

A new type of multi-Sugeno-inspired aggregation functions

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1 Introduction

Within the domain of aggregation functions [3], fuzzy integrals hold significant importance. Calculating them requires eliciting a fuzzy measure, a process that can be computationally demanding. Different applications necessitate different formulations of fuzzy integrals. The Sugeno integral operates solely with ordinal data [13]. It is designed to model situations involving uncertainty and imprecision, particularly when the primary concern is the relative position of the different options. When it is impossible to consistently measure the intensity of preferences, no numerical values derived from the evaluation of alternatives should be assigned meaning, thus prohibiting the use of arithmetic operations like addition or multiplication. Applications of Sugeno integration include scientometrics [10, 12] and fuzzy rule-based classification [15]. Several extensions have been explored [5, 6, 7, 11].

Relatedly, a Sugeno-inspired aggregation function [8] draws upon the principles of discrete Sugeno integration. However, it replaces the fuzzy measure with a single symmetric, monotonically increasing function that compares each input to the tuple formed by the remaining inputs. This structure avoids the computationally intensive precalculation of a fuzzy measure. Recent research [1] has introduced another variation of the Sugeno integral, which can be described as a component-sensitive extension of these Sugeno-inspired aggregation functions [8]. This model is referred to as multi-Sugeno-inspired aggregation functions. The use of multiple comparative functions (one for each component of the multivalued input) allows for comparisons independent of position, thereby enabling the use of more reliable Sugeno-inspired evaluations. Meanwhile, the model in [1] still avoids the precomputation of fuzzy measures and maintains symmetry.

The multi-Sugeno-inspired aggregation functions defined in [1] are a direct extension of the Sugeno-inspired aggregation function [8], which, in turn, are directly inspired by the disjunctive form of the classical discrete Sugeno integrals. While a conjunctive formula also exists [8, Proposition 3.4], it is not directly comparable to the standard conjunctive form of Sugeno integrals. We investigate the multi-Sugeno-inspired aggregation functions that arise from the generalization of such a conjunctive form upon the principles outlined in [8]. Comparisons with the model investigated in [1] are made too.

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2 Preliminary concepts and background

We adhere to the notation of [8]. We refer to $n \in \mathbb{N}$ with $n \geq 2$, and the finite set $N = \{1, \dots, n\}$. The set of subsets of N is $\mathcal{P}(N)$. Vectors in $[0, 1]^n$ will be denoted by bold symbols, for example, $\mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n$.

The notation $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ applies when $a, b \in [0, 1]$. It is exported to vectors in a natural way: we let $\mathbf{a} \wedge \mathbf{b} = (a_1 \wedge b_1, \dots, a_n \wedge b_n)$, $\mathbf{a} \vee \mathbf{b} = (a_1 \vee b_1, \dots, a_n \vee b_n)$, for $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in [0, 1]^n$.

We associate necessary concepts from every $\mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n$. First, we let $()$ be a permutation of N for which $a_{(1)} \leq a_{(2)} \leq \dots \leq a_{(n)}$. This permutation produces $\mathbf{a}_{\nearrow} = (a_{(1)}, a_{(2)}, \dots, a_{(n)})$, the non-decreasing sorted vector of $\mathbf{a}$. Then for each $k \in N$, the vector $\hat{\mathbf{a}}_{(k)} \in [0, 1]^{n-1}$ is the result of removing $\mathbf{a}_{\nearrow}$'s k -th component, i.e.,

$$(\hat{\mathbf{a}}_{(k)})_j = \begin{cases} a_{(j)} & \text{when } j < k \\ a_{(j+1)} & \text{when } j \geq k. \end{cases} \quad (1)$$

We will refer to aggregation functions defined as follows:

Definition [3, 9]. A mapping $f : [0, 1]^n \rightarrow [0, 1]$ is an aggregation function if it satisfies boundedness (i.e., $f(0, \dots, 0) = 0$ and $f(1, \dots, 1) = 1$) and monotonicity (i.e., $f(\mathbf{b}) \geq f(\mathbf{a})$ when $\mathbf{a}, \mathbf{b} \in [0, 1]^n, \mathbf{b} \geq \mathbf{a}$).

The following concept precedes the definition of fuzzy integrals:

Definition [4]. A mapping $m : \mathcal{P}(N) \rightarrow [0, 1]$ is a fuzzy measure (also, a capacity) on N if it is set-monotonic (i.e., $m(A) \leq m(A')$ when $A \subseteq A' \subseteq N$), $m(N) = 1$ (a requirement called the normalization condition), and $m(\emptyset) = 0$.

The Sugeno integral was designed for applications in settings defined by ordinal data [13]. Its discrete formula is as follows:

Definition [4, Sect. 5.3]. The Sugeno integral defined by m , a fuzzy measure on N , is the mapping $S_m : [0, 1]^n \rightarrow [0, 1]$ that with each $\mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n$ associates

$$S_m(\mathbf{a}) = \bigvee_{j=1}^n a_{(j)} \wedge m(H_j) = \bigwedge_{j=1}^n a_{(j)} \vee m(H_{j+1}), \quad (2)$$

where $H_j = \{(j), \dots, (n)\}$ for $j = 1, \dots, n$, and $H_{n+1} = \emptyset$.

The formulas in Eq. (2) are the classical disjunctive and conjunctive formulations of the discrete Sugeno integral, respectively. They produce monotonic or non-decreasing evaluations with an ordinal spirit. Remarkably, there is not yet a clear interpretation of their operational principles [14].

A crucial limitation of Eq. (2) is that it necessitates a fuzzy measure, but this concept is computationally demanding. It is well known that learning fuzzy measures is difficult in practical applications. This reliance is avoided in the model defined by [8], namely, Sugeno-inspired functions. This model keeps the spirit of the Sugeno integral, although now a single non-decreasing (i.e., monotonic) function F captures the importance of each input compared with the group formed by the other inputs. This strategy reduces computational complexity and simultaneously allows enough modeling flexibility through the choice of F , which still ensures that all inputs are involved in the calculations:

Definition [8]. Each non-decreasing function $F : [0, 1]^{n-1} \rightarrow [0, 1]$ defines a Sugeno-inspired function $SI_F : [0, 1]^n \rightarrow [0, 1]$ by the expression

$$SI_F(\mathbf{a}) = \bigvee_{j=1}^n a_{(j)} \wedge F(\hat{\mathbf{a}}_{(j)}), \text{ for each } \mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n. \quad (3)$$

Eq. (3) defines a symmetric function, therefore it extends OWMAX [8, Remark 3.2 (iii) and Sect. 3.3], but non-symmetric Sugeno integrals are never Sugeno-inspired functions.

A multi-Sugeno-inspired type of functions is defined and investigated in [1]. This model can be considered a component-dependent generalization of Sugeno-inspired functions that preserves the inspiration of Sugeno integrals in the disjunctive form:

Definition [1]. Let $\mathbf{M}_j : [0, 1]^{n-1} \rightarrow [0, 1]$ be a non-decreasing function ($j = 1, \dots, n$), and define $\mathbf{M} = (\mathbf{M}_1, \dots, \mathbf{M}_n)$. The multi-Sugeno-inspired function $SI_{\mathbf{M}} : [0, 1]^n \rightarrow [0, 1]$ is

$$SI_{\mathbf{M}}(\mathbf{a}) = \bigvee_{j=1}^n a_{(j)} \wedge \mathbf{M}_j(\hat{\mathbf{a}}_{(j)}), \text{ for each } \mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n. \quad (4)$$

Eq. (4) defines a monotonic and symmetric function too. It defines an aggregation function under mild conditions, for example, when $\min \leq \mathbf{M}_j$ for some $j \in N$ [1]. In this statement, $\min$ denotes the standard minimum function. The case $\mathbf{M}_1 = \dots = \mathbf{M}_n = F$ produces the Sugeno-inspired function SI_F .

3 Results

By inspiration of the alternative formulas in Eq. (2), we propose to investigate the following conjunctive variation of Sugeno-inspired functions [8] and their component-dependent generalization with a disjunctive structure [1]:

Definition. Let $\mathbf{M}_j : [0, 1]^{n-1} \rightarrow [0, 1]$ be a non-decreasing function ($j = 1, \dots, n$). Then $\mathbf{M} = (\mathbf{M}_1, \dots, \mathbf{M}_n)$ defines a conjunctive multi-Sugeno-inspired function $SI^{\mathbf{M}} : [0, 1]^n \rightarrow [0, 1]$ by

$$SI^{\mathbf{M}}(\mathbf{a}) = \bigwedge_{j=1}^n a_{(j)} \vee \mathbf{M}_j(\hat{\mathbf{a}}_{(j)}), \text{ for each } \mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n. \quad (5)$$

Proposition. Eq. (5) defines a monotonic and symmetric function.

Proof. Symmetry is obvious. Proving monotonicity is routine. Let $\mathbf{a}, \mathbf{b} \in [0, 1]^n$ such that $\mathbf{a} \geq \mathbf{b}$. Fix $(\cdot)$ and $(\cdot)'$, respective permutations of N such that $\mathbf{a}_{\nearrow} = (a_{(1)}, a_{(2)}, \dots, a_{(n)})$ and $\mathbf{b}_{\nearrow} = (b_{(1)'}, b_{(2)'}, \dots, b_{(n)'})$. Because $\mathbf{a}_{\nearrow} \geq \mathbf{b}_{\nearrow}$, one has $\mathbf{M}_j(\hat{\mathbf{a}}_{(j)}) \geq \mathbf{M}_j(\hat{\mathbf{b}}_{(j)'})$ ($j = 1, \dots, n$). Therefore $a_{(j)} \vee \mathbf{M}_j(\hat{\mathbf{a}}_{(j)}) \geq b_{(j)'} \vee \mathbf{M}_j(\hat{\mathbf{b}}_{(j)'})$ for $j = 1, \dots, n$, and $SI^{\mathbf{M}}(\mathbf{a}) = \bigwedge_{j=1}^n a_{(j)} \vee \mathbf{M}_j(\hat{\mathbf{a}}_{(j)}) \geq \bigwedge_{j=1}^n b_{(j)'} \vee \mathbf{M}_j(\hat{\mathbf{b}}_{(j)'}) = SI^{\mathbf{M}}(\mathbf{b})$. $\square$

Unlike the case of Eq. (2), the disjunctive and conjunctive variations of multi-Sugeno-inspired functions generally do not coincide.

Lemma. Fix $n = 2$ and $\mathbf{M}_j : [0, 1]^{2-1} \rightarrow [0, 1]$ ($j = 1, 2$) such that $a \leq \mathbf{M}_2(a) \leq \mathbf{M}_1(a)$, for all $a \in [0, 1]$. Then

$$SI^{\mathbf{M}}(\mathbf{a}) = \begin{cases} a_2 \vee \mathbf{M}_2(a_1), & \text{if } a_1 < a_2, \\ a_1 \vee \mathbf{M}_2(a_2), & \text{if } a_1 \geq a_2. \end{cases} \quad (6)$$

Equivalently, $SI^{\mathbf{M}}(\mathbf{a}) = \max \left\{ \mathbf{M}_2(\min\{a_1, a_2\}), \max\{a_1, a_2\} \right\}$.

Proof. It suffices to prove the first case, due to symmetry. When $a_1 < a_2$,

$$SI^{\mathbf{M}}(\mathbf{a}) = (a_1 \vee \mathbf{M}_1(a_2)) \wedge (a_2 \vee \mathbf{M}_2(a_1)) = a_2 \vee \mathbf{M}_2(a_1)$$

because $a_2 \leq \mathbf{M}_1(a_2) \leq a_1 \vee \mathbf{M}_1(a_2)$ and $\mathbf{M}_2(a_1) \leq \mathbf{M}_2(a_2) \leq \mathbf{M}_1(a_2) \leq a_1 \vee \mathbf{M}_1(a_2)$. $\square$

Example. Fix $n = 2$ and define $\mathbf{M}_j : [0, 1]^{2-1} \rightarrow [0, 1]$ ($j = 1, 2$) by $\mathbf{M}_1(a) = \sqrt[3]{a}$, $\mathbf{M}_2(a) = \sqrt{a}$ for each $a \in [0, 1]$. We can use the Lemma because $a \leq \mathbf{M}_2(a) \leq \mathbf{M}_1(a)$ for each $a \in [0, 1]$. Therefore, when $\mathbf{a} = (a_1, a_2) \in [0, 1]^2$, $SI^{\mathbf{M}}(\mathbf{a}) = \max \left\{ \sqrt{\min\{a_1, a_2\}}, \max\{a_1, a_2\} \right\}$

However, [1] proves $SI_{\mathbf{M}}(\mathbf{a}) = \min \left\{ \sqrt{\min\{a_1, a_2\}}, \max\{a_1, a_2\} \right\}$.

The case $\mathbf{M}_1 = \dots = \mathbf{M}_n = F$ produces a conjunctive multi-Sugeno-inspired function that is a variation of the Sugeno-inspired function SI_F , namely, the new function

$$SI^F(\mathbf{a}) = \bigwedge_{j=1}^n a_{(j)} \vee F(\hat{\mathbf{a}}_{(j)}), \text{ for each } \mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n. \quad (7)$$

Definition. We can consider Eq. (3) and (7) as disjunctive and conjunctive Sugeno-inspired functions defined from the non-decreasing function $F : [0, 1]^{n-1} \rightarrow [0, 1]$.

Remark. A conjunctive-type alternative expression for Sugeno-inspired functions exists [8, Propos. 3.4], namely, the function that for each $\mathbf{a} = (a_1, \dots, a_n) \in [0, 1]^n$ computes

$$SI_F(\mathbf{a}) = \begin{cases} \bigwedge_{j=1}^n a_{(j)} \vee F(\hat{\mathbf{a}}_{(j+1)}), & \text{if } a_{(1)} \leq F(\hat{\mathbf{a}}_{(1)}), \\ F(\hat{\mathbf{a}}_{(1)}), & \text{otherwise,} \end{cases}$$

where $F(\hat{\mathbf{a}}_{(n+1)}) = 0$. Therefore if $F : [0, 1]^{n-1} \rightarrow [0, 1]$ is a non-decreasing function such that $F \geq \min$, then $SI_F(\mathbf{a}) = \bigwedge_{j=1}^n a_{(j)} \vee F(\hat{\mathbf{a}}_{(j+1)})$ with the convention $F(\hat{\mathbf{a}}_{(n+1)}) = 0$. Even though this formulation emphasizes the structural similarity between Sugeno integrals and Sugeno-inspired functions, the resulting expression is not a satisfactory Sugeno-inspired function. This is because it fails to meet the fundamental requirement that the fuzzy measure is replaced with a function that compares each input to the tuple formed by the remaining inputs. We believe that the solution presented in Eq. (7) fully satisfies all the requirements to operate faithfully as a Sugeno-inspired function in a conjunctive form.

4 Conclusions

Research into Sugeno-type evaluative functions represents a promising area for future development. We have presented a comprehensive review of existing literature and new results that incorporate symmetric aggregation methods. Furthermore, it is possible to investigate non-symmetric Sugeno-inspired functions for aggregating unbalanced ordinal data. This is the focus of the pioneering work [2], which explores the trade-offs between disjunctive and conjunctive expressions. We anticipate continuing our investigation along these lines in the future.

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