QED_{2+1}: the Compton effect.

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Abstract

The Compton effect in a two-dimensional world is compared with the same process in ordinary three-dimensional space.

1 Introduction

QED, the strange theory of light and matter, [1], embraces in a unified dogma several of the most successful physical doctrines. Quantum electrodynamics describes the interactions of the electromagnetic field with electrons and positrons in a framework ruled by the laws of special relativity and quantum mechanics. According to Feynman, see Reference [1], QED is the jewel of physics: there is no significant difference between experiment and theory. Nevertheless, nobody understand why Nature works that way, quoting Feynman again. The lack of comprehension is partly due to the conceptual difficulties of both special relativity and quantum mechanics, so far from common sense, and partly due to the complexity of the phenomena involved. There are too many degrees of freedom entering the game and a nightmare of divergences must be tamed by proper physical insight.

During the eighties, interesting investigations were devoted to QED in a space-time of (2+1)-dimensions, see e.g. Reference [2]. The research was pushed forward by either purely theoretical reasons, to study conventional QED at the infinite temperature limit, or condensed matter experiments: both the Quantum Hall Effect [3] and High T_c Superconductivity [4] are many-body quantum phenomena including interactions of charged fermions with the electromagnetic field that essentially occur in two-dimensions.

Perturbation theory of planar quantum electrodynamics is rich enough to be compared with perturbative features of the theory of photons, electrons and positrons in three – dimensional space. The theoretical analysis of photon-electron scattering performed within the framework of QED_{2+1} allows for a comparative study with the same process in three dimensions. Besides the academic interest, we find it fruitful to enlarge the list of lowest order processes analyzed in QED. We compute the differential cross section, a length in two-dimensions [5], up to second order in perturbation theory starting from the usual plane wave

expansion of eld operators. We then compare the results in dierent physical situations with the outcome of the very well known analysis in three dimensions.

The organization of the paper is as follows: in Section x2 we briey present QED_{2+1} Perturbation Theory and its application to compute S-matrix elements. Section x3 is divided into two sub-sections: calculation of both the dierential and total cross-lengths of scattering for the Compton eect is performed in x3.1. The planar analogue of the Thomson and Klein-Nishina formulas are discussed in x3.2 and compared with the behaviour of the same expressions in three dimensions. Finally, in the Appendices the denition of the Dirac and electromagnetic elds in three-dimensional Minkowski space is given. Also, some useful formulas are collected and the conventions to be used throughout the paper are xed.

2 Quantum electrodynamics in the plane.

Quantum electrodynamics in the plane, QED_{2+1}, describes the interaction of two-dimensional electrons, positrons and photons by means of the quantum eld theory derived from the Lagrangian density

$$L = L_0 + L_I \tag{1}$$

with the free-eld Lagrangian density

$$L_0 = N \quad c \ (x) (ih \ @ \quad mc) \ (x) \quad \frac{1}{4} f \ (x) f \ (x) \tag{2}$$

where: (1) (x) and (x) = $Y(x)^0$ are the Dirac elds that have relativistic matter particles of spin $\frac{1}{2}$, electrons and positrons, as quanta. We take the charge of the electron as q = e < 0, and m is the mass of the particle. Study of the Dirac free-eld in (2+1) dimensions and its quantization can be found in Appendix C. (2) a (x), = 0;1;2, is the three-vector electromagnetic potential and f the associated antisymmetric tensor to the electromagnetic eld :

$$f = @ a \quad @ a :$$

A Lorentz-covariant formulation of the free electromagnetic eld in the plane, the quantization procedure leading to the identication of polarized photons as its quanta, is also developed in the Appendix D. (3) h and c are respectively the Planck constant and the speed of light in vacuum. The interaction Lagrangian density is

$$L_I = N \quad \overset{h}{e} \ (x) \ a \ (x) \ (x) \quad \overset{i}{N} \quad \frac{1}{c} j \ (x) a \ (x) \tag{3}$$

which couples the conserved current j (x) = (e)c (x) (x) (c (x);ĵ(x)) to the electromagnetic eld. We have dened the Lagrangian density as a normal product, N [ ], each creation operator standing to the left of any annihilation operator, to ensure that the vacuum expectation values of all observables vanish, (we follow the conventions of [6], chapters 4,5,6 and 7).

The action integral S for quantum electrodynamics in three-dimensional space-time is therefore

$$S = \overset{Z}{} d^3 x \ N \quad c \ (x) \quad ih @ + \frac{e}{c} a \ (x) \quad mc \quad (x) \quad \frac{1}{4} f \ (x) f \ (x) \tag{4}$$

In relativistic quantum eld theory it is convenient to work in natural units (n.u.), $h = c = 1$. In these units the fundamental dimensions are the mass (M), the action (A) and the velocity (V) instead of the mass (M), length (L) and time (T) that are the fundamental dimensions in c.g.s. (or S.I.) units. It is interesting to analyse the dimensions of the free-elds and constants that appear in the action S for QED_{2+1}, and to compare them with the dimensional features of the same magnitudes in QED_{3+1}. In general, given the dimension of the action integral,

$$S = \int d^{d+1}x \, L ; \tag{5}$$

where L is de ned by (2) and (3); the dimension of the elds is determined from the kinetic terms, and then the dimension of the coupling constants is xed. We nd that:

Quantity	c.g.s.	n.u.
Action S	$M L^2 T^{-1}$	1
Lagrangian density L	$M L^{2-d} T^{-2}$	M^{d+1}
Electromagnetic eld a (x)	$M^{\frac{1}{2}} L^{2-\frac{d}{2}} T^{-1}$	$M^{\frac{d-1}{2}}$
Dirac elds (x) and (x)	$L^{-\frac{d}{2}}$	$M^{\frac{d}{2}}$
Electric charge e	$M^{\frac{1}{2}} L^{\frac{d}{2}} T^{-1}$	$M^{\frac{3-d}{2}}$
Mass of the electron m	M	M

The c.g.s. dimensions of e^2 in $d = 3$ are $[e^2] = M L^3 T^{-2}$ $[hc]$ due to the fact that the Coulomb force decreases as $\frac{1}{r^2}$. In natural units, however, the electron charge is dimension-less. The ne structure constant $\frac{1}{137.04}$ is given by,

$$= \frac{e^2}{4 \, hc} \quad (c.g.s.) \quad \text{or} \quad = \frac{e^2}{4} \quad (n.u.) \tag{6}$$

This means that one can take or the electron charge as a good expansion parameter because both of them are dimensionless in n.u..

Things are di erent in a two-dimensional world: if $d = 2$, the electron charge is not dimensionless but $[e^2] = M L^2 T^{-2}$ in c.g.s. units, or $[e^2] = M$ in n.u. because the Coulomb force is proportional to $\frac{1}{r}$. The ne structure constant is still the expansion parameter for a perturbative treatment of QED_{2+1} but we must keep in mind that the electron charge has dimensions. The dimension of the product of e^2 times the Compton wave length $\frac{h}{mc}$ is $[e^2 \frac{h}{mc}] = M L^3 T^{-2}$ or $[\frac{e^2}{m}] = 1$, respectively in c.g.s. or n.u. systems. Therefore, we express the ne structure constant as:

$$= \frac{e^2}{4 \, m c^2} \quad (c.g.s.) \quad \text{or} \quad = \frac{e^2}{4 \, m} \quad (n.u.) \tag{7}$$

bearing in mind that e^2 is not dimensionless in natural units when d = 2. Thus, in (2+1)-dimensional Minkowski space the fine structure constant is $\frac{(\text{electron charge})^2}{\text{electron mass}}$, (up to 4π factors).

Perhaps a rapid comparison of the several systems of units used in electromagnetism will help to clarify this point: in the above formulas we have adopted the rationalized Lorentz-Heaviside system of electromagnetic units; that is, the 4π factors appear in the force equations rather than in the Maxwell equations, and the vacuum dielectric constant ϵ_0 is set equal to unity [7]. In rationalized mks units, the fine structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi \epsilon_0 \hbar c} \ (d = 3) \qquad \text{or} \qquad \alpha = \frac{e^2}{4\pi a_0 m c^2} \ (d = 2) \tag{8}$$

where a_0 has dimensions of permitivity by length. We define $a_0 = \epsilon_0 \frac{\hbar}{mc}$, the permitivity of vacuum times the fundamental length of the system. Then,

$$\alpha = \frac{e^2}{4\pi \epsilon_0 \hbar c} = \frac{e^2}{4\pi a_0 m c^2} \simeq \frac{1}{137.04} \tag{9}$$

where the rationalized electric charges $\frac{e^2}{\epsilon_0}$ and $\frac{e^2}{a_0}$ have different dimensions.

The Hamiltonian H of the system splits into the free H_0 and the interaction H_I Hamiltonians. H_I can be treated as a perturbation since the dimensionless coupling constant, characterizing the photon-electron interaction in the plane, is small enough: $\alpha \simeq \frac{1}{137.04}$. In the interaction picture the S-matrix expansion is

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^3x_1 d^3x_2 \cdots d^3x_n \ T\{H_I(x_1)H_I(x_2) \cdots H_I(x_n)\} \tag{10}$$

where $T\{\}$ is the time-ordered product. The QED_{2+1} interaction Hamiltonian density

$$H_I(x) = -L_I(x) = -eN\left[\bar{\psi}(x) \gamma^\mu a_\mu(x) \psi(x)\right] \tag{11}$$

determines the basic vertex part of the theory.

For the $|i\rangle \to |f\rangle$ transition, the S-matrix element is given by

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^3 \delta^{(3)}(P_f - P_i) \left[\prod_{ext.} \left(\frac{m}{AE}\right)^{1/2} \prod_{ext.} \left(\frac{1}{2A\omega}\right)^{1/2} \right] M. \tag{12}$$

Here, P_i and P_f are the total three-momenta of the initial and final states, the products extend over all external particles, and $A = L^2$ is a large but finite area in the plane. E and ω are the energies of the individual external fermions and photons, respectively. M is the Feynman amplitude such that: $M = \sum_{n=1}^{\infty} M^{(n)}$, and the contribution to $M^{(n)}$ (nth order in perturbation theory) from each topologically different graph is obtained from the Feynman rules [6]. We only enumerate the fundamental differences with respect to the Feynman rules in QED_{3+1}:

The four-momenta of the particles are now three-momenta.

For each initial and final electron or positron there is only one label, $s = 1$, that characterizes the spin state.

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For each initial and nal photon there is also only one label, $r = 1$, that characterizes the polarization state.

Initial and nal electrons and positrons have associated two-component spinors.

The -matrices arising at vertices and the S_F-functions, coming from propagation of internal fermion lines, are 2×2 matrices.

For each three-momentum q which is not xed by energy-momentum conservation one must carry out the integration $(2\pi)^{-3} \int d^3 q$.

3 A QED$_{2+1}$ Lowest Order Process: The Compton E ect

In this Section we shall discuss the scattering cross-section, a cross–length in $d = 2$, for planar Compton scattering up to second order in perturbation theory.

3.1 Compton Scattering

The S-matrix element for the transition

$$ji i = c^y(p) b^y(k)j0 i \quad ! \quad jf i = c^y(p^0) b^y(k^0)j0 i \tag{13}$$

to second order in e is:

$$S^{(2)} = e^2 \int d^3 x_1 d^3 x_2 N \left[(x_1) \quad a(x_1) iS_F(x_1 \quad x_2) \quad a(x_2) (x_2) \right] = S_a + S_b: \tag{14}$$

Here, $iS_F(x_1 \quad x_2)$ is the fermion propagator (64). Feynmann technology provides the formula

$$hf jS^{(2)} ji i = (2\pi)^3 \quad {}^{(3)}(p^0 + k^0 \quad p \quad k) \prod_{ext}^{Y} \left(\frac{m}{A E_p} \right)^{!1=2} \prod_{ext}^{Y} \left(\frac{1}{2A!_k} \right)^{!1=2} (M_a + M_b) \tag{15}$$

for the S-matrix element up to second order in perturbation, where the Feynman amplitudes are

$$M_a = e^2 u(p^0) \quad (k^0) iS_F(p + k) \quad (k) u(p)$$
$$M_b = e^2 u(p^0) \quad (k) iS_F(p \quad k^0) \quad (k^0) u(p): \tag{16}$$

The di erential cross–length for this process is therefore

$$d = (2\pi)^3 \quad {}^{(3)}(p^0 + k^0 \quad p \quad k) \frac{(2m)^2}{4E!v_{rel}} \frac{d^2 p^0}{(2\pi)^2 2E^0} \frac{d^2 k^0}{(2\pi)^2 2!^0} jM_{fi} j^2 \tag{17}$$

where $p = (E; p)$ and $k = (!; k)$ are the three-momenta for the initial electron and photon, and the corresponding quantities for the nal electron and photon are $p^0 = (E^0; p^0)$ and $k^0 = (!^0; k^0)$.

Analysis of the scattering of photons by electrons is easier in the laboratory frame, in which $p = (m;0;0)$ and $\hat{p}^0 = \hat{k} - \hat{k}^0$. The relative velocity in this system is unity, i. e., $v_{rel} = \frac{|k|}{!} = 1$.

From the energy-momentum consevation law $p + k = p^0 + k^0$, the Compton shift in wavelength for this process is easily deduced. In the laboratory system: $p = (m;0;0)$, $\hat{k} \cdot \hat{k}^0 = !!^0 \cos$, is the scattering angle, and

$$!^0 = \frac{m!}{m + !(1 - \cos)} \tag{18}$$

There are no differences with the three-dimensional case in this respect. The recoil energy of the electron is

$$E^0 = \sqrt{m^2 + !^2 + !^{02} - 2!!^0 \cos} \tag{19}$$

In (17) we can integrate with respect to the dependent variables $\hat{k}^0$ and $\hat{p}^0$ as a consecuence of the conservation of the initial and final momenta. Using (18) and (19), we obtain the differential cross-length in the laboratory frame

$$\frac{d}{d}\bigg|_{Lab} = \frac{1}{8!}\left(\frac{!^0}{!}\right)|M|^2 \tag{20}$$

In QED_{3+1} to obtain unpolarized cross-sections we must average $|M|^2$ over all the polarizations and spins of the initial state and sum it over all final polarizations and spin states. By doing this, we render the square of the Feynman amplitudes as the trace of products of -matrices. It is shown in Appendix E that despite the lack of polarization or spin degrees of freedom in QED_{2+1} the square of the Feynman amplitudes is also the trace of products of -matrices. The right-hand member of (20), $|M|^2$, is the sum of four terms:

$$X_{aa} = \frac{e^4}{16m^2(pk)^2}\text{Tr}\left[(\;(p+k)+m\;)(\;p+m\;)(\;(p+k)+m\;)(\;p^0+m\;)\right]$$

$$\tag{21}$$

$$X_{ab} = \frac{e^4}{16m^2(pk)(pk^0)}\text{Tr}\left[(\;(p+k)+m\;)(\;p+m\;)(\;(p-k^0)+m\;)(\;p^0+m\;)\right]$$

and $X_{bb} = X_{aa}(k \$ -k^0; \$ ^0)$, $X_{ba} = X_{ab}(k \$ -k^0; \$ ^0)$. Computation of the traces in (21) is considerably simplified by the use of the contraction identities, see Appendix A, because it involves products of up to eight -matrices. Note that the contraction identities for 2 2 -matrices are very different than the usual in (3+1)-dimensions. In short, in terms of the three linearly independent scalars $p^2 = p^{02} = m^2$, $pk = p^0k^0$ and $pk^0 = p^0k$ we have

$$X_{aa} = \frac{e^4}{4m^2(pk)^2}\left[4m^4 + 4m^2(pk) + (pk)(pk^0)\right]$$

$$X_{bb} = \frac{e^4}{4m^2(pk^0)^2}\left[4m^4 - 4m^2(pk^0) + (pk)(pk^0)\right] \tag{22}$$

$$X_{ab} = \frac{e^4}{4m^2(pk)(pk^0)}\left[4m^4 + 2m^2(pk - pk^0) - (pk)(pk^0) + i6m \; p \; k \; k^0\right]$$

$$X_{ba} = \frac{e^4}{4m^2(pk)(pk^0)}\left[4m^4 + 2m^2(pk - pk^0) - (pk)(pk^0) - i6m \; p \; k \; k^0\right]$$

In the laboratory system $pk = m\omega$, $pk^0 = m\omega^0$ and from (22) and (20) we obtain the differential cross-length for the Compton scattering in the plane

$$\frac{d\sigma}{d\Omega}\bigg|_{Lab} = \frac{\alpha^2}{2\omega}\frac{\omega^0}{\omega}\left(\frac{\omega}{\omega^0}+\frac{\omega^0}{\omega}+4\cos^2\Theta-2\right); \qquad (23)$$

$\frac{d\sigma}{d\Omega}$ has dimensions of length (or M^{-1} in the natural units system).

Plugging (18) into (23) and integrating the resulting equation over the scattering angle, we find the total cross-length

$$\sigma_{total}=\frac{\alpha^2}{m^2}\left(\frac{(1+\delta)}{(1+2\delta)^{3=2}}+1+\frac{4(1+\delta)(1+\frac{p}{1+2\delta})}{2\delta^2\frac{p}{1+2\delta}}\frac{2}{p1+2\delta}\right) \qquad (24)$$

where δ denotes the ratio of the photon initial energy to the electron rest energy, i.e., $\delta=\frac{\omega}{m}$ in natural units.

3.2 Planar Thomson and Klein-Nishina formulas

Contact between the experimental outcome of the Compton effect in the real world and the theory is established through the Klein-Nishina and Thomson formulas derived in QED_{3+1}. Also, the total cross-section offers a direct connection between theory and experiment, which is particularly fruitful at the non-relativistic and extreme relativistic limits. Very good information about the behaviour of photons when scattered by electrons can be obtained by studying the angular distribution of the unpolarized differential cross-section. In this sub-section we discuss the same aspects in QED_{2+1} and compare the results of the analysis with their higher-dimensional counterparts.

Starting with the famous Klein-Nishina formula for the polarized differential cross-section of Compton scattering:

$$\frac{d\sigma}{d\Omega}\bigg|_{Lab,pol} = \frac{\alpha^2}{4m^2}\frac{\omega^{0\,2}}{\omega}\left(\frac{\omega}{\omega^0}+\frac{\omega^0}{\omega}+4(\epsilon^{(\lambda)}\epsilon^{0(\lambda^0)})^2-2\right); \qquad (25)$$

we focus on the same magnitude in QED_{2+1}. Before, however, let us notice that $\epsilon^{(\lambda)}$ $\epsilon^{(\lambda)}(k)$ and $\epsilon^{0(\lambda^0)}$ $\epsilon^{0(\lambda^0)}(k^0)$ are the polarizations of the incident and scattered photons and $\lambda;\lambda^0=1;2$.

It is convenient to write (25) in the form

$$\frac{d\sigma}{d\Omega}\bigg|_{Lab,pol} = \frac{\alpha^2}{4m^2}\frac{\omega^{0\,2}}{\omega}\left(\frac{\omega}{\omega^0}+\frac{\omega^0}{\omega}+4\cos^2\Theta-2\right); \qquad (26)$$

where $(\epsilon^{(\lambda)}\epsilon^{0(\lambda^0)})^2=\cos^2\Theta$ and Θ is the angle formed by the polarization vectors of the incident and scattered photons. In QED_{2+1}, the polarized differential cross-length is given precisely by equation (23). We therefore call this expression the planar Klein-Nishina formula; choosing the polarization vectors of the incident, $\tilde\epsilon^{(1)}(k)$, and scattered, $\tilde\epsilon^{0(1)}(k^0)$, photons in

a reference frame where the wave vector $\vec{k}$ and $\tilde{\epsilon}^{(1)}(\vec{k})$ are respectively taken along the z–and x-axes, the formula reads:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab} = \frac{\alpha^2}{2\pi}\left(\frac{\omega'}{\omega}\right)^{\prime 0}\left(\frac{\omega}{\omega'^0} + \frac{\omega'^0}{\omega} + 4(\epsilon^{(1)}\cdot\epsilon'^{(1)})^2 - 2\right): \tag{27}$$

The angle between the polarization vectors and the scattering angle now coincide $(\epsilon^{(1)}\cdot\epsilon'^{(1)})^2 = \cos^2\theta$. It is remarkable that one could have derived the planar from the spatial Klein–Nishina formula: taking the first of the polarization vectors $\tilde{\epsilon}'^{(1)} = (\cos\theta\cos\varphi; \cos\theta \operatorname{sen}\varphi;$ $-\operatorname{sen}\theta)$, we obtain: $\cos^2\gamma = \cos^2\theta\cos^2\varphi$. For $\varphi = 0$ we almost recover the planar Klein–Nishina formula, but dimensional reasons forbid a perfect identity between both formulas and also differences between volume and area elements induce some distinct factors. The other polarization vector $\tilde{\epsilon}'^{(2)} = (\operatorname{sen}\varphi; -\cos\varphi; 0)$ is non-projectable to the plane, because when $\varphi = 0$ it points in the direction which disappears.

A subtle point; in (2+1)-dimensions there is no difference between polarized and unpolarized photon scattering because planar photons have only one polarization. The differential cross-length of scattering in QED_{2+1} can also be compared with the unpolarized differential cross-section in QED_{3+1}. At the non-relativistic limit it is given by the Thomson formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab;NR} = \frac{r_0^2}{2}(1 + \cos^2\theta) \tag{28}$$

where $r_0 = \frac{\alpha}{m}$ is the classical electron radius. At the NR limit, where $\omega \ll m$ and $\omega^0 \simeq \omega$, we find from (23) an analogous formula,

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab;NR} = \frac{2\pi\alpha^2}{\omega}\cos^2\theta \equiv \frac{l_T}{\xi}\cos^2\theta \tag{29}$$

that we shall call the planar Thomson formula. Besides the scattering angle, $\left(\frac{d\sigma}{d\Omega}\right)_{Lab;NR}$ depends on the parameter $\xi = \omega/m$, the ratio of the frequency of the incident photon to the rest electron mass. Also, we have introduced a constant with dimensions of length $l_T = (2\pi^2\alpha)r_0$ out of the two fundamental constants α and r_0, whose meaning for the problem will be clear later. Unlike the classical Thomson formula, the differential cross-length depends on the incident photon energy at the non-relativistic limit; in fact $(d\sigma/d\Omega)_{Lab;NR}$ diverges when $\omega \to 0$. $(d\sigma/d\Omega)_{Lab;NR}$, however, is ω–independent. The intensity of the scattered radiation is thus higher when ξ decreases in the planar Compton effect, but it does not change when energy varies in (3+1)-dimensions. A common point is that both differential scattering cross-legths and cross-sections of the Compton effect are backward-forward symmetric.

There are also noticeable differences in the behaviour of the scattering differential cross-length and unpolarized cross-section at the extreme relativistic limit $\omega \gg m$. If $\omega(1 - \cos\theta) \ll m$, in this regime occuring at very small scattering angles, $\omega^0 \simeq \omega$ and:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab;ER} \simeq \frac{l_T}{\xi}\cos^2\theta \quad ; \quad \left(\frac{d\sigma}{d\Omega}\right)_{Lab;ER} \simeq \frac{r_0^2}{2}(1 + \cos^2\theta) \tag{30}$$

In this case, the intensity distribution of scattered radiation obeys the same Thomson formulas as at the non-relativistic limit. The cross-lenght at stake in the planar Compton

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e ect is smaller than the classical one because the energy of the incoming photons is very high; in three dimensions the cross-section at low energy and the cross-section at high-energy and small enough angles are, however, the same. If $\lambda'^0 = \frac{m}{1-\cos\vartheta}$ and $\lambda(1-\cos\vartheta) >> m$, we are at the extreme relativistic limit looking at very large scattering angles. Thus,

$$-\frac{\pi}{2} + \vartheta \; ; \quad \left(\frac{d\lambda}{d\vartheta}\right)_{Lab,ER} \simeq \frac{l_T}{2} \; ; \quad \left(\frac{d\sigma}{d\vartheta}\right)_{Lab,ER} \simeq \frac{r_0^2}{2}\frac{1}{(1-\cos\vartheta)} \tag{31}$$

and the intensity of the planar Compton e ect decreases with increasing energy of the incoming photon, although it is independent of the scattering angle. The latter feature is not shared by the cross-section at the extreme relativistic limit of QED_{3+1}.
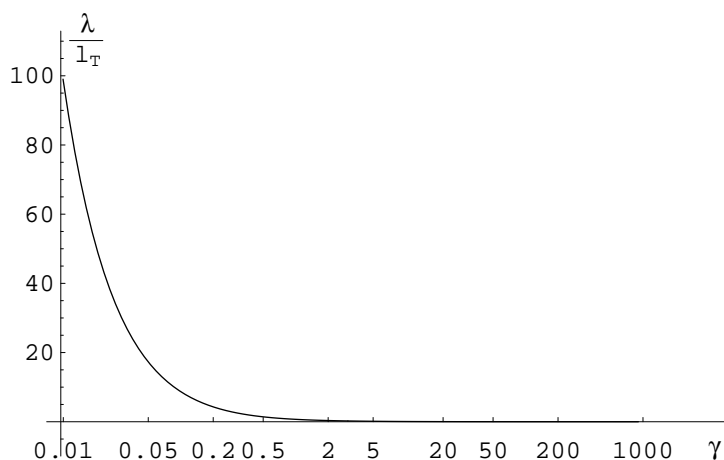

Figure 1: Total cross-length $\lambda = l_T$ for the planar Compton scattering as a function of the initial photon energy $\gamma = w/m$ on a logarithmic scale. l_T is the Thomson length de ned by $l_T = (2\pi^2)r_0$.

It is interesting to write the di erential cross-length and the unpolarized cross-section as functions of ϑ:

$$\frac{d\lambda}{d\vartheta}\bigg|_{Lab} = \frac{l_T}{4}\frac{1}{(1+\gamma(1-\cos\vartheta))}\left(\frac{1}{(1+\gamma(1-\cos\vartheta))} + (1-\cos\vartheta) + 4\cos^2\vartheta - 1\right) \tag{32}$$

$$\frac{d\sigma}{d\vartheta}\bigg|_{Lab} = \frac{r_0^2}{2}\frac{1}{(1+\gamma(1-\cos\vartheta))^2}\left(\frac{1}{(1+\gamma(1-\cos\vartheta))} + (1-\cos\vartheta) + \cos^2\vartheta\right) \tag{33}$$

From these expressions one computes respectively the total cross-length

$$\lambda = \frac{l_T}{2}\left(1 + \frac{1+\gamma}{(1+2\gamma)^{3=2}} + \frac{4(1+\gamma)(1+\frac{p}{\sqrt{1+2\gamma}})}{\gamma^2\frac{p}{\sqrt{1+2\gamma}}} - \frac{2\gamma^2}{\gamma^2}\right) \tag{34}$$

for the planar Compton e ect, and the total cross-section

$$\sigma = 2\pi r_0^2\left(\frac{1+\gamma}{\gamma^3}\left(\frac{2\gamma(1+\gamma)}{1+2\gamma} - \log(1+2\gamma)\right) + \frac{1}{2\gamma}\log(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2}\right) \tag{35}$$
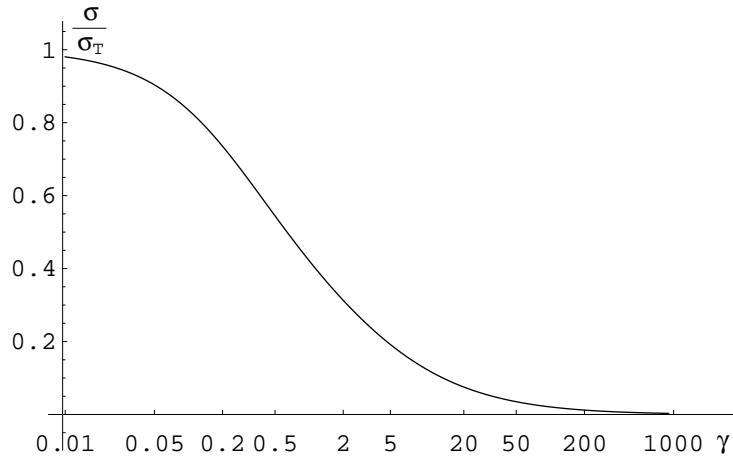
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Figure 2: Total cross-section $\sigma = \sigma_T$ for Compton scattering as a function of the initial photon energy $\gamma = w=m$ on a logarithmic scale in order to cover a large energy region. σ_T is the cross-section for the Thomson scattering, [8].

for the same process in space [8].

The non-relativistic limit, $\gamma \ll 1$, of σ and ℓ is easily obtained

$$\ell_{NR} = \frac{2\pi^2}{\gamma} r_0 \frac{l_T}{}$$ (36)

$$\sigma_{NR} = \frac{8\pi}{3} r_0^2 = 6.65 \times 10^{25} \, \text{cm}^2 :$$ (37)

The cross-section $\sigma_T \equiv \sigma_{NR}$ for Thomson scattering is constant and independent of the incoming photon frequency. ℓ_{NR}, however, depends on γ; we introduce a "natural" Thomson length $l_T = \frac{2\pi^2 \alpha^2}{m} = 4.06 \times 10^{14} \, \text{cm}$. It happens that for a photon such that $! = 0.05m = 0.0025 \, \text{Mev}$, $\ell_{NR} = 8.12 \times 10^{13} \, \text{cm} \overset{p}{\frac{}{T}}$. Other smaller values of γ lead to higher values of ℓ_{NR}: for $\gamma = 0.03$ we find $\ell_{NR} = 1.35 \times 10^{12} \, \text{cm}$, for $\gamma = 0.01$, $\ell_{NR} = 4.06 \times 10^{12} \, \text{cm}$, and so on.

At the other, extreme relativistic limit, $\gamma \gg 1$, we have

$$\ell_{ER} = \frac{l_T}{2} \left(\frac{1}{\gamma} + \frac{5}{2} \frac{1}{p_{\overline{2}}} \frac{1}{\gamma^{3=2}} \right)^!$$ (38)

$$\sigma_{ER} = \frac{3\sigma_T}{8} \left(\frac{1}{\gamma} \log 2 + \frac{1}{2} \right)^!$$ (39)

and we see that for very high energy the Compton effect is a negligible effect and pair production becomes dominant both in the plane and in the three-dimensional world. The logarithmic factor in σ_{ER} announces that ultraviolet divergences in higher order corrections will be more severe in QED$_{3+1}$.

The total cross-length ℓ and cross-section σ are respectively plotted as functions of γ in Figures 1 and 2. The intensity of the scattered radiation is very large for small energies in
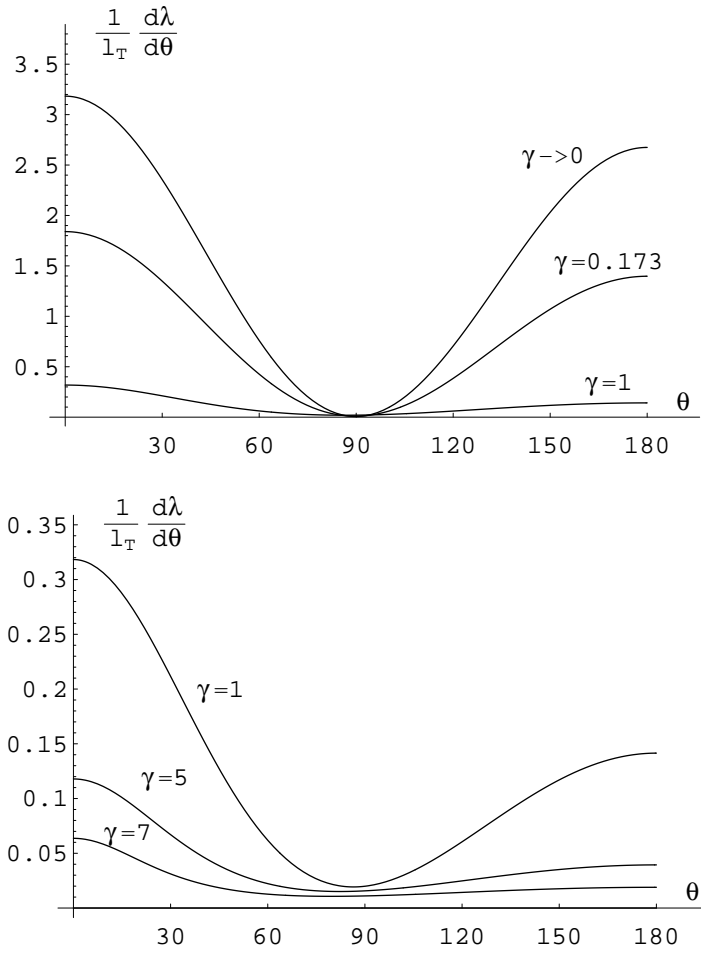
Figure 3: Angular distribution of the Planar Compton scattering as a function of the scattering angle θ for several values of the initial photon energy $\gamma = \frac{\omega}{m}$.

the plane whereas in space it is practically constant and equal to the classical value. For high energy of the incoming photons the intensity is very small in both cases, although it goes to zero faster than in the first case.

Finally, we study how $\frac{d\sigma}{d\theta}\Big|_{Lab}$ and $\frac{d\sigma}{d\Omega}\Big|_{Lab}$ depend on the scattering angle. The angular distribution of the differential cross–length is plotted in Figure 3 for several chosen values of γ between the non–relativistic limit and the high energy regime. A similar picture of $\frac{d\sigma}{d\Omega}\Big|_{Lab}$ is drawn in Figure 4. In both cases, the angular distribution is forward-backward symmetric in the non–relativistic limit ($\gamma \to 0$), whereas in the relativistic regime ($\gamma >> 1$) the forward direction becomes more and more preponderant. For small angles, the scattered intensity in the three-dimensional case has almost the classical (non–relativistic) value for all the incident energies; in the bi-dimensional case, however, we observe that the higher the incident energy, the smaller the intensity. Particularly, for $\gamma \to 0$ the planar intensity is larger than the classical value in Thomson scattering. For large angles the angular distribution is similar in both cases but whereas in the plane $\frac{d\sigma}{d\theta}\Big|_{Lab,NR}(\frac{\pi}{2}) = 0$, $\frac{d\sigma}{d\Omega}\Big|_{Lab,NR}(\frac{\pi}{2})$ is not zero. For high incident energies, the intensity is practically equal to the minimum in the range $\theta = (\pi/2;\pi)$.

A last comment on the pole found in $\frac{d\sigma}{d\theta}\Big|_{Lab}$ at $\theta = 0$: this is an infrared divergence due to soft photons. This infrared catastrophe is similar to that arising in bremsstrahlung processes in QED_{3+1}. Infrared divergences seem to be more dangerous in QED_{2+1}, but fortunately, a "topological" mass for the photons is generated by the vacuum polarization graph, see [2], and a natural infrared cut-off exists in the theory due to quantum corrections.
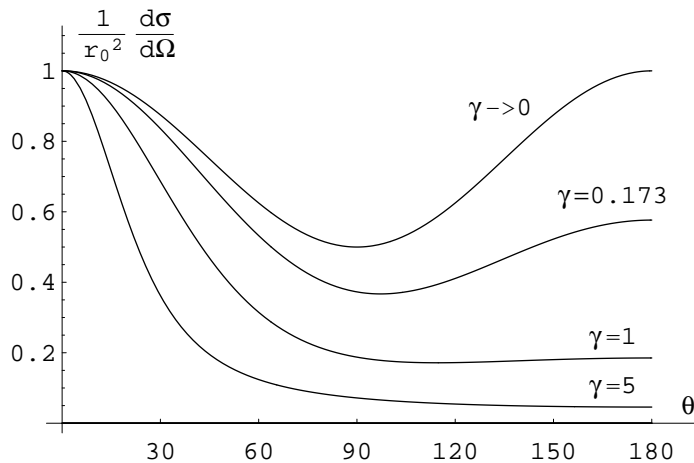

Figure 4: Angular distribution of the Compton scattering as a function of the scattering angle for several values of the initial photon energy $\gamma = \frac{\omega}{m}$ [8].

A Gamma Matrices in 3-dimensional Space-time

The Dirac (Clifford) algebra in the 3-dimensional Minkowski space $M_3 = R^{1,2}$ is built from the three gamma matrices γ^μ satisfying the anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{40}$$

$$\mu = 0,1,2 \quad ; \quad g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$$

and the hermiticity conditions $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. The tensors

$$1; \quad \gamma^\mu; \quad \gamma^{\mu_1}\gamma^{\mu_2}; \quad \gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}; \quad \mu_1 < \mu_2 < \mu_3 \tag{41}$$

with respect to the SO$(2,1)$-group, the piece connected to the identity of the Lorentz group in flatland, form the basis of the Dirac algebra, which is thus 2^3-dimensional. 1 and $\gamma^1\gamma^2\gamma^3 = i\gamma^1\gamma^2\gamma^3 1$ are respectively scalar and pseudo-scalar objects. γ^μ is a three-vector but $\gamma^1\gamma^2$ can be seen alternatively as a anti-symmetric tensor or a pseudo-vector, which are equivalent irreducible representations of the SO$(2,1)$-group. If we denote by $\epsilon^{\mu\nu\rho}$ the completely antisymmetric tensor, equal to $+1(-1)$ for an even (odd) permutation of $(0,1,2)$ and to 0 otherwise, the γ-matrices must also satisfy the commutation relations:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu ; \gamma^\nu] = \tag{42}$$

The σ-matrices are the Lie algebra generators of the spin$(1,2;R) = $ SL$(2;R)$-group, the universal covering of the connected piece of the Lorentz group.

The irreducible representations of the Lie SL$(2;R)$-group are the spinors and, before choosing a particular representation suitable for describing planar electrons and positrons, we list some algebraic identities that are useful in the evaluation of the trace of γ-matrix products:

γ-matrix contractions:

$$\gamma^\mu\gamma_\mu = 3 \quad ; \quad \gamma^\mu\gamma^\nu\gamma_\mu = 4g^{\nu} \tag{43}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = \quad ; \quad = 2 \quad + $$

γ-products:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu}g - g^{\mu}g \quad ; \quad = 2g \quad ; \quad = 6 \tag{44}$$

Traces of products of γ-matrices:

$$\mathrm{Tr}(\gamma^\mu) = 0 = \mathrm{Tr}(\gamma^\mu\gamma^\nu) \quad ; \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu} \tag{45}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = -2i\epsilon^{\mu\nu\rho} \quad ; \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

For an even number of γ-matrices,

$$\mathrm{Tr}(\gamma^\mu\cdots) = \mathrm{Tr}(\cdots) \tag{46}$$

but this identity is false for an odd number of γ-matrices. For instance, $\mathrm{Tr}(\quad) = -\mathrm{Tr}(\quad)$.

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B The Dirac Equation.

Massive classical Dirac fields satisfy the momentum space Dirac equations at rest:

$$(\gamma^0 - 1)u(0) = 0 \qquad (\gamma^0 + 1)v(0) = 0 \tag{47}$$

The spinors $u(0)$ and $v(0)$ are eigenfunctions of the spin matrix $\frac{\hbar}{2}\Sigma^{12} = \frac{\hbar}{2}\gamma^0$ with $\pm\frac{\hbar}{2}$ eigenvalues. In a spinor representation, a general Lorentz transformation, $p'^0 = \Lambda p$, $p = (\frac{E_p}{c};\vec{p})$ with $\vec{p} = (p_1;p_2)$ and $E_p = +\sqrt{m^2c^4 + c^2\vec{p}\cdot\vec{p}}$, is given by:

$$f'^0(p'^0) = \exp\left(-\frac{i}{2}\omega\cdot\Sigma\right)f(\Lambda^{-1}p) \tag{48}$$

Lorentz boosts arise when $\omega_{0i} \neq 0$ and Σ^{12} is thus the generator of rotations at the centre of mass of the system. Applying a pure Lorentz transformation to (47), we obtain the Dirac equations

$$(\gamma\cdot p - mc)u(p) = 0 \qquad (\gamma\cdot p + mc)v(p) = 0 \tag{49}$$

which are automatically Lorentz-invariant. We also write the conjugate equations satisfied by the Dirac adjoint spinors $\bar{u}(p) = u^y(p)\gamma^0$; $\bar{v}(p) = v^y(p)\gamma^0$:

$$\bar{u}(p)(\gamma\cdot p - mc) = 0 \qquad \bar{v}(p)(\gamma\cdot p + mc) = 0 \tag{50}$$

Choosing the normalizations as follows,

$$u^y(p)u(p) = v^y(p)v(p) = \frac{E_p}{mc^2} \qquad ; \qquad \bar{u}(p)u(p) = -\bar{v}(p)v(p) = 1$$

and given the orthogonality conditions,

$$u^y(p)v(-p) = 0 \qquad ; \qquad \bar{u}(p)v(p) = \bar{v}(p)u(p) = 0$$

these spinors satisfy the completeness relation:

$$u_\alpha(p)\bar{u}_\beta(p) - v_\alpha(p)\bar{v}_\beta(p) = \delta_{\alpha\beta} \qquad ; \qquad \alpha;\beta = 1;2 \tag{51}$$

in a representation of the Dirac algebra of minimal dimension.

In fact, in 3-dimensional Minkowski space (pseudo)-Majorana spinors do exist: in this representation of the Clifford Algebra, the γ-matrices are purely imaginary 2×2 matrices and the spinors have two real components. We wish to describe charged particles, so we choose the representation of the Dirac algebra as:

$$\gamma^0 = \sigma_3 \quad ; \quad \gamma^1 = i\sigma_1 \quad ; \quad \gamma^2 = i\sigma_2 \tag{52}$$

where the σ^a; $a = 1;2;3$ are the Pauli matrices. In this representation the normalized solutions of (49) are:

$$u(p) = \sqrt{\frac{E_p + mc^2}{2mc^2}}\begin{pmatrix} 1 \\ \frac{c(p_2 - ip_1)}{E_p + mc^2} \end{pmatrix} \quad ; \quad v(p) = \sqrt{\frac{E_p + mc^2}{2mc^2}}\begin{pmatrix} \frac{c(p_2 + ip_1)}{E_p + mc^2} \\ 1 \end{pmatrix} \tag{53}$$

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Putting the system in a normalization square of large but finite area, $A = L^2$, we can expand the classical Dirac field in a Fourier series:

$$\psi(x) = \psi^+(x) + \psi^-(x)$$

$$= \sum_p \left(\frac{mc^2}{AE_p}\right)^{1=2} \left[c(p)u(p)e^{\frac{ipx}{h}} + d^*(p)v(p)e^{\frac{ipx}{h}} \right] \qquad (54)$$

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x)$$

$$= \sum_p \left(\frac{mc^2}{AE_p}\right)^{1=2} \left[d(p)\bar{u}(p)e^{\frac{ipx}{h}} + c^*(p)\bar{v}(p)e^{\frac{ipx}{h}} \right]$$

The Fourier transform of equations (49) and (50) are the Dirac equation and its conjugate in the configuration space:

$$ih\gamma\frac{\partial\psi(x)}{\partial x} - mc\,\psi(x) = 0 \quad ; \quad ih\frac{\partial\bar\psi(x)}{\partial x}\gamma + mc\,\bar\psi(x) = 0 \qquad (55)$$

The plane wave expansions (54), where $c(p)$ and $d(p)$ are the Fourier coefficients, solve (55).

The symmetry group of classical $(CED)_{2+1}$ is the Poincare group, the semi-direct product of the Lorentz group times the abelian group of translations in Minkowski space. The two sheets of the hyperboloid $p^2 = m^2$,

$$\hat{O}_m^+ = \{p \ 2 \ \hat{M}_3 ; p^2 = m^2 ; p_0 > 0\} \ ; \ \hat{O}_m^- = \{p \ 2 \ \hat{M}_3 ; p^2 = m^2 ; p_0 < 0\};$$

are disconnected orbits of the Lorentz group in the dual of Minkowski space $\hat{M}_3$. Equations (47) hold at the points $p = (\pm m;0;0)$, respectively. Thus, solutions on $\hat{O}_m^-$ correspond to $E_p = -mc^2$ and, after quantization, $v(p)$ will be interpreted as the antiparticle spinor. There is a very important difference with the situation in 4-dimensional space-time: the isotropy group of the $p = (\pm m;0;0)$ points is now $SO(2)$, which has R as covering group. Therefore, the spin is a scalar in $(2+1)$-dimensions having any real value because the irreducible representations of an abelian group such as R are one-dimensional.

One can check that the Dirac equation is the Euler-Lagrange equation for the Lagrangian:

$$L = c\,\bar\psi(x)\left(ih\gamma\frac{\partial}{\partial x} - mc\right)\psi(x) \qquad (56)$$

which, besides the invariance with respect to the Poincare group transformations connected to the identity, is invariant under the discrete transformations of parity and time-reversal:
1)

$$\tilde{x}^0 = (x^{01};x^{02}) = (-x^1;x^2)$$

$$P\,\psi(x^0;\mathbf{x})P^{-1} = \gamma^1\psi(x^0;\tilde{\mathbf{x}})$$

2)

$$x^{00} = -x^0$$

$$T\,\psi(x^0;\mathbf{x})T^{-1} = \gamma^2\psi(x^{00};\mathbf{x})$$

Finally, the energy projection operators $\Lambda_\pm(p) = \frac{\pm\not{p} + mc}{2mc}$ are

$$\Lambda^+(p) = u(p)\bar{u}(p) \quad ; \quad \Lambda^-(p) = -v(p)\bar{v}(p) \qquad (57)$$

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C   The Dirac Field

From the Lagrangian, we obtain the Dirac Hamiltonian

$$H_D = \int d^2x\, \Psi^\dagger(x)\, c\left[\tilde{\alpha} \cdot (-i\hbar\tilde{\nabla}) + \beta m c^2 \right] \Psi(x) \tag{58}$$

where $\beta = \gamma^0$ and $\alpha^j = \gamma^0 \gamma^j$, $j = 1;2$, and quantize the system by promoting the Fourier coefficients to quantum operators which satisfy the anticommutation relations:

$$\{c(p)\,; c^\dagger(\tilde{p}^0)\} = \{d(p)\,; d^\dagger(\tilde{p}^0)\} = \delta_{p,\tilde{p}^0} \tag{59}$$

and all other anticommutators vanish. c and $c^\dagger$ are the annihilation and creation operators of electrons, while d and $d^\dagger$ play a similar role with respect to planar positrons. The fermionic Fock space is built out of the vacuum,

$$c(p)|0i = d(p)|0i = 0 \quad ; \quad 8p$$

by the action of strings of creation operators:

$$\prod(p_1^+)^{n(p_1)} \cdots \prod_N (p_N^+)|i \,/\, [d^\dagger(p_1^+)]^{n(p_1^+)} [c^\dagger(p_1)]^{n(p_1)} \cdots \Psi^\dagger (p_N^+)]^{n(p_N^+)} [c^\dagger(p_N^+)]^{n(p_N^+)}|0i$$

where $n(p_a) = 0$ or 1 due to the Fermi statistics coming from (59).

From (59) and the plane wave expansion (54) one obtains

$$\{\Psi_\alpha(x)\,; \Psi_\beta(y)\} = \{\Psi_\alpha^\dagger(x)\,; \Psi_\beta^\dagger(y)\} = 0 \quad ; \quad \{\Psi_\alpha(x)\,; \Psi_\beta^\dagger(y)\} = iS_{\alpha\beta}(x-y) \tag{60}$$

where the 2×2-matrix function $S(x) = S^+(x) + S^-(x)$ is given by

$$S^\pm(x) = \left(i\gamma \cdot \frac{\partial}{\partial x} + \frac{mc}{\hbar} \right) \Delta^\pm(x) \tag{61}$$

Here, $\Delta^\pm(x)$ are the invariant Δ-functions, see [6], that admit the integral representation

$$S^\pm(x) = \frac{\hbar}{(2\pi\hbar)^3} \int_{C^\pm} d^3p\, e^{\frac{ipx}{\hbar}} \frac{\gamma \cdot p + mc}{p^2 - m^2 c^2} \tag{62}$$

if $C^\pm$ are the contours in the complex p_0-plane that enclose the poles at $p_0 = \pm(E_p = c)$.

The fermion propagator $S_F(x-y)$ is the expectation value of the time-ordered product $T\{\Psi(x)\bar{\Psi}(y)\}$ at the vacuum state:

$$iS_F(x-y) = h0|T\{\Psi(x)\bar{\Psi}(y)\}|0i =$$
$$= i(\theta(x^0 - y^0) S^+(x-y) - \theta(y^0 - x^0) S^-(x-y)) = \tag{63}$$
$$= \frac{i\hbar}{(2\pi\hbar)^3} \int d^3p\, e^{\frac{ip(x-y)}{\hbar}} \frac{\gamma \cdot p + mc}{p^2 - m^2 c^2 + i\epsilon} \tag{64}$$

$\theta(x)$ is the step function, $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x > 0$.

The Dirac field at $p = 0$

$$\Psi(x) = \frac{c}{L}\left[c(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{mc^2}{\hbar}t} + d^\dagger(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\frac{mc^2}{\hbar}t} \right] \tag{65}$$

destroys an electron of spin $h=2$ and creates one positron of spin $h=2$. Quanta with any p are obtained from the center of mass states by the action of Lorentz boosts. Unlike in four-dimensional space-time, we cannot talk of helicity in a purely three-dimensional universe because the spin is a scalar.

D   The electromagnetic field in (2 + 1)-dimensions.

The canonical quantization of the electromagnetic field in (2 + 1)-dimensions is equivalent to the four-dimensional case. We shall follow the covariant formalism of Gupta and Bleuler, see [6]. We consider the Fermi Lagrangian density

$$L = -\frac{1}{2}(\partial_\nu a_\mu(x))(\partial^\nu a^\mu(x)) \tag{66}$$

where now $a_\mu(x)$; $\mu = 0;1;2$ is the three-vector potential. The fields equations are

$$\partial^2 a_\mu(x) = 0 \tag{67}$$

which are equivalent to Maxwell's equations if the potential satisfies the Lorentz condition $\partial_\mu a^\mu(x) = 0$. We expand the free electromagnetic field in a complete set of plane wave states:

$$
\begin{aligned}
a_\mu(x) &= a_\mu^+(x) + a_\mu^-(x) \\
&= \sum_{\vec{k},r} \frac{1}{\sqrt{2A\omega_{\vec{k}}}} \left[\epsilon_{\mu r}(\vec{k})b_r(\vec{k})e^{-ikx} + \epsilon_{\mu r}(\vec{k})b_r^y(\vec{k})e^{ikx} \right] \tag{68}
\end{aligned}
$$

Here, the summation is over wave vectors, allowed by the periodic boundary conditions in A, with $k^0 = \frac{1}{c}\omega_{\vec{k}} = |\vec{k}|$. The summation over $r = 0;1;2$ corresponds to the three linearly independent polarizations states that exist for each $\vec{k}$. The real polarization vectors $\epsilon_{\mu r}(\vec{k})$ satisfy the orthonormality and completeness relations

$$\sum_\mu \epsilon_\mu^r(\vec{k})\epsilon_{\mu s}(\vec{k}) = \zeta_r \delta_{rs}; \qquad r;s = 0;1;2 \tag{69}$$

$$\sum_r \zeta_r \epsilon_{\mu r}(\vec{k})\epsilon_{\nu r}(\vec{k}) = -g_{\mu\nu} \tag{70}$$

$$\zeta_0 = -1 ; \quad \zeta_1 = \zeta_2 = 1$$

The equal-time commutation relations for the fields $a_\mu(x)$ and their momenta $\pi_\mu(x) = \frac{1}{c^2}\dot{a}_\mu(x)$ are

$$[a_\mu(\mathbf{x};t);a_\nu(\tilde{\mathbf{x}}^0;t)] = [\dot{a}_\mu(\mathbf{x};t);\dot{a}_\nu(\tilde{\mathbf{x}}^0;t)] = 0$$

$$[a_\mu(\mathbf{x};t);\dot{a}_\nu(\tilde{\mathbf{x}}^0;t)] = -ihc^2 g_{\mu\nu}\delta^{(2)}(\mathbf{x} - \tilde{\mathbf{x}}^0) \tag{71}$$

The operators $b_r(\vec{k})$ and $b_r^y(\vec{k})$ satisfy

$$[b_r(\vec{k});b_s^y(\vec{k}^0)] = \zeta_r \delta_{rs}\delta_{\vec{k}\vec{k}^0} \tag{72}$$

and all other commutators vanish. For each value of r there are transverse ($r = 1$), longitudinal ($r = 2$) and scalar ($r = 0$) photons, but as result of the Lorentz condition, which in the Gupta-Bleuler theory is replaced by a restriction on the states, only transverse photons are observed as free particles. This is accomplished as follows: the states of the basis of the bosonic Fock space have the form,

$$|n_{r_1}(\vec{k}_1)n_{r_2}(\vec{k}_2)\cdots n_{r_N}(\vec{k}_N)i \propto \left[a_{r_1}^y(\vec{k}_1)\right]^{n_{r_1}(\vec{k}_1)}\left[a_{r_2}^y(\vec{k}_2)\right]^{n_{r_2}(\vec{k}_2)}\cdots\left[a_{r_N}^y(\vec{k}_N)\right]^{n_{r_N}(\vec{k}_N)}|0i;$$

where $n_{r_i}(\vec{k}_i) \in Z^+$, $\delta i = 1;2;\ldots;N$ and

$$a_r(\vec{k})|0\rangle = 0 \; ; \; r = 0;1;2$$

de nes the vacuum state. To avoid negative norm states the condition

$$\left[a_2(\vec{k}) - a_0(\vec{k}) \right] |\psi\rangle = 0 \; ; \; \forall \vec{k} \quad (\Leftrightarrow) \quad \langle\psi| N_2(\vec{k})|\psi\rangle = \langle\psi| N_0(\vec{k})|\psi\rangle$$

is required on the physical photon states of the Hilbert space. Therefore, in two dimensions, there is only one degree of freedom for each $\vec{k}$ of the radiation eld.

From the covariant commutation relations we derive the Feynman photon propagator:

$$\langle 0| T\{a_\mu(x) a_\nu(y)\}|0\rangle = i\hbar c D_{F\mu\nu}(x-y) \tag{73}$$

where

$$D_{F\mu\nu}(x) = \frac{1}{(2\pi)^3} \int d^3k \frac{-g_{\mu\nu}}{k^2 + i\epsilon} e^{-ikx} \tag{74}$$

Choosing the polarization vectors in a given frame of reference as

$$\epsilon_0(\vec{k}) = n = (1;0;0)$$
$$\epsilon_1(\vec{k}) = (0;\tilde{\epsilon}_1(\vec{k})) \; ; \; \tilde{\epsilon}_1(\vec{k}) \cdot \vec{k} = 0 \tag{75}$$
$$\epsilon_2(\vec{k}) = (0;\frac{\vec{k}}{|\vec{k}|}) = \frac{k_\mu - (kn)n_\mu}{((kn)^2 - k^2)^{1=2}}$$

it is possible to express the momentum space propagator from (74) as

$$\begin{aligned}
D_{F\mu\nu}(k) &= \frac{-g_{\mu\nu}}{k^2 + i\epsilon} \\
&= D_{FT}(k) + D_{FC}(k) + D_{FR}(k) \\
&= \frac{-1}{k^2 + i\epsilon} \epsilon_{1\mu}(\vec{k}) \epsilon_{1\nu}(\vec{k}) + \frac{n_\mu n_\nu}{(kn)^2 - k^2} + \frac{1}{k^2 + i\epsilon} \left[\frac{k_\mu k_\nu - (kn)(k_\mu n_\nu + k_\nu n_\mu)}{(kn)^2 - k^2} \right]
\end{aligned} \tag{76}$$

The rst term in (76) can be interpreted as the exchange of transverse photons. The remaining two terms follow from a linear combination of longitudinal and temporal photons such that

$$D_{FC\mu\nu}(x) = \frac{g_{\mu 0} g_{\nu 0}}{(2\pi)^3} \int \frac{d^2\vec{k}\, e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2} \int dk^0 e^{-ik^0 x^0} = -g_{\mu 0} g_{\nu 0} \frac{1}{4\pi} \ln \frac{1}{|\vec{x}|} \delta(x^0); \tag{77}$$

This term corresponds to the instantaneous Coulomb interaction between charges in the plane, and the contribution of the remainding term $D_{FR}(k)$ vanishes because the electromagnetic eld only interacts with the conserved charge-current density, [6].

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E   Spin and polarization sums in (2+1) dimensions.

In QED_{3+1} the unpolarized cross-section is obtained by averaging $|M|^2$ over all initial polarization states and summing it over all final polarization states. However, in QED_{2+1} we have seen on the one hand that there are not degrees of freedom concerning the spin of the particles and, on the other hand, that there is only one transverse polarization state for the photon: in the plane there is no need to average and sum over all the polarization states. However, it is possible to show that $|M|^2$ can be written in one of the two forms:

1. We consider a Feynman amplitude $M = \bar{u}(\tilde{p}^0)\, \Gamma\, u(p)$, where $u(p)$ and $u(\tilde{p}^0)$ are two component spinors that specify the momenta of the electron in the initial and final states, and Γ is a 2×2 matrix built up out of γ-matrices. Then

$$
\begin{aligned}
|M|^2 &= \bar{u}(\tilde{p}^0)\,\Gamma\, u(p)\overline{u(p)\,\tilde{\Gamma}\,u(\tilde{p}^0)} \\
&= u_\alpha(\tilde{p}^0)\bar{u}_\beta(\tilde{p}^0)\,\Gamma_{\beta\gamma}\,(u_\gamma(p)\bar{u}_\delta(p))\,\tilde{\Gamma}_{\delta\alpha} \\
&= \Lambda^+_{\alpha\beta}(p^0)\,\Gamma_{\beta\gamma}\,\Lambda^+_{\gamma\delta}(p)\,\tilde{\Gamma}_{\delta\alpha} \\
&= \mathrm{Tr}\left[\frac{\not{p}^0 + m}{2m}\,\Gamma\,\frac{\not{p} + m}{2m}\,\tilde{\Gamma}\right]
\end{aligned}
\tag{78}
$$

where we have used the positive energy projection operator (57) and $\tilde{\Gamma} = \gamma^0\,\Gamma^\dagger\,\gamma^0$. This can be extended to Feynman amplitudes with one or two spinors of antiparticles using the negative energy projection operator (57).

2. We consider a Feynman amplitude of the form $M = \epsilon_1^\mu(k)M_\mu(k)$, i. e., with one external photon. The gauge invariance requires $k^\mu M_\mu(k) = 0$ so that:

$$
|M|^2 = \epsilon_1^\mu(k)\,\epsilon_1^{*\nu}(k)M_\mu(k)M_\nu^*(k) = -M_\mu(k)M^{*\mu}(k);
\tag{79}
$$

Here we have used the relation

$$
\epsilon_1^\mu(k)\,\epsilon_1^{*\nu}(k) = -\left[g^{\mu\nu} - \frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + n^\mu k^\nu)}{(kn)^2}\right]
\tag{80}
$$

for a physical photon $k^2 = 0$. Once again, this formalism can be extended to several external photons.

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