A unifying model to measure consensus solutions in a society

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Abstract

In this work we contribute to the formal and computational analysis of the measurement of consensus in a society. We propose a unifying model that generates a consistent decision in terms of the individual preferences and then measures the consensus that arises from it. We focus our inspection on two relevant and specific cases: the Borda and the Copeland rules under a Kemeny-type measure. A computational analysis of these two proposals serves us to compare their respective performances.

Key words: Consensus, measurement, Borda rule, Copeland rule, Kemeny's distance

1. Introduction

A classical group decision making problem is established in a context where a group of voters or experts have to make a decision on a set of alternatives or candidates. The experts' opinions about alternatives are usually characterized by their ideas, principles, knowledge, etc., and the Arrovian position assumes that each expert constructs a preference binary relation (usually a weak order) on the set of alternatives by using some unspecified internal process. This causes difficulties when it comes to making a collective decision or selecting one alternative (see [1]) and many voting procedures have been proposed to account for different sets of compatible aggregation properties. Therefore it is important to provide distinctive properties of focal voting rules.

In this paper we are interested in measures of the consensus in a society, in the sense of the seminal Bosch [2], from an Arrovian perspective. Our motivation is that a large proportion of the settings where measuring consensus is relevant relate to aggregating individual information. Let us introduce our problem with a common situation, namely, the case of a (finite) committee that intends to offer vacancies to a (finite) list of candidates. Each member is assumed to produce a complete preorder on the candidates, that is, ties are allowed but all pairs of alternatives are comparable. For various plausible reasons the committee wants to agree on a complete preorder of the candidates, for example because the candidates may reject the offer, or because the number of candidates to be appointed is externally and independently decided. It is intuitively clear that some orderings convey “higher consensus” than others, irrespective of the formal meaning that we attach to that term. Thus a consensus measure could be interpreted as a social welfare function, where group satisfaction relates to the coherence between individual preferences and social decision. From this perspective we obtain evidence that the outcome after aggregation must not be isolated from the degree of agreement among the individual preferences. [1]

We here propose a model that considers both aspects of the process, namely, the social preference on the alternatives and the consensus that arises from it. Generally speaking, the question we pose ourselves is: How should the design be for the committee to reach a consistent decision (in the form of a complete preorder on the candidates) with regard to favouring consensus? The literature abounds with references about the decision making process under different positions such as the Theory of Decision Making (see [3], [4], [5], [6] and [7], among others) and the Social Choice Theory (see [8], [9], [2] and [10], among many others). In this paper we focus on measuring the degree of agreement between the voters and the final decision reached via voting systems.

Regarding the assessment of cohesiveness we separate from the main trend in the literature, that consists of proposing and axiomatizing particular formulations for an absolute intrinsic measure of consensus or coherence (v., e.g., Bosch [2] or Alcalde and Vorsatz [9]). We here provide an alternative

[1] To name an extreme instance: Suppose there are only two candidates x and y, all members of a huge committee agree that x is better than y except for one that thinks the opposite, but the social decision is dictatorial and as a result y is appointed. Can we detach this outcome from the composition of the preference profile and just claim that the measurement of the consensus is almost the highest possible? Under a welfarist point of view, certainly not because social welfare is a function of personal utilities and almost all members are disgruntled.

and practical methodology for approaching the measurement of consensus, which we call *referenced consensus measures*. A study of its analytic properties is beyond the scope of this paper and we refer to [11] for a thorough exposition. We then focus on two relevant cases whose explicit constructions are detailed from an algorithmic viewpoint. We compare their relative performance for a realistic situation, where the number of candidates and voters is small. This permits to assess which voting procedure should be invoked before facing the choice, as a function of its particular parameters.

This paper is organized as follows. Section 2 is devoted to introduce basic notation and definitions, as well as our proposal of measurement of consensus, the referenced consensus measure. A first result is that such apparent technicality does not exclude any standard consensus measure in the sense of [2]. Then we introduce the particular subclass of normal referenced consensus measures as a suitable framework where a better normative behavior can be guaranteed. In Section 3 we present two explicit proposals of our model, give operational characterizations and provide simple algorithms for their implementation. A computational comparison of these proposals is shown in Section 4. Finally, Section 5 concludes and poses questions for further research.

2. Notation and Definitions

We fix $X = \{x_1, ..., x_k\}$, a finite set of k options, alternatives or candidates. Abusing notation, on occasions we refer to option x_s as option s for convenience. A population of agents or voters is a finite subset $\mathbf{N} = \{1, 2, ..., N\}$ of natural numbers. We also denote $\mathbf{K} = \{\{i, j\} \subseteq \mathbb{N} : i, j \in \{1, 2, ..., k\}, i < j\}$.

Let $W(X)$ be the set of weak orders or complete preorders on X, that is, the set of complete and transitive binary relations on X. If $R \in W(X)$ is a weak order on X that reflects the preferences of a voter, then by $x_k R\, x_j$ we mean "R-voter thinks that alternative x_k is at last as good as x_j". $L(X)$ denotes the set of linear orders on X, where ties are not permitted.

A *profile* $\mathcal{R} = (R_1, \ldots, R_N) \in W(X) \times \overset{N}{\ldots} \times W(X)$ is a vector of weak orders, where $R_i \in W(X)$ represents the preferences of the individual i on the k alternatives or candidates for each $i = 1, \ldots, N$. We say that the profile $\mathcal{R}$ is *constant* to R if $\mathcal{R} = (R, \overset{N}{\ldots\ldots}, R)$.

Any permutation σ of the voters $\{1, 2, ..., N\}$ determines a permutation of $\mathcal{R}$ by $\mathcal{R}^\sigma = (R_{\sigma(1)}, ..., R_{\sigma(N)})$. Similarly, any permutation π of the candidates $\{1, 2, ..., k\}$ determines a permutation of every complete preorder $R \in W(X)$ via $x_s \,{}^{\pi}R_i\, x_t \Leftrightarrow x_{\pi^{-1}(s)}\, R_i\, x_{\pi^{-1}(t)}$ for all $s, t \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, N\}$. Then with $\mathcal{R}$ and π we can associate ${}^{\pi}\mathcal{R} = ({}^{\pi}R_1, ..., {}^{\pi} R_N)$.

Finally, given any profile of weak orders $\mathcal{R} = (R_1, \ldots, R_N) \in W(X)^N$ and any weak order R' on X, we denote $\mathcal{R} \uplus R'$ the profile $(R_1, \ldots, R_N, R')$ of $N + 1$ weak orders. We denote by $\mathcal{P}(X)$ the set of all profiles, that is, $\mathcal{P}(X) = \bigcup_{N \geqslant 1} W(X)^N$.

2.1. Basic Definitions

We start by defining the basic concept of consensus measure (Bosch [2]).

Definition 1. A (conventional) *consensus measure* is a mapping:

$$\mathcal{M} : \mathcal{P} \to [0, 1]$$

that assigns a real number $\mathcal{M}(\mathcal{R})$ to each profile of complete preorder $\mathcal{R}$ with the following properties:

i) $\mathcal{M}(\mathcal{R}) = 1$ if and only if $\mathcal{R}$ is a constant profile.
ii) $\mathcal{M}(\mathcal{R}^{\sigma}) = \mathcal{M}(\mathcal{R})$ for each permutation σ of the voters.
iii) $\mathcal{M}({}^{\pi}\mathcal{R}) = \mathcal{M}(\mathcal{R})$ for each permutation π of the candidates.

Example 1. Among other instances, Bosch [2] characterizes the *trivial measure* $\mathcal{T}(\mathcal{R})$ or the *simple measure* $\mathcal{S}(\mathcal{R})$ defined as follows: for each profile of complete preorders $\mathcal{R}$, let $\mathcal{A}(\mathcal{R})$ denote the set of alternatives that are ranked at the same position by all voters according to $\mathcal{R}$, then

$$\mathcal{T}(\mathcal{R}) = \begin{cases} 1 & \text{if } \mathcal{R} \text{ is a constant profile} \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathcal{S}(\mathcal{R}) = \frac{|\mathcal{A}(\mathcal{R})|}{\text{n. of alternatives}}$$

Our proposal is based on the following alternative concept.

Definition 2. A *Consensus measure with reference to a consensus function* (henceforth, *referenced consensus measure*, also RCM, when the consensus function is common knowledge) is a pair $\mathbf{M} = (\mathcal{C}, \partial)$ where:

1) $\mathcal{C}$ is a consensus function (cf., McMorris and Powers, [12]), that is, a mapping

$$\mathcal{C} : \mathcal{P}(X) \to W(X),$$

that associates a complete preorder $\mathcal{C}(\mathcal{R})$ with each profile of complete preorders $\mathcal{R}$. We speak of the *consensus preorder* $\mathcal{C}(\mathcal{R})$ associated with $\mathcal{R}$, and assume that it verifies

1.a) Unanimity: $\mathcal{C}(\mathcal{R}) = R$ for each profile $\mathcal{R}$ that is constant to the complete preorder R.

1.b) Anonymity: $\mathcal{C}(\mathcal{R}^\sigma) = \mathcal{C}(\mathcal{R})$ for each profile of complete preorders and σ permutation of the voters.

1.c) Neutrality: $\mathcal{C}(^\pi\mathcal{R}) =^\pi \mathcal{C}(\mathcal{R})$ for each profile of complete preorders and π permutation of the candidates or alternatives.

2) ∂ is a *referenced measure function* (RMF), that is, a mapping

$$\partial : \mathcal{P}(X) \times W(X) \to [0, 1],$$

that assigns a real number, $\partial(\mathcal{R}, R) \in [0, 1]$, to each pair formed by a profile of complete preorder $\mathcal{R}$ and a complete preorder R, with the following properties:

2.a) $\partial(\mathcal{R}, R) = 1$ if and only if $\mathcal{R}$ is constant to R.

2.b) $\partial(\mathcal{R}^\sigma, R) = \partial(\mathcal{R}, R)$ for each possible permutation σ of the voters.

2.c) $\partial(^\pi\mathcal{R},^\pi R) = \partial(\mathcal{R}, R)$ for each possible permutation π of the candidates.

With regard to $\mathbf{M} = (\mathcal{C}, \partial)$ each profile of complete preorders $\mathcal{R}$ on X has a consensus $\nabla_{\mathbf{M}}(\mathcal{R}) = \partial(\mathcal{R}, \mathcal{C}(\mathcal{R}))$. Figure 1 sketches how RCMs measure consensus.

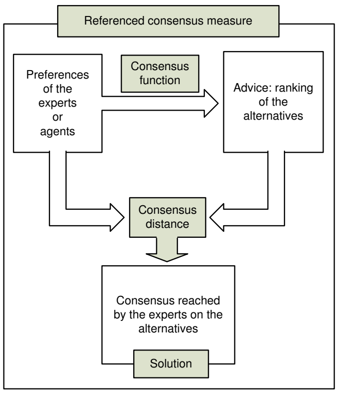

Figure 1: Consensus measure with reference to a consensus function

The following lemma reveals that despite its apparently restrictive formulation we do not lose generality by using the RCM concept to define consensus since it incorporates the usual model.

Lemma 1. Each conventional consensus measure can be interpreted as a referenced consensus measure, that is for every consensus measure $\mathcal{M}$, there are RCMs $(\mathcal{C}, \partial)$ such that both $\mathcal{M}$ and $(\mathcal{C}, \partial)$ associate the same number with every profile of weak orders, i.e., $\mathcal{M}(\mathcal{R}) = \nabla_{\mathbf{M}}(\mathcal{R})$ throughout.

Proof 1. Given a consensus measure $\mathcal{M}$ we define its associated RMF as

$$\partial_{\mathcal{M}}(\mathcal{R}, R) = \begin{cases} \mathcal{M}(\mathcal{R} \uplus R) & \text{if } \mathcal{R} \text{ is constant,} \\ \mathcal{M}(\mathcal{R}) & \text{otherwise.} \end{cases}$$

Now it is straightforward to check that $\partial_{\mathcal{M}}$ satisfies 2.a), 2.b) and 2.c), and that $\partial_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) = \mathcal{M}(\mathcal{R})$ holds for any consensus function $\mathcal{C}$.

Note that contrary to the spirit of our proposal, the role of $\mathcal{C}(\mathcal{R})$ is irrelevant in the previous construction. In order to enhance the influence of $\mathcal{C}(\mathcal{R})$ in the consensus measure and thus favour its analysis we restrict our attention to referenced measure functions that verify an additional property and introduce the corresponding new subclass of consensus measures.

Definition 3. A referenced consensus measure $\mathbf{M} = (\mathcal{C}, \partial)$ is called *normal referenced consensus measure* if its referenced measure function ∂ verifies

2.d) $\quad \partial(\mathcal{R}, R) > 0 \quad$ whenever $\quad R \in \mathcal{R}$.

If we adopt the position that overall welfare is an aggregate of individual satisfaction (under our approach, in terms of coherence) then property 2.d) can be regarded as natural. We emphasize that the subclass of normal referenced consensus measures does not coincide with the conventional ones. For example, the trivial measure is not a normal RCM. To see this, note that we can assume that there exists a non-constant profile $\mathcal{R}$ such that $\mathcal{C}(\mathcal{R}) \in \mathcal{R}$ (this forcefully holds e.g., when the number of voters is higher than the cardinality of $W(X)$). We conclude because $\mathcal{T}(\mathcal{R}) = 0$ and $\partial(\mathcal{R}, \mathcal{C}(\mathcal{R})) > 0$ for any normal RCM.

Concerning normative behavior, Alcantud et al. [11, Section 4] shows that normality permits to guarantee that consensus measures verify various interesting properties.

3. Some proposals for normal referenced consensus measures

In this section we detail the construction of two relevant normal RCM proposals. These models reach the consensus decision via the Borda and Copeland methods, thus 1.a) to 1.c) are ensured, and both measure consensus via the Kemeny's measure.

The Borda rule [13] is a classic procedure that is frequently used in group decision making problems where there are several alternatives or candidates (v., [14, 15, 16, 17]). In this work we consider the tie-breaking Borda rule given by Suzumura [18, pp. 107-108] (see also Bouyssou et al. [19]), which ranks the candidates according the following scores:

$$\beta(x_s) = \sum_{i=1}^{N} \left(\#\{\, x_t \in X \,:\, x_s \, R_i \, x_t \,\} - \#\{x_t \in X \,:\, x_t \, R_i \, x_s\}\right).$$

Henceforth, $\mathcal{C}_{\mathcal{B}}(\mathcal{R})$ denotes the complete preorder that the Borda rule produces from the profile $\mathcal{R}$.

Another classical rule that we take into account is the Copeland rule. We follow the Copeland method as described in Saari and Merlin [20] or Suzumura [18, p. 108]. It ranks the candidates according to their respective Copeland score defined as follows:

$$\begin{aligned}\kappa(x_s) = \#\{\, x_t \in X \,:\, x_s \text{ beats } x_t \text{ by s.s.m.}\} \\ - \#\{\, x_t \in X \,:\, x_t \text{ beats } x_s \text{ by s.s.m.}\},\end{aligned}$$

where s.s.m. stands for "strict simple majority". Henceforth $\mathcal{C}_{\mathcal{C}}(\mathcal{R})$ denotes the complete preorder that the Copeland rule produces from the profile $\mathcal{R}$.

Finally to measure the agreement between the individuals preferences and the final decision we use Kemeny's measure, that we now recall. Let $\mathcal{R} = (R_1, \ldots, R_N)$ be a profile of complete preorders, its Kemeny's measure, denoted $\mathcal{K}(\mathcal{R})$, is the probability that the binary ordering between a pair of randomly selected alternatives is the same for all voters. On this basis we define a distance between $\mathcal{R}$ and a complete preorder R as the average of the "individual" Kemeny's measures $\mathcal{K}(R_i \uplus R)$ given by: for each $i = 1, ..., N$,

$$\mathcal{K}(R_i \uplus R) = \frac{2}{k(k-1)} \sum_{(s,t)\in \mathbf{K}} \mathcal{K}^{s,t}(R_i \uplus R)$$

with

$$\mathcal{K}^{s,t}(R_i \uplus R) = \begin{cases} 1 & \text{if } R_i \text{ and } R \text{ coincide when comparing } x_s \text{ and } x_t, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently we henceforth refer to:

$$\partial_{\mathcal{K}}(\mathcal{R}, R) = \frac{\mathcal{K}(R_1 \uplus R) + \ldots + \mathcal{K}(R_N \uplus R)}{N}.$$

It is trivial to check that properties 2.a) to 2.d) hold true.

Remark 1. Let $\mathcal{M}$ be a consensus measure. Given a profile of complete preorders $\mathcal{R}$ and a complete preorder R we define the $\mu^p(\mathcal{M})$-*reference measure function* ($\mu^p(\mathcal{M})$-RMF) as the p-generalized mean of the $\mathbb{R}^N$ vector that has the i-th component equal to $\mathcal{M}(R_i \uplus R)$, that is

$$\partial^p_{\mathcal{M}}(\mathcal{R}, R) = \left(\sum_{i=1}^{N} \frac{1}{N} \mathcal{M}(R_i \uplus R)^p \right)^{1/p}.$$

Then Alcantud et al. [11] proves that most analytic properties of the models do not vary when RMFs from this family replace $\partial_{\mathcal{K}}$. Nevertheless a computational analysis calls for a focal specific instance.

3.1. Some Operational Characterizations

Let us fix a profile $\mathcal{R} = (R_1, ..., R_N)$ of complete preorders on X. Its Borda and Copeland scores can be reinterpreted in terms of simple matrix operations. For each complete preorder R_s its preference matrix $\mathbf{P}_s$ is defined as the $k \times k$ binary matrix whose (i, j) cell is 1 when $x_i \, R_s \, x_j$, and 0 otherwise. We say that $\mathcal{R}$ has an aggregate preference matrix $\mathbf{A}(\mathcal{R}) = \mathbf{P}_1 + ... + \mathbf{P}_N$.

If we define $\overline{\mathbf{A}}(\mathcal{R}) = \mathbf{A}(\mathcal{R}) - (\mathbf{A}(\mathcal{R}))^t$ then the sum of the cells in its i-th file is $\beta(x_i)$, the Borda score of x_i. If we further define $\widetilde{\mathbf{A}}(\mathcal{R}) = sig(\overline{\mathbf{A}}(\mathcal{R}))$ then the sum of the cells in its i-th file is $\kappa(x_i)$, the Copeland score of x_i (for a complete description, see Alcantud et al. [11]).

Calculating $\partial_{\mathcal{K}}(\mathcal{R}, R)$ for $\mathcal{R} = (R_1, ..., R_N)$ profile of complete preorders and R complete preorder is trivial from the numbers $\mathcal{K}^{s,t}(R_i \uplus R)$. These amounts can be computed with the assistance of basic matrix manipulations too. The following algorithm outputs these quantities:

Algorithm: Kemeny's measure $\mathcal{K}(R_1, R_2)$

```
Input: Two weak orders R_1 and R_2 on k options, and their respective
k × k preference matrices P_1 and P_2
d = 0

for i = 1 to k
    for j = i + 1 to k
      if ( P_1(i,j) + P_2(i,j) == 2 OR P_1(j,i) + P_2(j,i) == 2
             OR P_1(i,j) + P_2(i,j) + P_1(j,i) + P_2(j,i) == 0 )
      then
          d = d+1;
Output: K(R_1, R_2) = 2d / (k(k-1))
```

3.2. *The RCM-B proposal*

Our first proposal is the referenced consensus measure given by the tie-breaking Borda rule. It is defined as $\mathbf{M}_B = (\mathcal{C}_B, \partial_{\mathcal{K}})$, where $\mathcal{C}_B$ denotes the consensus function based on the Borda rule, and $\partial_{\mathcal{K}}$ is the Kemeny's measure. We now present an algorithm to compute it and a simple example:

Algorithm: RCM-B

```
Input: A profile R = (R_1, ..., R_N) of N weak orders on k options
(1) For each preorder compute its k × k preference matrix P_s(i,j)
(2) Calculate the aggregate preference matrix: A(i,j) = Σ_{s=1}^{N} P_s(i,j)
(3) Compute rule score A_B = A − A^t,      R_R(i) = Σ_{j=1}^{N} A_B(i,j)
(4) Compute rule preference matrix yielding R_R
(5) Compute 'individual' measures K(R_s, R_R) for each s = 1, ..., N
Output: Measure K(R, R_R) = Σ_{s=1}^{N} K(R_s, R_R) / N
```

Example 2. Suppose $X = \{x, y, z, w\}$ thus $k = 4$. Let $\mathcal{R} = (R_1, R_2, R_3)$ be the profile of three linear orders given by:

$$w\, R_1\, y\, R_1\, x\, R_1\, z, \;\; z\, R_2\, w\, R_2\, y\, R_2\, x, \;\; x\, R_3\, z\, R_3\, y\, R_3\, w.$$

Then simple computations yield $\nabla_{\mathbf{M}_B}(\mathcal{R}) = \partial_{\mathcal{K}}(\mathcal{R}, \mathcal{C}_{\mathcal{B}}(\mathcal{R})) = \frac{2+4+1}{3\times 6} = \frac{7}{18}$. This means that 7 out of 18 possible pairwise comparisons made by a member of the society $\{1, 2, 3\}$ coincide with the binary ordering given by the consensus function in the model.

3.3. *The RCM-C proposal*

Our second proposal is based on the Copeland method. We refer to this model as RCM-C, and it is given by $\mathbf{M}_C = (\mathcal{C}_C, \partial_{\mathcal{K}})$. The RCM-C algorithm is the same as the RCM-B above, except in that step (3) is replaced by:

(3) `Compute rule score` $A_C = \mathtt{sign}(A - A^t)$, $R_{\mathcal{R}}(i) = \sum_{j=1}^{N} A_C(i,j)$.

Example 3. Elaborating on the data of Example 2 one gets $\nabla_{\mathbf{M}_C}(\mathcal{R}) = \partial_{\mathcal{K}}(\mathcal{R}, \mathcal{C}_{\mathcal{C}}(\mathcal{R})) = \frac{7}{18}$. Thus for such situation, the referenced consensus measures obtained by the Borda and the Copeland rules are the same.

4. Computational comparison among different particular proposals

In this section we carry out a computational exploration of the behaviour of our proposals for $k = 3$ and $k = 4$ alternatives and small societies. We do not need to study the dichotomous case ($k = 2$), where the Borda and Copeland rules coincide, because it admits a purely analytic treatment (as shown in Alcantud et al. [11, Subsection 3.4]). Besides we restrict our study to the case where the voters linearly order the alternatives, both for expository and computational reasons. Of course, even though all voters have linear orders the models can produce ties in the consensus preorder.

Tables 1 and 2 show the respective performance when $k = 3$ and $k = 4$. The total number of cases that must be listed for k options and n experts is shown in the first row, and it is given by the number of n-combinations with repetition from a set of $k!$ elements (the number of linear orders), i.e. $\binom{k!+n-1}{n}$. We provide the number of cases where the Borda and Copeland methods convey the same consensus, resp., Borda gives a higher consensus than Copeland. The tables convey relative values (in larger types) and absolute values. We observe that for the cases that have been examined, RCM-C performs better than RCM-B when the number of experts is odd and the situation is the opposite for even-numbered groups (cf., Fig. 2). We do not have an analytic proof that this is a generalized property but we guess that it is partially due to the fact that even though all voters have linear orders the consensus preorder can produce ties, which *ceteris paribus* harms consensus. Although the combinatorial argument would be lengthy and tedious, we can observe that contrary to the case of the Borda rule, the proportion of consensus preorders under the Copeland rule that are (or are not) linear is very sensible to the parity of n for small societies (cf., Table 3 and Fig. 3).

5. Concluding remarks and future research

Alongside with normative approaches like the foundational Bosch [2] or Alcalde and Vorsatz [9], we have analysed the measurement of consensus from

Table 1: Comparing RCMs for 3 options

	Experts							
	$n=3$ 56	$n=4$ 126	$n=5$ 252	$n=6$ 462	$n=7$ 792	$n=8$ $1,287$	$n=9$ $2,002$	$n=10$ $3,003$
B=C	79% 44	90% 114	74% 186	82% 378	73% 582	78% 993	73% $1,468$	75% $2,253$
B>C	0% 0	10% 12	2% 6	13% 60	4% 30	14% 186	5% 90	15% 450
B<C	21% 12	0% 0	24% 60	5% 24	23% 180	8% 108	22% 444	10% 300

Table 2: Comparing RCMs for 4 options

	Experts							
	$n=3$ $2,600$	$n=4$ $17,550$	$n=5$ $98,280$	$n=6$ $475,020$	$n=7$ $2,035,800$	$n=8$ $7,888,725$	$n=9$ $28,048,800$	$n=10$ $92,561,040$
B=C	57% $1,472$	72% $12,726$	46% $45,312$	59% $281,304$	43% $882,456$	52% $4,134,345$	42% $11,942,136$	49% $45,046,752$
B>C	6% 168	18% $3,120$	14% $13,968$	24% $114,972$	18% $362,520$	27% $2,151,372$	20% $5,510,328$	29% $26,736,828$
B<C	37% 960	10% $1,704$	40% $39,000$	17% $78,744$	39% $790,824$	20% $1,603,008$	38% $10,596,336$	22% $20,777,460$

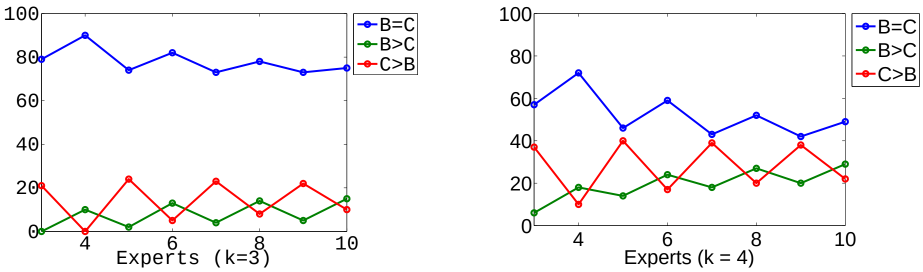

Figure 2: Comparison between RCM-B and RCM-C for $k = 3$ and $k = 4$.

a descriptive point of view. We have presented a general framework and given two particular specifications that link this proposal to voting theory.

Our formulation permits to compare a finite list of proposals on a common ground so that the society can decide which one conveys a higher consensus. Nonetheless its primary objective is to assess the coherence within a society with reference to a given voting rule. As is apparent, this may serve to discriminate among the voting rule that should be selected if we aim at

Table 3: Proportion of consensus preorders under Borda and Copeland rules that are linear.

		Experts							
		$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$k=3$	B	75%	76%	78%	79%	83%	82%	85%	85%
	C	96%	66%	95%	71%	94%	74%	94%	76%
$k=4$	B	56%	51%	59%	60%	64%	65%	68%	69%
	C	84%	46%	80%	51%	79%	55%	78%	57%

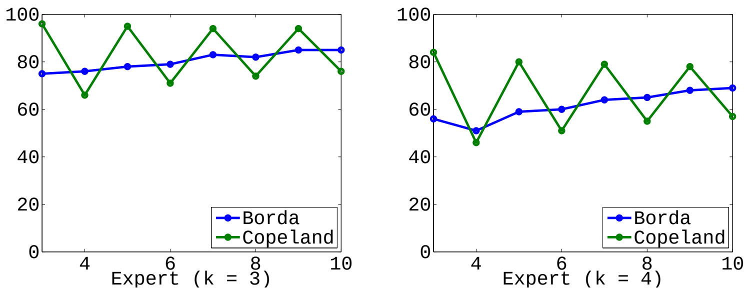

Figure 3: Proportion of consensus preorders under Borda and Copeland rules that are linear for $k = 3$ and $k = 4$.

producing indisputable results. For example, our tables show that for 3 candidates and 3 voters the Borda rule never produces higher consensus than the Copeland rule, and for 3 candidates and 4 voters the Copeland rule never produces higher consensus than the Borda rule.

Several questions remain open. Clearly, the performance of other measures with reference to alternative voting rules is a direct variation of our analysis. Also, different subclasses besides normal referenced consensus measures can yield a good normative performance. An ambitious project is the identification of the consensus function that yields the highest consensus as a function of the consensus distance (or at least, for focal examples like $\partial_{\mathcal{K}}$) [2]. Obviously when such procedure is used to make social decisions, the researcher can elaborate on manipulability issues too.

[2] This has slight resemblances to the approach by Meskanen and Nurmi [21].

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