On d-Choquet integrals and differential privacy

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Abstract

This manuscript overviews the contribution of the author to the utilization of d-Choquet integrals with a guarantee for differential privacy, a topic that has received considerable attention in 2025.

1 Introducción

These notes encompass the presentation of the paper titled "The use of d-Choquet integrals for differential privacy: experimental evidence," which was delivered by the author at the International congress Mathematical Modelling in Engineering & Human Behaviour (MME&HB2025) held in Valencia, Spain, on July 9th, 2025. The notes anticipate the contents of the forthcoming [2].

The research presented here is in continuation of Alcantud [1], a work that launched the investigation of differential privacy with aggregation operators extending the Choquet integral [10]. Recall that Choquet integrals extend weighted average means and OWA (for ordered weighted averaging) aggregation operators [5, 19], and that their applications are multiple [4, 12]. New theoretical results, plus a few experiments, are put forward in the slides presented below that supplement the contents of [1]. By doing so they amplify the scope of application of the pioneering Torra [17] and its sequel by Alcantud [3], who started the research of differential privacy issues with the Choquet integral. to the d-Choquet integrals defined by Bustince *et al.* [9]. The innovation of [9] was the utilization of an axiomatic notion of dissimilarity studied in Bustince *et al.* [8] as a replacement of the naive subtraction.

These works pave the way to the investigation of data privacy issues in other extensions of the discrete Choquet integral that have been recently proposed, e.g., the d-CC integrals [13, 15] that extend CC-integrals [14], d_G-Choquet integrals [16], and d-XC integrals [18].

2 The presentation

The research corresponding to the slides printed below will soon appear at the website https://imm.webs.upv.es/jornadas/2025/home.html

The use of d-Choquet integrals for differential privacy: experimental evidence

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Motivation and goal of this presentation

▷ Choquet integrals extend both Weighted average means (WAM) and Ordered weighted average (OWA) operators.

They have been generalized by Bustince *et al.* (2021) with the use of restricted dissimilarity functions (Bustince *et al.*, 2008).

▷ Within the study of differential privacy (a mathematical expression of privacy preservation for data analysis from databases):

Torra (2025) and A. (2025a) investigate Choquet integrals.

A. (2025b) has initiated the investigation of differentially private d-Choquet integrals (MDAI, September 2025, València).

Goal: explore this problem further, inclusive of numerical experiments.

Basic definitions

Preliminary notation

We work with an arbitrary set $N = \{e_1, \ldots, e_T\}$ with T elements.

For any $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}^T$, a vector $\mathbf{a}_{\nearrow} = (a_{\sigma(1)}, \ldots, a_{\sigma(T)})$ is presented with $a_{\sigma(1)} \leqslant \ldots \leqslant a_{\sigma(T)}$ for a bijective mapping $\sigma : \{1, \ldots, T\} \longrightarrow \{1, \ldots, T\}$ (i.e., a permutation of the indices).

Associated with $\mathbf{a}$ and j we refer to $L_j^{\mathbf{a}} = \{\sigma(j), \ldots, \sigma(T)\}$, the set of indices corresponding to the largest $T - j + 1$ components of $\mathbf{a}$.

As in Torra (2025) and A. (2025a,b), we consider the space of databases $\mathcal{D} = [0,1]^T = \{f | f : N \longrightarrow [0,1]\}$. We let $f_i = f(e_i) \in [0,1]$.

We write $f \sim \bar{f}$ when $f, \bar{f} \in \mathcal{D}$ differ in at most 1 element.

We denote by $rg(M)$ the range of any mapping M.

Capacities

We identify $N = \{1, \ldots, T\}$, it represents the set of indices too.

Definition. A discrete fuzzy measure (or a **capacity**) is a set function $\mu : 2^N \longrightarrow [0,1]$ that is monotonic (i.e., $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B \subseteq N$) and satisfies $\mu(\varnothing) = 0$, $\mu(N) = 1$.

- The smallest (or null, sometimes degenerate) fuzzy measure $\mu_{\mathbb{S}}$ assigns $\mu_{\mathbb{S}}(N) = 1$, and $\mu_{\mathbb{S}}(N') = 0$ when $N' \subsetneq N$.
- The largest (also called universal or maximal) fuzzy measure $\mu_{\mathbb{L}}$ assigns $\mu_{\mathbb{L}}(\varnothing) = 0$, and $\mu_{\mathbb{L}}(N') = 1$ when $\varnothing \neq N' \subseteq N$.

The discrete Choquet integral

Definition. Fix a discrete fuzzy measure μ on N. The discrete Choquet integral $C^{\mu} : \mathbb{R}_+^T \longrightarrow \mathbb{R}_+$ associated with μ evaluates each $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}_+^T$ by

$$C^{\mu}(\mathbf{a}) = \sum_{i=1}^{T} \left(a_{\sigma(i)} - a_{\sigma(i-1)}\right) \mu(L_i^{\mathbf{a}}),$$

where the permutation σ ensures $\mathbf{a}_{\nearrow} = \left(a_{\sigma(1)}, \ldots, a_{\sigma(T)}\right)$ and the convention $a_{\sigma(0)} = 0$ applies.

- ▷ The discrete Choquet integral associated with $\mu_{\mathbb{S}}$, resp., $\mu_{\mathbb{L}}$, concides with the minimum, resp., maximum, function.

Torra (2025) considers the Choquet integral on $\mathcal{D}$ –the space of databases– associated with μ: for any $f \in \mathcal{D}$, $C^{\mu}(f) = C^{\mu}(f_1, \ldots, f_T)$.

Reformulation of the discrete Choquet integral

Beliakov *et al.* (2020) report on a reformulation that sums up over all evaluations by the fuzzy measure: for each $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}_+^T$,

$$C^\mu(\mathbf{a}) = \sum_{S \subseteq N} g_S(\mathbf{a}) \mu(S),$$

where $g_S(\mathbf{a}) = \max\{\min_{j \in S} a_j - \max_{j \notin S} a_j, 0\}$.

Convention: minimum/maximum over the empty set is zero.

Usefulness of this expression e.g., in learning fuzzy measures: Having an index that runs over all subsets is very convenient to set up the corresponding programming problems (Beliakov *et al.*, 2020).

The d-discrete Choquet integral – preliminaries

It replaces the arithmetic subtraction of values with a restricted dissimilarity function:

Definition (Bustince *et al.*, 2008). The mapping $\delta : [0,1]^2 \to [0,1]$ is a *restricted dissimilarity function* when for all $a_1, a_2, a_3 \in [0,1]$:

a) $\delta(a_1, a_2) = \delta(a_2, a_1)$,
b) $\delta(a_1, a_2) = 0 \Leftrightarrow a_1 = a_2$, and $\delta(a_1, a_2) = 1 \Leftrightarrow \{a_1, a_2\} = \{0, 1\}$,
c) $a_1 \leqslant a_2 \leqslant a_3 \Rightarrow \delta(a_1, a_2) \leqslant \delta(a_1, a_3)$ and $\delta(a_2, a_3) \leqslant \delta(a_1, a_3)$.

Examples. $\delta_1(a, b) = |\sqrt{a} - \sqrt{b}|$, $\delta_2(a, b) = |a^2 - b^2|$, $\delta_3(a, b) = |a - b|$, for all $a, b \in [0,1]$.

The d-discrete Choquet integral

Definition (Bustince *et al.*, 2021). Fix a discrete fuzzy measure μ on N and $\delta : [0,1]^2 \to [0,1]$, a restricted dissimilarity function. The *discrete d-Choquet integral* (with T variables) with respect to μ and δ is

$$C^{\mu,\delta}(a_1,\ldots,a_T) = \sum_{i=1}^{T} \delta(a_{\sigma(i)}, a_{\sigma(i-1)})\mu(L_i^{\mathbf{a}}), \, \forall \mathbf{a} = (a_1,\ldots,a_T) \in \mathbb{R}_+^T,$$

where the permutation σ ensures $\mathbf{a}_{\nearrow} = (a_{\sigma(1)},\ldots,a_{\sigma(T)})$ and the convention $a_{\sigma(0)} = 0$ applies.

Example. If the dissimilarity is $\delta_3(a,b) = |a-b|$ then $C^{\mu,\delta_3} = C^{\mu}$.

A. (2025b) defines d-Choquet integrals on $\mathcal{D}$ by

$$C^{\mu,\delta}(f) = C^{\mu,\delta}(f(e_1),\ldots,f(e_T)) = C^{\mu,\delta}(f_1,\ldots,f_T) \in [0,T], \text{ each } f \in \mathcal{D}.$$

A theoretical result

for d-Choquet integrals

A new expression for d-Choquet integrals

A new compact expression for the d-Choquet integral that uses the standard sign (or signum) piecewise function:

Proposition. In the conditions above, the discrete d-Choquet integral defined is expressed as: for each $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}_+^T$,

$$C^{\mu,\delta}(a_1, \ldots, a_n) = \sum_{S \subseteq N} \gamma_S(\mathbf{a})\mu(S),$$

where $\gamma_S(\mathbf{a}) = \max\{\mathrm{sgn}(\min_{j \in S} a_j - \max_{j \notin S} a_j) \cdot \delta(\min_{j \in S} a_j, \max_{j \notin S} a_j), 0\}$ for all $S \subseteq N$.

Particular expression for d-Choquet integrals

A more restricted version with the help of the median function:

Proposition. In the conditions of the previous Proposition, suppose that δ is such that for each $a, b \in [0, 1]$, $\delta(a, b) \leqslant |a - b|$.

Then the discrete d-Choquet integral is: for each $\mathbf{a} = (a_1, \ldots, a_T) \in \mathbb{R}_+^T$,

$$C^{\mu,\delta}(a_1, \ldots, a_n) = \sum_{S \subseteq N} \gamma'_S(\mathbf{a})\mu(S),$$

where $\gamma'_S(\mathbf{a}) = \text{med}\{\min_{j \in S} a_j - \max_{j \notin S} a_j, \delta(\min_{j \in S} a_j, \max_{j \notin S} a_j), 0\}$ for all $S \subseteq N$.

Differential privacy

Differential privacy for our model

Dwork (2006) defined differential privacy in terms of a "privacy budget" ϵ. Smaller values of $\epsilon \Rightarrow$ better privacy; absolute privacy if $\epsilon = 0$.

The implementation of this concept by the Laplace mechanism uses the **sensitivity** of a query or function (Dwork, 2006).

We are concerned with disclosure of the **output of an aggregation of databases by a d-Choquet integral** $C^{\mu,\delta}$. Its sensitivity is

$$\Delta_{\mathcal{D}}(C^{\mu,\delta}) = \max\{|C^{\mu,\delta}(f) - C^{\mu,\delta}(\bar{f})| : f, \bar{f} \in \mathcal{D} \text{ and } f \sim \bar{f}\}.$$

Then $L_{C^{\mu,\delta}}(f) = C^{\mu,\delta}(f) + L\left(0, \frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}\right)$ is the **differentially private d-Choquet integral defined from μ and δ**.

$L\left(0, \frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}\right)$: Laplacian noise, 0 mean, scale parameter $\frac{\Delta_{\mathcal{D}}(C^{\mu,\delta})}{\epsilon}$.

$\Rightarrow Pr(L_{C^{\mu,\delta}}(f) \in S) \leqslant e^{\epsilon} \cdot Pr(L_{C^{\mu,\delta}}(\bar{f}) \in S)$ when $f \sim \bar{f}$, $S \subseteq rg(L_{C^{\mu,\delta}})$.

Experiments

General setup

We consistently use the fuzzy measure μ_0 and data from Torra (2025).

Table 1 Fuzzy measure for assessing the candidates in Table 2 (based on a measure identified from Table 8.5 in [28])

	$\emptyset$	$\{E_5\}$	$\{E_4\}$	$\{E_4, E_5\}$
$\emptyset$	0.0000	0.4400	0.1974	0.6292
$\{E_3\}$	0.0000	0.5809	0.5614	0.7045
$\{E_2\}$	0.2000	0.5709	0.2000	0.6957
$\{E_2, E_3\}$	0.7049	0.7049	0.7320	0.8157
$\{E_1\}$	0.4386	0.6084	0.6197	0.8067
$\{E_1, E_3\}$	0.6997	0.7485	0.7442	0.8594
$\{E_1, E_2\}$	0.5859	0.7064	0.6840	0.9191
$\{E_1, E_2, E_3\}$	0.8250	0.8250	0.8251	1.0000

Table 2 Experts evaluation for 10 candidates (based on Example 8.6 and Table 8.5 in [28] and the fuzzy measure in Table 1)

	E_1	E_2	E_3	E_4	E_5	asses
s_1	0.8	0.9	0.8	0.1	0.1	0.6975
s_2	0.7	0.6	0.9	0.2	0.3	0.6000
s_3	0.7	0.7	0.7	0.2	0.6	0.6125
s_4	0.6	0.9	0.9	0.4	0.4	0.7765
s_5	0.8	0.6	0.3	0.9	0.9	0.8000
s_6	0.2	0.4	0.2	0.8	0.1	0.3015
s_7	0.1	0.2	0.4	0.1	0.2	0.1705
s_8	0.3	0.3	0.3	0.8	0.3	0.3987
s_9	0.5	0.2	0.1	0.2	0.1	0.3000
s_{10}	0.8	0.2	0.2	0.5	0.1	0.5000

Two types of experiments compare the rankings produced from the aggregation by a d-Choquet integral with the ranking produced from aggregation with privacy preserving d-Choquet integrals (i.e., executions of an appropriate random modification of it described before).

Experimental setup (Torra, 2025), $\delta_3(x, y) = |x - y|$

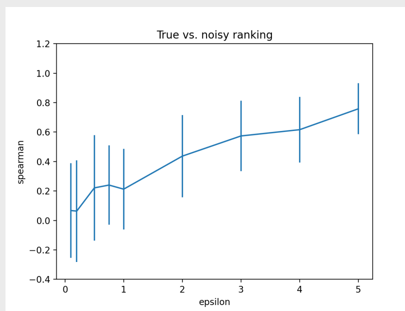

Figure: Spearman's correlation coefficients for the ranking derived from Choquet integral vs. 50 simulations with its Laplacian modification ensuring ϵ-differential privacy (replication of the experiment producing Figure 1 of Torra, 2025).

For each ϵ, we represent both average / standard deviation of Spearman's correlation coefficient.

Experiment 1 - setup

We compare the results of the first experiment in Torra (2025) with the case where other dissimilarities are used.

- Torra calculates $\Delta_{\mathcal{D}}(C^{\mu_0}) = 0.7049$. Recall $C^{\mu_0} = C^{\mu_0,\delta_3}$ with $\delta_3(a,b) = |a-b|$ for each $a, b \in [0,1]$.
- A. (2025b) proves a general argument that in particular, guarantees $\Delta_{\mathcal{D}}(C^{\mu_0,\delta_1}) = \Delta_{\mathcal{D}}(C^{\mu_0,\delta_2}) = \Delta_{\mathcal{D}}(C^{\mu_0})$.

We reproduce Torra's simulation with the two privacy preserving d-Choquet integrals defined from μ_0 and either δ_1 or δ_2.

We modify Torra's freely available Python software https://mdai.cat/code/.

Experiment 1 - results

δ_2 seems to be associated with a better performance.

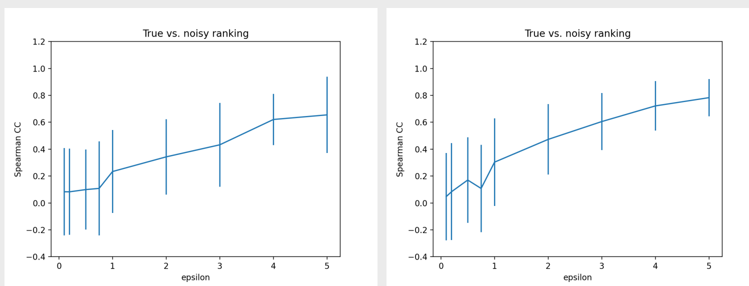

Figure: Spearman's correlation coefficients for the ranking derived from two d-Choquet integrals versus 50 simulations with its Laplacian modification ensuring ϵ-differential privacy. We use δ_1 (left) and δ_2 (right).

Experiment 2 - setup

We compare the performance of new privacy preserving d-Choquet integrals, with the results obtained with the dissimilarities δ_1 and δ_2 in the case of the μ_0 measure (Experiment 1).

We replicate the experiments **changing the fuzzy measure**.

Arguments in A. (2025b) prove that the sensitivity of the d-Choquet integrals that they define jointly with the smallest/largest fuzzy measures is 1: $\Delta_{\mathcal{D}}(C^{\mu_{\mathbb{S}},\delta_1}) = \Delta_{\mathcal{D}}(C^{\mu_{\mathbb{L}},\delta_1}) = \Delta_{\mathcal{D}}(C^{\mu_{\mathbb{S}},\delta_2}) = \Delta_{\mathcal{D}}(C^{\mu_{\mathbb{L}},\delta_2}) = 1.$

Experiment 2 - results

The d-Choquet integral is respectively defined **from δ_1 and the smallest and largest fuzzy measures**, and its privacy preserving modifications are defined with the appropriate sensitivity (namely, 1).

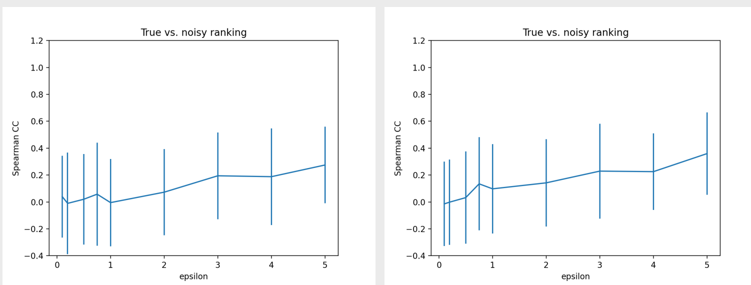

Figure: Spearman's correlation coefficients for the ranking derived from d-Choquet integral vs. 50 simulations with its Laplacian modification ensuring ϵ-differential privacy. We use δ_1 with smallest (left) and largest fuzzy measure.

Experiment 2 - results

The d-Choquet integral respectively defined **from δ_2 and the smallest and largest fuzzy measures**, with its privacy preserving modifications (defined with sensitivity 1).

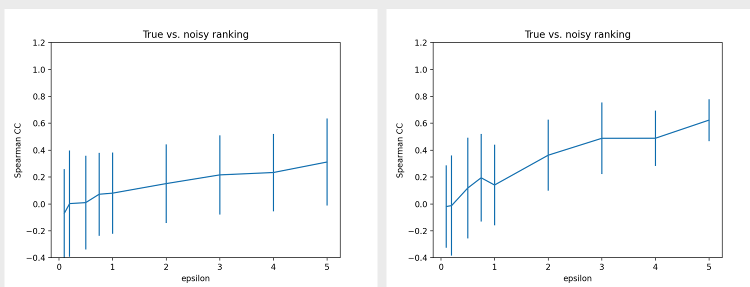

Figure: Spearman's correlation coefficients for the ranking derived from d-Choquet integral vs. 50 simulations with its Laplacian modification ensuring ϵ-differential privacy. We use δ_2 with smallest (left) and largest (right) fuzzy

Conclusions

Conclusions

The investigation of differential privacy in the context of generalized forms of the Choquet integral is a promising avenue for further research.

Our work paves the way to the investigation of data privacy issues in other extensions of the discrete Choquet integral that have been recently proposed, e.g., the d-CC integrals (Sartori *et al.*, 2023) that extend CC-integrals (Lucca *et al.*, 2017), d_G-Choquet integrals (Takáč *et al.*, 2022), and d-XC integrals (Wieczynski *et al.*, 2022).

References

[1] Alcantud, J.C.R.: The differentially private d-Choquet integral: an extension of differentially private Choquet integrals. In: V. Torra et al. (ed.) MDAI 2025. Springer Nature (2025). Proceedings of the 22nd International Conference on Modeling Decisions for Artificial Intelligence

[2] Alcantud, J.C.R.: Modelling for Engineering & Human Behaviour 2025, chap. “The use of d-Choquet integrals for differential privacy: experimental evidence”. Universitat Politécnica de València (2025)

[3] Alcantud, J.C.R.: New classes of differentially private Choquet integrals for privacy preservation (2025). Submitted

[4] Alcantud, J.C.R., de Andrés Calle, R.: The problem of collective identity in a fuzzy environment. Fuzzy Sets and Systems **315**, 57–75 (2017). DOI https://doi.org/10.1016/j.fss.2016.02.011

[5] Alcantud, J.C.R., Santos-García, G., Akram, M.: OWA aggregation operators and multi-agent decisions with N-soft sets. Expert Systems with Applications **203**, 117430 (2022). DOI 10.1016/j.eswa.2022.117430. URL `https://www.sciencedirect.com/science/article/pii/S0957417422007680`

[6] Beliakov, G., James, S., Wu, J.: Discrete fuzzy measures. Computational Aspects, *Studies in Fuzziness and Soft Computing*, vol. 382. Springer Nature Switzerland AG, Cham, Switzerland (2020)

[7] Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners, *Studies in Fuzziness and Soft Computing*, vol. 221. Springer-Verlag Berlin Heidelberg, Heidelberg (2007)

[8] Bustince, H., Barrenechea, E., Pagola, M.: Relationship between restricted dissimilarity functions, restricted equivalence functions and normal EN-functions: Image thresholding invariant. Pattern Recognition Letters **29**(4), 525–536 (2008). DOI 10.1016/j.patrec.2007.11.007. URL `https://www.sciencedirect.com/science/article/pii/S0167865507003686`

[9] Bustince, H., Mesiar, R., Fernández, J., Galar, M., Paternain, D., Altalhi, A., Dimuro, G., Bedregal, B., Takáč, Z.: *d*-Choquet integrals: Choquet integrals based on dissimilarities. Fuzzy Sets and Systems **414**, 1–27 (2021). DOI 10.1016/j.fss.2020.03.019. URL `https://www.sciencedirect.com/science/article/pii/S0165011420301032`

[10] Choquet, G.: Theory of capacities. Annales de l’institut Fourier **5**, 131–295 (1954)

[11] Dwork, C.: Differential privacy. In: M. Bugliesi, B. Preneel, V. Sassone, I. Wegener (eds.) International colloquium on automata, languages, and programming, *Lecture Notes in Computer Science*, vol. 4052, pp. 1–12. Springer (2006)

[12] Grabisch, M., Labreuche, C.: A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. Annals of Operations Research **175**(1), 247–290 (2010). DOI 10.1007/s10479-009-0655-8. URL `https://shs.hal.science/halshs-00496558`

[13] Lucca, G., Dalmazo, B.L., Ayres, D.G., Asmus, T., Sartori, J., Santos, H., Borges, E., Bedregal, B., Bustince, H., Dimuro, G.P.: dcc-integrals: A generalization of cc-integrals by restricted dissimilarity functions with applications on two different contexts. Applied Soft Computing **181**, 113517 (2025). DOI https://doi.org/10.1016/j.asoc.2025.113517. URL `https://www.sciencedirect.com/science/article/pii/S1568494625008282`

[14] Lucca, G., Sanz, J.A., Dimuro, G.P., Benjamín, Asiain, M.J., Elkano, M., Bustince, H.: CC-integrals: Choquet-like copula-based aggregation functions and its application in fuzzy rule-based classification systems. Knowledge-Based Systems **119**, 32–43 (2017). DOI https://doi.org/10.1016/j.knosys.2016.12.004. URL `https://www.sciencedirect.com/science/article/pii/S0950705116304889`

[15] Sartori, J., Lucca, G., Asmus, T., Santos, H., Borges, E., Bedregal, B., Bustince, H., Dimuro, G.P.: d-CC integrals: Generalizing CC-integrals by restricted dissimilarity functions with applications to fuzzy-rule based systems. In: M.C. Naldi, R.A.C. Bianchi (eds.) Intelligent Systems, pp. 243–258. Springer Nature Switzerland, Cham (2023)

[16] Takáč, Z., Uriz, M., Galar, M., Paternain, D., Bustince, H.: Discrete IV d_G-Choquet integrals with respect to admissible orders. Fuzzy Sets and Systems **441**, 169–195 (2022). DOI https://doi.org/10.1016/j.fss.2021.09.013. URL `https://www.sciencedirect.com/science/article/pii/S0165011421003432`

[17] Torra, V.: Differentially private Choquet integral: extending mean, median, and order statistics. International Journal of Information Security **24**, 68 (2025). DOI 10.1007/s10207-025-00984-7

[18] Wieczynski, J., Fumanal-Idocin, J., Lucca, G., Borges, E.N., Asmus, T.d.C., Emmendorfer, L.R., Bustince, H., Dimuro, G.P.: *d*-XC integrals: On the generalization of the expanded form of the Choquet integral by restricted dissimilarity functions and their applications. IEEE Transactions on Fuzzy Systems **30**(12), 5376–5389 (2022). DOI 10.1109/TFUZZ.2022.3176916

[19] Yager, R.R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Transactions on Systems, Man and Cybernetics **18**(1), 183–190 (1988). DOI 10.1109/21.87068