Orbit-based deformation procedure for two-field models

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We present a method for generating new deformed solutions starting from systems of two real scalar fields for which defect solutions and orbits are known. The procedure generalizes the approach introduced in a previous work [Phys. Rev. D 66, 101701(R) (2002)], in which it is shown how to construct new models altogether with its defect solutions, in terms of the original model and solutions. As an illustration, we work out an explicit example in detail.

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I. INTRODUCTION

Kinks, domain wall, vortices, strings and monopoles are all well known examples of defect solutions with topological profiles for field theories in different dimensions [1, 2, 3]. These solutions have been intensively studied since the seventies in high energy physics. The interest has been continuously renewed in different branches of physics since defect solutions in general appear in models of condensed matter as well as string theory. In particular, systems of real scalar fields have attracted attention with very distinct motivation, since they can be used to describe domain walls in supergravity [4] and braneworld scenarios with an extra dimension [5]. Therefore, it is important to find explicit analytic solutions for this kind of systems and is in this direction that the method conceived in ref. [6] has shown to be very useful. In that work it was shown that knowing a defect solution of a scalar field model for a single real field is enough for generating an infinity of new models with its solutions, all written in terms of the original model and solutions.

For field theories involving two real scalar fields, the mathematical problem concerning the integrability of the equations of motion is much harder, as one deals with a system of two coupled second order nonlinear ordinary differential equations, and the configuration space shows a distribution of minima that allows for a number of topological sectors. One way of simplifying the problem is to consider potentials belonging to the (wide) class corresponding to the bosonic sector of supersymmetric theories. This kind of systems can be studied, via the introduction of a superpotential, in a first order formalism which allows (stable) BPS configurations [7]. Even in this case finding explicit solutions can be a highly non-trivial task and therefore, any method for obtaining new solutions would be of great utility.

For models with two interacting components, the solutions on each topological sector determine orbits in the configuration space, which can be expressed as a constraint equation $O(\phi_1;\phi_2) = 0$. Based on this fact it was introduced in [8] a procedure called trial orbit method consisting in shooting an orbit and testing it on the equation corresponding to the model considered. Later, this method was adapted [9] for the searching of BPS states of systems of first order ODEs, leading to some advances. In recent years, other general methods for the investigation of complicated nonlinear problems arising in many-field systems which comprise multidefect solutions have been developed; see, for instance, [10, 11, 12, 13, 14, 15] and references therein. Models of two scalar fields have also been used to describe complex phenomena such as the entrapment of topological defects; see, for instance [16, 17, 18].

As will be shown below, the field deformation method introduced in [6] for one-field models also works for connecting ODE systems of two first-order equations in models with two scalar fields. The equations arising in this extended procedure are in general much more complicated than their counterpart for a single scalar field, given that not one but two deformation functions are now required, and it happens to be difficult to realize which pair of deformation functions would do the job in the right way, generating a well behaved deformed potential and solutions consistent with the equations of motion of the system. To overcome this difficulty, we take into account the fact that the actual solutions connecting two vacua of a topological sector live restricted to orbits in field space. Therefore, by deforming the first order equations for a two-field system while imposing the orbit constraint, we assure the consistency of the solutions of the deformed model at the level of the dynamical equations.

The paper is organized as follows. In the next Sec. II we briefly review the deformation method for one component systems, and we extend the deformation procedure to two-field models. In Secs. III and IV we show how the incorporation of orbit constraints allows to obtain consistent deformed solutions for two-field models. Then we work out an explicit example in detail to illustrate the procedure in Sec. V. We end this work with some comments and conclusions in Sec. IV.

II. THE DEFORMATION METHOD

We start with a system of a single real scalar field $\phi(x): R^{1,1} \to R$, on a bi-dimensional Minkowski spacetime ($\mu = 0,1$ with $x^0 = x_0 = t; x^1 = -x_1 = x$), described by a Lagrangian with the usual form

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi); \tag{1}$$

where the potential $V(\phi)$ specifies the model.

For static configurations ($\phi = \phi(x): R \to R$), the equation of motion reads

$$\frac{d^2\phi}{dx^2} = \frac{dV(\phi)}{d\phi}; \tag{2}$$

and the energy functional associated to the static solutions is given by

$$E[\phi] = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi) \right] \tag{3}$$

Requiring for the energy of the solutions to be finite results in the boundary conditions $d\phi/dx \to 0$ and $V(\phi) \to 0$ as $x \to \pm\infty$. Thus, the physical solutions are constant at infinity, and their asymptotic values are minima of the potential. Performing a first integral of the equation of motion under these conditions, we get to the first order equation

$$\left(\frac{d\phi}{dx}\right)^2 = 2V(\phi): \tag{4}$$

In this work we will restrict our study to solutions of this first order equation which present topological (kink-like) character, in the sense that they connect two different minima of the potential.

It is convenient to consider potentials of the form $V(\phi) = \frac{1}{2}[W'(\phi)]^2$, where the prime means derivative with respect to the argument, and the functional $W(\phi)$ is the superpotential. This allows us to write equation (4) as the gradient flow equations of W

$$\frac{d\phi}{dx} = \pm W'(\phi) \tag{5}$$

Let us now describe the deformation procedure for a single real scalar field model, as introduced in [6]. The prescription is the following. First, we define the deformed potential as

$$U(\chi) = \frac{V(f(\chi))}{[f'(\chi)]^2}; \tag{6}$$

where f is the deformation function.

This new potential determines the model for the deformed field χ through the deformed Lagrangian

$$L = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - U(\chi): \tag{7}$$

The first-order ODE determining the defect profile in the deformed model reads

$$\left(\frac{d\chi}{dx}\right)^2 = 2U(\chi); \tag{8}$$

with $U(\chi)$ given by (6). This can also be seen as the gradient flow equation of the deformed superpotential $\widetilde{W}(\chi)$ defined by

$$\frac{d\chi}{dx} = \pm \widetilde{W}'(\chi) \quad ; \quad \widetilde{W}'(\chi) = \frac{W'(f(\chi))}{f'(\chi)} \tag{9}$$

Second, we connect solutions of the original and deformed models through the deformation function

$$\phi(x) = f[\chi(x)]; \tag{10}$$

Here the deformation function $f(\chi)$ is assumed to be bijective – see however [11].

Therefore, if defect solutions $\phi(x)$ of (4) are known, the link between the two models provides the defect solutions $\chi(x)$ of (8) by just inverting the field transformation (10)

$$\chi(x) = f^{-1}(\phi(x)): \tag{11}$$

III. EXTENSION TO TWO-FIELD MODELS

At the level of the first order equations, the deformation method can be directly generalized to models with two scalar fields. We start with the model

$$L = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}); \tag{12}$$

where $\vec{\phi}$ is an isospinorial real field $\vec{\phi} = (\phi_1;\phi_2)$. Suppose that the potential energy density can be written in the form

$$V(\phi_1;\phi_2) = \frac{1}{2}\left(\frac{\partial W}{\partial \phi_1}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \phi_2}\right)^2 \tag{13}$$

where the superpotential W is a well behaved function in the space of scalar fields $\vec{\phi}(x;t) \in \mathrm{Maps}(\mathbb{R}^{1;1};\mathbb{R}^2)$. A subtle point is the following: Because (13) is a PDE equation there can be several independent solutions for W – not merely a global change of sign as in the one-field case – see [19]. Then the static finite energy solutions (topological defects) of this model satisfy the first-order ODE system

$$\begin{cases} \frac{d\phi_1}{dx} = \frac{\partial W}{\partial \phi_1} \\ \frac{d\phi_2}{dx} = \frac{\partial W}{\partial \phi_2} \end{cases} \tag{14}$$

Now we choose a deformation function $\tilde{f}:\mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\vec{\phi}(x) = \tilde{f}(\tilde{\chi}(x)) \quad , \quad \begin{cases} \phi_1(x) = f_1(\chi_1(x);\chi_2(x)) \\ \phi_2(x) = f_2(\chi_1(x);\chi_2(x)) \end{cases} ; \tag{15}$$

where χ_1 and χ_2 are the deformed fields. Then the first-order ODE system (14) becomes

$$\begin{cases} \partial_1 f_1 \frac{d\chi_1}{dx} + \partial_2 f_1 \frac{d\chi_2}{dx} = \partial_{f_1} W \\ \partial_1 f_2 \frac{d\chi_1}{dx} + \partial_2 f_2 \frac{d\chi_2}{dx} = \partial_{f_2} W \end{cases} ; \tag{16}$$

where

$$\partial_j f_i \equiv \frac{\partial f_i(\chi_1;\chi_2)}{\partial \chi_j} \qquad \text{and} \qquad \partial_{f_i} W \equiv \frac{\partial W(f_1(\chi_1;\chi_2);f_2(\chi_1;\chi_2))}{\partial f_i}:$$

This system can be rewritten as

$$\begin{cases} \dfrac{d\phi_1}{dx} = \dfrac{1}{J(\vec f)}\left(\partial_2 f_2\, \partial_{f_1} W - \partial_2 f_1\, \partial_{f_2} W\right) \\[2ex] \dfrac{d\phi_2}{dx} = \dfrac{1}{J(\vec f)}\left(\partial_1 f_1\, \partial_{f_2} W - \partial_1 f_2\, \partial_{f_1} W\right) \end{cases} \tag{17}$$

where $J(\vec f) = \partial_1 f_1 \partial_2 f_2 - \partial_2 f_1 \partial_1 f_2$ (the Jacobian of $\vec f$), and we assume that $J(\vec f) \neq 0$. (Note that this fact can be relaxed by restricting the deformation method to act on a family of open sets in R^2 where $J(\vec f) \neq 0$). Equations (17) can be interpreted as the first-order ODE system

$$\begin{cases} \dfrac{d\phi_1}{dx} = \partial_1 W \\[2ex] \dfrac{d\phi_2}{dx} = \partial_2 W \end{cases} \tag{18}$$

where $W(\phi_1;\phi_2)$ is the superpotential of the deformed system and the right hand side derivatives denote $\partial_j W = \frac{\partial W(\phi_1;\phi_2)}{\partial \phi_j}$.

In terms of the original superpotential, the deformed one is determined by the PDE system

$$\begin{cases} \dfrac{1}{J(\vec f)}\left(\partial_2 f_2\, \partial_{f_1} W - \partial_2 f_1\, \partial_{f_2} W\right) = \partial_1 W \\[2ex] \dfrac{1}{J(\vec f)}\left(\partial_1 f_1\, \partial_{f_2} W - \partial_1 f_2\, \partial_{f_1} W\right) = \partial_2 W \end{cases} \tag{19}$$

Therefore, if any two-component defect solution $\vec\phi(x)$ of (14) is known, the link between the two models provides two-component defect solutions $\tilde{\vec\phi}(x)$ of (18) by calculating

$$\tilde{\vec\phi}(x) = \vec f^{\,-1}(\vec\phi(x))$$

Vice versa, if two-component defect solutions $\tilde{\vec\phi}(x)$ of (18) are known, the link between both models provides two-component defect solutions $\vec\phi(x)$ of (14) by applying the transformation

$$\vec\phi(x) = \vec f(\tilde{\vec\phi}(x))$$

The existence of $\vec f^{\,-1}$ is associated to the above mentioned precisions about the zeros of $J(\vec f)$ via the Inverse Function Theorem.

At this point, the differences between working with one or more scalar fields appear. First, the ODE system (18) is supposed to give the solutions for the deformed system. This will be true only when those solutions satisfy, besides the first order system, the dynamical equations for the deformed potential

$$V(\phi_1;\phi_2) = \frac{1}{2}\left[(\partial_1 W)^2 + (\partial_2 W)^2\right]: \tag{20}$$

This is automatically satisfied by any solution of (18). But beside this, one is assuming that the deformation leads to well-behaved, smooth potentials. Therefore, we should ask for the first order derivatives of the deformed superpotential $W(\phi_1;\phi_2)$ to be continue or, alternatively, for its second order cross-derivatives to be identical $\partial_{12} W = \partial_{21} W$. Imposing this condition on the systems (18) and (19) leads to a very complicated constraint which suggest no obvious choice of the deformation functions f_1 and f_2. For this reason, in order to make progress we need to introduce some assumptions to simplify the situation to a tractable case. We will consider functions of the form $f_1 = f_1(\phi_1)$ and $f_2 = f_2(\phi_2)$. This significantly reduces the complexity of the constraint, which becomes the simple condition

$$(\partial_1 f_1)^2 - (\partial_2 f_2)^2 \; \partial_{f_1 f_2} W \; \partial_1 f_1 \partial_2 f_2 = 0 \tag{21}$$

This is true whenever f_1 and f_2 satisfy $\partial_1 f_1 = \pm\partial_2 f_2$ but, as $f_1(\phi_1)$ and $f_2(\phi_2)$ are functions of different fields, this expression seems to be nonsense. However, since the solutions $\phi_1(x)$ and $\phi_2(x)$ live on an orbit in configuration space, condition (21) must be understood as a function of the solutions, therefore we rewrite it as

$$\frac{df_1(\chi_1)}{d\chi_1} = \left.\frac{df_2(\chi_2)}{d\chi_2}\right|_{orbit} \tag{22}$$

When the deformation functions depend on one field only, the PDE system (19) takes the simpler form

$$\begin{cases} \partial_1 W(\chi_1;\chi_2) = \dfrac{\partial_{f_1} W[f_1;f_2]}{\partial_1 f_1} \\ \partial_2 W(\chi_1;\chi_2) = \dfrac{\partial_{f_2} W[f_1;f_2]}{\partial_2 f_2}. \end{cases} \tag{23}$$

Each one of these equations resembles the deformation recipe applied to the case of a single scalar field model, so one could think that it is just a duplication of the one-field procedure. However, the superpotential W now depends on both deformation functions $f_1(\chi_1)$ and $f_2(\chi_2)$, as a consequence of the interaction between the fields, and then the deformed superpotential W depends on both χ_1 and χ_2, and the resulting deformed model describes interacting fields.

As we have shown, requiring that the deformed model has a smooth potential leads to a condition on the deformation functions. Using this, and the fact that the fields are enforced to obey an orbit we are able to construct a consistent deformation procedure, detailed in the following section.

IV. THE ORBIT-BASED DEFORMATION

Taking into account the considerations above, we present an orbit-based procedure for constructing the deformation pair of functions. The steps to be followed for deforming two-field interacting models are the following:

1. Choose a deformation function for one of the fields, for example a function $f_1(\chi_1)$. Then, as stated in (15) we define the deformed field by $\phi_1 = f_1(\chi_1)$ (or, equivalently, $\chi_1 = f_1^{-1}(\phi_1)$). For the other field, $\phi_2 = f_2(\chi_2)$ we can write
$$\chi_2 = \int \partial_2 f_2^{-1}(\phi_2)\,d\phi_2 = \int \frac{d\phi_2}{\partial_2 f_2\left(f_2^{-1}(\phi_2)\right)} \tag{24}$$

2. Choose the topological sector to be deformed and an orbit $O(\phi_1;\phi_2) = 0$, associated to this sector. Use this equation to write ϕ_1 as a function of ϕ_2, i.e. $\phi_1 = F(\phi_2)$.

3. Then impose the condition (22) on (24) and use the expression $\phi_1 = F(\phi_2)$ to obtain
$$\chi_2 = \int \frac{d\phi_2}{\partial_1 f_1\left(f_1^{-1}(F(\phi_2))\right)} = \int \left.\partial_1 f_1^{-1}(\phi_1)\right|_{\phi_1 = F(\phi_2)} d\phi_2. \tag{25}$$

After integration we obtain χ_2 as a function of ϕ_2, which is nothing but the inverse of the deformation function f_2. This is the key result of the present work.

Thus, the chosen function $f_1(\chi_1)$ and the constructed one $f_2(\chi_2)$ form a pair that takes the original model and solutions to a deformed model with a smooth potential, and solutions satisfying the first order equations as well as the equations of motion.

Note that the procedure described above is orbit-dependent and the possibility of finding the second deformation function is restricted to the ability of explicitly integrating eq. (25).

V. A DETAILED EXAMPLE

As an example of application of the extended deformation method we consider the model [20, 21]

$$W = \phi_1 - \frac{1}{3}\phi_1^3 - r\,\phi_1\phi_2^2; \qquad r \in \mathbb{R} \tag{26}$$

It presents 4 minima: $v_{AA} = [\pm 1; 0]$ on the χ_1 axis, and $v_{BB} = [0; \pm\frac{1}{\sqrt{r}}]$ on the χ_2 axis.

The corresponding first order system of equations is

$$\begin{cases} \frac{d\chi_1}{dx} = 1 - \chi_1^2 - r\chi_2^2 \\ \frac{d\chi_2}{dx} = -2r\chi_1\chi_2 : \end{cases} \tag{27}$$

The integrating factor for this system can be found explicitly, and this allows to find the flow-line family of curves

$$r\chi_2^2 - (1-2r)(1-\chi_1^2) + C\chi_2^{\frac{1}{r}} = 0; \qquad r \neq \frac{1}{2} \tag{28}$$

where C is an integration constant. Real values of C give orbits starting or ending in v_{AA} minima, but there exist some critical values. When $C = C^S = -2r(\sqrt{r})^{\frac{1}{r}}$ the orbits start and end at different axis, joining v_{AA} and v_{BB} minima. The existence of these critical values determines several ranges of C in $\mathbb{R}$ for which the corresponding solutions of (28) are not kink orbits - see [21].

So we have a two-field model and its general orbit equation depending on two parameters (r and C). In order to apply the orbit-based procedure described in the previous section, we will consider separately the different kind of orbits, corresponding to different regions in parameter space.

A. Elliptic orbit deformation

Let us first consider the simplest case in which the integration constant C is taken to be zero and r restricted to the interval $(0; \frac{1}{2})$. In this case, the orbits are ellipses and (28) can be rewritten as

$$\chi_2 = F(\chi_1) = \sqrt{\frac{1-2r}{r}(1-\chi_1^2)} \qquad (C = 0; \ 0 < r < \tfrac{1}{2}) \tag{29}$$

A two-field static solution for the system (27), which satisfies this constraint is [20]

$$\chi_1(x) = \tanh(2rx); \qquad \chi_2(x) = \sqrt{\frac{1-2r}{r}}\,\text{sech}(2rx): \tag{30}$$

In figure 1 we show the vacua structure and some orbits of the model for different values of the r parameter. While the two minima in the horizontal axis form a topological sector (AA-sector), the two minima in the vertical axis (BB-sector) cannot be connected by solutions of the first order system (27).

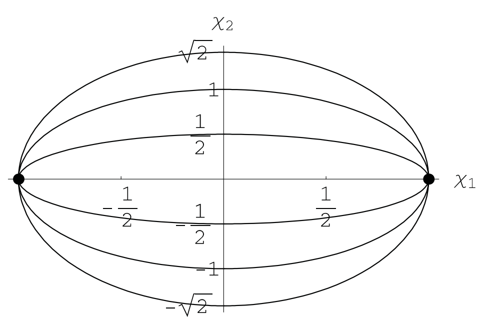


Figure 1: Minima and orbits of the undeformed model for $C = 0$ and different r values ($r = \frac{4}{9}; \frac{1}{3}; \frac{1}{4}$).

Now, following the prescription established in the preceding section, we choose a deformation function for χ_1

$$\phi_1 = f_1^{-1}(\chi_1) = \text{arcsinh}(\chi_1) \tag{31}$$

For constructing the deformation function for the other field, χ_2, we calculate the integral (25) using (29) and (31)

$$\phi_2 = \int \frac{d\chi_2}{f_1'\left(f_1^{-1}(F(\chi_2))\right)} = \int \frac{d\chi_2}{\sqrt{2 - \frac{r}{1-2r}\chi_2^2}} \tag{32}$$

We obtain

$$\phi_2 = f_2^{-1}(\chi_2) = \sqrt{\frac{1-2r}{r}}\arcsin\left(\sqrt{\frac{r}{2(1-2r)}}\,\chi_2\right): \tag{33}$$

Thus

$$f_2(\phi_2) = \sqrt{\frac{2(1-2r)}{r}}\sin\left(\sqrt{\frac{r}{1-2r}}\,\phi_2\right) \tag{34}$$

With the deformation functions at hand, and making use of (23), we are able to write down the deformed potential, which reads

$$\begin{aligned} U(\chi_1;\chi_2;r) = {} & \tfrac{1}{2}\left[1 - \sinh^2(\chi_1) - 2(1-2r)\sin^2\left(\sqrt{\tfrac{r}{1-2r}}\,\chi_2\right)\right]^2 \operatorname{sech}^2(\chi_1) \\ & + 2r(1-2r)\sinh^2(\chi_1)\tan^2\left(\sqrt{\tfrac{r}{1-2r}}\,\chi_2\right) \end{aligned} \tag{35}$$

Evaluating the fields χ_1 and χ_2 at the solution of the original model we obtain a solitonic solution of the deformed model specified by $U(\chi_1;\chi_2)$

$$\chi_1(x) = \operatorname{arcsinh}[\tanh(2rx)]; \qquad \chi_2(x) = \sqrt{\frac{1-2r}{r}}\arcsin\left(\frac{\sqrt{2}}{2}\operatorname{sech}(2rx)\right) \tag{36}$$

Condition (22) is automatically satisfied by this solution. This makes it consistent with the second order equations of the deformed system, as can be explicitly verified.

To show how this deformation acts, in figure 2 we plot both, the deformed and the original solutions.

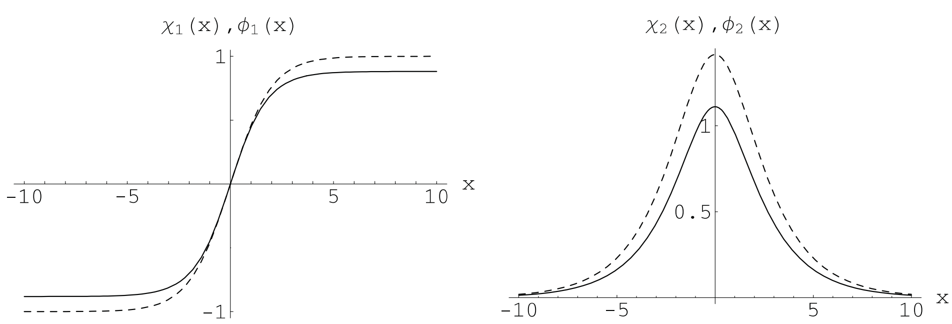


Figure 2: Solutions of the undeformed (dotted line) and deformed (solid line) models for $C = 0$, $r = \frac{1}{4}$.

Substituting the original fields by its corresponding deformed partners in (28) we obtain the deformed orbit

$$\cosh^2(\chi_1) - 2\cos^2\left(\sqrt{\frac{r}{1-2r}}\,\chi_2\right) = 0: \tag{37}$$

which allows writing χ_2 as a function of χ_1

$$\chi_2 = \sqrt{\frac{1-2r}{r}}\left[\arccos\left(\frac{\sqrt{2}}{2}\cosh(\chi_1)\right) + k\pi\right] \qquad (k \in \mathbb{Z}); \tag{38}$$

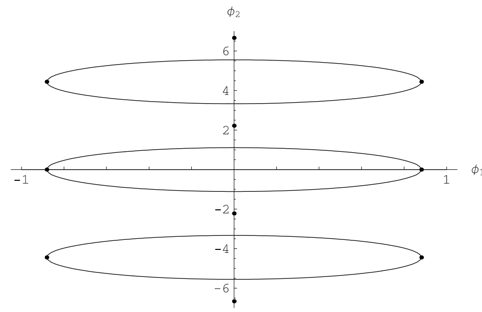


Figure 3: Some vacua and orbits of the deformed model with $C = 0$, $r = 1/4$.

and this explicitly shows that the new model presents a periodic vacua structure. Such structure and some orbits of the deformed model are shown in fig.3.

We can summarize the results obtained for the AA-sector as follows

$$\begin{cases} orb_{AA} & : r\phi_2^2 - (1-2r)(1-\phi_1^2) = 0; \qquad v_{AA} = \{[1,0],[-1,0]\} \\ \tilde{\phi}(x) & = \left(\tanh 2rx,\ \pm\sqrt{\frac{1-2r}{r}}\,\mathrm{sech}\,2rx\right) \\ \tilde{f} & = \left(\sinh\chi_1,\ \sqrt{\frac{2(1-2r)}{r}}\sin\left(\frac{\sqrt{r}}{\sqrt{1-2r}}\chi_2\right)\right) \\ orb_{AA\,def} & : \cosh^2\chi_1 - 2\cos^2\left(\frac{\sqrt{r}}{\sqrt{1-2r}}\chi_2\right) = 0. \\ v_{AA\,def} & = \left(0,\ \sqrt{\frac{1-2r}{r}}\arcsin\frac{1}{\sqrt{2(1-r)}} + k\pi\right) \\ \tilde{\chi}(x) & = \left(\mathrm{arcsinh}(\tanh 2rx),\ \pm\sqrt{\frac{1-2r}{r}}\arcsin\left(\frac{\sqrt{2}}{2}\mathrm{sech}\,2rx\right)\right) \end{cases} \tag{39}$$

B. Linear orbits deformation

It is also possible to find explicit solutions for the BNRT model (26) in other regions in parameter space. For example, for specific values of the integration constant C and parameter r, there are orbits connecting one minimun on the $\phi_1 = 0$ axis with one on the $\phi_2 = 0$ axis (AB-sectors). As an illustration we address now the $r = 1$ case, and deform orbits for integration constants $C = C^S = -2$. There are four linear orbits, one for each sector, that we label a, b, c and d, starting from the first quadrant and moving forward clockwise, as shown in figure4.

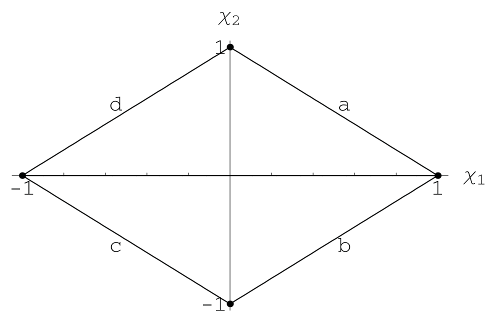
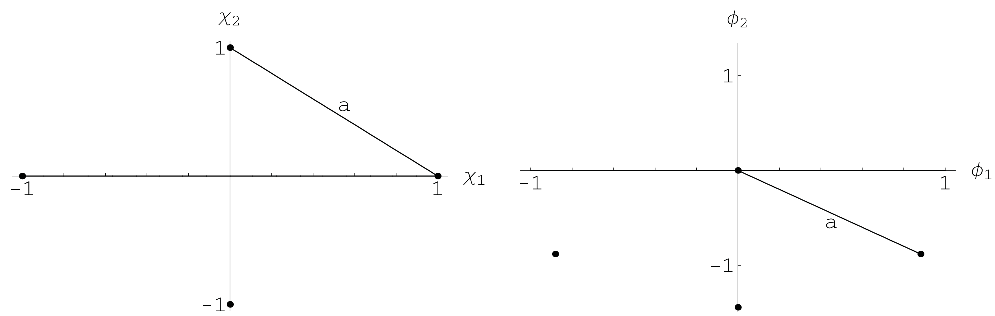


Figure 4: AB-orbits ($C = -2$, $r = 1$).

In order to deform these linear sectors we choose again the deformation function for one of the fields as $\phi_1 = f_1(\chi_1) = \sinh(\chi_1)$, and construct the $f_2(\chi_2)$ by using the corresponding orbits. The resulting vacua structure and orbits for the deformed models are shown below.

For the a sector we obtain

$$\begin{cases} orb^{(a)}_{AB} & : \ \chi_2 + \chi_1 - 1 = 0; \qquad v^{(a)}_{AB} = \{[1,0],[0,1]\} \\ \tilde{\chi}^{(a)}(x) & = \left(\frac{1}{2}(1 - \tanh x), \frac{1}{2}(1 + \tanh x)\right) \\ \tilde{f}^{(a)} & = (\sinh \phi_1, 1 + \sinh \phi_2) \\ orb^{(a)}_{AB\,def} & : \ \sinh \phi_2 + \sinh \phi_1 = 0 \\ v^{(a)}_{AB\,def} & = \{[0,0],[0,-\sinh^{-1}(2)],[\sinh^{-1}(1),\sinh^{-1}(-1)],[\sinh^{-1}(-1),\sinh^{-1}(-1)]\} \\ \tilde{\phi}^{(a)}(x) & = \left(-\sinh^{-1}\frac{1}{2}(-1 + \tanh x), \sinh^{-1}\frac{1}{2}(-1 + \tanh x)\right) \end{cases} \tag{40}$$



(a) Original orbit (b) Deformed orbit

Figure 5: Deformation of $AB^{(a)}$ sector ($C = -2$, $r = 1$).

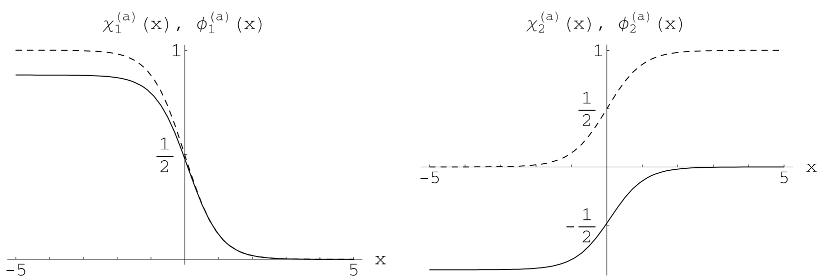
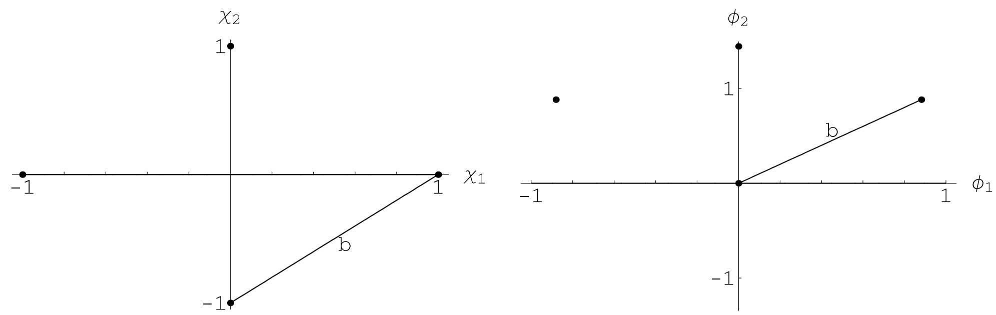


Figure 6: Compared profiles of original (dashed line) and deformed (solid line) defect solutions. $AB^{(a)}$ sector.

For the b sector we get

$$\begin{cases} orb^{(b)}_{AB} & : \ \chi_2 - \chi_1 + 1 = 0; \qquad v^{(b)}_{AB} = \{[1,0],[0,-1]\} \\ \tilde{\chi}^{(b)}(x) & = \left(\frac{1}{2}(1 - \tanh x), -\frac{1}{2}(1 + \tanh x)\right) \\ \tilde{f}^{(b)} & = (\sinh \phi_1, -1 + \sinh \phi_2) \\ orb^{(b)}_{AB\,def} & : \ \sinh \phi_2 - \sinh \phi_1 = 0 \\ v^{(b)}_{AB\,def} & = \{[0,0],[0,\sinh^{-1}2],[\sinh^{-1}(1),\sinh^{-1}(1)],[\sinh^{-1}(-1),\sinh^{-1}(1)]\} \\ \tilde{\phi}^{(b)}(x) & = \left(-\sinh^{-1}\frac{1}{2}(-1 + \tanh x), -\sinh^{-1}\frac{1}{2}(-1 + \tanh x)\right) \end{cases} \tag{41}$$



(a) Original orbit (b) Deformed orbit

Figure 7: Deformation of $AB^{(b)}$ sector ($C = 2, r = 1$).

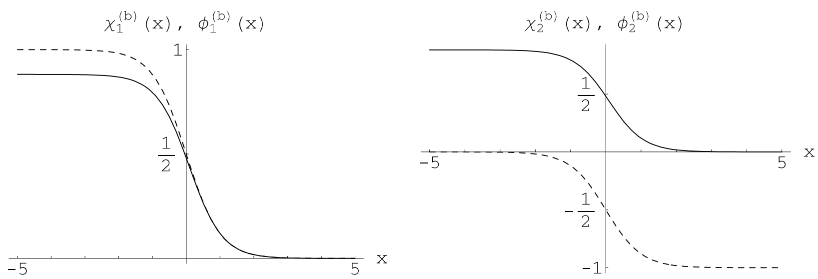
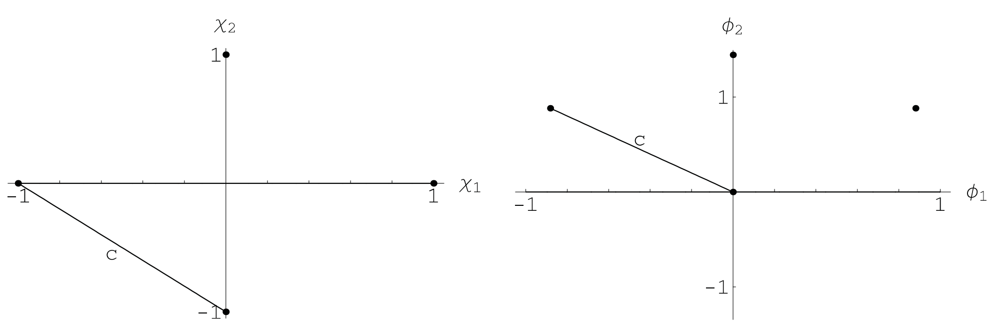


Figure 8: Compared profiles of original (dashed) and deformed (solid) defect solutions. $AB^{(b)}$ sector

In the case of the c sector we obtain

$$\begin{cases} \mathrm{orb}^{(c)}_{AB} : \ \chi_2 + \chi_1 + 1 = 0; \qquad v^{(c)}_{AB} = \{[-1;0];[0;-1]\} \\ \tilde{\chi}^{(c)}(x) = \left(-\frac{1}{2}(1 - \tanh x); -\frac{1}{2}(1 + \tanh x)\right); \\ \tilde{f}^{(c)} = (\sinh\phi_1; 1 + \sinh\phi_2) \\ \mathrm{orb}^{(c)}_{AB\,def} : \ \sinh\phi_2 + \sinh\phi_1 = 0 \\ v^{(c)}_{AB\,def} = \{[0;0];[0;\sinh^{-1}2];[\sinh^{-1}(1);\sinh^{-1}(1)];[\sinh^{-1}(-1);\sinh^{-1}(1)]\} \\ \tilde{\phi}^{(c)}(x) = \left(\sinh^{-1}\frac{1}{2}(-1 + \tanh x); -\sinh^{-1}\frac{1}{2}(-1 + \tanh x)\right) \end{cases} \tag{42}$$



(a) Original orbit. (b) Deformed orbit.

Figure 9: Deformation of $AB^{(c)}$ sector ($C = 2, r = 1$).

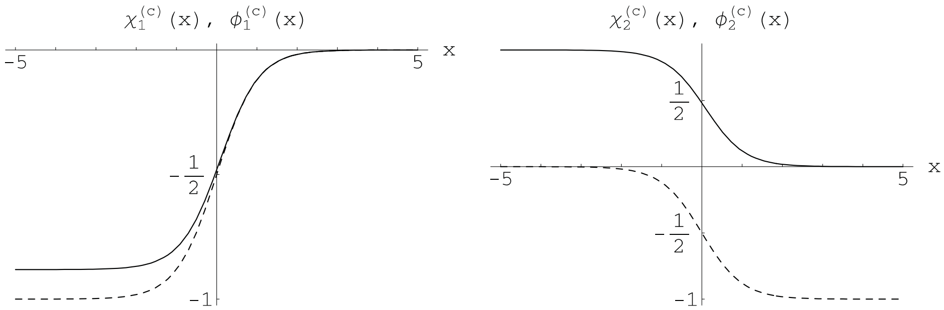
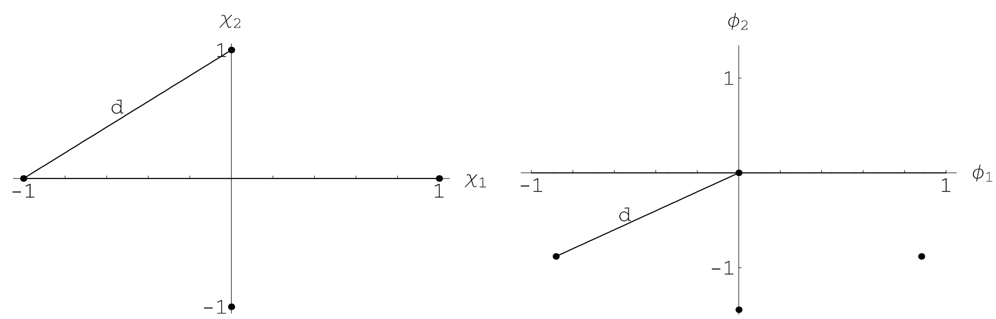


Figure 10: Compared profiles of original (dashed line) and deformed (solid line) defect solutions. $AB^{(c)}$ sector.

Finally, for the d sector we have

$$\begin{cases} orb^{(d)}_{AB} & : \chi_2 - \chi_1 - 1 = 0; \qquad v^{(d)}_{AB} = \{[-1;0];[0;1]\} \\ \tilde{\chi}^{(d)}(x) & = \left(-\frac{1}{2}(1 - \tanh x); \frac{1}{2}(1 + \tanh x)\right); \\ \tilde{f}^{(d)} & = (\sinh \phi_1; 1 + \sinh \phi_2) \\ orb^{(d)}_{AB\,def} & : \sinh \phi_2 - \sinh \phi_1 = 0 \\ v^{(d)}_{AB\,def} & = \{[0;0];[0;-\sinh^{-1} 2];[\sinh^{-1}(1);\sinh^{-1}(-1)];[\sinh^{-1}(-1);\sinh^{-1}(-1)]\} \\ \tilde{\phi}^{(d)}(x) & = \left(\sinh^{-1}\frac{1}{2}(-1 + \tanh x); \sinh^{-1}\frac{1}{2}(-1 + \tanh x)\right) \end{cases} \qquad (43)$$



(a) Original orbit. (b) Deformed orbit.

Figure 11: Deformation of $AB^{(d)}$ sector ($C = -2$, $r = 1$).

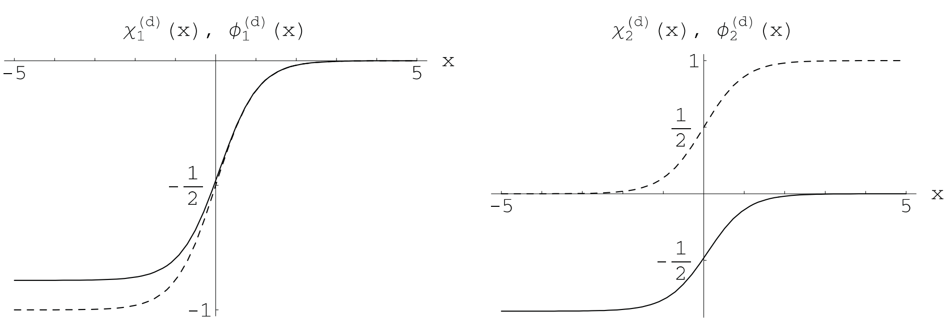


Figure 12: Compared profiles of original (dashed line) and deformed (solid line) defects. $AB^{(d)}$ sector.

In the above examples, we have considered two cases in which C has been fixed. Let us now move to other regions in parameter space, letting C to be undetermined. This will of course increment the complexity of the problem but, as we will show below, there is still a lot of possibilities of generating new models. This illustrates the richness of results that can be obtained by exploiting the deformation method extended to interacting field models.

C. An integrable case

Recall the general orbit equation (28)

$$\phi_1^2 = 1 - \frac{r}{1-2r}\phi_2^2 - \frac{C}{1-2r}\phi_2^{\frac{1}{r}}. \tag{44}$$

Now, following the prescription for generating the deformed model, we are led to

$$\phi_1 = f_1(\chi_1) = \sinh\chi_1; \qquad \chi_1 = f_1^{-1}(\phi_1) = \text{arcsinh}\,\phi_1 \tag{45}$$

and

$$\chi_2 = \int \frac{d\phi_2}{\sqrt{2 - \frac{r}{1-2r}\phi_2^2 - \frac{C}{1-2r}\phi_2^{\frac{1}{r}}}} \tag{46}$$

Integral (46) is highly non trivial for arbitrary values of C and r. However it is integrable for some values of r, in particular for $r = \frac{1}{4}$. For this value there is only one critical constant $C^S = -\frac{1}{32}$, and kink orbits arise if $C \in [C^S;\infty)$. In this case, (46) is an elliptic integral, that can be written in the form

$$\chi_2 = \frac{1}{\sqrt{-2C}}\int \frac{d\phi_2}{\sqrt{(\phi_2^2-\alpha_1)(\phi_2^2-\alpha_2)}}; \tag{47}$$

with $\alpha_1 = \frac{1}{8C}\left(-1+\sqrt{1+64C}\right)$ and $\alpha_2 = -\frac{1}{8C}\left(1+\sqrt{1+64C}\right)$.

By an appropriate change of the integration variable, integral (47) can be solved in terms of Jacobian elliptic functions [22]. The result is

$$\chi_2 = \frac{1}{\sqrt{\alpha_1}\sqrt{-2C}}\,\text{sn}^{-1}\left(\frac{\phi_2}{\sqrt{\alpha_2}}; \sqrt{\frac{\alpha_2}{\alpha_1}}\right). \tag{48}$$

We note that for all the Jacobi elliptic functions appearing in this work we have to take its real part, as we are dealing with real scalar fields and they are solutions of a physical problem.

This solution presents distinguishable behaviors depending on C taking values on the regions $(-\frac{1}{32}; -\frac{1}{64})$, $(-\frac{1}{64};0)$ or $(0;\infty)$. For the special values $C = 0$ and $C = -\frac{1}{64}$, the deformed system presents field solutions in terms of elementary functions rather than Jacobi elliptic functions.

Putting the original field solutions in terms of the deformed ones

$$\phi_1 = \sinh\chi_1; \qquad \phi_2 = \sqrt{\alpha_2}\,\text{sn}\left(\sqrt{\alpha_1}\sqrt{-2C}\,\chi_2; \sqrt{\frac{\alpha_2}{\alpha_1}}\right); \tag{49}$$

we can write the explicit form the deformed model potential. We obtain

$$\begin{aligned} U\left(\chi_1;\chi_2;\frac{1}{4};C\right) = {} & \frac{1}{2}\left[1 - \sinh^2\chi_1 - \alpha_2\,\text{sn}^2\left(\sqrt{\alpha_1}\sqrt{-2C}\,\chi_2; \sqrt{\alpha_2/\alpha_1}\right)\right]^2 \text{sech}^2\chi_1 \\ & - \frac{\alpha_1 C}{2}\sinh^2\chi_1\,\frac{\text{sc}^2\left(\sqrt{\alpha_1}\sqrt{-2C}\,\chi_2; \sqrt{\alpha_2/\alpha_1}\right)}{\text{dn}^2\left(\sqrt{\alpha_1}\sqrt{-2C}\,\chi_2; \sqrt{\alpha_2/\alpha_1}\right)} \end{aligned} \tag{50}$$

A field solution of the undeformed system (27), satisfying the orbit (44) for $r = \frac{1}{4}$ is

$$\phi_1(x) = \frac{\sinh(x)}{\cosh(x)+b^2}; \qquad \phi_2(x) = 2\sqrt{\frac{b}{\cosh(x)+b^2}}; \tag{51}$$

where $b^2 = 1/\sqrt{1+32C}$. Therefore, using the deformation functions (49), the solutions for the deformed system read

$$\chi_1(x) = \operatorname{arcsinh}\left(\frac{\sinh(x)}{b^2+\cosh(x)}\right) \; ; \qquad \chi_2(x) = \frac{1}{\sqrt{2\lambda_1 C}}\,\mathrm{sn}^{-1}\left[\frac{2\sqrt{b/\sqrt{2}}}{\cosh(x)+b^2} ; \sqrt{\frac{\lambda_2}{\lambda_1}}\right] \tag{52}$$

In figure13 we plot the original and deformed profiles of the field solution (taking $C = 1$). It is remarkable how similar is the behavior of these solutions and the ones plotted in figure2, despite the much more involved analytical expressions in this last case (compare formula (50) with formula (35)).

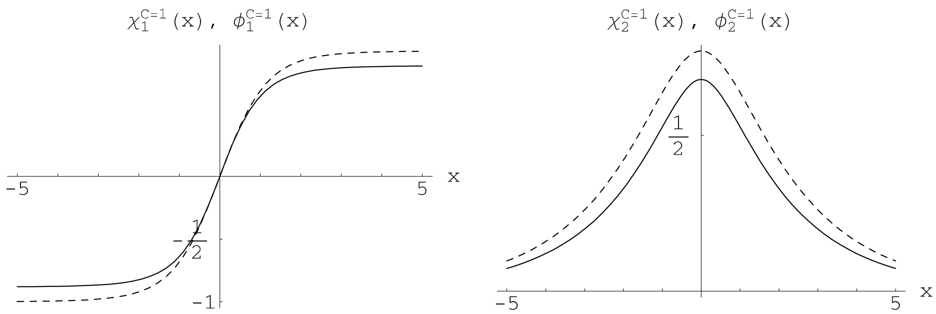


Figure 13: Deformed (solid) and undeformed (dotted) solutions for BNRT model ($r = \frac{1}{4}$, $C = 1$).

From the results obtained above we stress that the orbit-based deformations can be applied to every kink orbit of the BNRT model (26) for the integrable case of $r = 1/4$. For almost any value of the constant C we obtain different models in terms of elliptic Jacobi functions, with two exceptions: $C = 0$ and $C = -1/64$. Also, we note that there is only one kink solution for each member of the family of deformed models, coming from the appropriate kink orbit in the original model.

D. Another integrable case.

For $r = 1$ the deformed model for every kink orbit can also be found, given that, as for the former case, it is possible to integrate (46) for any value of C. In fact, setting $r = 1$ in (46) and taking $\chi_1 = \operatorname{arcsinh}(\phi_1)$, we obtain the deformed χ_2 field

$$\chi_2 = \ln\left[\frac{1}{2}C + \phi_2 + \sqrt{2 + C\phi_2 + \phi_2^2}\right] \tag{53}$$

Writing the original field solutions in terms of the deformed ones, we can find the explicit form for the deformed model potential, which turns out to be

$$\begin{aligned} U(\chi_1;\chi_2;r=1;C) = {} & \tfrac{1}{2}\operatorname{sech}(\chi_1)^2\left[1-\sinh(\chi_1)^2-\tfrac{1}{64}e^{-2\chi_2}\left(-8+4e^{2\chi_2}-4Ce^{\chi_2}+C^2\right)^2\right]^2 \\ & +\tfrac{1}{4}\sinh(\chi_1)^2 e^{-2\chi_2}\left[\frac{-8+4e^{2\chi_2}-4Ce^{\chi_2}+C^2}{8e^{-\chi_2}+4e^{\chi_2}-C^2e^{-\chi_2}}\right]^2 \end{aligned} \tag{54}$$

The solutions of the BNRT model for $r = 1$ can be written as

$$\phi_1 = \frac{(e^{2x})^2 - C_1^2 + C_2^2}{(e^{2x}-C_1)^2 - C_2^2}; \qquad \phi_2 = \frac{2C_2e^{2x}}{(e^{2x}-C_1)^2-C_2^2} \tag{55}$$

with $C_2 = 2C_1/C$ and $|C| \geq C^S = 2$. As was said before, the solutions given in formula (55) are not kinks if $|C| < 2$, as both ϕ_1 and ϕ_2 become singular for this values of C.

The corresponding deformed solutions read

$$\begin{aligned} \chi_1 &= \operatorname{arcsinh}\left(\frac{(e^{2x})^2-C_1^2+C_2^2}{(e^{2x}-C_1)^2-C_2^2}\right) \\ \chi_2 &= \ln\left[\frac{1}{2}C - \frac{2C_2e^{2x}}{(e^{2x}-C_1)^2-C_2^2} + \sqrt{2+\frac{4C_2^2(e^{2x})^2}{((e^{2x}-C_1)^2-C_2^2)^2} - \frac{4C_1e^{2x}}{(e^{2x}-C_1)^2-C_2^2}}\right] \end{aligned} \tag{56}$$

For the specific value $C = 2$, the solutions (55) are both kink-like, as well as the deformed solutions (56), and its profiles are very similar to that of the case $C = -2$, plotted in Fig. 10.

For $C > 2$ we find that the profiles of ϕ_1 and ϕ_2 in (55) are kink and lump-like respectively, as shown in Fig. 14.

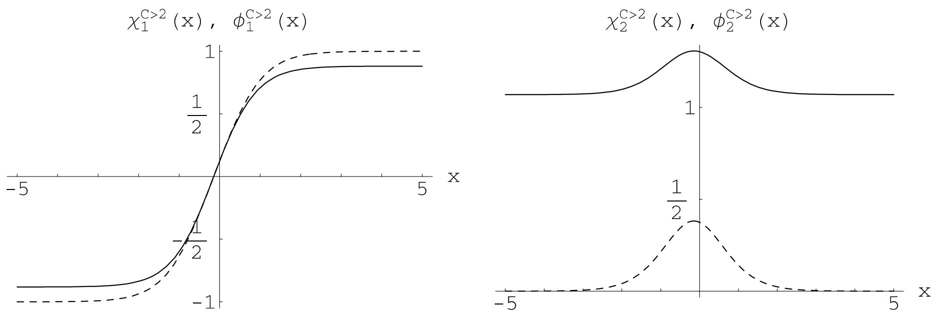


Figure 14: Deformed (solid) and original (dotted) solutions for BNRT model for $r = 1$, $C_1 = -1$ and $C = 3$.

Before ending this section, let us recall that the values $r = 1/4$ and $r = 1$; for which the integral (46) can be analyticaly implemented, present special features: kinks of the $r = 1/4$ family are separatrix trajectories of a mechanical problem which is Hamilton-Jacobi separable in parabolic coordinates, and for $r = 1$; the mechanical problem is separable in Cartesian coordinates; these problems are known as Liouville type III and IV, respectivelly [23]. However, even though the separable Liouville type problems seem to lead to integrable expressions in equation (25), in practice not all the kink orbits are algebraic, and this may preclude the presence of analytical solutions.

VI. COMMENTS AND CONCLUSIONS

In this work we have presented a generalization of the deformation method, first introduced in [6], which allows to generate deformed potentials and solutions given a model of two real scalar field and its solutions. The main new ingredient consist in the need of imposing a constraint on the functions used to deform the fields, required to preserve the relation between the original solutions, that live in orbits of the configuration space. As the construction of the deformed solutions involves the orbit constraint, the deformation must be implemented independently on each topological sector. Consequently, different orbits (even when belonging to the same sector) can lead to different full deformed models and solutions.

The present version of the method applies to models with an associated superpotential $W(\phi_1;\phi_2)$: Although we have considered deformation functions depending on a single field, more general functions can be used, and reduced to the former case with the use of the orbit itself. Deformation functions depending on both scalar fields are now being considered in the context of models with holomorphic superpotentials [24], as is the case of the bosonic sector of the (1+1)-dimensional N = 2 SUSY Wess-Zumino model. We will further report on this possibility in future work.

As the described procedure is orbit dependent, whenever deforming a two-field model, the integrability of equation (25) has to be analyzed separately in each case. However, in case of two-field theories which are associated to separable mechanical systems of the Liouville type, it seems that the present method will work very nicely, at least when one is restricted to consider algebraic kink orbits. Interesting examples of this kind are known, as the celebrated MSTB model [25] and other models proposed recently in Ref. [26].

This new version of the deformation procedure provides a tool for studying more sophisticated systems. It allows one to generate a diversity of new systems with their corresponding solutions, which may contribute to improve the understanding of complex problems.

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