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Arithmetic properties of algebraic curves from a pseudo-characteristic polynomial of a finite potent endomorphism

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Abstract
The aim of this work is to offer the definition and the main properties of a pseudo-characteristic polynomial of a finite potent endomorphism. From this polynomial we can characterize the spectrum of a bounded finite potent linear operator on a Hilbert space and we study arithmetic properties of complete algebraic curves. In particular, we provide a new algebraic proof of the Residue Theorem.

Keywords Finite potent endomorphism · Tate's trace · Arithmetic properties · Algebraic curves

Mathematics Subject Classification 15A03 · 15A15 · 19F15

1 Introduction

The notion of finite potent endomorphism on an arbitrary vector space was introduced by Tate in [10] as a basic tool for his elegant definition of Abstract Residues.

During the last decade the theory of finite potent endomorphisms has been applied to study different topics related to Algebra, Arithmetic and Algebraic Geometry. Thus, Yekutieli in [11] and Braunling in [2] and [3] have addressed problems of arithmetic symbols by using properties of finite potent endomorphisms; Debry in [4] and Taelman in [9] have offered results about Drinfeld modules from these linear operators and the author of this work has given explicit solutions of non-homogeneous difference equations associated with finite potent endomorphisms in [7].

The concept of "finite potent endomorphism" makes sense on infinite dimensional vector spaces and it is clear that for these operators it is not possible to define a classical characteristic polynomial.

To Professor Adolfo Quirós Gracián, for his 65th birthday.

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The aim of this work is to give the definition and the main properties of a pseudo-characteristic polynomial of a finite potent endomorphism on an arbitrary vector space. Although the defined polynomial does not coincide, in general, with the classical characteristic polynomial of an endomorphism in a finite-dimensional vector space, it does verify most of its properties. Moreover, from this polynomial we can characterize the spectrum of a bounded finite potent linear operator on a Hilbert space and we study arithmetic properties of complete algebraic curves.

The paper is organized as follows. In Sect. 2 we recall the basic definitions of this work and a summary of the statements of the articles [1, 6, 8] and [10].

Section 3 contains the main results of this work: the definition and the proofs of the main properties of the offered pseudo-characteristic polynomial of a finite potent endomorphism on an arbitrary vector space. An illustrative example of the results obtained is also incorporated.

Finally, Sect. 4 is devoted to apply the statements proved in Sect. 3 to obtain arithmetic expressions for algebraic curves. In particular, we provide a new algebraic proof of the Residue Theorem ((Theorem 4.9) from a Reciprocity Law associated with the pseudo-characteristic polynomial of a finite potent endomorphism (Theorem 4.8).

We hope that more applications of the results of this work can be achieved in the near future.

2 Preliminaries

This section is added for the sake of completeness.

2.1 Finite potent endomorphisms

Let k be an arbitrary field, let V be a k-vector space and let us consider an endomorphism φ of V. We say that φ is "finite potent" if $\varphi^n V$ is finite dimensional for some n, where the power means the composition $\varphi \circ \overset{(n)}{\dots} \circ \varphi$. This definition was introduced by Tate in [10] as a basic tool for his elegant definition of Abstract Residues.

In 2007, Argerami, Szechtman and Tifenbach showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, W_φ is finite dimensional and $\varphi_{|W_\varphi} \colon W_\varphi \overset{\sim}{\longrightarrow} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some } m\}$;
- $W_\varphi = \left\{ \begin{array}{l} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{array} \right\}$.

Note that if the annihilator polynomial of φ is $a_\varphi(x) = x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Moreover, we shall call "index of φ", $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector spaces and finite rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

2.2 CN-decomposition of a finite potent endomorphism

Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

The decomposition $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition of φ (core-nilpotent). According to [6, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the **core part** of φ. Also, φ_2 is named the **nilpotent part** of φ and one has that

$$\varphi = \varphi_1 \iff U_\varphi = \operatorname{Ker} \varphi \iff W_\varphi = \operatorname{Im} \varphi \iff (\varphi^D)^D = \varphi \iff i(\varphi) \leq 1 . \tag{1}$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases} . \tag{2}$$

Note that, from (2) we have that $U_\varphi = U_{\varphi_1}$, $W_\varphi = W_{\varphi_1}$ and $\varphi_{|W_\varphi} = (\varphi_1)_{|W_{\varphi_1}}$.

2.3 Bounded finite potent linear operators of a Hilbert space

Given a Hilbert space $\mathcal{H}$ and an endomorphism $\varphi \in \operatorname{End}_{\mathbb{C}}(\mathcal{H})$, according to [8, Theorem 3.7] the following conditions are equivalent:

1. $\varphi \in B_{fp}(\mathcal{H})$ (φ is a bounded finite potent linear operator);
2. $\mathcal{H}$ admits a decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$ where W_φ and U_φ are closed φ-invariant subspaces of $\mathcal{H}$, W_φ is finite-dimensional, $\varphi_{|W_\varphi}$ is an homeomorphism of W_φ and $\varphi_{|U_\varphi}$ is a bounded nilpotent operator.
3. φ has a decomposition $\varphi = \psi + \phi$, where ψ is a bounded finite rank operator, ϕ is a bounded nilpotent operator and $\psi \circ \phi = \phi \circ \psi = 0$.

From [8, Proposition 4.1] it is known that, if $\varphi \in B_{fp}(\mathcal{H})$, then the adjoint φ^* is also a bounded finite potent endomorphism.

Moreover, if $\mathcal{H} = W_{\varphi^*} \oplus U_{\varphi^*}$ is the AST-decomposition of $\mathcal{H}$ determined by φ^*, one has that $U_{\varphi^*} = W_\varphi^\perp$ [8, Proposition 4.3] and that $W_{\varphi^*} = U_\varphi^\perp$ [8, Proposition 4.5].

From the statements of [8] we also know that the spectrum $\sigma(\varphi)$ of $\varphi \in B_{fp}(\mathcal{H})$ satisfies the following properties:

1. $\sigma(\varphi)$ is finite;
2. $\lambda \in \sigma(\varphi)$ if and only if λ is an eigenvalue of φ;
3. $\dim_{\mathbb{C}} \operatorname{Ker}(\varphi - \lambda \operatorname{Id}) < \infty$ for every $0 \neq \lambda \in \sigma(\varphi)$.
4. If $0 \neq \lambda \in \sigma(\varphi)$, then the algebraic multiplicity of λ as an eigenvalue of φ_1 coincides with the algebraic multiplicity of λ as an eigenvalue of $\varphi_{|W_\varphi}$.

2.4 Trace of a finite potent endomorphism

For an arbitrary k-vector space and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, a trace $\mathrm{Tr}_V(\varphi) \in k$ may be defined from the following properties:

1. if V is finite dimensional, then $\mathrm{Tr}_V(\varphi)$ is the ordinary trace;
2. if W is a subspace of V such that $\varphi W \subset W$, then

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_W(\varphi) + \mathrm{Tr}_{V/W}(\varphi)\,;$$

3. if φ is nilpotent, then $\mathrm{Tr}_V(\varphi) = 0$.

Usually, Tr_V is named "Tate's trace". If $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by $\varphi \in \mathrm{End}_k(V)$ and $\varphi = \varphi_1 + \varphi_2$ is its CN-decomposition, one has that

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_V(\varphi_1) = \mathrm{Tr}_{W_\varphi}(\varphi_{|W_\varphi})\,.$$

It is known that in general Tr_V is not linear; that is, it is possible to find finite potent endomorphisms $\theta_1, \theta_2 \in \mathrm{End}_k(V)$ such that

$$\mathrm{Tr}_V(\theta_1 + \theta_2) \neq \mathrm{Tr}_V(\theta_1) + \mathrm{Tr}_V(\theta_2)\,.$$

However, the linearity of the trace holds if we consider endomorphisms of a finite potent subspace of $\mathrm{End}_k(V)$ according to the following definition: a subspace $F \subset \mathrm{End}_k(V)$ is named "finite potent" when there exists an $n \in \mathbb{N}$ such that for any family of n elements $\varphi_1, \dots, \varphi_n \in F$ the space $\varphi_1 \dots \varphi_n V$ is finite dimensional.

2.5 Abstract residue

Let k again be an arbitrary field, let V be a k-vector space and let A, B be two k-vector subspaces of V.

Definition 2.1 A and B are said to be commensurable if $(A + B)/(A \cap B)$ is a finite dimensional vector space over k. We write $A \sim B$ to denote commensurable subspaces - [10]-.

If we set $A < B$ when $(A + B)/B$ is finite dimensional, it is clear that A and B are commesurable, $A \sim B$, if and only if $A < B$ and $B < A$. Commensurability is an equivalence relation on the set of k-vector subspaces of V.

Let us now fix a subspace A of V, and then define subspaces E, E_0, E_1, E_2 of $\mathrm{End}_k(V)$ by

- $\varphi \in E \iff \varphi A < A$,
- $\varphi \in E_1 \iff \varphi V < A$,
- $\varphi \in E_2 \iff \varphi A < (0)$,
- $\varphi \in E_0 \iff \varphi V < A$ and $\varphi A < (0)$.

Tate [10] showed that E is a k-subalgebra of $\mathrm{End}(V)$, the E_i are two-sided ideals in E, the E's depend only on the $\sim$-equivalence class of A, $E_1 \cap E_2 = E_0$, $E_1 + E_2 = E$, and E_0 is finite potent. Moreover, if we assume either $f \in E_0$ and $g \in E$, or $f \in E_1$ and $g \in E_2$, then the commutator $[f, g]_A^V = fg - gf$ is in E_0 and has zero trace.

Let K be a commutative k-algebra, V a K-module, and A a k-subspace of V such that $fA < A$ for all $f \in K$. With the above notations, K operates on V through $E \subset \mathrm{End}_k(V)$.

Definition 2.2 (Abstract Residue) There exists a unique k-linear "residue map"

$$\operatorname{Res}_A^V : \Omega^1_{K/k} \longrightarrow k$$

such that for each pair of elements f and g in K one has that

$$\operatorname{Res}_A^V(f dg) = \operatorname{Tr}_V([f_1, g_1]_A^V)$$

for every pair of endomorphisms f_1 and g_1 in E satisfying the following conditions:

- Both $f \equiv f_1 \pmod{E_2}$ and $g \equiv g_1 \pmod{E_2}$;
- Either $f_1 \in E_1$ or $g_1 \in E_1$.

This residue is well-defined because $[f_1, g_1]_A^V \in \operatorname{End}_k(V)$ is a finite potent endomorphism.

2.6 Residues on algebraic curves

Let C be now a non-singular and irreducible curve over a perfect field k and let Σ_C be its function field. Each closed point $p \in C$ corresponds to a discrete valuation ring $\mathcal{O}_p$ with field of fractions $K = \Sigma_C$ that contains k. Write $A_p = \hat{\mathcal{O}_p}$, the completion of $\mathcal{O}_p$, and write K_p for the field of fractions of A_p (which is the completion of Σ_C with respect to the valuation defined by $\mathcal{O}_p$). Thus, we have a map

$$\operatorname{Res}_{A_p}^{K_p} : \Omega^1_{K/k} \to k\,.$$

When p is a rational point of C, one has that $A_p \simeq k[[t]]$, $K_p \simeq k((t))$ and $\operatorname{Res}_p(f dg)$ is the classical residue; that is, it is the coefficient of t^{-1} in $f(t)g'(t)$.

In general, if $k(p)$ is the residue field of the closed point p, the classical residue $\operatorname{Res}_p(f dg)$ values in $k(p)$, because $A_p \simeq k(p)[[t]]$ and $K_p \simeq k(p)((t))$, and the explicit expression of this map is

$$\operatorname{Res}_{A_p}^{K_p}(f dg) = \operatorname{Tr}_{k(p)/k}[\operatorname{Res}_p(f dg)]\,,$$

where $\operatorname{Tr}_{k(p)/k}$ is the trace of the finite extension of fields $k \hookrightarrow k(p)$.

If we set

$$V_C = \prod_{p \in C}{}' K_p = \{f = (f_p) \text{ such that } f_p \in K_p \text{ and } f_p \in A_p \text{ for almost all } p\}\,,$$

then Σ_C can be regarded as a subspace of V by means of the diagonal embedding.

Moreover, putting $A_C = \prod_{p \in C} A_p \subset V_C$, it is clear that $f A_C < A_C$ for every $f \in \Sigma_C$.

When C is complete, it follows from the statements of [10] that $\operatorname{Res}_{A_C}^{V_C}(f dg) = 0$ for all functions $f, g \in \Sigma_C$.

3 Pseudo-characteristic polynomial of a finite potent endomorphism

This section is devoted to introduce the notion of the pseudo-characteristic polynomial of a finite potent endomorphism and to prove its main properties.

Let V be an arbitrary k-vector space and let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism that induces the AST-decomposition $V = W_\varphi \oplus U_\varphi$.

Definition 3.1 We shall call "pseudo-characteristic polynomial" of a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ to the polynomial $\tilde{c}_\varphi(x) \in k[x]$ defined by:

$$\tilde{c}_\varphi(x) = \begin{cases} c_{\varphi_{|W_\varphi}}(x) & \text{if } W_\varphi \neq 0 \\ 1 & \text{if } W_\varphi = 0 \end{cases},$$

where $c_{\varphi_{|W_\varphi}}(x)$ is the characteristic polynomial of $\varphi_{|W_\varphi} \in \mathrm{End}_k(W_\varphi)$.

It is clear that Definition 3.1 makes sense because W_φ is unique and is a finite-dimensional k-vector space for every finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$. Moreover, if E is a finite-dimensional vector space and $f \in \mathrm{End}_k(E)$ is an endomorphism, the pseudo-characteristic polynomial $\tilde{c}_f(x)$ coincides with the classical characteristic polynomial $c_f(x)$ if and only if f is an automorphism of E, that is, when f is a finite potent endomorphism with $i(f) = 0$. In general, if the dimension of E is n, $i(f) = m$ and the dimension of $\mathrm{Im}\, f^m$ is r, then one has that

$$c_f(x) = x^{n-r} \cdot \tilde{c}_f(x)\,.$$

Note also that a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ satisfies that $W_\varphi = 0$ if and only if φ is nilpotent.

Remark 3.2 The motivation of the Definition 3.1 is the following: in the finite-dimensional situation, if $f \in \mathrm{End}_k(E)$ is an endomorphism and $E = E_1 \oplus E_2$, where E_1 and E_2 are f-invariant subspaces of E, it is known that

$$c_f(x) = c_{f_{|E_1}}(x) \cdot c_{f_{|E_2}}(x)\,.$$

Hence, this property is valid for every two f-invariant complementary subspaces of E only when we assume that $c_{f_{|\{0\}}}(x) = 1$, and this fact justifies the way of defining the polynomial $\tilde{c}_\varphi(x)$.

Lemma 3.3 *If V is an arbitrary k-vector space and $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism with CN-decomposition $\varphi = \varphi_1 + \varphi_2$, then $\tilde{c}_\varphi(x) = \tilde{c}_{\varphi_1}(x)$.*

Proof Since $W_\varphi = W_{\varphi_1}$ and $\varphi_{|W_\varphi} = (\varphi_1)_{|W_{\varphi_1}}$, then the statement is a direct consequence of Definition 3.1. □

We shall now prove several properties of the polynomial $\tilde{c}_\varphi(x)$ that are, somehow, the generalization of the main properties of the classical characteristic polynomial.

Lemma 3.4 *If V is an arbitrary k-vector space and $\varphi, \psi \in \mathrm{End}_k(V)$ are two finite potent endomorphisms such that $\psi = \tau \circ \varphi \circ \tau^{-1}$ for a certain automorphism $\tau \in \mathrm{Aut}_k(V)$, then $\tilde{c}_\psi(x) = \tilde{c}_\varphi(x)$.*

Proof Since, under the hypothesis of this lemma, φ is nilpotent if and only if ψ is nilpotent, in this case we have that $\tilde{c}_\psi(x) = \tilde{c}_\varphi(x) = 1$.

Let us now assume that φ and ψ are not nilpotent. If $V = W_\varphi \oplus U_\varphi$ and $V = W_\psi \oplus U_\psi$ are the AST-decompostions of V induced by φ and ψ respectively, it is clear that $W_\psi = \tau(W_\varphi)$.

Hence, if $B = \{w_1, \dots, w_r\}$ is a basis of W_φ, then $B_\tau = \{\tau(w_1), \dots, \tau(w_r)\}$ is a basis of W_ψ and an easy computation shows that the matrix associated with $\varphi_{|W_\varphi}$ in the basis B coincides with the matrix associated with $\psi_{|W_\psi}$ in the basis B_τ, from where the claim is deduced. □

In [5] the author of this work has offered the classification of finite potent endomorphisms of an arbitrary vector space V under the action by conjugation of the group $\mathrm{Aut}_k(V)$, that is: if $\varphi, \psi \in \mathrm{End}_k(V)$ are finite potent, then $\varphi \sim \psi$ if and only if $\psi = \tau \circ \varphi \circ \tau^{-1}$ for a certain automorphism $\tau \in \mathrm{Aut}_k(V)$. If $\varphi \sim \psi$, from Lemma 3.4 we deduce that $\tilde{c}_\psi(x) = \tilde{c}_\varphi(x)$.

Lemma 3.5 *Given an arbitrary k-vector space V and a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, such that φ is not nilpotent, if*

$$\tilde{c}_\varphi(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i} \, ,$$

then $a_i = (-1)^i \operatorname{Tr}_{\bigwedge^i W_\varphi}(\bigwedge^i \varphi_{|W_\varphi})$ and $a_1 = -\operatorname{Tr}_V \varphi$.

Proof The first assertion is immediately deduced from Definition 3.1 and the well-known explicit expression of the classical characteristic polynomial of an endomorphism on a finite-dimensional vector space.

Moreover, since $\operatorname{Tr}_V \varphi = \operatorname{Tr}_{W_\varphi}(\varphi_{|W_\varphi})$, we have that $a_1 = -\operatorname{Tr}_V \varphi$. □

Given a finite-dimensional k-vector space E, an endomorphism $f \in \mathrm{End}_k(E)$ and a r-cycle permutation σ_r, we shall denote $\operatorname{Tr}_E^{\sigma_r} f = \operatorname{Tr}_E(f^r)$. In general, for a permutation $\sigma \in S_n$ with $\sigma = \sigma_{r_1} \circ \cdots \circ \sigma_{r_s}$, we shall write

$$\operatorname{Tr}_E^{\sigma} f = [\operatorname{Tr}_E^{\sigma_{r_1}} f] \cdots [\operatorname{Tr}_E^{\sigma_{r_s}} f] = [\operatorname{Tr}_E(f^{r_1})] \cdots [\operatorname{Tr}_E(f^{r_s})] \, .$$

Bearing in mind that it is known that

$$\operatorname{Tr}_{\bigwedge^i E}\left(\bigwedge^i f\right) = \frac{1}{i!} \sum_{\sigma \in S_i} sig\,(\sigma) \operatorname{Tr}_E^{\sigma}(f) \, ,$$

it follows immediately from Lemma 3.5 that

Corollary 3.6 *If V is an arbitrary k-vector space, $\varphi \in \mathrm{End}_k(V)$ is a non-nilpotent finite potent endomorphism and*

$$\tilde{c}_\varphi(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i} \, ,$$

then

$$a_i = (-1)^i \frac{1}{i!} \sum_{\sigma \in S_i} sig\,(\sigma) \operatorname{Tr}_{W_\varphi}^{\sigma}(\varphi_{|W_\varphi}) \, ,$$

for every $i \in \{1, \dots, n\}$.

Similar to above, given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, we now write $\operatorname{Tr}_V^{\sigma_r} \varphi = \operatorname{Tr}_V(\varphi^r)$ for a r-cycle σ_r and

$$\operatorname{Tr}_V^{\sigma} \varphi = [\operatorname{Tr}_V^{\sigma_{r_1}} \varphi] \cdots [\operatorname{Tr}_V^{\sigma_{r_s}} \varphi] = [\operatorname{Tr}_V(\varphi^{r_1})] \cdots [\operatorname{Tr}_V(\varphi^{r_s})] \, ,$$

for a permutation $\sigma = \sigma_{r_1} \circ \cdots \circ \sigma_{r_s}$.

Proposition 3.7 *With the hypothesis of Corollary 3.6, one has that*

$$a_i = (-1)^i \frac{1}{i!} \sum_{\sigma \in S_i} sig\,(\sigma)\, \mathrm{Tr}_V^{\sigma}(\varphi)\,,$$

for each $i \in \{1, \dots, n\}$.

Proof The statement is a direct consequence of Corollary 3.6, because if $\varphi \in \mathrm{End}_k(V)$ is finite potent, then φ^n is also finite potent for all $n \in \mathbb{N}$ and $W_\varphi = W_{\varphi^n}$. □

Note that, from Proposition 3.7, the first coefficients of the pseudo-characteristic polynomial

$$\tilde{c}_\varphi(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$$

of a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ are:

- $a_1 = -\mathrm{Tr}_V\, \varphi$;
- $a_2 = \frac{1}{2}[(\mathrm{Tr}_V\, \varphi)^2 - \mathrm{Tr}_V\, \varphi^2]$;
- $a_3 = -\frac{1}{6}[(\mathrm{Tr}_V\, \varphi)^3 - 3(\mathrm{Tr}_V\, \varphi^2) \cdot (\mathrm{Tr}_V\, \varphi) + 2(\mathrm{Tr}_V\, \varphi^3)]$.

Proposition 3.8 *If V is an arbitrary k-vector space and $\varphi \in \mathrm{End}_k(V)$ is a non-nilpotent finite potent endomorphism with index $m = i(\varphi)$, then one has that the annihilator polynomial $a_\varphi(x)$ divides $x^m \cdot [\tilde{c}_\varphi(x)]$ and $\lambda \in k$ is a non-zero root of $a_\varphi(x)$ if and only if λ is a root of $\tilde{c}_\varphi(x)$.*

Proof If $a_\varphi(x) = x^m \cdot p(x)$ with $(x, p(x)) = 1$ and $m = i(\varphi)$, we have that $W_\varphi = \mathrm{Ker}\, p(\varphi)$ and $p(x)$ is the annihilator polynomial of $\varphi_{|W_\varphi}$.

Hence, it follows from Definition 3.1 and the Hamilton-Cayley Theorem for endomorphisms on finite-dimensional vector spaces that $p(x)$ divides $\tilde{c}_\varphi(x)$ and both polynomials have the same roots.

Therefore, the claim holds. □

If $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism and $V' \subset V$ is a φ-invariant subspace of V, henceforth we shall denote by φ' to the finite potent endomorphism $\varphi_{|_{V'}} \in \mathrm{End}_k(V')$ and by $V' = W_{\varphi'} \oplus U_{\varphi'}$ its corresponding AST-decomposition.

Lemma 3.9 *Given an arbitrary k-vector space V, a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ with AST-decomposition $V = W_\varphi \oplus U_\varphi$ and a φ-invariant subspace $V' \subset V$, with the previous notation we have that $W_{\varphi'} \subseteq W_\varphi$.*

Proof If $\varphi \in \mathrm{End}_k(V)$ is nilpotent, then $\varphi' \in \mathrm{End}_k(V')$ is also nilpotent, and it is clear that $W_{\varphi'} = \{0\} = W_\varphi$.

Let us now assume that φ is not nilpotent and that its annihilator polynomial is $a_\varphi(x) = x^m \cdot p(x)$, with $(x, p(x)) = 1$. So, $W_\varphi = \mathrm{Ker}\, p(\varphi)$.

If $a_{\varphi'}(x) = x^r \cdot q(x) \in k[x]$, with $(x, q(x)) = 1$, is the annihilator polynomial of φ' in V', we have that $W_{\varphi'} = \mathrm{Ker}\, q(\varphi') \subseteq V'$ and, therefore, $W_{\varphi'} \subseteq \mathrm{Ker}\, q(\varphi) \subseteq V$.

Hence, bearing in mind that under these hypothesis $q(x)$ divides $p(x)$, we conclude that

$$W_{\varphi'} \subseteq \mathrm{Ker}\, q(\varphi) \subseteq \mathrm{Ker}\, p(\varphi) \subseteq W_\varphi\,.$$

□

Lemma 3.10 *Given an arbitrary k-vector space V, a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with AST-decomposition $V = W_\varphi \oplus U_\varphi$ and a φ-invariant subspace $V' \subset V$, one has that $U_{\varphi'} \subseteq U_\varphi$.*

Proof If $i(\varphi') = m'$, for every $u' \in U_{\varphi'}$ we have that

$$0 = (\varphi')^{m'}(u') = \varphi^{m'}(u')$$

that implies that $u' \in U_\varphi$. Thus, one has that $U_{\varphi'} \subseteq U_\varphi$ and the assertion is proved. □

Note that from Lemma 3.10 we obtain that $i(\varphi') \leq i(\varphi)$.

Lemma 3.11 *With the previous notation, one has that $V' \cap W_\varphi = W_{\varphi'}$.*

Proof It follows from Lemma 3.9 that $W_{\varphi'} \subseteq V' \cap W_\varphi$.

Conversely, let us consider $v' \in V' \cap W_\varphi$. If $v' = w' + u'$ with $w' \in W_{\varphi'}$ and $u' \in U_{\varphi'}$, since it follows from Lemma 3.9 that $w' \in W_\varphi$ and from Lemma 3.10 that $u' \in U_\varphi$, we have that

$$u' = v' - w' \in U_\varphi \cap W_\varphi = \{0\}$$

and we conclude that $v' \in W_{\varphi'}$. □

Lemma 3.12 *Given an arbitrary k-vector space V, a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with AST-decomposition $V = W_\varphi \oplus U_\varphi$, and two φ-invariant subspaces $V', V'' \subseteq V$ such that $V = V' \oplus V''$, if $\varphi' := \varphi_{|_{V'}}$, $\varphi'' := \varphi_{|_{V''}}$, and $V' = W_{\varphi'} \oplus U_{\varphi'}$ and $V'' = W_{\varphi''} \oplus U_{\varphi''}$ are the respective AST-decompositions, one has that*

$$W_\varphi = W_{\varphi'} \oplus W_{\varphi''} .$$

Proof From the hypothesis of this lemma we have that $W_{\varphi'} \cap W_{\varphi''} = \{0\}$.

Moreover, from Lemma 3.11 we know that $W_{\varphi'} \subseteq W_\varphi$ and $W_{\varphi''} \subseteq W_\varphi$.

Accordingly, to prove the claim we only need to check that $W_\varphi = W_{\varphi'} + W_{\varphi''}$.

Let us now consider $w \in W_\varphi$ such that $w = (w' + u') + (w'' + u'') \in V' + V''$, with $w' \in W_{\varphi'}$, $w'' \in W_{\varphi''}$, $u' \in U_{\varphi'}$ and $u'' \in U_{\varphi''}$. Bearing in mind that

$$u' + u'' = w - w' - w'' \in W_\varphi \cap U_\varphi = \{0\} ,$$

we have that

$$w = w' + w'' \in W_{\varphi'} + W_{\varphi''} ,$$

from where the statement is deduced. □

Proposition 3.13 *Let V be an arbitrary k-vector space, let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism and let $V', V'' \subseteq V$ be two φ-invariant subspaces of V such that $V = V' \oplus V''$. If we denote $\varphi' := \varphi_{|_{V'}}$ and $\varphi'' := \varphi_{|_{V''}}$, then*

$$\tilde{c}_\varphi(x) = [\tilde{c}_{\varphi'}(x)] \cdot [\tilde{c}_{\varphi''}(x)] .$$

Proof The claim is immediately deduced from Definition 3.1 and Lemma 3.12. □

Let us consider again an arbitrary k-vector space V, a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ and a φ-invariant subspace $V' \subseteq V$. We can define $\overline{\varphi} \in \mathrm{End}_k(V/V')$ as the linear map

$$\begin{aligned} \overline{\varphi}\colon V/V' &\longrightarrow V/V' \\ [v] &\longmapsto [\varphi(v)] \end{aligned} ,$$

which is clearly a finite potent endomorphism. Our goal now is to demonstrate the relationship between $\tilde{c}_\varphi(x)$, $\tilde{c}_{\varphi'}(x)$ and $\tilde{c}_{\overline{\varphi}}(x)$.

Lemma 3.14 *Let us consider an arbitrary k-vector space V, a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ and a φ-invariant subspace $V' \subseteq V$. If $\varphi' = \varphi_{|_{V'}}$, and $V = W_\varphi \oplus U_\varphi$ and $V' = W_{\varphi'} \oplus U_{\varphi'}$ are the AST-decompositions induced by φ and φ' respectively, for every $v \in V$ such that $\varphi^m(v) \in V'$ for a certain $m \in \mathbb{N}$, one has that $v \in W_{\varphi'} + U_\varphi$.*

Proof Given $v \in V$ such that $\varphi^m(v) \in V'$, let us write $\varphi^m(v) = w' + u'$, with $w' \in W_{\varphi'}$ and $u' \in U_{\varphi'}$. Since $w' = \varphi^m(\tilde{w}')$ for a unique $\tilde{w}' \in W_{\varphi'}$, then

$$\varphi^m(v - \tilde{w}') = u' \in U_\varphi$$

and, therefore, $v - \tilde{w}' \in U_\varphi$, because $\varphi(\tilde{v}) \in U_\varphi$ if and only if $\tilde{v} \in U_\varphi$.

Accordingly, the statement is proved. □

Lemma 3.15 *With the previous notation, if $V = W_\varphi \oplus U_\varphi$, $V' = W_{\varphi'} \oplus U_{\varphi'}$ and $V/V' = W_{\overline{\varphi}} \oplus U_{\overline{\varphi}}$ are the AST-decompositions induced by φ, $\varphi_{|_{V'}}$ and $\overline{\varphi}$ respectively, one has that*

$$U_{\overline{\varphi}} = (W_{\varphi'} + U_\varphi)/V' .$$

Proof If $[v] \in U_{\overline{\varphi}}$, it is known that $\overline{\varphi}^m([v]) = [0]$ for a certain $m \in \mathbb{N}$, which is equivalent to $\varphi^m(v) \in V'$. Hence, it follows from Lemma 3.14 that $v \in W_{\varphi'} + U_\varphi$ and $[v] \in (W_{\varphi'} + U_\varphi)/V'$.

Conversely, given $[\bar{v}] \in (W_{\varphi'} + U_\varphi)/V'$, we have that $\bar{v} = w' + u + v'$, with $w' \in W_{\varphi'}$, $u \in U_\varphi$ and $v' \in V'$. Thus, if $\varphi^n(u) = 0$, one has that $\varphi^n(\bar{v}) \in V'$, that implies $[\bar{v}] \in U_{\overline{\varphi}}$ and the assertion is deduced. □

Proposition 3.16 *Keeping the above notation, one has that*

$$W_{\overline{\varphi}} = (U_{\varphi'} + W_\varphi)/V' .$$

Proof Since $(U_{\varphi'} + W_\varphi)/V'$ is clearly invariant by $\overline{\varphi}$, bearing in mind Lemma 3.15, to prove this claim we need to check that

$$V/V' = [(U_{\varphi'} + W_\varphi)/V'] \oplus [(W_{\varphi'} + U_\varphi)/V']$$

and that the restriction of $\overline{\varphi}$ to $(U_{\varphi'} + W_\varphi)/V'$ is an automorphism.

Let us first consider $[v] \in V/V'$ such that

$$[v] \in [(U_{\varphi'} + W_\varphi)/V'] \cap [(W_{\varphi'} + U_\varphi)/V'] ,$$

that is, $[v] = [w' + u] = [u' + w]$ with $w' \in W_{\varphi'}$, $w \in W_\varphi$, $u' \in U_{\varphi'}$ and $u \in U_\varphi$. Under this hypothesis, one has that

$$(w' + u) - (u' + w) = \tilde{w}' + \tilde{u}' ,$$

with $\tilde{w}' \in W_{\varphi'}$ and $\tilde{u}' \in U_{\varphi'}$. Then,

$$w' - w - \tilde{w}' = u' - u + \tilde{u}' \in W_{\varphi'} \cap U_{\varphi'} = \{0\} ,$$

from where we deduce that $w = w' - \tilde{w}' \in V'$, $u = u' + \tilde{u}' \in V'$ and $[v] = [0]$.

Moreover, given $[v] \in V/V'$, with $v = w + u$, $w \in W_\varphi$ and $u \in U_\varphi$, since $[w] \in (W_{\varphi'} + U_\varphi)/V'$ and $[u] \in (U_{\varphi'} + W_\varphi)/V'$, we obtain that

$$V/V' = [(U_{\varphi'} + W_\varphi)/V'] + [(W_{\varphi'} + U_\varphi)/V'] .$$

Accordingly, we have proved that

$$V/V' = [(U_{\varphi'} + W_\varphi)/V'] \oplus [(W_{\varphi'} + U_\varphi)/V'] .$$

Finally, to check that the restriction of $\overline{\varphi}$ to $(U_{\varphi'} + W_\varphi)/V'$ is an automorphism, we only have to see that this linear map is injective, because $(U_{\varphi'} + W_\varphi)/V'$ is a finite dimensional vector space.

Indeed, if $\overline{\varphi}([u' + w]) = [0]$, where $u' \in U_{\varphi'}$ and $w \in W_\varphi$, one has that $\varphi(u' + w) = v' \in V'$ and

$$\varphi(w) = v' - \varphi(u') \in V' .$$

Then, since from Lemma 3.11 we know that $\varphi(w) \in W_\varphi \cap V' = W_{\varphi'}$, we deduce that $w \in W_{\varphi'}$, $[u' + w] = [0]$, the restriction of $\overline{\varphi}$ to $(U_{\varphi'} + W_\varphi)/V'$ is injective and it is an automorphism. □

Proposition 3.17 *Let V be an arbitrary k-vector space, let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism and let $V' \subseteq V$ be a φ-invariant subspace. If $\varphi' \in \mathrm{End}_k(V')$ is the restriction of φ to V', $\overline{\varphi} \in \mathrm{End}_k(V/V')$ is the linear map induced by φ and $V = W_\varphi \oplus U_\varphi$, $V' = W_{\varphi'} \oplus U_{\varphi'}$ and $V/V' = W_{\overline{\varphi}} \oplus U_{\overline{\varphi}}$ are the corresponding AST-decompositions, then there exists an exact sequence of finite-dimensional vector spaces*

$$0 \longrightarrow W_{\varphi'} \xrightarrow{i} W_\varphi \xrightarrow{\pi} W_{\overline{\varphi}} \longrightarrow 0 .$$

where $i \colon W_{\varphi'} \hookrightarrow W_\varphi$ is the natural immersion and

$$\begin{aligned} \pi \colon W_\varphi &\longrightarrow W_{\overline{\varphi}} = (U_{\varphi'} + W_\varphi)/V' \\ w &\longmapsto [w] \end{aligned} .$$

Proof It is clear that i is injective and it follows from Lemma 3.11 that

$$\mathrm{Ker}\, \pi = W_\varphi \cap V' = W_{\varphi'} = \mathrm{Im}\, i .$$

Hence, to prove the claim we only have to check that π is surjective. To do it, let us consider a vector $[v] \in W_{\overline{\varphi}}$. Since $v = u' + w + v'$, with $u' \in U_{\varphi'}$, $w \in W_\varphi$ and $v' \in V'$, one has that $\pi(w) = [w] = [v]$, from where we obtain that π is surjective and the statement is deduced. □

Theorem 3.18 *With the hypothesis of Proposition 3.17, we have that*

$$\tilde{c}_\varphi(x) = [\tilde{c}_{\varphi'}(x)] \cdot [\tilde{c}_{\overline{\varphi}}(x)] .$$

Proof This assertion is immediately deduced from Definition 3.1 and Proposition 3.17, because from the properties of the characteristic polynomial of an endomorphism on a finite-dimensional vector space we know that

$$c_{\varphi_{|W_\varphi}}(x) = [c_{\varphi_{|W_{\varphi'}}}(x)] \cdot [c_{\overline{\varphi}_{|W_{\overline{\varphi}}}}(x)]$$

and $\varphi_{|W_{\varphi'}} = (\varphi')_{|W_{\varphi'}}$. □

Let us now consider a Hilbert space $\mathcal{H}$ and a bounded finite potent linear operator $\varphi \in B_{fp}(\mathcal{H})$ with spectrum $\sigma(\varphi)$. If $\mathcal{H} = W_\varphi \oplus U_\varphi$ is the AST-decomposition determined by φ, it follows from [8, Proposition 3.14] that the spectrum of φ is:

- $\sigma(\varphi) = \{\lambda_1, \dots, \lambda_m\}$ when $i(\varphi) = 0$;
- $\sigma(\varphi) = \{0, \lambda_1, \dots, \lambda_m\}$ when $i(\varphi) \geq 1$,

where $\{\lambda_1, \dots, \lambda_m\}$ are the eigenvalues of $\varphi_{|W_\varphi}$.

Lemma 3.19 *If $\mathcal{H}$ is a Hilbert space and $\varphi \in B_{fp}(\mathcal{H})$ is a bounded finite potent linear operator with spectrum $\sigma(\varphi)$, then $0 \neq \lambda \in \sigma(\varphi)$ if and only if λ is a root of $\tilde{c}_\varphi(x)$.*

Proof The assertion is a direct consequence of the fact that the set of eigenvalues on an endomorphism of a finite dimensional vector space coincides with the set of roots of its characteristic polynomial. □

Given now a non-zero $\lambda \in \mathbb{C}$, from [8, Proposition 4.8] we know that $\lambda \in \sigma(\varphi^*)$ if and only if $\overline{\lambda} \in \sigma(\varphi)$, where $\overline{\lambda}$ is the conjugate of λ. In particular, $\sigma(\varphi^*) = \overline{\sigma(\varphi)}$. Moreover, the algebraic multiplicity of a non-zero eigenvalue λ of φ^* coincides with the algebraic multiplicity of $\overline{\lambda}$ as an eigenvalue of φ.

From these results one immediately deduces that $\mathrm{Tr}_{\mathcal{H}}(\varphi^*) = \overline{\mathrm{Tr}_{\mathcal{H}}(\varphi)}$.

Lemma 3.20 *Given a Hilbert space $\mathcal{H}$ and a bounded finite potent endomorphism $\varphi \in B_{fp}(\mathcal{H})$, if*

$$\tilde{c}_\varphi(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}\,,$$

one has that

$$\tilde{c}_{\varphi^*}(x) = x^n + \sum_{i=1}^{n} \overline{a}_i x^{n-i}\,.$$

Proof If $\{\lambda_1, \dots, \lambda_n\}$ is the listing of all roots of $\tilde{c}_\varphi(x)$, counted up to algebraic multiplicity, it is known that

$$\tilde{c}_\varphi(x) = x^n + \sum_{i=1}^{n} S_i(\lambda_1, \dots, \lambda_n) x^{n-i}\,,$$

where $\{S_i(\lambda_1, \dots, \lambda_n)\}_{i \in \{1,\dots,n\}}$ are the elementary symmetric polynomials.

Hence, from the characterization of the spectrums $\sigma(\varphi)$ and $\sigma(\varphi^*)$, we have that

$$\tilde{c}_{\varphi^*}(x) = x^n + \sum_{i=1}^{n} S_i(\overline{\lambda}_1, \dots, \overline{\lambda}_n) x^{n-i}\,,$$

from where the claim is deduced. □

Remark 3.21 Note that the assertion of Lemma 3.20 can not be proved from the properties of the characteristic polynomial of a linear operator on a finite dimensional Hilbert space because, in general, W_φ and $W_{\varphi^*} = [U_\varphi]^\perp$ are different subspaces of $\mathcal{H}$.

To finish this section we shall offer an illustrative example of the previous results.

Example 1 Let $\{u_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$. If we consider $\varphi \in B_{fp}(\mathcal{H})$ determined by the conditions

$$\varphi(u_j) = \begin{cases} (5-9i)u_1 + (4-9i)u_2 - u_3 + (4-9i)u_4 & \text{if } j = 1 \\ (-1+6i)u_1 + (1+6i)u_2 + u_3 + (1+6i)u_4 & \text{if } j = 2 \\ 5u_1 + (8+2i)u_2 + (3+i)u_3 + (8+2i)u_4 & \text{if } j = 3 \\ 0 & \text{if } j = 4 \\ \frac{1}{j^2}u_4 & \text{if } j \geq 5 \end{cases},$$

it is clear that the AST-decomposition of V induced by φ is $V = W_\varphi \oplus U_\varphi$, with $W_\varphi = \langle u_1, u_2 + u_4, u_3\rangle$ and $U_\varphi = \overline{\langle u_i\rangle_{i\geq 4}}$.

Then, since the matrix associated with $\varphi_{|W_\varphi}$ in this basis of W_{φ^*} is

$$\varphi_{|W_\varphi} \equiv \begin{pmatrix} 5-9i & -1+6i & 5 \\ 4-9i & 1+6i & 8+2i \\ -1 & 1 & 3+i \end{pmatrix},$$

one has that

$$\begin{aligned}\tilde{c}_\varphi(x) &= x^3 - (9-2i)x^2 + (27-17i)x - (26-26i) \\ &= (x-2)(x-(2-3i))(x-(5+i)).\end{aligned}$$

Thus, the spectrum of φ is

$$\sigma(\varphi) = \{0, 2, 2-3i, 5+i\}.$$

Moreover, an easy computation shows that

$$\varphi^*(u_j) = \begin{cases} (5+9i)u_1 - (1+6i)u_2 - 5u_3 & \text{if } j = 1 \\ (4+9i)u_1 + (1-6i)u_2 + (8-2i)u_3 & \text{if } j = 2 \\ -u_1 + u_2 + (3-i)u_3 & \text{if } j = 3 \\ (4+9i)u_1 + (1-6i)u_2 + (8-2i)u_3 + \sum_{h\geq 5}\frac{1}{h^2}u_h & \text{if } j = 4 \\ 0 & \text{if } j \geq 5 \end{cases},$$

$W_{\varphi^*} = U_\varphi^\perp = \langle u_1, u_2, u_3\rangle$ and $U_{\varphi^*} = W_\varphi^\perp = \langle u_2 - u_4\rangle \oplus \overline{\langle u_j\rangle_{j\geq 5}}$.

Accordingly, bearing in mind that the matrix associated with ${\varphi^*}_{|W_{\varphi^*}}$ in the referred basis of W_{φ^*} is

$$(\varphi^*)_{|W_{\varphi^*}} \equiv \begin{pmatrix} 5+9i & 4+9i & -1 \\ -1-6i & 1-6i & 1 \\ -5 & 8-2i & 3-i \end{pmatrix},$$

we obtain that

$$\begin{aligned}\tilde{c}_{\varphi^*}(x) &= x^3 - (9+2i)x^2 + (27+17i)x - (26+26i) \\ &= (x-2)(x-(2+3i))(x-(5-i)),\end{aligned}$$

from where we deduce that $\sigma(\varphi^*) = \{0, 2, 2+3i, 5-i\}$.

Furthermore, from the explicit expressions of $\tilde{c}_\varphi(x)$ and $\tilde{c}_{\varphi^*}(x)$, we get that $\mathrm{Tr}_{\mathcal{H}}(\varphi) = 9-2i$ and $\mathrm{Tr}_{\mathcal{H}}(\varphi^*) = 9+2i$, that coincide with the sums of the respective eigenvalues.

4 Arithmetic properties of algebraic curves

This final section is devoted to apply the results of the previous one to obtain arithmetic properties of algebraic curves.

Let K be again a commutative k-algebra, V a K-module, and A a k-subspace of V such that, for all $f \in K$, one has that $fA < A$, that is: $(A + fA)/fA$ is finite dimensional.

Recall from Sect. 2.5 that, for each pair of elements f and g in K, one has that $[f_1, g_1]_A^V \in \operatorname{End}_k(V)$ is a finite potent endomorphism when f_1 and g_1 are endomorphisms in E satisfying the following conditions:

- Both $f \equiv f_1 \pmod{E_2}$ and $g \equiv g_1 \pmod{E_2}$;
- Either $f_1 \in E_1$ or $g_1 \in E_1$.

Moreover, we have that $\operatorname{Res}_A^V(fdg) = \operatorname{Tr}_V([f_1, g_1]_A^V)$ and it is independent of the choices made.

According to [10, Remark, page 152], if $f, g \in K$ we can compute $\operatorname{Res}_A^V(fdg)$ in finite terms as follows. Let

$$
\begin{aligned}
B &= A + gA \\
C &= B \cap f^{-1}A + (fg)^{-1}A = \{v \in B \text{ such that } fv \in A \text{ and } fgv \in A\}
\end{aligned}
$$

and let π_A be a k-linear projection of $A + fA + fgA$ onto A, then the dimension of B/C is finite and

$$
\operatorname{Res}_A^V(fdg) = \operatorname{Tr}_{B/C}([\pi_A f, g]) \,. \tag{3}
$$

Indeed, if we extend π_A to a projection of V onto A, then $\pi_A f \in E_1$ and $\pi_A f \equiv f(\text{ mod.} E_2)$, so $\operatorname{Res}_A^V(fdg) = \operatorname{Tr}_V([\pi_A f, g])$. On the other hand, $[\pi_A f, g] = \pi_A fg - g\pi_A f$ maps V into B, and C into 0 (because $fg = gf$). Hence (3) holds by the properties of the trace of a finite potent endomorphism.

In particular, with these assumptions we have that $[\pi_A f, g] \in \operatorname{End}_k(V)$ is a finite potent endomorphism.

Lemma 4.1 *If K is a commutative k-algebra, V a K-module, and A a k-subspace of V such that, for all $f \in K$, one has that $fA < A$, for every $f, g \in K$ such that $fA \subset A$ and $gA \subset A$, one has that $[\pi_A f, g] = 0$ for each projection $\pi_A \in \operatorname{End}_k(V)$.*

Proof If $f, g \in K$ are such that $fA \subset A$ and $gA \subset A$, then $A = B = C$ and, therefore, $[\pi_A f, g] = 0$. □

Corollary 4.2 *With the previous notation, if A is a K-submodule of V, then $[\pi_A f, g] = 0$ for every $f, g \in K$ and for all projection $\pi_A \in \operatorname{End}_k(V)$.*

Proof The assertion is a direct consequence of Lemma 4.1. □

Lemma 4.3 *If K is a commutative k-algebra, V a K-module, A a k-subspace of V such that, for all $f \in K$, one has that $fA < A$, and $V' \subset V$ is a K-submodule, if $A' = A \cap V$, then we have that $fA' < A'$ for every $f \in K$.*

Proof If $f \in K$ and we consider the natural map

$$
\begin{aligned}
h\colon fA' + A' &\longrightarrow (fA + A)/A \\
v &\longmapsto [v]
\end{aligned}
\,,
$$

since $\operatorname{Ker} h = (fA' + A') \cap A$ and

$$A' \subseteq (fA' + A') \cap A \subseteq V' \cap A = A',$$

we obtain that the induced linear map

$$\bar{h} \colon (fA' + A')/A' \hookrightarrow (fA + A)/A$$

is injective and, therefore, $(fA' + A')/A'$ has finite dimension, from where we deduce that $fA' < A'$ for every $f \in K$. □

Lemma 4.4 *Keeping the notation of Lemma 4.3, if $\pi_A \in \operatorname{End}_k(V)$ is a projection of V onto A such that $(\pi_A)_{|_{V'}} \in \operatorname{End}_k(V')$, then $(\pi_A)_{|_{V'}}$ is a projection of V' onto A'.*

Proof Bearing in mind that π_A is a projection, then $(\pi_A)_{|_{V'}}$ is also a projection because

$$(\pi_A)_{|_{V'}} \circ (\pi_A)_{|_{V'}} = (\pi_A \circ \pi_A)_{|_{V'}} = (\pi_A)_{|_{V'}},$$

and $\operatorname{Im}((\pi_A)_{|_{V'}}) = V' \cap A = A'$. □

Proposition 4.5 *If K is a commutative k-algebra, V a K-module, $V' \subset V$ is a K-submodule, A a k-subspace of V such that, for all $f \in K$, one has that $fA < A$, and $A' = A \cap V$, if $\pi_A \in \operatorname{End}_k(V)$ is such that $(\pi_A)_{V'} \in \operatorname{End}_k(V')$, then one has that $[\pi_A f, g]_{|_{V'}} = [\pi_{A'} f, g]$, with $\pi_{A'} = (\pi_A)_{|_{V'}}$, for every $f, g \in K$.*

Proof Bearing in mind that V' is a K-submodule of V, the claim is immediately deduced from Lemma 4.4. □

Similar to Sect. 2.6, let us now consider a non-singular and irreducible algebraic curve C over a perfect field k with function field Σ_C. For each closed point $p \in C$ we again write $A_p = \hat{\mathcal{O}}_p$ and $K_p = (\hat{\mathcal{O}}_p)_0$.

Moreover, setting

$$V_C = \prod_{p \in C}{}' K_p = \{f = (f_p) \text{ such that } f_p \in K_p \text{ and } f_p \in A_p \text{ for almost all } p\},$$

one has that Σ_C can be regarded as a subspace of V_C by means of the diagonal embedding and, for all $f \in \Sigma_C$, we have that $fA_C < A_C$, with $A_C = \prod_{p \in C} A_p \subset V_C$.

Given $f, g \in \Sigma_C$, let us consider the set S of C consisting of the poles of f and g. Since $S = \{p_1, \dots, p_r\}$ is finite, we have that

$$V_S = \prod_{i=1}^{r} K_{p_i} \quad \text{and} \quad A_S = \prod_{i=1}^{r} A_{p_i}.$$

Denoting $T = C - S$, if we write $V_T = \prod_{p \in T}{}' K_p$ and $A_T = \prod_{p \in T} A_p$, it is clear that V_S and V_T are Σ_C-submodules of V_C, that $V_C = V_S \oplus V_T$ and $A_C = A_S \oplus A_T$. Furthermore, $A_C \cap V_S = A_S$ and $A_C \cap V_T = A_T$.

Lemma 4.6 *If $f, g \in \Sigma_C$ and all poles of f and g are in $C - T$, one has that $[\pi_{A_T} f, g] = 0$ for each projection $\pi_{A_T} \in \operatorname{End}_k(V_T)$.*

Proof The statement is a direct consequence of Lemma 4.1, because $fA_C \subseteq A_C$ and $gA_C \subseteq A_C$. □

Proposition 4.7 *With the previous notation, if $\pi_{A_C} \in \mathrm{End}_k(V_C)$ is a projection such that $\pi_{A_S} = (\pi_{A_C})_{|A_S} \in \mathrm{End}_k(V_S)$ and $(\pi_{A_C})_{|A_T} \in \mathrm{End}_k(V_T)$, then*

$$\tilde{c}_{[\pi_{A_C} f, g]}(x) = \tilde{c}_{[\pi_{A_S} f, g]}(x) \,.$$

Proof This claim is immediately deduced from Proposition 3.13, Proposition 4.5 and Lemma 4.6. □

For each closed point, we fix now an isomorphism $K_p \simeq k(p)((t_p))$ and, since $k((t_p)) = k(p)[[t_p]] \oplus t_p^{-1}k(p)[t_p^{-1}]$, we have a decomposition $K_p = A_p \oplus (A_p)_-$, that induces a projection $\pi_{A_p} \in \mathrm{End}_k(K_p)$.

If $f, g \in K$ and $S = \{p_1, \dots, p_r\} \subset C$ is again a set containing all the poles of f and g, we shall denote by $\pi_{A_C} \in \mathrm{End}_k(V_C)$ to the projection induced by the decomposition

$$V_C = A_C \oplus [(A_S)_- \oplus (A_T)_-] \quad \text{with} \quad V_S = A_S \oplus (A_S)_- \quad \text{and} \quad V_T = A_T \oplus (A_T)_- \,.$$

Moreover, fixing a projection $\pi_{A_T} \in \mathrm{End}_k(V_T)$ induced by a decomposition $V_T = A_T \oplus (A_T)_-$, henceforth, we shall write $\pi_{A_C} \in \mathrm{End}_k(V_C)$ to refer to the projection induced by the decomposition

$$V_C = A_C \oplus [(A_{p_1})_- \oplus \cdots \oplus (A_{p_r})_- \oplus (A_T)_-] \,.$$

It is clear that $(\pi_{A_C})_{|K_p} = \pi_{A_p}$ and $(\pi_{A_C})_{|V_T} = \pi_{A_T}$.

Theorem 4.8 (Pseudo-characteristic Reciprocity Law) *If C is a non-singular and irreducible algebraic curve over a perfect field k with function field Σ_C, for every $f, g \in \Sigma_C$ we have that*

$$\tilde{c}_{[\pi_{A_C} f, g]}(x) = \prod_{p \in C} \tilde{c}_{[\pi_{A_p} f, g]}(x) \,. \tag{4}$$

Proof The statement is obtained from Definition 3.1, Proposition 3.13 Proposition 4.5, Lemma 4.6 and Proposition 4.7, because if $q \in C$ is such that q is not a pole of f or g, then it follows from Lemma 4.1 that $[\pi_{A_q} f, g] = 0$ and, therefore, $\tilde{c}_{[\pi_{A_q} f, g]}(x) = 1$. □

From this reciprocity law we obtain a new proof of the Residue Theorem:

Theorem 4.9 (Residue Theorem) *If C is a non-singular and irreducible algebraic curve over a perfect field k with function field Σ_C, for every $f, g \in \Sigma_C$ we have that*

$$\sum_{p \in C} \mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f dg)] = \mathrm{Res}_{A_C}^{V_C}(f dg) \,.$$

Proof Bearing in mind that $\mathrm{Res}_{A_p}^{K_p}(f dg) = \mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f dg)]$ for every $p \in C$, the assertion is immediately deduced from Theorem 4.8 because, when $\tilde{c}_{[\pi_{A_p} f, g]}(x) \neq 1$, we have that

$$\tilde{c}_{[\pi_{A_p} f, g]}(x) = x^{n_p} - [\mathrm{Res}_{A_p}^{K_p}(f dg)] x^{n_p - 1} + \cdots$$

and

$$\tilde{c}_{[\pi_{A_C} f, g]}(x) = x^{n_C} - [\mathrm{Res}_{A_C}^{V_C}(f dg)] x^{n_C - 1} + \cdots \,.$$

□

Corollary 4.10 *If C is a non-singular, complete and irreducible algebraic curve over a perfect field k with function field Σ_C, for every $f, g \in \Sigma_C$ we have that*

$$\sum_{p\in C} \mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f\,dg)] = 0\,.$$

Proof Since $\mathrm{Res}_{A_C}^{V_C}(f\,dg) = 0$ when C is non-singular, complete and irreducible, this classical result is deduced from Theorem 4.9. □

Remark 4.11 A remaining question is to relate Taelman's Trace Formula proved in [9, Theorem 3] with the above Pseudo-characteristic Reciprocity Law.

Let us now write

$$\tilde{c}_{[\pi_{A_p} f,g]}(x) = x^{n_p} + \sum_{i=1}^{n_p} R^i_{\pi_{A_p}}(f,g)x^{n_p-i}$$

and

$$\tilde{c}_{[\pi_{A_C} f,g]}(x) = x^{n_C} + \sum_{i=1}^{n_C} R^i_{\pi_{A_C}}(f,g)x^{n_C-i}.$$

If C is a non-singular and irreducible curve, from Theorem 4.8 we have an arithmetic formula for Σ_C for each coefficient of $\tilde{c}_{[\pi_{A_C} f,g]}(x)$.

Thus, for $R^2_{\pi_{A_C}}(f,g)$, we have that

$$R^2_{\pi_{A_C}}(f,g) = \sum_{p\in C} R^2_{\pi_{A_p}}(f,g) + \prod_{\substack{p,q\in C\\ p\neq q}} (\mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f\,dg)])(\mathrm{Tr}_{k(q)/k}[\mathrm{Res}_q(f\,dg)])\,. \tag{5}$$

When C is complete, readers can check that (5) is equivalent to the expression

$$\sum_{p\in C} \mathrm{Tr}_{A_p}([\pi_{A_p} f,g]^2) = \mathrm{Tr}_{V_C}([\pi_{A_C} f,g]^2)\,.$$

Moreover, for $R^3_{\pi_{A_C}}(f,g)$, one has that

$$R^3_{\pi_{A_C}}(f,g) = \sum_{p\in C} R^3_{\pi_{A_p}}(f,g) - \prod_{\substack{p,q\in C\\ p\neq q}} (\mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f\,dg)])R^2_{\pi_{A_q}}(f,g)\,. \tag{6}$$

And, if C is complete, one has that (6) is analogous to

$$R^3_{\pi_{A_C}}(f,g) = \sum_{p\in C} R^3_{\pi_{A_p}}(f,g) + \prod_{p\in C} [(\mathrm{Tr}_{k(p)/k}[\mathrm{Res}_p(f\,dg)])R^2_{\pi_{A_p}}(f,g)]\,.$$

Similarly, we can obtain arithmetic expressions related to two functions $f, g \in \Sigma_C$ for every $R^i_{\pi_{A_C}}(f,g)$, with $i \geq 4$.

Conjecture 4.12 *The polynomials $\tilde{c}_{[\pi_{A_C} f,g]}(x)$ and $\tilde{c}_{[\pi_{A_p} f,g]}(x)$ are independent of the projections π_{A_C} and π_{A_p} chosen.*

It should be noted that the assertion of Conjecture 4.12 is equivalent to stating that $\mathrm{Tr}_{V_C}([\pi_{A_C} f, g]^i)$ and $\mathrm{Tr}_{K_p}([\pi_{A_p} f, g]^i)$ are independent of the projections π_{A_C} and π_{A_p} chosen for all $i \geq 1$. It is known from [10] that this claim holds for $i = 1$.

Remark 4.13 (Final Remark) If Conjecture 4.12 is true, we wish to point out that the expressions (5) and (6) will only depend of the functions $f, g \in \Sigma_C$ (and the same for the induced formulas by the coefficients $R^i_{\pi_{A_C}}(f, g)$ for each $i \geq 4$).

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