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Working paper No.: 1402
September 2014

http://borda.usal.es

MAJORIZATION, MATRIX TRANSFORMATIONS AND ELECTORAL SYSTEMS

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ABSTRACT. Let $\mathcal{M}$ be the space of all the $\tau \times n$ matrices with pairwise distinct entries and with both rows and columns sorted in descending order. If $X = (x_{ij}) \in \mathcal{M}$ and X_n is the set of the n greatest entries of X, we denote by ψ_j the number of elements of X_n in the column j of X and by ψ^i the number of elements of Ψ in the row i of X. If a new matrix $X' = (x'_{ij}) \in \mathcal{M}$ is obtained from X through a so defined *elementary transformation* (i.e. $X' \leq X$, the first ν columns are left unchanged, and in the last $(n - \nu)$ columns it holds that $x_{ij} \leq x_{uv}$ if and only if $x'_{ij} \leq x'_{uv}$), then there is a relation of majorization between $(\psi^1, \psi^2, ..., \psi^\tau)$ and the corresponding $(\psi'^1, \psi'^2, ..., \psi'^\tau)$ of X', and between $(\psi'_1, \psi'_2, ..., \psi'_n)$ of X' and $(\psi_1, \psi_2, ..., \psi_n)$. This result can be applied to the comparison of closed list electoral systems, providing a unified proof of the standard hierarchy of these electoral systems according to whether they are more or less favourable to larger parties.

1. INTRODUCTION

Given two vectors $a = (a_1, a_2, ..., a_\tau)$ and $a' = (a'_1, a'_2, ..., a'_\tau)$ of $\mathbb{R}^\tau$ such that $a_1 \geq a_2 \geq ... \geq a_\tau$ and $a'_1 \geq a'_2 \geq ... \geq a'_\tau$, and with identical component sum $a_1 + a_2 + ... + a_\tau = a'_1 + a'_2 + ... + a'_\tau = M$, we say that a *majorizes* a' when

$$
\begin{aligned}
a_1 &\geq a'_1 \\
a_1 + a_2 &\geq a'_1 + a'_2 \\
&... \\
a_1 + a_2 + ... + a_{\tau-1} &\geq a'_1 + a'_2 + ... + a'_{\tau-1}
\end{aligned}
$$

holds, and denote this ordering by $a \succ a'$. This notation and terminology was introduced in the classical [4]. The relation is transitive. Majorization has useful applications (see [5]).

Let $X = (x_{ij}) \in \mathcal{M}$, where $\mathcal{M}$ is the space of all the $\tau \times n$ matrices with pairwise distinct entries and with both rows and columns sorted in descending order (i.e. $x_{i1} > x_{i2} > ... > x_{in}$ for $i = 1, ..., \tau$ and $x_{1j} > x_{2j} > ... > x_{\tau j}$ for $j = 1, ..., n$). If X_n is the set of the n greatest entries of X, we denote by ψ_j the number of elements of X_n in the column j of X and by ψ^i the number of elements of Ψ in the row i of X. We consider the distribution of the n greatest entries of X among the columns, $(\psi_1, \psi_2, ..., \psi_n)$, and among the rows, $(\psi^1, \psi^2, ..., \psi^\tau)$. If a new matrix $X' = (x'_{ij}) \in \mathcal{M}$ is obtained from X through a so defined *elementary transformation*, we prove that there is a relation of majorization between $(\psi^1, \psi^2, ..., \psi^\tau)$ and

the corresponding $(\psi'^1, \psi'^2, ..., \psi'^\tau)$ of X', and between $(\psi'_1, \psi'_2, ..., \psi'_n)$ of X' and $(\psi_1, \psi_2, ..., \psi_n)$. We say that the matrix X' is an elementary transformation of X if $X' \leq X$, the first ν columns are left unchanged, and in the last $(n - \nu)$ columns it holds that $x_{ij} \leq x_{uv}$ if and only if $x'_{ij} \leq x'_{uv}$.

In both highest average and largest remainders closed list electoral systems, the n greatest elements of a certain matrix are calculated. The theorem just mentioned can be applied to the comparison of closed list electoral systems (or apportionment methods), providing a unified proof of the standard hierarchy of these electoral systems according to whether they are more or less favourable to larger parties.

2. Closed list electoral systems

In closed list electoral systems voters may only vote for closed lists proposed by political parties (first-past-the-post is a particular case, with $n = 1$). We consider a constituency with n seats, and let τ be the number of competing closed lists ($\tau \geq n \geq 1$). After counting, the final outcome of the election in the constituency is the set E of elected candidates. A particular candidate is identified by the party list p where they figure and their rank r in that list, where $p = 1, .., \tau$ and $r = 1, ..., n$.

We introduce what we shall call the matrix of a closed list electoral system. As a motivation, consider the usual table to allocate the seats by D'Hondt system, in a numerical example (with $n = 10$) taken from [3]:

	Votes (V)	$(1/2) \cdot V$	$(1/3) \cdot V$	$(1/4) \cdot V$	$(1/5) \cdot V$	Seats
Socialist Party	34,000	17,000	11,333	8,500	6,800	4
Center-right Party	25,000	12,500	8,333			3
Liberal Party	15,000	7,500				1
Green Party	12,000	6,000				1
Radical Right Party	10,000	5,000				1
Regionalist Party	4,000					
TOTAL	100,000					10

This table, taking aside the last row and the last column, provides a 6×5 matrix, making up the first 5 columns of an instance of the *matrix of the electoral system*, as we shall call it. The n largest entries of the matrix (here underlined) have been worked out also; only the entries of the matrix that may be relevant for this purpose are calculated.

In general, we can define a closed list electoral system through what we call its *matrix*. Given a distribution of votes $y = (y_1, y_2, ..., y_\tau)$, where y_i is the number of votes obtained by party i, we shall introduce the *matrix of the electoral system (for y)*, $Y = (y_{ij})$, $i = 1, ..., \tau$, $j = 1, ..., n$. The first column is always vector y, and so $y_{i1} := y_i$ for every party $i = 1, ..., \tau$. The n highest entries of the matrix are found, and the corresponding candidates are those selected through the electoral system to represent the constituency. Although the matrix of the electoral system is not unique, we shall speak of *the* matrix of the electoral system (i.e. that one used to define the electoral system).

Let us introduce matrix Y in the two most important cases: the highest average systems and the largest remainders systems.

- In the *highest average* systems the columns of Y are proportional to the first column (the distribution of votes y), and the coefficients of proportionality are provided by a sequence of divisors. Alternatively, we consider the following parameters c_r, $r = 1, ..., n$. For example, in the D'Hondt system (sequence of "divisors" 1, 2, 3, 4, 5, ...)

$$c_r = \frac{r}{r+1} \ ,$$

in the Sainte-Laguë system (sequence of "divisors" 1, 3, 5, 7, 9, ...)

$$c_r = \frac{2r-1}{2r+1} \ ,$$

in the Imperiali system (sequence of "divisors" 1, 1.5, 2, 2.5, 3, ...)

$$c_r = \frac{r+1}{r+2} \ ,$$

and in the so-called Danish system (sequence of "divisors" 1, 4, 7, 10, 13, ...)

$$c_r = \frac{3r-2}{3r+1}$$

Let us take a particular highest average system. Given a distribution of votes $y = (y_1, y_2, ..., y_\tau)$, where y_i is the number of votes obtained by party i , we introduce the *matrix of the electoral system (for y)*, $Y = (y_{ij})$, $i = 1, ..., \tau$, $j = 1, ..., n$. The first column is just vector y, and so $y_{i1} := y_i$ for every party $i = 1, ..., \tau$. The second column is $y_{i2} := c_1 \cdot y_i$, the third one $y_{i3} := c_2 \cdot c_1 \cdot y_i$, the forth one $y_{i4} := c_3 \cdot c_2 \cdot c_1 \cdot y_i$. In general,

(1) $$y_{ij} := c_{j-1} \cdot c_{j-2} \cdot ... \cdot c_2 \cdot c_1 \cdot y_i \ , \text{ for } i = 1, ..., \tau, \ j = 2, ..., n$$

- The *largest remainders* systems are based on a quota q defined from the total number $|B|$ of valid ballots cast; so $q = |B|/(n+1)$ is the Droop quota, $q = |B|/n$ is the Hare quota and $q = |B|/(n+2)$ is the Imperiali quota. Roughly, each party is then awarded as many seats as it has full quotas, and the remaining seats (if any) go to the parties with the highest remainders (after the assigned quotas have been taken off). Alternatively, we define in the following way the matrix of the electoral system:

(2) $$y_{ij} := y_i - (j-1)q \ , \text{ for } i = 1, ..., \tau, \ j = 2, ..., n$$

Note that every column of Y is the result of adding a constant to all the elements of the first column.

Obviously to every row corresponds a party list, and to every column a rank in a party list; thus the entry y_{ij} is assigned to the candidate in rank j of the list of party i. In every case the n highest entries of the matrix are found, and the corresponding candidates are selected to represent the constituency (usually not all the entries of the matrix have to be calculated to determine the n highest among them). Rules to break the possible ties are to be given with every electoral system. Note that the two main sorts of closed list electoral systems, highest average and largest remainders systems, represent the two simplest and more "natural" ways of deriving the matrix Y from vector y, the former multiplicative and the latter additive.

As a further example, consider the following table (not the usual one appearing in books) to allocate the seats by the Hare largest remainders system, with the same data as above by the D'Hondt system:

	Votes (V)	V−q	V−2q	V−3q	Seats
Socialist Party	34,000	24,000	14,000	4,000	3
Center-right Party	25,000	15,000	5,000		3
Liberal Party	15,000	5,000			2
Green Party	12,000	2,000			1
Radical Right Party	10,000	0			1
Regionalist Party	4,000				
TOTAL	100,000				10

This table, taking aside the last row and the last column, provides the first 4 columns of an instance of the matrix of the electoral system (only the relevant entries are calculated).

3. MAJORIZATION AND MATRIX TRANSFORMATIONS

If closed list electoral systems are considered, we can consider whether an electoral system is more or less favourable to large or small parties. More precisely, if n and τ are given, two electoral systems may be compared according to how favourable they are to larger or smaller parties.

In this section we shall assume that $n \geq 2$ and consider closed list electoral systems satisfying the following: given two parties i and i', if the number of votes of i is greater than the number of votes of i', then the number of seats assigned to i will be greater than or equal to the number of seats assigned to i'. If y_i is the number of votes obtained by party i and $y := (y_1, y_2, ..., y_\tau)$, we suppose that $y_1 \geq y_2 \geq ... \geq y_\tau$ in order to avoid a cumbersome notation (without introducing new symbols for this re-ordered vector). Moreover, ties appear rarely in practice and the rules to break them are varied (by lot, criteria related with the age of the candidates, etc). We shall not discuss these rules in our comparison of electoral systems, and we take $y_1 > y_2 > ... > y_\tau$.

Firstly we need a precise definition of what it is to be understood by an electoral system being more or less favourable to larger parties. We follow [6].

Definition 3.1. *Let A and A' be two closed list electoral systems. We consider the distributions of votes $y = (y_1, y_2, ..., y_\tau)$ with $y_1 > y_2 > ... > y_\tau$ and not leading to ties in the matrices of A and A'. We say that the electoral system A is more favourable to larger parties than the electoral system A' if, for any such distribution of votes $y = (y_1, y_2, ..., y_\tau)$,*

$$a \succ a'$$

where a_i is the number of seats obtained by party i with the electoral system A and a_i' the number of seats obtained by party i with the electoral system A'.

Note that always

$$(3) \qquad\qquad a_1 + a_2 + ... + a_\tau = a_1' + a_2' + ... + a_\tau' = n$$

The following relation is discussed in [1]. Given two vectors $a = (a_1, a_2, ..., a_\tau)$ and $a' = (a_1', a_2', ..., a_\tau')$ of $\mathbb{R}^\tau$ such that $a_1 \geq a_2 \geq ... \geq a_\tau$ and $a_1' \geq a_2' \geq ... \geq a_\tau'$, and with identical component sum $a_1 + a_2 + ... + a_\tau = a_1' + a_2' + ... + a_\tau' = M$,

we say that a' *gives up to* a if, for all $i < j$, $a'_i \leq a_i$ or $a'_j \geq a_j$. This relation implies majorization and it is not transitive (see [6]). The definition of a closed list electoral system A' *giving up to* a closed list electoral system A is the obvious one; now in every comparison (from A' to A) of a larger party i with a smaller party j (i.e. $i < j$) it cannot happen that the larger party i loses seats and at the same time the smaller party j gains seats.

Consider the matrix Y of the D'Hondt electoral system ($c_r = \frac{r}{r+1}$) and the matrix Y' of the Sainte-Laguë system ($c_r = \frac{2r-1}{2r+1}$). We can obtain Y' from Y stepwise, by a sequence of $(n-1)$ transformations. The first columns of both matrices are equal (i.e. the vector y of the distribution of votes). Consider the factor $\lambda_j := (\frac{2j-1}{2j+1})/(\frac{j}{j+1})$, $j = 1, ..., (n-1)$. In a first step, we multiply the columns 2 through n of Y by λ_1 (and leave unchanged the first column). In a second step, we multiply the columns 3 through n of the matrix resulting of step 1 by λ_2 (and leave unchanged the first two columns)... In step j, we multiply the columns (j+1) through n of the matrix resulting of step (j-1) by λ_{j-1} (and leave unchanged the first j columns). After (n-1) steps, we obtain matrix Y' (see (1)). We have applied an *elementary transformation* in every step, as defined below.

Consider now two largest remainders systems, determined by quotas q and q', with $q < q'$, and let Y and Y' be their matrices. We are to obtain Y' from Y stepwise, by a sequence of $(n-1)$ transformations. In a first step, we add the constant $(q - q')$ to the columns 2 through n of Y (and leave unchanged the first column). In a second step, we add again the constant $(q - q')$ to the columns 3 through n of the matrix resulting of step 1 (and leave unchanged the first two columns)... In step j, we add the constant $(q - q')$ to the columns (j+1) through n of the matrix resulting of step (j-1) (and leave unchanged the first j columns). After (n-1) steps, we obtain matrix Y' (see (2)). We have again applied an elementary transformation in every step, as defined in the following paragraph. Recall that $\mathcal{M}$ is the space of all the $\tau \times n$ matrices with pairwise distinct entries and with rows and columns sorted in descending order.

Definition 3.2. *Given $X = (x_{ij}) \in \mathcal{M}$, we say that $X' = (x'_{ij}) \in \mathcal{M}$ is an elementary transformation of X if the following two conditions hold for some integer ν, $1 \leq \nu \leq n$:*

(i) The first ν columns are left unchanged, i.e. $x'_{ij} = x_{ij}$ for $i = 1, ..., \tau$ and $j = 1, ..., \nu$.

(ii) In the last $(n - \nu)$ columns it holds that $x_{ij} \geq x'_{ij}$ for $i = 1, ..., \tau$ and $j = \nu + 1, ..., n$, and also that $x_{ij} \leq x_{uv}$ if and only if $x'_{ij} \leq x'_{uv}$, for $i, u = 1, ..., \tau$ and $j, v = \nu + 1, ..., n$.

We have the following theorem.

Theorem 3.1. *Let $X = (x_{ij}) \in \mathcal{M}$ and let X_n be the set of the n greatest entries of X. We denote by ψ_j the number of elements of X_n in the column j of X and by ψ^i the number of elements of Ψ in the row i of X. Let $X' = (x'_{ij}) \in \mathcal{M}$ be an elementary transformation of X. The symbols X'_n, ψ'_j and ψ'^i have the obvious meanings. Then we have:*

(i) $(\psi'_1, \psi'_2, ..., \psi'_n) \succ (\psi_1, \psi_2, ..., \psi_n)$

(ii) $(\psi^1, \psi^2, ..., \psi^\tau) \succ (\psi'^1, \psi'^2, ..., \psi'^\tau)$

Proof. (i) Obviously $\psi_1 + \psi_2 + ... + \psi_n = \psi'_1 + \psi'_2 + ... + \psi'_n = n$. We have to prove:

$$\begin{aligned} \psi_1 &\leq \psi'_1 \\ \psi_1 + \psi_2 &\leq \psi'_1 + \psi'_2 \\ &... \\ \psi_1 + \psi_2 + ... + \psi_{n-1} &\leq \psi'_1 + \psi'_2 + ... + \psi'_{n-1} \end{aligned}$$

Let $\Psi \subseteq \{(i,j) : i = 1, ..., \tau , j = 1, ..., n\}$ be the set of the coordinates of the n greatest entries of X and let $\Psi' \subseteq \{(i,j) : i = 1, ..., \tau , j = 1, ..., n\}$ be the set of the coordinates of the n greatest entries of X'. On the one hand, $\{(i,j) \in \Psi : i = 1, ..., \tau, j = 1, ..., \nu\} \subseteq \{(i,j) \in \Psi' : i = 1, ..., \tau, j = 1, ..., \nu\}$, where ν is as in Definition 3.2. Thus $\psi_j \leq \psi'_j$ for $j = 1, ..., \nu$. Therefore the first ν inequalities hold. On the other hand, $\{(i,j) \in \Psi' : j = \nu + 1, ..., n\} \subseteq \{(i,j) \in \Psi : j = \nu + 1, ..., n\}$, and $\psi_j \geq \psi'_j$ for $j = \nu + 1, ..., n$. Also

$$(\psi_1 + ... + \psi_\nu) + (\psi_{\nu+1} + ... + \psi_n) = n = (\psi'_1 + ... + \psi'_\nu) + (\psi'_{\nu+1} + ... + \psi'_n)$$

$$(\psi_{\nu+1} + ... + \psi_n) - (\psi'_{\nu+1} + ... + \psi'_n) = (\psi'_1 + ... + \psi'_\nu) - (\psi_1 + ... + \psi_\nu)$$

$$(\psi_{\nu+1} - \psi'_{\nu+1}) + ... + (\psi_n - \psi'_n) = (\psi'_1 + ... + \psi'_\nu) - (\psi_1 + ... + \psi_\nu)$$

We consider now the inequality μ, with $\mu > \nu$. We have

$$(\psi_{\nu+1} - \psi'_{\nu+1}) + ... + (\psi_\mu - \psi'_\mu) \leq (\psi'_1 + ... + \psi'_\nu) - (\psi_1 + ... + \psi_\nu)$$

and it follows that

$$\psi_1 + ... + \psi_\nu + ... + \psi_\mu \leq \psi'_1 + ... + \psi'_\nu + ... + \psi'_\mu$$

(ii) Clearly $\psi^1 + \psi^2 + ... + \psi^\tau = \psi'^1 + \psi'^2 + ... + \psi'^\tau = n$. We have to prove:

$$\begin{aligned} \psi^1 &\geq \psi'^1 \\ \psi^1 + \psi^2 &\geq \psi'^1 + \psi'^2 \\ &... \\ \psi^1 + \psi^2 + ... + \psi^{\tau-1} &\geq \psi'^1 + \psi'^2 + ... + \psi'^{\tau-1} \end{aligned}$$

Note that $\psi^1 = \max\{j : (1,j) \in \Psi , j = 1, ..., n\}$ and $\psi'^1 = \max\{j : (1,j) \in \Psi' , j = 1, ..., n\}$. Also $\psi_\beta = 0$ for $\beta > \alpha := \psi^1$, and thus, from (i), we have

$$n = \psi_1 + \psi_2 + ... + \psi_\alpha \leq \psi'_1 + \psi'_2 + ... + \psi'_\alpha$$

Therefore $\psi'_1 + \psi'_2 + ... + \psi'_\alpha = n$, and $\psi'_\beta = 0$ for $\beta > \alpha$. We conclude that $\psi'^1 \leq \alpha = \psi^1$. In order to prove the inequality γ, with $\gamma > 1$, let us consider the matrices $\overline{X}$ and $\overline{X}'$ obtained eliminating the first $(\gamma - 1)$ rows of X and X', respectively. The notations $\overline{\Psi}, \overline{\psi}_j$ and $\overline{\Psi}', \overline{\psi}'_j$ have the obvious meanings. Now $\psi^\gamma = \max\{j : (1,j) \in \overline{\Psi} , j = 1, ..., n\}$ and $\psi'^\gamma = \max\{j : (1,j) \in \overline{\Psi}' , j = 1, ..., n\}$. Besides, $\overline{\psi}_\beta = 0$ for $\beta > \alpha := \psi^\gamma$, and $\overline{\psi}_1 + \overline{\psi}_2 + ... + \overline{\psi}_\alpha = \overline{\psi}_1 + \overline{\psi}_2 + ... + \overline{\psi}_n = n - (\psi^1 + \psi^2 + ... + \psi^{\gamma-1})$. Also, applying (i) to $\overline{X}$ and $\overline{X}'$, we have

$$\overline{\psi}_1 + \overline{\psi}_2 + ... + \overline{\psi}_\alpha \leq \overline{\psi}'_1 + \overline{\psi}'_2 + ... + \overline{\psi}'_\alpha$$

Therefore $n - (\psi^1 + \psi^2 + ... + \psi^{\gamma-1}) \leq \overline{\psi}'_1 + \overline{\psi}'_2 + ... + \overline{\psi}'_\alpha$. Since $\overline{\psi}'_1 + \overline{\psi}'_2 + ... + \overline{\psi}'_n = n - (\psi'^1 + \psi'^2 + ... + \psi'^{\gamma-1})$, it follows that

$$\begin{aligned} \overline{\psi}'_{\alpha+1} + ... + \overline{\psi}'_n &= n - (\psi'^1 + \psi'^2 + ... + \psi'^{\gamma-1}) - (\overline{\psi}'_1 + \overline{\psi}'_2 + ... + \overline{\psi}'_\alpha) \leq \\ &\leq (\psi^1 + \psi^2 + ... + \psi^{\gamma-1}) - (\psi'^1 + \psi'^2 + ... + \psi'^{\gamma-1}) \end{aligned}$$

The elements of the first row of $\overline{\Psi}'$ are either in the columns $1, ..., \alpha$ or in the columns $(\alpha + 1), ..., n$, and consequently

$$\psi'^\gamma \leq \psi^\gamma + (\psi^1 + \psi^2 + ... + \psi^{\gamma-1}) - (\psi'^1 + \psi'^2 + ... + \psi'^{\gamma-1})$$

We conclude that $\psi'^1 + \psi'^2 + ... + \psi'^\gamma \leq \psi^1 + \psi^2 + ... + \psi^\gamma$. $\blacksquare$

The following Corollary is immediate from Theorem 3.1 (ii). In order to prove the statement (i) of the Corollary, consider the inequalities

$$\frac{r+1}{r+2} > \frac{r}{r+1} > \frac{r - \frac{1}{2}}{r + \frac{1}{2}} = \frac{2r-1}{2r+1} > \frac{r - \frac{2}{3}}{r + \frac{1}{3}} = \frac{3r-2}{3r+1}$$

which follow from the simple fact that the quotient of two consecutive positive numbers, $h/(h+1)$, increases (strictly) as h increases.

Corollary 3.2. *(i) Among the highest average closed list systems, from the most favourable to larger parties to the least favourable, the order is:*

Imperiali, D'Hondt, Sainte-Laguë, Danish.

(ii) Among the largest remainders closed list systems, the smaller the quota, the more favourable to larger parties.

Following approaches different from ours, statement (i) was proved in [1] and statement (ii) in [2], in both cases for the "give up to" relation mentioned above.

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