QED_{2+1}: Compton effect on Dirac-Landau electrons

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Abstract

Planar Quantum Electrodynamics is developed when charged fermions are under the influence of a constant and homogeneous external magnetic field. We compute the cross-length for the scattering of optical/ultraviolet photons by Dirac-Landau electrons.

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1 Introduction

The second quantization method and the associated occupation number formalism are basic pillars in quantum field theory. Both in "fundamental" elementary particle physics and condensed matter systems one deals with many-particle ensembles and the number of particles is not conserved [1]. The fermionic/bosonic Fock space is built out of the one particle Hilbert space of states by the antisymmetric/symmetric tensor product: $\mathcal{F} = \sum_{N=0}^{\infty} \mathcal{A}/\mathcal{S} \otimes^N L^2(\mathbb{R}^n)$, where N and n are respectively the number of particles and the dimension of the configuration space. Usually, eigenfunctions of either the momentum or position operators are taken as a "basis" in $L^2(\mathbb{R}^n)$ and thus plane-waves or δ-functions are the one particle wave-functions on which the procedure is based.

The interaction of photons, electrons and positrons when fermions are subjected to a constant external magnetic field is essentially described by quantum electrodynamics on a plane orthogonal to the direction of the magnetic field $\vec{B}$. Thus, here we shall discuss QED_{2+1} starting from a basis of Landau states in $L^2(\mathbb{R}^n)$, [2]. Apart from providing an example of the occupation number formalism not dependent on plane-wave states, the Dirac equation in an external magnetic field also presents important novelties with respect to the zero field case, e.g. spectral asymmetry.

We shall analyze the scattering of photons by electrons under the action of an external homogeneous magnetic field in perturbation theory. We set thus forth a physical situation closely related to that occurring in planar Hall devices at very low temperatures and very high magnetic fields. To fit this in with the relativistic approach, we shall compare the theoretical outcome with electromagnetic radiation scattering in samples where the electron effective mass is very small. Such a Hall device could be the Hg Cd Te MISFET (metal-insulator-semiconductor-field-effect-transistor), see [3], which works at very low temperatures of around 1 degree Kelvin; this setting therefore also allows for a zero-temperature field-theoretical treatment. Measures of the optic Hall angle in this system, similar to those performed in high-Tc superconductors [4], might be addressed within the framework of planar QED.

The organization of the paper is as follows: in Section §2 we study the Dirac equation in a homogeneous magnetic field and quantize the Dirac-Landau field. Section §3 is devoted to developing perturbation theory and its application to the understanding of Compton scattering. Finally, in Section §4 we compute the cross-length for the scattering of optical/ultraviolet photons by Dirac-Landau electrons and comment on several issues.

2 Field expansion in Dirac–Landau states

2.1 The Dirac equation in a homogeneous magnetic field

The Dirac equation governing the quantum mechanics of a relativistic charged particle of mass m and spin $1/2$ is,

$$\left(i\hbar\gamma^\mu\partial_\mu + \frac{e}{c}\gamma^\mu A^{\rm ext}_\mu(x)\right)\psi(x) = mc\,\psi(x) \qquad (1)$$

if the fermion moves in a plane under a time-independet and homogeneous magnetic field. Our conventions for the metric, 2×2 Dirac matrices and the like are explained in appendix A and planar Dirac fermions in external fields are described in Reference [6]. We work in the Weyl and Landau gauges where the three-vector external potential reads as $A^{\rm ext}_\mu(x) = (0,\vec{A}(x))$ and $A_1(x) = Bx_2$, $A_2(x) = 0$. This produces a constant and uniform magnetic field $\vec{B} = -B\vec{k}$ and the stationary states $\psi_E(\vec{x})e^{-\frac{iEt}{\hbar}}$ satisfy the spectral equation $H\psi_E(\vec{x}) = E\psi_E(\vec{x})$ for the Hamiltonian Dirac operator:

$$H = \begin{pmatrix} mc^2 & D \\ D^\dagger & -mc^2 \end{pmatrix} \qquad (2)$$

$$D = -\sqrt{2eB\hbar c}\,a^\dagger\ ;\ D^\dagger = -\sqrt{2eB\hbar c}\,a$$

The solution of the non-relativistic Landau problem in the plane is well known, see [2]. In terms of the annihilation and creation operators a and $a^\dagger$,

$$a = \frac{1}{\sqrt{2}l}\left[l^2\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_1}\right) + x_2\right]\ ;\ a^\dagger = \frac{1}{\sqrt{2}l}\left[-l^2\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1}\right) + x_2\right]$$

that do not commute, $[a,a^\dagger] = 1$, the Schrodinger operator is: $H_S = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$. H_S commutes in the Landau gauge with $p_1 = -i\hbar\frac{\partial}{\partial x_1}$; thus, there exists a complete set of eigenfunctions common to H_S and p_1 formed by products of Hermite polynomials and plane-waves. $\omega = \frac{eB}{mc}$ and $l^2 = \frac{\hbar}{m\omega}$ are the cyclotron frequency and the magnetic length.

Given the $N = \frac{eBA}{2\pi\hbar c}$ eigenfunctions of H_S,

$$\psi_{n,p_1}(\vec{x}) = \frac{1}{\sqrt{L_1}} e^{\frac{ip_1x_1}{\hbar}} \phi_{n,p_1}(x_2) \tag{3}$$

$$\phi_{n,p_1}(x_2) = \frac{1}{\pi^{1/4}\sqrt{2^n n! l}} H_n\left[\frac{x_2 + \frac{p_1}{\hbar} l^2}{l}\right] e^{-\frac{1}{2l^2}(x_2 + \frac{p_1}{\hbar} l^2)^2} \tag{4}$$

with the center of the orbit $x_2^0 = -\frac{p_1 l^2}{\hbar}$ located in a rectangular enclosure of area $A = L_1 L_2$, one can easily find the eigenfunctions for the Dirac operator H. In (3), (4) $n = 0, 1, 2, \ldots$ label the Landau levels, $p_1/\hbar = 2\pi q_1/\hbar$, so that $q_1 \in \mathbb{Z}$ is the "discrete" momentum in the OX_1-direction and $H_n[x]$ are the Hermite polynomials. Therefore, the energy eigenvalues of H are

$$\begin{aligned} E_n^{\pm} &= \pm\sqrt{2eB\hbar c n + m^2c^4} \; ; n = 1, 2, \ldots \\ E_0^{+} &= +mc^2 \end{aligned} \tag{5}$$

whereas the corresponding eigenspinors read

$$\Psi^{\pm}_{n,p_1}(\vec{x}) = \sqrt{\frac{|E_n| \pm mc^2}{2|E_n|}} \begin{pmatrix} \psi_{n,p_1}(\vec{x}) \\ \frac{E_n^0}{E_n + mc^2}\, \psi_{n-1,p_1}(\vec{x}) \end{pmatrix} \; ; n = 1, 2, \ldots \tag{6}$$

$$\Psi^{+}_{0,p_1}(\vec{x}) = \begin{pmatrix} \psi_{0,p_1}(\vec{x}) \\ 0 \end{pmatrix}$$

if $E_n = \pm|E_n|$ and $E_n^0 = +\sqrt{2eB\hbar c n}$. Accordingly, the Dirac-Landau spectral problem shares the following properties with the non-relativistic counterpart: (1) The spectrum is discrete and the Dirac-Landau energy levels are labeled by a non-negative integer. (2) Each energy level is degenerated and the eigenvalues of p_1 characterize the degeneracy. Nevertheless, there are two important differences: (1) Infinite negative energy levels appear and we can talk of a Dirac-Landau sea. (2) The spectrum shows a spectral assymmetry associated with the fundamental or ground state; for $n = 0$ there are states with positive energy which are not paired with others of negative energy. It is remarkable that the energy of these states, which form what we shall call the first Landau level, is independent of B. At the zero mass limit, the first Landau level is spanned by "zero modes" of the Dirac operator.

For later convenience we introduce the notation:

$$\Psi^{\pm}_{n,p_1}(\vec{x}) = \frac{1}{\sqrt{L_1}} e^{\frac{ip_1x_1}{\hbar}} u^{\pm}_{np_1}(x_2) \; ; \; u^{\pm}_{np_1}(x_2) = \sqrt{\frac{|E_n| \pm mc^2}{2|E_n|}} \begin{pmatrix} \phi_{n,p_1}(x_2) \\ \frac{E_n^0}{E_n + mc^2}\, \phi_{n-1,p_1}(x_2) \end{pmatrix} \; ; n \neq 0$$

$$\Psi^{+}_{0,p_1}(\vec{x}) = \frac{1}{\sqrt{L_1}} e^{\frac{ip_1x_1}{\hbar}} u^{+}_{0p_1}(x_2) \; ; \; u^{+}_{0p_1}(x_2) = \begin{pmatrix} \phi_{0,p_1}(x_2) \\ 0 \end{pmatrix} \tag{7}$$

and define the Fourier transform and its inverse for the two-spinors $u^{\pm}_{np_1}$ and $u^{+}_{op_1}$:

$$u^{\pm}_{np_1}(x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U^{\pm}_{np_1}(k_2) e^{ik_2x_2} dk_2 \; ; \; u^{+}_{0p_1}(x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U^{+}_{0p_1}(k_2) e^{ik_2x_2} dk_2 \tag{8}$$

$$U_{np_1}(k_2) = \frac{1}{l}\int_{-\infty}^{\infty} u_{np_1}(x_2)e^{-ik_2x_2}dx_2 \; ; \; U^+_{0p_1}(k_2) = \frac{1}{l}\int_{-\infty}^{\infty} u^+_{0p_1}(x_2)e^{-ik_2x_2}dx_2$$

Bearing in mind that $k_2 = p_2/\hbar$, we find

$$U_{np_1}(k_2) = \sqrt{\frac{E_n - mc^2}{2E_n}}\begin{pmatrix} \varphi_{n,p_1}(k_2) \\ \frac{E^0_n}{E_n+mc^2}\varphi_{n-1,p_1}(k_2)\end{pmatrix} \; ; \; U^+_{0p_1}(k_2) = \begin{pmatrix}\varphi_{0,p_1}(k_2) \\ 0\end{pmatrix} \tag{9}$$

where $\varphi_{n,p_1}(k_2)$ are the Fourier transforms of the non-relativistic Landau wave-functions:

$$\varphi_{n,p_1}(k_2) = \frac{1}{l}\int_{-\infty}^{\infty}\psi_{n,p_1}(x_2)e^{-ik_2x_2}dx_2 = \frac{\sqrt{2}}{\pi^{1/4}i^n\sqrt{2^n n!\,l}}H_n[k_2 l]e^{i\frac{p_1k_2l^2}{\hbar}-\frac{k_2^2l^2}{2}} \tag{10}$$

2.2 The Dirac-Landau field

Wave-particle duality, $E = \hbar\omega$, $\vec{p} = \hbar\vec{k}$ and $\hbar e_{\rm Field} = e$, allows the understanding of equation (1) in classical field theory instead of relativistic quantum mechanics. In this framework the Dirac-Landau equation appears as the Euler-Lagrange equation for the Lagrangian:

$$\mathcal{L} = c\,\bar\psi(x)\left[\gamma^\mu\left(i\hbar\partial_\mu + \frac{e}{c}A^{\rm ext}_\mu(x)\right) - mc\right]\psi(x) \tag{11}$$

Thus, in the Weyl gauge the field theoretical Dirac Hamiltonian is:

$$H = \int d^2x\,\psi^\dagger(x)\left[\vec\alpha\cdot\left(-i\hbar c\vec\nabla + e\vec{A}^{\rm ext}(x)\right) + \beta mc^2\right]\psi(x) \tag{12}$$

where $\beta = \gamma^0$ and $\alpha^j = \beta\gamma^j$, $j = 1,2$, and $\bar\psi(x) = \psi^\dagger(x)\gamma^0$.

In order to quantize this system, see [7], we impose the anti-commutation relations at equal times:

$$\begin{aligned} \{\psi_\alpha(t,\vec{x}),\psi^\dagger_\beta(t,\vec{y})\} &= \delta_{\alpha\beta}\delta^{(2)}(\vec{x}-\vec{y}) \\ \{\psi_\alpha(t,\vec{x}),\psi_\beta(t,\vec{y})\} &= \{\psi^\dagger_\alpha(t,\vec{x}),\psi^\dagger_\beta(t,\vec{y})\} = 0 \end{aligned} \tag{13}$$

The expansion of the Dirac field ψ and its adjoint $\bar\psi$

$$\begin{aligned} \psi(x) &= \psi^+(x) + \psi^-(x) + \psi^0(x) \\ &= \sum_{n,p_1}\left[C_{np_1}\psi^+_{np_1}(\vec{x})e^{-\frac{iE_nt}{\hbar}} + D^\dagger_{np_1}\psi^-_{np_1}(\vec{x})e^{\frac{iE_nt}{\hbar}}\right] + \sum_{p_1}A_{0p_1}\psi_{0p_1}(\vec{x})e^{-\frac{iE_0t}{\hbar}} \\ \bar\psi(x) &= \bar\psi^+(x) + \bar\psi^-(x) + \bar\psi^0(x) \\ &= \sum_{n,p_1}\left[D_{np_1}\bar\psi^-_{np_1}(\vec{x})e^{-\frac{iE_nt}{\hbar}} + C^\dagger_{np_1}\bar\psi^+_{np_1}(\vec{x})e^{\frac{iE_nt}{\hbar}}\right] + \sum_{p_1}A^\dagger_{0p_1}\bar\psi_{0p_1}(\vec{x})e^{+\frac{iE_0t}{\hbar}} \end{aligned} \tag{14}$$

is compatible with (13) if the coefficients C_{np_1}, D_{np_1} and A_{0p_1} are operators satisfying the anticommutation relations:

$$\{C_{np_1},C^\dagger_{n',p'_1}\} = \{D_{np_1},D^\dagger_{n',p'_1}\} = \delta_{n,n'}\delta_{p_1,p'_1} \; ; \; \{A_{0p_1},A^\dagger_{0,p'_1}\} = \delta_{p_1,p'_1} \tag{15}$$

and every other anticommutator between these operators vanishes.

The fermionic Fock-Landau space admits a basis built out of the vacuum

$$C_{np_1}|0\rangle = D_{np_1}|0\rangle = A_{0p_1}|0\rangle = 0\,;\ \forall n\,;\forall p_1$$

by the action of strings of creation operators

$$\begin{aligned} &|\alpha_{n^1p_1^1}\,\beta_{n^1p_1^1}\,\gamma_{0p_1^1}\cdots\alpha_{n^Np_1^N}\,\beta_{n^Np_1^N}\,\gamma_{0p_1^N}\rangle \\ &\propto [C^\dagger_{n^1p_1^1}]^{\alpha_{n^1p_1^1}}[D^\dagger_{n^1p_1^1}]^{\beta_{n^1p_1^1}}[A^\dagger_{0p_1^1}]^{\gamma_{0p_1^1}}\cdots[C^\dagger_{n^Np_1^N}]^{\alpha_{n^Np_1^N}}[D^\dagger_{n^Np_1^N}]^{\beta_{n^Np_1^N}}[A^\dagger_{0p_1^N}]^{\gamma_{0p_1^N}}|0\rangle \end{aligned} \tag{16}$$

where $\alpha_{n^ip_1^i};\ \beta_{n^ip_1^i};\ \gamma_{0p_1^i}$ are 0 or 1, complying with Fermi statistics. Therefore, the states of the basis are eigenvectors of the number operators $N^+_{np_1} = C^\dagger_{np_1}C_{np_1}$, $N^-_{np_1} = D^\dagger_{np_1}D_{np_1}$ and $N_{0p_1} = A^\dagger_{0p_1}A_{0p_1}$. From (15) one can easily deduce that C_{np_1} and $C^\dagger_{np_1}$ annihilate and create respectively, electrons in the n<u>th</u> Landau level, whereas D_{np_1} and $D^\dagger_{np_1}$ do the same job with positrons. A_{0p_1} and $A^\dagger_{0p_1}$ are the annihilation and creation operators of electrons occupying the first Landau level. The spectral asymmetry of the Dirac-Landau operator forbids the ocupation of the first Landau level by positrons in the second quantization framework for this system.

Our states of the basis are eigenvalues of the Hamiltonian, component one of the momentum and charge operators, which properly normal-ordered read:

$$\begin{aligned} H &= \sum_{n=1}^{\infty}\sum_{p_1=-\infty}^{\infty} E_n(C^\dagger_{np_1}C_{np_1} + D^\dagger_{np_1}D_{np_1}) + \sum_{p_1=-\infty}^{\infty} E_0\left(A^\dagger_{0p_1}A_{0p_1} - \frac{1}{2}\right) \\ P_1 &= \sum_{n=1}^{\infty}\sum_{p_1=-\infty}^{\infty} \hbar q_1(C^\dagger_{np_1}C_{np_1} + D^\dagger_{np_1}D_{np_1}) + \sum_{p_1=-\infty}^{\infty} \hbar q_1 A^\dagger_{0p_1}A_{0p_1} \\ Q &= -e\sum_{n=1}^{\infty}\sum_{p_1=-\infty}^{\infty}(C^\dagger_{np_1}C_{np_1} - D^\dagger_{np_1}D_{np_1}) - e\sum_{p_1=-\infty}^{\infty}\left(A^\dagger_{0p_1}A_{0p_1} - \frac{1}{2}\right) \end{aligned} \tag{17}$$

Notice that $\langle 0|Q|0\rangle = \frac{e}{2}\zeta(0)$, $\zeta(s) = \sum_{n\in Z\setminus\{0\}}\frac{1}{n^s} + 1$, because we do not have the A_{0p_1} and $A^\dagger_{0p_1}$ normally ordered. This choice is made to distinguish the Dirac-Landau sea from the normal situation where all the particles have their anti-particles. The states (16) however, do not have definite spin because we are working in the Landau gauge.

The anticommutation relations at different times are

$$\{\psi(x);\psi(y)\} = \{\bar\psi(x);\bar\psi(y)\} = 0\,;\ \{\psi(x);\bar\psi(y)\} = iS(x-y) \tag{18}$$

where the 2×2 matrix function $S(x) = S^+(x) + S^-(x) + S^0(x)$ is

$$\begin{aligned} iS^+(x-y) &= \sum_{n,p_1}\psi^+_{np_1}(\vec{x})\,\bar\psi^+_{np_1}(\vec{y})\,e^{-\frac{iE_n(x_0-y_0)}{\hbar c}} \\ iS^-(x-y) &= \sum_{n,p_1}\psi^-_{np_1}(\vec{x})\,\bar\psi^-_{np_1}(\vec{y})\,e^{\frac{iE_n(x_0-y_0)}{\hbar c}} \\ iS^0(x-y) &= \sum_{p_1}\psi_{0p_1}(\vec{x})\,\bar\psi_{0p_1}(\vec{y})\,e^{-\frac{iE_0(x_0-y_0)}{\hbar c}} \end{aligned} \tag{19}$$

The fermion propagator in a magnetic field is the expectation value of the time ordered product $T\{\psi(x),\bar\psi(y)\}$ at the vacuum state:

$$
\begin{aligned}
iS_F(x-y) &= \langle 0|T\{\psi(x),\bar\psi(y)\}|0\rangle \qquad (20)\\
&= i\,\theta(x_0-y_0)S^+(x-y) - \theta(y_0-x_0)S^-(x-y) + i\,\delta(x_0-y_0)S^0(x-y)
\end{aligned}
$$

($\theta(x)$ is the step function, $\theta(x)=1$ if $x>0$, $\theta(x)=0$ if $x<0$). Taking into account finite temperature effects requires that one must define the propagator as:

$$
iS_F(\beta;x-y) = \frac{\mathrm{Tr}(e^{-\beta H}\,T\{\psi(x),\bar\psi(y)\})}{\mathrm{Tr}\,e^{-\beta H}} \qquad (21)
$$

where β is the inverse temperature. Temperature Green functions like this can be computed in the canonical formalism, see [8]: $T\neq 0$ effects are included by considering a complex Minkowski time and (anti)-periodic fields in the imaginary component of period $i\beta$. One considers a contour C between 0 and $i\beta$ containing the real axis and defines the path ordering along C. This amounts to doubling the fields: $\psi_1=\psi(x_0;\vec{x})$; $\psi_2=\psi(x_0-i\frac{\beta}{2};\vec{x})$. If $\beta\to\infty$, ψ_2 decouples and, equivalently, only the vaccum state contributes to $S_F(x-y)$, which is given by (20). Thus, at very low temperatures we reach a very good approximation by considering $\beta=\infty$ QED.

3 Planar quantum electrodynamics in a magnetic background

3.1 QED_{2+1} in external homogeneous magnetic fields

Our goal is to describe the interactions of two-dimensional electrons and positrons with photons when there is a constant homogeneous magnetic field in the background. We choose the free-field Lagrangian density in the form,

$$
\mathcal{L}_0 = N\left[c\,\bar\psi(x)\gamma^\mu\left(i\hbar\partial_\mu+\frac{e}{c}A^{ext}_\mu(x)\right)\psi(x) - mc\,\bar\psi(x)\psi(x) - \frac{1}{4}f_{\mu\nu}(x)f^{\mu\nu}(x) - \frac{1}{4}F^{\mu\nu}_{ext}(x)F^{ext}_{\mu\nu}(x)\right] \qquad (22)
$$

after the descomposition of the electromagnetic three-vector potential in terms of the radiation and external fields: $A_\mu(x)=a_\mu(x)+A^{ext}_\mu(x)$. The associated antisymmetric tensor $F_{\mu\nu}=f_{\mu\nu}+F^{ext}_{\mu\nu}$ also splits and the quantas associated with the field $a_\mu(x)$ are the planar photons discussed in appendix A. The quanta corresponding to $\psi(x)$ and $\bar\psi(x)$ have been analyzed in Section §2.

The interaction Lagrangian density is

$$
\mathcal{L}_I = N\left[e\,\bar\psi(x)\gamma^\mu a_\mu(x)\psi(x)\right] \qquad (23)
$$

and the action integral reads

$$
\begin{aligned}
S &= \int d^3x\, N\left[c\,\bar\psi(x)\gamma^\mu\left(i\hbar\partial_\mu+\frac{e}{c}A_\mu(x)\right)\psi(x) - mc\,\bar\psi(x)\psi(x) - \frac{1}{4}f_{\mu\nu}(x)f^{\mu\nu}(x)\right]\\
&\quad - \frac{1}{4}\int d^3x\, N\left[F^{\mu\nu}_{ext}(x)F^{ext}_{\mu\nu}(x)\right] \qquad (24)
\end{aligned}
$$

There is a natural length scale to the problem; the magnetic length l and the product $e^2 l = \frac{e^2}{\sqrt{eB}}$ is dimensionless if $d = 2$ in the n.u. system. Thus, we express the fine structure constant in the form:

$$\alpha = \frac{e^2}{4\pi\sqrt{eB\hbar c}} \quad (c.g.s.) \qquad \text{or} \qquad \alpha = \frac{e^2}{4\pi\sqrt{eB}} \quad (n.u.) \tag{25}$$

This is consistent: in rationalized mks units the fine structure constant is defined as

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \quad (d = 3) \qquad \text{or} \qquad \alpha = \frac{e^2}{4\pi c_0\sqrt{eB\hbar c}} \quad (d = 2)$$

where c_0 has dimensions of permitivity by length. The natural choice $c_0 = \epsilon_0\sqrt{\frac{\hbar c}{eB}}$, the permitivity of vacuum times magnetic length, means that

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \equiv \frac{e^2}{4\pi c_0\sqrt{eB\hbar c}} \simeq \frac{1}{137.04};$$

although the rationalized charges $\frac{e^2}{\epsilon_0}$ and $\frac{e^2}{c_0}$ have different dimensions. We could also have defined the rationalized charge as $\frac{e^2}{a_0}$ where $a_0 = \epsilon_0\frac{\hbar}{mc}$, the vacuum permitivity times the electron Compton wavelength, but using the magnetic length as length scale makes it possible to take the limit of massless fermions in this problem.

Perturbation theory is based on the S-matrix expansion in powers of α. In the interaction picture the n^{th} term is the chronological product of the interaction Hamiltonian densities at n different points, integrated to every possible value in $R^{1,2}$:

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int d^3x_1 d^3x_2\cdots d^3x_n T\{H_I(x_1)H_I(x_2)\cdots H_I(x_n)\} \tag{26}$$

Here, $H_I(x) = -L_I(x)$ and the differences with the $\vec{B} = 0$ case, see reference [5], lie on the initial, final and intermediate states in the expectation values of the S-matrix at given orders of Perturbation Theory. From now on we shall work in natural units.

3.2 A process in lowest order: Compton scattering

The transition

$$|i\rangle = C^{\dagger}_{np_1} b^{\dagger}(\vec{k})|0\rangle \;\to\; |f\rangle = C^{\dagger}_{n'p'_1} b^{\dagger}(\vec{k}')|0\rangle$$

from one electron in the n^{th} Dirac-Landau level with momentum p_1, (E_n, p_1), and one photon with three-momentum $k = (\omega, \vec{k})$ in the initial state to one electron in the n'^{th} Dirac-Landau level with momentum p'_1, $(E_{n'}, p'_1)$ and one photon with $k' = (\omega', \vec{k}')$ in the final state, is a scattering process of amplitude $\langle f|S|i\rangle$. The dominant contribution to this matrix element comes from the operator

$$S^{(2)} = -e^2\int d^3x\, d^3y\, N\left[\bar{\psi}(x)\gamma^a A_a(x) iS_F(x-y)\gamma^a A_a(y)\psi(y)\right] = S_a + S_b \tag{27}$$

which appears up to second order in α in the S-matrix. A subtle point is that to apply Wick's theorem we also normal-order the creation/annihilation electron operators in the first Landau level. This is what is meant by the symbol N and avoids tadpole photon graphs with fermions running around the loop.

Using,

$$\psi^+(x)|e^-;E_n;p_1\rangle = |0\rangle\frac{1}{\sqrt{L_1}}u^+_{n,p_1}(x_2)e^{ip_1x_1}e^{-iE_nx_0}$$

$$a^+_\mu(x)|\gamma;\vec{k}\rangle = |0\rangle\frac{1}{\sqrt{2A\omega}}\epsilon_\mu(\vec{k})e^{-ikx}$$

$$\bar\psi(x)|0\rangle = \sum_{n,p_1}|e^-;E_n;p_1\rangle\frac{1}{\sqrt{L_1}}\bar{u}^+_{n,p_1}(x_2)e^{-ip_1x_1}e^{iE_nx_0}$$

$$a_\mu(x)|0\rangle = \sum_{\vec{k}}|\gamma;\vec{k}\rangle\frac{1}{\sqrt{2A\omega}}\epsilon_\mu(\vec{k})e^{ikx}$$

one obtains

$$\begin{aligned}\langle f|S_a|i\rangle = -e^2\int\int d^3x\,d^3y&\left[\frac{1}{\sqrt{L_1}}\bar{u}^+_{n',p_1'}(x_2)e^{-ip_1'x_1+iE_{n'}x_0}\gamma^\mu\right]\left[\frac{1}{\sqrt{2A\omega'}}\epsilon_\mu(\vec{k}')e^{ik'x}\right]\\ &\left[iS_F(x-y)\frac{1}{\sqrt{2A\omega}}\gamma^\nu\epsilon_\nu(\vec{k})e^{-iky}\right]\left[\frac{1}{\sqrt{L_1}}u^+_{n,p_1}(y_2)e^{ip_1y_1-iE_ny_0}\right]\end{aligned} \quad (28)$$

Taking into account formulas (7), (8) and

$$\theta(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}d\tau\,\frac{e^{i\tau z}}{\tau - i\epsilon};\ \lim\ \epsilon \to 0;\ \epsilon > 0$$

the propagator can be written in the form:

$$iS_F(x-y) = \frac{1}{(2\pi)^3 i}\int d^2q\int d\tau\left\{\sum_{r=0}^{\infty}\frac{e^{iq_+^{(r)}(x-y)}}{\tau - i\epsilon}U^+_{rq_1}(q_2)\bar{U}^+_{rq_1}(q_2) - \sum_{r=1}^{\infty}\frac{e^{iq^{(r)}(x-y)}}{\tau - i\epsilon}U^-_{rq_1}(q_2)\bar{U}^-_{rq_1}(q_2)\right\}; \quad (29)$$

where $q^{(r)} = (-q_0^{(r)}; -\vec{q})$ with $q_0^{(r)} = -\tau - E_r$ and $\vec{q} = (q_1; q_2)$. Also plugging the Fourier transform of the ongoing and outgoing spinors in (28), we are ready to perform the x- and y-integrations. The outcome of the x_0- and y_0-integrations is energy conservation:

$$\begin{aligned}&\int dx_0\,\exp[ix_0(E_{n'}+\omega' - E_r - \tau)]\int dy_0\,\exp[iy_0(E_r + \tau - E_n - \omega)]\\ &= (2\pi)^2\,\delta(E_{n'}+\omega' - E_r - \tau)\,\delta(E_r + \tau - E_n - \omega)\end{aligned}$$

Simili modo, the spatial integrations

$$\begin{aligned}&\int dx_j\,\exp[ix_j(q_j - p_j' - k_j')]\int dy_j\,\exp[iy_j(p_j + k_j - q_j)]\\ &= (2\pi)^2\,\delta(q_j - p_j' - k_j')\,\delta(p_j + k_j - q_j);\end{aligned}$$

lead to momentum conservation.

The δ-functions allow us to compute the ω'-, q_1- and q_2-integrations. We obtain:

$$\langle f|S_a|i\rangle = 2\pi l^2 \int\!\!\int dp_2\, dp_2'\, \delta(E_{n'}+\omega'-E_n-\omega)\, \delta^{(2)}(\vec{p}'+\vec{k}'-\vec{p}-\vec{k})\frac{1}{L_1}\frac{1}{2A\omega}\mathcal{M}_a(p_2,p_2') \tag{30}$$

where

$$\mathcal{M}_a(p_2,p_2') = -e^2 \bar{U}^+_{n'p_1'}(p_2')\, \not\epsilon(\vec{k}')\, iS_F(E_n+\omega,\vec{p}+\vec{k})\, \not\epsilon(\vec{k})\, U^+_{np_1}(p_2) \tag{31}$$

is the Feynman. The momentum space propagator reads:

$$\begin{aligned} &iS_F(E_n+\omega,\vec{p}+\vec{k}) \\ &= \frac{1}{i}\left\{\sum_{r=0}^{\infty}\frac{U^+_{r,(p_1+k_1)}(p_2+k_2)\bar{U}^+_{r,(p_1+k_1)}(p_2+k_2)}{E_r-(E_n+\omega)-i\epsilon} - \sum_{r=1}^{\infty}\frac{U^-_{r,(p_1+k_1)}(p_2+k_2)\bar{U}^-_{r,(p_1+k_1)}(p_2+k_2)}{E_r+(E_n+\omega)-i\epsilon}\right\}; \end{aligned} \tag{32}$$

The matrix element $\langle f|S_b|i\rangle$ corresponding to the exchange graph is given by an analogous expression to (30), $\mathcal{M}_a(p_2,p_2')$ being replaced by the Feynman amplitude:

$$\mathcal{M}_b(p_2,p_2') = -e^2 \bar{U}^+_{n'p_1'}(p_2')\, \not\epsilon(\vec{k})\, iS_F(E_n-\omega',\vec{p}-\vec{k}')\, \not\epsilon(\vec{k}')\, U^+_{np_1}(p_2) \tag{33}$$

$$\begin{aligned} &iS_F(E_n-\omega',\vec{p}-\vec{k}') \\ &= \frac{1}{i}\left\{\sum_{r=0}^{\infty}\frac{U^+_{r,(p_1-k_1')}(p_2-k_2')\bar{U}^+_{r,(p_1-k_1')}(p_2-k_2')}{E_r-(E_n-\omega')-i\epsilon} - \sum_{r=1}^{\infty}\frac{U^-_{r,(p_1-k_1')}(p_2-k_2')\bar{U}^-_{r,(p_1-k_1')}(p_2-k_2')}{E_r+(E_n-\omega')-i\epsilon}\right\}; \end{aligned} \tag{34}$$

We express the result for the S-matrix element in the form

$$\langle f|S^{(2)}|i\rangle \equiv S^{(2)}_{fi} = \frac{l^2}{(2\pi)}\int dp_2\, S^{(2)}_{fi}(p_2,p_2+k_2-k_2')$$

$$\begin{aligned} S^{(2)}_{fi}(p_2,p_2+k_2-k_2') = {}& (2\pi)^2\,\delta(E_{n'}+\omega'-E_n-\omega)\,\delta(p_1'+k_1'-p_1-k_1)\prod_{ext}\left(\frac{1}{L_1}\right)^{1/2} \\ & \prod_{ext}\left(\frac{1}{2A\omega_{\vec{k}}}\right)^{1/2}\left(\mathcal{M}_a(p_2,p_2+k_2-k_2')+\mathcal{M}_b(p_2,p_2+k_2-k_2')\right) \end{aligned} \tag{35}$$

Note that despite the formal identity with the scattering amplitudes for the planar Compton effect at zero external magnetic field, there are profound differences: (1) The Dirac-Landau spinors and the fermion propagator include Hermite polynomials that depends on the momentum in the OX_2-direction. (2) Because there is no invariance with respect to translations in the OX_2-direction, the initial and final states are not eigenvalues of the $\hat{p}_2$ operator; thus, we obtain contributions from all the possible eigenvalues p_2 and p_2' and we need to integrate over their full range. Nevertheless, there is invariance under magnetic traslations, see [9], and because of this p_2 and p_2' are not completely independent but related by the condition: $p_2' = p_2 + k_2 - k_2'$.

3.3 Feynman rules for QED_{2+1} in a magnetic field

Following the pattern shown in the derivation of the planar Compton effect, it is possible to establish a set of Feynman rules for writing the S-matrix elements directly for the Feynman graphs in QED_{2+1} when there is an external magnetic field such that $\vec{B}(\vec{x}) = -B\hat{k}$.

The initial and final states for any process are tensor products of photons occupying plane wave states and fermions in Dirac-Landau states. Thus, the "quantum" numbers are the photon momenta and the energies and momenta in the OX_1-direction of the fermions. For the transition $|i\rangle \to |f\rangle$, the S-matrix element takes the form:

$$\langle f|S|i\rangle = \frac{1}{2\pi}\int dp_2^{(1)} \frac{1}{2\pi}\int dp_2^{(2)} \cdots \frac{1}{2\pi}\int dp_2^{(n-1)}\, S_{fi}(p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n-1)}, P_2^i - P_2^f) \tag{36}$$

where $p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n)}$ are the momenta in the OX_2-direction of the external fermions. Here, we have:

$$\begin{aligned}
P_1^i &= \sum_{a=1}^{n} p_1^{i(a)} + \sum_{a=1}^{N} k_1^{i(a)}\,;\quad P_2^i = \sum_{a=1}^{n} p_2^{i(a)} + \sum_{a=1}^{N} k_2^{i(a)} \\
P_1^f &= \sum_{a=1}^{m} p_1^{f(a)} + \sum_{a=1}^{M} k_1^{f(a)}\,;\quad P_2^f = \sum_{a=1}^{m-1} p_2^{f(a)} + \sum_{a=1}^{M} k_2^{f(a)}
\end{aligned} \tag{37}$$

$$p_2^{(n)} = p_2^{f(m)} = P_2^i - P_2^f\,;$$

and,

$$\begin{aligned}
S_{fi}(p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n-1)}, P_2^i - P_2^f) &= \delta_{fi} + \left[(2\pi)^2\,\delta(E_f - E_i)\,\delta(P_1^f - P_1^i)\right. \\
&\left.\prod_{ext}\frac{1}{\sqrt{L_1}}\prod_{ext}\frac{1}{\sqrt{2A\omega}}\right] M(p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n-1)}, P_2^i - P_2^f)
\end{aligned} \tag{38}$$

E_i and E_f are the total energies of the initial and final states; the products extend over all external fermions and photons with normalization factors $1/\sqrt{L_1}$ and $1/\sqrt{2A\omega}$, respectively.

The Feynman amplitude $M(p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n-1)}, P_2^i - P_2^f)$ is the sum of the contributions $M^{(m)}(p_2^{(1)}, p_2^{(2)}, \ldots, p_2^{(n-1)}, P_2^i - P_2^f)$ for all orders in perturbation theory. The contribution to $M^{(m)}$ from each topologically different graph is obtained from the following Feynman rules:

- For each vertex, write a factor $ie\gamma^\mu$.
- For each internal photon line, labelled by the three-momentum k, write a factor: $iD_{F\mu\nu}(k) = i\frac{-g_{\mu\nu}}{k^2+i\epsilon}$.
- For each internal fermion line labelled by the energy E and the momentum $\vec{q} = (q_1, q_2)$, write a factor

$$iS_F(E, \vec{q}) = \frac{1}{i}\left[\sum_{r=0}^{\infty}\frac{U^+_{r,q_1}(q_2)\bar{U}^+_{r,q_1}(q_2)}{E_r - E - i\epsilon} - \sum_{r=1}^{\infty}\frac{U^-_{r,q_1}(q_2)\bar{U}^-_{r,q_1}(q_2)}{E_r + E - i\epsilon}\right] \tag{39}$$

For each external line, write one of the following factors:

- for each initial electron: $U^{+}_{n,p_1}(p_2)$ or $U^{+}_{0,p_1}(p_2)$
- for each initial positron: $U_{n,p_1}(p_2)$
- for each final electron: $U^{+}_{n,p_1}(p_2)$ or $U^{0}_{0,p_1}(p_2)$
- for each final positron: $U_{n,p_1}(p_2)$
- for each initial or final photon: $\epsilon_{1}(\vec{k})$

There is a $\frac{1}{2\pi}\int dp_2$ integration for each external fermion.

Because of energy conservation, $E = \sum_a E_a$, where E_a is the energy of each particle created or annihilated at any given vertex. There is also momentum conservation at each vertex.

For each photon three-momentum that is not fixed by energy-momentum conservation carry out the integration $\frac{1}{(2\pi)^3}\int d^3q$. One such integration with respect to an internal photon momentum occurs for each closed loop.

For each closed fermion loop, there is one fermion energy and one momentum which are not fixed by energy-momentum conservation. One must perform the integration over these variables and also take the trace and multiply by a factor (-1). For instance, for the graph of vacuum polarization by Dirac-Landau electrons one obtains:

$$\Pi^{\mu\nu}[k] = \frac{e^2}{(2\pi)^3}\int d^2\vec{p}\int d\omega\; \mathrm{Tr}\left[\sum_{m=0}^{\infty}\gamma^\mu iS_F(E_m-\omega;\vec{p}-\vec{k})\,\gamma^\nu iS^{+}_{F_m}(E_m-\omega;\vec{p})\right. \tag{40}$$

$$\left.-\sum_{m=1}^{\infty}\gamma^\mu iS_F(-E_m+\omega;\vec{p}-\vec{k})\,\gamma^\nu iS^{-}_{F_m}(-E_m+\omega;\vec{p})\right] \tag{41}$$

where $iS_F(-E_m-\omega;\vec{p}-\vec{k})$ is given by (39) with $E = -E_m-\omega$ and $\vec{q} = \vec{p}-\vec{k}$ and

$$iS^{-}_{F_m}(-E_m-\omega;\vec{p}) = \frac{1}{i}\,\frac{U_{m,p_1}(p_2)\bar{U}_{m,p_1}(p_2)}{\omega - i\epsilon}$$

4 Compton effect on Dirac-Landau electrons

In this Section we shall discuss the scattering cross-length for the Compton effect on electrons in a constant magnetic field up to second order in perturbation theory. We bear in mind the quantum Hall effect where a two-dimensional gas of electrons at very low temperatures is subjected to a strong magnetic field, from 6 to 26 Teslas, which is constant, uniform and perpendicular to the plane where the electrons move.

4.1 The scattering amplitude of photons by Dirac–Landau electrons

Under these circumstances we expect the electrons to be in the lowest (non-filled) Landau level. Thus, we consider the initial and final electrons occupying two states in the first Landau level. The S-matrix element and the Feynman amplitude for the transition $|i\rangle = |e^-(E_0, p_1); \gamma(k)\rangle \to |f\rangle = |e^-(E_0, p_1^0); \gamma(k^0)\rangle$ are given by:

$$\langle f|S|i\rangle = \frac{l^2\,\delta(\omega^0-\omega)\,\delta(p_1^0+k_1^0-p_1-k_1)}{AL_1\sqrt{\omega\omega^0}}\int\!\!\int dp_2\,dp_2^0\,\delta(p_2^0+k_2^0-p_2-k_2)\,[\mathcal{M}_a(p_2,p_2^0)+\mathcal{M}_b(p_2,p_2^0)]$$

$$\mathcal{M}_a(p_2,p_2^0) = ie^2\omega l\,\epsilon(\vec k^0)\,\bar\epsilon(\vec k)\,\varphi_{0p_1^0}(p_2^0)\,\varphi_{0p_1}(p_2)\sum_{r=1}^{\infty}\frac{|\varphi_{r-1,p_1+k_1}(p_2+k_2)|^2}{E_r^2-(m+\omega)^2} \tag{42}$$

$$\mathcal{M}_b(p_2,p_2^0) = -ie^2\omega l\,\epsilon(\vec k)\,\bar\epsilon(\vec k^0)\,\varphi_{0p_1^0}(p_2^0)\,\varphi_{0p_1}(p_2)\sum_{r=1}^{\infty}\frac{|\varphi_{r-1,p_1-k_1^0}(p_2-k_2^0)|^2}{E_r^2-(m-\omega^0)^2}$$

To obtain these formulas we have re-written (31) and (33) according to the following information:

$$U^+_{0p_1^0}(p_2^0) = \begin{pmatrix}\varphi_{0p_1^0}(p_2^0) & 0\end{pmatrix}\quad ; \quad U^+_{0p_1}(p_2) = \begin{pmatrix}\varphi_{0p_1}(p_2)\\ 0\end{pmatrix}$$

$\epsilon^{(1)}(\vec k) = (0, \tilde\epsilon^{(1)}(\vec k))$ such that $\tilde\epsilon^{(1)}(\vec k)\cdot\vec k = 0$ is the only transversal polarization vector of the incoming planar photon. We define $\epsilon(\vec k) = \epsilon_2^{(1)}(\vec k) + i\epsilon_1^{(1)}(\vec k)$ and its complex conjugate: $\bar\epsilon(\vec k) = \epsilon_2^{(1)}(\vec k) - i\epsilon_1^{(1)}(\vec k)$. Then,

$$\not\epsilon^{(1)}(\vec k) = \begin{pmatrix}0 & \epsilon(\vec k)\\ \bar\epsilon(\vec k) & 0\end{pmatrix}$$

There is an identical formula for the outgoing photon.

The fermion propagator splits into three parts:

$$iS_F(E, q_1, q_2) = i\left[S_F^+(E, q_1, q_2) - S_F^-(E, q_1, q_2) + S_F^0(E, q_1, q_2)\right]$$

where

$$S_F^{\pm}(E, q_1, q_2) = l\sum_{r=1}^{\infty}\frac{E_r \pm m}{2E_r(E_r \mp E \mp i\epsilon)}\begin{pmatrix}\varphi_{rq_1}(q_2)\varphi_{rq_1}(q_2) & \frac{E_r^0}{E_r\pm m}\varphi_{rq_1}(q_2)\varphi_{r-1q_1}(q_2)\\ \frac{E_r^0}{E_r\pm m}\varphi_{r-1q_1}(q_2)\varphi_{rq_1}(q_2) & \left(\frac{E_r^0}{E_r\pm m}\right)^2\varphi_{r-1q_1}(q_2)\varphi_{r-1q_1}(q_2)\end{pmatrix}$$

and

$$S_F^0(E, q_1, q_2) = \frac{1}{E_0 - E - i\epsilon}\begin{pmatrix}\varphi_{0q_1}(q_2)\varphi_{0q_1}(q_2) & 0\\ 0 & 0\end{pmatrix}$$

In our process we have $E = m + \omega$, $q_1 = p_1 + k_1$ and $q_2 = p_2 + k_2$ for $\mathcal{M}_a(p_2, p_2^0)$ and $E = m - \omega^0$, $q_1 = p_1 - k_1^0$ and $q_2 = p_2 - k_2^0$ for $\mathcal{M}_b(p_2, p_2^0)$; the factor $E_r^0 = +\sqrt{E_r^2 - m^2}$ appears in the normalization.

Now replacing,

$$\varphi_{0p_1^0}(p_2^0)\varphi_{0p_1}(p_2) = \frac{2\sqrt{\pi}}{l}\exp\left[-i(p_1^0p_2^0 - p_1p_2)l^2\right]\exp\left[-\tfrac{1}{2}(p_2^2 + p_2^{0\,2})l^2\right]$$

$$E_r^2 - (m+\omega)^2 = 2eB\,[r+c]\,;\; c = \frac{\omega(\omega+2m)}{2eB}$$

$$E_r^2 - (m-\omega^0)^2 = 2eB\,[r+c^0]\,;\; c^0 = \frac{\omega^0(\omega^0-2m)}{2eB}$$

$$|\varphi_{r-1,q_1}(q_2)|^2 = \frac{2\sqrt{\pi}}{l}\,\frac{1}{2^{(r-1)}(r-1)!}H^2_{r-1}[q_2 l]\,e^{-q_2^2l^2}$$

in M_a, where $q_1 = p_1 + k_1$, $q_2 = p_2 + k_2$, and in M_b, where $q_1 = p_1 - k_1^0$, $q_2 = p_2 - k_2^0$, we obtain:

$$\begin{aligned}
M_a(p_2;p_2^0) &= \frac{i2\sqrt{\pi}}{lB}e\,\omega\,\varepsilon(k^0)\,\varepsilon(k)\exp\left[-i(p_1^0p_2^0 - p_1p_2)l^2\right]\exp\left[-\frac{1}{2}(p_2^2 + p_2^{0\,2})l^2\right]\\
&\quad\sum_{r=1}^{\infty}\frac{H^2_{r-1}[(p_2+k_2)l]}{\sqrt{\pi}\,2^{(r-1)}(r-1)!}\,\frac{e^{-(p_2+k_2)^2l^2}}{r+c}\\
M_b(p_2;p_2^0) &= \frac{i2\sqrt{\pi}}{lB}e\,\omega\,\varepsilon(k)\,\varepsilon(k^0)\exp\left[-i(p_1^0p_2^0 - p_1p_2)l^2\right]\exp\left[-\frac{1}{2}(p_2^2 + p_2^{0\,2})l^2\right]\\
&\quad\sum_{r=1}^{\infty}\frac{H^2_{r-1}[(p_2-k_2^0)l]}{\sqrt{\pi}\,2^{(r-1)}(r-1)!}\,\frac{e^{-(p_2-k_2^0)^2l^2}}{r+c^0}
\end{aligned} \tag{43}$$

In order to sum the series in (43), we consider the spectral problem

$$\Delta_c\,\psi_n(x) = \lambda_n^c\,\psi_n(x)$$

$$\lambda_n^c = n + c + 1\,;\; n = 0,1,2,\ldots\,;\quad \psi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}H_n[x]e^{-\frac{1}{2}x^2}\,;$$

for the elliptic differential operator $\Delta_c = \frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2 + 2c + 1\right]$ on $L^2(\mathbb{R})$. The Green function for the s-power of Δ_c is defined as:

$$G_{\Delta_c}(x,y;s) = \sum_{n=0}^{\infty}\frac{\psi_n(x)\psi_n(y)}{(\lambda_n^c)^s}$$

We immediately notice that:

$$\begin{aligned}
G_{\Delta_c}(x,x;1) &= \sum_{r=1}^{\infty}\frac{H^2_{r-1}[(p_2+k_2)l]}{\sqrt{\pi}\,2^{(r-1)}(r-1)!}\,\frac{e^{-(p_2+k_2)^2l^2}}{r+c}\,;\; x = (p_2+k_2)l\\
G_{\Delta_{c^0}}(x,x;1) &= \sum_{r=1}^{\infty}\frac{H^2_{r-1}[(p_2-k_2^0)l]}{\sqrt{\pi}\,2^{(r-1)}(r-1)!}\,\frac{e^{-(p_2-k_2^0)^2l^2}}{r+c^0}\,;\; x = (p_2-k_2^0)l
\end{aligned} \tag{44}$$

This is related to the heat kernel

$$K_{\Delta_c}(x,y;\beta) = \sum_{n=0}^{\infty}e^{-\beta\lambda_n^c}\psi_n(x)\psi_n(y)$$

of the operator Δ_c through a Mellin transform:

$$G_{\Delta_c}(x,y;s)=\frac{1}{\Gamma[s]}\int_0^1 d\tau\, K_{\Delta_c}(x,y;\tau)$$

We write $\Delta_c = \Delta + c + 1/2$, so that,

$$K_{\Delta_c}(x,y;\tau)= e^{-(c+\frac{1}{2})\tau}K_{\Delta}(x,y;\tau)$$

The heat kernel for the differential operator Δ of the Harmonic oscillator is very well known [10] and we obtain:

$$\begin{aligned} G_{\Delta_c}(x,x;1) &= \frac{1}{\Gamma[1]}\int_0^1 d\tau\, K_{\Delta_c}(x,x;\tau) \\ K_{\Delta_c}(x,x;\tau) &= \frac{e^{-(c+\frac{1}{2})\tau}}{\sqrt{2\pi\sinh\tau}}\exp\left[-x^2\tanh\frac{\tau}{2}\right]; \end{aligned} \tag{45}$$

and a similar expression for $G_{\Delta_{c'}}(x,x;1)$. Before performing the integration we plug the integral form of G_{Δ_c} and $G_{\Delta_{c'}}$ into

$$M = \int dp_2 \int dp_2'\, \delta(p_2'+k_2'-p_2-k_2)\,[M_a(p_2,p_2')+M_b(p_2,p_2')]:$$

We first integrate in the variables p_2 and p_2' and then in τ to reach the finite answer:

$$\begin{aligned} M_a &= \frac{i\sqrt{2}\,\lambda^2}{l^2B}e\,\omega\,\theta(\tilde{k}')\,\theta(-\tilde{k})\exp\left[i(p_1'k_2'-p_1k_2)^2l^2-\frac{1}{2}(k_2^2+k_2'^2)l^2\right] \\ &\exp\left[-\frac{[(p_1'-p_1)+i(k_2+k_2')]^2l^2}{8}\right]\frac{\Gamma[c+1]}{\Gamma[c+3/2]}\,{}_1F_1\left[c+1;c+\frac{3}{2};\frac{[(k_2+k_2')-i(p_1'-p_1)]^2l^2}{8}\right] \\ M_b &= \frac{i\sqrt{2}\,\lambda^2}{l^2B}e\,\omega\,\theta(\tilde{k})\,\theta(-\tilde{k}')\exp\left[-i(p_1'k_2-p_1k_2')^2l^2-\frac{1}{2}(k_2^2+k_2'^2)l^2\right] \\ &\exp\left[-\frac{[(p_1'-p_1)-i(k_2+k_2')]^2l^2}{8}\right]\frac{\Gamma[c'+1]}{\Gamma[c'+3/2]}\,{}_1F_1\left[c'+1;c'+\frac{3}{2};\frac{[(k_2+k_2')+i(p_1'-p_1)]^2l^2}{8}\right] \end{aligned} \tag{46}$$

where ${}_1F_1[a;b;z]$ is a degenerated confluent hypergeometric function, see [13]. The integration in the variable p_2' is performed immediately because of the δ-function; we are left with Gaussian integrals in p_2 which can be easily calculated. Finally, the integration in the τ variable produces a degenerated hypergeometric function. If we had chosen to integrate first in the τ variable, we would have used some regularization procedure to avoid the dangerous singularity at $\tau = 0$ that renders $G_{\Delta_c}(x,x;1)$ strictly divergent.

4.2 The differential cross-length

The differential cross-length is the transition rate into a group of final states for a scattering centre and unit incident flux. If $\omega = \frac{|S_{fi}|^2}{T}$ is the transition probability per unit of time for our process, we have:

$$d\sigma = \omega\frac{1}{v_{rel}/A}\frac{dp_1'}{(2\pi)}\frac{d^2\tilde{k}'}{(2\pi)^2}$$

where we use finite normalization time T and area A. There are $\frac{A\, d^2\vec{k}'}{(2\pi)^2}$ photon final states with momentum belonging to the interval $(\vec{k}', \vec{k}' + d\vec{k}')$ and $\frac{L_1 dp_1'}{(2\pi)}$ electron final states in the first Landau level and $p_1^f \in (p_1', p_1' + dp_1')$. v_{rel}/A is the incident flux of incoming particles. Using

$$[\delta(\omega - \omega')\,\delta(p_1' + k_1' - p_1 - k_1)]^2 = \frac{TL_1}{(2\pi)^2}\,\delta(\omega - \omega')\,\delta(p_1' + k_1' - p_1 - k_1)$$

we obtain

$$d\sigma = (2\pi)^2\,\delta(\omega' - \omega)\,\delta(p_1' + k_1' - p_1 - k_1)\,\frac{l^4}{16\pi^2 \omega v_{rel}}\,\frac{dp_1'}{(2\pi)}\,\frac{d^2\vec{k}'}{(2\pi)^2 2\omega'}\,|M|^2$$

Because of the conservation of energy and momentum it is easy to integrate this expression with respect to p_1' and $|\vec{k}'|$. Note that $d^2\vec{k}' = |\vec{k}'|\,d|\vec{k}'|\,d\theta$, and

$$\frac{d\sigma}{d\theta} = \frac{l^4}{32\pi^2 \omega v_{rel}}|M|^2$$

bearing in mind that $p_1' = p_1 + k_1 - k_1'$ and $\omega' = \omega$.

The above result is referred to a general reference frame. We choose the laboratory frame characterized by:

1) initial state; one electron in the lowest Dirac-Landau state with energy $E_0 = m$ and momentum $p_1 = 0$ plus a photon of momentum $k = (\omega; \vec{k})$, $\vec{k} = (k_1; 0)$ and polarization $\vec{\epsilon}^{(1)}(\vec{k}) = (0; 1)$.

2) final state; one electron also in the lowest Dirac-Landau state but with momentum $p_1' \neq 0$ plus a photon with momentum $k' = (\omega'; \vec{k}')$, such that $\vec{k}\cdot\vec{k}' = \omega\omega'\cos\theta$, and polarization $\vec{\epsilon}^{(1)}(\vec{k}') = (-\sin\theta; \cos\theta)$.

Conservation of energy and momentum requires that,

$$E_0 + \omega = E_0 + \omega' \;;\; \omega' = \omega$$
$$p_1 + k_1 = p_1' + k_1' \;;\; p_1' = \omega(1 - \cos\theta)$$

There is no Compton shift in wavelength for the photon because the energies of the final and initial electrons are the same; however, there is a shift in the electron momentum. The recoil angle of the electron is given by $\cos\theta_r = \frac{\vec{p}'\cdot\vec{k}}{|\vec{p}'||\vec{k}|} = \frac{p_1'}{|\vec{p}'|}$, and the relative velocity is $v_{rel} = |\vec{k}|/\omega = 1$. Inelastic scattering would require a different Dirac-Landau state for the outgoing electron.

In the laboratory frame the calculation of $|M|^2 = M_a^* M_a + M_a^* M_b + M_b^* M_a + M_b^* M_b = X_{aa} + X_{ab} + X_{ba} + X_{bb}$, gives:

$$X_{aa} = \frac{2\pi^4 e^2}{l^4}\frac{\omega^2}{B^2}\exp\left[-\frac{\omega^2 l^2}{2}\left(\sin^2\theta + 2\sin^2\frac{\theta}{2}\right)\right]\left[\frac{\Gamma[c+1]}{\Gamma[c+3/2]}\right]^2$$
$$\left[{}_1F_1\left(c+1; c+\frac{3}{2}; \frac{\omega^2 l^2}{2}\left(\frac{\sin\theta - i(1-\cos\theta)}{2}\right)^2\right)\right]^2$$

$$X_{bb} = \frac{2\pi^4 e^2}{\ell^4}\frac{\omega^2}{B^2}\exp\left[-\frac{\omega^2\ell^2}{2}\left(\sin^2\theta + 2\sin^2\frac{\phi}{2}\right)\right]\left[\frac{\Gamma[c'+1]}{\Gamma[c'+3/2]}\right]^2$$
$$\left[{}_1F_1\left(c'+1;c'+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta + i(1-\cos\phi)}{2}\right)^2\right)\right]^2$$

$$X_{ab} = X_{ba} = \frac{2\pi^4 e^2}{\ell^4}\frac{\omega^2}{B^2}\exp\left[-\frac{\omega^2\ell^2}{2}\left(\sin^2\theta + 2\sin^2\frac{\phi}{2}\right)\right]\frac{\Gamma[c+1]}{\Gamma[c+3/2]}\frac{\Gamma[c'+1]}{\Gamma[c'+3/2]}$$
$$\exp\left[2i\phi + i\omega^2\ell^2\sin\theta\sin^2\frac{\phi}{2}\right]{}_1F_1\left(c+1;c+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta - i(1-\cos\phi)}{2}\right)^2\right)$$
$${}_1F_1\left(c'+1;c'+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta + i(1-\cos\phi)}{2}\right)^2\right)$$

We finally write the differential cross-length for the scattering of photons by Dirac-Landau electrons in the plane:

$$\left(\frac{d\sigma}{d\theta}\right)_{Lab} = \frac{2\pi^3\omega}{eB}\exp\left[-\frac{\omega^2\ell^2}{2}\left(\sin^2\theta + 2\sin^2\frac{\phi}{2}\right)\right]$$
$$\left\{\left[\frac{\Gamma[c+1]}{\Gamma[c+3/2]}\right]^2\left[{}_1F_1\left(c+1;c+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta - i(1-\cos\phi)}{2}\right)^2\right)\right]^2 + \left[\frac{\Gamma[c'+1]}{\Gamma[c'+3/2]}\right]^2\right.$$
$$\left[{}_1F_1\left(c'+1;c'+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta + i(1-\cos\phi)}{2}\right)^2\right)\right]^2 - 2\frac{\Gamma[c+1]}{\Gamma[c+3/2]}\frac{\Gamma[c'+1]}{\Gamma[c'+3/2]}$$
$$\mathrm{Re}\left[\exp\left[2i\phi + i\omega^2\ell^2\sin\theta\sin^2\frac{\phi}{2}\right]{}_1F_1\left(c+1;c+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta - i(1-\cos\phi)}{2}\right)^2\right)\right.$$
$$\left.\left.{}_1F_1\left(c'+1;c'+\frac{3}{2};\frac{\omega^2\ell^2}{2}\left(\frac{\sin\theta + i(1-\cos\phi)}{2}\right)^2\right)\right]\right\}; \tag{47}$$

4.3 Angular distribution and total cross-length

In this Section we shall discuss the physical meaning of the important formula (47). It is convenient to express the differential cross-length in terms of the dimensionless constants $\epsilon = \frac{\omega}{m}$ and $\beta = \frac{\sqrt{2eB}}{m}$, and also to introduce a new constant, $L_T = \frac{2\pi^2\sqrt{2}}{\sqrt{2eB}}$, which is a length associated with the system. Equation (47) becomes:

$$\frac{1}{L_T}\left(\frac{d\sigma}{d\theta}\right)_{Lab} = \frac{\epsilon}{\beta}\exp\left[-\left(\frac{\epsilon}{\beta}\right)^2\left(\sin^2\theta + 2\sin^2\frac{\phi}{2}\right)\right]$$
$$\left\{\left[\frac{\Gamma[1-\frac{\alpha(\alpha+2)}{2}]}{\Gamma[3/2-\frac{\alpha(\alpha+2)}{2}]}\right]^2\left[{}_1F_1\left(1-\frac{\alpha(\alpha+2)}{2};\frac{3}{2}-\frac{\alpha(\alpha+2)}{2};-\left(\frac{\epsilon}{\beta}\right)^2\left(\frac{\sin\theta - i(1-\cos\phi)}{2}\right)^2\right)\right]^2 + \right.$$

$$
\begin{aligned}
&\left[\frac{[1 \quad \frac{(\ \ 2)}{2}]}{[3=2 \quad \frac{(\ \ 2)}{2}]}\right]^{2}\ {}_1F_1\left[1 \quad \frac{(\quad 2)}{2};\frac{3}{2} \quad \frac{(\quad 2)}{2};\ -\left(\frac{!}{}\right)^{2}\left(\frac{\sin \ + i(1 \quad \cos\ )}{2}\right)^{2}\right]^{2} \\
&2\frac{[1 \quad \frac{(\ +2)}{2}]}{[3=2 \quad \frac{(\ +2)}{2}]}\frac{[1 \quad \frac{(\ \ 2)}{2}]}{[3=2 \quad \frac{(\ \ 2)}{2}]}\mathrm{Re}\left[\exp\left[2i \ + 2i - \left(\frac{!}{}\right)^{2}\sin \ \sin^{2}\frac{}{2}\right]\right. \\
&\quad {}_1F_1\left[1 \quad \frac{(\ +2)}{2};\frac{3}{2} \quad \frac{(\ +2)}{2};\ -\left(\frac{!}{}\right)^{2}\left(\frac{\sin \quad i(1 \quad \cos\ )}{2}\right)^{2}\right] \\
&\quad \left.{}_1F_1\left[1 \quad \frac{(\quad 2)}{2};\frac{3}{2} \quad \frac{(\quad 2)}{2};\ -\left(\frac{!}{}\right)^{2}\left(\frac{\sin \ + i(1 \quad \cos\ )}{2}\right)^{2}\right]\right]\Bigg\} ;
\end{aligned}
\tag{48}
$$

In a Hall device, the electron effective mass and the constant homogeneous magnetic field are given. This means that is fixed and $\frac{d}{d}\,_{\mathrm{Lab}}$ is a function of two variables: $= !=m$, the ratio of the energy of the incoming photon to the electron effective mass, and , the scattering angle. We analyse in turn the dependence of the differential scattering cross-length on the photon energy for fixed and the angular distribution of the scattering for fixed .

In order to unveil the values of for which $\frac{d}{d}\,_{\mathrm{Lab}}$ diverges, we summmarize the properties of the confluent hypergeometric functions and the Gamma function.

A. ${}_1F_1[a;b;z]$ is:

- i. A convergent series for all values of a, b and z if $a \neq n$ and $b \neq n^0$, with n, n^0 positive integers.
- ii. A polynomial of degree n in z if $a = n$ and $b \neq n^0$. ${}_1F_1[a;b;z]$ has a simple pole at $b = n^0$ if $a = n$, $b = n^0$ and $n > n^0$. ${}_1F_1[a;b;z]$ is undefined if $a = n$, $b = n^0$ and $n^0 \quad n$.

B. $[z]$ is single-valued and analytic over the entire complex plane, except for the points $z = n$, $n = 0;1;2;$ where it has simple poles.

In our formula $a = n$ is tantamount to $! = m \sqrt{2eB(n+1)+m^2}$ whereas $b = n^0$ requires $! = m \sqrt{2eB(n^0+3=2)+m^2}$. Both identities cannot happen simultaneously for any values of n and n^0 and there are no divergences in $\frac{d}{d}\,_{\mathrm{Lab}}$ due to singularities of ${}_1F_1$. The Gamma function entering the term in $\frac{d}{d}\,_{\mathrm{Lab}}$ due to direct scattering has poles at the values $! = m \sqrt{2eB(n+1)+m^2}$. Thus, $\frac{d}{d}\,_{\mathrm{Lab}}$ becomes infinite when $! = E_{n+1} \quad E_0$, corresponding the physical situation when the incoming photon is captured by the electron to jump to the $(n+1)$ Landau level. The other sign cannot arise in this physical process (! would be negative) but this divergence is the signal for the opposite phenomenon: photon emission. In the exchange term, however, the singularities in the Gamma function appear when $! = m \quad \sqrt{2eB(n+1)+m^2}$. Again, only the plus sign for ! is acceptable and we have a divergent $\frac{d}{d}\,_{\mathrm{Lab}}$ when $! = E_0 + E_{n+1} = E_0 \quad E_{n\ 1}$. The incoming photon is captured by an electron that occupies a negative energy state in the Dirac-Landau sea and jumps to the first Landau level. Of course, this process can be re-interpreted as "pair" creation and, also, the other sign, which is not compatible with the incoming photons, would correspond to "pair" annihilation of electrons and positrons. Observe that there is a $2m$ gap with respect to the other "divergences". It should be noticed that in the interference term of $\left(\frac{d\sigma}{d\Omega}\right)_{Lab}$ the two kinds of divergences enter together.

In short, the differential and total cross-lengths present divergences at values of the ongoing photon energies corresponding to the energy gaps between the lowest and the other (positive & negative) Dirac-Landau states. For these energies there is no scattering but the absorption of photons and transitions from one Dirac-Landau state to another takes place. We encounter a phenomenon well known in the quantum theory of radiation: resonance fluorescence. In the scattering of light by atoms described by the Kramers-Heisenberg formula, similar divergences appear, see reference [11]. We are also wrongly assuming that the life-time of the intermediate states is infinity. These states are indeed unstable due to spontaneous emission of photons. The energy picks an imaginary contribution that measures the resonance width $\Gamma_r = 1/\tau_r$, the inverse of the life-time; replacing E_r by $E_r - i\Gamma_r/2$ in formula (42), c and c' become imaginary in such a way that the products of the Gamma function entering in (47) are regular and $\frac{d\sigma}{d\Omega}$ reaches finite maxima for $\omega = \pm(E_0 - E_{n_1})$.

In practice, for other values of ω Γ_r can be ignored. The differential cross-length of scattering is regular and a study of the angular distribution of $\left(\frac{d\sigma}{d\Omega}\right)_{Lab}$ is possible. A MATHEMATICA plot of the antenna pattern encoded by formula (48) for $\frac{d\sigma}{d\Omega}$ is depicted in Figure 1 for incoming photon energies in the ultraviolet/infrared range of the electromagnetic spectrum. Here we are thinking of a MISFET, at 1.2 Kelvin degrees of temperature and a very low filling factor; also, $m = 0.006\, m_e$, $B/c = 6$ Teslas, i.e. $L_T = 7.43 \times 10^{10}$ cm. in the c.g.s. system.

In this range of photon energies, far from the pair creation zone, the graphic work reveal a general pattern which can be explained as follows:

1. $\omega < E_1 - E_0$. The photon comes through the x_2-axis toward the electron, which is in one state of the E_0-level. The charge distribution is accelerated up and down the x_1-axis in a motion of very low amplitude by the incoming transverse electric field. The antenna pattern of the electromagnetic field emitted by this oscillatory shaking of the electron is similar to the same distribution in the $B = 0$ case: we find maximum probability of photon emission in forward and backward scattering.

2. $\omega = E_1 - E_0$. The ongoing photon is absorbed by the E_0-level electron and a resonance in the E_1-level is formed. In the excited level the electron oscillates up and down the x_2-axis; recall the $H_1(x_2 - x_2^0)$ factor in the wave function. Thus, the angular distribution of the spontaneously emitted photons undergoes an abrupt change: there is now maximum probability of finding the scattered photons at the angles $\theta = 90°$ and $270°$.

Before going on, notice also that the photon energy values

$$\omega = -m + \sqrt{(2n_1 + 3)eB + m^2} \equiv \omega_{n_1} \;;\; n_1 = 0,1,2,\ldots$$

are special. If $\omega = \omega_{n_1}$, the contribution of the direct and interference terms to the differential cross-length is zero; the scattering is solely due to the exchange diagram. As a function of ω and θ, $\left(\frac{d\sigma}{d\Omega}\right)_{Lab}$ shows saddle points at $\omega = \omega_{n_1}$.

3. $E_1 - E_0 < \omega < \omega_0$. The probability of resonance fluorescence decreases with increasing

ω in this interval. Non-resonant amplitudes become more and more important and interfere with the resonant one. There are two competing effects of the photon collision: first, the resonance fluorescence induces an oscillatory motion of the electron on the x_2-axis; second, the non-resonant amplitudes shake the electron up and down the x_1-axis. The more to the left on the energy interval the more preponderant is the first movement over the second. Thus, 90° and 270° are favoured, although the maxima are flattened throughout the interval from left to right. It is amusing to note that for these energies photons scatter out of electrons in a Hall device just like the quasi-particle anyonic excitations in the quantum Hall effect do between themselves.

4. $\omega_0 \leq \omega \leq E_2 - E_0$. If $\omega = \omega_0$, the angular distribution is isotropic within one part in a million. This is due to the perfect balance of the resonant and non-resonant amplitudes in the direct scattering, leaving only the contribution of the exchange graph; in this bremsstrahlung there are no preferred directions. Beyond this point, the non-resonant amplitudes are preponderant and the antenna pattern in the range $\omega_0 < \omega < E_2 - E_0$ is as in the $B = 0$ case. When $\omega = E_2 - E_0$, the next resonance is reached and a new change in the angular distribution appear.

Below the pair creation threshold $\omega = 2m$, this behaviour is periodically repeated. The forward-backward and left-right symmetries, however, cease to be almost perfect for higher values of ω due to stronger quantum fluctuations. Instead, $\theta = 0$ scattering in the first regime and $\theta = \frac{\pi}{6}$ in the second become dominant. For lighter efective mass, this behaviour is reached before. Figure 2 shows plots of the differential cross-length as a function of θ for $\theta = 0$. In the second graph the effective mass has been chosen in such a way that the threshold for pair creation occurs at values of ω for soft X-rays. The $\omega = 2.35$ angular distribution of photon emission is due to pair annihilation and thus shows a maximum at $\theta = 0$. Beyond this energy, the resonances are so short-lived that the angular distribution does not change when they are formed. It seems that rather than two quantum mechanical processes of absorption/emission, a single resonant scattering takes place when $\omega > 2m$. There are also no changes in the antenna pattern, either in the saddle points $\omega = \omega_{n_1}$ or in another type of saddle point reached when:

$$\omega = m + \sqrt{(2n_2 + 3)eB + m^2} = \omega^+_{n_2}; n_2 = 0, 1, 2, \ldots \tag{49}$$

In these last points there is no contribution of the exchange diagram to $\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}}$ and only the direct graph contributes to a very weak light/X-ray scattering.

Numerical integration of the differential cross-length provides us with the total cross-length of scattering. A picture of the function $\sigma_T(\omega)$ is shown in Figure 3. As expected, divergences appear at the values of ω that coincides with the Landau energy levels. In contrast to the ordinary planar Compton effect, no infrared divergence due to soft photons arises in σ_T because the magnetic field supplies an infrared cut-off.

A Gamma Matrices and the Electromagnetic Field in 3-dimensional Space-time

The Dirac (Clifford) algebra in the 3-dimensional Minkowski space $M_3 = R^{1,2}$ is built from the three gamma matrices γ^μ satisfying the anticommutation relations:

$$\{\gamma^\mu , \gamma^\nu\} = 2g^{\mu\nu} \tag{50}$$

$$\mu,\nu = 0,1,2 \quad ; \quad g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$$

and the hermiticity conditions $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. The tensors

$$1 ; \ \gamma^\mu ; \ \gamma^{\mu_1}\gamma^{\mu_2} ; \ \gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} ; \ \mu_1 < \mu_2 < \mu_3 \tag{51}$$

with respect to the $SO(2,1)$-group, the piece connected to the identity of the Lorentz group in flatland, form the basis of the Dirac algebra, which is thus 2^3-dimensional. 1 and $\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} = -i\epsilon^{\mu_1\mu_2\mu_3} 1$ are respectively scalar and pseudo-scalar objects. γ^μ is a three-vector but $\gamma^{\mu_1}\gamma^{\mu_2}$ can be seen alternatively as a anti-symmetric tensor or a pseudo-vector, which are equivalent irreducible representations of the $SO(2,1)$-group. If we denote by $\epsilon^{\mu\nu\rho}$ the completely antisymmetric tensor, equal to $+1(-1)$ for an even (odd) permutation of $(0,1,2)$ and to 0 otherwise, the γ-matrices must also satisfy the commutation relations:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu , \gamma^\nu] = \epsilon^{\mu\nu\rho}\gamma_\rho \tag{52}$$

The $\sigma^{\mu\nu}$-matrices are the Lie algebra generators of the $spin(1,2;R) = SL(2;R)$-group, the universal covering of the connected piece of the Lorentz group and the irreducible representations of the Lie $SL(2;R)$-group are the spinors. Our choice of the representation of the Dirac algebra is as follows:

$$\gamma^0 = \sigma^3 ; \ \gamma^1 = i\sigma^1 ; \ \gamma^2 = i\sigma^2$$

where the σ^a, $a = 1,2,3$ are the Pauli matrices.

The canonical quantization of the electromagnetic field in $(2+1)$-dimensions is equivalent to the four-dimensional case. We shall follow the covariant formalism of Gupta and Bleuler, see [12]. We consider the Fermi Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu a_\nu(x))(\partial^\mu a^\nu(x)) \tag{53}$$

where now $a_\mu(x)$; $\mu = 0,1,2$ is the three-vector potential. The fields equations are

$$\Box a_\mu(x) = 0 \tag{54}$$

which are equivalent to Maxwell's equations if the potential satisfies the Lorentz condition $\partial^\mu a_\mu(x) = 0$. We expand the free electromagnetic field in a complete set of plane wave states:

$$\begin{aligned} a_\mu(x) &= a_\mu^+(x) + a_\mu^-(x) \\ &= \sum_{\vec{k},r} \frac{1}{\sqrt{2A\,\omega_{\vec{k}}}} \left[\epsilon_\mu^r(\vec{k})\, b_r(\vec{k})\, e^{-ikx} + \epsilon_\mu^r(\vec{k})\, b_r^\dagger(\vec{k})\, e^{ikx} \right] \end{aligned} \tag{55}$$

Here, the summation is over wave vectors, allowed by the periodic boundary conditions in A, with $k^0 = \frac{1}{c}\omega_{\vec{k}} = |\vec{k}|$. The summation over $r = 0,1,2$ corresponds to the three linearly independent polarizations states that exist for each $\vec{k}$. The real polarization vectors $\epsilon_r^\mu(\vec{k})$ satisfy the orthonormality and completeness relations

$$\sum_\mu \epsilon_r^\mu(\vec{k})\,\epsilon_{s\mu}(\vec{k}) = -\eta_r\delta_{rs}; \qquad r,s = 0,1,2 \tag{56}$$

$$\sum_r \eta_r\,\epsilon_r^\mu(\vec{k})\,\epsilon_r^\nu(\vec{k}) = -g^{\mu\nu} \tag{57}$$

$$\eta_0 = -1;\ \eta_1 = \eta_2 = 1$$

The equal-time commutation relations for the fields $a^\mu(x)$ and their momenta $\pi^\mu(x) = -\frac{1}{c^2}\dot{a}^\mu(x)$ are

$$[a^\mu(\vec{x},t), a^\nu(\vec{x}',t)] = [\dot{a}^\mu(\vec{x},t), \dot{a}^\nu(\vec{x}',t)] = 0$$
$$[a^\mu(\vec{x},t), \dot{a}^\nu(\vec{x}',t)] = -i\hbar c^2 g^{\mu\nu}\delta^{(2)}(\vec{x}-\vec{x}') \tag{58}$$

The operators $b_r(\vec{k})$ and $b_r^\dagger(\vec{k})$ satisfy

$$[b_r(\vec{k}), b_s^\dagger(\vec{k}')] = \eta_r\delta_{rs}\delta_{\vec{k}\vec{k}'} \tag{59}$$

and all other commutators vanish. For each value of r there are transverse ($r = 1$), longitudinal ($r = 2$) and scalar ($r = 0$) photons, but as result of the Lorentz condition, which in the Gupta-Bleuler theory is replaced by a restriction on the states, only transverse photons are observed as free particles. This is accomplished as follows: the states of the basis of the bosonic Fock space have the form,

$$|n_{r_1}(\vec{k}_1)n_{r_2}(\vec{k}_2)\cdots n_{r_N}(\vec{k}_N)\rangle \propto \left[a^\dagger_{r_1}(\vec{k}_1)\right]^{n_{r_1}(\vec{k}_1)}\left[a^\dagger_{r_2}(\vec{k}_2)\right]^{n_{r_2}(\vec{k}_2)}\cdots\left[a^\dagger_{r_N}(\vec{k}_N)\right]^{n_{r_N}(\vec{k}_N)}|0\rangle;$$

where $n_{r_i}(\vec{k}_i) \in \mathbb{Z}^+$, $\forall i = 1,2,\cdots,N$ and

$$a_r(\vec{k})|0\rangle = 0\ ;\ r = 0,1,2$$

defines the vacuum state. To avoid negative norm states the condition

$$\left[a_2(\vec{k}) - a_0(\vec{k})\right]|\Psi\rangle = 0\ ;\ \forall\,\vec{k} \iff \langle\Psi|N_2(\vec{k})|\Psi\rangle = \langle\Psi|N_0(\vec{k})|\Psi\rangle$$

is required on the physical photon states of the Hilbert space. Therefore, in two dimensions, there is only one degree of freedom for each $\vec{k}$ of the radiation field.

From the covariant commutation relations we derive the Feynman photon propagator:

$$\langle 0|T\{a^\mu(x)a^\nu(y)\}|0\rangle = i\hbar c D_F^{\mu\nu}(x-y) \tag{60}$$

where

$$D_F^{\mu\nu}(x) = \frac{1}{(2\pi)^3}\int d^3k\,\frac{-g^{\mu\nu}}{k^2+i\epsilon}e^{-ikx} \tag{61}$$

Choosing the polarization vectors in a given frame of reference as

$$\begin{aligned} \epsilon_0(\vec{k}) &= n = (1;0;0) \\ \epsilon_1(\vec{k}) &= (0;\tilde{\epsilon}_1(\vec{k}))\,;\ \tilde{\epsilon}_1(\vec{k})\cdot\vec{k} = 0 \\ \epsilon_2(\vec{k}) &= (0;\frac{\vec{k}}{|\vec{k}|}) = \frac{k - (kn)n}{((kn)^2 - k^2)^{1/2}} \end{aligned} \tag{62}$$

it is possible to express the momentum space propagator from (61) as

$$\begin{aligned} D_F^{\mu\nu}(k) &= \frac{-g^{\mu\nu}}{k^2 + i\epsilon} \\ &= D_{FT}^{\mu\nu}(k) + D_{FC}^{\mu\nu}(k) + D_{FR}^{\mu\nu}(k) \\ &= \frac{1}{k^2 + i\epsilon}\,\epsilon_1^{\mu}(\vec{k})\,\epsilon_1^{\nu}(\vec{k}) + \frac{n^\mu n^\nu}{(kn)^2 - k^2} + \frac{1}{k^2 + i\epsilon}\left[\frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2 - k^2}\right] \end{aligned} \tag{63}$$

The first term in (63) can be interpreted as the exchange of transverse photons. The remaining two terms follow from a linear combination of longitudinal and temporal photons such that

$$D_{FC}^{\mu\nu}(x) = \frac{g^{\mu 0}g^{\nu 0}}{(2\pi)^3}\int \frac{d^2\vec{k}\,e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2}\int dk^0 e^{-ik^0x^0} = g^{\mu 0}g^{\nu 0}\frac{1}{4\pi}\ln\frac{1}{|\vec{x}|}\,\delta(x^0); \tag{64}$$

This term corresponds to the instantaneous Coulomb interaction between charges in the plane, and the contribution of the remaining term $D_{FR}^{\mu\nu}(k)$ vanishes because the electromagnetic field only interacts with the conserved charge-current density, [12].

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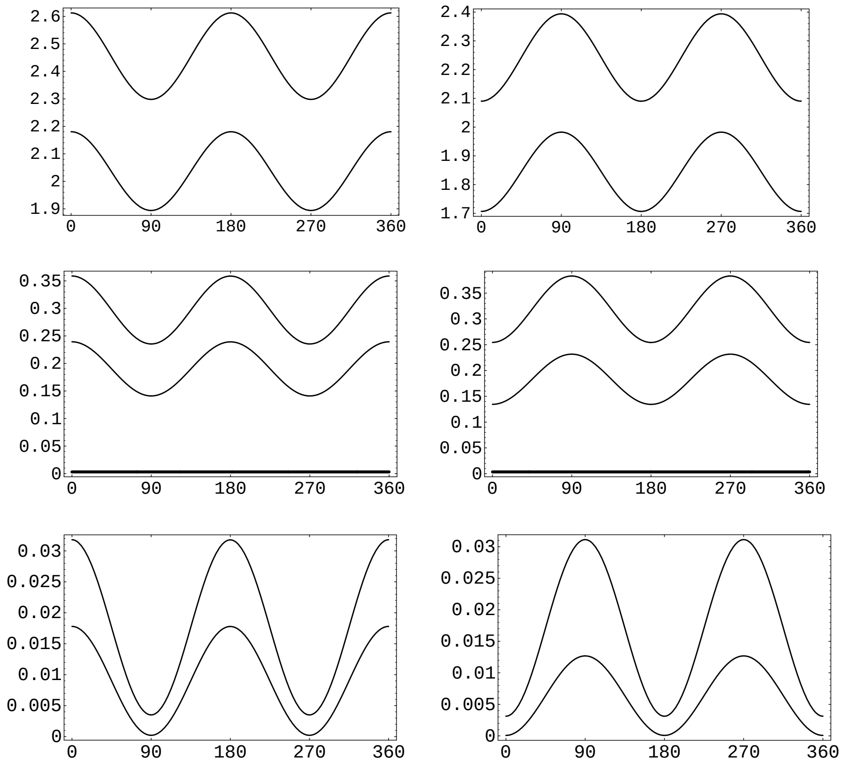

Figure 1: Angular distribution $\frac{1}{L_T}\ \frac{d}{d}$ as a function of the scattering angle for several values of : (a) In this case, $! < E_1 \ E_{\ 0}$. (b) $! > E_1 \ E_{\ 0}$. (c) $! < E_2 \ E_{\ 0}$ and the straight line corresponds to $! = \ _0$ the rst saddle point. (d) $! > E_2 \ E_{\ 0}$ and the straight line for $! = \ _1$. (e) $! < E_3 \ E_{\ 0}$ and nally (f) $! > E_3 \ E_{\ 0}$. In this case, $= 0{:}009$ and $L_T = 7{:}43 \ 10^{10}$ cm. We have chosen the $[0;2\ ]$ interval because the $\ ! \ $ symmetry is not evident from the formula.

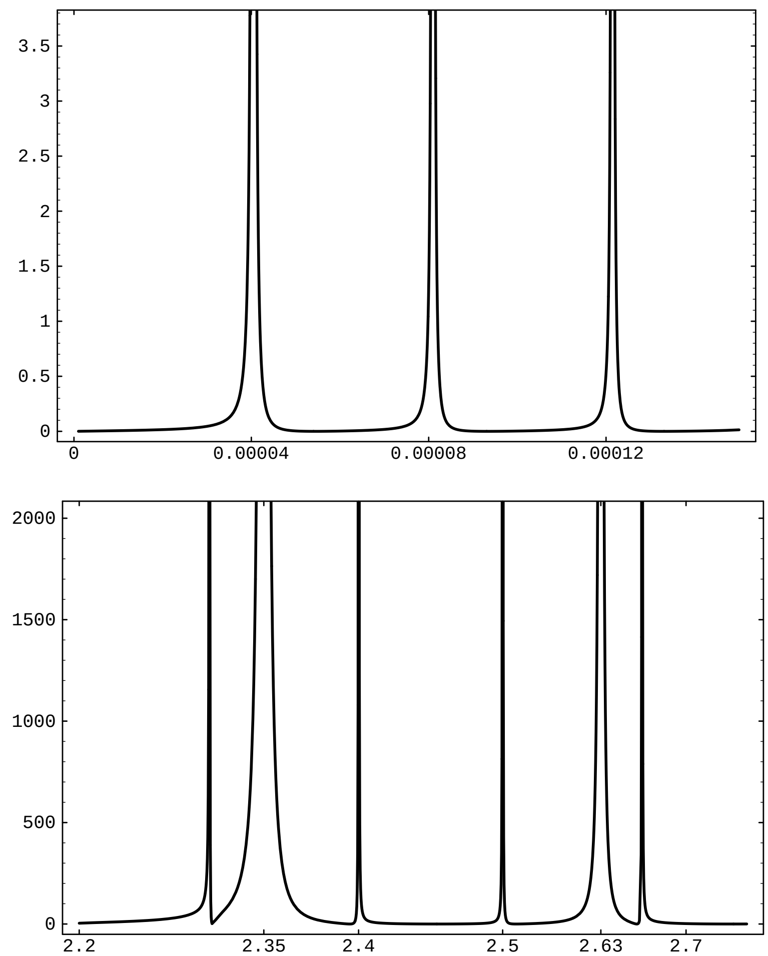

Figure 2: The differential cross-length distribution $\frac{1}{L_T}\ \frac{d}{d}$ as a function of for = 0. (a) For = 0.009 and $L_T = 7.43\ 10^{10}$ cm. (b) For = 0.9 and the same L_T.

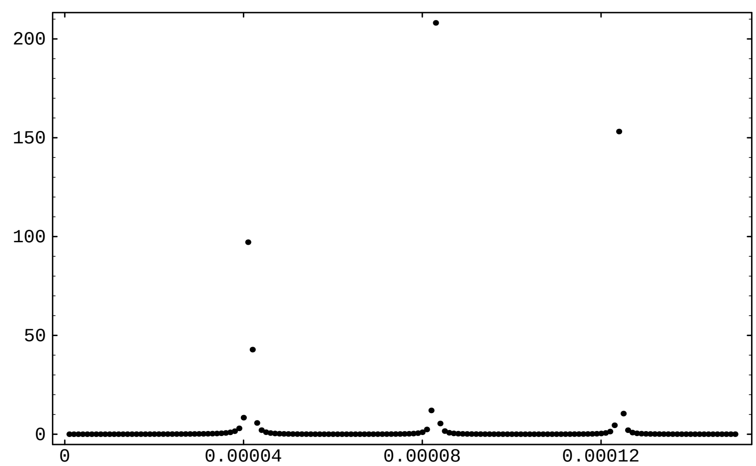

Figure 3: Total cross-length $\frac{T}{L_T}$ as a function of . = 0:009 and L_T = 7:43 10 ^{10} cm .