A note on “d-Choquet integrals: Choquet integrals based on dissimilarities”

José Carlos R. Alcantud[a]

[a] *BORDA Research Group and IME, University of Salamanca, 37007 Salamanca, Spain*

Abstract

The d-Choquet integral extends the classical Choquet integral by substituting the difference operator with a restricted dissimilarity function. Bustince *et al.* introduced Property (P1) to ensure the boundedness of this generalized integral within the unit interval. In this note, we emphasize the necessity of revising this property. Such a revision is crucial for maintaining the formal accuracy of the theoretical framework for d-Choquet integrals.

Keywords: Choquet integral, d-Choquet integral, Aggregation function, Restricted dissimilarity, Property (P1)

1. Introduction

Bustince *et al.* [8] introduced the d-Choquet integral as a generalization of the Choquet integral [5, 9] that extends weighted average means and OWAS [6, 11], wherein the standard difference $x_{\sigma(i)} - x_{\sigma(i-1)}$ is replaced by a restricted dissimilarity function δ. This innovation facilitates the extension of aggregation concepts to domains where subtraction is not the most suitable method for assessing differences between pairs of evaluations. To ensure that the output of such an integral remains within the interval $[0, 1]$, Bustince *et al.* proposed a structural constraint, which they termed Property (P1). This work demonstrates that their seminal article [8] implicitly utilized an infinite number of distinct versions of a general property, which remained undefined, without a clear differentiation. Consequently, it is imperative to clarify this matter to ensure the formal consistency of subsequent theoretical results pertaining to d-Choquet integrals.

2. The new properties $(\mathbf{P1})_T$ and $(\mathbf{P1})_\infty$

The following notion serves as the origin of the concepts investigated in this technical note:

Email address: jcr@usal.es (José Carlos R. Alcantud)

Definition 2.1. *[7] A restricted dissimilarity function is a mapping* $\delta : [0,1]^2 \to [0,1]$ *such that for all* $a, b, c \in [0,1]$*, the following properties hold:*

a) $\delta(a,b) = \delta(b,a)$*,*

b) $\delta(a,b) = 1 \Leftrightarrow \{a,b\} = \{0,1\}$*,*

c) $\delta(a,b) = 0 \Leftrightarrow a = b$*,*

d) $a \leqslant b \leqslant c \Rightarrow \delta(a,b) \leqslant \delta(a,c),\ \delta(b,c) \leqslant \delta(a,c)$*.*

Now, let us consider the following infinite set of properties, inspired by the arguments presented in [8]:

Definition 2.2. *Fix* $T \in \mathbb{N}$*. The restricted dissimilarity function* δ *satisfies property* $(P1)_T$ *when:*

$$\text{if } a_1, \ldots, a_T \in [0,1] \text{ are such that } a_1 \leqslant \cdots \leqslant a_T \text{, then } \sum_{j=1}^{T} \delta(a_{j-1}, a_j) \leqslant 1.$$

It is straightforward to demonstrate that if a restricted dissimilarity function δ satisfies property $(\mathrm{P1})_T$, then it also satisfies $(\mathrm{P1})_{T-1}, \ldots, (\mathrm{P1})_1$. However, *the converse is not necessarily true*:

Proposition 1. *For each* $T \in \mathbb{N}$*,* $T > 1$*, there exists a restricted dissimilarity function that satisfies* $(P1)_1$*,* $(P1)_2$*, …,* $(P1)_T$ *but not* $(P1)_{T+1}$*.*

We prove this Proposition using a basic family of restricted dissimilarity functions.

Example 2.1. *Fix* $a \in (0,1)$*. Consider the following restricted dissimilarity function* δ_a *defined in [8]:* $\delta_a(x,y) = \begin{cases} 1, & \text{when } \{x,y\} = \{0,1\}, \\ 0, & \text{when } x = y, \\ a, & \text{otherwise.} \end{cases}$

Then δ_a *satisfies* $(P1)_T$ *if and only if* $T \cdot a \leqslant 1$*. Therefore,* δ_a *satisfies* $(P1)_1$*,* $(P1)_2$*, …,* $(P1)_{\lfloor 1/a \rfloor}$*, but it fails to satisfy* $(P1)_{\lfloor 1/a \rfloor + 1}$*.*

Observe that δ_a with $a = \frac{1}{T} \in (0,1)$ (because $T > 1$) proves Proposition 1.

In light of the behavior exhibited by the family of properties described in Definition 2.2, the next concept becomes pertinent:

Definition 2.3. *Fix* $T \in \mathbb{N}$*. The restricted dissimilarity function* δ *satisfies property* $(P1)_\infty$ *when for each* $T \in \mathbb{N}$*,* δ *satisfies property* $(P1)_T$*.*

3. Implications for the foundational theory

Our scrutiny of the precise structure of (P1) as defined in [8] reveals inconsistencies and conceptual ambiguities in the statements proven in that article. A comparison of the statements of [8, Propositions 3.3 and 3.5] is particularly illustrative:

- The primary motivation for (P1) was to demonstrate that the range of the d-Choquet integral, with respect to any measure (on a finite set of cardinality n) and a restricted dissimilarity function δ, is contained within $[0, 1]$. This corresponds to [8, Proposition 3.3], a significant contribution whose proof is straightforward.

  We note that its statement only necessitates that δ satisfies $(\text{P1})_n$ to hold true.

  For illustration, this clarification implies that when $n = 3$, [8, Proposition 3.3] applies to $\delta_{0.3}$ but not to $\delta_{0.4}$. Without an explicit differentiation of these properties, this critical detail could be overlooked. Subsequent applications of this fundamental Proposition might be incomplete or erroneous due to this subtlety.

- A property that allows us to guarantee (P1) is articulated in [8, Proposition 3.5]. However, a cursory inspection of its proof is sufficient to observe that for [8, Proposition 3.5] to hold, its statement must refer to $(\text{P1})_\infty$.

It is not only [8, Proposition 3.5] that implicitly uses $(\text{P1})_\infty$ instead of the strictly weaker $(\text{P1})_n$ for some n. Other properties contingent upon that result are, in fact, stated in terms of $(\text{P1})_\infty$, e.g., [8, Theorems 3.8 and 3.17]. The proof of [8, Proposition 3.9] indicates that this result is also stated for $(\text{P1})_\infty$.

4. Conclusion

While Property (P1) represents a well-motivated attempt to constrain the range of d-Choquet integrals to the interval $[0, 1]$, further analysis has underscored the necessity of re-examining its formulation. Mathematical rigor demands a clear distinction among an infinite family of properties. This family is bound to play a role in future applications, such as the investigation of differential privacy for d-Choquet integrals [1, 2, 3, 4, 10], where bounding sensitivity is a critical step. Our contribution in this domain fortifies the mathematical robustness of the theoretical framework of d-Choquet integration, thereby ensuring that subsequent developments are established on a solid foundation.

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