Quantum fluctuations of topological S^3-kinks

A. Alonso Izquierdo[(a)], M. A. Gonzalez Leon[(a)],
J. Mateos Guilarte[(b)], M. J. Senosiain[(c)]
[(a)] Departamento de Matematica Aplicada and IUFFyM
[(b)] Departamento de Fisica Fundamental and IUFFyM
[(c)] Departamento de Matematicas, Universidad de Salamanca, SPAIN

The kink Casimir effect in the massive non-linear S^3-sigma model is analyzed.

Keywords: Kink Casimir effect, spectral zeta function, non-linear sigma model

1. Introduction

Quantum fluctuations around background kink fields are sophisticated cousins of vacuum fluctuations. Van Nieuwenhuizen et al.[1] reported on the state of the art in this topic in QFEXT03 for susy solitons as pointed out by Milton[2]. More recently, new results have been achieved by (almost) the same Stony Brook/Wien group in the analysis of the quantum fluctuations of susy solitons of non-linear sigma models[3,4]. Almost in parallel, we developed a similar program [5–8] for the kinks of the massive non-linear S^2-sigma model in a purely bosonic framework. Our goal in this work is to describe the quantum fluctuations of the S^3-kinks. The bosonic sector of the non-linear version of the Gell-Mann/Levy σ-model[9] is precisely the system that we are going to address. Being non renormalizable in (3+1)-dimensions, it was conceived as an effective theory describing the low energy interactions of nucleons and pions. In (1+1)-dimensions, however, the pion dynamics can be re-interpreted as the dynamics of a linear chain of $O(4)$ spin fields, which was renormalized by Brezin et al[10]. We just merely add quadratic terms in the fields to escape from infrared divergences.

2. Massive non-linear S^3-sigma model and topological kinks

Let us consider $\phi_a(t;x)$; $a = 1;2;3;4$, four scalar fields in the $(1+1)$-dimensional Minkowski space-time $R^{1,1}$. The action of the massive non-linear S^3-sigma model looks very simple

$$S[\phi_1;\phi_2;\phi_3;\phi_4] = \int dt dx \left\{ \frac{1}{2} g^{\mu\nu} \sum_{a=1}^{4} \frac{\partial \phi_a}{\partial x^\mu}\frac{\partial \phi_a}{\partial x^\nu} - \frac{1}{2}\sum_{a=1}^{4} \nu_a^2 \phi_a^2(x^\mu) \right\} ; \quad (1)$$

where $\nu_1^2 > \nu_2^2 > \nu_3^2 > \nu_4^2$, but the fields are constrained to live in the S^3-sphere, $\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2 = R^2$ forming the infinite dimensional space: $\mathrm{Maps}(\mathbb{R}^{1,1};S^3)$. We take $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and the natural system of units $\hbar = c = 1$. We select $\sigma_2^2 = \frac{\nu_2^2-\nu_4^2}{\nu_1^2-\nu_4^2} = \frac{\gamma_2^2}{\gamma^2}$, $\sigma_3^2 = \frac{\nu_3^2-\nu_4^2}{\nu_1^2-\nu_4^2} = \frac{\gamma_3^2}{\gamma^2}$, such that $0 < \sigma_3^2 < \sigma_2^2 < \sigma_1^2 = 1$, and define non-dimensional coordinates: $x^\mu \to \frac{x^\mu}{\gamma}$.

The extremely non-linear dynamics implied by (1) plus the constraint is unveiled if one solves ϕ_4 in favor of $\phi_1;\phi_2;\phi_3$ in the action and introduce the power expansion of the non-polynomial term. This process shows that: (a) There are an infinite number of vertices determining the interactions between the three pseudo-Nambu-Goldstone bosons. (b) $\frac{1}{R^2}$ is the coupling constant. Vertices with different numbers of legs belong to different orders of perturbation theory: $\frac{1}{R^{2n-2}}$ arises as a factor in the vertices with $2n$ legs. (c) In $(1+1)$-dimensions massless bosons are discarded due to the infrared asymptotics. We consider the situation when the three masses are different. The one-loop self-energy graphs of ϕ_1, ϕ_2 and ϕ_3: $\delta_2(\sigma_2;\sigma_3) = \sigma_2^2\,\delta_1(\sigma_2;\sigma_3)$, $\delta_3(\sigma_2;\sigma_3) = \sigma_3^2\,\delta_1(\sigma_2;\sigma_3)$ are divergent because $\delta_1(\sigma_2;\sigma_3) = \frac{2i}{R^2}\left[I(1)+I(\sigma_2^2)+I(\sigma_3^2)\right]$ with $I(c^2) = \int \frac{dk}{4\pi}\frac{1}{\sqrt{k^2+c^2}}$. To tame these infinities the one-loop mass renormalization counter-terms

$$\mathcal{L}_{CT} = \frac{1}{R^2}\,\delta_1(\sigma_2;\sigma_3)\left[\phi_1^2(x^\mu) + \sigma_2^2\phi_2^2(x^\mu) + \sigma_3^2\phi_3^2(x^\mu)\right]$$

must be added to the bare Lagrangian. Searching only for semi-classical effects we do not need to care about other divergent graphs.

The classical minima of the action are the static and homogeneous configurations that annihilate the integrand in (1), i.e., the North and South Poles of S^3. There is the possibility of the existence of topological kinks and to search for them it is convenient to use polar hyper-spherical coordinates: $\phi_1 = R\sin\chi\sin\theta\cos\varphi$, $\phi_2 = R\sin\chi\sin\theta\sin\varphi$, $\phi_3 = R\sin\chi\cos\theta$, $\phi_4 = R\cos\chi$, $\chi \in [0;\pi)$, $\theta \in [0;\pi)$, $\varphi \in [0;2\pi)$. There are three types of these kinks: (1) in the meridians on the $\phi_3-\phi_4$ plane, $\theta = 0$ or π, the non-trivial field equation is: $\frac{\partial^2\chi}{\partial t^2} - \frac{\partial^2\chi}{\partial x^2} + \frac{\sigma_3^2}{2}\sin 2\chi = 0$ and the kink solutions, that we shall denote generically as K_1, can be written in the form $\chi_{K_1}(t;x) = 2\arctan e^{\pm\sigma_3\bar{x}}$ where $\bar{x} = \frac{x-x_0-vt}{\sqrt{1-v^2}}$; (2) analogously in the meridians on the $\phi_2-\phi_4$ plane, $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the kink solutions will be referred to as K_2 and are given by $\chi_{K_2}(t;x) = 2\arctan e^{\pm\sigma_2\bar{x}}$ and (3)

K_3 kinks, which live in the meridians on the $_1$ $_4$ plane, $= \frac{}{2}$, $' = 0$ or , are $_{K_3}(t;x) = 2\arctan e^{x}$. The topological S^3-kink classical energies are: $E(K_1) = 2 R^2 {}_3 < E(K_2) = 2 R^2 {}_2 < E(K_3) = 2 R^2$.

Changing slightly the notation by denoting $= {}^1$, $= {}^2$, $' = {}^3$, small uctuations around the kink solution $(x) = {}_K(x) + (x) = ({}^1_K(x); {}^2_K(x); {}^3_K(x)) + ({}^1(x); {}^2(x); {}^3(x))$ modify the action as:

$$S[{}^1; {}^2; {}^3] = S[{}^1_K; {}^2_K; {}^3_K] + \frac{R^2}{2}\int dtdx\ (x)\ (K)\ (x) + O({}^3) \quad :$$

The second-order operator governing the kink small uctuations is the geodesic deviation operator plus the Hessian of the potential: $(K) = r_{{}^0_K} r_{{}^0_K} + R({}^0_K;){}^0_K + r\ \mathrm{grad}V$. Standard geometric calculations allow us to conclude that K_1 small uctuations are governed by the matrix of Schrodinger operators:

$$(K_1) = \frac{d^2}{dx^2} \quad \frac{2\ {}^2}{\cosh^2 {}_3 x}\ I + \mathrm{diag}\ {}^2_3; 1; {}^2_2 \qquad (2)$$

provided that a \parallel frame" to the kink orbit, i.e., uctuations of the form ${}^2(x) = \cosh {}_3x\ {}^2(x)$, ${}^3(x) = \cosh {}_3x\ {}^3(x)$, is chosen.

Therefore, the meson spectrum in the K_1 kink sector has three branches that share a perfectly transmitting Posch-Teller well but have di erent thresholds. The rst branch corresponds to uctuations tangent to the kink orbit. There is a bound state, ${}^1_0(x) = \frac{1}{\cosh {}_3x}$, of zero eigenvalue and one-particle scattering states ${}^1_k(x) = e^{ik\ {}_3x}(\tanh {}_3x \quad ik)$ with frequencies $!^2(k) = {}^2_3(k^2+1)$. In the orthogonal directions the eigenfunctions are the same but the bound state energies and thresholds of the continuous spectra are shifted respectively to: $1 \quad {}^2_3$, ${}^2_2 \quad {}^2_3$, 1 and ${}^2_2$.

3. Spectral zeta function and kink mass quantum correction

We choose a normalization interval of length $l = L$ and impose periodic boundary conditions on the uctuations: $(\frac{l}{2}) = (\frac{l}{2})$. At the end of the computations we will send the length l of the normalization interval to in nity. (K) acts on the Hilbert space $L^2 = L^2_1(S^1) \bigoplus L^2_2(S^1) \bigoplus L^2_3(S^1)$. The heat trace (is a ctitious inverse temperature or Euclidean time) is:

$$\mathrm{Tr}_{L^2} e^{\ (K_1)} = \frac{lA}{\sqrt{4}} + \tanh\frac{{}_3 l}{2}\quad 1 + e^{(1\ \ {}^2_3)} + e^{({}^2_2\ \ {}^2_3)}\quad \mathrm{Erf}[{}_3\sqrt{\ }]$$

where $A = e^{\ {}^2_3} + e + e^{\ {}^2_2}$. It is interesting also to use the short time asymptotics of the heat trace. Due to the structure of the second-order uctuations operator (2), a power expansion of the heat trace is sensible[7]:

$$\mathrm{Tr}_{L^2} e^{\ (K_1)} = \mathrm{Tr}_{L^2} e^{\ _0(K_1)} \sum_{n=0}^{\infty} c_n(K_1)\ ^n = \frac{A}{\sqrt{4}} \sum_{n=0}^{\infty} c_n(K_1)\ ^n ;$$

where the coefficients are: $c_0(K_1) = 1$, $c_n(K_1) = \frac{2^{n+1}\ ^{2n-1}_{\ 3}}{(2n-1)!!}$.
The Casimir energy $E^C = E - E_0 = \frac{}{2}\left[\mathrm{Tr}_{L^2}\ ^{\frac{1}{2}}(K_1) - \mathrm{Tr}_{L^2}\ ^{\frac{1}{2}}_0(K_1)\right]$ is ultra-violet divergent. We shall regularize these divergences by using the zeta function method. The zeta functions are the Mellin transform of the heat traces, $(s) = \frac{1}{(s)}\int_0^1 d\ \ ^{s-1}\mathrm{Tr}_{L^2} e$ and thus we regularize the divergence by assigning to it the value of the spectral zeta function at a regular point of the s-complex plane: $E^C(s) = \frac{}{2}\left(\frac{^2}{^2}\right)^s\left[\ _{(K_1)}(s) - \ _{0(K_1)}(s)\right]$. The behaviour of the kink Casimir energy near the physical pole $s = -\frac{1}{2} + "$ is:

$$E^C = -\frac{3}{2}\left[\frac{3}{"} + 3\ln\frac{^2}{^2} + \ln\frac{2^6}{\ ^2_3\ ^2_{13}\ ^2_{23}} - 4 + F\left[-\frac{^2_3}{^2_{13}}\right] + F\left[-\frac{^2_3}{^2_{23}}\right]\right] \quad (3)$$

where we denote $F[x] = {}_2F_1^{(0;1;0;0)}[\frac{1}{2};0;\frac{3}{2};x]$, $^2_{13} = 1 - ^2_3$ and $^2_{23} = ^2_2 - ^2_3$.

The kink energy due to the mass renormalization counter-terms that must be added, $E^{MR} = \frac{^2_3}{R^2}[I(1) + I(^2_2) + I(^2_3)]\int dx\ ^{K_1}_3(x)\ ^{K_1}_3(x) = 2\ _3[I(1) + I(^2_2) + I(^2_3)]$ is also ultra-violet divergent. The loop integrals become in the finite length normalization interval divergent series susceptible of being regularized as spectral zeta functions:

$$I(c^2) = \frac{1}{2l}\sum_{n=-1}^{\infty}\frac{1}{(\ ^2_3 n^2 + c^2)^{\frac{1}{2}}} = \frac{}{l}\lim_{s\to-\frac{1}{2}}\left(\frac{^2}{^2}\right)^{s+1}\frac{(s+1)}{(s)}\ _{-\frac{d^2}{dx^2}+c^2}(s)$$

The regularized mass renormalization kink energy

$$E^{MR}(s) = \frac{2\ _3\ ^2}{\sqrt{4}}\left(\frac{^2}{^2}\right)^{s+1}\frac{(s+\frac{1}{2})}{(s)}\left[1 + \frac{1}{\ ^{2s+1}_2} + \frac{1}{\ ^{2s+1}_3}\right]$$

behaves near the physical pole as:

$$E^{MR}(-\frac{1}{2} + ") = \frac{3}{2}\left[\frac{3}{"} + 3\ln\frac{^2}{^2} + 3(\ln 4 - 2) - \ln\ ^2_2\ ^2_3\right] \quad (4)$$

From the short-time asymptotics of the heat trace we obtain an approximated formula for the kink Casimir energy by means of the partial Mellin transform on the $[0;b]$ integration interval of the truncated to N_0 terms heat trace expansion:

$$E^C(b;N_0) = -\frac{}{2\sqrt{b}} - \frac{}{8}\sum_{n=1}^{N_0} c_n(K)\left[\frac{^2_3}{\ ^{2n}_3}[\ ^2_3 b] + [b] + \frac{^2_2}{\ ^{2n}_2}[\ ^2_2 b]\right]$$

where [c] = [n 1;c] and [z;c] is the incomplete Euler gamma function. The contribution $E^{C}_{(1)}$ of the term with $c_1(K_1) = 4\ _3$ to this approximation to the kink Casimir energy is divergent because $z = 0$ is a pole of [z;c]. Fortunately, the divergent mass renormalization kink energy E^{MR} exactly cancels $E^{C}_{(1)}$.

Finally, the K_1 semiclassical mass, $E(K_1) = 2\ \ ^2_3 R^2 + \ E + O(\frac{1}{R^2})$, is obtained by adding (3) and (4):

$$E = \frac{3}{2}\ \ 2 + F[\frac{^2_3}{^2_{13}}] + F[\frac{^2_3}{^2_{23}}] + \ln\frac{^2_2}{^2_{13}\ ^2_{23}} \ : \qquad (5)$$

Because the wells in the second-order uctuation operator are transparent the Cahill-Comtet-Glauber formula[11], $E(K_1) = \frac{3}{}[\sin\ _1 + \frac{1}{3}\sin\ _2 + \frac{2}{3}\sin\ _3\ \ \ _1\cos\ _1\ \ \frac{1}{3}\ _2\cos\ _2\ \ \frac{2}{3}\ _3\cos\ _3]$, with $_1 = \arccos(0) = \frac{}{2}$, $_2 = \arccos\ _{13}$, $_3 = \arccos\frac{_{23}}{_2}$, giving the one-loop mass shift in terms only of the bound state eigenvalues and the thresholds of the continuous spectra, can be applied[11]. Despite appearances, the result

$$E(K_1) = \frac{3}{}\ \ 3\ \ \frac{_{13}}{3}\arccos(\ _{13})\ \ \frac{_{23}}{3}\arccos(\frac{_{23}}{_2}) \qquad (6)$$

is identical to (5) as one can check by plotting of both expressions. A third (approximate) formula, useful in the cases when the spectral information on the kink uctuations is unknown, is derived from the asymptotic expansion:

$$E(b;N_0) = \frac{}{2\sqrt{b}}\ \frac{}{8}\sum_{n=2}^{N_0} c_n(K)\ \frac{^2_3}{^{2n}_3}\ [\ ^2_3 b] + \ [b] + \frac{^2_2}{^{2n}_2}\ [\ ^2_2 b] \qquad (7)$$

References

1. A. Rebhan, P. van Nieuwenhuizen, R. Wimmer, [arXiv:hep-th/0401127]
2. K. Milton, Journal of Physics A 37(2004)R209, [arXiv: hep-th/0406024]
3. C. Mayrhofer, A. Rebhan, P. van Nieuwenhuizen, R. Wimmer, JHEP (2007) 0709:069, [arXiv:0706.4476]
4. A. Rebhan, A. Schmitt, P. van Nieuwenhuizen, Phys. Rev. D 80:045012,2009, [arXiv: 0903.5242]
5. A. Alonso, M. A. G. Leon, J. M. Guilarte, Phys. Rev. Lett. 101131602,2008.
6. A. Alonso, M. A. G. Leon, J. M. Guilarte, Phys. Rev. D 79:125003,2009.
7. A. Alonso, M. A. G. Leon, J. M. Guilarte, M. J. Senosiain, Jour. Phys. A: Math. Theor. 42 (2009)385403,
8. A. Alonso, W. G. Fuertes, J. M. Guilarte, M. de la Torre Mayado, M. J. Senosiain, PoS(ISFTG)013,2009, [arXiv:0909.2107]
9. M. Gell-Mann, M. Levy, Il Nuovo Cimento XVI (1960) 705
10. E. Brezin, J. Zinn-Justin, J. C. Le Guillou, Phys. Rev. D 14 (1976) 2615
11. K. Cahill, A. Comtet, R. Glauber, Phys. Lett. 64B (1976) 283–285