde{\Sigma^*_C})^{\varphi}_{A_{x_1} \oplus \cdots \oplus A_{x_k}}} = (h_{-1})^{v_{v_{\mathscr{F}}, X}(f,g,h)} \cdot \prod_{i=1}^{k} \{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, v_{x_i}},$$

*with*

$$v_{v_{\mathscr{F}}, X}(f, g, h) = \sum_{x_i \in X} \deg(x_i) \deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f) v^z_{x_i}(g) v^z_{x_i}(h) + v_{\mathscr{F}}(g) v^z_{x_i}(f) v^z_{x_i}(h) + v_{\mathscr{F}}(h) v^z_{x_i}(f) v^z_{x_i}(g)].$$

**Proof.** Using induction over $\#X$ and bearing in mind that $\deg(x)^2 \equiv \deg(x)$ mod. 2, the formula holds by Lemma 3.1 and the property of complete curves

$$\sum_{p \in C} \deg(p) v_p(\phi) = \sum_{p \in X} \deg(p) v_p(\phi) = 0$$

for all $\phi \in \Sigma^*_C$ such that $X$ contains all zeros and poles of $\phi$. □

**Theorem 3.3.** *If* $\mathscr{F}$ *is an arbitrary discrete valuation field and there exists a tame central extension associated with* $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$ *such that* $\mathscr{K} = \Sigma_C$, *C being a complete, irreducible and non-singular curve over a perfect field k, and* $f, g, h \in \mathscr{F}^*$, *for each morphism of commutative groups* $\varphi: k^* \to G$, *one has that*

$$\prod_{x \in C} \{f, g, h\}^{\varphi}_{v_{\mathscr{F}}, v_x} = (h_{-1})^{v_{v_{\mathscr{F}}, C}(f,g,h)},$$

*with*

$$v_{v_{\mathscr{F}}, C}(f, g, h) = \sum_{x \in C} \deg(x) \deg(v_{\mathscr{F}})[v_{\mathscr{F}}(f) v^z_x(g) v^z_x(h) + v_{\mathscr{F}}(g) v^z_x(f) v^z_x(h)] + v_{\mathscr{F}}(h) v^z_x(f) v^z_x(g)].$$

**Proof.** Using the theory of adeles and with a similar method to Tate's proof of the Residue Theorem, the statement follows from the characterization of the commutator $\{\cdot, \cdot\}^{\varphi}_{\prod_{x \in C} A_x}$. This result is a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathscr{O}_C)$ and $H^1(C, \mathscr{O}_C)$. □

**Corollary 3.4** (*First Reciprocity Law*). *If* $\mathscr{F}$ *is an arbitrary discrete valuation field and there exists a tame central extension associated with* $(\mathscr{F}, v_{\mathscr{F}}, \mathscr{K})$ *such that* $\mathscr{K} = \Sigma_C$, *C being* *a complete, irreducible and non-singular curve over a perfect field k, and $f, g, h \in \mathscr{F}^*$, for each morphism of commutative groups $\varphi\colon k^* \to G$ one has that*

$$\prod_{x\in C}(f,g,h)^{\varphi}_{v_{\mathscr{F}},v_x} = 1.$$

**Proof.** The claim follows immediately from Theorem 3.3 because

$$v_{v_{\mathscr{F}},C}(f,g,h) = \sum_{x\in C}\gamma_{v_{\mathscr{F}},v_x}(f,g,h). \qquad \square$$

**Remark 3.5.** The Parshin Symbol $\langle\cdot,\cdot,\cdot\rangle_{(x,C)}$, associated with a sequence of complete varieties $x \in C \subset S$, is a particular case of our construction (Example 2). Hence the formula

$$\prod_{x\in C}\langle f,g,h\rangle_{(x,C)} = 1 \quad \text{for all } f,g,h \in \Sigma_S^*$$

is a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathscr{O}_C)$ and $H^1(C, \mathscr{O}_C)$ ([10]). We should note that this reciprocity law is already known also in the case of algebraic varieties over a perfect field [6, Chapter 7].

**Remark 3.6.** With the hypothesis of Example 3, one has a symbol

$$(\cdot,\cdot,\cdot)^{\phi_m}_{v_C,v_x}\colon \Sigma_S^* \times \Sigma_S^* \times \Sigma_S^* \to \mu_m,$$

which satisfies the law

$$\prod_{x\in C}(f,g,h)^{\phi_m}_{v_C,v_x} = 1 \quad \text{for all } f,g,h \in \Sigma_S^*.$$

**Remark 3.7.** If $k$ is a field of characteristic 0, one has that $k(t)$ is a perfect field and the projective line $\widetilde{Y} = \mathbb{P}^1_{k(t)}$ is a curve over $k(t)$ satisfying the conditions of Example 5, $X$ being the projective line $\mathbb{P}^1_k \to \operatorname{Spec} k$. Thus, for each closed point $\widetilde{y} \in \mathbb{P}^1_{k(t)}$ and for each morphism of commutative groups $\varphi\colon k^* \to G$, one has that

$$\prod_{x\in\mathbb{P}^1_k}(f,g,h)^{\varphi}_{v_{\widetilde{y}},v_x} = 1 \quad \text{for all } f,g,h \in \Sigma^*_{\mathbb{P}^1_{k(t)}}.$$

Explicitly, if $k$ is an algebraically closed field, $\Sigma_{\mathbb{P}^1_{k(t)}} = k(t)(s)$; $\widetilde{y}$ is a rational point with $\widetilde{y} \equiv \{s = \alpha(t)\}$, and $z = s - \alpha(t)$, for each morphism of commutative groups $\varphi\colon k^* \to G$

one has that

$$\prod_{\beta\in k}\varphi\left[(-1)^{\alpha_{v_{\tilde{y}},v_{t-\beta}}(f,g,h)}\left(\frac{f(t,s)^{v_{\tilde{y}}(g)v_{t-\beta}^{s-\alpha(t)}(h)-v_{\tilde{y}}(h)v_{t-\beta}^{s-\alpha(t)}(g)}}{g(t,s)^{v_{\tilde{y}}(f)v_{t-\beta}^{s-\alpha(t)}(h)-v_{\tilde{y}}(h)v_{t-\beta}^{s-\alpha(t)}(f)}}\cdot h(t,s)^{v_{\tilde{y}}(f)v_{t-\beta}^{s-\alpha(t)}(g)-v_{\tilde{y}}(g)v_{t-\beta}^{s-\alpha(t)}(f)}\right)_{|s=\alpha(t),t=\beta}\right]$$

$$=\varphi\left[(-1)^{\alpha_{v_{\tilde{y}},v_{\frac{1}{t}-0}}(f,g,h)}\left(\frac{f\left(\frac{1}{t},s\right)^{v_{\tilde{y}}(g)v_{\frac{1}{t}-0}^{s-\alpha(t)}(h)-v_{\tilde{y}}(h)v_{\frac{1}{t}-0}^{s-\alpha(t)}(g)}}{g\left(\frac{1}{t},s\right)^{v_{\tilde{y}}(f)v_{\frac{1}{t}-0}^{s-\alpha(t)}(h)-v_{\tilde{y}}(h)v_{\frac{1}{t}-0}^{s-\alpha(t)}(f)}}\cdot h\left(\frac{1}{t},s\right)^{v_{\tilde{y}}(f)v_{\frac{1}{t}-0}^{s-\alpha(t)}(g)-v_{\tilde{y}}(g)v_{\frac{1}{t}-0}^{s-\alpha(t)}(f)}\right)_{|s=\alpha(t),\frac{1}{t}=0}\right]$$

for all $f,g,h\in k(t)(s)^*$, $\alpha_{v_{\tilde{y}},v_x}(f,g,h)$ being the integer number

$$v_{\tilde{y}}(f)v_{\tilde{y}}(g)v_x^{s-\alpha(t)}(h)+v_{\tilde{y}}(f)v_x^{s-\alpha(t)}(g)v_{\tilde{y}}(h)$$
$$+v_{\tilde{y}}(f)v_x^{s-\alpha(t)}(g)v_x^{s-\alpha(t)}(h)+v_x^{s-\alpha(t)}(f)v_{\tilde{y}}(g)v_{\tilde{y}}(h)$$
$$+v_x^{s-\alpha(t)}(f)v_{\tilde{y}}(g)v_x^{s-\alpha(t)}(h)+v_x^{s-\alpha(t)}(f)v_x^{s-\alpha(t)}(g)v_{\tilde{y}}(h),$$

with $v_x^{s-\alpha(t)}(f(t,s))=v_x(\{f(t,s),s-\alpha(t)\}_{A_{\tilde{y}}}^{K_{\tilde{y}}})$, for a closed point $x\in\mathbb{P}_k^1$.

### 3.2. *Reciprocity law for $\mathscr{F}=\Sigma_C$*

Finally, we formulate the following conjecture:

**Conjecture 3.8** (*Second Reciprocity Law*). *Let us assume that $\mathscr{K}$ is an arbitrary discrete valuation perfect field and there exists a tame central extension associated with $(\mathscr{K},v_{\mathscr{K}},k)$. If $C$ is a complete, irreducible and non-singular curve over $\mathscr{K}$, and $f,g,h\in\Sigma_C^*$, for each morphism of commutative groups $\varphi\colon k^*\to G$ one has that*

$$\prod_{x\in C}(f,g,h)_{v_x,v_{\mathscr{K}}}^{\varphi}=1.$$

**Remark 3.9.** It is clear that proving this conjecture directly implies several new reciprocity laws on curves. In particular, reciprocity laws for the symbols defined in Example 4 are deduced immediately from the statement of the conjecture.

## References

[1] E. Arbarello, C. de Concini, V.G. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, in: Proceedings of the Symposia in Pure Mathematics, American Mathematical Society, vol. 49, Part I, 1989, pp. 171–190.

[2] J.L. Brylinsky, D.A. McLaughlin, The geometry of two dimensional symbols, K-Theory 10 (1996) 215–237.
[3] S. Eilenberg, S. McLane, Cohomology theory in abstract groups. I, Ann. Math. 48 (1) (1947) 51–78.
[4] I. Fesenko, A generalized symbol in multidimensional local fields, Rings and modules, Limit Theorems of Probability Theory, vol. 2, Leningrad. Universitat, vol. 214, 1988, pp. 88–92 (in Russian).
[5] I. Fesenko, S. Vostokov, On torsion in higher Milnor functors for multidimensional local fields, Amer. Math. Soc. Transl. 154 (2) (1992) 25–35.
[6] T. Fimmel, A.N. Parshin, An Introduction to the Higher Adelic Theory, preprint, Moscow.
[7] J. Milnor, Introduction to Algebraic K-Theory, Annals of Mathematics Studies, Princeton University Press, Princeton, 1971.
[8] F. Pablos Romo, On the Tame symbol of an algebraic curve, Comm. Algebra 30 (9) (2002) 4349–4368.
[9] F. Pablos Romo, A note on Steinberg symbols on algebraic curves, Comm. Algebra 31 (2) (2003) 981–990.
[10] F. Pablos Romo, Algebraic construction of the Tame symbol and the Parshin symbol on a surface, J. Algebra 274 (1) (2004) 335–346.
[11] A.N. Parshin, Local class field theory, Proceedings of the Steklov Inst. Math. 3 (1985) 157–185.
[12] A.N. Parshin, I.R. Shafarevich, Number theory II, Encyclopaedia of Mathematical Sciences, vol. 62, Springer, Berlin, Heidelberg, 1992.
[13] J. Tate, Residues of differentials on curves, Ann. Scient. Éc. Norm. Sup. 1968, 4a série 1 (1968) 149–159.