

LIE SYSTEMS, LIE SYMMETRIES AND RECIPROCAL TRANSFORMATIONS.

Departamento de Física Fundamental
Área Física Teórica
Universidad de Salamanca



MEMORIA PARA OPTAR AL TÍTULO DE DOCTOR

Cristina Sardón Muñoz

PhD Thesis, 2015

Dña. Pilar García Estévez, Catedrática de la Universidad de Salamanca y
D. Javier de Lucas Araújo, ayudante Doctor de la Universidad de Varsovia,

CERTIFICAN:

Que el trabajo de investigación que se recoge en la siguiente memoria titulado “Sistemas de Lie, simetrías de Lie y transformaciones recíprocas”, presentada por **Dña. Cristina Sardón Muñoz** para optar al título de doctor y la Mención de “Doctorado Internacional”, ha sido realizada en su totalidad bajo su dirección y autorizan su presentación.

Abril, 2015

Dña. PILAR GARCÍA ESTÉVEZ
Catedrática
Universidad de Salamanca

D. JAVIER DE LUCAS ARAÚJO
Ayudante Doctor
Universidad de Varsovia

My spine just squinted and my eye is weak
I can't find peace in a form of speech
My views change color red hot, ice cold
Black ain't a color, happiness ain't gold
It's hard for me to feel normal easy to feel free
It's hard for you to understand if you can't feel me
I'm a municipal and the mundane
I try to keep weird keep away from the same
I haven't read an outline on how to pass youth
Rather than get passed, I pass doobs
I'm out, I'm in, it's hard to live in this given culture
All the hers and hymns
All of the rats, snakes, and youth vultures around
My heart's out of shape and my head's in a cast
I am tired, I am weary
I could sleep for a thousand years
A thousand dreams that would awake me
different colors, made of tears.

-The Growlers and The Velvet Underground-

La esencia de todo arte bello, es la gratitud

-Friederich Nietzsche

Cuando expresamos nuestra gratitud, no debemos olvidar que la mayor apreciación no va en las palabras, sino en las acciones, parafraseando a J.F. Kennedy. Las personas han sido muchas, las acciones fueron más. Intentar hacer un resumen de cuatro años de trabajo y apoyo en un simple folio, son incompatibles. Sin embargo, en un breve y futil intento, siguiendo un orden cronológico, me dispongo a agradecer debidamente:

A la Universidad de Salamanca, por cubrir económicamente este trabajo y a Pilar García Estévez por aceptarme como alumna. Al Departamento de Física Fundamental y todos sus integrantes, por la grata acogida y hacerme sentir parte de ellos. En especial, a Jose María Cerveró por sus sabias aportaciones en la investigación asociada a esta tesis.

A mis dos directores de tesis, Pilar García Estévez y Javier de Lucas Araújo por depositar su confianza en mí como futura investigadora y tener la paciencia suficiente para formar a una novata, muy despistada. La discusión científica, las explicaciones, las demostraciones y consejos en temas distendidos y/o personales, han sido de lo más necesarios y bienvenidos. Pero sobre todo, es de digno agradecimiento final, el esfuerzo final. La elaboración de este manuscrito y sabias correcciones.

A la Cátedra de Métodos Matemáticos en Física de la Universidad de Varsovia, al IMPAN (Instituto de Matemáticas de la Academia Polaca de las Ciencias) y al Polish National Science Center, por todo tipo de acondicionamiento y apoyo económico para realizar mi estancia internacional. En particular, querría agradecer la financiación del proyecto HARMONIA¹ y al Profesor Grabowski por la aceptación de mi colaboración y resolver la burocracia pertinente.

¹DEC-2012/04/M/ST1/00523

A la Universidad Roma Tré en Roma, por aceptarme en una estancia breve y gratificante para la discusión con los profesores de mi más suma admiración, Decio Levi, Orlando Ragnisco y Pavel Winternitz, con quienes espero mantener un estrecho contacto científico de ahora en adelante. A Pepín Cariñena, por transmitirme sus conocimientos y contribuir a mi formación en mis estancias en la Universidad de Zaragoza. A la red de Geometría, Mecánica y Control por permitirme ser miembro de ella y sufragar mis gastos en viajes nacionales, que han contribuido muy provechosamente durante mi doctorado, permitiéndome conocer a un gran número de científicos de renombre internacional.

A Ángel Ballesteros por invitarme a la Universidad de Burgos y a Alfonso Blasco por trabajar conmigo. A mi colega Francisco José Herranz, colega en sentido científico, porque en el plano personal le considero un gran amigo, que me ha servido de consejero, apoyo incondicional y el mejor compañero de viajes internacionales. Además de lo obvio, por transmitirme sus conocimientos y enseñarme el perfeccionismo que pone en la investigación. A Juan Domingo Lejarreta, por su metodismo y meticulosidad en sus cálculos, ayudándome en la realización y comprobación de los míos. A Silvia Vilariño, porque es admirable en su trabajo y ofrece verdadera confianza y profesionalidad y me ha enseñado a dar una buena charla y en la organización de mi desorden matemático.

A Cuchi, que ha contribuido en la estructuración y bonita presentación de esta tesis. Además de nuestra amistad. A los marsopas Álvaro, Cuchi, Teresa, Alberto y Edu, porque no sólo de ciencia vive el hombre, nos pegamos nuestras juergas. A Lorena, estimada amiga y compañera de promoción, de alegrías y penas, de exploración internacional, cuya amistad y su profesionalidad investigadora no han sido para mí sino el acicate para intentar conseguir parecerme y mantenerme junto a ella. A Bea, estimada amiga y compañera de joven adolescencia, de instituto, de promoción, de alegrías y penas, de exploración internacional, cuya amistad, profesionalidad y buen humor son el incentivo de júbilo y buenos ratos en mi vida, si me encuentro con ella. A Rafa, Aser y Luis, porque son mis mejores amigos. Encuentro en ellos la mayor afinidad de cualquier índole. Porque me han enseñado que la vida puede ser simple pero insisto en hacerla complicada. Un diez por ciento es lo que nos ocurre, el noventa es cómo actuamos. La despreocupación ha contribuido muy positivamente estos últimos cuatro años. Y que lo pasamos de muerte.

Finalmente, me gustaría agradecer a mis queridos padres el haberme acompañado siempre. Hoy me siento dichosa mujer, que gracias a ellos, no he sentido nunca la soledad en este mundo. Que su ejemplo de sacrificio me ha servido en esforzarme en la finalización de un doctorado y que me recuerden mi valía, en períodos de privación de mi ocio y su comprensión en la ansiedad, han sido ineludibles en esta etapa.

A todos vosotros, gracias de todo corazón.

Contents

1	Introduction	1
1	The notion of integrability	1
1.1	Integrability of dynamical systems	2
1.2	Integrability of Hamiltonian systems	2
2	Methods to guarantee integrability	3
2.1	Lax pairs and the IST	3
2.2	The Hirota bilinear method	5
2.3	The Painlevé test	5
2.4	The singular manifold method	9
2.5	Lie systems	10
2.6	Lie symmetries	13
2.7	Reciprocal transformations	14
2	Geometric Fundamentals	17
1	Tangent spaces, vector fields and integral curves	19
2	Time-dependent vector fields	22
3	Higher-order tangent spaces and related notions	25
4	Jet bundles	26
5	Poisson algebras	30
6	Symplectic, presymplectic and Poisson manifolds	32
7	Dirac manifolds	34
8	Jacobi manifolds	37
3	Lie systems	39
1	The interest of Lie systems	39

2	Lie systems	46
2.1	Lie systems and superposition rules	46
2.2	Lie systems on Lie groups	49
2.3	An algorithm to derive superposition rules	50
2.4	New applications of Lie systems	51
3	Lie–Hamilton systems	66
3.1	On the necessity of Lie–Hamilton systems	66
3.2	Lie–Hamilton structures	69
3.3	t -independent constants of motion	73
3.4	Symmetries, linearization and comomentum maps	76
3.5	On t -dependent constants of motion	79
3.6	Lie integrals	81
3.7	Polynomial Lie integrals	82
3.8	The coalgebra method, constants of motion and superposition rules	86
4	Applications of the geometric theory of Lie–Hamilton systems	90
4.1	The Ermakov system	90
4.2	A superposition rule for Riccati equations	91
4.3	Kummer–Schwarz equations in Hamilton form	93
4.4	The n -dimensional Smorodinsky–Winternitz systems	95
4.5	A trigonometric system	98
5	Classification of Lie–Hamilton systems on the plane	99
5.1	On general definitions and properties	100
5.2	Lie–Hamilton algebras	105
5.3	Local classification	108
6	Applications of Lie–Hamilton systems on the plane	116
6.1	$\mathfrak{sl}(2, \mathbb{R})$ -Lie–Hamilton systems	117
6.2	Lie–Hamilton biological models	122
6.3	Other Lie–Hamilton systems on the plane	129
6.4	Two-photon Lie–Hamilton systems	137
6.5	\mathfrak{h}_2 -Lie–Hamilton systems	139
7	Dirac–Lie systems	142
7.1	Motivation for Dirac–Lie systems	142
7.2	Dirac–Lie Hamiltonians	145
7.3	Diagonal prolongations	149
7.4	Superposition rules and t -independent constants of motion	153
7.5	Bi–Dirac–Lie systems	157
7.6	Dirac–Lie systems and Schwarzian–KdV equations	159
8	Jacobi–Lie systems	160
8.1	Jacobi–Lie systems	161
8.2	Jacobi–Lie Hamiltonians	163

8.3	Jacobi–Lie systems on low dimensional manifolds	164
4	Lie symmetry for differential equations	169
1	What can a symmetry transformation achieve?	169
2	The CLS for ODEs	172
2.1	Algorithmic computation of Lie point symmetry for systems of ODEs	174
3	Lie symmetries for Lie systems	179
3.1	On certain Lie symmetries for Lie systems	179
3.2	Lie algebras of Lie symmetries for Lie systems	183
3.3	Applications to systems of ODEs and HODEs	185
3.4	Lie symmetries for $\text{Aff}(\mathbb{R})$ -Lie systems	193
3.5	Lie symmetries for PDE Lie systems	196
3.6	Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -PDE Lie systems	199
4	The CLS and NSM for PDEs	201
4.1	Algorithmic computation of Lie point symmetries for systems of PDEs	202
4.2	Reduction method of systems of PDEs	203
5	Applications to PDEs in Physics	207
5.1	The 2 + 1 dimensional BKP equation and corresponding Lax pair	208
5.2	A reduction of the 2+1-BKP equation and corresponding Lax pair to 1 + 1 dimensions	215
6	Explicit calculation for hierarchies of PDEs	221
6.1	The negative Camassa-Holm hierarchy in 2+1 dimensions and corresponding Lax pair	222
6.2	The Qiao hierarchy in 2+1 dimensions and corresponding Lax pair	236
5	Reciprocal transformations	255
1	What is a reciprocal transformation?	255
2	Quasi-algorithmical construction of reciprocal transformations	259
3	CHH(2+1) versus CBS	262
4	mCHH(2+1) versus mCBS	264
5	Reciprocal-Miura transformations	266
5.1	Particular case 1: The Qiao equation	267
5.2	Particular case 2: The Camassa-Holm equation	268
6	Conclusions and outlook	271
1	Lie systems	271
2	Lie symmetries	273
3	Reciprocal transformations	275

<i>Appendix 1: Tables</i>	277
<i>Appendix 2: Auxiliar calculations for Chapter 5</i>	283
Bibliography	285

1 The notion of integrability

On a fairly imprecise, first approximation, the universal definition of integrability is understood as *the exact solvability or regular behavior of solutions* of a system. Nevertheless, in Mathematics and Physics, there are various distinct notions of integrable systems [5, 13, 25, 248, 280, 339]. The characterization and unified definition of integrable systems are two nontrivial matters. The study of integrability from several different perspectives has led to an apparent fragmented notion of them. The aim of this introduction is to give a comprehensive account of the variety of approaches to such an important and difficult concept as *integrability*. In particular, we focus on those approaches applicable to mathematical and physical models described through ordinary differential equations (ODEs) and partial differential equations (PDEs).

The field of integrable systems appeared with Classical Mechanics with a quest for exact solutions to Newton's equations of motion [30]. Integrable systems usually present regular behavior of solutions or *conserved quantities* in time, such as energy, angular momentum, etc. Indeed, some of such systems present an infinite number of conserved quantities [30]. But having a big amount of conserved quantities is more of an exception rather than a rule.

Then, we are in need of a mathematically rigorous, constructive approach based on the study of algebraic or analytic structures related to integrable systems. The requirement of existence of such structure could be taken as the notion of integrability [324].

In the general theory of differential equations, we can endow different definitions and interpretations of integrability.

1.1 Integrability of dynamical systems

From a very geometrical point of view, dynamical systems are formulated based on their underlying geometric or algebraic structure. Differential equations can be interpreted in terms of a system of differential one-forms, or a *Pfaffian system*. The idea is to see how these differential one-forms restrict to a submanifold and how this restriction is compatible with the exterior derivative.

Given a collection of one-forms over a general manifold N , an integral submanifold is a submanifold M whose tangent space at every point is annihilated by each one-form at every point $p \in M$. We call a *maximal integral manifold* a submanifold M such that the kernel of the restriction $\iota^* : \Omega_p^{(1)}(N) \rightarrow \Omega_p^{(1)}(M)$, with $\iota : M \hookrightarrow N$, is spanned by the one-forms of the Pfaffian system in every point of M . Additionally, the one-forms are linearly independent. A Pfaffian system is said to be *completely integrable* if N admits a foliation by maximal integral manifolds (the foliation need not be regular, the leaves of the foliation might not be embedded submanifolds). The integrability condition is a condition on these one-forms that guarantees that there will be integral submanifolds of sufficiently high dimension. In these terms, the *Fröbenius theorem* states the integrability condition for such systems: if the ideal algebraically generated by the collection of one-forms is closed under exterior differentiation, then, the system admits a foliation by maximal integral manifolds [13, 196].

In the context of differentiable dynamical systems, the notion of integrability refers to *Liouville integrability* [304], or the existence of invariant, regular foliations, i.e., ones whose leaves are embedded submanifolds of the smallest possible dimension that are invariant under the flow.

1.2 Integrability of Hamiltonian systems

In the case of integrable Hamiltonian systems with an associated Hamiltonian function, there exists a regular foliation of the phase space by invariant manifolds, such that the Hamiltonian vector fields associated with the invariants of the foliation, span the tangent distribution. Similarly, we can say that there exists a maximal set of invariants that commute with the Hamiltonian function H under the Poisson bracket $\{\cdot, \cdot\}$, that is a Lie bracket satisfying the Leibnitz rule

$$\{H, I\} = 0, \tag{1.1}$$

for a certain number of first-integrals I .

- If the space is symplectic, it has an even dimension $2n$ and the maximal number of independent Poisson commuting invariants (including H) is n .
- When the number of independent Poisson commuting invariants is less than n , we say that the system is *partially integrable* [13, 25].

- When there exist further functionally independent invariants, beyond the maximal number that can be Poisson commuting, it is *superintegrable* [13, 25]. This is equivalent to saying that the dimension of the leaves of the invariant manifold is less than n . If there is a regular foliation with one-dimensional leaves (curves), this is called *maximally superintegrable* [13, 184, 208].

When the Hamiltonian system is completely integrable in the Liouville sense and the level sets are compact, the flows are complete and the leaves of the invariant foliation are tori. Then, there exist local canonical coordinates on the phase space known as *action-angle* variables [280], such that the invariant tori are the joint level set of the action variables. The Liouville integrability provides a complete set of invariants of the Hamiltonian that are constants of motion. The angle variables are periodic and the motion on the tori is linear in these coordinates.

2 Methods to guarantee integrability

In 1960, the discovery of *solitons* as strongly stable, localized solutions of certain PDEs could be understood by viewing these equations as infinite-dimensional Hamiltonian systems. The study of soliton solutions led to multiple methods for solving PDEs. A fundamental one is the inverse scattering method/transform (IST) [5, 8, 9], which appeared as a result of the derivation of Lax pairs [281, 282].

2.1 Lax pairs and the IST

A *Lax pair* (LP) or *spectral problem* is a pair of linear operators $L(t)$ and $P(t)$, acting on a fixed Hilbert space \mathcal{H} , that satisfy a corresponding differential equation, the so called *Lax equation*

$$\frac{dL}{dt} = [P, L], \quad (2.1)$$

where $[P, L] = PL - LP$. The operator $L(t)$ is said to be *isospectral* if its spectrum of eigenvalues is independent of the evolution variable. We call *eigenvalue problem* the relation

$$L\psi = \lambda\psi, \quad (2.2)$$

where $\psi \in \mathcal{H}$, henceforth called a *spectral function or eigenfunction*, and λ is a *spectral value or eigenvalue*.

The core observation is that there exists a unitary operator $u(s, t)$ such that

$$L(t) = u(s, t)L(s)u^{-1}(s, t), \quad (2.3)$$

where $u(s, t)$ is the general solution of the Cauchy problem

$$\frac{du(s, t)}{dt} = P(t)u(s, t), \quad u(s, s) = 1. \quad (2.4)$$

Notice that $L(t)$ is self-adjoint and $P(t)$ is skew-adjoint. In other words, to solve the eigenvalue problem (2.2) we can use the value $L(0)$ and for obtaining the solution at time t , we use the following condition and ODE

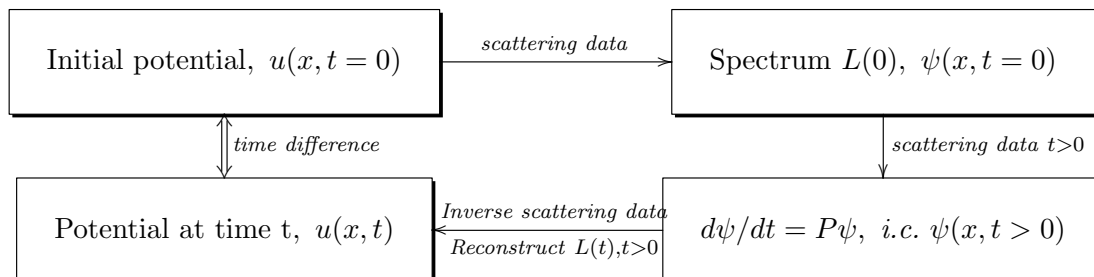
$$\begin{aligned}\lambda(t) &= \lambda(0), \\ \frac{\partial \psi}{\partial t} &= P\psi.\end{aligned}\tag{2.5}$$

Lax pairs are interesting because they guarantee the integrability of certain differential equations [5, 9]. Generally, nonlinear PDEs that are integrable, can equivalently be rewritten as the compatibility condition (2.1) of a spectral problem [164, 171]. Sometimes, it is easier to solve the equations provided by the LP rather than the nonlinear problem. In this way, the IST is based on the aforementioned properties [5].

The *inverse scattering transform* is a method for solving certain nonlinear PDEs with an associated LP. The IST guarantees the existence of analytical solutions of the PDE (when it can be applied). The name inverse transform comes from the idea of recovering the time evolution of the potential $u(x, t)$ from the time evolution of its scattering data, opposed to the direct scattering which finds the scattering matrix from the evolution of the potential. This method has been applied to many exactly solvable models or completely integrable infinite-dimensional systems as: the Korteweg de Vries equation [273], to be soon extended to the nonlinear Schrödinger equation, Sine-Gordon equation, etc. [170, 426]. It was first introduced by Gardner, Greene, Krustal, Miura *et al* in 1967 [199, 200] and it can be summarized in a number of steps, in the case of $1 + 1$ dimensions. This process could be generalized to higher dimensions [272]

- Consider L and P acting on \mathcal{H} , where L depends on an unknown function $u(x, t)$ and P is independent of it in the scattering region.
- We can compute the spectrum of eigenvalues λ for $L(0)$ and obtain $\psi(x, 0)$.
- If P is known, we can propagate the eigenfunction with the equation $\frac{\partial \psi}{\partial t}(x, t) = P\psi(x, t)$ with initial condition $\psi(x, 0)$.
- Knowing $\psi(x, t)$ in the scattering region, we construct $L(t)$ and reconstruct $u(x, t)$ by means of the Gelfand–Levitan–Marchenko equation [316]

The following diagram shows the procedure



2.2 The Hirota bilinear method

Another method guaranteeing the integrability of a nonlinear PDE, together with the IST, is the *Hirota's bilinear method* (HBM) [242]. The major advantage of the HBM over the IST is the obtainance of possible *multi-soliton solutions* [242, 243, 323] by imposing Ansätze. Additionally, the HBM can be applied to a greater number of equations and it is solved algebraically instead of relying on nontrivial analysis, as the IST [239, 240]. Most PDEs are not initially expressed in a tractable format for subsequent mathematical analysis. Hirota noticed that the best dependent variables for constructing soliton solutions are those in which the soliton appears as a finite number of exponentials. To apply this method it is necessary that the equation is quadratic and that the derivatives can be expressed using Hirota's *D-operator* [244] defined by

$$D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2) \Big|_{x_2=x_1=x}.$$

Unfortunately, the process of bilinearization is far from being algorithmic, and it is hard to know how many variables are needed for bilinearization.

2.3 The Painlevé test

To discern whether our equation is integrable or not, involves a deep inspection of its geometrical properties and aforementioned methods. Now, a question arises: Is there any algorithmical method to check the integrability of a differential equation? The answer is affirmative.

The Painlevé test establishes an algorithmical integrability criteria for differential equations [131, 171, 417]. It focuses on the *singularity analysis* of the differential equation, attending to a fundamental property: being a *fixed* or a *movable singularity*, or a singularity not depending or depending on the initial conditions, respectively.

These concepts were initiated by Fuchs and Sophia Kovalevskaya [134]. The latter centered herself in the study of equations for solid rigid dynamics: singularities

and properties of single-valuedness of poles of PDEs on the complex plane, etc. Eventually, she expanded her results to other physical systems [322].

Let us illustrate the concept of fixed and movable singularity. Consider a manifold N locally diffeomorphic to $\mathbb{R} \times \mathbb{T}\mathbb{R}$, with local coordinates $\{t, u(t), u_t\}$, where $\mathbb{T}N$ represents the tangent space. Consider the differential equation

$$(t - c)u_t = bu \quad (2.6)$$

and $c, b \in \mathbb{R}$. Its general solution reads

$$u(t) = k_0(t - c)^b, \quad (2.7)$$

where k_0 is a constant of integration. Depending on the value of the exponent b , we have different types of singularities

- If b is a positive integer, then, $u(t)$ is a holomorphic function.
- If b is a negative integer, then c is a *pole singularity*.
- In case of b rational, c is a branch point.

Nevertheless, the singularity $t = c$ does not depend on initial conditions. We say that the singularity is *fixed*.

Let us now consider an ODE on $\mathbb{R} \times \mathbb{T}^2\mathbb{R}$, which reads

$$buu_{tt} + (1 - b)u_t^2 = 0, \quad (2.8)$$

with $a \in \mathbb{R}$. The general solution to this equation is

$$u(t) = k_0(t - t_0)^b. \quad (2.9)$$

If b is a negative integer, the singularity $t = t_0$ is a singularity that depends on the initial conditions through the constant of integration t_0 . In this case, we say that the singularity is *movable*.

Painlevé, Gambier *et al* oriented their study towards second-order differential equations. In particular, Painlevé focused on differential equations on $\mathbb{R} \times \mathbb{T}^2\mathbb{R}$ with local coordinates $\{t, u, u_t, u_{tt}\}$, of the type

$$u_{tt} = F(t, u, u_t), \quad (2.10)$$

where F is a rational function in u, u_t and analytic in t . He found that there were 50 different equations of type (2.10) whose unique movable singularities were poles [347, 348]. Out of the 50 types, 44 were integrated in terms of known functions as Riccati, elliptic, linear, etc., and the 6 remaining, although having meromorphic solutions, they do not possess algebraic integrals that permit us to reduce them by

quadratures. These six functions are the nonlinear analogues of special functions and their solutions are called *Painlevé transcendents* ($P_I - P_{VI}$), because they cannot be expressed in terms of elementary or rational functions or solutions expressible in terms of special functions [125].

From here we can establish the *Painlevé Property* (PP) for an algebraic ODE. We say that an ODE has the PP if all the movable singularities of its solution are poles. In other words, if the solution is single-valued anywhere but in the fixed singularities of the coefficients. The PP does not attend to other type of singularities as branch points, essential singularities, etc.

To establish the nature of the singularities and a criteria of integrability, we can make use of the *Painlevé test* (PT) for ODEs [417]. Given a general ODE on $\mathbb{R} \times \mathbb{T}^p \mathbb{R}$ with local coordinates $\{t, u, u_t, \dots, \underbrace{u_{t, \dots, t}}_{p \text{ times}}\}$,

$$F = F(t, u(t), \dots, \underbrace{u_{t, \dots, t}}_{p \text{ times}}), \quad (2.11)$$

the PT analyzes local properties by proposing solutions in the form

$$u(t) = \sum_{j=0}^{\infty} a_j (t - t_0)^{(j-\alpha)}, \quad (2.12)$$

where t_0 is the singularity, $a_j, \forall j$ are constants and α is necessarily a positive integer. If (2.12) is a solution of an ODE, then, the ODE is conjectured integrable. To prove this, we have to follow a number of steps

1. We first need to determine the value of α by balance of dominant terms [387], which will simultaneously permit us to obtain a_0 . The values of α and a_0 are not necessarily unique, and α must be a positive integer. If there is more than one possibility, we say that there are branches of expansion and the forthcoming steps need to be satisfied for every possible value of α .
2. Having introduced (2.12) into the differential equation (2.11), we obtain a relation of recurrence for the rest of coefficients a_j that can be written as

$$(j - \beta_1) \cdots (j - \beta_n) a_j = F_j(t, \dots, u_k, (u_k)_t, \dots), \quad k < j \quad (2.13)$$

and which arises from setting equal to zero different orders in $(t - t_0)$. This gives us a_j in terms of a_k for $k < j$. Observe that when $j = \beta_l$ with $l = 1, \dots, n$, the left-hand side of the equation is null and the associated a_{β_l} is arbitrary. Those values of j , are called *resonances* and the equation (2.13) turns into a relation for a_k for $k < \beta_l$ which is known as the *resonance condition*. Considering that the number of arbitrary constants of motion that an ODE must have is equal to its order, we must find the order of the equation minus one, as t_0 is one of the constants.

3. Make sure that the resonance conditions are satisfied identically, $F_j = 0$ for every $j = \beta_l$. If so, we say that the ODE possesses the PP and that it is integrable. The resonances have to be positive except $j = -1$, which is associated with the arbitrariness of t_0 .
4. For systems of ODEs with k dependent variables u_1, \dots, u_k , this procedure needs the addition of expansions for every u_i ,

$$u_i(t) = \sum_{j=0}^{\infty} a_j^i (t - t_0)^{(j-\gamma)}, \quad \forall i. \quad (2.14)$$

In the case of PDEs, Weiss, Tabor and Carnevale carried out the generalization of the Painlevé method, the so called *WTC method* [417].

Consider a system of q partial differential equations of p -order, defined over a general manifold with local coordinates

$$\left(x_i, u_j, (u_j)_{x_i}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, \dots, x_{i_n}^{j_n}} \right) \quad (2.15)$$

which reads

$$\Psi^l = \Psi^l \left(x_i, u_j, (u_j)_{x_i}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}} \right), \quad l = 1, \dots, q, \quad (2.16)$$

such that $j_1 + \dots + j_n = p$, $1 \leq i_1, \dots, i_n \leq n$ and $x_1 \leq x_i \leq x_n$, $j = 1, \dots, k$.

The Ablowitz-Ramani-Segur conjecture, *ARS conjecture*, says that a PDE is integrable in the Painlevé sense [348], if all of its reductions have the Painlevé property [415]. This is equivalent to saying that if a PDE is reducible to an ODE that is not one of the Painlevé transcendents [194], then the ARS conjecture predicts that the PDE is not completely integrable [8]. In other words, a PDE enjoys the PP if $u(t)$ is single-valued in the movable singularity manifold ϕ and $\alpha > 0$ and integer. McLeod and Olver [322] tested a weaker version of this conjecture. No rigorous proof was given, but the extended study of its reduced versions suggests the validity as a method to prove integrability.

Similarly, we can extend the Painlevé test to PDEs by substituting the function $(t - t_0)$ in (2.12) by an arbitrary function $\phi(x_i)$ for all $i = 1, \dots, n$, which receives the name of *movable singularity manifold*. We propose a Laurent expansion series

$$u^l(x_i) = \sum_{j=1}^n u_j^{(l)}(x_i) \phi(x_i)^{j-\alpha}, \quad \forall x_i, \quad i = 1, \dots, n, \quad \forall l = 1, \dots, k, \quad (2.17)$$

which incorporates $u^l(x_i)$ as functions of the coordinates x_i , instead of being constants as a_j in (2.12). Steps 1–3 for ODEs can be reenacted in order to prove the

integrability of a PDE. It is significant to notice that the resonance condition and recursion equation (2.13) have now more general expressions as

$$(j - \beta_1) \cdots (j - \beta_n) u_j(x_i) = F_j(\phi_{x_i}, \dots, u_k, (u_k)_{x_i}, \dots), \quad k < j, \quad \forall i = 1, \dots, n. \quad (2.18)$$

It is important to mention that the PT is not invariant under changes of coordinates. This means that an equation can be integrable in the Painlevé sense in certain variables, but not when expressed in others, i.e., it is not a geometrical property. There exist certain changes of coordinates to check the Painlevé integrability when the equation does not possess the PP in its initial ones. In particular, we deal with a change of coordinates, the known as *reciprocal transformations* [138, 158, 372].

2.4 The singular manifold method

The *singular manifold method* (SMM) focuses on solutions which arise from truncated series of the generalized PP method (2.17). We require the solutions of the PDE written in the form (2.17) to select the truncated terms [164]

$$u^l(x_i) \simeq u_0^{(l)}(x_i) \phi(x_i)^{-\alpha} + u_1^{(l)}(x_i) \phi(x_i)^{1-\alpha} + \cdots + u_\alpha^{(l)}(x_i), \quad (2.19)$$

for every l . In the case of several branches of expansion, this truncation needs to be formulated for every value of α . Here, the function $\phi(x_i)$ is no longer arbitrary, but a *singular manifold* with the expression arising from (2.19) being a truncated solution of the PDE under study.

Generally, a lot of physical models appear as equations of this type, in $1 + 1$ dimensions (one spatial and one temporal dimension).

The SMM is interesting because it contributes substantially in the derivation of a Lax pair. This procedure can be devised in [170, 171, 172].

Some of the described methods, apart from guaranteeing the integrability of a differential equation, provide us with possible solutions. In the following chapters, we shall focus on three primordial ones. The first, different, but not unrelated approach to integrability of differential equations, begins with the seminal contribution of Sophus Lie in classical symmetries of differential equations. Later on, this method was generalized by Bluman and Cole [64] and Olver and Rosenau [341, 342]. They searched for a particular type of symmetries that leaves invariant a subset of all the possible solutions of the PDE. These symmetries do not form a Lie group in themselves but they are very worth of study for the analysis of properties of the PDE.

Another way to achieve integrability is finding symmetries not through the usual group theoretical approach [397], but through *similarity transformations*. One chooses an Ansatz that permits the reduction of the PDE to an ODE, which can be

studied through the Painlevé analysis. The difficult part resides in choosing proper Ansätze. This work was deeply inspected by Clarkson and Kruskal [126, 229, 357, 410].

2.5 Lie systems

In short, a Lie system is a system of ODEs describing the integral curves of a t -dependent vector field taking values in a finite-dimensional Lie algebra V of vector fields: a so-called *Vessiot–Guldberg (VG) Lie algebra* for the Lie system [89, 91, 102, 109, 419].

Equivalently, a Lie system is a system of ODEs admitting a *superposition principle* or *superposition rule*, i.e., a map allowing us to express the general solution of the system of ODEs in terms of a family of particular solutions and a set of constants related to initial conditions [89, 91, 102, 303, 419]. These superposition principles are, in general, no linear. Among other reasons, superposition rules are interesting because they allow us to integrate systems of ODEs from the knowledge of a small set of particular solutions [18, 19, 102].

Lie systems enjoy a plethora of mathematical properties. Geometrically, the Lie–Scheffers theorem states that a Lie system amounts to a curve within a Lie algebra of vector fields. Additionally, Cariñena, Grabowski and Marmo proved that superposition rules can be understood as a certain type of projective foliation on an appropriate bundle [91]. This idea gave rise to a geometric method for determining superposition rules that has been employed, along with the new techniques described in the thesis, to obtain superposition rules in an algebraic and geometric manner for certain types of Lie systems admitting compatible geometric structures [35, 88].

From the point of view of their applications, Lie systems play a relevant rôle in Physics, Mathematics, and other fields of research (see [102] which details more than 400 references on Lie systems and related topics). Some of the most representative Lie systems are the Riccati equations and their multiple derived versions (matrix Riccati equations, projective Riccati equations, etc.) [228, 278, 338]. These latter differential equations frequently appear in Cosmology, Financial Mathematics, Control Theory and other disciplines [28, 102, 292, 386]. It is also worth noting that Lie systems appear in the study of Wei–Norman equations [110], quantum mechanical problems [100, 102] and Biology [32].

Additionally, many other differential equations can be studied through the theory of Lie systems, even though they are not Lie systems. This is the case of the Kummer–Schwarz [309] and Milne–Pinney equations [374], the Ermakov system [97], t -dependent frequency Winternitz–Smorodinsky oscillators [35], Buchdahl equations [236], among others [108, 109, 309].

Many of the previously mentioned examples have been discovered by the author of this thesis and her collaborators. We have also added the Hamiltonian version of the second-order Riccati equations [108], systems of ODEs that appear in the study

of diffusion equations [236], new types of Riccati equations on different types of composition algebras, as the complex, quaternions, Study numbers [166], and many others [236]. Other new Lie systems concern viral models [32], certain reductions of Yang–Mills equations [166] and complex Bernoulli equations [236].

As a consequence of the new applications of Lie systems, we have significantly enlarged the field of potential applications of such systems. Moreover, most applications of Lie systems on the study of PDEs, e.g., in analyzing flat \mathfrak{g} -valued connections and partial Riccati equations [166], are described in this thesis [88, 166]. Due to the special geometric structure of Lie systems, our findings open a new research approach to the study of the above-mentioned differential equations and their related problems.

Very surprisingly, it was proved that the found new Lie systems admitted Vessiot–Guldberg algebras of Hamiltonian vector fields with respect to some symplectic or Poisson structure [109]. This led to the study of an important particular case of Lie systems, the so-called *Lie–Hamilton systems*.

Lie–Hamilton systems are Lie systems that admit Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to a Poisson structure [35, 94, 109]. The author and collaborators proved that Lie–Hamilton systems possess a time-dependent Hamiltonian given by a curve in a finite-dimensional Lie algebra of functions with respect to a Poisson bracket related with the Poisson structure: a *Lie–Hamilton algebra* [35]. For Lie systems, this structure plays the analogue rôle of Hamiltonians in Hamiltonian Mechanics. From this analogy, the writer has found numerous properties relative to the description of constants of motion, momentum maps and Lie symmetries of Lie–Hamilton systems [109].

Among the developed methods for Lie–Hamilton systems, we consider of upmost importance the computation of superposition principles through Poisson coalgebras [33, 34, 36, 39, 40]. The traditional method for the computation of superposition principles for Lie systems relies in the integration of systems of ordinary or partial differential equations [18, 89, 91, 419]. Nevertheless, we can obtain such principles for Lie–Hamilton systems by methods of algebraic and geometric nature [35].

Let us explain the last statement more precisely. Every Lie–Hamilton system induces a Lie–Hamilton algebra. We construct a Poisson algebra of polynomials in the elements of a basis of the Lie–Hamilton algebra. In obtaining superposition rules for a Lie–Hamilton system, we have to define the so-called diagonal prolongations of the Lie–Hamilton system, which are again Lie–Hamilton systems [35]. Hence, the diagonal prolongations are endowed with Poisson algebras in the same way. The point is that there exists a primitive coproduct passing from the Poisson algebra of the initial Lie–Hamilton system to the Poisson algebras of the successive diagonal prolongations giving rise to a Poisson coalgebra structure [33, 34, 36, 39]. The coproduct and the Casimir elements of the Lie–Hamilton algebra of a Lie–Hamilton system enable us to obtain conserved quantities and Lie symmetries of the Lie–Hamilton system

and their diagonal prolongations. Such constants of motion are employed to obtain superposition principles [35]. From now on, we call this procedure the *coalgebra method for obtaining superposition principles*.

The coalgebra method has been developed by the author of this thesis and her collaborators in an original way and has shown its efficiency [35], by avoiding the long integration of systems of ODEs or PDEs arising from traditional methods [91, 102, 419]. Moreover, it allows us to provide geometrical and algebraic interpretations of superposition rules. A very relevant outcome has been the achievement of the superposition principle for the Riccati equations describing that the constant of motion that leads to such a principle is basically given by the image, via a coproduct, of a Casimir function for $\mathfrak{sl}(2, \mathbb{R})$ [35].

Due to the interest shown in Lie–Hamilton systems, we have classified all the Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on the plane with respect to a Poisson structure and have analyzed their properties [32]. We obtained twelve different non diffeomorphic classes of Lie algebras of Hamiltonian vector fields. Furthermore, relationships among all the finite-dimensional Lie algebras of Hamiltonian vector fields on the plane were established (see Table 4), which completed several details (See Table 4) in the previous study carried out by Olver, Kamran and González [210].

Our classification permitted the identification of physical and mathematical properties of the Lie–Hamilton systems on the plane [236]. For example, it helped in the study of trigonometric Lie systems appearing in the study of integrable systems [21], diffusion equations [32, 236, 389, 390], Smorodinsky–Winternitz oscillators [314], systems with medical applications [153], Lotka–Volterra systems [32], systems with periodic trajectories, etcetera.

But not every Lie system is a Lie–Hamilton system. The so-called no-go theorem [88] shows a very general condition that helps us identify when a Lie system is not a Lie–Hamilton one. We have found that numerous Lie systems admit a Vessiot–Guldberg Lie algebra which consists of Hamiltonian vector fields with respect to a certain geometric structure different from Poisson structures. If the vector fields of the Vessiot–Guldberg Lie algebra are Hamiltonian with respect to a Dirac or Jacobi structure, we say that they are Dirac–Lie [88] or Jacobi–Lie [237] systems, correspondingly. Along this thesis, we will give numerous new examples of Lie systems over different geometries, which do not only have great importance from the mathematical point of view, but actually model the Physics of nature [35, 88, 237, 309].

A notorious example of Lie system compatible with another geometrical structure is that of Dirac–Lie systems [88]. These are Lie systems that possess a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Dirac structure. As Dirac structures describe Poisson structures as particular cases, Dirac–Lie systems cover Lie–Hamilton systems.

In this thesis we show how Dirac–Lie systems describe, through similar techniques to those of the Poisson geometry, systems which cannot be described through Lie–Hamilton systems. For example, the third-order Kummer–Schwarz equations and certain systems appearing in the study of diffusion equations [88, 389]. Also, some generalizations of the results for Lie–Hamilton systems were applied to the realm of Dirac–Lie systems. Moreover, it is interesting that Dirac structures give rise to a particular type of Lie algebroid structures [319]. As a consequence, Lie algebroid structures can also be employed to study Dirac–Lie systems [88]. Due to the fact that the use of Lie algebroid structures in Physics is much discussed, we expect to continue this line of research in the future so as to analyze its relevance.

The last type of geometry that has been employed in this thesis is the Jacobi geometry [237]. A Jacobi manifold is another generalization of a Poisson manifold. We have defined and studied the denominated *Jacobi–Lie systems*, namely Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Jacobi manifold. We generalize the theory of Lie–Hamilton systems to the realm of Jacobi–Lie systems and we detail a classification of Jacobi–Lie systems on the line and on the plane (see Table 5). This entails the study of the structure of the space of constants of motion for such systems.

2.6 Lie symmetries

One of the most common techniques for the study of differential equations is the Lie symmetry method [331], initiated by Sophus Lie in the XIX century [296, 298] and all its variant versions developed along the last decades. The Lie symmetry method can be summarized by determining a transformation that leaves invariant a group of equations [339, 358, 388]. Invariance under a transformation implies the possibility of reducing the number of independent variables by one, for each symmetry. A number of symmetries equal to the number of degrees of freedom of the equation, leads us to its integration until it is reduced to an ordinary equation. Given an ordinary equation, a symmetry would lead to the complete integration (by quadrature) of the differential equation.

The classic method for symmetry computation was generalized along the XX century. The development of computers has greatly helped in the generalization of the Lie symmetry method and its application to more complicated equations.

New software orientated towards symbolic calculus, for example *Maple*, has become very useful in the treatment of cumbersome intermediate calculation.

An important generalization of the classical method is the nonclassical approach by Bluman and Cole in 1969 [64], Olver and Rosenau [341, 342]. We look for a particular transformation that leaves invariant a subset of all the possible solutions of the equation. The proposal of this type of symmetry started with the search of solutions for the heat equation [181], which were not deducible through the classic method. Since then, the nonclassical method was popularized [26, 127, 332]. A

notorious difference between the classical and nonclassical method is that the later provides us with no linear systems of differential equations for the resolution of the symmetries. In the last few years, the classical and nonclassical analysis have shown their efficiency in the treatment of equations of hydrodynamic origin, in the context of Plasma Physics, cosmological models, Fluid Mechanics [5, 79, 171, 291, 330, 334].

In our research, we find of particular interest the application of the Lie symmetry method (classical and nonclassical) to Lax pairs associated with the nonlinear differential equations. The inspection of symmetries of equations has been a very treated topic by lots of authors. Notwithstanding, the symmetries of their associated Lax pairs have been less investigated.

Our aim is to see how the Lax pairs reduce under the symmetry and in the case of nonisospectral problems, to see if their nonisospectrality condition is propagated to lower dimensions [167, 168, 169].

2.7 Reciprocal transformations

Reciprocal transformations can be suitably used in the field of PDEs. These transformations consist on the mixing of the role of the dependent and independent variables [245], to achieve simpler versions or even linearized versions of the initial, nonlinear PDE [158, 172]. Reciprocal transformations, by experience of many worked examples, help in the identification of the plethora of PDEs available in the Physics and Mathematics literature. Two different equations, although seemingly unrelated at first sight, happen to be equivalent versions of a same equation, after a reciprocal transformation [176]. In this way, the big number of integrable equations in the literature, could be greatly diminished by establishing a method to discern which equations are disguised versions of common problem. Then, the next question comes out: Is there a way to identify different versions of a common nonlinear problem?

In principle, the only way to ascertain is by proposing different transformations and obtain results by recurrent trial and error. It is desirable to derive the aforementioned canonical form from the SMM, but it is still a conjecture to be proven.

The content of this thesis has either been published or it is on its way for soon publication. Here we show a list of our results.

PUBLICATIONS

1. Similarity reductions arising from nonisospectral Camassa Holm hierarchy in 2+1 dimensions,
P.G. Estévez, J.D. Lejarreta, C. Sardón,
J. Nonl. Math. Phys. **18**, 9–28 (2011).

2. A new Lie systems approach to second-order Riccati equations,
J.F. Cariñena, J. de Lucas, C. Sardón,
Int. J. Geom. Methods Mod. Phys. **9**, 1260007 (2012).
3. Miura reciprocal Transformations for hierarchies in 2+1 dimensions,
P.G. Estévez, C. Sardón,
J. Nonl. Math. Phys. **20**, 552–564 (2013).
4. Integrable 1+1 dimensional hierarchies arising from reduction of a non-isospectral problem in 2+1 dimensions,
P.G. Estévez, J.D. Lejarreta, C. Sardón,
Appl. Math. Comput. **224**, 311–324 (2013).
5. On Lie systems and Kummer–Schwarz equations,
J. de Lucas, C. Sardón,
J. Math. Phys. **54**, 033505 (2013).
6. From constants of motion to superposition rules of Lie–Hamilton systems,
A. Ballesteros, J.F. Cariñena, F.J. Herranz, J. de Lucas, C. Sardón,
J. Phys. A: Math. Theor. **46**, 285203 (2013).
7. Lie–Hamilton systems: theory and applications,
J.F. Cariñena, J. de Lucas, C. Sardón,
Int. Geom. Methods Mod. Phys. **10**, 0912982 (2013).
8. Dirac–Lie systems and Schwarzian equations,
J.F. Cariñena, J. Grabowski, J. de Lucas, C. Sardón,
J. Differential Equations **257**, 2303–2340 (2014).
9. Lie–Hamilton systems on the plane: theory, classification and applications,
A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, C. Sardón,
J. Differential Equations **258**, 2873–2907 (2015).
10. Lie symmetries for Lie systems: applications to systems of ODEs and PDEs,
P.G. Estévez, F.J. Herranz, J. de Lucas, C. Sardón,
submitted to *Appl. Math. Comput.*
arXiv:1404.2740
11. Lie–Hamilton systems on the plane: applications and superposition rules,
A. Blasco, F.J. Herranz, J. de Lucas, C. Sardón,
submitted to *J. Phys. A: Math. Theor.*
arXiv:1410.7336

1. Miura reciprocal transformations for two integrable hierarchies in 1+1 dimensions,
P.G. Estevez, C. Sardón,
Proceedings GADEIS, Protaras, Cyprus (2012),
arXiv:1301.3636
2. Jacobi–Lie systems: fundamentals and low-dimensional classification,
F.J. Herranz, J. de Lucas, C. Sardón,
Proceedings AIMS, Madrid, Spain (2014),
arXiv:1412.0300

PUBLICATIONS IN PREPARATION

1. Classical and nonclassical approach for a wave model in 2+1 dimensions,
P.G. Estévez, J.D. Lejarreta, C. Sardón,
In preparation.

Geometric Fundamentals

In short, Differential Geometry is the mathematical discipline generalizing the standard differential and integral calculus on vector spaces to more general spaces, the manifolds, in order to study the geometric properties of the structures defined on them.

During the XVIII and XIX century, Differential geometry was mostly concerned with the study of the intrinsic properties of curves and surfaces in \mathbb{R}^n , e.g., the *Gauss' Egregium Theorem*, and the notion of parallelism [201, 202, 203].

The first mature notion of manifold appears in Riemann's Habilitation thesis in 1854 [366]. Riemann defined manifolds, term coming from the german *Mannigfaltigkeit*, as a topological space that resembles the Euclidean space near each point, but not necessarily globally. Since then, manifolds became a central concept for Geometry and modern Mathematical Physics. Posteriorly, manifolds arose as solution sets of systems of equations, graphs of functions, surfaces, etc. They allow us to describe locally very complicated structures in terms of an Euclidean space and techniques from linear algebra.

Riemannian settled metrics on manifolds as a way to allow us measure distances and angles. This carried in a natural way the existence of non-Euclidean geometries. It is worth noting that in Riemann's time it was a fundamental problem to understand the existence of such non-Euclidean geometries which have then just appeared [76, 48].

Moreover, the Gauss' Egregium Theorem is based on the notion of a metric: *“the curvature of a surface can be determined entirely by measuring angles, distances and the rates on the surface itself”*, without further reference on how the surface is embedded in the 3-dimensional Euclidean space. In more modern geometrical terms, the Gaussian curvature of a surface is invariant under local isometry. These were ideas fully established by Riemann: *relevant properties of geometric structures are independent of the spaces in which they might be embedded.*

Since the XIX, Differential Geometry began involved with geometric aspects of the theory of differential equations and geometric structures on manifolds. Indeed, Lie pioneered this line of research by developing a theory of invariance of differential equations under groups of continuous transformations, as a way to achieve conserved quantities [295, 297, 300, 301, 302, 299].

Historically, the formulation of Differential Geometry and symmetry groups remained still from Euler's theory description of the plane sound waves in fluids until Birkhoff retook the subject during World War II, working on Fluid Dynamics including bazooka changes and problems of air-launched missiles entering water [59, 60]. He emphasized Group Theory for handling symmetries in Hydrodynamics and urged innovative numerical methods relying more heavily on computing. Indeed, he started working on the solution of partial differential equations by successive approximations.

By the end of the XX century, a big explosion in research of geometrical aspects of differential equations had taken place. In order to study the Hamiltonian formalism of Classical Mechanics, the introduction of symplectic manifolds served as the mathematical description of the phase space. More relevant manifolds were henceforth introduced, as the Lorentzian manifold model for space-time in General Relativity [48], etc.

Properties of nonlinear systems of partial differential equations with geometrical origin and natural description in the language of infinite-dimensional Differential Geometry were settled. This helped the appearance of their Lagrangian and Hamiltonian formalisms, and the obtention of low-dimensional conservation laws [429]. Many problems in Physics, Medicine, Finance and industry required descriptions by means of nonlinear partial differential equations. So, their investigation through Differential Geometry began an independent field of research, with lots of different directions: optimal transport problems, free boundary problems, nonlinear diffusive systems, singular perturbations, stochastic partial differential equations, regularity issues and so on [102, 113, 232].

For all the above reasons, we find of importance to formulate the content of this thesis in geometrical terms. This introductory chapter provides an overview of the geometrical and standard notation we shall use along the forthcoming chapters. We will consider differential equations as submanifolds in jets bundles. Investigation of singularities is a very subtle theme which we shall skip by assuming every geometric structure to be real, smooth and well defined globally. Minor technical details will be specified when strictly needed.

Additionally, this section provides a survey in other many differential geometric structures that will appear during this thesis. In particular, we will describe symplectic, presymplectic, Poisson, Dirac and Jacobi manifolds. In following chapters we will illustrate how these structures will allow us to describe constants of motion, symmetries, superposition rules and other properties of differential equations.

1 Tangent spaces, vector fields and integral curves

Let N be an n -dimensional differentiable manifold and let $\{y_i \mid i = 1, \dots, n\}$ be a coordinate system on N . This allows us to denote each point $y \in N$ by an n -tuple $(y_1, \dots, y_n) \in \mathbb{R}^n$. We assume ϵ to be a coordinate system on \mathbb{R} .

We write curves in N in the form $\gamma : \epsilon \in \mathbb{R} \mapsto \gamma(\epsilon) = (\gamma_1(\epsilon), \dots, \gamma_n(\epsilon)) \in N$. The space of curves in N can be endowed with an equivalence relation R given by

$$\gamma R \bar{\gamma} \iff \gamma(0) = \bar{\gamma}(0) \quad \text{and} \quad \frac{d\gamma_i}{d\epsilon}(0) = \frac{d\bar{\gamma}_i}{d\epsilon}(0), \quad i = 1, \dots, n. \quad (1.1)$$

In other words, two curves in N are related if their first-order Taylor polynomials around $\epsilon = 0$ coincide. Observe that the above equivalence relation is independent of the chosen coordinate system. That is, if $\gamma R \bar{\gamma}$ for the coordinate systems ϵ on \mathbb{R} and $\{y_1, \dots, y_n\}$ on N , then $\gamma R \bar{\gamma}$ also for any other coordinate systems on \mathbb{R} and N . Hence, the equivalence relation (1.1) is independent of the chosen coordinate system and it therefore has a geometrical meaning.

Every equivalence class of R is called a *tangent vector*. We write v_y for an equivalence class of curves passing through $y \in N$ for $\epsilon = 0$. We call *tangent space at $y \in N$* the space $T_y N$ of equivalence classes of the form v_y . In view of (1.1) each tangent vector $v_y \in T_y N$ can be identified with a set of numbers (v_1, \dots, v_n) and vice versa. Hence, the tangent space to each point $y \in N$ can be written as

$$T_y N = \{(v_1, \dots, v_n) \mid (v_1, \dots, v_n) \in \mathbb{R}^n\}. \quad (1.2)$$

The space $T_y N$ is naturally endowed with an n -dimensional vectorial space structure and there exists an isomorphism $T_y N \simeq \mathbb{R}^n$. Consequently, given a local coordinate system $\{y_1, \dots, y_n\}$ on N around $y \in N$, we can identify each vector $(v_1, \dots, v_n) \in \mathbb{R}^n$ with the tangent vector v_y associated to those curves $\gamma : \epsilon \in \mathbb{R} \mapsto (\gamma_1(\epsilon), \dots, \gamma_n(\epsilon)) \in N$ satisfying that $d\gamma_i/d\epsilon(0) = v_i$ for $i = 1, \dots, n$. It is worth noting that the above identification depends on the chosen coordinate systems due to the fact that $\gamma_i(\epsilon) = y_i(\gamma(\epsilon))$ for $i = 1, \dots, n$ and $\epsilon \in \mathbb{R}$.

Alternatively, tangent vectors can be understood as derivations on a certain \mathbb{R} -algebra, i.e., a real vector space along with an \mathbb{R} -bilinear commutative multiplication. More specifically, let $C^\infty(U)$ be the space of differentiable functions defined on an open subset $U \subset N$ containing y . The space $C^\infty(U)$ admits an equivalence relation R_g given by

$$f R_g g \iff f, g \text{ coincide on an open subset containing } y.$$

The equivalence classes of this relation, let us say $[f]$ with $f \in C^\infty(U)$, are called *germs*. It is easy to verify that the quotient space, $C_g^\infty(U)$, related to the equivalence relation R_g becomes a vectorial space with the sum $[f] + [g] = [f + g]$ and scalar

multiplication by scalars $\lambda \in \mathbb{R}$ given by $\lambda[f] = [\lambda f]$. Additionally, $C^\infty(U)$ forms an \mathbb{R} -algebra with the \mathbb{R} -bilinear commutative multiplication $[f][g] = [fg]$.

Each v_y induces a mapping $D_{v_y} : [f] \in C_g^\infty(U) \mapsto D_{v_y}[f] \in \mathbb{R}$ given by

$$D_{v_y}[f] = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} f \circ \gamma, \quad \gamma \in v_y. \quad (1.3)$$

Since the first-order Taylor polynomials around 0 of each $\gamma \in v_y$ coincide by definition of v_y , the above mapping is well defined and it does not depend on the chosen $\gamma \in v_y$. It can be proved that each D_{v_y} is a derivation on $C_g^\infty(U)$, i.e., D_{v_y} is linear and satisfies the Leibniz rule $[D_{v_y}([f][g])] = D_{v_y}([f])g(y) + f(y)D_{v_y}([g])$ for each $[f], [g] \in C_g^\infty(U)$. Conversely, it can be proved that every derivation on $C_g^\infty(U)$ is of the form (1.3) for a certain $v_y \in \mathbb{T}_y N$. This motivates to identify each v_y with a derivation D_{v_y} and vice versa. In particular, the equivalence class v_{v_y} related to the vector $(0, \dots, 1 (i\text{-position}), \dots, 0) \in \mathbb{R}^n$ induces the derivation $(\partial/\partial y_i)_y : [f] \in C_g^\infty(U) \mapsto \frac{\partial f}{\partial y_i}(y) \in \mathbb{R}$.

We call *tangent space* to N the space $\mathbb{T}N \equiv \bigcup_{y \in N} \mathbb{T}_y N$. We can endow $\mathbb{T}N$ with a differentiable structure. To do so, we related each coordinate system $\{y_1, \dots, y_n\}$ on N to a coordinate system on $\mathbb{T}N$ of the form

$$y_i(v_y) = y_i(y), \quad v_i(v_y) = \frac{d\gamma_i}{d\epsilon}(0), \quad i = 1, \dots, n, \quad (1.4)$$

where $\epsilon \mapsto (\gamma_1(\epsilon), \dots, \gamma_n(\epsilon))$ is a curve belonging to the class v_y . This gives rise to a coordinate system $\{y_1, \dots, y_n, v_1, \dots, v_n\}$ on $\mathbb{T}N$. In consequence, if N is an n -dimensional manifold, then $\mathbb{T}N$ becomes a $2n$ -dimensional manifold. We call *tangent bundle* the bundle $(\mathbb{T}N, N, \pi_N)$, where $\pi_N : v_y \in \mathbb{T}N \mapsto y \in N$.

A *vector field* on N is a section of the tangent bundle $\mathbb{T}N$, i.e., a mapping $X : N \rightarrow \mathbb{T}N$ such that $\pi_N \circ X = \text{Id}_N$, with Id_N being the identity on N . In other words, a vector field is a mapping assigning to every $y \in N$ a tangent vector $v_y \in \mathbb{T}_y N$. In the coordinate system $\{y_1, \dots, y_n\}$, a vector field can be brought into the form

$$X(y) = \sum_{i=1}^n \eta_i(y) \frac{\partial}{\partial y_i}, \quad (1.5)$$

for certain functions $\eta_i \in C^\infty(N)$ with $i = 1, \dots, n$.

An *integral curve* $\gamma : \epsilon \in \mathbb{R} \mapsto \gamma(\epsilon) \in N$ for X with initial condition $y_0 \in N$ is the curve $\gamma : \epsilon \in \mathbb{R} \mapsto \gamma(\epsilon) \in N$ satisfying that

$$\begin{cases} \gamma(0) = y_0, \\ \frac{d\gamma_i}{d\epsilon}(\epsilon) = \eta_i(\gamma(\epsilon)), \end{cases} \quad \forall \epsilon \in \mathbb{R}, \quad i = 1, \dots, n. \quad (1.6)$$

In view of the theorem of existence and uniqueness of solutions of systems of first-order ordinary differential equations, the solution to the above Cauchy problem

exists and is unique. So, there exists a unique integral curve of a vector field at each point.

Observe that if $\gamma : \epsilon \in \mathbb{R} \mapsto (\gamma_1(\epsilon), \dots, \gamma_n(\epsilon)) \in N$ is a particular solution to system (1.6), then the coordinate expression for γ in other coordinate system is a particular solution for (1.6) in the new coordinate system. This justifies to write system (1.6) in an intrinsic form as

$$\begin{cases} \gamma(0) = y_0, \\ \frac{d\gamma}{d\epsilon}(\epsilon) = X(\gamma(\epsilon)), \end{cases} \quad (1.7)$$

where $d\gamma/d\epsilon(\epsilon_0)$ is understood to be the tangent vector associated with γ at $\gamma(\epsilon_0)$, i.e., the equivalence class related to the curve $\epsilon \mapsto \bar{\gamma}(\epsilon) \equiv \gamma(\epsilon + \epsilon_0)$. Therefore, an integral curve γ of a vector field X on N is a curve in N whose tangent vector at each point $y \in \text{Im } \gamma$ coincides with $X(y)$. We write γ_{y_0} for the integral curve of X with initial condition $\gamma_{y_0}(0) = y_0 \in N$.

By solving system (1.6) for each $y_0 \in N$, we obtain a family of integral curves $\{\gamma_{y_0}\}_{y_0 \in N}$. This gives rise to the so-called *flow* of X , namely the mapping $\Phi : (\epsilon, y) \in \mathbb{R} \times N \mapsto \gamma_y(\epsilon) \in N$. Since $X(y) = d\gamma_y/d\epsilon(0)$ for each $y \in N$, the flow uniquely determines the vector field X .

The flow allows us to define an ϵ -parametric set of diffeomorphisms $\Phi_\epsilon : y \in N \mapsto \Phi(\epsilon, y) \equiv \gamma_y(\epsilon) \in N$. Observe that $\gamma_{\gamma_y(\bar{\epsilon})}(\epsilon) = \gamma_y(\epsilon + \bar{\epsilon}), \forall \epsilon, \bar{\epsilon} \in \mathbb{R}, \forall y \in N$. Indeed, fixed $\bar{\epsilon}$, the curve $\epsilon \mapsto \gamma_y(\epsilon + \bar{\epsilon})$ takes the value $\gamma_y(\bar{\epsilon})$ for $\epsilon = 0$ and, defining $\tilde{\epsilon} \equiv \epsilon + \bar{\epsilon}$, we obtain

$$\frac{d\gamma_y(\epsilon + \bar{\epsilon})}{d\epsilon} = \frac{d\gamma_y(\tilde{\epsilon})}{d\tilde{\epsilon}} = X(\gamma_y(\tilde{\epsilon})) = X(\gamma_y(\epsilon + \bar{\epsilon})), \quad \gamma_y(0 + \bar{\epsilon}) = \gamma_y(\bar{\epsilon}). \quad (1.8)$$

Therefore, the curve $\epsilon \mapsto \gamma_y(\epsilon + \bar{\epsilon})$ is an integral curve of X taking the value $\gamma_y(\bar{\epsilon})$ for $\epsilon = 0$. So, it must coincide with $\gamma_{\gamma_y(\bar{\epsilon})}(\epsilon)$ and $\gamma_{\gamma_y(\bar{\epsilon})}(\epsilon) = \gamma_y(\epsilon + \bar{\epsilon})$. In consequence, we have

$$\Phi_\epsilon(\Phi_{\bar{\epsilon}}(y)) = \Phi_\epsilon(\gamma_y(\bar{\epsilon})) = \gamma_{\gamma_y(\bar{\epsilon})}(\epsilon) = \gamma_y(\epsilon + \bar{\epsilon}) = \Phi_{\epsilon + \bar{\epsilon}}(y), \quad \forall y \in N, \quad \epsilon, \bar{\epsilon} \in \mathbb{R}. \quad (1.9)$$

On the other hand, $\Phi_0(y) = \gamma_y(0) = y$ for all $y \in N$. Thus,

$$\Phi_\epsilon \circ \Phi_{\bar{\epsilon}} = \Phi_{\epsilon + \bar{\epsilon}}, \quad \Phi_0 = \text{Id}_N, \quad \forall \epsilon, \bar{\epsilon} \in \mathbb{R}, \quad (1.10)$$

where Id_N is the identity on N . The properties (1.10) tell us that $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ is an ϵ -parametric group of diffeomorphisms on N . Indeed, the composition of elements of that family belongs to the family in virtue of (1.10). The element $\Phi_0 = \text{Id}_N$ is the neutral element and $\Phi_{-\epsilon} \circ \Phi_\epsilon = \text{Id}_N$. In this way, Φ_ϵ has inverse $\Phi_{-\epsilon}$, which makes each Φ_ϵ , with $\epsilon \in \mathbb{R}$, into a diffeomorphism. We can also understand the

uniparametric group of diffeomorphisms $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ as a *Lie group action* $\Phi : (\epsilon, y) \in \mathbb{R} \times N \mapsto \Phi_\epsilon(y) \in N$.

Alternatively, every ϵ -parametric group of diffeomorphisms $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ on N enables us to define a vector field on N of the form

$$X_\Phi(y) \equiv \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi_\epsilon(y), \quad \forall y \in N, \quad (1.11)$$

whose integral curves are $\gamma_y(\epsilon) = \Phi_\epsilon(y)$, with $y \in N$. If $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ is coming from the flow of a vector field X , then $X_\Phi = X$. This shows that each vector field is equivalent to an ϵ -parametric group of diffeomorphisms.

Assuming X to be given by (1.5), the corresponding flow gives rise to a set of ϵ -parametric diffeomorphisms which, for infinitesimal ϵ , takes the form

$$y_i \mapsto y_i + \epsilon \eta_i(y) + O(\epsilon^2), \quad i = 1, \dots, n, \quad (1.12)$$

where $O(\epsilon^2)$ stands, as usual, for a function of ϵ satisfying that $\lim_{\epsilon \rightarrow 0} O(\epsilon^2)/\epsilon = 0$. This transformation is invertible for each fixed value of the parameter ϵ . The set of all transformations for different values of ϵ forms a *group*.

2 Time-dependent vector fields

In order to study systems of non-autonomous first-order ordinary differential equations from a geometrical viewpoint, we introduce t -dependent vector fields. In this section we present some of their basic properties.

Let $\pi_2 : (t, y) \in \mathbb{R} \times N \mapsto y \in N$ be the projection onto the second factor with t being the natural coordinate system on \mathbb{R} . We now write t for the variable in \mathbb{R} because this variable mostly stands for the physical time in the applications described in this thesis. A *t -dependent vector field* X on N is a map $X : (t, y) \in \mathbb{R} \times N \mapsto X(t, y) \in TN$ satisfying the commutative diagram

$$\begin{array}{ccc} & & TN \\ & \nearrow X & \downarrow \pi_N \\ \mathbb{R} \times N & \xrightarrow{\pi_2} & N \end{array}$$

that is, $\pi_N \circ X = \pi_2$. Hence, $X(t, y) \in \pi_N^{-1}(y) = T_y N$ and $X_t : y \in N \mapsto X_t(y) \equiv X(t, y) \in T_y N \subset TN$ is a vector field on N for every $t \in \mathbb{R}$. Conversely, every t -parametric family $\{X_t\}_{t \in \mathbb{R}}$ of vector fields on N gives rise to a unique t -dependent vector field $X : \mathbb{R} \times N \mapsto X_t \in TN$. Thus, each t -dependent vector field X is equivalent to a family $\{X_t\}_{t \in \mathbb{R}}$ of vector fields on N .

The t -dependent vector fields enable us to describe, as a particular instance, standard vector fields. Indeed, every vector field $X : y \in N \mapsto X(y) \in TN$ can naturally be regarded as a t -dependent vector field $X : (t, y) \in \mathbb{R} \times N \mapsto X(y) \in TN$.

Conversely, a ‘constant’ t -dependent vector field X on N , i.e., $X_t = X_{t'}$ for every $t, t' \in \mathbb{R}$, can be understood as a vector field $X = X_0$ on N .

Each t -dependent vector field X is equivalent to a linear morphism $X : f \in C^\infty(N) \mapsto (Xf) \in C^\infty(\mathbb{R} \times N)$, with $(Xf)(t, y) \equiv (X_t f)(y)$ for every $(t, y) \in \mathbb{R} \times N$, satisfying a Leibniz rule, namely $[X(fg)](t, y) = (X_t f)(y)g(y) + f(y)(X_t g)(y)$ for arbitrary $f, g \in C^\infty(N)$ and $(t, y) \in \mathbb{R} \times N$.

Similarly to vector fields, t -dependent vector fields also admit local integral curves. To define them, we make use of the so-called autonomization of X . The *autonomization* of a vector field X on N is the only vector field \bar{X} on $\mathbb{R} \times N$ satisfying that $\iota_{\bar{X}} dt = 1$ and $(\bar{X}\pi_2^* f)(t, y) = (Xf)(t, y)$ for an arbitrary function $f \in C^\infty(N)$ and $(t, y) \in \mathbb{R} \times N$. In coordinates, if $X = \sum_{i=1}^n X_i(t, y)\partial/\partial y_i$, then

$$\bar{X}(t, y) = \frac{\partial}{\partial t} + \sum_{i=1}^n X_i(t, y) \frac{\partial}{\partial y_i}. \quad (2.1)$$

Now we say that $\gamma : \mathbb{R} \rightarrow \mathbb{R} \times N$ is an integral curve of the t -dependent vector field X if γ is an integral curve of \bar{X} . If $\gamma : s \in \mathbb{R} \mapsto (t(s), y(s)) \in \mathbb{R} \times N$ is an integral curve of X , then

$$\begin{cases} \frac{dy_i}{ds} = X_i(t, y), \\ \frac{dt}{ds} = 1, \end{cases} \quad i = 1, \dots, n. \quad (2.2)$$

By using the reparametrization $t = t(s)$, we obtain

$$\frac{dy_i}{dt} = X_i(t, y), \quad i = 1, \dots, n. \quad (2.3)$$

The particular solutions to this system can be considered as integral curves of X with a preferred parametrization $t \rightarrow (t, y(t))$. Above system is called the *associated system* to X . Conversely, given a first-order system in normal form (2.3), we can define a t -dependent vector field of the form $X(t, y) = \sum_{i=1}^n X_i(t, y)\partial/\partial y_i$ whose integral curves $t \rightarrow (t, y(t))$ are the particular solutions to (2.3). This motivates to employ X to represent a nonautonomous system of ordinary differential equations and its related t -dependent vector field.

Every t -dependent vector field X on N gives rise to its *generalized flow* g^X , i.e., the map $g^X : \mathbb{R} \times N \rightarrow N$ such that $g^X(t, y) \equiv g_t^X(y) = \gamma_y(t)$ with γ_y being the particular solution to X such that $\gamma_y(0) = y$. We already showed that for X being a standard vector field the particular solutions satisfy that $\gamma_{\gamma_y(s)}(t) = \gamma_y(t + s)$ for each $t, s \in \mathbb{R}$ and $y \in N$. As a consequence, the generalized flow for the t -dependent vector field X associated to an autonomous vector field leads to a uni-parametric group of diffeomorphisms $\{g_t^X\}_{t \in \mathbb{R}}$. If X is not related to an autonomous vector field, this property is not longer valid.

Given a fiber vector bundle $\text{pr} : P \rightarrow N$, we denote by $\Gamma(\text{pr})$ the $C^\infty(N)$ -module of its smooth sections. So, if $\tau_N : TN \rightarrow N$ and $\pi_N : T^*N \rightarrow N$ are the canonical projections associated with the tangent and cotangent bundle to N , respectively, then $\Gamma(\tau_N)$ and $\Gamma(\pi_N)$ designate the $C^\infty(N)$ -modules of vector fields and one-forms on N , correspondingly.

We call *generalized distribution* \mathcal{D} on N , a correspondence relating each $y \in N$ to a linear subspace $\mathcal{D}_y \subset T_yN$. A generalized distribution is said to be *regular at* $y' \in N$ when the function $r : y \in N \mapsto \dim \mathcal{D}_y \in \mathbb{N} \cup \{0\}$ is locally constant around y' . Similarly, \mathcal{D} is regular on an open $U \subset N$ when r is constant on U . Finally, a vector field $Y \in \Gamma(\tau_N)$ is said to take values in \mathcal{D} , in short $Y \in \mathcal{D}$, when $Y_y \in \mathcal{D}_y$ for all $y \in N$. Likewise, similar notions can be defined for a *generalized codistribution*, namely a correspondence mapping relating every $y \in N$ to a linear subspace of T_y^*N .

It will be very important to our purposes to relate t -dependent vector fields to the so-called Lie algebras as follows. A *Lie algebra* is a pair $(V, [\cdot, \cdot])$, where V stands for a real linear space endowed with a Lie bracket $[\cdot, \cdot] : V \times V \rightarrow V$, namely an \mathbb{R} -bilinear antisymmetric mapping satisfying the Jacobi identity. Given two subsets $\mathcal{A}, \mathcal{B} \subset V$, we write $[\mathcal{A}, \mathcal{B}]$ for the real vector space spanned by the Lie brackets between elements of \mathcal{A} and \mathcal{B} , and we define $\text{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ to be the smallest Lie subalgebra of V containing \mathcal{B} . Note that $\text{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ is expanded by

$$\mathcal{B}, [\mathcal{B}, \mathcal{B}], [\mathcal{B}, [\mathcal{B}, \mathcal{B}]], [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{B}]]], [[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]], \dots \quad (2.4)$$

From now on, we use $\text{Lie}(\mathcal{B})$ and V to represent $\text{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ and $(V, [\cdot, \cdot])$, correspondingly, when their meaning is clear from the context.

Given a t -dependent vector field X , we call *minimal algebra* of X the smallest Lie algebra, V^X , of vector fields (relative to the Lie bracket of vector fields) containing all the vector fields $\{X_t\}_{t \in \mathbb{R}}$, namely $V^X = \text{Lie}(\{X_t\}_{t \in \mathbb{R}})$.

Given a t -dependent vector field X on N , its *associated distribution*, \mathcal{D}^X , is the generalized distribution on N spanned by the vector fields of V^X , i.e.,

$$\mathcal{D}_x^X = \{Y_x \mid Y \in V^X\} \subset T_xN, \quad (2.5)$$

and its *associated co-distribution*, \mathcal{V}^X , is the generalized co-distribution on N of the form

$$\mathcal{V}_x^X = \{\vartheta \in T_x^*N \mid \vartheta(Z_x) = 0, \forall Z_x \in \mathcal{D}_x^X\} = (\mathcal{D}_x^X)^\circ \subset T_x^*N, \quad (2.6)$$

where $(\mathcal{D}_x^X)^\circ$ is the *annihilator* of \mathcal{D}_x^X .

Proposition 1. *A function $f : U \rightarrow \mathbb{R}$ is a local t -independent constant of motion for a system X if and only if $df \in \mathcal{V}^X|_U$.*

Proof. If f is assumed to be a t -independent constant of motion, then $X_t f|_U = df(X_t)|_U = 0$ for all $t \in \mathbb{R}$. Consequently, df also vanishes on the successive Lie brackets of elements from $\{X_t\}_{t \in \mathbb{R}}$ and hence

$$df(Y)|_U = Yf|_U = 0, \quad \forall Y \in \text{Lie}(\{X_t\}_{t \in \mathbb{R}}). \quad (2.7)$$

Since the elements of V^X span the generalised distribution \mathcal{D}^X , then $df_x(Z_x) = 0$ for all $x \in U$ and $Z_x \in \mathcal{D}_x^X$, i.e., $df \in \mathcal{V}^X|_U$. The converse directly follows from the above considerations. \square

The following lemma can easily be proved [109].

Lemma 2. *Given a system X , its associated co-distribution \mathcal{V}^X admits a local basis around every $x \in U^X$ of the form $df_1, \dots, df_{p(x)}$, with $p(x) = r^X(x)$ and $f_1, \dots, f_{p(x)} : U \subset U^X \rightarrow \mathbb{R}$ being a family of (local) t -independent constants of motion for X . Furthermore, the \mathbb{R} -linear space $\mathcal{I}^X|_U$ of t -independent constants of motion of X on U can be written as*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{p(x)} \rightarrow \mathbb{R}, g = F(f_1, \dots, f_{p(x)})\}. \quad (2.8)$$

3 Higher-order tangent spaces and related notions

Given two curves $\rho, \sigma : \mathbb{R} \mapsto N$ such that $\rho(0) = \sigma(0) = y \in N$, we say that they have a *contact of order p at y* , with $p \in \mathbb{N} \cup \{0\}$, if they satisfy

$$\frac{d^p(f \circ \rho)}{dt^p}(0) = \frac{d^p(f \circ \sigma)}{dt^p}(0), \quad (3.1)$$

for every function $f \in C^\infty(N)$. The relation ‘to have a contact of order p at y ’ is an equivalence relation. Observe that this relation is purely geometrical, i.e., if two curves are related with respect to a certain coordinate system, then they do so in any other coordinate system. Note also that the above relation amounts to saying that the two curves have the same Taylor expansion around 0 up to order p .

Each equivalence class of the previous equivalence relation, let us say \mathbf{t}_y^p , is called a *p -tangent vector* at y . More specifically, \mathbf{t}_y^p stands for an equivalence class of contacts of order p with a curve $\sigma(t)$ with $\sigma(0) = y$. We write $\mathbb{T}_y^p N$ for the space of all p -tangent vectors at p and we define

$$\mathbb{T}^p N = \bigcup_{y \in N} \mathbb{T}_y^p N.$$

It can be proved that $\mathbb{T}^p N$ can be endowed with a differential structure turning it into a differential manifold. Additionally, $(\mathbb{T}^p N, \pi, N)$, with $\pi : \mathbf{t}_y^p \in \mathbb{T}^p N \mapsto y \in N$, is a fiber bundle. Let us briefly analyze these facts.

Every coordinate system t on \mathbb{R} and $\{y_1, \dots, y_n\}$ on N induces a natural coordinate system on the space $\mathbb{T}^p N$. Indeed, consider again a curve $\rho : t \in \mathbb{R} \mapsto \rho(t) \in N$ with coordinates $\rho_1(t), \dots, \rho_n(t)$. The p -tangent vector, $\mathbf{t}_{\rho(0)}^p$, associated with this curve admits a representative

$$\rho_i(0) + \frac{t}{1!} \frac{d\rho_i}{dt}(0) + \dots + \frac{t^p}{p!} \frac{d^p \rho_i}{dt^p}(0), \quad i = 1, \dots, n, \quad (3.2)$$

which can be characterized by the coefficients

$$y_i(0) = \rho_i(0), \quad y_i^1(0) = \frac{d\rho_i}{dt}(0), \quad \dots, \quad y_i^p(0) = \frac{d^p \rho_i}{dt^p}(0), \quad i = 1, \dots, n. \quad (3.3)$$

In consequence, the mapping $\varphi : \mathbf{t}_{y(0)}^p \in T^p N \mapsto (y_i(0), y_i^1(0), \dots, y_i^p(0)) \in \mathbb{R}^{(n+1)p}$ gives a coordinate system for $T^p N$. Obviously, the map π becomes a smooth submersion which makes $T^p N$ into a fiber bundle with base N . We hereby denote each element of $T^p N$ by $\mathbf{t}_y^p = (y, y^1, \dots, y^p)$.

Now, given a curve $c : t \in \mathbb{R} \mapsto c(t) \in N$, we call *prolongation to $T^p N$ of c* the curve $\mathbf{t}^p c : t \in \mathbb{R} \mapsto \mathbf{t}^p c(t) \in T^p N$, associating with every t_0 the corresponding equivalence class of $c(t + t_0)$ given in coordinates by

$$\mathbf{t}^p c(t_0) = \left(c(t_0), c^1(t_0), \dots, c^p(t_0) \right). \quad (3.4)$$

4 Jet bundles

Jet bundles are certain types of bundles constructed out of the sections of a fiber bundle. Our interest in them is due to the geometrical description of systems of higher-order ordinary and partial differential equations and their Lie symmetries as structures on an appropriate jet bundle.

For simplicity, consider a projection $\pi : (x, u) \in \mathbb{R}^n \times N \equiv N_{\mathbb{R}^n} \mapsto x \in \mathbb{R}^n$ giving rise to a trivial bundle $(N_{\mathbb{R}^n}, \mathbb{R}^n, \pi)$. Let us hereafter assume N to be a k -dimensional manifold and let $\{x_1, \dots, x_n\}$ be a global coordinate system on \mathbb{R}^k .

We say that two sections $\sigma_1, \sigma_2 : \mathbb{R}^n \rightarrow N_{\mathbb{R}^n}$ are *p -equivalent at a point $x \in \mathbb{R}^n$* or they have a *contact of order p at x* if they have the same Taylor expansion of order p at $x \in \mathbb{R}^n$. Equivalently,

$$\sigma_1(x) = \sigma_2(x), \quad \frac{\partial^{|\mathbf{J}|}(\sigma_1)_j}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x) = \frac{\partial^{|\mathbf{J}|}(\sigma_2)_j}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x), \quad (4.1)$$

for every multi-index $\mathbf{J} = (j_1, \dots, j_n)$ such that $0 < |\mathbf{J}| \equiv j_1 + \dots + j_n \leq p$. Being p -equivalent induces an equivalence relation in the space $\Gamma(\pi)$ of sections of the bundle $(N_{\mathbb{R}^n}, \mathbb{R}^n, \pi)$. Observe that if two sections have a contact of order p at a point x , then they do have a contact at that point of the same type for any other coordinate systems on \mathbb{R}^n and N , i.e., this equivalence relation is geometric.

We write $j_x^p \sigma$ for the equivalence class of sections that have a *contact of p -order at $x \in \mathbb{R}^n$* with a section σ . Every such an equivalence class is called a *p -jet*. We write $J_x^p \pi$ for the space of all jets of order p of sections at x . We will denote by $J^p \pi$ the space of all jets of order p . Alternatively, we will write $J^p(\mathbb{R}^n, \mathbb{R}^k)$ for the jet bundle of sections of the bundle $\pi : (x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mapsto x \in \mathbb{R}^n$.

Given a section $\sigma : \mathbb{R}^n \rightarrow \mathbb{J}^p\pi$, we can define the functions

$$(u_j)_J(j_x^p\sigma) = \frac{\partial^{|J|}\sigma_j}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x), \quad \forall j, \quad |J| \leq p. \quad (4.2)$$

For $|J| = 0$, we define $u_J(x) \equiv u(x)$. Coordinate systems on \mathbb{R}^n and N along with the previous functions give rise to a local coordinate system on $\mathbb{J}^p\pi$. We will also hereafter denote the n -tuple and k -tuple, respectively, by $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_k)$, then

$$(u_j)_J = u_{x_{i_1}^{j_1} \dots x_{i_n}^{j_n}} = \frac{\partial^{|J|}u_j}{\partial x_{i_1}^{j_1} \dots \partial x_{i_n}^{j_n}}, \quad \forall j, \quad |J| \leq 0. \quad (4.3)$$

All such local coordinate systems give rise to a manifold structure on $\mathbb{J}^p\pi$. In this way, every point of $\mathbb{J}^p\pi$ can be written as

$$\left(x_i, u_j, (u_j)_{x_i}, (u_j)_{x_{i_1}^{j_1} x_{i_2}^{2-j_1}}, (u_j)_{x_{i_1}^{j_1} x_{i_2}^{j_2} x_{i_3}^{3-j_1-j_2}}, \dots, (u_j)_{x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_n}^{p-\sum_{i=1}^{n-1} j_i}} \right), \quad (4.4)$$

where the numb indices run $i_1, \dots, i_p = 1, \dots, n$, $j = 1, \dots, k$, $j_1 + \dots + j_n \leq p$.

For small values of p , jet bundles have simple descriptions: $\mathbb{J}^0\pi = N_{\mathbb{R}^n}$ and $\mathbb{J}^1\pi \simeq \mathbb{R}^n \times \text{TN}$.

The projections $\pi_{p,l} : j_x^p\sigma \in \mathbb{J}^p\pi \mapsto j_x^l\sigma \in \mathbb{J}^l\pi$ with $l < p$ lead to define the smooth bundles $(\mathbb{J}^p\pi, \mathbb{J}^l\pi, \pi_{p,l})$. Conversely, for each section $\sigma : \mathbb{R}^n \rightarrow N_{\mathbb{R}^n}$, we have a natural embedding $j_x^p\sigma : \mathbb{R}^n \ni x \mapsto j_x^p\sigma \in \mathbb{J}^p\pi$.

Consider now a vector field on $N_{\mathbb{R}^n}$ of the form

$$X(x, u) = \sum_{i=1}^n \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^k \eta_j(x, u) \frac{\partial}{\partial u_j}. \quad (4.5)$$

This gives rise to an ϵ -parametric group of transformations

$$\begin{cases} x_i \mapsto x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \\ u_j \mapsto u_j + \epsilon \eta_j(x, u) + O(\epsilon^2), \end{cases} \quad i = 1, \dots, n, \quad j = 1, \dots, k. \quad (4.6)$$

We can extend this infinitesimal transformation up to p -order derivatives

$$\begin{cases} \bar{x}_i = x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \\ \bar{u}_j = u_j + \epsilon \eta_j(x, u) + O(\epsilon^2), \\ (\bar{u}_j)_J = (u_j)_J + \epsilon (\eta_j)_J(x, u) + O(\epsilon^2), \end{cases} \quad (4.7)$$

for $i = 1, \dots, n$, $j = 1, \dots, k$, $0 < |J| \leq p$ and $(u_j)_J$ represents (4.3) and the $(\eta_j)_J$ are given in the forthcoming steps.

In coordinates, the above ϵ -parametric group of diffeomorphisms has the associated vector field

$$X^p = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k \eta_j \frac{\partial}{\partial u_j} + \sum_{0 < |J| \leq p} \sum_{j=1}^k \mathfrak{Pr}_J \eta_j \frac{\partial}{\partial (u_j)_J}, \quad (4.8)$$

where $\mathfrak{Pr}_J \eta_j$ denotes the prolongations of η_j for the multi-index J .

Lemma 1. *Let $\{\Phi_\epsilon : (x, u) \in N_{\mathbb{R}^n} \mapsto (\bar{x}, \bar{u}) \in N_{\mathbb{R}^n}\}_{\epsilon \in \mathbb{R}}$ be the one-parametric group of transformations induced by the vector field $X = \sum_{i=1}^n \xi_i(t, u) \partial / \partial x_i + \sum_{j=1}^k \eta_j(t, u) \partial / \partial u_j$ and let $\sigma : s \in \mathbb{R}^n \mapsto (x(s), u(s)) \in N_{\mathbb{R}^n}$ be a section of the bundle $\pi : N_{\mathbb{R}^k} \rightarrow \mathbb{R}^k$. The section $\bar{\sigma}_\epsilon = \Phi_\epsilon \circ \sigma$ has slopes $\partial \bar{u}_j / \partial \bar{x}_i = \partial u_j / \partial x_i + \epsilon (\eta_j)_{x_i} + O(\epsilon^2)$, for any two fixed values of $1 \leq i \leq n$ and $1 \leq j \leq k$, where*

$$\mathfrak{Pr}_{J \equiv x_i} \eta_j = (\eta_j)_{x_i} = \frac{D\eta_j}{Dx_i} - \sum_{q=1}^n (u_j)_{x_q} \frac{D\xi_q}{Dx_i}, \quad (4.9)$$

and D/Dx_i is the total derivative with respect to x_i , namely

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{l=1}^k (u_l)_{x_i} \frac{\partial}{\partial u_l}. \quad (4.10)$$

Proof. Observe that

$$\delta \bar{u}_j = \delta u_j + \epsilon \delta \eta_j + O(\epsilon^2) = \sum_{i=1}^n \left[\frac{\partial u_j}{\partial x_i} + \epsilon \left(\frac{\partial \eta_j}{\partial x_i} + \sum_{l=1}^k \frac{\partial \eta_j}{\partial u_l} \frac{\partial u_l}{\partial x_i} \right) \right] \delta x_i + O(\epsilon^2) \quad (4.11)$$

and

$$\delta \bar{x}_i = \delta x_i + \epsilon \delta \xi_i + O(\epsilon^2) = \sum_{m=1}^n \left[\delta_m^i + \epsilon \left(\frac{\partial \xi_i}{\partial x_m} + \sum_{l=1}^k \frac{\partial \xi_i}{\partial u_l} \frac{\partial u_l}{\partial x_m} \right) \right] \delta x_m + O(\epsilon^2). \quad (4.12)$$

Using the operator defined in (4.10), we can rewrite these two expressions as

$$\delta \bar{u}_j = \sum_{l=1}^k \left((u_j)_{x_l} + \epsilon \frac{D\eta_j}{Dx_l} \right) \delta x_l + O(\epsilon^2), \quad \delta \bar{x}_i = \sum_{m=1}^n \left(\delta_m^i + \epsilon \frac{D\xi_i}{Dx_m} \right) \delta x_m + O(\epsilon^2). \quad (4.13)$$

Hence,

$$(\bar{u}_j)_{\bar{x}_i} = \frac{\sum_{l=1}^k \left((u_j)_{x_l} + \epsilon \frac{D\eta_j}{Dx_l} \right)}{\sum_{m=1}^n \left(\delta_m^i + \epsilon \frac{D\xi_i}{Dx_m} \right)} \delta_m^l + O(\epsilon^2)$$

$$\begin{aligned}
&= \sum_{l=1}^n \left((u_j)_{x_l} + \epsilon \frac{D\eta_j}{Dx_l} + O(\epsilon^2) \right) \cdot \left(\delta_i^l - \epsilon \frac{D\xi_l}{Dx_i} + O(\epsilon^2) \right) \\
&= (u_j)_{x_i} + \epsilon \left(\frac{D\eta_j}{Dx_i} - \sum_{q=1}^n (u_j)_{x_q} \frac{D\xi_q}{Dx_i} \right) + O(\epsilon^2), \tag{4.14}
\end{aligned}$$

which finishes the proof. \square

In a similar way as we deduced the expression for the first-order prolongation, we can deduce it for higher-order prolongations. We will deduce the expression by generalization of results in lower dimensional cases.

Lemma 2. *Let $\sigma : s \in \mathbb{R}^n \mapsto (x(s), u(s)) \in N_{\mathbb{R}^n}$ and let $\Phi_\epsilon : (x, u) \in N_{\mathbb{R}^n} \mapsto (\bar{x}, \bar{u}) \in N_{\mathbb{R}^n}$ be a one-parametric group of transformations induced by the vector field $X = \sum_{i=1}^n \xi_i(t, u) \partial / \partial x_i + \sum_{j=1}^k \eta_j(t, u) \partial / \partial u_j$. We obtain that the section $\bar{\sigma}_\epsilon = \Phi_\epsilon \circ \sigma$ of $\pi : \mathbb{R}^k \times N \rightarrow \mathbb{R}^k$ has slope $\partial^2 \bar{u}_j / \partial \bar{x}_{i_1} \partial \bar{x}_{i_2} = \partial^2 u_j / \partial x_{i_1} \partial x_{i_2} + \epsilon (\eta_j)_{x_{i_1} x_{i_2}} + O(\epsilon^2)$, where*

$$\begin{aligned}
(\eta_j)_{x_{i_1} x_{i_2}} &= \frac{D^2 \eta_j}{Dx_{i_1} Dx_{i_2}} - \sum_{l=1}^n (u_j)_{x_{i_1} x_l} \frac{D\xi_l}{Dx_{i_2}} \\
&\quad - \sum_{l=1}^n (u_j)_{x_{i_2} x_l} \frac{D\xi_l}{Dx_{i_1}} - \sum_{l=1}^n (u_j)_{x_l} \frac{D\xi_l}{Dx_{i_1} Dx_{i_2}} \tag{4.15}
\end{aligned}$$

and D/Dx_i , when acting on functions of $J^1\pi$, stands for

$$\frac{Df}{Dx_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^k \left[(u_j)_{x_i} \frac{\partial f}{\partial u_j} + \sum_{l=1}^n (u_j)_{x_l x_i} \frac{\partial f}{\partial (u_j)_{x_l}} \right]. \tag{4.16}$$

Proof. We have $(\bar{u}_j)_{\bar{x}_{i_1}} = (u_j)_{x_{i_1}} + \epsilon (\eta_j)_{x_{i_1}}$. Therefore,

$$\delta[(\bar{u}_j)_{\bar{x}_{i_1}}] = \delta[(u_j)_{x_{i_1}}] + \epsilon \delta[(\eta_j)_{x_{i_1}}] = \sum_{l=1}^n \left(\frac{D[(u_j)_{x_{i_1}}]}{Dx_l} + \epsilon \frac{D[(\eta_j)_{x_{i_1}}]}{Dx_l} \right) \delta x_l \tag{4.17}$$

and

$$\delta \bar{x}_{i_2} = \delta x_{i_2} + \epsilon \delta \xi_{i_2} = \sum_{q=1}^n \left(\delta_q^{i_2} + \epsilon \frac{D\xi_{i_2}}{Dx_q} \right) \delta x_q. \tag{4.18}$$

Therefore, proceeding in similar fashion as for the first-order prolongation,

$$(\bar{u}_j)_{\bar{x}_{i_1} \bar{x}_{i_2}} = \frac{\delta[(\bar{u}_j)_{\bar{x}_{i_1}}]}{\delta \bar{x}_{i_2}} = \frac{\sum_{l=1}^n \left(\frac{D[(u_j)_{x_{i_1}}]}{Dx_l} + \epsilon \frac{D[(\eta_j)_{x_{i_1}}]}{Dx_l} \right)}{\sum_{q=1}^n \left(\delta_q^{i_2} + \epsilon \frac{D\xi_{i_2}}{Dx_q} \right)} \delta_q^l + O(\epsilon^2)$$

$$\begin{aligned}
&= \sum_{l=1}^n \left(\frac{D[(u_j)_{x_{i_1}}]}{Dx_l} + \epsilon \frac{D[(\eta_j)_{x_{i_1}}]}{Dx_l} \right) \cdot \left(\delta_l^{i_2} - \epsilon \frac{D\xi_l}{Dx_{i_2}} \right) + O(\epsilon^2) \\
&= (u_j)_{x_{i_1}x_{i_2}} + \epsilon \left(\frac{D[(\eta_j)_{x_{i_1}}]}{Dx_{i_2}} - \sum_{l=1}^n (u_j)_{x_{i_1}x_l} \frac{D\xi_l}{Dx_{i_2}} \right) + O(\epsilon^2). \quad (4.19)
\end{aligned}$$

Introducing the value of $(\eta_j)_{x_{i_1}}$ in (4.19), using (4.9), we arrive at (4.15). \square

Applying this process recursively, we obtain the p -order prolongations

$$(\eta_j)_{x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_n}^{j_n}} = \frac{D \left((\eta_j)_{x_{i_1}^{j_1} \dots x_{i_{n-1}}^{j_{n-1}} x_{i_n}^{(j_n)-1}} \right)}{Dx_{i_n}} - \sum_{l=1}^n (u_j)_{x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_{n-1}}^{j_{n-1}} x_{i_n}^{(j_n)-1} x_l} \frac{D\xi_l}{Dx_{i_n}}, \quad (4.20)$$

or equivalently,

$$\begin{aligned}
\mathfrak{Pr}_J \eta_j &= (\eta_j)_{x_{i_1}^{j_1} x_{i_2}^{j_2} \dots x_{i_n}^{j_n}} = \frac{D^{|J|} \eta_j}{Dx_{i_1}^{j_1} Dx_{i_2}^{j_2} \dots Dx_{i_n}^{j_n}} - \sum_{l=1}^n (u_j)_{x_l} \frac{D^{|J|} \eta_j}{Dx_{i_1}^{j_1} Dx_{i_2}^{j_2} \dots Dx_{i_n}^{j_n}} - \dots \\
&\quad - \sum_{i,i'=1}^n \sum_{l=1}^n (u_j)_{x_l \dots x_{i_n-2}^{j_{n-2}}} \frac{D^2 \xi_l}{Dx_i Dx_{i'}} - \sum_{i=1}^n \sum_{l=1}^n (u_j)_{x_l \dots x_{i_n-1}^{j_{n-1}}} \frac{D\xi_l}{Dx_i}, \quad (4.21)
\end{aligned}$$

where $x_1 \leq x_{i_1}, \dots, x_{i_n} \leq x_n$ and $j_1 + \dots + j_n \leq p$. The associated vector field for this transformation reads

$$X^p = \xi \frac{\partial}{\partial t} + \sum_{j=1}^n \eta_j \frac{\partial}{\partial u_j} + \sum_{j=1}^n \sum_{|J|} \mathfrak{Pr}_J \eta_j \frac{\partial}{\partial (u_j)_J}, \quad (4.22)$$

5 Poisson algebras

A *Poisson algebra* is a triple $(A, \star, \{\cdot, \cdot\})$, where A is a real vector space endowed with two bilinear maps, namely \star and $\{\cdot, \cdot\}$, such that \star is a commutative and associative real algebra and $(A, \{\cdot, \cdot\})$ is a real Lie algebra whose Lie bracket, the *Poisson bracket*, satisfies the *Leibnitz rule* with respect to \star

$$\{f \star g, h\} = f \star \{g, h\} + \{f, h\} \star g, \quad \forall f, g, h \in A.$$

In other words, $\{\cdot, h\}$ is a derivation.

A *Poisson algebra morphism* is a morphism $\mathcal{T} : (A, \star_A, \{\cdot, \cdot\}_A) \rightarrow (B, \star_B, \{\cdot, \cdot\}_B)$ of \mathbb{R} -algebras $\mathcal{T} : (A, \star_A) \rightarrow (B, \star_B)$ that satisfies that $\mathcal{T}(\{a, b\}_A) = \{\mathcal{T}(a), \mathcal{T}(b)\}_B$ for every $a, b \in A$.

One of the types of Poisson algebras to be used in this paper are symmetric and universal algebras. Let us describe their main characteristics. Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, its *universal algebra*, $U_{\mathfrak{g}}$, is obtained from the quotient $T_{\mathfrak{g}}/\mathcal{R}$ of the tensor algebra $(T_{\mathfrak{g}}, \otimes)$ of \mathfrak{g} by the bilateral ideal \mathcal{R} spanned by the elements $v \otimes w - w \otimes v - [v, w]$, with $v, w \in \mathfrak{g}$. Given the quotient map $\pi : T_{\mathfrak{g}} \rightarrow U_{\mathfrak{g}}$, the space $U_{\mathfrak{g}}$ becomes an \mathbb{R} -algebra $(U_{\mathfrak{g}}, \tilde{\otimes})$ when endowed with the product $\tilde{\otimes} : U_{\mathfrak{g}} \times U_{\mathfrak{g}} \rightarrow U_{\mathfrak{g}}$ given by $\pi(P) \tilde{\otimes} \pi(Q) \equiv \pi(P \otimes Q)$, for every $P, Q \in T_{\mathfrak{g}}$. The Lie bracket on \mathfrak{g} can be extended to a Lie bracket $\{\cdot, \cdot\}_{U_{\mathfrak{g}}}$ on $U_{\mathfrak{g}}$ by imposing it to be a derivation of $(U_{\mathfrak{g}}, \tilde{\otimes})$ on each factor. This turns $U_{\mathfrak{g}}$ into a Poisson algebra $(U_{\mathfrak{g}}, \tilde{\otimes}, \{\cdot, \cdot\}_{U_{\mathfrak{g}}})$ [96]. The elements of its Casimir subalgebra are henceforth dubbed as *Casimir elements* of \mathfrak{g} [50].

If we set \mathcal{R} to be the bilateral ideal spanned by the elements $v \otimes w - w \otimes v$ in the above procedure, we obtain a new commutative Poisson algebra $S_{\mathfrak{g}}$ called *symmetric algebra* of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. The elements of $S_{\mathfrak{g}}$ are polynomials on the elements of \mathfrak{g} . Via the isomorphism $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$, they can naturally be understood as polynomial functions on \mathfrak{g}^* [1, 96]. The Casimir elements of this Poisson algebra are called *Casimir functions* of \mathfrak{g} .

The Poisson algebras $U_{\mathfrak{g}}$ and $S_{\mathfrak{g}}$ are related by the *symmetrizer map* [1, 50, 402], i.e. the linear isomorphism $\lambda : S_{\mathfrak{g}} \rightarrow U_{\mathfrak{g}}$ of the form

$$\lambda(v_{i_1}) = \pi(v_{i_1}), \quad \lambda(v_{i_1}v_{i_2} \dots v_{i_l}) = \frac{1}{l!} \sum_{s \in \Pi_l} \lambda(v_{s(i_1)}) \tilde{\otimes} \dots \tilde{\otimes} \lambda(v_{s(i_l)}), \quad (5.1)$$

for all $v_{i_1}, \dots, v_{i_l} \in \mathfrak{g}$ and with Π_l being the set of permutations of l elements. Moreover,

$$\lambda^{-1}(\{v, P\}_{U_{\mathfrak{g}}}) = \{v, \lambda^{-1}(P)\}_{S_{\mathfrak{g}}}, \quad \forall P \in U_{\mathfrak{g}}, \quad \forall v \in \mathfrak{g}. \quad (5.2)$$

So, λ^{-1} maps the Casimir elements of \mathfrak{g} into Casimir elements of $S_{\mathfrak{g}}$. If $(A, \star_A, \{\cdot, \cdot\}_A)$ and $(B, \star_B, \{\cdot, \cdot\}_B)$ are Poisson algebras and operations: \star_A and \star_B are commutative, then $A \otimes B$ becomes a Poisson algebra $(A \otimes B, \star_{A \otimes B}, \{\cdot, \cdot\}_{A \otimes B})$ by defining

$$\begin{aligned} (a \otimes b) \star_{A \otimes B} (c \otimes d) &= (a \star_A c) \otimes (b \star_B d), \quad \forall a, c \in A, \quad \forall b, d \in B, \\ \{a \otimes b, c \otimes d\}_{A \otimes B} &= \{a, c\}_A \otimes b \star_B d + a \star_A c \otimes \{b, d\}_B. \end{aligned} \quad (5.3)$$

Similarly, a Poisson structure on $A^{(m)} \equiv \overbrace{A \otimes \dots \otimes A}^{m\text{-times}}$ can be constructed by induction.

We say that $(A, \star_A, \{\cdot, \cdot\}_A, \Delta)$ is a *Poisson coalgebra* if $(A, \star_A, \{\cdot, \cdot\}_A)$ is a Poisson algebra and $\Delta : (A, \star_A, \{\cdot, \cdot\}_A) \rightarrow (A \otimes A, \star_{A \otimes A}, \{\cdot, \cdot\}_{A \otimes A})$, the so-called *coproduct*, is a Poisson algebra homomorphism which is *coassociative* [119], i.e. $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$. Then, the m -th coproduct map $\Delta^{(m)} : A \rightarrow A^{(m)}$ can be defined

recursively as follows

$$\Delta^{(m)} = \overbrace{(\text{Id} \otimes \dots \otimes \text{Id})}^{(m-2)\text{-times}} \otimes \Delta^{(2)} \circ \Delta^{(m-1)}, \quad m > 2, \quad (5.4)$$

and $\Delta \equiv \Delta^{(2)}$. Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

In particular, $S_{\mathfrak{g}}$ is a Poisson coalgebra with *primitive coproduct map* given by $\Delta(v) = v \otimes 1 + 1 \otimes v$, for all $v \in \mathfrak{g} \subset S_{\mathfrak{g}}$. The coassociativity of Δ is straightforward, and its m -th generalization reads

$$\Delta^{(m)}(v) = v \otimes \overbrace{1 \otimes \dots \otimes 1}^{(m-1)\text{-times}} + 1 \otimes v \otimes \overbrace{1 \otimes \dots \otimes 1}^{(m-2)\text{-times}} + \dots + \overbrace{1 \otimes \dots \otimes 1}^{(m-1)\text{-times}} \otimes v, \quad \forall v \in \mathfrak{g} \subset S_{\mathfrak{g}}. \quad (5.5)$$

6 Symplectic, presymplectic and Poisson manifolds

A *symplectic manifold* is a pair (N, ω) , where N stands for a manifold and ω is a non-degenerate closed two-form on N . We say that a vector field X on N is Hamiltonian with respect to (N, ω) if there exists a function $f \in C^\infty(N)$ such that¹

$$\iota_X \omega = -df. \quad (6.1)$$

In this case, we say that f is a *Hamiltonian function* for X . Conversely, given a function f , there exists a unique vector field X_f on N , the so-called *Hamiltonian vector field* of f , satisfying (6.1). This allows us to define a bracket $\{\cdot, \cdot\} : C^\infty(N) \times C^\infty(N) \rightarrow C^\infty(N)$ given by²

$$\{f, g\} = \omega(X_f, X_g) = X_f(g). \quad (6.2)$$

This bracket turns $C^\infty(N)$ into a *Poisson algebra* $(C^\infty(N), \cdot, \{\cdot, \cdot\})$, i.e., $\{\cdot, \cdot\}$ is a Lie bracket on $C^\infty(N)$ which additionally holds the *Leibniz rule* with respect to the standard product ‘ \cdot ’ of functions

$$\{\{f, g\}, h\} = \{\{f, h\}, g\} + \{g, \{f, h\}\} \quad \forall f, g, h \in C^\infty(N). \quad (6.3)$$

The Leibniz rule can be rephrased by saying that $\{f, \cdot\}$ is a derivation of the associative algebra $(C^\infty(N), \cdot)$ for each $f \in C^\infty(N)$. Actually, this derivation is represented by the Hamiltonian vector field X_f . The bracket $\{\cdot, \cdot\}$ is called the *Poisson bracket* of $(C^\infty(N), \cdot, \{\cdot, \cdot\})$. Note that if (N, ω) is a symplectic manifold, the non-degeneracy condition for ω implies that N is even dimensional [13].

¹In geometric mechanics it is used the convention $\iota_X \omega = df$.

²In geometric mechanics $\{f, g\} = X_g f$. So, the mapping $f \rightarrow X_f$ becomes a Lie algebra anti-homomorphism.

The above observations lead to the concept of a Poisson manifold which is a natural generalization of the symplectic one. A *Poisson manifold* is a pair $(N, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\} : C^\infty(N) \times C^\infty(N) \rightarrow C^\infty(N)$ is the Poisson bracket of $(C^\infty(N), \cdot, \{\cdot, \cdot\})$ which is also referred to as a *Poisson structure* on N . In view of this and (6.2), every symplectic manifold is a particular type of Poisson manifold. Moreover, by noting that $\{f, \cdot\}$ is a derivation on $(C^\infty(N), \cdot)$ for every $f \in C^\infty(N)$, we can associate with every function f a single vector field X_f , called the *Hamiltonian vector field* of f , such that $\{f, g\} = X_f g$ for all $g \in C^\infty(N)$, like in the symplectic case.

As the Poisson structure is a derivation in each entry, it gives rise to a bivector field Λ , i.e., an element of $\Gamma(\wedge^2 TN)$, the referred to as *Poisson bivector*, such that $\{f, g\} = \Lambda(df, dg)$. It is known that the Jacobi identity for $\{\cdot, \cdot\}$ amounts to $[\Lambda, \Lambda]_{SN} = 0$, with $[\cdot, \cdot]_{SN}$ being the *Schouten–Nijenhuis bracket* [401]. Conversely, a bivector Λ satisfying $[\Lambda, \Lambda]_{SN} = 0$ gives rise to a Poisson bracket on $C^\infty(N)$ by setting $\{f, g\} = \Lambda(df, dg)$. Hence, a Poisson manifold can be considered, equivalently, as $(N, \{\cdot, \cdot\})$ or (N, Λ) . It is remarkable that Λ induces a bundle morphism $\widehat{\Lambda} : \alpha_x \in T^*N \rightarrow \widehat{\Lambda}(\alpha_x) \in TN$, where $\bar{\alpha}_x(\widehat{\Lambda}(\alpha_x)) = \Lambda_x(\alpha_x, \bar{\alpha}_x)$ for all $\bar{\alpha}_x \in T_x^*N$, which enables us to write $X_f = \widehat{\Lambda}(df)$ for every $f \in C^\infty(N)$.

Another way of generalizing a symplectic structure is to consider a two-form ω which is merely closed (not necessarily of constant rank), forgetting the non-degeneracy assumption. In this case, ω is said to be a *presymplectic form* and the pair (N, ω) is called a *presymplectic manifold* [293]. Like in the symplectic case, we call a vector field X on N *Hamiltonian* if there exists a function $f \in C^\infty(N)$, a *Hamiltonian function* for X , such that (6.1) holds for the presymplectic form ω .

The possible degeneracy of ω introduces several differences with respect to the symplectic setting. For example, given an $f \in C^\infty(N)$, we cannot ensure neither the existence nor the uniqueness of a vector field X_f satisfying $\iota_{X_f}\omega = -df$. If it exists, we say that f is an *admissible function* with respect to (N, ω) . Since the linear combinations and multiplications of admissible functions are also admissible functions, the space $\text{Adm}(N, \omega)$ of admissible functions of (N, ω) is a real associative algebra. It is canonically also a Poisson algebra. Indeed, observe that every $f \in \text{Adm}(N, \omega)$ is associated to a family of Hamiltonian vector fields of the form $X_f + Z$, with Z being a vector field taking values in $\ker \omega$. Hence, (6.2) does not depend on the representatives X_f and X_g and becomes a Poisson bracket on the space $\text{Adm}(N, \omega)$, making the latter into a Poisson algebra. It is also remarkable that

$$\iota_{[X_f, X_g]}\omega = \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathcal{L}_{X_f}\omega = -\mathcal{L}_{X_f}dg = -d\{f, g\}. \quad (6.4)$$

In consequence, $[X_f, X_g]$ is a Hamiltonian vector field with a Hamiltonian function $\{f, g\}$.

Proposition 1. *The Casimir co-distribution of a Poisson manifold is involutive.*

Proof. Given two sections $\omega, \omega' \in \mathcal{C}^\Lambda$, we have that $\widehat{\Lambda}(\omega) = \widehat{\Lambda}(\omega') = 0$ and then $\Lambda(\omega, \omega') = 0$. In consequence,

$$[\omega, \omega']_\Lambda = \mathcal{L}_{\widehat{\Lambda}(\omega)}\omega' - \mathcal{L}_{\widehat{\Lambda}(\omega')}\omega - d\Lambda(\omega, \omega') = 0. \quad (6.5)$$

□

7 Dirac manifolds

The concept of *Dirac structure* was proposed by Dorfman [147] in the Hamiltonian framework of integrable evolution equations and defined in [136] as a subbundle of the *Whitney sum*, $TN \oplus_N T^*N$, called the *extended tangent* or *Pontryagin bundle*. It was thought-out as a common generalization of Poisson and presymplectic structures. It was designed also to deal with constrained systems, including constraints induced by degenerate Lagrangians investigated by Dirac [146], hence their name. See [74, 136, 137, 139, 264, 401, 430] for details.

Definition 1. We call a *Pontryagin bundle* $\mathcal{P}N$ a vector bundle $TN \oplus_N T^*N$ on N .

Definition 2. An *almost-Dirac manifold* is a pair (N, L) , where L is a maximally isotropic subbundle of $\mathcal{P}N$ with respect to the pairing

$$\langle X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \rangle_+ \equiv \frac{1}{2}(\bar{\alpha}_x(X_x) + \alpha_x(\bar{X}_x)), \quad (7.1)$$

where $X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \in T_xN \oplus T_x^*N = \mathcal{P}_xN$. In other words, L is isotropic and has rank $n = \dim N$.

Definition 3. A *Dirac manifold* is an almost-Dirac manifold (N, L) whose subbundle L , its *Dirac structure*, is involutive relative to the *Courant–Dorfman bracket* [136, 147, 212, 264], namely

$$[[X + \alpha, \bar{X} + \bar{\alpha}]]_C \equiv [X, \bar{X}] + \mathcal{L}_X\bar{\alpha} - \iota_{\bar{X}}d\alpha, \quad (7.2)$$

where $X + \alpha, \bar{X} + \bar{\alpha} \in \Gamma(TN \oplus_N T^*N)$.

Note that the Courant–Dorfman bracket satisfies the Jacobi identity in the form

$$[[[e_1, e_2]]_C, e_3]_C = [[e_1, [[e_2, e_3]]_C]]_C - [[e_2, [[e_1, e_3]]_C]]_C, \quad \forall e_1, e_2, e_3 \in \Gamma(\mathcal{P}N), \quad (7.3)$$

but is not skew-symmetric. It is, however, skew-symmetric on sections of the Dirac subbundle L , defining a *Lie algebroid* structure $(L, [[\cdot, \cdot]]_C, \rho)$, where $\rho : L \ni X_x + \alpha_x \mapsto X_x \in TN$. This means that $(\Gamma(L), [[\cdot, \cdot]]_C)$ is a Lie algebra and the vector bundle morphism $\rho : L \rightarrow TN$, the *anchor*, satisfies

$$[[e_1, f e_2]]_C = (\rho(e_1)f)e_2 + f[[e_1, e_2]]_C \quad (7.4)$$

for all $e_1, e_2 \in \Gamma(L)$ and $f \in C^\infty(N)$ [136]. One can prove that, automatically, ρ induces a Lie algebra morphism of $(\Gamma(L), [[\cdot, \cdot]]_C)$ into the Lie algebra of vector fields on N . The generalized distribution $\rho(L)$, called the *characteristic distribution* of the Dirac structure, is therefore integrable in the sense of Stefan–Sussmann [391].

Definition 4. A vector field X on N is said to be an *L-Hamiltonian vector field* (or simply a *Hamiltonian vector field* if L is fixed) if there exists an $f \in C^\infty(N)$ such that $X + df \in \Gamma(L)$. In this case, f is an *L-Hamiltonian function* for X and an *admissible function* of (N, L) . Let us denote by $\text{Ham}(N, L)$ and $\text{Adm}(N, L)$ the spaces of Hamiltonian vector fields and admissible functions of (N, L) , respectively.

The space $\text{Adm}(N, L)$ becomes a Poisson algebra $(\text{Adm}(N, L), \cdot, \{\cdot, \cdot\}_L)$ relative to the standard product of functions and the Lie bracket given by $\{f, \bar{f}\}_L = X\bar{f}$ where X is an L -Hamiltonian vector field for f . Since L is isotropic, $\{f, \bar{f}\}_L$ is well defined, i.e., its value is independent on the choice of the L -Hamiltonian vector field associated to f . The elements $f \in \text{Adm}(N, L)$ possessing trivial Hamiltonian vector fields are called the *Casimir functions* of (N, L) [430]. We write $\text{Cas}(N, L)$ for the set of Casimir functions of (N, L) . We can also distinguish the space $G(N, L)$ of L -Hamiltonian vector fields which admit zero (or, equivalently, any constant) as an L -Hamiltonian function. We call them *gauge vector fields* of the Dirac structure.

Note that, if X and \bar{X} are L -Hamiltonian vector fields with Hamiltonian functions f and \bar{f} , then $\{f, \bar{f}\}_L$ is a Hamiltonian for $[X, \bar{X}]$

$$[[X + df, \bar{X} + d\bar{f}]]_C = [X, \bar{X}] + \mathcal{L}_X d\bar{f} - \iota_{\bar{X}} d^2 f = [X, \bar{X}] + d\{f, \bar{f}\}_L. \quad (7.5)$$

This implies that $(\text{Ham}(N, L), [\cdot, \cdot])$ is a Lie algebra in which $G(N, L)$ is a Lie ideal. Denote the quotient Lie algebra $\text{Ham}(N, L)/G(N, L)$ by $\widehat{\text{Ham}}(N, L)$.

Proposition 5. *If (N, L) is a Dirac manifold, then $\{\text{Cas}(N, L), \text{Adm}(N, L)\}_L = 0$, i.e., $\text{Cas}(N, L)$ is an ideal of the Lie algebra $(\text{Adm}(N, L), \{\cdot, \cdot\}_L)$. Moreover, we have the following exact sequence of Lie algebra homomorphisms*

$$0 \hookrightarrow \text{Cas}(N, L) \hookrightarrow \text{Adm}(N, L) \xrightarrow{B_L} \widehat{\text{Ham}}(N, L) \rightarrow 0, \quad (7.6)$$

with $B_L(f) = \pi(X_f)$, where the vector field X_f is an L -Hamiltonian vector field of f , and π is the canonical projection $\pi : \text{Ham}(N, L) \rightarrow \widehat{\text{Ham}}(N, L)$.

For every Dirac manifold (N, L) , we have a canonical linear map $\Omega_x^L : \rho(L)_x \subset T_x^*N \rightarrow \rho(L)_x^* \subset T_x^*N$ given by

$$[\Omega_x^L(X_x)](\bar{X}_x) = -\alpha_x(\bar{X}_x), \quad X_x, \bar{X}_x \in \rho(L), \quad (7.7)$$

where $\alpha_x \in T_x^*N$ is such that $X_x + \alpha_x \in L$. Note that, as L is isotropic, Ω_x^L is well defined, i.e., the value of

$$\Omega_x^L(X_x, \bar{X}_x) = [\Omega_x^L(X_x)](\bar{X}_x) \quad (7.8)$$

is independent of the particular α_x and defines a skew-symmetric bilinear form Ω^L on the (generalized) distribution $\rho(L)$. Indeed, given $X_x + \bar{\alpha}_x \in L$, we have that $\alpha_x - \bar{\alpha}_x \in L$. Since L is isotropic, $\langle \alpha_x - \bar{\alpha}_x, \bar{X}_x + \bar{\alpha}_x \rangle_+ = (\alpha_x - \bar{\alpha}_x)\bar{X}_x/2 = 0$ for all $\bar{X}_x + \bar{\alpha}_x \in L$. Then, $[\Omega_x^L(X_x)](\bar{X}_x) = -\bar{\alpha}_x(\bar{X}_x) = -\alpha_x(\bar{X}_x)$ for all $\bar{X}_x \in \rho(L)$ and Ω^L is well defined.

It is easy to see that gauge vector fields generate the *gauge distribution* $\ker \Omega^L$. Moreover, the involutivity of L ensures that $\rho(L)$ is an integrable generalized distribution in the sense of Stefan–Sussmann [391]. Therefore, it induces a (generalized) foliation $\mathfrak{F}^L = \{\mathfrak{F}_x^L : x \in N\}$ on N .

Since $\rho(L_x) = T_x \mathfrak{F}_x^L$, if the elements $X_x + \alpha_x$ and $X_x + \bar{\alpha}_x$, with $X_x \in T_x \mathfrak{F}_x^L$, are in $L_x \subset \mathcal{P}_x N = T_x N \oplus T_x^* N$, then $\alpha_x - \bar{\alpha}_x$ is in the annihilator of $T_x \mathfrak{F}_x^L$, so the image of α_x under the canonical restriction $\sigma : \alpha_x \in T_x^* N \mapsto \alpha_x|_{T_x \mathfrak{F}_x^L} \in T_x^* \mathfrak{F}_x^L$ is uniquely determined. One can verify that $\sigma(\alpha_x) = -\Omega_x^L(X_x)$. The two-form Ω^L restricted to \mathfrak{F}_x^L turns out to be closed, so that \mathfrak{F}_x^L is canonically a presymplectic manifold, and the canonical restriction of L to \mathfrak{F}_x^L is the graph of this form [136].

As particular instances, Poisson and presymplectic manifolds are particular cases of Dirac manifolds. On one hand, consider a presymplectic manifold (N, ω) and define L^ω to be the graph of minus the fiber bundle morphism $\widehat{\omega} : X_x \in \text{TN} \mapsto \omega_x(X_x, \cdot) \in T_x^* N$. The generalized distribution L^ω is isotropic, as

$$\langle X_x - \widehat{\omega}(X_x), \bar{X}_x - \widehat{\omega}(\bar{X}_x) \rangle_+ = -(\omega_x(X_x, \bar{X}_x) + \omega_x(\bar{X}_x, X_x))/2 = 0. \quad (7.9)$$

As L^ω is the graph of $-\widehat{\omega}$, then $\dim L_x^\omega = \dim N$ and L^ω is a maximally isotropic subbundle of $\mathcal{P}N$. In addition, its integrability relative to the Courant–Dorfman bracket comes from the fact that $d\omega = 0$. Indeed, for arbitrary $X, X' \in \Gamma(\text{TN})$, we have

$$[[X - \iota_X \omega, X' - \iota_{X'} \omega]]_C = [X, X'] - \mathcal{L}_X \iota_{X'} \omega + \iota_{X'} d\iota_X \omega = [X, X'] - \iota_{[X, X']} \omega, \quad (7.10)$$

since

$$\mathcal{L}_X \iota_{X'} \omega - \iota_{X'} d\iota_X \omega = \mathcal{L}_X \iota_{X'} \omega - \iota_{X'} \mathcal{L}_X \omega = \iota_{[X, X']} \omega. \quad (7.11)$$

In this case, $\rho : L^\omega \rightarrow \text{TN}$ is a bundle isomorphism. Conversely, given a Dirac manifold whose $\rho : L \rightarrow \text{TN}$ is a bundle isomorphism, its characteristic distribution satisfies $\rho(L) = \text{TN}$ and it admits a unique integral leaf, namely N , on which Ω^L is a closed two-form, i.e., (N, Ω^L) is a presymplectic manifold.

On the other hand, every Poisson manifold (N, Λ) induces a subbundle L^Λ given by the graph of $\widehat{\Lambda}$. It is isotropic,

$$\langle \widehat{\Lambda}(\alpha_x) + \alpha_x, \widehat{\Lambda}(\bar{\alpha}_x) + \bar{\alpha}_x \rangle_+ = (\Lambda_x(\bar{\alpha}_x, \alpha_x) + \Lambda_x(\alpha_x, \bar{\alpha}_x))/2 = 0, \quad (7.12)$$

for all $\alpha_x, \bar{\alpha}_x \in T_x^* N$ and $x \in N$, and of rank $\dim N$ as the graph of $\widehat{\Lambda}$ is a map from T^*N . Additionally, L^Λ is integrable. Indeed, as $\widehat{\Lambda}(d\{f, g\}) = [\widehat{\Lambda}(df), \widehat{\Lambda}(dg)]$

for every $f, g \in C^\infty(N)$ [401], we have

$$[[\widehat{\Lambda}(df) + df, \widehat{\Lambda}(dg) + dg]]_C = [\widehat{\Lambda}(df), \widehat{\Lambda}(dg)] + \mathcal{L}_{\widehat{\Lambda}(df)} dg - \iota_{\widehat{\Lambda}(dg)} d^2 f = \quad (7.13)$$

$$\widehat{\Lambda}(d\{f, g\}) + d\{f, g\} \quad (7.14)$$

and the involutivity follows from the fact that the module of 1-forms is generated locally by exact 1-forms.

Conversely, every Dirac manifold (N, L) such that $\rho^* : L \rightarrow T^*N$ is a bundle isomorphism is the graph of $\widehat{\Lambda}$ of a Poisson bivector.

Let us motivate our terminology. We call $\rho(L)$ the characteristic distribution of (N, L) , which follows the terminology of [430] instead of the original one by Courant [136]. This is done because when L comes from a Poisson manifold, $\rho(L)$ coincides with the characteristic distribution of the Poisson structure [401]. Meanwhile, the vector fields taking values in $\ker \Omega^L$ are called *gauge vector fields*. In this way, when L is the graph of a presymplectic structure, such vector fields are its gauge vector fields [152].

From here, we see that the Dirac structure incorporates the presymplectic and Poisson structures as particular cases. Courant [136, 137] provided the theory to this statement.

Recall that every presymplectic manifold (N, ω) gives rise to a Dirac manifold (N, L^ω) whose distribution L^ω is spanned by elements of $\Gamma(TN \oplus_N T^*N)$ of the form $X - \iota_X \omega$ with $X \in \Gamma(TN)$. Obviously, this shows that the Hamiltonian vector fields for (N, ω) are L -Hamiltonian vector fields relative to (N, L) .

8 Jacobi manifolds

Jacobi manifolds were independently introduced by Kirillov and Lichnerowicz [270, 294]. We now briefly survey their most fundamental properties. Several known results will be illustrated during the proof of the main results of the paper.

Definition 1. A *Jacobi manifold* is a triple (N, Λ, R) , where Λ is a bivector field on N and R is a vector field, the referred to as *Reeb vector field*, satisfying:

$$[\Lambda, \Lambda]_{SN} = 2R \wedge \Lambda, \quad [R, \Lambda]_{SN} = 0. \quad (8.1)$$

Example 1. Every Poisson manifold (N, Λ) can be considered as a Jacobi manifold $(N, \Lambda, R = 0)$.

Example 2. The continuous Heisenberg group [413] can be described as the space of matrices

$$\mathbb{H} = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\}, \quad (8.2)$$

endowed with the standard matrix multiplication, where $\{x, y, z\}$ is the natural coordinate system on \mathbb{H} induced by (8.2). Consider the bivector field on \mathbb{H} given by

$$\Lambda_{\mathbb{H}} \equiv -y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \quad (8.3)$$

and the vector field $R_{\mathbb{H}} \equiv \partial/\partial z$. After a simple calculation, we obtain that

$$[\Lambda_{\mathbb{H}}, \Lambda_{\mathbb{H}}]_{SN} = 2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} = 2R_{\mathbb{H}} \wedge \Lambda_{\mathbb{H}}, \quad [R_{\mathbb{H}}, \Lambda_{\mathbb{H}}]_{SN} = 0. \quad (8.4)$$

So, $(\mathbb{H}, \Lambda_{\mathbb{H}}, R_{\mathbb{H}})$ is a Jacobi manifold.

Definition 2. We say that X is a *Hamiltonian vector field* with respect to the Jacobi manifold (N, Λ, R) if there exists a function $f \in C^\infty(N)$ such that

$$X = [\Lambda, f]_{SN} + fR = \widehat{\Lambda}(df) + fR. \quad (8.5)$$

In this case, f is said to be a *Hamiltonian function* of X and we write $X = X_f$. If f is also a first-integral of R , we say that f is a *good Hamiltonian function* and we call X_f a *good Hamiltonian vector field*.

Example 3. Given the Jacobi manifold $(\mathbb{H}, \Lambda_{\mathbb{H}}, R_{\mathbb{H}})$ and the vector field $X_1^L \equiv \partial/\partial x$, we have that

$$X_1^L = [\Lambda_{\mathbb{H}}, -y]_{SN} - yR_{\mathbb{H}} = \widehat{\Lambda}_{\mathbb{H}}(-dy) - yR_{\mathbb{H}}. \quad (8.6)$$

Hence, X_1^L is a Hamiltonian vector field with Hamiltonian function $h_1^L = -y$ with respect to $(\mathbb{H}, \Lambda_{\mathbb{H}}, R_{\mathbb{H}})$.

Each function gives rise to a unique Hamiltonian vector field. Nevertheless, each vector field may admit several Hamiltonian functions. This last result will be illustrated afterwards within relevant results concerning Lie–Jacobi systems. We write $\text{Ham}(\Lambda, R)$ for the space of Hamiltonian vector fields relative to (N, Λ, R) . It can be proved that $\text{Ham}(\Lambda, R)$ is a Lie algebra with respect to the standard Lie bracket of vector fields. Additionally, a Jacobi manifold allow us to define a Lie bracket on $C^\infty(N)$ given by

$$\{f, g\}_{\Lambda, R} = \Lambda(df, dg) + fRg - gRf. \quad (8.7)$$

This Lie bracket becomes a Poisson bracket if and only if $R = 0$. Moreover, the morphism $\phi_{\Lambda, R} : f \in C^\infty(N) \mapsto X_f \in \text{Ham}(\Lambda, R)$ is a Lie algebra morphism. It is important to emphasize that it may not be injective.

1 The interest of Lie systems

Lie systems constitute a very specific class of nonautonomous systems of first-order ordinary differential equations admitting a *superposition rule*, i.e., a t -independent function allowing us to write its general solution in terms of a generic family of its particular solutions and a set of constants to be related to initial conditions [102, 297, 300, 301, 303, 403, 404, 405].

The analysis of Lie systems dates back to the end of the XIX century, when Vessiot, Guldberg and Lie [225, 303, 403] found the fundamental properties of such systems. In that time, Lie proved the nowadays called *Lie–Scheffers Theorem* [303], which states that a nonautonomous system of first-order ordinary differential equations in normal form admits a superposition rule if and only if it describes the integral curves of a t -dependent vector field taking values in a finite-dimensional real Lie algebra V of vector fields [91]. We call V a *Vessiot–Guldberg Lie algebra* [102] of the Lie system.

The theory of Lie systems was deeply analyzed in its origins only to be forgotten soon after. In the XX century, Winternitz and coworkers revived the theory by deriving superposition rules for many relevant systems of first-order ordinary differential and superdifferential equations and studied the existence of particular types of Lie systems on \mathbb{R}^n [18, 44, 45, 250, 419].

Posteriorly, there has been strong interest in using modern geometrical techniques [35, 61, 62, 363, 419] to investigate Lie systems [99, 107, 108, 236, 284, 286, 308]. For instance, Cariñena, Grabowski and Marmo interpreted superposition rules as a particular type of projective foliations [91]. Also, the outlook of Lie systems as curves within finite-dimensional real Lie algebras of vector fields [303, 384] suggests the inspection of generalized distributions of vector fields and Lie group actions to study their properties [89, 91, 102].

The Lie–Scheffers Theorem shows that there exists a quite restricted number of Lie systems in the literature: most differential equations cannot be described as a curve in a finite-dimensional Lie algebras of vector fields [102, 258]. Nevertheless, this fact does not downplay their great importance. For instance, Lie systems appear in the study of relevant physical models, Mathematics and Control theory [113, 114, 283, 386]. More specifically, Lie systems have also been employed to study the integrability of Riccati and matrix Riccati equations [112, 228, 278], Floquet theory [190, 191, 192, 193], etc. They can also be employed in Quantum Mechanics [100, 113] or in the computation of geometric phases in [192]. Furthermore, they have served as a way for the geometric investigation of stochastic equations [283], superequations [249, 250], and other topics [87, 113].

In recent years, the concepts of Lie systems and superposition rules have been extrapolated to higher-order systems of ordinary differential equations [87]. This was employed to study systems of higher-order ordinary differential equations appearing in Classical Mechanics and Mathematics. For instance, the higher-order Riccati equation [108, 216], the second- and third-order Kummer–Schwarz [308], Milne–Pinney and dissipative Milne–Pinney equations [99, 286], etcetera [106].

The theory of Lie systems has also been expanded to the realm of PDE Lie systems by Odziejewicz and Grundland in [335] and Cariñena, Grabowski and Marmo in [91]. Additionally, other extensions of Lie systems to other realms have been carried out during latter decades, e.g., to stochastic equations [283].

Many new examples of Lie systems have been found along the research for this thesis. Concerning PDEs Lie systems [91], the findings of this thesis are almost the only applications of this theory existing in the actual literature [166]. Additionally, we unveil new applications of Lie systems. Among other systems, we analyze through our methods the Hamilton equations describing certain second-order Riccati equations, Cayley–Klein Riccati equations [166], planar Riccati equations, coupled Riccati equations, Smorodinsky–Winternitz oscillators [35], systems of trigonometric nonlinearities [35], among many others [109]. A plethora of other Lie systems and applications can be consulted in [35, 87, 109, 236].

Although the Lie–Scheffers theorem determines whether a system is a Lie system, it can be difficult to determine when a system is a Lie system using this theorem. This is due to the fact that it is complicated to distinguish whether a system possesses a Vessiot–Guldberg Lie algebra of very high dimension or it has no Vessiot–Guldberg Lie algebra at all. In such cases, it is helpful to know all the possible Vessiot–Guldberg Lie algebras on the manifold where the system is defined as this delimites the dimension and structures of the possible Vessiot–Guldberg algebras for the system (see [216]).

Considering the definition of a Lie system, a classification of finite-dimensional Lie algebras of vector fields on a manifold N amounts to a classification of Lie systems on N [210]. There exists a classification of finite-dimensional Lie algebras

on the real plane by Lie [299, 302]. This classification was retaken and refurbished by Gonzalez, Kamran and Olver giving rise to the henceforth called *GKO classification* [210].

As a byproduct of the big number of new applications of Lie systems found in recent years by the author of this thesis and her collaborators, it was found that many of them possess a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields [109, 32, 35]. This led us to focus on studying geometric structures compatible with Lie systems. The idea consists in identifying Lie systems admitting Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to a certain geometric structure [88, 109]. The geometric structures that we used simplified very much the study of Lie systems. For example, the case of Hamiltonian vector fields with respect to a Poisson structure led to the derivation of a superposition rule for Riccati equations by using the Casimir function of $\mathfrak{sl}(2, \mathbb{R})$ [35].

Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Poisson structure, are known as *Lie–Hamilton systems*. Some of the abovementioned systems, as some second-order Riccati equations or the second-order Kummer–Schwarz equations, are Lie–Hamilton systems. On the other hand, the third-order Kummer–Schwarz equation is not a Lie–Hamilton system. Our *no-go theorem* explains how a Lie–Hamilton system is always a particular type of Lie system but the converse is not true [88].

It is also possible to classify Lie–Hamilton systems attending to their underlying Vessiot–Guldberg Lie algebras. We give such a classification in this part of this thesis. Based on the previous GKO classification, we classify Vessiot–Guldberg Lie algebras of Hamiltonian planar vector fields relative to a symplectic structure. Their associated Poisson or Hamiltonian functions shall be obtained [32]. Table 2 in Appendix 3 summarizes our results.

One of the main advantages of Lie–Hamilton systems is that they allow us to use a Poisson coalgebra approach to derive algebraically their superposition rules. An analogue of the structure of a co-algebra symmetry, first noticed by Ragnisco, Ballesteros and coworkers for the study of integrable systems [33, 34, 36, 41], naturally appears in the study of Lie–Hamilton systems. Each Lie–Hamilton system is endowed with a finite-dimensional Lie algebra of Hamiltonian functions. The polynomials on the elements of a basis of this Lie algebra form a Poisson algebra. Next, the diagonal prolongations of the Lie–Hamilton system, which are employed to obtain its superposition rules [91], naturally induces another Poisson algebra for each diagonal prolongation. There also exists a primitive coproduct passing from the initial Poisson algebra to the prolonged ones. Using this structure, the Casimir operators of the Vessiot–Guldberg Lie algebra for the Lie–Hamilton system induce, via the coproduct, naturally constants of motion for the diagonal prolongations of the Lie–Hamilton system. These constants of motion allow us to derive superposition rules. This method is much easier and geometrically more clarifying than

previous methods to obtain superposition rules based on integrating vector fields that involved solving ODEs or PDEs [89, 91, 419].

Meanwhile, there are certain systems which do not admit a symplectic nor a Poisson structure turning any of their Vessiot–Guldberg Lie algebras into Hamiltonian vector fields. Instead, they do admit a *presymplectic form*, a closed two form, turning one of their Vessiot–Guldberg Lie algebras into Hamiltonian vector fields [88]. Indeed, the third-order Kummer–Schwarz, which is not a Lie–Hamilton system, admit such a property.

A class of systems including all the previous types of Lie systems with respect to the mentioned structures are the so-called *Dirac–Lie systems*. This type of Lie systems admits Vessiot–Guldberg Lie algebras of Hamiltonian vector fields of a very general kind. Most properties of other Hamiltonian vector fields with respect to the other structures can naturally be extended to these “generalized” vector fields [88]. *Generalized Hamiltonian vector fields* imply *generalized admissible functions* [88] with respect to a Dirac structure and will enjoy a plethora of properties that will be explained in this thesis. Examples of physical interest will be accompanying as well.

It is worth of mentioning *Jacobi–Lie systems*. In a similar fashion, Jacobi–Lie systems are Lie systems admitting Hamiltonian vector fields with respect to a *Jacobi structure*. Furthermore, we can find *k-symplectic Lie systems*, which admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to the presymplectic forms of a *k-symplectic structure* [310, 311].

The content of this third Chapter by sections corresponds to:

- **Section 1: Lie systems:** We define Lie systems, superposition rules and Vessiot–Guldberg Lie algebras. We describe the Lie–Scheffers theorem. Subsequently, four important examples of Lie systems are contemplated: the Riccati equation, a planar Riccati equation with t -dependent real coefficients, a system of coupled Riccati equations and a matrix Riccati equation. Next, we review Lie systems on Lie groups whose superposition rules are given through the action of the Lie group on itself. The theorem of reduction of Lie systems is briefly commented to solve Lie systems defined on Lie groups. We show how to derive superposition rules by using the so-called diagonal prolongations of the vector fields of a Vessiot–Guldberg Lie algebra. As a new application, we study three Lie systems: the second-order Riccati equation and the Kummer–Schwarz equations of second- and third-order. We obtain a solution in terms of a Lie group action for all these Lie systems, which have Vessiot–Guldberg Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. To conclude, we show the link between the Schwarzian derivative and the third-order Kummer–Schwarz equation and we find a superposition rule for the latter.
- **Section 2: Lie–Hamilton systems:** We introduce Lie–Hamilton systems.

We show the planar Riccati equations are Lie–Hamilton systems and we study their properties. Some new examples of Lie–Hamilton systems are given: the coupled Riccati equations, the Hamilton equations of the second-order Riccati equations, the Hamilton equations of the second-order Kummer–Schwarz equations, and the Hamilton equations of the Smorodinsky–Winternitz oscillator.

We study t -dependent Hamiltonians for Lie–Hamilton systems. We analyze the properties of constants of motion for Lie–Hamilton systems. We use Poisson coalgebras to devise an algebraic and geometric method for the construction of superposition rules. We will explicitly construct superposition rules for certain systems of physical and mathematical relevance: the classical Ermakov system, four coupled Riccati equations, the Hamilton equations of the second-order Kummer–Schwarz equations, the Hamilton equations of the Smorodinsky–Winternitz oscillator and a system with trigonometric nonlinearities.

- **Section 3: Classification of Lie–Hamilton systems on the plane:** First, we introduce the notion generic points of Lie algebras, primitive and imprimitive Lie algebras, integrating factors and modular generating systems. From these concepts, a number of new results give rise to classify Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on the plane (with respect to a Poisson structure). To do so, we rely on the GKO classification for finite-dimensional Lie algebras on the plane. We obtain that, out of the initial non-diffeomorphic 28 classes of finite-dimensional Lie algebras of vector fields on \mathbb{R}^2 given by the GKO classification, 12 of them consist of Hamiltonian vector fields. We will inspect all of them and describe their properties.
- **Section 4: Applications of Lie–Hamilton systems on the plane:** The classification of Lie–Hamilton systems on the plane enables us to classify Lie–Hamilton systems on the plane with physical applications. In this section we aim to give examples of Lie–Hamilton systems on the plane of physical and mathematical interest.

We start by Lie–Hamilton systems with Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Some examples are: the Milne–Pinney equation, the second-order Kummer–Schwarz equations, and the complex Riccati equation with t -dependent real coefficients.

Then, we study Lie–Hamilton systems with biological and physical applications: the generalized Buchdahl equations, time-dependent Lotka–Volterra systems, quadratic polynomial models and viral infection models. More Lie–Hamilton systems on the plane with relevance are displayed: the Cayley–Klein Riccati equations which is a compendium of Double-Clifford or split-complex

Riccati equations and the Dual-Study Riccati equation. Next, we present other Lie–Hamilton systems with Vessiot–Guldberg Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and we study the equivalence between them. More specifically, we analyze coupled Riccati equations, the Milne–Pinney equations, the second-order Kummer–Schwarz equation and a planar diffusion Riccati system. Later, we present Lie–Hamilton systems of two-photon type, which encompasses: the dissipative harmonic oscillator and the second-order Riccati equation. To conclude, we study Lie systems with Vessiot–Guldberg Lie algebras isomorphic to \mathfrak{h}_2 . This covers: the complex Bernoulli equation with t -dependent real coefficients, generalized Buchdahl equations t -dependent Lotka–Volterra systems, etc.

- **Section 6: Dirac–Lie systems:** We motivate the study of Dirac–Lie systems by showing that third-order Kummer–Schwarz equations and a Riccati system cannot be studied through Lie–Hamilton systems. Nevertheless, they admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a presymplectic structure. This has led us to make use of Dirac’s structures and its corresponding geometry. The non Lie–Hamiltonian nature of the mentioned two examples also suggests to enunciate the so-called “no-go” theorem, which tells us when a Lie system cannot be a Lie–Hamilton one.

We state how a Dirac–Lie system covers Lie systems with Hamiltonian Vessiot–Guldberg Lie algebras with respect to presymplectic and Poisson structures as particular cases. In this way, we generalize Lie–Hamilton systems to a more general class of Lie systems covering the aforementioned structures.

In a similar fashion as for Lie–Hamilton systems, we derive Dirac–Lie Hamiltonians and all their corresponding concepts are reformulated: the concept of diagonal prolongation is redefined to the realm of Dirac–Lie systems in order to work out their superposition rules. Certain properties for their constants of motion are reconsidered. To conclude, we show a superposition rule for third-order Kummer–Schwarz equations treated with Dirac–Lie systems theory. It is obvious that the previously studied coalgebra method to derive superposition rules can also be extrapolated to Dirac–Lie systems. To conclude, we introduce bi-Lie-Dirac systems, or Lie systems admitting two compatible Dirac structures. A method for generating bi-Lie-Dirac systems is given. We find out that our techniques can be applied to Schwarzian Korteweg-de Vries (SKdV) equations [27, 46]. This provides a new approach to the study of these equations.

We derive soliton-type solutions for Schwarzian–KdV equations, namely shape-preserving traveling wave solutions. Moreover, we show how Lie systems and our methods can be applied to provide Bäcklund transformations for certain solutions of these equations. This can be considered as one of the first applica-

tions of Dirac–Lie systems in the study of PDEs of physical and mathematical interest from the point of view of the theory of Lie systems.

- **Section 7: Jacobi–Lie systems:** In this section we search for Vessiot–Guldberg Lie algebras of Hamiltonian vector fields relative to a Jacobi structure on the plane.

As in past sections, we introduce the concept of Hamiltonian vector field with respect to a Jacobi structure and the consequent definition of Jacobi–Lie systems. Also, it is possible to define Jacobi–Lie Hamiltonians and to find a Lie algebra of Hamiltonian functions with respect to a defined Jacobi–Lie bracket.

It is our purpose to classify all Jacobi–Lie systems on the plane (up to diffeomorphisms). We rely on our previous classification of Lie–Hamilton systems on the plane and the GKO classification. The classification of Jacobi–Lie systems on the plane is displayed in the appendix for Tables, in Table 5 in Appendix 3. An explicit example is given: the continuous Heisenberg group is understood as a Jacobi manifold and a Jacobi–Lie system is constructed out of the left-invariant vector fields associated with the continuous Heisenberg group.

2 Lie systems

On a first approximation, a *Lie system* is a first-order system of ODEs that admits a superposition rule. In this section we will describe Lie systems, their main properties and the most relevant results used to study them.

2.1 Lie systems and superposition rules

Let us now turn to describing some fundamental notions appearing in the theory of Lie systems.

Definition 1. A *superposition rule* depending on m particular solutions for a system X on N is a function $\Phi : N^m \times N \rightarrow N$, $x = \Phi(x_{(1)}, \dots, x_{(m)}; \lambda)$, such that the general solution $x(t)$ of X can be brought into the form $x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); \lambda)$, where $x_{(1)}(t), \dots, x_{(m)}(t)$ is any generic family of particular solutions and λ is a point of N to be related to initial conditions.

Example 1. The Riccati equation on the real line [419]

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad (2.1)$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary t -dependent functions, admits the superposition rule $\Phi : (x_{(1)}, x_{(2)}, x_{(3)}; k) \in \mathbb{R}^3 \times \mathbb{R} \mapsto x \in \mathbb{R}$ given by

$$x = \frac{x_{(1)}(x_{(3)} - x_{(2)}) + kx_{(2)}(x_{(1)} - x_{(3)})}{(x_{(3)} - x_{(2)}) + k(x_{(1)} - x_{(3)})}. \quad (2.2)$$

In other words, the general solution, $x(t)$, to (2.1) can be brought into the form

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{(x_{(3)}(t) - x_{(2)}(t)) + k(x_{(1)}(t) - x_{(3)}(t))}. \quad (2.3)$$

where $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ are three different particular solutions and k is an arbitrary constant.

The conditions ensuring that a system X possesses a superposition rule are stated by the *Lie–Scheffers Theorem* [303, Theorem 44]. A modern statement of this relevant result is described next (for a modern geometric description see [91, Theorem 1]).

Theorem 2. (The Lie–Scheffers Theorem) *A first-order system*

$$\frac{dx}{dt} = F(t, x), \quad x \in N, \quad (2.4)$$

admits a superposition rule if and only if X can be written as

$$X_t = \sum_{\alpha=1}^r b_{\alpha}(t)X_{\alpha} \quad (2.5)$$

for a certain family $b_1(t), \dots, b_r(t)$ of t -dependent functions and a family X_1, \dots, X_r of vector fields on N spanning an r -dimensional real Lie algebra of vector fields.

The Lie–Scheffers Theorem yields that every Lie system X is related to (at least) one finite-dimensional real Lie algebra of vector fields V , a so-called *Vessiot–Guldberg Lie algebra*, satisfying that $\{X_t\}_{t \in \mathbb{R}} \subset V$. This implies that V^X must be finite-dimensional. Conversely, if V^X is finite-dimensional, this Lie algebra can be chosen as a Vessiot–Guldberg Lie algebra for X . This proves the following theorem, which motivates, among other reasons, the definition of V^X [102].

Theorem 3. (The abbreviated Lie–Scheffers Theorem) *A system X admits a superposition rule if and only if V^X is finite-dimensional.*

Example 2. In order to illustrate the above concepts, let us consider the Riccati equation (2.1). Observe that (2.1) is the system associated to the t -dependent vector field $X = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x}, \quad X_3 = x^2 \frac{\partial}{\partial x} \quad (2.6)$$

span a Vessiot–Guldberg Lie algebra V for (2.1) isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The Lie–Scheffers Theorem shows that Riccati equations must admit a superposition rule. Indeed, the general solution of Riccati equation (2.1) can be brought into the form $\Phi : (x_{(1)}, x_{(2)}, x_{(3)}; k) \in \mathbb{R}^3 \times \mathbb{R} \mapsto x \in \mathbb{R}$ given by (2.2) enabling us to write their general solutions as $x(t) = \Phi(x_{(1)}(t), x_{(2)}(t), x_{(3)}(t); k)$.

Example 3. Consider the system of differential equations

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)(x^2 - y^2), \quad \frac{dy}{dt} = a_1(t)y + a_2(t)2xy, \quad (2.7)$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary t -dependent real functions. This system is a particular type of planar Riccati equation briefly studied in [154]. By writing $z = x + iy$, we find that (2.7) is equivalent to

$$\frac{dz}{dt} = a_0(t) + a_1(t)z + a_2(t)z^2, \quad z \in \mathbb{C}, \quad (2.8)$$

which is a particular type of complex Riccati equations, whose study has attracted some attention. Particular solutions of periodic equations of this type have been investigated in [83, 343].

Every particular solution $(x(t), y(t))$ of (2.7) obeying that $y(t_0) = 0$ at $t_0 \in \mathbb{R}$ satisfies that $y(t) = 0$ for every $t \in \mathbb{R}$. In such a case, $x(t)$ is a particular solution of a real Riccati equation [419]. This suggests us to restrict ourselves to studying (2.7) on $\mathbb{R}_{y \neq 0}^2 \equiv \{(x, y) \mid y \neq 0\} \subset \mathbb{R}^2$.

Let us show that (2.7) on $\mathbb{R}_{y \neq 0}^2$ is a Lie system. This is related to the t -dependent vector field

$$X_t = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3, \quad (2.9)$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \quad (2.10)$$

span a Vessiot–Guldberg real Lie algebra $V \simeq \mathfrak{sl}(2)$ with commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \quad (2.11)$$

Hence, $\{X_t\}_{t \in \mathbb{R}} \subset V^X \subset V$ and V^X is finite-dimensional, which makes X into a Lie system. Hence, it admits a superposition rule.

Example 4. Consider the following system of Riccati equations [102]

$$\frac{dx_i}{dt} = a_0(t) + a_1(t)x_i + a_2(t)x_i^2, \quad i = 1, \dots, n, \quad (2.12)$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary t -dependent functions. System (2.12) is associated to the t -dependent vector field $X_R = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \sum_{i=1}^n \frac{\partial}{\partial x_i}, \quad X_2 = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}, \quad X_3 = \sum_{i=1}^n x_i^2 \frac{\partial}{\partial x_i}. \quad (2.13)$$

Hence, X_R takes values in a Vessiot–Guldberg Lie algebra $\langle X_1, X_2, X_3 \rangle \simeq \mathfrak{sl}(2, \mathbb{R})$. Lie proved that each Lie system on the real line is, up to a local change of variables, a particular case of system (2.12) for $n = 1$ [102, 295].

Example 5. Consider the matrix Riccati equations on \mathbb{R}^2 [10, 236, 292], namely

$$\begin{cases} \frac{dx}{dt} = a_0(t) + a_{11}(t)x + a_{12}(t)y + b_0(t)x^2 + b_1(t)xy, \\ \frac{dy}{dt} = a_1(t) + a_{21}(t)x + a_{22}(t)y + b_0(t)xy + b_1(t)y^2, \end{cases} \quad (2.14)$$

with $a_0(t), a_1(t), a_{11}(t), a_{12}(t), a_{21}(t), a_{22}(t), b_0(t), b_1(t)$ being any t -dependent functions. The associated t -dependent vector field reads

$$X = a_0(t)X_0 + a_1(t)X_1 + a_{11}(t)X_2 + a_{12}(t)X_3 + a_{21}(t)X_4 + a_{22}(t)X_5 + b_0(t)X_6 + b_1(t)X_7, \quad (2.15)$$

where the vector fields

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x}, & X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial x}, & X_3 &= y \frac{\partial}{\partial x}, & X_4 &= x \frac{\partial}{\partial y}, \\ X_5 &= y \frac{\partial}{\partial y}, & X_6 &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, & X_7 &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \end{aligned} \quad (2.16)$$

span an eight-dimensional Lie algebra V^{MR} with non-vanishing commutation relations

$$\begin{aligned} [X_0, X_2] &= X_0, & [X_0, X_4] &= X_1, & [X_0, X_6] &= 2X_2 + X_5, & [X_0, X_7] &= X_3, \\ [X_1, X_3] &= X_0, & [X_1, X_5] &= X_1, & [X_1, X_6] &= X_4, & [X_1, X_7] &= X_2 + 2X_5, \\ [X_2, X_3] &= -X_3, & [X_2, X_4] &= X_4, & [X_2, X_6] &= X_6, & [X_3, X_4] &= X_5 - X_2, \\ [X_3, X_5] &= -X_3, & [X_3, X_6] &= X_7, & [X_4, X_5] &= X_4, & [X_4, X_7] &= X_6, \\ [X_5, X_7] &= X_7. \end{aligned} \quad (2.17)$$

Since X takes value in the Lie algebra V^{MR} , it becomes a Lie system [236]. Indeed, it can be proven that V^{MR} is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ (see Table 1 in Appendix 1 [210]).

2.2 Lie systems on Lie groups

Every Lie system X associated with a Vessiot–Guldberg Lie algebra V gives rise by integrating V to a (generally local) Lie group action $\varphi : G \times N \rightarrow N$ whose fundamental vector fields are the elements of V and such that $T_e G \simeq V$ with e being the neutral element of G [349]. If we assume X to be of the form (2.5), its associated Lie group action allows us to write the general solution of X as

$$x(t) = \varphi(g_1(t), x_0), \quad x_0 \in \mathbb{R}^n, \quad (2.18)$$

with $g_1(t)$ being a particular solution of

$$\frac{dg}{dt} = - \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g), \quad (2.19)$$

where X_1^R, \dots, X_r^R is a certain basis of right-invariant vector fields on G such that $X_\alpha^R(e) = \mathfrak{a}_\alpha \in T_e G$, with $\alpha = 1, \dots, r$, and each \mathfrak{a}_α is the element of $T_e G$ associated to the fundamental vector field X_α (see [89] for details).

Since X_1^R, \dots, X_r^R span a finite-dimensional real Lie algebra, the Lie–Scheffers Theorem guarantees that (2.19) admits a superposition rule and becomes a Lie system. Indeed, as the right-hand side of (2.19) is invariant under every $R_{g_0^*}$ with $R_{g_0} : g' \in G \mapsto g' \cdot g_0 \in G$ being the multiplication on the right on G by $g_0 \in G$, the general solution to (2.19) can be brought into the form

$$g(t) = R_{g_0} g_1(t), \quad g_0 \in G, \quad (2.20)$$

where $g_1(t)$ is a particular solution of (2.19) [89]. In other words, (2.19) admits a superposition rule.

Several methods can now be applied to solve (2.19). If $\mathfrak{g} \simeq T_e G$ is solvable, the theory of reduction of Lie systems allows us to integrate (2.19) by quadratures. More generally, we can use a *Levi decomposition* of \mathfrak{g} to write $\mathfrak{g} \simeq \mathfrak{r} \oplus_s (\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_l)$, where \mathfrak{r} is the radical of \mathfrak{g} , the Lie algebra $\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_l$ is the direct sum of a family of simple Lie subalgebras of \mathfrak{g} , and \oplus_s denotes a semi-direct sum of \mathfrak{r} and $\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_l$. The *theorem of reduction of Lie systems* [92, Theorem 2] yields, with the aid of the previous decomposition, that the solution of a Lie system (2.19) defined on a Lie group with Lie algebra $\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_l$ enables us to construct a t -dependent change of variables that transforms (2.19) into a Lie system on a Lie group with Lie algebra \mathfrak{r} , which is integrable by quadratures (see [92] for details). Summarizing, the explicit integration of (2.19) reduces to providing a particular solution of a Lie system related to $\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_l$.

2.3 An algorithm to derive superposition rules

General solutions of Lie systems can also be investigated through superposition rules. There exist different procedures to derive them [18, 91, 419], but we hereafter use the method devised in [91], which is based on the *diagonal prolongation* notion [91, 102].

Definition 4. Given a t -dependent vector field X on N , its *diagonal prolongation* \tilde{X} to $N^{(m+1)}$ is the unique t -dependent vector field on $N^{(m+1)}$ such that

- Given $\text{pr} : (x_{(0)}, \dots, x_{(m)}) \in N^{(m+1)} \mapsto x_{(0)} \in N$, we have that $\text{pr}_* \tilde{X}_t = X_t \forall t \in \mathbb{R}$.
- \tilde{X} is invariant under the permutations $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \dots, m$.

In coordinates, we have that

$$X = \sum_{i=1}^n X^i(t, x) \frac{\partial}{\partial x_i} \Rightarrow \tilde{X} = \sum_{j=0}^{m+1} X_{(j)} = \sum_{j=0}^{m+1} \sum_{i=1}^n X^i(t, x_{(j)}) \frac{\partial}{\partial (x_i)_{(j)}}. \quad (2.21)$$

The procedure to determine superposition rules described in [91] goes as follows

- Take a basis X_1, \dots, X_r of a Vessiot–Guldberg Lie algebra V associated with the Lie system.
- Choose the minimum integer m so that the diagonal prolongations to N^m of X_1, \dots, X_r are linearly independent at a generic point.
- Obtain n functionally independent first-integrals F_1, \dots, F_n common to all the diagonal prolongations, $\tilde{X}_1, \dots, \tilde{X}_r$, to $N^{(m+1)}$, for instance, by the *method of characteristics*. We require such functions to hold that

$$\frac{\partial(F_1, \dots, F_n)}{\partial((x_1)_{(0)}, \dots, (x_n)_{(0)})} \neq 0. \quad (2.22)$$

- Assume that these integrals take certain constant values, i.e., $F_i = k_i$ with $i = 1, \dots, n$, and employ these equalities to express the variables $(x_1)_{(0)}, \dots, (x_n)_{(0)}$ in terms of the variables of the other copies of N within $N^{(m+1)}$ and the constants k_1, \dots, k_n . The obtained expressions constitute a superposition rule in terms of any generic family of m particular solutions and n constants.

There exists another method to obtain superposition rules frequently used in applications [419]. This second method goes as follows. The general solution $x(t)$ of X can be written as $x(t) = \varphi(g_1(t), x(0))$, where $g_1(t)$ is the solution of the associated Lie system (2.19) with $g_1(0) = e$ and φ is the Lie group action obtained by integrating a Vessiot–Guldberg Lie algebra of X . The curve $g_1(t)$ can be characterized as the unique solution to the algebraic system $x_p(t) = \varphi(g_1(t), x_p(0))$, where $x_p(t)$ ranges over a “sufficient large set” of particular solutions of X [419]. Obtaining $g_1(t)$ from these equations and substituting in $x(t) = \varphi(g_1(t), x(0))$, we can write $x(t)$ in terms of the particular solutions $x_p(t)$ and the parameter $x(0)$ giving rise to a superposition rule for X . In general, this and previous method are convenient to derive superposition rules for different types of systems.

2.4 New applications of Lie systems

Here we show a few new applications of Lie systems through examples of physical relevance. These applications were obtained by the author of this thesis for the first time.

The second order Riccati equation

The most general class of second-order Riccati equations is given by the family of second-order differential equations of the form

$$\frac{d^2x}{dt^2} + (f_0(t) + f_1(t)x) \frac{dx}{dt} + c_0(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 = 0, \quad (2.23)$$

with

$$f_1(t) = 3\sqrt{c_3(t)}, \quad f_0(t) = \frac{c_2(t)}{\sqrt{c_3(t)}} - \frac{1}{2c_3(t)} \frac{dc_3}{dt}(t), \quad c_3(t) \neq 0 \quad (2.24)$$

and $c_0(t), c_1(t), c_2(t)$ being arbitrary t -dependent functions with $c_3(t) > 0$ for every $t \in \mathbb{R}$. These equations arise by reducing third-order linear differential equations through a dilation symmetry and a t -reparametrization [115]. Their interest is due to their use in the study of several physical and mathematical problems [101, 115, 120, 197, 223].

It was recently discovered that a quite general family of second-order Riccati equation (2.23) admits a t -dependent non-natural regular Lagrangian of the form

$$L(t, x, v) = \frac{1}{v + U(t, x)}, \quad (2.25)$$

with $U(t, x) = a_0(t) + a_1(t)x + a_2(t)x^2$ and $a_0(t), a_1(t), a_2(t)$ being certain functions related to the t -dependent coefficients of (2.23), see [115]. Therefore,

$$p = \frac{\partial L}{\partial v} = \frac{-1}{(v + U(t, x))^2}, \quad (2.26)$$

and the image of the Legendre transform $\mathbb{F}L : (t, x, v) \in \mathcal{W} \subset \mathbb{R} \times \mathbb{T}\mathbb{R} \mapsto (t, x, p) \in \mathbb{R} \times \mathbb{T}^*\mathbb{R}$, where $\mathcal{W} = \{(t, x, v) \in \mathbb{R} \times \mathbb{T}\mathbb{R} \mid v + U(t, x) \neq 0\}$, is the open submanifold $\mathbb{R} \times \mathcal{O}$ where $\mathcal{O} = \{(x, p) \in \mathbb{T}^*\mathbb{R} \mid p < 0\}$. The Legendre transform is not injective, as $(t, x, p) = \mathbb{F}L(t, x, v)$ for $v = \pm 1/\sqrt{-p} - U(t, x)$. Nevertheless, it can become so by restricting it to the open $\mathcal{W}_+ = \{(t, x, v) \in \mathbb{R} \times \mathbb{T}\mathbb{R} \mid v + U(t, x) > 0\}$. In such a case, $v = 1/\sqrt{-p} - U(t, x)$ and we can define over $\mathbb{R} \times \mathcal{O}$ the t -dependent Hamiltonian

$$h(t, x, p) = p \left(\frac{1}{\sqrt{-p}} - U(t, x) \right) - \sqrt{-p} = -2\sqrt{-p} - pU(t, x). \quad (2.27)$$

Its Hamilton equations read

$$\begin{cases} \frac{dx}{dt} = \frac{\partial h}{\partial p} = \frac{1}{\sqrt{-p}} - U(t, x) = \frac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\ \frac{dp}{dt} = -\frac{\partial h}{\partial x} = p \frac{\partial U}{\partial x}(t, x) = p(a_1(t) + 2a_2(t)x). \end{cases} \quad (2.28)$$

Since the general solution $x(t)$ of a second-order Riccati equation (2.23) can be recovered from the general solution $(x(t), p(t))$ of its corresponding system (2.28), the analysis of the latter provides information about general solutions of such second-order Riccati equations.

The important point now is that system (2.28) is a Lie system. Indeed, consider the vector fields on \mathcal{O} of the form

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{-p}} \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}, & X_4 &= x^2 \frac{\partial}{\partial x} - 2xp \frac{\partial}{\partial p}, \\ X_5 &= \frac{x}{\sqrt{-p}} \frac{\partial}{\partial x} + 2\sqrt{-p} \frac{\partial}{\partial p}. \end{aligned} \quad (2.29)$$

Their non-vanishing commutation relations read

$$\begin{aligned} [X_1, X_3] &= \frac{1}{2}X_1, & [X_1, X_4] &= X_5, & [X_2, X_3] &= X_2, & [X_2, X_4] &= 2X_3, \\ [X_2, X_5] &= X_1, & [X_3, X_4] &= X_4, & [X_3, X_5] &= \frac{1}{2}X_5, \end{aligned} \quad (2.30)$$

and therefore span a five-dimensional Lie algebra V of vector fields. Additionally, the t -dependent vector field X_t associated with (2.28) holds

$$X_t = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4. \quad (2.31)$$

In view of (2.31), system (2.28) is a Lie system. Note also that a similar result would have been obtained by restricting the Legendre transform over the open $\mathcal{W}_- = \{(t, x, v) \in \mathbb{R} \times \mathbb{T}\mathbb{R} \mid v + U(t, x) < 0\}$.

Let us use the theory of Lie systems to reduce the integration of (2.28) to solving a Lie system on a Lie group. Using a Levi decomposition of V , we get $V \simeq V_1 \oplus_s V_2$, with $V_2 = \langle X_2, X_3, X_4 \rangle$ being a semisimple Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and $V_1 = \langle X_1, X_5 \rangle$ being the radical of V . Hence, V is isomorphic to the Lie algebra of a Lie group $G \equiv \mathbb{R}^2 \rtimes SL(2, \mathbb{R})$, where \rtimes denotes a semidirect product of \mathbb{R}^2 by $SL(2, \mathbb{R})$, and there exists a local action $\Phi : G \times \mathcal{O} \rightarrow \mathcal{O}$ whose fundamental vector fields are those of V . It is a long, but simple, computation to show that

$$\Phi \left(\left((\lambda_1, \lambda_5), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right), (x, p) \right) = \left(\frac{\sqrt{-\bar{p}}x - \lambda_1}{\sqrt{-\bar{p}} + \lambda_5}, -(\sqrt{-\bar{p}} + \lambda_5)^2 \right), \quad (2.32)$$

where $\bar{x} = (\alpha x + \beta)/(\gamma x + \delta)$, $\bar{p} = p(\gamma x + \delta)^2$ and $\alpha\delta - \beta\gamma = 1$, is one of such actions (for a detailed example of how to derive these actions see [102, Ch. 2]).

The above action enables us to write the general solution $\xi(t)$ of system (2.28) in the form $\xi(t) = \Phi(g(t), \xi_0)$, where $\xi_0 \in G$ and $g(t)$ is a particular solution of

$$\frac{dg}{dt} = - (X_1^R(g) - a_0(t)X_2^R(g) - a_1(t)X_3^R(g) - a_2(t)X_4^R(g)), \quad g(0) = e, \quad (2.33)$$

on G , with the X_α^R being a family of right-invariant vector fields on G whose vectors $X_\alpha^R(e) \in T_e G$ close on the opposite commutation relations of the vector fields X_α (cf. [102]).

We now turn to apply to Lie systems the theory of reduction for Lie systems. Since $T_e G \simeq \mathbb{R}^2 \oplus_s \mathfrak{sl}(2, \mathbb{R})$, a particular solution of a Lie system of the form (2.33) but on $SL(2, \mathbb{R})$, which amounts to integrating (first-order) Riccati equations (cf. [18, 102]), provides us with a transformation which maps system (2.33) into an easily integrable Lie system on \mathbb{R}^2 . In short, the explicit determination of the general solution of a second-order Riccati equation reduces to solving Riccati equations.

Another way of analyzing the solutions of (2.28) is based on the determination of a superposition rule. According to the method sketched in Subsection 2.3, a superposition rule for a Lie system (2.28), which admits a decomposition of the form (2.31), can be obtained through two common functionally independent first-integrals I_1, I_2 , for the diagonal prolongations $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4, \tilde{X}_5$ to a certain $T^*N^{(m+1)}$ provided their prolongations to $T^*\mathbb{R}^m$ are linearly independent at a generic point and

$$\frac{\partial(I_1, I_2)}{\partial(x(0), p(0))} \neq 0. \quad (2.34)$$

In our case, it can easily be verified that $m = 3$. The resulting first-integrals, derived through a long but easy calculation (see [101] for a similar procedure), read

$$\begin{aligned} F_0 &= (x_{(2)} - x_{(3)})\sqrt{p_{(2)}p_{(3)}} + (x_{(3)} - x_{(1)})\sqrt{p_{(3)}p_{(1)}} + (x_{(1)} - x_{(2)})\sqrt{p_{(1)}p_{(2)}}, \\ F_1 &= (x_{(1)} - x_{(2)})\sqrt{p_{(1)}p_{(2)}} + (x_{(2)} - x_{(0)})\sqrt{p_{(2)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(0)}p_{(1)}}, \\ F_2 &= (x_{(1)} - x_{(3)})\sqrt{p_{(1)}p_{(3)}} + (x_{(3)} - x_{(0)})\sqrt{p_{(3)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(0)}p_{(1)}}. \end{aligned} \quad (2.35)$$

Note that given a family of solutions $(x_{(i)}(t), p_{(i)}(t))$, with $i = 0, \dots, 3$, of (2.28), then $d\bar{F}_j/dt = \tilde{X}_t F_j = 0$ for $j = 0, 1, 2$ and $\bar{F}_j = F_j(x_{(0)}(t), p_{(0)}(t), \dots, x_{(3)}(t), p_{(3)}(t))$.

In order to derive a superposition rule, we just need to obtain the value of $p_{(0)}$ from the equation $k_1 = F_1$, where k_1 is a real constant, to get

$$\sqrt{-p_{(0)}} = \frac{k_1 + (x_{(2)} - x_{(1)})\sqrt{p_{(1)}p_{(2)}}}{(x_{(2)} - x_{(0)})\sqrt{-p_{(2)}} + (x_{(0)} - x_{(1)})\sqrt{-p_{(1)}}}, \quad (2.36)$$

and then plug this value into the equation $k_2 = F_2$ to have

$$\begin{aligned} x_{(0)} &= \frac{k_1\Gamma(x_{(1)}, p_{(1)}, x_{(3)}, p_{(3)}) + k_2\Gamma(x_{(2)}, p_{(2)}, x_{(1)}, p_{(1)}) - F_0 x_{(1)}\sqrt{-p_{(1)}}}{k_1(\sqrt{-p_{(1)}} - \sqrt{-p_{(3)}}) + k_2(\sqrt{-p_{(2)}} - \sqrt{-p_{(1)}}) - \sqrt{-p_{(1)}}F_0}, \\ p_{(0)} &= -\left[k_1/F_0(\sqrt{-p_{(3)}} - \sqrt{-p_{(1)}}) + k_2/F_0(\sqrt{-p_{(1)}} - \sqrt{-p_{(2)}}) + \sqrt{-p_{(1)}}\right]^2, \end{aligned} \quad (2.37)$$

where $\Gamma(x_{(i)}, p_{(i)}, x_{(j)}, p_{(j)}) = \sqrt{-p_{(i)}}x_{(i)} - \sqrt{-p_{(j)}}x_{(j)}$. The above expressions give us a superposition rule $\Phi : (x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}; k_1, k_2) \in T^*\mathbb{R}^3 \times \mathbb{R}^2 \mapsto (x_{(0)}, p_{(0)}) \in T^*\mathbb{R}$ for system (2.28). Finally, since every $x_{(i)}(t)$ is a particular solution for (2.23), the map $\Upsilon = \tau \circ \Phi$ gives the general solution of second-order Riccati equations in terms of three generic particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ of (2.23), the corresponding $p_{(1)}(t), p_{(2)}(t), p_{(3)}(t)$ and two real constants k_1, k_2 .

Kummer–Schwarz equations

Here we analyze second- and third-order Kummer–Schwarz equations (KS-2 and KS-3, respectively), [51, 52, 224, 287], with the aid of Lie systems.

The KS-2 equations take the form

$$\frac{d^2x}{dt^2} = \frac{3}{2x} \left(\frac{dx}{dt} \right)^2 - 2c_0x^3 + 2b_1(t)x, \quad (2.38)$$

with c_0 being a real constant and $b_1(t)$ a t -dependent function. KS-2 equations are a particular case of second-order Gambier equations [87, 224] and appear in the study of cosmological models [329]. In addition, their relations to other differential equations like Milne–Pinney equations [224], make them an alternative approach to the analysis of many physical problems [87, 259, 287].

In the case of KS-3 equations, i.e., we have

$$\frac{d^3x}{dt^3} = \frac{3}{2} \left(\frac{dx}{dt} \right)^{-1} \left(\frac{d^2x}{dt^2} \right)^2 - 2c_0(x) \left(\frac{dx}{dt} \right)^3 + 2b_1(t) \frac{dx}{dt}, \quad (2.39)$$

where $c_0 = c_0(x)$ and $b_1 = b_1(t)$ are arbitrary.

The relevance of KS-3 equations resides in their relation to the Kummer’s problem [51, 52, 55], Milne–Pinney [287] and Riccati equations [130, 178, 287]. Such relations can be useful in the interpretation of physical systems through KS-3 equations, e.g., the case of quantum non-equilibrium dynamics of many body systems [218]. Furthermore, KS-3 equations with $c_0 = 0$ can be rewritten as $\{x, t\} = 2b_1(t)$, where $\{x, t\}$ is the *Schwarzian derivative* [288] of the function $x(t)$ relative to t .

KS-2 and KS-3 appear in other related themes [27, 51, 52, 57, 320, 393, 392]. Moreover, there is some interest on studying of particular solutions of KS-2 and KS-3 equations, which have been analyzed in several manners in the literature, e.g., through non-local transformations or in terms of solutions to other differential equations [51, 55, 57, 224, 287]. From a physical viewpoint, KS-2 and KS-3 equations occur in the study of Milne–Pinney equations, Riccati equations, and time-dependent frequency harmonic oscillators [87, 130, 224, 287], which are of certain relevancy in two-body problems [47, 50], Quantum Mechanics [259, 276], Classical Mechanics [329], etcetera [352].

We show that KS-2 equations can be studied through two $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems [353], i.e., Lie systems that describe the integral curves of a t -dependent vector field taking values in Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. This new result slightly generalizes previous findings about these equations [87].

Afterwards, we obtain two Lie group actions whose fundamental vector fields correspond to those of the abovementioned Vessiot–Guldberg Lie algebras. These actions allow us to prove that the explicit integration of KS-2 and KS-3 equations is

equivalent to working out a particular solution of a Lie system on $SL(2, \mathbb{R})$. Further, we will see that Riccati and Milne–Pinney equations exhibit similar features.

We show that the knowledge of the general solution of any of the reviewed equations enables us to solve simultaneously any other related to the same equation on $SL(2, \mathbb{R})$. This fact provides a new powerful and general way of linking solutions of these equations, which were previously known to be related through ad hoc expressions in certain cases [130, 287].

Second-order Kummer Schwarz equations

Consider the first-order system associated with (2.38)

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = \frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 + 2b_1(t)x, \end{cases} \quad (2.40)$$

on $\mathbb{T}\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$, obtained by adding the new variable $v \equiv dx/dt$ to the KS-2 equation (2.38). This system describes the integral curves of the t -dependent vector field

$$X_t = v \frac{\partial}{\partial x} + \left(\frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 + 2b_1(t)x \right) \frac{\partial}{\partial v} = M_3 + b_1(t)M_1, \quad (2.41)$$

where

$$M_1 = 2x \frac{\partial}{\partial v}, \quad M_2 = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad M_3 = v \frac{\partial}{\partial x} + \left(\frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 \right) \frac{\partial}{\partial v} \quad (2.42)$$

satisfy the commutation relations

$$[M_1, M_3] = 2M_2, \quad [M_1, M_2] = M_1, \quad [M_2, M_3] = M_3. \quad (2.43)$$

These vector fields span a three-dimensional real Lie algebra V of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ [87, 102]. Hence, in view of (2.41) and the Lie–Scheffers Theorem, X admits a superposition rule and becomes a Lie system associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, i.e., an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system.

The knowledge of a Lie group action $\varphi_{2KS} : G \times \mathbb{T}\mathbb{R}_0 \rightarrow \mathbb{T}\mathbb{R}_0$ whose fundamental vector fields are V and $T_e G \simeq V$ allows us to express the general solution of X in the form (2.18), in terms of a particular solution of a Lie system (2.19) on G . Let us determine φ_{2KS} in such a way that our procedure can easily be extended to third-order Kummer–Schwarz equations.

Consider the basis of matrices of $\mathfrak{sl}(2, \mathbb{R})$ given by

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad (2.44)$$

satisfying the commutation relations

$$[a_1, a_3] = 2a_2, \quad [a_1, a_2] = a_1, \quad [a_2, a_3] = a_3, \quad (2.45)$$

which match those satisfied by M_1, M_2 and M_3 . So, the linear function $\rho : \mathfrak{sl}(2, \mathbb{R}) \rightarrow V$ mapping a_α into M_α , with $\alpha = 1, 2, 3$, is a Lie algebra isomorphism. If we consider it as an infinitesimal Lie group action, we can then ensure that there exists a local Lie group action $\varphi_{2KS} : SL(2, \mathbb{R}) \times \mathbb{T}\mathbb{R}_0 \rightarrow \mathbb{T}\mathbb{R}_0$ obeying the required properties. In particular,

$$\frac{d}{ds} \varphi_{2KS}(\exp(-sa_\alpha), \mathbf{t}_x) = M_\alpha(\varphi_{2KS}(\exp(-sa_\alpha), \mathbf{t}_x)), \quad (2.46)$$

where $\mathbf{t}_x \equiv (x, v) \in \mathbb{T}_x\mathbb{R}_0 \subset \mathbb{T}\mathbb{R}_0$, $\alpha = 1, 2, 3$, and $s \in \mathbb{R}$. This condition determines the action on $\mathbb{T}\mathbb{R}_0$ of the elements of $SL(2, \mathbb{R})$ of the form $\exp(-sa_\alpha)$, with $\alpha = 1, 2, 3$ and $s \in \mathbb{R}$. By integrating M_1 and M_2 , we obtain

$$\begin{aligned} \varphi(\exp_{2KS}(-\lambda_1 a_1), \mathbf{t}_x) &= (x, v + 2x\lambda_1), \\ \varphi(\exp_{2KS}(-\lambda_2 a_2), \mathbf{t}_x) &= (xe^{\lambda_2}, ve^{2\lambda_2}). \end{aligned} \quad (2.47)$$

Observe that M_3 is not defined on $\mathbb{T}\mathbb{R}_0$. Hence, its integral curves, let us say $(x(\lambda_3), v(\lambda_3))$, must be fully contained in either $\mathbb{T}\mathbb{R}^+$ or $\mathbb{T}\mathbb{R}^-$. These integral curves are determined by the system

$$\frac{dx}{d\lambda_3} = v, \quad \frac{dv}{d\lambda_3} = \frac{3v^2}{2x} - 2c_0x^3. \quad (2.48)$$

When $v \neq 0$, we obtain

$$\frac{dv^2}{dx} = \frac{3v^2}{x} - 4c_0x^3 \implies v^2(\lambda_3) = x^3(\lambda_3)\Gamma - 4c_0x^4(\lambda_3), \quad (2.49)$$

for a real constant Γ . Hence, for each integral curve $(x(\lambda_3), v(\lambda_3))$, we have

$$\Gamma = \frac{v^2(\lambda_3) + 4c_0x^4(\lambda_3)}{x^3(\lambda_3)}. \quad (2.50)$$

Moreover, $d\Gamma/d\lambda_3 = 0$ not only for solutions of (2.48) with $v(\lambda_3) \neq 0$ for every λ_3 , but for any solution of (2.48). Using the above results and (2.48), we see that

$$\frac{dx}{d\lambda_3} = \text{sg}(v)\sqrt{\Gamma x^3 - 4c_0x^4} \implies x(\lambda_3) = \frac{x(0)}{F_{\lambda_3}(x(0), v(0))}, \quad (2.51)$$

where sg is the well-known *sign function* and

$$F_{\lambda_3}(\mathbf{t}_x) = \left(1 - \frac{v\lambda_3}{2x}\right)^2 + c_0x^2\lambda_3^2. \quad (2.52)$$

Now, from (2.51) and taking into account the first equation within (2.48), it immediately follows that

$$\varphi_{2KS}(\exp(-\lambda_3 \mathbf{a}_3), \mathbf{t}_x) = \left(\frac{x}{F_{\lambda_3}(\mathbf{t}_x)}, \frac{v - \frac{v^2 + 4c_0 x^4}{2x} \lambda_3}{F_{\lambda_3}^2(\mathbf{t}_x)} \right). \quad (2.53)$$

Let us employ previous results to determine the action on $\mathfrak{sl}(2, \mathbb{R})$ of those elements g close to the neutral element $e \in SL(2, \mathbb{R})$. Using the so-called *canonical coordinates of the second kind* [261], we can write g within an open neighborhood U of e in a unique form as

$$g = \exp(-\lambda_3 \mathbf{a}_3) \exp(-\lambda_2 \mathbf{a}_2) \exp(-\lambda_1 \mathbf{a}_1), \quad (2.54)$$

for real constants λ_1, λ_2 and λ_3 . This allows us to obtain the action of every $g \in U$ on $\mathbb{T}\mathbb{R}_0$ through the composition of the actions of elements $\exp(-\lambda_\alpha \mathbf{a}_\alpha)$, with $\lambda_\alpha \in \mathbb{R}$ for $\alpha = 1, 2, 3$. To do so, we determine the constants λ_1, λ_2 and λ_3 associated to each $g \in U$ in (2.54).

Considering the standard matrix representation of $SL(2, \mathbb{R})$, we can express every $g \in SL(2, \mathbb{R})$ as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \quad (2.55)$$

In view of (2.44), and comparing (2.54) and (2.55), we obtain

$$\alpha = e^{\lambda_2/2}, \quad \beta = -e^{\lambda_2/2} \lambda_1, \quad \gamma = e^{\lambda_2/2} \lambda_3. \quad (2.56)$$

Consequently,

$$\lambda_1 = -\beta/\alpha, \quad \lambda_2 = 2 \log \alpha, \quad \lambda_3 = \gamma/\alpha, \quad (2.57)$$

and, from the basis (2.44), the decomposition (2.54) and expressions (2.47) and (2.53), the action reads

$$\varphi_{2KS}(g, \mathbf{t}_x) = \left(\frac{x}{F_g(\mathbf{t}_x)}, \frac{1}{F_g^2(\mathbf{t}_x)} \left[(v\alpha - 2x\beta) \left(\delta - \frac{\gamma v}{2x} \right) - 2c_0 x^3 \alpha \gamma \right] \right), \quad (2.58)$$

where

$$F_g(\mathbf{t}_x) = \left(\delta - \frac{\gamma v}{2x} \right)^2 + c_0 x^2 \gamma^2. \quad (2.59)$$

Although this expression has been derived for g being close to e , it can be proved that the action is properly defined at points (g, \mathbf{t}_x) such that $F_g(\mathbf{t}_x) \neq 0$. If $c_0 > 0$, then $F_g(\mathbf{t}_x) > 0$ for all $g \in SL(2, \mathbb{R})$ and $\mathbf{t}_x \in \mathbb{T}\mathbb{R}_0$. So, φ_{2KS} becomes globally defined. Otherwise, $F_g(\mathbf{t}_x) > 0$ for g close enough to e . Then, φ_{2KS} is only defined on a neighborhood of e .

The action φ_{2KS} also permits us to write the general solution of system (2.40) in the form $(x(t), v(t)) = \varphi_{2KS}(g(t), \mathbf{t}_x)$, with $g(t)$ being a particular solution of

$$\frac{dg}{dt} = -X_3^R(g) - b_1(t)X_1^R(g), \quad (2.60)$$

where X_α^R , with $\alpha = 1, 2, 3$, are the right-invariant vector fields on $SL(2, \mathbb{R})$ satisfying $X_\alpha^R(e) = \mathbf{a}_\alpha$ [89, 102]. Additionally, as $x(t)$ is the general solution of KS-2 equation (2.38), we readily see that

$$x(t) = \tau \circ \varphi_{2KS}(g(t), \mathbf{t}_x), \quad (2.61)$$

with $\tau : (x, v) \in \mathbb{T}\mathbb{R} \mapsto x \in \mathbb{R}$, the natural tangent bundle projection, provides us with the general solution of (2.38) in terms of a particular solution of (2.60).

Conversely, we prove that we can recover a particular solution to (2.60) from the knowledge of the general solution of (2.38). For simplicity, we will determine the particular solution $g_1(t)$ with $g_1(0) = e$. Given two particular solutions $x_1(t)$ and $x_2(t)$ of (2.38) with $dx_1/dt(t) = dx_2/dt(t) = 0$, the expression (2.61) implies that

$$(x_i(t), v_i(t)) = \varphi_{2KS}(g_1(t), (x_i(0), 0)), \quad i = 1, 2. \quad (2.62)$$

Writing the above expression explicitly, we get

$$\begin{aligned} -\frac{x_i(0)v_i(t)}{2x_i^2(t)} &= \beta(t)\delta(t) + c_0x_i^2(0)\alpha(t)\gamma(t), \\ \frac{x_i(0)}{x_i(t)} &= \delta^2(t) + c_0x_i^2(0)\gamma^2(t), \end{aligned} \quad (2.63)$$

for $i = 1, 2$. The first two equations allow us to determine the value of $\beta(t)\delta(t)$ and $\alpha(t)\gamma(t)$. Meanwhile, we can obtain the value of $\delta^2(t)$ and $\gamma^2(t)$ from the other two ones. As $\delta(0) = 1$, we know that $\delta(t)$ is positive when close to $t = 0$. Taking into account that we have already worked out $\delta^2(t)$, we can determine $\delta(t)$ for small values of t . Since we have already obtained $\beta(t)\delta(t)$, we can also derive $\beta(t)$ for small values of t by using $\delta(t)$. Note that $\alpha(0) = 1$. So, $\alpha(t)$ is positive for small values of t , and the sign of $\alpha(t)\gamma(t)$ determines the sign of $\gamma(t)$ around $t = 0$. In view of this, the value of $\gamma(t)$ can be determined from $\gamma^2(t)$ in the interval around $t = 0$. Summing up, we can obtain algebraically a particular solution of (2.63) with $g_1(0) = e$ from the general solution of (2.38).

Third-order Kummer–Schwarz equation

Let us write KS-3 equations (2.39) as a first-order system

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = a, \\ \frac{da}{dt} = \frac{3}{2} \frac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v, \end{cases} \quad (2.64)$$

in the open submanifold $\mathcal{O}_2 = \{(x, v, a) \in \mathbb{T}^2\mathbb{R} \mid v \neq 0\}$ of $\mathbb{T}^2\mathbb{R} \simeq \mathbb{R}^3$, the referred to as *second-order tangent bundle* of \mathbb{R} [13].

Consider now the set of vector fields on \mathcal{O}_2 given by

$$\begin{aligned} N_1 &= 2v \frac{\partial}{\partial a}, \\ N_2 &= v \frac{\partial}{\partial v} + 2a \frac{\partial}{\partial a}, \\ N_3 &= v \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + \left(\frac{3}{2} \frac{a^2}{v} - 2c_0(x)v^3 \right) \frac{\partial}{\partial a}, \end{aligned} \quad (2.65)$$

which satisfy the commutation relations

$$[N_1, N_3] = 2N_2, \quad [N_1, N_2] = N_1, \quad [N_2, N_3] = N_3. \quad (2.66)$$

Thus, they span a three-dimensional Lie algebra of vector fields V isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since (2.64) is determined by the t -dependent vector field

$$X_t = v \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + \left(\frac{3}{2} \frac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v \right) \frac{\partial}{\partial a}, \quad (2.67)$$

we can write $X_t = N_3 + b_1(t)N_1$. Consequently, X takes values in the finite-dimensional Vessiot–Guldberg Lie algebra V and becomes an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. This generalizes the result provided in [216] for $c_0(x) = \text{const.}$

We shall now reduce the integration of (2.64) with $c_0(x) = \text{const.}$, and in consequence the integration of the related (2.39), to working out a particular solution of the Lie system (2.60). To do so, we employ the Lie group action $\varphi_{3KS} : SL(2, \mathbb{R}) \times \mathcal{O}_2 \rightarrow \mathcal{O}_2$ whose infinitesimal action is given by the Lie algebra isomorphism $\rho : \mathfrak{sl}(2, \mathbb{R}) \rightarrow V$ satisfying that $\rho(\mathfrak{a}_\alpha) = N_\alpha$, with $\alpha = 1, 2, 3$. This Lie group action holds that

$$\frac{d}{ds} \varphi_{3KS}(\exp(-s\mathfrak{a}_\alpha), \mathfrak{t}_x^2) = N_\alpha(\varphi_{3KS}(\exp(-s\mathfrak{a}_\alpha), \mathfrak{t}_x^2)), \quad (2.68)$$

with $\mathbf{t}_x^2 \equiv (x, v, a) \in \mathcal{O}_2$ and $\alpha = 1, 2, 3$. Integrating N_1 and N_2 , we obtain that

$$\varphi_{3KS}(\exp(-\lambda_1 a_1), \mathbf{t}_x^2) = \begin{pmatrix} x \\ v \\ a + 2v\lambda_1 \end{pmatrix} \quad (2.69)$$

and

$$\varphi_{3KS}(\exp(-\lambda_2 a_2), \mathbf{t}_x^2) = \begin{pmatrix} x \\ ve^{\lambda_2} \\ ae^{2\lambda_2} \end{pmatrix}. \quad (2.70)$$

To integrate N_3 , we need to obtain the solutions of

$$\frac{dx}{d\lambda_3} = v, \quad \frac{dv}{d\lambda_3} = a, \quad \frac{da}{d\lambda_3} = \frac{3}{2} \frac{a^2}{v} - 2c_0 v^3. \quad (2.71)$$

Proceeding mutatis mutandis, as in the analysis of system (2.48), we obtain

$$v(\lambda_3) = \frac{v(0)}{F_{\lambda_3}(x(0), v(0), a(0))}, \quad (2.72)$$

with

$$F_{\lambda_3}(\mathbf{t}_x^2) = \left(1 - \frac{a\lambda_3}{2v}\right)^2 + c_0 v^2 \lambda_3^2. \quad (2.73)$$

Taking into account this and the first two equations within (2.71), we see that

$$\varphi_{3KS}(e^{-\lambda_3 a_3}, \mathbf{t}_x^2) = \begin{pmatrix} x + v \int_0^{\lambda_3} F_{\lambda'_3}^{-1}(\mathbf{t}_x^2) d\lambda'_3 \\ F_{\lambda_3}^{-1}(\mathbf{t}_x^2) v \\ v \partial(F_{\lambda_3}^{-1}(\mathbf{t}_x^2)) / \partial \lambda_3 \end{pmatrix}. \quad (2.74)$$

Using decomposition (2.54), we can reconstruct the new action

$$\varphi_{3KS}(g, \mathbf{t}_x^2) = \begin{pmatrix} x + v \int_0^{\gamma/\alpha} \bar{F}_{\lambda_3, g}^{-1}(\mathbf{t}_x^2) d\lambda_3 \\ \bar{F}_{\gamma/\alpha, g}^{-1}(\mathbf{t}_x^2) v \\ v \frac{\partial(\bar{F}_{\lambda_3, g}^{-1}(\mathbf{t}_x^2))}{\partial \lambda_3} \Big|_{\lambda_3 = \gamma/\alpha} \end{pmatrix}, \quad (2.75)$$

with $\bar{F}_{\lambda_3, g}(\mathbf{t}_x^2) = \alpha^{-2} F_{\lambda_3}(x, v\alpha^2, (a\alpha - 2v\beta)\alpha^3 \lambda_3)$, i.e.,

$$\bar{F}_{\lambda_3, g}(\mathbf{t}_x^2) = \left(\frac{1}{\alpha} - \frac{a\alpha - 2v\beta}{2v} \lambda_3\right)^2 + c_0 v^2 \alpha^2 \lambda_3^2. \quad (2.76)$$

This action enables us to write the general solution of (2.64) as

$$(x(t), v(t), a(t)) = \varphi_{3KS}(g(t), \mathbf{t}_x^2), \quad (2.77)$$

where $\mathbf{t}_x^2 \in \mathcal{O}_2$ and $g(t)$ is a particular solution of the equation on $SL(2, \mathbb{R})$ given by (2.60). Hence, if $\tau^2 : (x, v, a) \in \mathbb{T}^2\mathbb{R} \mapsto x \in \mathbb{R}$ is the fiber bundle projection corresponding to the second-order tangent bundle on \mathbb{R} , we can write the general solution of (2.39) in the form

$$x(t) = \tau^2 \circ \varphi_{3KS}(g(t), \mathbf{t}_x^2), \quad (2.78)$$

where $g(t)$ is any particular solution of (2.60).

Conversely, given the general solution of (2.39), we can obtain a particular solution of (2.60). As before, we focus on obtaining the particular solution $g_1(t)$, with $g_1(0) = e$. In this case, given two particular solution $x_1(t), x_2(t)$ of (2.39) with $d^2x_1/dt^2(0) = d^2x_2/dt^2(0) = 0$, we obtain that the t -dependent coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ corresponding to the matrix expression of $g_1(t)$ obey a system similar to (2.63) where v and x have been replaced by a and v , respectively.

The Schwarzian derivative and Kummer–Schwarz

The Schwarzian derivative of a real function $f = f(t)$ is defined by

$$\{f, t\} = \frac{d^3 f}{dt^3} \left(\frac{df}{dt} \right)^{-1} - \frac{3}{2} \left[\frac{d^2 f}{dt^2} \left(\frac{df}{dt} \right)^{-1} \right]^2. \quad (2.79)$$

This derivative is clearly related to KS-3 equations (2.39) with $c_0 = 0$, which can be written as $\{x, t\} = 2b_1(t)$.

Although a superposition rule for studying KS-3 equations was developed in [87], the result provided in there was not valid when $c_0 = 0$, which retrieves the relevant equation $\{x, t\} = 2b_1(t)$. This is why we aim to reconsider this case and its important connection to the Schwarzian derivative.

The vector fields N_1, N_2, N_3 are linearly independent at a generic point of $\mathcal{O}_2 \subset \mathbb{T}^2\mathbb{R}_0$. Therefore, obtaining a superposition rule for (2.64) amounts to obtaining three functionally independent first-integrals F_1, F_2, F_3 common to all diagonal prolongations $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$ in $(\mathcal{O}_2)^2$, such that

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x_0, v_0, a_0)} \neq 0. \quad (2.80)$$

As $[\tilde{N}_1, \tilde{N}_3] = 2\tilde{N}_2$, it suffices to obtain common first-integrals for \tilde{N}_1, \tilde{N}_3 to describe first-integrals common to the integrable distribution \mathcal{D} spanned by $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$.

Let us start by solving $\tilde{N}_1 F = 0$, with $F : \mathcal{O}_2 \rightarrow \mathbb{R}$, i.e.,

$$v_0 \frac{\partial F}{\partial a_0} + v_1 \frac{\partial F}{\partial a_1} = 0. \quad (2.81)$$

The method of characteristics shows that F must be constant along the solutions of the associated *Lagrange–Charpit equations* [144], namely

$$\frac{da_0}{v_0} = \frac{da_1}{v_1}, \quad dx_0 = dx_1 = dv_0 = dv_1 = 0. \quad (2.82)$$

Such solutions are the curves $(x_0(\lambda), v_0(\lambda), a_0(\lambda), x_1(\lambda), v_1(\lambda), a_1(\lambda))$ within \mathcal{O}_2 with $\Delta = v_1(\lambda)a_0(\lambda) - a_1(\lambda)v_0(\lambda)$, for a real constant $\Delta \in \mathbb{R}$, and constant $x_i(\lambda)$ and $v_i(\lambda)$, with $i = 0, 1$. In other words, there exists a function $F_2 : \mathbb{R}^5 \rightarrow \mathbb{R}$ such that $F(x_0, v_0, a_0, x_1, v_1, a_1) = F_2(\Delta, x_0, x_1, v_0, v_1)$.

If we now impose $\tilde{N}_3 F = 0$, we obtain

$$\begin{aligned} \tilde{N}_3 F = \tilde{N}_3 F_2 = & \frac{\Delta + a_1 v_0}{v_1} \frac{\partial F_2}{\partial v_0} + a_1 \frac{\partial F_2}{\partial v_1} \\ & + v_0 \frac{\partial F_2}{\partial x_0} + v_1 \frac{\partial F_2}{\partial x_1} + \frac{3\Delta^2 + 6\Delta a_1 v_0}{2v_1 v_0} \frac{\partial F_2}{\partial \Delta} = 0. \end{aligned} \quad (2.83)$$

We can then write that $\tilde{N}_2 F_2 = (a_1/v_1)\Xi_1 F_2 + \Xi_2 F_2 = 0$, where

$$\begin{aligned} \Xi_1 &= v_0 \frac{\partial}{\partial v_0} + v_1 \frac{\partial}{\partial v_1} + 3\Delta \frac{\partial}{\partial \Delta}, \\ \Xi_2 &= v_0 \frac{\partial}{\partial x_0} + v_1 \frac{\partial}{\partial x_1} + \frac{\Delta}{v_1} \frac{\partial}{\partial v_0} + \frac{3\Delta^2}{2v_0 v_1} \frac{\partial}{\partial \Delta}. \end{aligned} \quad (2.84)$$

As F_2 does not depend on a_1 in the chosen coordinate system, it follows $\Xi_1 F_2 = \Xi_2 F_2 = 0$. Using the characteristics method again, we obtain that $\Xi_1 F_2 = 0$ implies the existence of a new function $F_3 : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that $F_2(\Delta, x_0, x_1, v_0, v_1) = F_3(K_1 \equiv v_1/v_0, K_2 \equiv v_0^3/\Delta, x_0, x_1)$.

The only condition remaining is $\Xi_2 F_3 = 0$. In the local coordinate system $\{K_1, K_2, x_0, x_1\}$, this equation reads

$$v_0 \left(\frac{3}{2K_1} \frac{\partial F_3}{\partial K_2} - \frac{1}{K_2} \frac{\partial F_3}{\partial K_1} + \frac{\partial F_3}{\partial x_0} + K_1 \frac{\partial F_3}{\partial x_1} \right) = 0, \quad (2.85)$$

and its Lagrange–Charpit equations becomes

$$-K_2 dK_1 = \frac{2K_1 dK_2}{3} = dx_0 = \frac{dx_1}{K_1}. \quad (2.86)$$

From the first equality, we obtain that $K_1^3 K_2^2 = \Upsilon_1$ for a certain real constant Υ_1 . In view of this and with the aid of the above system, it turns out

$$\frac{2}{3} K_1^2 dK_2 = dx_1 \longrightarrow \frac{2}{3} \Upsilon_1^{2/3} K_2^{-4/3} dK_2 = dx_1. \quad (2.87)$$

Integrating, we see that $-2K_2K_1^2 - x_1 = \Upsilon_2$ for a certain real constant Υ_2 . Finally, these previous results are used to solve the last part of the Lagrange–Charpit system, i.e.,

$$dx_0 = \frac{dx_1}{K_1} = \frac{4\Upsilon_1 dx_1}{(x_1 + \Upsilon_2)^2} \longrightarrow \Upsilon_3 = x_0 + \frac{4\Upsilon_1}{x_1 + \Upsilon_2}. \quad (2.88)$$

Note that $\partial(\Upsilon_1, \Upsilon_2, \Upsilon_3)/\partial(x_0, v_0, a_0) \neq 0$. Therefore, considering $\Upsilon_1 = k_1$, $\Upsilon_2 = k_2$ and $\Upsilon_3 = k_3$, we can obtain a mixed superposition rule. From these equations, we easily obtain

$$x_0 = \frac{x_1 k_3 + k_2 k_3 - 4k_1}{x_1 + k_2}. \quad (2.89)$$

Multiplying numerator and denominator of the right-hand side by a non-null constant Υ_4 , the above expression can be rewritten as

$$x_0 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}, \quad (2.90)$$

with $\alpha = \Upsilon_4 k_3$, $\beta = \Upsilon_4(k_2 k_3 - 4k_1)$, $\gamma = \Upsilon_4$, $\delta = k_2 \Upsilon_4$. Observe that

$$\alpha\delta - \gamma\beta = 4\Upsilon_4^2 \Upsilon_1 = \frac{4\Upsilon_4^2 v_0^3 v_1^3}{(v_1 a_0 - a_1 v_0)^2} \neq 0. \quad (2.91)$$

Then, choosing an appropriate Υ_4 , we obtain that (2.89) can be rewritten as (2.90) for a family of constants $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \gamma\beta = \pm 1$. It is important to recall that the matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad I = \alpha\delta - \beta\gamma = \pm 1, \quad (2.92)$$

are the matrix description of the Lie group $PGL(2, \mathbb{R})$. Recall that we understand each matrix as the equivalence class related to it.

Operating, we also obtain that

$$v_0 = \frac{I v_1}{(\gamma x_1 + \delta)}, \quad a_0 = I \left[\frac{a_1}{(\gamma x_1 + \delta)^2} - \frac{2v_1^2 \gamma}{(\gamma x_1 + \delta)^3} \right]. \quad (2.93)$$

The above expression together with (2.90) become a superposition rule for KS-3 equations with $c_0 = 0$ (written as a first-order system). In other words, the general solution $(x(t), v(t), a(t))$ of (2.39) with $c_0 = 0$ can be written as

$$(x(t), v(t), a(t)) = \Phi(A, x_1(t), v_1(t), a_1(t)), \quad (2.94)$$

with $(x_1(t), v_1(t), a_1(t))$ being a particular solution, $A \in PGL(2, \mathbb{R})$ and

$$\Phi(A, x_1, v_1, a_1) = \left(\frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}, \frac{I v_1}{(\gamma x_1 + \delta)^2}, I \left[\frac{a_1(\gamma x_1 + \delta) - 2v_1^2 \gamma}{(\gamma x_1 + \delta)^3} \right] \right). \quad (2.95)$$

Moreover, $x(t)$, which is the general solution of a KS-3 equation with $c_0 = 0$, can be determined out of a particular solution $x_1(t)$ and three constants through

$$x(t) = \tau^2 \circ \Phi \left(A, x_1(t), \frac{dx_1}{dt}(t), \frac{d^2x_1}{dt^2}(t) \right), \quad (2.96)$$

where we see that the right-hand part does merely depend on A and $x_1(t)$. This constitutes a *basic superposition rule* [87] for equations $\{x(t), t\} = 2b_1(t)$, i.e., it is an expression that allows us to describe the general solution of any of these equations in terms of a particular solution (without involving its derivatives) and some constants to be related to initial conditions. We shall now employ this superposition rule to describe some properties of the Schwarzian derivative.

From the equation above, we analyze the relation between two particular solutions $x_1(t)$ and $x_2(t)$ of the same equation $\{x, t\} = 2b_1(t)$, i.e., $\{x_1(t), t\} = \{x_2(t), t\}$. Our basic superposition rule (2.96) tells us that from $x_1(t)$ we can generate every other solution of the equation. In particular, there must exist certain real constants c_1, c_2, c_3, c_4 such that

$$x_2(t) = \frac{c_1x_1(t) + c_2}{c_3x_1(t) + c_4}, \quad c_1c_4 - c_2c_3 \neq 0. \quad (2.97)$$

In this way, we recover a relevant property of this type of equations [346].

Our superposition rule (2.96) also provides us with information about the Lie symmetries of $\{x(t), t\} = 2b_1(t)$. Indeed, note that (2.96) implies that the local Lie group action $\varphi : PGL(2, \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}$

$$\varphi(A, x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad (2.98)$$

transforms solutions of $\{x(t), t\} = 2b_1(t)$ into solutions of the same equation. The prolongation [72, 340] $\widehat{\varphi} : PGL(2, \mathbb{R}) \times T^2\mathbb{R}_0 \rightarrow T^2\mathbb{R}_0$ of φ to $T^2\mathbb{R}_0$, i.e.,

$$\widehat{\varphi}(A, \mathbf{t}_x^2) = \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{Iv}{(\gamma x + \delta)^2}, I \frac{a(\gamma x + \delta) - 2\gamma v^2}{(\gamma x + \delta)^3} \right), \quad (2.99)$$

gives rise to a group of symmetries $\varphi(A, \cdot)$ of (2.64) when $c_0 = 0$. The fundamental vector fields of this action are spanned by

$$\begin{aligned} Z_1 &= -\frac{\partial}{\partial x}, & Z_2 &= x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v} + a \frac{\partial}{\partial a}, \\ Z_3 &= -\left[x^2 \frac{\partial}{\partial x} + 2vx \frac{\partial}{\partial v} + 2(ax + v^2) \frac{\partial}{\partial a} \right], \end{aligned} \quad (2.100)$$

which close on a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and commute with X_t for every $t \in \mathbb{R}$. In addition, their projections onto \mathbb{R} must be Lie symmetries

of $\{x(t), t\} = 2b_1(t)$. Indeed, they read

$$S_1 = -\frac{\partial}{\partial x}, \quad S_2 = x \frac{\partial}{\partial x}, \quad S_3 = -x^2 \frac{\partial}{\partial x}, \quad (2.101)$$

which are the known Lie symmetries for these equations [345].

Consider now the equation $\{x(t), t\} = 0$. Obviously, this equation admits the particular solution $x(t) = t$. This, together with our basic superposition rule, show that the general solution of this equation is

$$x(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\delta - \gamma\beta \neq 0, \quad (2.102)$$

recovering another relevant known solution of these equations.

3 Lie–Hamilton systems

We have found that many instances of relevant Lie systems possess Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to a Poisson structure. Such Lie systems are hereafter called *Lie–Hamilton systems*.

Lie–Hamilton systems admit a plethora of geometric properties. For instance, we proved that every Lie–Hamilton system admits a t -dependent Hamiltonian which can be understood as a curve in finite-dimensional Lie algebra of functions (with respect to a certain Poisson structure). These t -dependent Hamiltonians, the here called *Lie–Hamiltonian structures*, are the key to understand the properties of Lie–Hamilton systems [21, 89, 109, 191]. Additionally, Lie–Hamilton systems appear in the analysis of relevant physical and mathematical problems, like second-order Kummer–Schwarz equations and t -dependent Winternitz–Smorodinsky oscillators [109].

In this section we study Lie–Hamilton systems and some of their applications. In particular, our achievements are employed to study superposition rules, Lie symmetries and constants of motion for these systems.

3.1 On the necessity of Lie–Hamilton systems

In this section we start by showing how several relevant Lie systems admit a Vessiot–Guldberg Lie algebra of Hamilton vector fields with respect to a Poisson or symplectic structure. This will allow us to justify the interest of defining a new particular type of Lie systems enjoying such a property: the Lie–Hamilton systems.

Example 1. Let us show that **planar Riccati equations** (2.7), described geometrically by the t -dependent vector field $X = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$ given by (2.9), admit a Vessiot–Guldberg Lie algebra consisting of Hamiltonian vector

fields with respect to a symplectic structure. To do so, we search for a symplectic form, let us say $\omega = f(x, y)dx \wedge dy$, turning the Vessiot–Guldberg Lie algebra $V = \langle X_1, X_2, X_3 \rangle$ into a Lie algebra of Hamiltonian vector fields with respect to it. To ensure that the basis for V given by the vector fields X_1, X_2 and X_3 , (see (2.10)), are locally Hamiltonian relative to ω , we impose $\mathcal{L}_{X_i}\omega = 0$ ($i = 1, 2, 3$), where $\mathcal{L}_{X_i}\omega$ stands for the Lie derivative of ω with respect to X_i . In coordinates, these conditions read

$$\frac{\partial f}{\partial x} = 0, \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0, \quad (x^2 - y^2) \frac{\partial f}{\partial x} + 2xy \frac{\partial f}{\partial y} + 4xf = 0. \quad (3.1)$$

From the first equation, we obtain $f = f(y)$. Using this in the second equation, we find that $f = y^{-2}$ is a particular solution of both equations (the third one is therefore automatically fulfilled). This leads to a closed and non-degenerate two-form on $\mathbb{R}^2_{y \neq 0} \equiv \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$, namely

$$\omega = \frac{dx \wedge dy}{y^2}. \quad (3.2)$$

Using the relation $\iota_X\omega = dh$ between a Hamiltonian vector field X and one of its corresponding Hamiltonian functions h , we observe that X_1, X_2 and X_3 are Hamiltonian vector fields with Hamiltonian functions

$$h_1 = -\frac{1}{y}, \quad h_2 = -\frac{x}{y}, \quad h_3 = -\frac{x^2 + y^2}{y}, \quad (3.3)$$

respectively. Obviously, the remaining vector fields of V become also Hamiltonian. Thus, planar Riccati equations (2.7) admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to (3.2).

Example 2. Let us now focus on analyzing the Hamilton equations for an n -dimensional **Smorodinsky–Winternitz oscillator** [420] on $T^*\mathbb{R}_0^n$ of the form

$$\begin{cases} \frac{dx_i}{dt} = p_i, \\ \frac{dp_i}{dt} = -\omega^2(t)x_i + \frac{k}{x_i^3}, \end{cases} \quad i = 1, \dots, n, \quad (3.4)$$

with $\omega(t)$ being any t -dependent function, $\mathbb{R}_0^n \equiv \{(x_1, \dots, x_n) \in \mathbb{R}_0^n \mid x_1, \dots, x_n \neq 0\}$ and $k \in \mathbb{R}$. These oscillators have attracted quite much attention in Classical and Quantum Mechanics for their special properties [220, 235, 424]. In addition, observe that Winternitz–Smorodinsky oscillators reduce to t -dependent isotropic harmonic oscillators when $k = 0$.

System (3.4) describes the integral curves of the t -dependent vector field

$$X_t = \sum_{i=1}^n \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3} \right) \frac{\partial}{\partial p_i} \right] \quad (3.5)$$

on $T^*\mathbb{R}_0^n$. This cotangent bundle admits a natural Poisson bivector Λ related to the restriction to $T^*\mathbb{R}_0^n$ of the canonical symplectic structure on $T^*\mathbb{R}^n$, namely $\Lambda = \sum_{i=1}^n \partial/\partial x_i \wedge \partial/\partial p_i$. If we consider the vector fields

$$\begin{aligned} X_1 &= -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, & X_2 &= \sum_{i=1}^n \frac{1}{2} \left(p_i \frac{\partial}{\partial p_i} - x_i \frac{\partial}{\partial x_i} \right), \\ X_3 &= \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3} \frac{\partial}{\partial p_i} \right), \end{aligned} \quad (3.6)$$

we can write $X_t = X_3 + \omega^2(t)X_1$. Additionally, since

$$[X_1, X_3] = 2X_2, \quad [X_1, X_2] = X_1, \quad [X_2, X_3] = X_3, \quad (3.7)$$

it follows that (3.4) is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In addition, this Lie algebra is again made of Hamiltonian vector fields. In fact, it is easy to check that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{1}{2} \sum_{i=1}^n x_i^2, \quad h_2 = -\frac{1}{2} \sum_{i=1}^n x_i p_i, \quad h_3 = \frac{1}{2} \sum_{i=1}^n \left(p_i^2 + \frac{k}{x_i^2} \right). \quad (3.8)$$

Consequently, system (3.4) admits a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to the Poisson structure Λ .

Example 3. Finally, let us turn to the system of Riccati equations (2.12) for $n = 4$. The relevance of this system is due to the fact that it appears in the calculation of superposition rules for Riccati equations [102].

We already know that **coupled Riccati equations** system (2.12) is a Lie system possessing a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ spanned by the vector fields X_1, X_2, X_3 given by (2.13). Let us prove that this Vessiot–Guldberg Lie algebra consists of Hamiltonian vector fields. Indeed, we can consider the Poisson bivector

$$\Lambda_R = (x_1 - x_2)^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (x_3 - x_4)^2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}, \quad (3.9)$$

on $\mathcal{O} = \{(x_1, x_2, x_3, x_4) | (x_1 - x_2)(x_2 - x_3)(x_3 - x_4) \neq 0\} \subset \mathbb{R}^4$. We have that $X_i = -\widehat{\Lambda}(dh_i)$ on (\mathcal{O}, ω_R) for

$$\begin{aligned} h_1 &= \frac{1}{x_1 - x_2} + \frac{1}{x_3 - x_4}, & h_2 &= \frac{1}{2} \left(\frac{x_1 + x_2}{x_1 - x_2} + \frac{x_3 + x_4}{x_3 - x_4} \right), \\ h_3 &= \frac{x_1 x_2}{x_1 - x_2} + \frac{x_3 x_4}{x_3 - x_4}. \end{aligned} \quad (3.10)$$

Hence, the vector fields X_1, X_2, X_3 given in (2.13) are Hamiltonian with respect to (\mathcal{O}, Λ_R) and our system of Riccati equations admits a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields. The relevance of this result is due to the fact

that it allowed us to obtain the superposition rule for Riccati equations through a Casimir of $\mathfrak{sl}(2, \mathbb{R})$ [35].

All previous examples lead to propose the following definition.

Definition 1. A system X on N is said to be a *Lie–Hamilton system* if N can be endowed with a Poisson bivector Λ in such a way that V^X becomes a finite-dimensional real Lie algebra of Hamiltonian vector fields relative to Λ .

Although Lie–Hamilton systems form a particular class of Lie systems, it turns out that it is easier to find applications of Lie–Hamilton systems than applications of other types of Lie systems. Moreover, Lie–Hamilton systems admit much more geometrical properties than general Lie systems.

3.2 Lie–Hamilton structures

Standard Hamiltonian systems can be described through a Hamiltonian. Likewise, Lie–Hamilton systems admit a similar structure playing an analogous rôle: the Lie–Hamiltonian structures.

Definition 2. A *Lie–Hamiltonian structure* is a triple (N, Λ, h) , where (N, Λ) stands for a Poisson manifold and h represents a t -parametrized family of functions $h_t : N \rightarrow \mathbb{R}$ such that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ is a finite-dimensional real Lie algebra.

Definition 3. A t -dependent vector field X is said to admit, or to possess, a Lie–Hamiltonian structure (N, Λ, h) if X_t is the Hamiltonian vector field corresponding to h_t for each $t \in \mathbb{R}$. The Lie algebra $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ is called a *Lie–Hamilton algebra* for X .

Example 4. Recall that the vector fields X_1 , X_2 and X_3 for planar Riccati equations (2.7) are Hamiltonian with respect to the symplectic form ω given by (3.2) and form a basis for a Vessiot–Guldberg Lie algebra $V \simeq \mathfrak{sl}(2, \mathbb{R})$. Assume that the minimal Lie algebra for a planar Riccati equation, X , is V , i.e., $V^X = V$. If $\{\cdot, \cdot\}_\omega : C^\infty(\mathbb{R}_{y \neq 0}^2) \times C^\infty(\mathbb{R}_{y \neq 0}^2) \rightarrow C^\infty(\mathbb{R}_{y \neq 0}^2)$ stands for the Poisson bracket induced by ω (see [401]), then

$$\{h_1, h_2\}_\omega = -h_1, \quad \{h_1, h_3\}_\omega = -2h_2, \quad \{h_2, h_3\}_\omega = -h_3. \quad (3.11)$$

Hence, the planar Riccati equation X possesses a Lie–Hamiltonian structure of the form $(\mathbb{R}_{y \neq 0}^2, \omega, h = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3)$ and, as $V^X \simeq \mathfrak{sl}(2)$, we have that $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega) \equiv ((h_1, h_2, h_3), \{\cdot, \cdot\}_\omega)$ is a Lie–Hamilton algebra for X isomorphic to $\mathfrak{sl}(2)$.

Proposition 4. *If a system X admits a Lie–Hamiltonian structure, then X is a Lie–Hamilton system.*

Proof. Let (N, Λ, h) be a Lie–Hamiltonian structure for X . Thus, $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$ is a finite-dimensional Lie algebra. Moreover, $\{X_t\}_{t \in \mathbb{R}} \subset \widehat{\Lambda} \circ d[\text{Lie}(\{h_t\}_{t \in \mathbb{R}})]$, and as $\widehat{\Lambda} \circ d$ is a Lie algebra morphism, it follows that $V = \widehat{\Lambda} \circ d[\text{Lie}(\{h_t\}_{t \in \mathbb{R}})]$ is a finite-dimensional Lie algebra of Hamiltonian vector fields containing $\{X_t\}_{t \in \mathbb{R}}$. Therefore, $V^X \subset V$ and X is a Lie–Hamilton system. \square

Example 5. Consider again the second-order Riccati equation (2.23) in Hamiltonian form (2.28) and the vector fields (2.29) which span a five-dimensional Vessiot–Guldberg Lie algebra for such equations. All the vector fields of this Lie algebra are Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on \mathcal{O} . Indeed, note that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, \dots, 5$ and

$$\begin{aligned} h_1(x, p) &= -2\sqrt{-p}, & h_2(x, p) &= p, & h_3(x, p) &= xp, & h_4(x, p) &= x^2p, \\ & & h_5(x, p) &= -2x\sqrt{-p}. \end{aligned} \quad (3.12)$$

This system is therefore a Lie–Hamilton system. Moreover, the previous Hamiltonian functions span along with $\tilde{h}_0 = 1$ a Lie algebra of functions isomorphic to the two-photon Lie algebra \mathfrak{h}_6 [35, 109] with non-vanishing Lie brackets given by

$$\begin{aligned} \{h_1, h_3\}_\omega &= -\frac{1}{2}\tilde{h}_1, & \{h_1, h_4\}_\omega &= -h_5, & \{h_1, h_5\}_\omega &= 2h_0, \\ \{h_2, h_3\}_\omega &= -h_2, & \{h_2, h_4\}_\omega &= -2h_3, & \{h_2, h_5\}_\omega &= -h_1, \\ \{h_3, h_4\}_\omega &= -h_4, & \{h_3, h_5\}_\omega &= -\frac{1}{2}h_5. \end{aligned} \quad (3.13)$$

In consequence, system (2.23) admits a t -dependent Hamiltonian structure given by $(T^*\mathbb{R}, \Lambda, h = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2(t)h_4)$.

Consider X to be a Lie–Hamilton system admitting a Lie–Hamiltonian structure (N, Λ, h) leading to a Lie–Hamilton algebra $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$. Let us now analyze the relations between all the previous geometric structures.

Lemma 5. *Given a system X on N possessing a Lie–Hamiltonian structure (N, Λ, h) , we have that*

$$0 \hookrightarrow \text{Cas}(N, \Lambda) \cap \text{Lie}(\{h_t\}_{t \in \mathbb{R}}) \hookrightarrow \text{Lie}(\{h_t\}_{t \in \mathbb{R}}) \xrightarrow{\mathcal{J}_\Lambda} V^X \rightarrow 0, \quad (3.14)$$

where $\mathcal{J}_\Lambda : f \in \text{Lie}(\{h_t\}_{t \in \mathbb{R}}) \mapsto \widehat{\Lambda} \circ df \in V^X$, is an exact sequence of Lie algebras.

Proof. Consider the exact sequence of (generally) infinite-dimensional real Lie algebras

$$0 \hookrightarrow \text{Cas}(N, \Lambda) \hookrightarrow C^\infty(N) \xrightarrow{\widehat{\Lambda} \circ d} \text{Ham}(N, \Lambda) \rightarrow 0. \quad (3.15)$$

Since $X_t = -\widehat{\Lambda} \circ dh_t$, we see that $V^X = \text{Lie}(\widehat{\Lambda} \circ d(\{h_t\}_{t \in \mathbb{R}}))$. Using that $\widehat{\Lambda} \circ d$ is a Lie algebra morphism, we have $V^X = \widehat{\Lambda} \circ d[\text{Lie}(\{h_t\}_{t \in \mathbb{R}})] = \mathcal{J}_\Lambda(\text{Lie}(\{h_t\}_{t \in \mathbb{R}}))$.

Additionally, as \mathcal{J}_Λ is the restriction to $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$ of $\widehat{\Lambda} \circ d$, we obtain that its kernel consists of Casimir functions belonging to $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, i.e. $\ker \mathcal{J}_\Lambda = \text{Lie}(\{h_t\}_{t \in \mathbb{R}}) \cap \text{Cas}(N, \Lambda)$. The exactness of sequence (3.14) easily follows from these results. \square

The above proposition entails that every X that possesses a Lie–Hamiltonian structure (N, Λ, h) is such that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$ is a Lie algebra extension of V^X by $\text{Cas}(N, \Lambda) \cap \text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, i.e., the sequence of Lie algebras (3.14) is exact. Note that if $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$ is finite-dimensional by assumption, all the Lie algebras appearing in such a sequence are finite-dimensional. For instance, the first-order system (2.28) associated to second-order Riccati equations in Hamiltonian form admits a Lie–Hamiltonian structure

$$\left(\mathcal{O}, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p}, h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2(t)h_4 \right), \quad (3.16)$$

where h_1, h_2, h_3, h_4 are given by (3.12). Note that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, for generic functions $a_0(t), a_1(t), a_2(t)$, is a six-dimensional Lie algebra of functions $\mathfrak{W} \simeq V^X \oplus \mathbb{R}$. Hence, we see that \mathfrak{W} is an extension of V^X .

It is worth noting that every t -dependent vector field admitting a Lie–Hamiltonian structure necessarily possesses many other Lie–Hamiltonian structures. For instance, if a system X admits (N, Λ, h) , then it also admits a Lie–Hamiltonian structure (N, Λ, h') , with $h' : (t, x) \in \mathbb{R} \times N \mapsto h(t, x) + f_C(x) \in \mathbb{R}$, where f_C is any Casimir function with respect to Λ . Indeed, it is easy to see that if h_1, \dots, h_r is a basis for $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, then h_1, \dots, h_r, f_C span $\text{Lie}(\{h'_t\}_{t \in \mathbb{R}})$, which also becomes a finite-dimensional real Lie algebra. As shown later, this has relevant implications for the linearization of Lie–Hamilton systems.

We have already proved that every system X admitting a Lie–Hamiltonian structure must possess several ones. Nevertheless, we have not yet studied the conditions ensuring that a Lie–Hamilton system X possesses a Lie–Hamiltonian structure. Let us answer this question.

Proposition 6. *Every Lie–Hamilton system admits a Lie–Hamiltonian structure.*

Proof. Assume X to be a Lie–Hamilton system on a manifold N with respect to a Poisson bivector Λ . Since $V^X \subset \text{Ham}(N, \Lambda)$ is finite-dimensional, there exists a finite-dimensional linear space $\mathfrak{W}_0 \subset C^\infty(N)$ isomorphic to V^X and such that $\widehat{\Lambda} \circ d(\mathfrak{W}_0) = V^X$. Consequently, there exists a curve h_t in \mathfrak{W}_0 such that $X_t = -\widehat{\Lambda} \circ d(h_t)$. To ensure that h_t gives rise to a Lie–Hamiltonian structure, we need to demonstrate that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ is finite-dimensional. This will be done by constructing a finite-dimensional Lie algebra of functions containing the curve h_t .

Define the linear isomorphism $T : X_f \in V^X \mapsto -f \in \mathfrak{W}_0 \subset C^\infty(N)$ associating each vector field in V^X with minus its unique Hamiltonian function within \mathfrak{W}_0 . This can be done by choosing a representative for each element of a basis of V^X and extending the map by linearity.

Note that this mapping needs not be a Lie algebra morphism and hence $\text{Im } T = \mathfrak{W}_0$ does not need to be a Lie algebra. Indeed, we can define a bilinear map $\Upsilon : V^X \times V^X \rightarrow C^\infty(N)$ of the form

$$\Upsilon(X_f, X_g) = \{f, g\}_\Lambda - T[X_f, X_g], \quad (3.17)$$

measuring the obstruction for T to be a Lie algebra morphism, i.e. Υ is identically null if and only if T is a Lie algebra morphism. In fact, if \mathfrak{W}_0 were a Lie algebra, then $\{f, g\}_\Lambda$ would be the only element of \mathfrak{W}_0 with Hamiltonian vector field $-[X_f, X_g]$, i.e. $T[X_f, X_g]$, and Υ would be a zero function.

Note that $\Upsilon(X_f, X_g)$ is the difference between two functions, namely $\{f, g\}_\Lambda$ and $T[X_f, X_g]$, sharing the same Hamiltonian vector field. Consequently, $\text{Im } \Upsilon \subset \text{Cas}(N, \Lambda)$ and it can be injected into a finite-dimensional Lie algebra of Casimir functions of the form

$$\mathfrak{W}_C \equiv \langle \Upsilon(X_i, X_j) \rangle, \quad i, j = 1, \dots, r, \quad (3.18)$$

where X_1, \dots, X_r is a basis for V^X . From here, it follows that

$$\{\mathfrak{W}_C, \mathfrak{W}_C\}_\Lambda = 0, \quad \{\mathfrak{W}_C, \mathfrak{W}_0\}_\Lambda = 0, \quad \{\mathfrak{W}_0, \mathfrak{W}_0\}_\Lambda \subset \mathfrak{W}_C + \mathfrak{W}_0. \quad (3.19)$$

Hence, $\mathfrak{W} \equiv \mathfrak{W}_0 + \mathfrak{W}_C$ is a finite-dimensional Lie algebra of functions containing the curve h_t . From here, it readily follows that X admits a Lie–Hamiltonian structure $(N, \Lambda, -TX_t)$. \square

Since every Lie–Hamilton system possesses a Lie–Hamiltonian structure and every Lie–Hamiltonian structure determines a Lie–Hamilton system, we obtain the following theorem.

Theorem 7. *A system X admits a Lie–Hamiltonian structure if and only if it is a Lie–Hamilton system.*

Example 6. Let us consider the Hamilton equations for the second-order Kummer–Schwarz equations (2.38) in Hamiltonian form [55, 87]. They read

$$\begin{cases} \frac{dx}{dt} = \frac{px^3}{2}, \\ \frac{dp}{dt} = -\frac{3p^2x^2}{4} - 4c_0 + \frac{4b_1(t)}{x^2}, \end{cases} \quad (3.20)$$

on $T^*\mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Once again, the above system is a Lie system as it describes the integral curves of the t -dependent vector field $X_t = X_3 + b_1(t)X_1$, where

$$X_1 = \frac{4}{x^2} \frac{\partial}{\partial p}, \quad X_2 = x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}, \quad X_3 = \frac{px^3}{2} \frac{\partial}{\partial x} - \left(\frac{3p^2x^2}{4} + 4c_0 \right) \frac{\partial}{\partial p}, \quad (3.21)$$

span a three-dimensional Lie algebra V^{2KS} isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Indeed,

$$[X_1, X_3] = 2X_2, \quad [X_1, X_2] = X_1, \quad [X_2, X_3] = X_3. \quad (3.22)$$

Apart from providing a new approach to Kummer–Schwarz equations (see [87] for a related method), our new description gives an additional relevant property: V^{2KS} consists of Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $T^*\mathbb{R}_0$. In fact, $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$ with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{4}{x}, \quad h_2 = xp, \quad h_3 = \frac{1}{4}p^2x^3 + 4c_0x. \quad (3.23)$$

Therefore (3.20) is a Lie–Hamilton system. Moreover, we have that

$$\{h_1, h_2\} = -h_1, \quad \{h_1, h_3\} = -2h_2, \quad \{h_2, h_3\} = -h_3. \quad (3.24)$$

Therefore, (3.20) admits a Lie–Hamiltonian structure $(T^*\mathbb{R}_0, \Lambda, h = h_3 + b_1(t)h_1)$ as ensured by Theorem 7.

3.3 t -independent constants of motion

Let us now study the structure of the space of t -independent constants of motion for Lie–Hamilton systems. In particular, we are interested in studying the use of Poisson structures to study such constants of motion.

Proposition 8. *Given a system X admitting a Lie–Hamiltonian structure (N, Λ, h) , then $\mathcal{C}^\Lambda \subset \mathcal{V}^X$, where we recall that \mathcal{C}^Λ is the Casimir distribution relative to Λ .*

Proof. Consider a $\theta_x \in \mathcal{C}_x^\Lambda$, with $x \in N$. As X is a Lie–Hamilton system, for every $Y \in V^X$ there exists a function $f \in C^\infty(N)$ such that $Y = -\widehat{\Lambda}(df)$. Then,

$$\theta_x(Y_x) = -\theta_x(\widehat{\Lambda}_x(df_x)) = -\Lambda_x(df_x, \theta_x) = 0, \quad (3.25)$$

where $\widehat{\Lambda}_x$ is the restriction of $\widehat{\Lambda}$ to T_x^*N . As the vectors Y_x , with $Y \in V^X$, span \mathcal{D}_x^X , then $\theta_x \in \mathcal{V}_x^X$ and $\mathcal{C}^\Lambda \subset \mathcal{V}^X$. \square

Observe that different Lie–Hamiltonian structures for a Lie–Hamilton system X may lead to different families of Casimir functions, which may determine different constants of motion for X .

Theorem 9. *Let X be a system admitting a Lie–Hamiltonian structure (N, Λ, h) , the space $\mathcal{I}^X|_U$ of t -independent constants of motion of X on an open $U \subset U^X$ is a Poisson algebra. Additionally, the codistribution $\mathcal{V}^X|_{U^X}$ is involutive with respect to the Lie bracket $[\cdot, \cdot]_\Lambda$ induced by Λ on the space $\Gamma(\pi_N)$ of smooth one-forms on N .*

Proof. Let $f_1, f_2 : U \rightarrow \mathbb{R}$ be two t -independent functions constants of motion for X , i.e., $X_t f_i = 0$, for $i = 1, 2$ and $t \in \mathbb{R}$. As X is a Lie–Hamilton system, all the elements of V^X are Hamiltonian vector fields and we can write $Y\{f, g\}_\Lambda = \{Yf, g\}_\Lambda + \{f, Yg\}_\Lambda$ for every $f, g \in C^\infty(N)$ and $Y \in V^X$. In particular, $X_t(\{f_1, f_2\}_\Lambda) = \{X_t f_1, f_2\}_\Lambda + \{f_1, X_t f_2\}_\Lambda = 0$. Hence, the Poisson bracket of t -independent constants of motion is a new one. As $\lambda f_1 + \mu f_2$ and $f_1 \cdot f_2$ are also t -independent constants of motion for every $\lambda, \mu \in \mathbb{R}$, it easily follows that $\mathcal{I}^X|_U$ is a Poisson algebra.

In view of Lemma 2 in Chapter 2, the co-distribution \mathcal{V}^X admits a local basis of exact forms $df_1, \dots, df_{p(x)}$ for every point $x \in U^X$, where \mathcal{V}^X has local constant rank $p(x) \equiv \dim N - \dim \mathcal{D}_x^X$. Now, $[df_i, df_j]_\Lambda = d(\{f_i, f_j\}_\Lambda)$ for $i, j = 1, \dots, p(x)$. We already proved that the function $\{f_i, f_j\}_\Lambda$ is another first-integral. Therefore, from Lemma 2 in Chapter 2, it easily follows that $\{f_i, f_j\}_\Lambda = G(f_1, \dots, f_{p(x)})$. Thus, $[df_i, df_j]_\Lambda \in \mathcal{V}^X|_{U^X}$. Using the latter and the properties of the Lie bracket $[\cdot, \cdot]_\Lambda$, it directly turns out that the Lie bracket of two one-forms taking values in $\mathcal{V}^X|_{U^X}$ belongs to $\mathcal{V}^X|_{U^X}$. Hence, $\mathcal{V}^X|_{U^X}$ is involutive. \square

Corollary 10. *Given a Lie–Hamilton system X , the space $\mathcal{I}^X|_U$, where $U \subset U^X$ is such that \mathcal{V}^X admits a local basis of exact forms, is a function group, that is*

1. *The space $\mathcal{I}^X|_U$ is a Poisson algebra.*
2. *There exists a family of functions $f_1, \dots, f_s \in \mathcal{I}^X|_U$ such that every element f of $\mathcal{I}^X|_U$ can be put in the form $f = F(f_1, \dots, f_s)$ for a certain function $F : \mathbb{R}^s \rightarrow \mathbb{R}$.*

Proof. In view of the previous theorem, $\mathcal{I}^X|_U$ is a Poisson algebra with respect to a certain Poisson bracket. Taking into account Proposition 1 in Chapter 2 and the form of $\mathcal{I}^X|_U$ given by Lemma 2 in Chapter 2, we obtain that this space becomes a function group. \square

The above properties do not necessarily hold for systems other than Lie–Hamilton systems, as they do not need to admit any *a priori* relation among a Poisson bracket of functions and the t -dependent vector field describing the system. Let us exemplify this.

Example 7. Consider the Poisson manifold $(\mathbb{R}^3, \Lambda_{GM})$, where

$$\Lambda_{GM} = \sigma_3 \frac{\partial}{\partial \sigma_2} \wedge \frac{\partial}{\partial \sigma_1} - \sigma_1 \frac{\partial}{\partial \sigma_2} \wedge \frac{\partial}{\partial \sigma_3} + \sigma_2 \frac{\partial}{\partial \sigma_3} \wedge \frac{\partial}{\partial \sigma_1} \quad (3.26)$$

and $(\sigma_1, \sigma_2, \sigma_3)$ is a coordinate basis for \mathbb{R}^3 , appearing in the study of Classical XYZ Gaudin Magnets [41]. The system $X = \partial/\partial \sigma_3$ is not a Lie–Hamilton system with respect to this Poisson structure as X is not Hamiltonian, namely $\mathcal{L}_X \Lambda_{GM} \neq 0$. In

addition, this system admits two first-integrals σ_1 and σ_2 . Nevertheless, their Lie bracket reads $\{\sigma_1, \sigma_2\} = -\sigma_3$, which is not a first-integral for X . On the other hand, consider the system

$$Y = \sigma_3 \frac{\partial}{\partial \sigma_2} + \sigma_2 \frac{\partial}{\partial \sigma_3}. \quad (3.27)$$

This system is a Lie–Hamilton system, as it can be written in the form $Y = -\widehat{\Lambda}_{GM}(d\sigma_1)$, and it possesses two first-integrals given by σ_1 and $\sigma_2^2 - \sigma_3^2$. Unsurprisingly, $Y\{\sigma_1, \sigma_2^2 - \sigma_3^2\} = 0$, i.e., the Lie bracket of two t -independent constants of motion is also a constant of motion.

Let us prove some final interesting results about the t -independent constants of motion for Lie–Hamilton systems.

Proposition 11. *Let X be a Lie–Hamilton system that admits a Lie–Hamiltonian structure (N, Λ, h) . The function $f : N \rightarrow \mathbb{R}$ is a constant of motion for X if and only if f Poisson commutes with all elements of $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$.*

Proof. The function f is a t -independent constant of motion for X if and only if

$$0 = X_t f = \{f, h_t\}_\Lambda, \quad \forall t \in \mathbb{R}. \quad (3.28)$$

From here,

$$\{f, \{h_t, h_{t'}\}_\Lambda\}_\Lambda = \{\{f, h_t\}_\Lambda, h_{t'}\}_\Lambda + \{h_t, \{f, h_{t'}\}_\Lambda\}_\Lambda = 0, \quad \forall t, t' \in \mathbb{R}, \quad (3.29)$$

and inductively follows that f Poisson commutes with all successive Poisson brackets of elements of $\{h_t\}_{t \in \mathbb{R}}$ and their linear combinations. As these elements span $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, we get that f Poisson commutes with $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$.

Conversely, if f Poisson commutes with $\text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, it Poisson commutes with the elements $\{h_t\}_{t \in \mathbb{R}}$, and, in view of (3.28), it becomes a constant of motion for X . \square

In order to illustrate the above proposition, let us show an example.

Example 8. Consider a Smorodinsky–Winternitz system (3.4) with $n = 2$. Recall that this system admits a Lie–Hamiltonian structure $(\mathbb{T}^*\mathbb{R}_0^2, \Lambda, h = h_3 + \omega^2(t)h_1)$, where $\Lambda = \sum_{i=1}^2 \partial/\partial x_i \wedge \partial/\partial p_i$ is a Poisson bivector on $\mathbb{T}^*\mathbb{R}_0^2$ and the functions h_1, h_3 are given within (3.8). For non-constant $\omega(t)$, we obtain that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ is a real Lie algebra of functions isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ generated by the functions h_1, h_2 and h_3 detailed in (3.8). When $\omega(t) = \omega_0 \in \mathbb{R}$, the Lie algebra $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ becomes a one-dimensional Lie subalgebra of the previous one. In any case, it is known that

$$I = (x_1 p_2 - p_1 x_2)^2 + k \left[\left(\frac{x_1}{x_2} \right)^2 + \left(\frac{x_2}{x_1} \right)^2 \right] \quad (3.30)$$

is a t -independent constant of motion (cf. [105]). A simple calculation shows that

$$\{I, h_\alpha\}_\Lambda = 0, \quad \alpha = 1, 2, 3. \quad (3.31)$$

Then, the function I always Poisson commutes with the whole $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$, as expected.

Obviously, every autonomous Hamiltonian system is a Lie–Hamilton system possessing a Lie–Hamiltonian structure (N, Λ, h) , with h being a t -independent Hamiltonian. Consequently, above proposition shows that the t -independent first-integrals for a Hamiltonian system are those functions that Poisson commute with its Hamiltonian, recovering as a particular case this wide-known result.

Moreover, the above proposition suggests us that the rôle played by autonomous Hamiltonians for Hamiltonian systems is performed by the finite-dimensional Lie algebras of functions associated with Lie–Hamiltonian structures in the case of Lie–Hamilton systems. This can be employed to study t -independent first-integrals of Lie–Hamilton systems or analyze the maximal number of such first-integrals in involution, which would lead to the interesting analysis of integrability/superintegrability of Lie–Hamilton systems.

3.4 Symmetries, linearization and comomentum maps

Definition 12. We say that a Lie system X admitting a Lie–Hamilton structure (N, Λ, h) possesses a compatible *strong comomentum map* with respect to this Lie–Hamilton structure if there exists a Lie algebra morphism $\lambda : V^X \rightarrow \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda)$ such that the following diagram

$$\begin{array}{ccc} & & V^X \\ & \swarrow \lambda & \downarrow \iota \\ \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\Lambda) & \xrightarrow{\widehat{\Lambda}^{od}} & \text{Ham}(N, \Lambda) \end{array}$$

where $\iota : V^X \hookrightarrow \text{Ham}(N, \Lambda)$ is the natural injection of V^X into $\text{Ham}(N, \Lambda)$, is commutative.

Proposition 13. *Let X be a Lie system possessing a Lie–Hamiltonian structure (N, Λ, h) compatible with a strong comomentum map λ such that $\dim \mathcal{D}_x^X = \dim N = \dim V^X$ at a generic $x \in N$. Then, there exists a local coordinate system defined on a neighbourhood of each x such that X and Λ are simultaneously linearisable and where X possesses a linear superposition rule.*

Proof. As it is assumed that $n \equiv \dim N = \dim V^X = \dim \mathcal{D}_x^X$ at a generic x , every basis X_1, \dots, X_n of V^X gives rise to a basis for the tangent bundle TN on a

neighbourhood of x . Since X admits a strong comomentum map compatible with (N, Λ, h) , we have $(V^X, [\cdot, \cdot]) \simeq (\lambda(V^X), \{\cdot, \cdot\}_\Lambda)$ and the family of functions, $h_\alpha = \lambda(X_\alpha)$, with $\alpha = 1, \dots, n$, form a basis for the Lie subalgebra $\lambda(V^X)$. Moreover, since $\widehat{\Lambda} \circ d \circ \lambda(V^X) = V^X$ and $\dim V^X = \dim \mathcal{D}_{x'}^X$ for x' in a neighbourhood of x , then $\widehat{\Lambda}_{x'} \circ d(\lambda(V^X)) \simeq T_{x'}N$ and $dh_1 \wedge \dots \wedge dh_n \neq 0$ at a generic point. Hence, the set (h_1, \dots, h_n) is a coordinate system on an open dense subset of N . Now, using again that $(\lambda(V^X), \{\cdot, \cdot\}_\Lambda)$ is a real Lie algebra, the Poisson bivector Λ can be put in the form

$$\Lambda = \frac{1}{2} \sum_{i,j=1}^n \{h_i, h_j\}_\Lambda \frac{\partial}{\partial h_i} \wedge \frac{\partial}{\partial h_j} = \frac{1}{2} \sum_{i,j,k=1}^n c_{ijk} h_k \frac{\partial}{\partial h_i} \wedge \frac{\partial}{\partial h_j}, \quad (3.32)$$

for certain real n^3 constants c_{ijk} . In other words, the Poisson bivector Λ becomes linear in the chosen coordinate system.

Since we can write $X_t = -\widehat{\Lambda}(d\bar{h}_t)$, with $\bar{h}_t = -\lambda(X_t)$ being a curve in the Lie algebra $\lambda(V^X) \subset \text{Lie}(\{h_t\}_{t \in \mathbb{R}})$, expression (3.32) yields

$$\begin{aligned} X_t &= -\widehat{\Lambda}(d\bar{h}_t) = -\widehat{\Lambda} \circ d \left(\sum_{l=1}^n b_l(t) h_l \right) \\ &= - \sum_{l=1}^n b_l(t) (\widehat{\Lambda} \circ dh_l) = - \sum_{l,j,k=1}^n b_l(t) c_{ljk} h_k \frac{\partial}{\partial h_j}, \end{aligned} \quad (3.33)$$

and X_t is linear in this coordinate system. Consequently, as every linear system, X admits a linear superposition rule in the coordinate system (h_1, \dots, h_n) . \square

Let us turn to describing some features of t -independent Lie symmetries for Lie–Hamilton systems. Our exposition will be based upon the properties of the hereafter called *symmetry distribution*.

Definition 14. Given a Lie–Hamilton system X that possesses a Lie–Hamiltonian structure (N, Λ, h) , we define its *symmetry distribution*, \mathcal{S}_Λ^X , by

$$(\mathcal{S}_\Lambda^X)_x = \widehat{\Lambda}_x(\mathcal{V}_x^X) \in T_x N, \quad x \in N. \quad (3.34)$$

As its name indicates, the symmetry distribution can be employed to investigate the t -independent Lie symmetries of a Lie–Hamilton system. Let us give some basic examples of how this can be done.

Proposition 15. *Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N, Λ, h) , then*

1. *The symmetry distribution \mathcal{S}_Λ^X associated with X and Λ is involutive on an open subset of U^X , where U^X is the open dense subset of N where \mathcal{V}^X is differentiable.*

2. If f is a t -independent constant of motion for X , then $\widehat{\Lambda}(df)$ is a t -independent Lie symmetry of X .
3. The distribution \mathcal{S}_Λ^X admits a local basis of t -independent Lie symmetries of X defined around a generic point of N . The elements of such a basis are Hamiltonian vector fields of t -independent constants of motion of X .

Proof. By definition of \mathcal{S}_Λ^X and using that \mathcal{V}^X has constant rank on the connected components of an open $U \subset U_X$, where Λ has locally constant rank, we can ensure that given two vector fields in $Y_1, Y_2 \in \mathcal{S}_\Lambda^X|_U$, there exist two forms $\omega, \omega' \in \mathcal{V}^X|_U$ such that $Y_1 = \widehat{\Lambda}(\omega)$, $Y_2 = \widehat{\Lambda}(\omega')$. Since X is a Lie–Hamilton system, $\mathcal{V}^X|_U$ is involutive and $\widehat{\Lambda}$ is an anchor, i.e., a Lie algebra morphism from $(\Gamma(\pi_N), [\cdot, \cdot]_\Lambda)$ to $(\Gamma(\tau_N), [\cdot, \cdot])$, then

$$[Y_1, Y_2] = [\widehat{\Lambda}(\omega), \widehat{\Lambda}(\omega')] = \widehat{\Lambda}([\omega, \omega']_\Lambda) \in \mathcal{S}_\Lambda^X. \quad (3.35)$$

In other words, since \mathcal{V}^X is involutive on U , then \mathcal{S}_Λ^X is so, which proves (1).

To prove (2), note that

$$[X_t, \widehat{\Lambda}(df)] = -[\widehat{\Lambda}(dh_t), \widehat{\Lambda}(df)] = -\widehat{\Lambda}(d\{h_t, f\}_\Lambda) = \widehat{\Lambda}[d(X_t f)] = 0. \quad (3.36)$$

Finally, the proof of (3) is based upon the fact that \mathcal{V}^X admits, around a point $x \in U^X \subset N$, a local basis of one-forms $df_1, \dots, df_{p(x)}$, with $f_1, \dots, f_{p(x)}$ being a family of t -independent constants of motion for X and $p(x) = \dim N - \dim \mathcal{D}_x^X$. From (2), the vector fields $X_{f_1}, \dots, X_{f_{p(x)}}$ form a family of Lie symmetries of X locally spanning \mathcal{S}_Λ^X . Hence, we can easily choose among them a local basis for \mathcal{S}_Λ^X . \square

Example 9. As a particular example of the usefulness of the above result, let us turn to a two-dimensional Smorodinsky–Winternitz oscillator X given by (3.4) and its known constant of motion (3.30). In view of the previous proposition, $Y = \widehat{\Lambda}(dI)$ must be a Lie symmetry for these systems. A little calculation leads to

$$Y = 2(x_1 p_2 - p_1 x_2) \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) + 2 \left[(x_1 p_2 - p_1 x_2) p_2 + k \frac{x_1^4 - x_2^4}{x_1^3 x_2^2} \right] \frac{\partial}{\partial p_1} - 2 \left[(x_1 p_2 - p_1 x_2) p_1 + k \frac{x_1^4 - x_2^4}{x_2^3 x_1^2} \right] \frac{\partial}{\partial p_2}, \quad (3.37)$$

and it is straightforward to verify that Y commutes with X_1, X_2, X_3 , given by (3.6), and therefore with every X_t , with $t \in \mathbb{R}$, i.e., Y is a Lie symmetry for X .

Proposition 16. Let X be a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) . If $[V^X, V^X] = V^X$ and $Y \in \text{Ham}(N, \Lambda)$ is a Lie symmetry of X , then $Y \in \mathcal{S}_\Lambda^X$.

Proof. As Y is a t -independent Lie symmetry, then $[Y, X_t] = 0$ for every $t \in \mathbb{R}$. Since Y is a Hamiltonian vector field, then $Y = -\widehat{\Lambda} \circ df$ for a certain $f \in C^\infty(N)$. Using that $X_t = -\widehat{\Lambda}(dh_t)$, we obtain

$$0 = [Y, X_t] = [\widehat{\Lambda}(df), \widehat{\Lambda}(dh_t)] = \widehat{\Lambda}(d\{f, h_t\}_\Lambda) = \widehat{\Lambda}[d(X_t f)]. \quad (3.38)$$

Hence, $X_t f$ is a Casimir function. Therefore, as every $X_{t'}$ is a Hamiltonian vector field for all $t' \in \mathbb{R}$, it turns out that $X_{t'} X_t f = 0$ for every $t, t' \in \mathbb{R}$ and, in consequence, $Z_1 f$ is a Casimir function for every $Z_1 \in V^X$. Moreover, as every $Z_2 \in V^X$ is Hamiltonian, we have

$$Z_2 Z_1 f = Z_1 Z_2 f = 0 \implies (Z_2 Z_1 - Z_1 Z_2) f = [Z_2, Z_1] f = 0. \quad (3.39)$$

As $[V^X, V^X] = V^X$, every element Z of V^X can be written as the commutator of two elements of V^X and, in view of the above expression, $Zf = 0$ which shows that f is a t -independent constant of motion for X . Finally, as $Y = -\widehat{\Lambda}(df)$, then $Y \in \mathcal{S}_\Lambda^X$. \square

Note that, roughly speaking, the above proposition ensures that, when V^X is *perfect*, i.e., $[V^X, V^X] = V^X$ (see [81]), then \mathcal{S}_Λ^X contains all Hamiltonian Lie symmetries of X . This is the case for Smorodinsky–Winternitz systems (3.4) with a non-constant $\omega(t)$, whose V^X was already shown to be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

3.5 On t -dependent constants of motion

The aim of this section is to present an analysis of the algebraic properties of the t -dependent constants of motion for Lie–Hamilton systems. More specifically, we prove that a Lie–Hamiltonian structure (N, Λ, h) for a Lie–Hamilton system X induces a Poisson bivector on $\mathbb{R} \times N$. This allows us to endow the space of constants of motion for X with a Poisson algebra structure, which can be used to produce new constants of motion from known ones.

Definition 17. We call *autonomization* of a vector field X , to the explicit t -dependent vector field $\bar{X} = X + d/dt$.

Given a system X on N , a constant of motion for X is a first-integral $f \in C^\infty(\mathbb{R} \times N)$ of the autonomization \bar{X} .

$$\frac{\partial f}{\partial t} + Xf = \bar{X}f = 0, \quad (3.40)$$

where X is understood as a vector field on $\mathbb{R} \times N$. Using this, we can straightforwardly prove the following proposition.

Proposition 18. *The space $\bar{\mathcal{I}}^X$ of t -dependent constants of motion for a system X forms an \mathbb{R} -algebra $(\bar{\mathcal{I}}^X, \cdot)$.*

Lemma 19. *Every Poisson manifold (N, Λ) induces a Poisson manifold $(\mathbb{R} \times N, \bar{\Lambda})$ with Poisson structure*

$$\{f, g\}_{\bar{\Lambda}}(t, x) \equiv \{f_t, g_t\}_{\Lambda}(x), \quad (t, x) \in \mathbb{R} \times N. \quad (3.41)$$

Definition 20. Given a Poisson manifold (N, Λ) , the associated Poisson manifold $(\mathbb{R} \times N, \bar{\Lambda})$ is called the *autonomization* of (N, Λ) . Likewise, the Poisson bivector $\bar{\Lambda}$ is called the *autonomization* of Λ .

The following lemma allows us to prove that $(\bar{\mathcal{I}}^X, \cdot, \{\cdot, \cdot\}_{\bar{\Lambda}})$ is a Poisson algebra.

Lemma 21. *Let (N, Λ) be a Poisson manifold and X be a Hamiltonian vector field on N relative to Λ . Then, $\mathcal{L}_{\bar{X}}\bar{\Lambda} = 0$.*

Proof. Given a coordinate system $\{x_1, \dots, x_n\}$ for N and $x_0 \equiv t$ in \mathbb{R} , we can naturally define a coordinate system $\{x_0, x_1, \dots, x_n\}$ on $\mathbb{R} \times N$. Since $(x_0)_t = t$ is constant as a function on N , then $\bar{\Lambda}(dx_0, df) = \{(x_0)_t, f_t\}_{\Lambda} = 0$ for every $f \in C^\infty(\mathbb{R} \times N)$. Additionally, $(x_i)_t = x_i$ for $i = 1, \dots, n$. Hence, we have that $\bar{\Lambda}(dx_i, dx_j) = \{x_i, x_j\}_{\bar{\Lambda}}$ is a x_0 -independent function for $i, j = 0, \dots, n$. So,

$$(\mathcal{L}_{\bar{X}}\bar{\Lambda})(t, x) = \left[\mathcal{L}_{\frac{\partial}{\partial x_0} + X} \left(\sum_{i < j=1}^n \{x_i, x_j\}_{\bar{\Lambda}} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) \right] (x) = (\mathcal{L}_X \Lambda)(x). \quad (3.42)$$

Since X is Hamiltonian, we obtain $0 = (\mathcal{L}_X \Lambda)(x) = (\mathcal{L}_{\bar{X}}\bar{\Lambda})(t, x) = 0$. \square

Now, we can establish the main result of this section.

Proposition 22. *Let X be a Lie-Hamilton system on N with a Lie-Hamiltonian structure (N, Λ, h) , then the space $(\bar{\mathcal{I}}^X, \cdot, \{\cdot, \cdot\}_{\bar{\Lambda}})$ is a Poisson algebra.*

Proof. From Proposition 18, we see that $(\bar{\mathcal{I}}^X, \cdot)$ is an \mathbb{R} -algebra. To demonstrate that $(\bar{\mathcal{I}}^X, \cdot, \{\cdot, \cdot\}_{\bar{\Lambda}})$ is a Poisson algebra, it remains to prove that the Poisson bracket of any two elements f, g of $\bar{\mathcal{I}}^X$ remains in it, i.e., $X\{f, g\}_{\bar{\Lambda}} = 0$. By taking into account that the vector fields $\{X_t\}_{t \in \mathbb{R}}$ are Hamiltonian relative to (N, Λ) and Lemma 21, we find that $\bar{\Lambda}$ is invariant under the autonomization of each vector field $X_{t'}$ with $t' \in \mathbb{R}$, i.e., $\mathcal{L}_{\bar{X}_{t'}}\bar{\Lambda} = 0$. Therefore,

$$\begin{aligned} \bar{X}\{f, g\}_{\bar{\Lambda}}(t', x) &= \overline{X_{t'}}\{f, g\}_{\bar{\Lambda}}(t', x) = \{\overline{X_{t'}}f, g\}_{\bar{\Lambda}}(t', x) + \{f, \overline{X_{t'}}g\}_{\bar{\Lambda}}(t', x) \\ &= \{\bar{X}f, g\}_{\bar{\Lambda}}(t', x) + \{f, \bar{X}g\}_{\bar{\Lambda}}(t', x) = 0. \end{aligned} \quad (3.43)$$

That is, $\{f, g\}_{\bar{\Lambda}}$ is a t -dependent constant of motion for X . \square

3.6 Lie integrals

Definition 23. Given a Lie–Hamilton system X on N possessing a Lie–Hamiltonian structure (N, Λ, h) , a *Lie integral* of X with respect to (N, Λ, h) is a (generally t -dependent) constant of motion f of X such that $\{f_t\}_{t \in \mathbb{R}} \subset \mathcal{H}_\Lambda$. In other words, given a basis h_1, \dots, h_r of the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, we have that $\bar{X}f = 0$ and $f_t = \sum_{\alpha=1}^r f_\alpha(t)h_\alpha$ for every $t \in \mathbb{R}$ and certain t -dependent functions f_1, \dots, f_r .

The Lie integrals of a Lie–Hamilton system X relative to a Lie–Hamiltonian structure (N, Λ, h) are the solutions of the equation

$$0 = \bar{X}f = \frac{\partial f}{\partial t} + Xf = \frac{\partial f}{\partial t} + \{f, h\}_\Lambda \implies \frac{\partial f}{\partial t} = \{h, f\}_\Lambda. \quad (3.44)$$

Since f and h can be understood as curves $t \mapsto f_t$ and $t \mapsto g_t$ within \mathcal{H}_Λ , the above equation can be rewritten as

$$\frac{df_t}{dt} = \{h_t, f_t\}_\Lambda, \quad (3.45)$$

which can be thought of as an Euler equation on the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$. Equations of this type quite frequently appear in the literature such as in the Lewis–Riesenfeld method and works concerning Lie–Hamilton systems [191, 307, 313].

Proposition 24. *Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N, Λ, h) , the space \mathfrak{L}_h^Λ of Lie integrals relative to (N, Λ, h) gives rise to a Lie algebra $(\mathfrak{L}_h^\Lambda, \{\cdot, \cdot\}_\Lambda)$ isomorphic to $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$.*

Proof. Since the Lie integrals of X are the solutions of the system of ODEs (3.45) on \mathcal{H}_Λ , they span an \mathbb{R} -linear space of dimension $\dim \mathcal{H}_\Lambda$. In view of Proposition 22, the Poisson bracket $\{f, g\}_\Lambda$ of two constants of motion f, g for X is another constant of motion. If f and g are Lie integrals, the function $\{f, g\}_\Lambda$ is then a new constant of motion that can additionally be considered as a curve $t \mapsto \{f_t, g_t\}_\Lambda$ taking values in \mathcal{H}_Λ , i.e., a new Lie integral.

Consider now the linear morphism $\mathcal{E}_0 : f \in \mathfrak{L}_h^\Lambda \mapsto f_0 \in \mathcal{H}_\Lambda$ relating every Lie integral to its value in \mathcal{H}_Λ at $t = 0$. As every initial condition in \mathcal{H}_Λ is related to a single solution of (3.45), we can relate every $v \in \mathcal{H}_\Lambda$ to a unique Lie integral f of X such that $f_0 = v$. Therefore, \mathcal{E}_0 is an isomorphism. Indeed, it is a Lie algebra isomorphism since $(\{f, g\}_\Lambda)_0 = \{f_0, g_0\}_\Lambda$ for every $f, g \in \mathfrak{L}_h^\Lambda$. \square

Proposition 25. *Given a Lie–Hamilton system X on N which possesses a Lie–Hamiltonian structure (N, Λ, h) , then \mathfrak{L}_h^Λ consists of t -independent constants of motion if and only if \mathcal{H}_Λ is Abelian.*

Proof. If $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$ is Abelian, then $\{f_t, h_t\}_\Lambda = 0$ and the system (3.45) reduces to $df_t/dt = 0$, whose solutions are of the form $f_t = g \in \mathcal{H}_\Lambda$, i.e., $\mathfrak{L}_h^\Lambda = \mathcal{H}_\Lambda$. Conversely, if $\mathfrak{L}_h^\Lambda = \mathcal{H}_\Lambda$, then every $g \in \mathcal{H}_\Lambda$ is a solution of (3.45) and $\{g, h_t\}_\Lambda = 0 \forall t \in \mathbb{R}$. Hence, every $g \in \mathcal{H}_\Lambda$ commutes with the whole \mathcal{H}_Λ , which becomes Abelian. \square

3.7 Polynomial Lie integrals

Let us formally define and investigate a remarkable class of constants of motion for Lie–Hamilton systems appearing in the literature [191, 313], the hereafter called *Lie integrals*, and a relevant generalization of them, the *polynomial Lie integrals*. We first prove that Lie integrals can be characterized by an Euler equation on a finite-dimensional real Lie algebra of functions, retrieving as a particular case a result given in [191]. Then, we show that Lie integrals form a finite-dimensional real Lie algebra and we devise several methods to determine them. Our results can easily be extended to investigate certain quantum mechanical systems [307]. Finally, we investigate polynomial Lie integrals and the relevance of Casimir functions to derive them.

Definition 26. Given a Lie–Hamilton system X on N possessing a Lie–Hamiltonian structure (N, Λ, h) , a *Lie integral* of X with respect to (N, Λ, h) is a constant of motion f of X such that $\{f_t\}_{t \in \mathbb{R}} \subset \mathcal{H}_\Lambda$. In other words, given a basis h_1, \dots, h_r of the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, we have that $\bar{X}f = 0$ and $f_t = \sum_{\alpha=1}^r f_\alpha(t)h_\alpha$ for every $t \in \mathbb{R}$ and certain t -dependent functions f_1, \dots, f_r .

The Lie integrals of a Lie–Hamilton system X relative to a Lie–Hamiltonian structure (N, Λ, h) are the solutions of the equation

$$0 = \bar{X}f = \frac{\partial f}{\partial t} + Xf = \frac{\partial f}{\partial t} + \{f, h\}_\Lambda \implies \frac{\partial f}{\partial t} = \{h, f\}_\Lambda. \quad (3.46)$$

Since f and h can be understood as curves $t \mapsto f_t$ and $t \mapsto g_t$ within \mathcal{H}_Λ , the above equation can be rewritten as

$$\frac{df_t}{dt} = \{h_t, f_t\}_\Lambda, \quad (3.47)$$

which can be thought of as an Euler equation on the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$ [191]. Equations of this type quite frequently appear in the literature such as in the Lewis–Riesenfeld method and works concerning Lie–Hamilton systems [191, 307, 313].

Polynomial Lie integrals and Casimir functions

We here investigate a generalization of Lie integrals: the hereafter called *polynomial Lie integrals*. Although we prove that these constants of motion can be determined by Lie integrals, we also show that their determination can be simpler in some cases. In particular, we can obtain polynomial Lie integrals algebraically by means of the Casimir functions related to the Lie algebra of Lie integrals.

Definition 27. Let X be a Lie–Hamilton system admitting a compatible Lie–Hamiltonian structure (N, Λ, h) . A *polynomial Lie integral* for X with respect to (N, Λ, h) is a constant of motion f for X of the form $f_t = \sum_{I \in M} \lambda_I(t)h^I$, where

the I 's are r -multi-indexes, i.e., sets (i_1, \dots, i_r) of nonnegative integers, the set M is a finite family of multi-indexes, the $\lambda_I(t)$ are certain t -dependent functions, and $h^I = h_1^{i_1} \cdot \dots \cdot h_r^{i_r}$ for a fixed basis $\{h_1, \dots, h_r\}$ for \mathcal{H}_Λ .

The study of polynomial Lie integrals can be approached through the symmetric Lie algebra $S_{\mathfrak{g}}$, where $\mathfrak{g} \simeq \mathcal{H}_\Lambda$.

Lemma 28. *Every Lie algebra isomorphism $\phi : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$ can be extended in a unique way to a Poisson algebra morphism $D : (S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}) \rightarrow (C^\infty(N), \cdot, \{\cdot, \cdot\}_\Lambda)$. Indeed, if $\{v_1, \dots, v_r\}$ is a basis for \mathfrak{g} , then $D(P(v_1, \dots, v_r)) = P(\phi(v_1), \dots, \phi(v_r))$, for every polynomial $P \in S_{\mathfrak{g}}$.*

Proof. The elements of $S_{\mathfrak{g}}$ given by $v^I \equiv v_1^{i_1} \cdot \dots \cdot v_r^{i_r}$, where the I 's are r -multi-indexes, “ \cdot ” denotes the product of elements of \mathfrak{g} as functions on \mathfrak{g}^* and $\{v_1, \dots, v_r\}$ is a basis for \mathfrak{g} , form a basis of $S_{\mathfrak{g}}$. Then, every $P \in S_{\mathfrak{g}}$ can be written in a unique way as $P = \sum_{I \in M} \lambda_I v^I$, where M is a finite family of multi-indexes and each $\lambda_I \in \mathbb{R}$. Hence, the \mathbb{R} -algebra morphism $D : (S_{\mathfrak{g}}, \cdot) \rightarrow (C^\infty(N), \cdot)$ extending $\phi : \mathfrak{g} \rightarrow \mathcal{H}_\Lambda$ is determined by the image of the elements of a basis for \mathfrak{g} . Indeed,

$$D(P) = \sum_I \lambda_I D(v^I) = \sum_I \lambda_I \phi(v_1^{i_1}) \cdot \dots \cdot \phi(v_r^{i_r}). \quad (3.48)$$

Let us prove that D is also an \mathbb{R} -algebra morphism. From (3.48), we see that D is linear. Moreover, $D(PQ) = D(P)D(Q)$ for every $P, Q \in S_{\mathfrak{g}}$. In fact, if we write $Q = \sum_{J \in M} \lambda_J v^J$, we obtain

$$\begin{aligned} D(PQ) &= D\left(\sum_I \lambda_I v^I \sum_J \lambda_J v^J\right) = \sum_K \sum_{I+J=K} \lambda_I \lambda_J D(v^K) \\ &= \sum_I \lambda_I D(v^I) \sum_J \lambda_J D(v^J) = D(P)D(Q), \end{aligned} \quad (3.49)$$

where $I + J = (i_1 + j_1, \dots, i_r + j_r)$ with $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$.

Let us show that D is also a Poisson morphism. By linearity, this reduces to proving that $D(\{v^I, v^J\}_{S_{\mathfrak{g}}}) = \{D(v^I), D(v^J)\}_\Lambda$ for arbitrary I and J . Define $|I| = i_1 + \dots + i_r$. If $|J| = 0$ or $|I| = 0$ this is satisfied, as a Poisson bracket vanishes when any entry is a constant. We now prove by induction the remaining cases. For $|I| + |J| = 2$, we have

$$D(\{v_\alpha, v_\beta\}_{S_{\mathfrak{g}}}) = \phi([v_\alpha, v_\beta]_{\mathfrak{g}}) = \{\phi(v_\alpha), \phi(v_\beta)\}_\Lambda = \{D(v_\alpha), D(v_\beta)\}_\Lambda, \quad (3.50)$$

for all $\alpha, \beta = 1, \dots, r$. If D is a Poisson morphism for $|I| + |J| = m > 2$, then for $|I| + |J| = m + 1$ we can set $v^I = v^{\bar{I}} v_\gamma^{i_\gamma}$ for $i_\gamma \neq 0$ and some γ to obtain

$$D(\{v^I, v^J\}_{S_{\mathfrak{g}}}) = D(\{v^{\bar{I}} v_\gamma^{i_\gamma}, v^J\}_{S_{\mathfrak{g}}}) = D(\{v^{\bar{I}}, v^J\}_{S_{\mathfrak{g}}} v_\gamma^{i_\gamma} + v^{\bar{I}} \{v_\gamma^{i_\gamma}, v^J\}_{S_{\mathfrak{g}}})$$

$$\begin{aligned}
&= \{D(v^{\bar{I}}), D(v^J)\}_\Lambda D(v^{i_\gamma}) + D(v^{\bar{I}})\{D(v^{i_\gamma}), D(v^J)\}_\Lambda \\
&= \{D(v^{\bar{I}})D(v^{i_\gamma}), D(v^J)\}_\Lambda = \{D(v^I), D(v^J)\}_\Lambda.
\end{aligned} \tag{3.51}$$

By induction, $D(\{v^I, v^J\}_{S_{\mathfrak{g}}}) = \{D(v^I), D(v^J)\}_\Lambda$ for any I and J .

□

Recall that “ \cdot ” denotes the standard product of elements of $S_{\mathfrak{g}}$ understood as polynomial functions on $S_{\mathfrak{g}}$. It is remarkable that D does not need to be injective, which causes that $S_{\mathfrak{g}}$ is not in general isomorphic to the space of polynomials on the elements of a basis of \mathcal{H}_Λ . For instance, consider the Lie algebra isomorphism $\phi : (\mathfrak{sl}(2, \mathbb{R}), [\cdot, \cdot]_{\mathfrak{sl}(2, \mathbb{R})}) \rightarrow (\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, with $\{v_1, v_2, v_3\}$ being a basis of $\mathfrak{sl}(2, \mathbb{R})$, of the form $\phi(v_1) = p^2$, $\phi(v_2) = xp$ and $\phi(v_3) = x^2$ and $\{\cdot, \cdot\}_\Lambda$ being the standard Poisson structure on $T^*\mathbb{R}$. Then, $D(v_1 v_3 - v_2^2) = \phi(v_1)\phi(v_3) - \phi^2(v_2) = 0$.

The following notion enables us to simplify the statement and proofs of our results.

Definition 29. Given a curve P_t in $S_{\mathfrak{g}}$, its *degree*, $\deg(P_t)$, is the highest degree of the polynomials $\{P_t\}_{t \in \mathbb{R}}$. If there exists no finite highest degree, we say that $\deg(P_t) = \infty$.

Proposition 30. *A function f is a polynomial Lie integral for a Lie–Hamilton system X with respect to the Lie–Hamiltonian structure (N, Λ, h) if and only if for every $t \in \mathbb{R}$ we have $f_t = D(P_t)$, where D is the Poisson algebra morphism $D : (S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}) \rightarrow (C^\infty(N), \cdot, \{\cdot, \cdot\}_\Lambda)$ induced by $\phi : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightarrow (\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, and the curve P_t is a solution of finite degree for*

$$\frac{dP}{dt} + \{P, w_t\}_{S_{\mathfrak{g}}} = 0, \quad P \in S_{\mathfrak{g}}, \tag{3.52}$$

where w_t stands for a curve in \mathfrak{g} such that $D(w_t) = h_t$ for every $t \in \mathbb{R}$.

Proof. Let P_t be a particular solution of (3.52). Since D is a Poisson algebra morphism and $h_t = D(w_t)$ for every $t \in \mathbb{R}$, we obtain by applying D to (3.52) that

$$\frac{\partial D(P_t)}{\partial t} + \{D(P_t), D(w_t)\}_{\bar{\Lambda}} = \frac{\partial D(P_t)}{\partial t} + \{D(P_t), h_t\}_{\bar{\Lambda}} = 0. \tag{3.53}$$

So, $D(P_t)$ is a Lie integral for X . Note that this does not depend on the chosen curve w_t satisfying $D(w_t) = h_t$.

Conversely, given a polynomial Lie integral f for X , there exists a curve P_t of finite degree such that $D(P_t) = f_t$ for every $t \in \mathbb{R}$. Hence, we see that

$$D\left(\frac{dP_t}{dt} + \{P_t, w_t\}_{S_{\mathfrak{g}}}\right) = \frac{\partial D(P_t)}{\partial t} + \{D(P_t), D(w_t)\}_{\bar{\Lambda}} = 0$$

$$\implies \frac{dP_t}{dt} + \{P_t, w_t\}_{S_{\mathfrak{g}}} = \xi_t, \quad (3.54)$$

where ξ_t is a curve in $\ker D$. As $\deg(dP_t/dt)$ and $\deg(\{P_t, w_t\}_{S_{\mathfrak{g}}})$ are at most $\deg(P_t)$, then $\deg(\xi_t) \leq \deg(P_t)$. Next, consider the equation

$$\frac{d\eta}{dt} + \{\eta, w_t\}_{S_{\mathfrak{g}}} = \xi_t, \quad \deg(\eta) \leq \deg(P) \quad \text{and} \quad \eta \subset \ker D. \quad (3.55)$$

Note that this equation is well defined. Indeed, since $\deg(\eta) \leq \deg(P)$ and $\deg(w_t) \leq 1$ for every $t \in \mathbb{R}$, then $\deg(\{\eta, w_t\}_{S_{\mathfrak{g}}}) \leq \deg(P)$ for all $t \in \mathbb{R}$. In addition, as $D(\eta_t) \subset \ker D$, then $\{\eta, w_t\}_{S_{\mathfrak{g}}} \in \ker D$. Then, the above equation can be restricted to the finite-dimensional space of elements of $\ker D$ with degree at most $\deg(P_t)$. Given a particular solution η_t of this equation, which exists for the Theorem of existence and uniqueness, we have that $P_t - \eta_t$ is a solution of (3.52) projecting into f_t . \square

Proposition 31. *Every polynomial Lie integral f of a Lie–Hamilton system X admitting a Lie–Hamiltonian structure (N, Λ, h) can be brought into the form $f = \sum_{I \in M} c_I l^I$, where M is a finite set of multi-indices, the c_I 's are certain real constants, and $l^I = f_1^{i_1} \cdots f_r^{i_r}$, with f_1, \dots, f_r being a basis of the space \mathfrak{L}_h^Λ .*

Proof. From Proposition 30, we have that $f_t = D(P_t)$ for a solution P_t of finite degree p for (3.52). So, it is a solution of the restriction of this system to $\mathbb{P}(p, \mathfrak{g})$, i.e., the elements of $S_{\mathfrak{g}}$ of degree at most p . Given the isomorphism $\phi : \mathfrak{g} \rightarrow \mathcal{H}_\Lambda$, define $\phi^{-1}(f_j)$, with $j = 1, \dots, r$, to be the curve in \mathfrak{g} of the form $t \mapsto \phi^{-1}(f_j)_t$. Note that $v_1 \equiv \phi^{-1}(f_1)_0, \dots, v_r \equiv \phi^{-1}(f_r)_0$ form a basis of \mathfrak{g} . Hence, their polynomials up to order p span a basis for $\mathbb{P}(p, \mathfrak{g})$ and we can write $P_0 = \sum_{I \in M} c_I v^I$, where $v^I = v_1^{i_1} \cdots v_r^{i_r}$. As $P'_t = \sum_{I \in M} c_I [\phi^{-1}(f_1)_t]^{i_1} \cdots [\phi^{-1}(f_r)_t]^{i_r}$ and P_t are solutions with the same initial condition of the restriction of (3.52) to $\mathbb{P}(p, \mathfrak{g})$, they must be the same in virtue of the Theorem of existence and uniqueness of systems of differential equations. Applying D , we obtain that $f_t = D(P_t) = D(\sum_{I \in M} c_I [\phi^{-1}(f_1)_t]^{i_1} \cdots [\phi^{-1}(f_r)_t]^{i_r}) = \sum_{I \in M} c_I l_t^I$. \square

Corollary 32. *Let X be a Lie–Hamilton system that possesses a Lie–Hamiltonian structure (N, Λ, h) inducing a Lie algebra isomorphism $\phi : \mathfrak{g} \rightarrow \mathcal{H}_\Lambda$ and a Poisson algebra morphism $D : S_{\mathfrak{g}} \rightarrow C^\infty(N)$. The function $F = D(C)$, where C is a Casimir element of $S_{\mathfrak{g}}$, is a t -independent constant of motion of X . If \mathcal{C} is a Casimir element of $U_{\mathfrak{g}}$, then $F = D(\lambda^{-1}(\mathcal{C}))$ is t -independent constant of motion for X .*

Note that if C is a constant of motion for X , it is also so for any other X' whose $V^{X'} \subset V^X$. From Proposition 31 and Corollary 32, it follows that $F = D(C) = \sum_{I \in M} c_I l^I$. Therefore, the knowledge of Casimir elements provides not only constants of motion for Lie–Hamilton systems, but also information about the Lie integrals of the system.

As Casimir functions are known for many Lie algebras, we can use them to derive constants of motion for the corresponding Lie–Hamilton systems algebraically instead of applying the usual procedure, i.e., by solving a system of PDEs or ODEs.

In particular, Casimir functions for (semi)simple Lie algebras of arbitrary dimension are known [222, 351]. The same is true for the so-called “quasi-simple” Lie algebras, which can be obtained from simple Lie algebras through contraction techniques [238]. Moreover, the Casimir invariants (Casimir elements of the Poisson algebra $(C^\infty(\mathfrak{g}^*), \{\cdot, \cdot\}_{\mathfrak{g}^*})$, being $\{\cdot, \cdot\}_{\mathfrak{g}^*}$ the Poisson structure induced by the Lie bracket for all real Lie algebras with dimension $d \leq 5$ were given in [350] (recall that the Casimir invariants for some of the solvable cases are not polynomial, i.e., they cannot be considered as elements of $S_{\mathfrak{g}}$), and the literature dealing with Casimir invariants for solvable and nilpotent Lie algebras is not scarce (see, e.g., [15, 80, 82]).

3.8 The coalgebra method, constants of motion and superposition rules

We here prove that each Lie–Hamiltonian structure (N, Λ, h) for a Lie–Hamilton system X gives rise in a natural way to a Poisson coalgebra $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}, \Delta)$ where $\mathfrak{g} \simeq (\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$. This allows us to use the coproduct of this coalgebra to construct new Lie–Hamiltonian structures for all the diagonal prolongations of X and to derive algebraically t -independent constants of motion for such diagonal prolongations. In turn, these constants can further be employed to obtain a superposition rule for the initial system. Our findings, which are only applicable to “primitive” Poisson coalgebras, give rigorous proof and generalizations of previous achievements established in [33, 36, 41].

Lemma 33. *If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) , then the space $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}, \Delta)$, with $\mathfrak{g} \simeq (\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, is a Poisson coalgebra with a coproduct $\Delta : S_{\mathfrak{g}} \rightarrow S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}$ satisfying*

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \forall v \in \mathfrak{g} \subset S_{\mathfrak{g}}. \quad (3.56)$$

Proof. We know that $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}})$ and $(S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}})$ are Poisson algebras. The coassociativity property for the coproduct map (3.56) is straightforward. Therefore, let us prove that there exists a Poisson algebra morphism $\Delta : (S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}) \rightarrow (S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}})$ satisfying (3.56), which turns $(S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}, \Delta)$ into a Poisson coalgebra.

The elements of $S_{\mathfrak{g}}$ of the form $v^I \equiv v_1^{i_1} \cdots v_r^{i_r}$, where the I 's are r -multi-index with $r = \dim \mathfrak{g}$, form a basis for $S_{\mathfrak{g}}$ (considered as a linear space). Then, every $P \in S_{\mathfrak{g}}$ can be written in a unique way as $P = \sum_{I \in M} \lambda_I v^I$, where the λ_I are real constants and I runs all the elements of a finite set M . In view of this, an \mathbb{R} -algebra

morphism $\Delta : S_{\mathfrak{g}} \rightarrow S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}$ is determined by the image of the elements of a basis for \mathfrak{g} , i.e.,

$$\Delta(P) = \sum_I \lambda_I \Delta(v^I) = \sum_I \lambda_I \Delta(v_1^{i_1}) \cdot \dots \cdot \Delta(v_r^{i_r}). \quad (3.57)$$

Therefore, two \mathbb{R} -algebra morphisms that coincide on the elements on \mathfrak{g} are necessarily the same. Hence, if there exists such a morphism, it is unique. Let us prove that there exists an \mathbb{R} -algebra morphism Δ satisfying (3.56).

From (3.57), we easily see that Δ is \mathbb{R} -linear. Let us also prove that $\Delta(PQ) = \Delta(P)\Delta(Q)$ for every $P, Q \in S_{\mathfrak{g}}$, which shows that Δ is an \mathbb{R} -algebra morphism. If we write $Q = \sum_{J \in M} \lambda_J v^J$, we obtain that

$$\begin{aligned} \Delta(PQ) &= \sum_K \left(\sum_{I+J=K} \lambda_I \lambda_J \right) \Delta(v^K) = \sum_I \lambda_I \Delta(v^I) \sum_J \lambda_J \Delta(v^J) \\ &= \Delta(P)\Delta(Q). \end{aligned} \quad (3.58)$$

Finally, we show that Δ is also a Poisson morphism. By linearity, this reduces to proving that $\Delta(\{v^I, v^J\}_{S_{\mathfrak{g}}}) = \{\Delta(v^I), \Delta(v^J)\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}}$. If $|I| = 0$ or $|J| = 0$, this result is immediate as the Poisson bracket involving a constant is zero. For the remaining cases and starting by $|I| + |J| = 2$, we have that $\Delta(\{v_{\alpha}, v_{\beta}\}_{S_{\mathfrak{g}}}) = \{\Delta(v_{\alpha}), \Delta(v_{\beta})\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}}$, $\forall \alpha, \beta = 1, \dots, r$. Proceeding by induction, we prove that this holds for every value of $|I| + |J|$; by writing $v^I = v^{\bar{I}} v_{\gamma}^{i_{\gamma}}$ and using induction hypothesis, we get

$$\begin{aligned} \Delta(\{v^I, v^J\}_{S_{\mathfrak{g}}}) &= \Delta(\{v^{\bar{I}} v_{\gamma}^{i_{\gamma}}, v^J\}_{S_{\mathfrak{g}}}) = \Delta(\{v^{\bar{I}}, v^J\}_{S_{\mathfrak{g}}} v_{\gamma}^{i_{\gamma}} + v^{\bar{I}} \{v_{\gamma}^{i_{\gamma}}, v^J\}_{S_{\mathfrak{g}}}) \\ &= \left\{ \Delta(v^{\bar{I}}), \Delta(v^J) \right\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}} \Delta(v_{\gamma}^{i_{\gamma}}) + \Delta(v^{\bar{I}}) \left\{ \Delta(v_{\gamma}^{i_{\gamma}}), \Delta(v^J) \right\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}} \\ &= \left\{ \Delta(v^{\bar{I}}) \Delta(v_{\gamma}^{i_{\gamma}}), \Delta(v^J) \right\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}} = \left\{ \Delta(v^I), \Delta(v^J) \right\}_{S_{\mathfrak{g}} \otimes S_{\mathfrak{g}}}. \end{aligned} \quad (3.59)$$

□

The coproduct defined in the previous lemma gives rise to a new Poisson algebra morphism as stated in the following immediate lemma.

Lemma 34. *The map $\Delta^{(m)} : (S_{\mathfrak{g}}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}}) \rightarrow (S_{\mathfrak{g}}^{(m)}, \cdot, \{\cdot, \cdot\}_{S_{\mathfrak{g}}^{(m)}})$, with $m > 1$, defined by the recursion rule (5.4) with $\Delta^{(2)} = \Delta$ given by (3.56), is a Poisson algebra morphism.*

The injection $\iota : \mathfrak{g} \rightarrow \mathcal{H}_{\Lambda} \subset C^{\infty}(N)$ is a Lie algebra morphism that can be extended to a Poisson algebra morphism $D : S_{\mathfrak{g}} \rightarrow C^{\infty}(N)$ given by $D(P(v_1, \dots, v_r)) = P(\iota(v_1), \dots, \iota(v_r))$. Recall that this map need not to be injective.

Lemma 35. *The Lie algebra morphism $\mathfrak{g} \hookrightarrow C^\infty(N)$ gives rise to a family of Poisson algebra morphisms $D^{(m)} : S_{\mathfrak{g}}^{(m)} \hookrightarrow C^\infty(N)^{(m)} \subset C^\infty(N^m)$ satisfying, for all $v_1, \dots, v_m \in \mathfrak{g} \subset S_{\mathfrak{g}}$, that*

$$\left[D^{(m)}(v_1 \otimes \dots \otimes v_m) \right] (x_{(1)}, \dots, x_{(m)}) = [D(v_1)](x_{(1)}) \cdot \dots \cdot [D(v_m)](x_{(m)}), \quad (3.60)$$

where $x_{(i)}$ is a point of the manifold N placed in the i -position within the product $N \times \dots \times N \equiv N^m$.

From the above results, we can easily demonstrate the following statement which shows that the diagonal prolongations of a Lie–Hamilton system X are also Lie–Hamilton ones admitting a structure induced by that of X .

Proposition 36. *If X is a Lie–Hamilton system on N with a Lie–Hamiltonian structure (N, Λ, h) , then the diagonal prolongation \tilde{X} to each N^{m+1} is also a Lie–Hamilton system endowed with a Lie–Hamiltonian structure $(N^{m+1}, \Lambda^{m+1}, \tilde{h})$ given by $\Lambda^{m+1}(x_{(0)}, \dots, x_{(m)}) = \sum_{a=0}^m \Lambda(x_{(a)})$, where we make use of the vector bundle isomorphism $\mathrm{TN}^{m+1} \simeq \mathrm{TN} \oplus \dots \oplus \mathrm{TN}$, and $\tilde{h}_t = D^{(m+1)}(\Delta^{(m+1)}(h_t))$, where $D^{(m+1)}$ is the Poisson algebra morphism (3.60) induced by the Lie algebra morphism $\mathfrak{g} \hookrightarrow \mathcal{H}_\Lambda \subset C^\infty(N)$.*

The above results enable us to prove the following theorem that provides a method to obtain t -independent constants of motion for the diagonal prolongations of a Lie–Hamilton system. From this theorem, one may obtain superposition rules for Lie–Hamilton systems in an algebraic way. Additionally, this theorem is a generalization, only valid in the case of primitive coproduct maps, of the integrability theorem for coalgebra symmetric systems given in [41].

Theorem 37. *If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) and C is a Casimir element of $(S_{\mathfrak{g}}, \cdot, \{, \}_{S_{\mathfrak{g}}})$, where $\mathfrak{g} \simeq \mathcal{H}_\Lambda$, then*

(i) *The functions defined as*

$$F^{(k)} = D^{(k)}(\Delta^{(k)}(C)), \quad k = 2, \dots, m, \quad (3.61)$$

are t -independent constants of motion for the diagonal prolongation \tilde{X} to N^m . Furthermore, if all the $F^{(k)}$ are non-constant functions, they form a set of $(m - 1)$ functionally independent functions in involution.

(ii) *The functions given by*

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \quad 1 \leq i < j \leq k, \quad k = 2, \dots, m, \quad (3.62)$$

where S_{ij} is the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, are t -independent constants of motion for the diagonal prolongation \tilde{X} to N^m .

Proof. Every $P \in S_{\mathfrak{g}}^{(j)}$ can naturally be considered as an element $P \otimes \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{(k-j)\text{-times}} \in S_{\mathfrak{g}}^{(k)}$. Since $j \leq k$, we have that $\{\Delta^{(j)}(\bar{v}), \Delta^{(k)}(v)\}_{S_{\mathfrak{g}}^{(k)}} = \{\Delta^{(j)}(\bar{v}), \Delta^{(j)}(v)\}_{S_{\mathfrak{g}}^{(j)}}$, $\forall \bar{v}, v \in \mathfrak{g}$. So,

$$\{\Delta^{(j)}(C), \Delta^{(k)}(v)\}_{S_{\mathfrak{g}}^{(k)}} = \{\Delta^{(j)}(C), \Delta^{(j)}(v)\}_{S_{\mathfrak{g}}^{(j)}} = \Delta^{(j)}\left(\{C, v\}_{S_{\mathfrak{g}}^{(j)}}\right) = 0. \quad (3.63)$$

Hence, by using that every function $f \in C^\infty(N^j)$ can be understood as a function $\pi^* f \in C^\infty(N^k)$, being $\pi : N^j \times N^{k-j} \rightarrow N^j$ the projection onto the first factor, and by applying the Poisson algebra morphisms introduced in Lemma 35 we get

$$\left\{D^{(j)}(\Delta^{(j)}(C)), D^{(k)}(\Delta^{(k)}(v))\right\}_{\Lambda^k} = \left\{F^{(j)}, D^{(k)}(\Delta^{(k)}(v))\right\}_{\Lambda^k} = 0, \quad \forall v \in \mathfrak{g}, \quad (3.64)$$

which leads to $\{F^{(j)}, F^{(k)}\}_{\Lambda^k} = 0$, that is, the functions (3.61) are in involution. By construction (see Lemma 34), if these are non-constant, then they are functionally independent functions since $F^{(j)}$ lives in N^j , meanwhile $F^{(k)}$ is defined on N^k .

Let us prove now that all the functions (3.61) and (3.62) are t -independent constants of motion for \tilde{X} . Using that $\mathcal{H}_\Lambda \simeq \mathfrak{g}$ and $X_t = -\hat{\Lambda} \circ dh_t$, we see that X can be brought into the form $X_t = -\hat{\Lambda} \circ d \circ D(v_t)$ for a unique curve $t \rightarrow v_t$ in \mathfrak{g} . From this and Proposition 36, it follows

$$\begin{aligned} \tilde{X}_t &= -\Lambda^m \circ dD^{(m)}(\Delta^{(m)}(v_t)) \implies \\ \tilde{X}_t(F^{(k)}) &= \left\{D^{(k)}(\Delta^{(k)}(C)), D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} = 0. \end{aligned} \quad (3.65)$$

Then, $F^{(k)}$ is a common first-integral for every \tilde{X}_t . Finally, consider the permutation operators S_{ij} , with $1 \leq i < j \leq k$ for $k = 2, \dots, m$. Note that

$$\begin{aligned} 0 &= S_{ij} \left\{F^{(k)}, D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} = \left\{S_{ij}(F^{(k)}), S_{ij}\left(D^{(m)}(\Delta^{(m)}(v_t))\right)\right\}_{\Lambda^m} \\ &= \left\{F_{ij}^{(k)}, D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} = \tilde{X}_t(F_{ij}^{(k)}). \end{aligned} \quad (3.66)$$

Consequently, the functions $F_{ij}^{(k)}$ are t -independent constants of motion for \tilde{X} . \square

Note that the ‘‘omitted’’ case with $k = 1$ in the set of constants (3.61) is, precisely, the one provided by Corollary 32 as $F^{(1)} \equiv F = D(C)$. Depending on the system X , or more specifically, on the associated \mathcal{H}_Λ , the function F can be either a useless trivial constant or a relevant function. It is also worth noting that constants (3.62) need not be functionally independent, but we can always choose those fulfilling such a property. Finally, observe that if X' is such that $V^{X'} \subset V^X$, then the functions (3.61) and (3.62) are also constants of motion for the diagonal prolongation of X' to N^m .

4 Applications of the geometric theory of Lie–Hamilton systems

In this section we aim to achieve superposition rules and constants of motion through the coalgebra formalism. We find of special interest our achievement of finding a superposition rule for Riccati equations with the aid of the coalgebras.

4.1 The Ermakov system

Let us consider the classical Ermakov system [102]

$$\begin{cases} \frac{d^2x}{dt^2} = -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{d^2y}{dt^2} = -\omega^2(t)y, \end{cases} \quad (4.1)$$

with a non-constant t -dependent frequency $\omega(t)$, being b a real constant. This system appears in a number of applications related to problems in Quantum and Classical Mechanics [287]. By writing this system as a first-order one

$$\begin{cases} \frac{dx}{dt} = v_x, & \frac{dv_x}{dt} = -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{dy}{dt} = v_y, & \frac{dv_y}{dt} = -\omega^2(t)y, \end{cases} \quad (4.2)$$

we can apply the theory of Lie systems. Indeed, this is a Lie system related to a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ [97]. In fact, system (4.2) describes the integral curves of the t -dependent vector field $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$\begin{aligned} X_1 &= -x \frac{\partial}{\partial v_x} - y \frac{\partial}{\partial v_y}, & X_2 &= \frac{1}{2} \left(v_x \frac{\partial}{\partial v_x} + v_y \frac{\partial}{\partial v_y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right), \\ X_3 &= v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{b}{x^3} \frac{\partial}{\partial v_x}, \end{aligned} \quad (4.3)$$

satisfy the commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \quad (4.4)$$

As a first new result we show that this is a Lie–Hamilton system. The vector fields are Hamiltonian with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial v_x + \partial/\partial y \wedge \partial/\partial v_y$ provided that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$ for $\alpha = 1, 2, 3$. Thus, we find the following Hamiltonian functions which form a basis for $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda) \simeq (\mathfrak{sl}(2, \mathbb{R}), [\cdot, \cdot])$

$$h_1 = \frac{1}{2}(x^2 + y^2), \quad h_2 = -\frac{1}{2}(xv_x + yv_y), \quad h_3 = \frac{1}{2} \left(v_x^2 + v_y^2 + \frac{b}{x^2} \right), \quad (4.5)$$

as they fulfil

$$\{h_1, h_2\} = -h_1, \quad \{h_1, h_3\} = -2h_2, \quad \{h_2, h_3\} = -h_3. \quad (4.6)$$

Since $X = X_3 + \omega^2(t)X_1$ and $\omega(t)$ is not a constant, every t -independent constant of motion f for X is a common first-integral for X_1, X_2, X_3 . Instead of searching an f by solving the system of PDEs given by $X_1f = X_2f = X_3f = 0$, we use Corollary 32. This easily provides such a first integral through the Casimir element of the symmetric algebra of $\mathfrak{sl}(2, \mathbb{R})$. Explicitly, given a basis $\{v_1, v_2, v_3\}$ for $\mathfrak{sl}(2, \mathbb{R})$ satisfying

$$[v_1, v_2] = -v_1, \quad [v_1, v_3] = -2v_2, \quad [v_2, v_3] = -v_3, \quad (4.7)$$

the Casimir element of $\mathfrak{sl}(2, \mathbb{R})$ reads $\mathcal{C} = \frac{1}{2}(v_1 \tilde{\otimes} v_3 + v_3 \tilde{\otimes} v_1) - v_2 \tilde{\otimes} v_2 \in U_{\mathfrak{sl}(2, \mathbb{R})}$. Then, the inverse of symmetrizer morphism (5.1), $\lambda^{-1} : U_{\mathfrak{sl}(2, \mathbb{R})} \rightarrow S_{\mathfrak{sl}(2, \mathbb{R})}$, gives rise to the Casimir element of $S_{\mathfrak{sl}(2, \mathbb{R})}$

$$C = \lambda^{-1}(\mathcal{C}) = v_1 v_3 - v_2^2. \quad (4.8)$$

According to Lemma 28 we consider the Poisson algebra morphism D induced by the isomorphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathcal{H}_\Lambda$ defined by $\phi(v_\alpha) = h_\alpha$ for $\alpha = 1, 2, 3$. Subsequently, via Corollary 32, we obtain

$$\begin{aligned} F = D(C) &= \phi(v_1)\phi(v_3) - \phi^2(v_2) = h_1 h_3 - h_2^2 = \\ &= (v_y x - v_x y)^2 + b \left(1 + \frac{y^2}{x^2}\right). \end{aligned} \quad (4.9)$$

In this way, we recover, up to an additive and multiplicative constant, the well-known Lewis–Riesenfeld invariant [287]. Note that when $\omega(t)$ is a constant, then $V^X \subset V$ and the function F is also a constant of motion for X (4.2).

4.2 A superposition rule for Riccati equations

Let us turn to the system of Riccati equations on $\mathcal{O} = \{(x_1, x_2, x_3, x_4) \mid x_i \neq x_j, i \neq j = 1, \dots, 4\} \subset \mathbb{R}^4$, given by

$$\frac{dx_i}{dt} = a_0(t) + a_1(t)x_i + a_2(t)x_i^2, \quad i = 1, \dots, 4, \quad (4.10)$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary t -dependent functions. The knowledge of a non-constant t -independent constant of motion for any system of this type leads to obtaining a superposition rule for Riccati equations [91]. Usually, this requires the integration of a system of PDEs [91] or ODEs [419]. We here obtain such a t -independent constant of motion through algebraic methods by showing that (4.10) is a Lie–Hamilton system with a given Lie–Hamiltonian structure and obtaining an associated polynomial Lie integral.

Observe that (4.10) is a Lie system related to a t -dependent vector field $X_R = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \sum_{i=1}^4 \frac{\partial}{\partial x_i}, \quad X_2 = \sum_{i=1}^4 x_i \frac{\partial}{\partial x_i}, \quad X_3 = \sum_{i=1}^4 x_i^2 \frac{\partial}{\partial x_i} \quad (4.11)$$

span a Vessiot–Guldberg Lie algebra V for (4.10) isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ satisfying the same commutation relations (4.4). For simplicity, we assume $V^X = V$. Nevertheless, our final results are valid for any other case.

To show that (4.10) is a Lie–Hamilton system for arbitrary functions $a_0(t)$, $a_1(t)$, $a_2(t)$, we need to search for a symplectic form ω such that V consists of Hamiltonian vector fields. By imposing $\mathcal{L}_{X_\alpha} \omega = 0$, for $\alpha = 1, 2, 3$, we obtain the 2-form

$$\omega_R = \frac{dx_1 \wedge dx_2}{(x_1 - x_2)^2} + \frac{dx_3 \wedge dx_4}{(x_3 - x_4)^2}, \quad (4.12)$$

which is closed and non-degenerate on \mathcal{O} . Now, observe that $\iota_{X_\alpha} \omega = dh_\alpha$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{1}{x_1 - x_2} + \frac{1}{x_3 - x_4}, \quad h_2 = \frac{1}{2} \left(\frac{x_1 + x_2}{x_1 - x_2} + \frac{x_3 + x_4}{x_3 - x_4} \right), \\ h_3 = \frac{x_1 x_2}{x_1 - x_2} + \frac{x_3 x_4}{x_3 - x_4}.$$

So, h_1, h_2 and h_3 are Hamiltonian functions for X_1, X_2 and X_3 , correspondingly. Using the Poisson bracket $\{\cdot, \cdot\}_\omega$ induced by ω , we obtain that h_1, h_2 and h_3 satisfy the commutation relations (4.6), and $(\langle h_1, h_2, h_3 \rangle, \{\cdot, \cdot\}_\omega) \simeq \mathfrak{sl}(2, \mathbb{R})$. Next, we again express $\mathfrak{sl}(2, \mathbb{R})$ in the basis $\{v_1, v_2, v_3\}$ with Lie brackets (4.7) and Casimir function (4.8), and we consider the Poisson algebra morphism $D : S_{\mathfrak{sl}(2, \mathbb{R})} \rightarrow C^\infty(\mathcal{O})$ given by the isomorphism $\phi(v_\alpha) = h_\alpha$ for $\alpha = 1, 2, 3$. As $(\mathcal{O}, \{\cdot, \cdot\}_\omega, h_t = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3)$ is a Lie–Hamiltonian structure for X and applying Corollary 32, we obtain the t -independent constant of motion for X

$$F = D(C) = h_1 h_3 - h_2^2 = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)}. \quad (4.13)$$

As in the previous example, if $V^X \subset V$, then F is also a constant of motion for X . It is worth noting that F is the known constant of motion obtained for deriving a superposition rule for Riccati equations [91, 419], which is here deduced through a simple algebraic calculation.

It is also interesting that V also becomes a Lie algebra of Hamiltonian vector fields with respect to a second symplectic structure given by $\omega = \sum_{1 \leq i < j \leq 4} \frac{dx_i \wedge dx_j}{(x_i - x_j)^2}$. Consequently, the system (4.10) can be considered, in fact, as a *bi-Lie–Hamilton system*.

4.3 Kummer–Schwarz equations in Hamilton form

It was proved in [109] that the second-order Kummer–Schwarz equations [56, 87] admit a t -dependent Hamiltonian which can be used to work out their Hamilton’s equations, namely

$$\begin{cases} \frac{dx}{dt} = \frac{px^3}{2}, \\ \frac{dp}{dt} = -\frac{3p^2x^2}{4} - \frac{b_0}{4} + \frac{4b_1(t)}{x^2}, \end{cases} \quad (4.14)$$

where $b_1(t)$ is assumed to be a non-constant t -dependent function, $(x, p) \in \mathbb{T}^*\mathbb{R}_0$ with $\mathbb{R}_0 \equiv \mathbb{R} - \{0\}$, and b_0 is a real constant. This is a Lie system associated to the t -dependent vector field $X = X_3 + b_1(t)X_1$ [109], where the vector fields

$$\begin{aligned} X_1 &= \frac{4}{x^2} \frac{\partial}{\partial p}, & X_2 &= x \frac{\partial}{\partial x} - p \frac{\partial}{\partial p}, \\ X_3 &= \frac{px^3}{2} \frac{\partial}{\partial x} - \frac{1}{4} (3p^2x^2 + b_0) \frac{\partial}{\partial p} \end{aligned} \quad (4.15)$$

span a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ fulfilling (4.4). Moreover, X is a Lie–Hamilton system as V consists of Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathbb{T}^*\mathbb{R}_0$. Indeed, $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{4}{x}, \quad h_2 = xp, \quad h_3 = \frac{1}{4} (p^2x^3 + b_0x) \quad (4.16)$$

are a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ satisfying the commutation relations (4.6). Therefore, (4.14) is a Lie–Hamilton system possessing a Lie–Hamiltonian structure $(\mathbb{T}^*\mathbb{R}_0, \Lambda, h)$, where $h_t = h_3 + b_1(t)h_1$.

To obtain a superposition rule for X we need to determine an integer m so that the diagonal prolongations of X_α to $\mathbb{T}^*\mathbb{R}_0^m$ ($\alpha = 1, 2, 3$) become linearly independent at a generic point (see [91]). This happens for $m = 2$. We consider a coordinate system in $\mathbb{T}\mathbb{R}_0^3$, namely $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$. A superposition rule for X can be obtained by determining two common first integrals for the diagonal prolongations \widetilde{X}_α to $\mathbb{T}^*\mathbb{R}_0^3$ satisfying

$$\frac{\partial(F_1, F_2)}{\partial(x_{(1)}, p_{(1)})} \neq 0. \quad (4.17)$$

Instead of searching F_1, F_2 in the standard way, i.e., by solving the system of PDEs given by $\widetilde{X}_\alpha f = 0$, we make use of Theorem 37. This provides such first integrals through the Casimir element C (4.8) of the symmetric algebra of $\mathcal{H}_\Lambda \simeq \mathfrak{sl}(2, \mathbb{R})$.

Indeed, the coproduct (3.56) enables us to define the elements

$$\begin{aligned}
\Delta(C) &= \Delta(v_1)\Delta(v_3) - \Delta(v_2)^2 \\
&= (v_1 \otimes 1 + 1 \otimes v_1)(v_3 \otimes 1 + 1 \otimes v_3) - (v_2 \otimes 1 + 1 \otimes v_2)^2, \\
\Delta^{(3)}(C) &= \Delta^{(3)}(v_1)\Delta^{(3)}(v_3) - \Delta^{(3)}(v_2)^2 \\
&= (v_1 \otimes 1 \otimes 1 + 1 \otimes v_1 \otimes 1 + 1 \otimes 1 \otimes v_1)(v_3 \otimes 1 \otimes 1 + 1 \otimes v_3 \otimes 1 \\
&\quad + 1 \otimes 1 \otimes v_3) - (v_2 \otimes 1 \otimes 1 + 1 \otimes v_2 \otimes 1 + 1 \otimes 1 \otimes v_2)^2,
\end{aligned} \tag{4.18}$$

for $S_{\mathfrak{sl}(2, \mathbb{R})}^{(2)}$ and $S_{\mathfrak{sl}(2, \mathbb{R})}^{(3)}$, respectively. By applying D , $D^{(2)}$ and $D^{(3)}$ coming from the isomorphism $\phi(v_\alpha) = h_\alpha$ for the Hamiltonian functions (4.16), we obtain, via Theorem 37, the following constants of motion of the type (3.61)

$$\begin{aligned}
F &= D(C) = h_1(x_1, p_1)h_3(x_1, p_1) - h_2^2(x_1, p_1) = b_0, \\
F^{(2)} &= D^{(2)}(\Delta(C)) \\
&= [h_1(x_1, p_1) + h_1(x_2, p_2)] [h_3(x_1, p_1) + h_3(x_2, p_2)] \\
&\quad - [h_2(x_1, p_1) + h_2(x_2, p_2)]^2 \\
&= \frac{b_0(x_1 + x_2)^2 + (p_1x_1^2 - p_2x_2^2)^2}{x_1x_2} = \frac{b_0(x_1^2 + x_2^2) + (p_1x_1^2 - p_2x_2^2)^2}{x_1x_2} + 2b_0, \\
F^{(3)} &= D^{(3)}(\Delta(C)) = \sum_{i=1}^3 h_1(x_i, p_i) \sum_{j=1}^3 h_3(x_j, p_j) - \left(\sum_{i=1}^3 h_2(x_i, p_i) \right)^2 \\
&= \sum_{1 \leq i < j}^3 \frac{b_0(x_i + x_j)^2 + (p_i x_i^2 - p_j x_j^2)^2}{x_i x_j} - 3b_0,
\end{aligned} \tag{4.19}$$

where, for the sake of simplicity, hereafter we denote (x_i, p_i) the coordinates $(x_{(i)}, p_{(i)})$. Thus F simply gives rise to the constant b_0 , while $F^{(2)}$ and $F^{(3)}$ are, by construction, two functionally independent constants of motion for \tilde{X} fulfilling (4.17) which, in turn, allows us to derive a superposition rule for X . Furthermore, the function $F^{(2)} \equiv F_{12}^{(2)}$ provides two other constants of the type (3.62) given by $F_{13}^{(2)} = S_{13}(F^{(2)})$ and $F_{23}^{(2)} = S_{23}(F^{(2)})$ that verify $F^{(3)} = F^{(2)} + F_{13}^{(2)} + F_{23}^{(2)} - 3b_0$. Since it is simpler to work with $F_{23}^{(2)}$ than with $F^{(3)}$, we choose the pair $F^{(2)}, F_{23}^{(2)}$ as the two functionally independent first integrals to obtain a superposition rule. We set

$$F^{(2)} = k_1 + 2b_0, \quad F_{23}^{(2)} = k_2 + 2b_0, \tag{4.20}$$

and compute x_1, p_1 in terms of the other variables and k_1, k_2 . From (4.20), we have

$$p_1 = p_1(x_1, x_2, p_2, x_3, p_3, k_1) = \frac{p_2 x_2^2 \pm \sqrt{k_1 x_1 x_2 - b_0(x_1^2 + x_2^2)}}{x_1^2}. \tag{4.21}$$

Substituting in the second relation within (4.20), we obtain

$$x_1 = x_1(x_2, p_2, x_3, p_3, k_1, k_2) = \frac{A^2 B_+ + b_0 B_- (x_2^2 - x_3^2) \pm 2A\sqrt{\Upsilon}}{B_-^2 + 4b_0 A^2}, \quad (4.22)$$

provided that the functions A, B_{\pm}, Υ are defined by

$$\begin{aligned} A &= p_2 x_2^2 - p_3 x_3^2, & B_{\pm} &= k_1 x_2 \pm k_2 x_3, \\ \Upsilon &= A^2 [k_1 k_2 x_2 x_3 - 2b_0^2 (x_2^2 + x_3^2) - b_0 A^2] \\ &\quad + b_0 x_2 x_3 B_- (k_2 x_2 - k_1 x_3) - b_0^3 (x_2^2 - x_3^2)^2. \end{aligned} \quad (4.23)$$

By introducing this result into (4.21), we obtain $p_1 = p_1(x_2, p_2, x_3, p_3, k_1, k_2)$ which, along with $x_1 = x_1(x_2, p_2, x_3, p_3, k_1, k_2)$, provides a superposition rule for X .

In particular for (4.14) with $b_0 = 0$, it results

$$\begin{aligned} x_1 &= \frac{A^2 (B_+ \pm 2\sqrt{k_1 k_2 x_2 x_3})}{B_-^2}, \\ p_1 &= B_-^3 \frac{\left[B_- p_2 x_2^2 \pm A \sqrt{k_1 x_2 (B_+ \pm 2\sqrt{k_1 k_2 x_2 x_3})} \right]}{A^4 (B_+ \pm 2\sqrt{k_1 k_2 x_2 x_3})^2}, \end{aligned} \quad (4.24)$$

where the functions A, B_{\pm} remain in the above same form. As the constants of motion were derived for non-constant $b_1(t)$, when $b_1(t)$ is constant we have $V^X \subset V$. As a consequence, the functions $F, F^{(2)}, F^{(3)}$ and so on are still constants of motion for the diagonal prolongation \tilde{X} and the superposition rules are still valid for any system (4.14).

4.4 The n -dimensional Smorodinsky–Winternitz systems

Let us focus on the n -dimensional Smorodinsky–Winternitz systems [195, 420] with unit mass and a non-constant t -dependent frequency $\omega(t)$ whose Hamiltonian is given by

$$h = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \omega^2(t) \sum_{i=1}^n x_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{b_i}{x_i^2}, \quad (4.25)$$

where the b_i 's are n real constants. The corresponding Hamilton's equations read

$$\begin{cases} \frac{dx_i}{dt} = p_i, \\ \frac{dp_i}{dt} = -\omega^2(t)x_i + \frac{b_i}{x_i^3}, \end{cases} \quad i = 1, \dots, n. \quad (4.26)$$

These systems have been recently attracting quite much attention in Classical and Quantum Mechanics for their special properties [109, 220, 234, 424]. Observe that

Ermakov systems (4.2) arise as the particular case of (4.26) for $n = 2$ and $b_2 = 0$. For $n = 1$ the above system maps into the Milne–Pinney equations, which are of interest in the study of several cosmological models [354, 102], through the diffeomorphism $(x, p) \in \mathbb{T}^*\mathbb{R}_0 \mapsto (x, v = p) \in \mathbb{T}\mathbb{R}_0$.

Let us show that the system (4.26) can be endowed with a Lie–Hamiltonian structure. This system describes the integral curves of the t -dependent vector field on $\mathbb{T}^*\mathbb{R}_0^n$ given by $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$\begin{aligned} X_1 &= -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, & X_2 &= \frac{1}{2} \sum_{i=1}^n \left(p_i \frac{\partial}{\partial p_i} - x_i \frac{\partial}{\partial x_i} \right), \\ X_3 &= \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{b_i}{x_i^3} \frac{\partial}{\partial p_i} \right), \end{aligned} \quad (4.27)$$

fulfil the commutation rules (4.4). Hence, (4.26) is a Lie system. The space $\mathbb{T}^*\mathbb{R}_0^n$ admits a natural Poisson bivector $\Lambda = \sum_{i=1}^n \partial/\partial x_i \wedge \partial/\partial p_i$ related to the restriction to this space of the canonical symplectic structure on $\mathbb{T}^*\mathbb{R}^n$. Moreover, the preceding vector fields are Hamiltonian vector fields with Hamiltonian functions

$$h_1 = \frac{1}{2} \sum_{i=1}^n x_i^2, \quad h_2 = -\frac{1}{2} \sum_{i=1}^n x_i p_i, \quad h_3 = \frac{1}{2} \sum_{i=1}^n \left(p_i^2 + \frac{b_i}{x_i^2} \right) \quad (4.28)$$

which satisfy the commutation relations (4.6), so that $\mathcal{H}_\Lambda \simeq \mathfrak{sl}(2, \mathbb{R})$. Consequently, every curve h_t that takes values in the Lie algebra spanned by h_1, h_2 and h_3 gives rise to a Lie–Hamiltonian structure $(\mathbb{T}^*\mathbb{R}_0^n, \Lambda, h)$. Then, the system (4.26), described by the t -dependent vector field $X = X_3 + \omega^2(t)X_1 = -\widehat{\Lambda}(dh_3 + \omega^2(t)dh_1)$, is a Lie–Hamilton system with a Lie–Hamiltonian structure $(\mathbb{T}^*\mathbb{R}_0^n, \Lambda, h_t = h_3 + \omega^2(t)h_1)$.

Subsequently, we derive an explicit superposition rule for the simplest case of the system (4.26) corresponding to $n = 1$, and proceed as in the previous subsection. The prolongations of X_α ($\alpha = 1, 2, 3$) again become linearly independent for $m = 2$ and we need to obtain two first integrals for the diagonal prolongations \widetilde{X}_α of $\mathbb{T}^*\mathbb{R}_0^3$ fulfilling (4.17) for the coordinate system $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$ of $\mathbb{T}^*\mathbb{R}_0^3$. Similarly to the previous example, we have an injection $D : \mathfrak{sl}(2, \mathbb{R}) \rightarrow C^\infty(\mathbb{T}^*\mathbb{R}_0)$ which leads to the morphisms $D^{(2)}$ and $D^{(3)}$. Then, by taking into account the Casimir function (4.8) and the Hamiltonians (4.28), we apply Theorem 37 obtaining the following first integrals:

$$\begin{aligned}
F^{(2)} &= D^{(2)}(\Delta(C)) = \frac{1}{4}(x_1 p_2 - x_2 p_1)^2 + \frac{b(x_1^2 + x_2^2)^2}{4x_1^2 x_2^2}, \\
F^{(3)} &= D^{(3)}(\Delta(C)) = \frac{1}{4} \sum_{1 \leq i < j}^3 \left[(x_i p_j - x_j p_i)^2 + \frac{b(x_i^2 + x_j^2)^2}{x_i^2 x_j^2} \right] - \frac{3}{4}b, \\
F_{13}^{(2)} &= S_{13}(F^{(2)}), \\
F_{23}^{(2)} &= S_{23}(F^{(2)}), \\
F^{(3)} &= F^{(2)} + F_{13}^{(2)} + F_{23}^{(2)} - 3b/4,
\end{aligned} \tag{4.29}$$

where (x_i, p_i) denote the coordinates $(x_{(i)}, p_{(i)})$; notice that $F = D(C) = b/4$. We choose $F^{(2)}$ and $F_{23}^{(2)}$ as the two functionally independent constants of motion and we shall use $F_{13}^{(2)}$ in order to simplify the results. Recall that these functions are exactly the first integrals obtained in other works, e.g., [97], for describing superposition rules of dissipative Milne–Pinney equations (up to the diffeomorphism $\varphi : (x, p) \in T^*\mathbb{R}_0 \mapsto (x, v) = (x, p) \in T\mathbb{R}_0$ to system (4.26) with $n = 1$), and lead straightforwardly to deriving a superposition rule for these equations [98].

Indeed, we set

$$F^{(2)} = \frac{k_1}{4} + \frac{b}{2}, \quad F_{23}^{(2)} = \frac{k_2}{4} + \frac{b}{2}, \quad F_{13}^{(2)} = \frac{k_3}{4} + \frac{b}{2}, \tag{4.30}$$

and from the first equation we obtain p_1 in terms of the remaining variables and k_1

$$p_1 = p_1(x_1, x_2, p_2, x_3, p_3, k_1) = \frac{p_2 x_1^2 x_2 \pm \sqrt{k_1 x_1^2 x_2^2 - b(x_1^4 + x_2^4)}}{x_1 x_2^2}. \tag{4.31}$$

By introducing this value in the second expression of (4.30), one can determine the expression of x_1 as a function of x_2, p_2, x_3, p_3 and the constants k_1, k_2 . Such a result is rather simplified when the third constant of (4.30) enters, yielding

$$\begin{aligned}
x_1 &= x_1(x_2, p_2, x_3, p_3, k_1, k_2) = x_1(x_2, x_3, k_1, k_2, k_3) \\
&= \left\{ \mu_1 x_2^2 + \mu_2 x_3^2 \pm \sqrt{\mu [k_3 x_2^2 x_3^2 - b(x_2^4 + x_3^4)]} \right\}^{1/2},
\end{aligned} \tag{4.32}$$

where the constants μ_1, μ_2, μ are defined in terms of k_1, k_2, k_3 and b as

$$\begin{aligned}
\mu_1 &= \frac{2bk_1 - k_2 k_3}{4b^2 - k_3^2}, & \mu_2 &= \frac{2bk_2 - k_1 k_3}{4b^2 - k_3^2}, \\
\mu &= \frac{4[4b^3 + k_1 k_2 k_3 - b(k_1^2 + k_2^2 + k_3^2)]}{(4b^2 - k_3^2)^2}.
\end{aligned} \tag{4.33}$$

And by introducing (4.32) into (4.31), we obtain

$$p_1 = p_1(x_2, p_2, x_3, p_3, k_1, k_2) = p_1(x_2, p_2, x_3, k_1, k_2, k_3), \quad (4.34)$$

which together with (4.32) provide a superposition rule for (4.26) with $n = 1$. These expressions constitute the known superposition rule for Milne–Pinney equations [98]. Observe that, instead of solving systems of PDEs for obtaining the first integrals as in [97, 98], we have obtained them algebraically in a simpler way. When $b = 0$ we recover, as expected, the superposition rule for the harmonic oscillator with a t -dependent frequency. Similarly to previous examples, the above superposition rule is also valid when $\omega(t)$ is constant.

4.5 A trigonometric system

Let us study a final example appearing in the study of integrability of classical systems [21, 22]. Consider the system

$$\begin{cases} \frac{dx}{dt} = \sqrt{1-x^2} (B_x(t) \sin p - B_y(t) \cos p), \\ \frac{dp}{dt} = -(B_x(t) \cos p + B_y(t) \sin p) \frac{x}{\sqrt{1-x^2}} - B_z(t), \end{cases} \quad (4.35)$$

where $B_x(t), B_y(t), B_z(t)$ are arbitrary t -dependent functions and $(x, p) \in \mathbb{T}^*I$, with $I = (-1, 1)$. This system describes the integral curves of the t -dependent vector field

$$X = \sqrt{1-x^2} (B_x(t) \sin p - B_y(t) \cos p) \frac{\partial}{\partial x} - \left[\frac{(B_x(t) \cos p + B_y(t) \sin p)x}{\sqrt{1-x^2}} + B_z(t) \right] \frac{\partial}{\partial p}, \quad (4.36)$$

which can be brought into the form $X = B_x(t)X_1 + B_y(t)X_2 + B_z(t)X_3$, where

$$\begin{aligned} X_1 &= \sqrt{1-x^2} \sin p \frac{\partial}{\partial x} - \frac{x}{\sqrt{1-x^2}} \cos p \frac{\partial}{\partial p}, \\ X_2 &= -\sqrt{1-x^2} \cos p \frac{\partial}{\partial x} - \frac{x}{\sqrt{1-x^2}} \sin p \frac{\partial}{\partial p}, \end{aligned} \quad (4.37)$$

and $X_3 = -\partial/\partial p$ satisfy the commutation relations

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = X_2, \quad [X_2, X_3] = X_1. \quad (4.38)$$

In other words, X describes a Lie system associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{su}(2)$. As in the previous examples, we assume $V^X = V$. Now,

the vector fields X_α ($\alpha = 1, 2, 3$) are Hamiltonian ones with Hamiltonian functions given by

$$h_1 = -\sqrt{1-x^2} \cos p, \quad h_2 = -\sqrt{1-x^2} \sin p, \quad h_3 = x, \quad (4.39)$$

thus spanning a real Lie algebra isomorphic to $\mathfrak{su}(2)$. Indeed,

$$\{h_1, h_2\} = -h_3, \quad \{h_3, h_1\} = -h_2, \quad \{h_2, h_3\} = -h_1. \quad (4.40)$$

Next we consider a basis $\{v_1, v_2, v_3\}$ for $\mathfrak{su}(2)$ satisfying

$$[v_1, v_2] = -v_3, \quad [v_3, v_1] = -v_2, \quad [v_2, v_3] = -v_1, \quad (4.41)$$

so that $\mathfrak{su}(2)$ admits the Casimir $\mathcal{C} = v_1 \tilde{\otimes} v_1 + v_2 \tilde{\otimes} v_2 + v_3 \tilde{\otimes} v_3 \in U_{\mathfrak{su}(2)}$. Then, the Casimir element of $S_{\mathfrak{su}(2)}$ reads $C = \lambda^{-1}(\mathcal{C}) = v_1^2 + v_2^2 + v_3^2$.

The diagonal prolongations of X_1, X_2, X_3 are linearly independent at a generic point for $m = 2$ and we have to derive two first integrals for the diagonal prolongations $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ on $T^*\Gamma^3$ satisfying (4.17) working with the coordinate system $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$ of $T^*\Gamma^3$. Then, by taking into account the Casimir function C , the Hamiltonians (4.39), the isomorphism $\phi(v_\alpha) = h_\alpha$ and the injection $D : \mathfrak{sl}(2, \mathbb{R}) \rightarrow C^\infty(T^*\Gamma)$, we apply Theorem 37 obtaining the following first integrals:

$$\begin{aligned} F^{(2)} &= 2 \left[\sqrt{1-x_1^2} \sqrt{1-x_2^2} \cos(p_1 - p_2) + x_1 x_2 + 1 \right], \\ F^{(3)} &= 2 \sum_{1 \leq i < j}^3 \left[\sqrt{1-x_i^2} \sqrt{1-x_j^2} \cos(p_i - p_j) + x_i x_j \right] + 3, \\ F_{13}^{(2)} &= S_{13}(F^{(2)}), \\ F_{23}^{(2)} &= S_{23}(F^{(2)}), \\ F^{(3)} &= F^{(2)} + F_{13}^{(2)} + F_{23}^{(2)} - 3, \end{aligned} \quad (4.42)$$

and $F = D(C) = 1$. We again choose $F^{(2)}$ and $F_{23}^{(2)}$ as the two functionally independent constants of motion, which provide us, after cumbersome but straightforward computations, with a superposition rule for these systems. This leads to a quartic equation, whose solution can be obtained through known methods. All our results are also valid for the case when $V^X \subset V$.

5 Classification of Lie–Hamilton systems on the plane

In this section we classify finite-dimensional real Lie algebras of Hamiltonian vector fields on the plane. In view of the definition of a Lie–Hamilton system, such a classification provides a classification of Lie–Hamilton systems on the plane, as well.

This classification will be of primordial importance, given the number of physical system which can be interpreted in terms of Lie–Hamilton systems. This will be illustrated in the following chapter.

5.1 On general definitions and properties

A natural problem in the theory of Lie systems is the classification of Lie systems on a fixed manifold, which amounts to classifying finite-dimensional Lie algebras of vector fields on it. Lie accomplished the local classification of finite-dimensional real Lie algebras of vector fields on the real line. More precisely, he showed that each such a Lie algebra is locally diffeomorphic to a Lie subalgebra of $\langle \partial_x, x\partial_x, x^2\partial_x \rangle \simeq \mathfrak{sl}(2)$ on a neighborhood of each *generic point* x_0 of the Lie algebra [210]. He also performed the local classification of finite-dimensional real Lie algebras of planar vector fields and started the study of the analogous problem on \mathbb{C}^3 [14].

Lie’s local classification on the plane presented some unclear points which were misunderstood by several authors during the following decades. To solve this, González-López, Kamran and Olver retook the problem and provided a clearer insight in [210]. Precisely, they proved that every non-zero Lie algebra of vector fields on the plane is locally diffeomorphic around each generic point to one of the finite-dimensional real Lie algebras (the GKO classification) given in Table 1 in Appendix 1.

As every Vessiot–Guldberg Lie algebra on the plane has to be locally diffeomorphic to one of the Lie algebras of the GKO classification, every Lie system on the plane is locally diffeomorphic to a Lie system taking values in a Vessiot–Guldberg Lie algebra within the GKO classification. So, the local properties of all Lie systems on the plane can be studied through the Lie systems related to the GKO classification. As a consequence, we say that the GKO classification gives the local classification of Lie systems on the plane. Their classification divides finite-dimensional Lie algebras of vector fields on \mathbb{R}^2 into 28 non-diffeomorphic classes, which, in fact, can be regarded as a local classification of Lie systems on the plane [32, 210].

For instance, we see from the matrix Riccati equations (2.17) that system (2.14) is related to a non-solvable eight-dimensional Vessiot–Guldberg Lie algebra. In view of the GKO classification in Table 1 (cf. [32, 210]), all Lie algebras of this type are locally diffeomorphic to $\mathbb{P}_8 \simeq \mathfrak{sl}(3)$. That is why we say that the system (2.14) is a *Lie system of class* \mathbb{P}_8 .

The GKO classification of finite-dimensional Lie algebras of vector fields X_i ($i = 1, \dots, n$) on the plane covers two subclasses called *primitive* (8 cases \mathbb{P}_x) and *imprimitive* (20 cases \mathbb{I}_x) ones [210].

Definition 1. A finite-dimensional real Lie algebra V of vector fields on an open subset $U \subset \mathbb{R}^2$ is *imprimitive* when there exists a one-dimensional distribution \mathcal{D} on \mathbb{R}^2 invariant under the action of V by Lie brackets, i.e., for every $X \in V$ and every

vector field Y taking values in \mathcal{D} , we have that $[X, Y]$ takes values in \mathcal{D} . Otherwise, V is called *primitive*.

Example 1. Consider the Lie algebra I_4 which is spanned by the vector fields X_1, X_2 and X_3 given in Table 1 in Appendix 1. If we define \mathcal{D} to be the distribution on \mathbb{R}^2 generated by $Y = \partial_x$, we see that:

$$[X_1, Y] = 0, \quad [X_2, Y] = -Y, \quad [X_3, Y] = -2xY. \quad (5.1)$$

We infer from this that \mathcal{D} is a one-dimensional distribution invariant under the action of I_4 . Hence, I_4 is an imprimitive Lie algebra of vector fields.

To determine which of the 28 classes can be considered as Vessiot–Guldberg Lie algebras of Hamiltonian vector fields, a symplectic form

$$\omega = f(x, y)dx \wedge dy \quad (5.2)$$

must be found so that each X_i , belonging to the basis of the considered Lie algebra, becomes Hamiltonian (see [32] for details). In particular, X_i are Hamiltonian vector fields with respect to ω whenever the Lie derivative of ω relative to X_i vanishes, that is, $\mathcal{L}_{X_i}\omega = 0$. If ω exists, then X_i become Hamiltonian vector fields and their corresponding Hamiltonian functions h_i are obtained by using the relation $\iota_{X_i}\omega = dh_i$. The functions $\langle h_1, \dots, h_n \rangle$ and their successive brackets with respect to the Lie bracket

$$\{\cdot, \cdot\}_\omega : C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2) \quad (5.3)$$

induced by ω span a finite-dimensional Lie algebra of functions $(\mathfrak{W}, \{\cdot, \cdot\}_\omega)$. We will call this Lie algebra a *Lie–Hamilton algebra* (LH algebra, in short).

In the case of the Vessiot–Guldberg Lie algebra V^{MR} for matrix Riccati equations, we see that there exists no such a structure. Indeed, if $X_0 = \partial_x$ and $X_2 = x\partial_x$ which belong to V^{MP} were Hamiltonian relative to a symplectic structure, let us say $\omega = f(x, y)dx \wedge dy$, we would have $\mathcal{L}_{X_0}\omega = \mathcal{L}_{X_2}\omega = 0$. This implies that $\partial_x f = x\partial_x f + f = 0$. Hence, $f = 0$ and ω cannot be symplectic. In other words, V^{MP} is not a Lie algebra of Hamiltonian vector fields.

This example illustrates that not every Lie system admits a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Poisson structure [401]. When a Lie system does, we call it a Lie–Hamilton system (LH system) [109].

The problem of which of the Lie algebras within the GKO classification consist of Hamiltonian vector fields with respect to some Poisson structure has recently been solved by the present writer and collaborators in [32], showing that, among the 28 classes of the GKO classification, only 12 remain as Lie algebras of Hamiltonian vector fields. Furthermore, examples of LH systems having applications in the above mentioned fields have also been worked out in [32]. We recall that one advantage of

LH systems is that their superposition rules might be obtained straightforwardly by applying a coalgebra approach as it has been formulated in the past section [35].

It has been recently proven that the initial 8 + 20 cases of the GKO classification are reduced to 4 + 8 classes of finite-dimensional Lie algebras of Hamiltonian vector fields [32]. The final result is summarized in Table 1 in Appendix 1, where we indicate the Lie algebra \mathfrak{g} of Hamiltonian vector fields X_i , a family of their corresponding Hamiltonian functions h_i and an associated symplectic form ω .

Minimal Lie algebras of Lie–Hamilton systems on the plane

In this section we study the local structure of the minimal Lie algebras of Lie–Hamilton systems on the plane around their generic points.

Our main result, Theorem 6, and the remaining findings of this section enable us to give the local classification of Lie–Hamilton systems on the plane in the following subsections. To simplify the notation, U will hereafter stand for a contractible open subset of \mathbb{R}^2 .

Definition 2. Given a finite-dimensional Lie algebra V of vector fields on a manifold N , we say that $\xi_0 \in N$ is a *generic point* of V when the rank of the generalized distribution

$$\mathcal{D}_\xi^V = \{X(\xi) \mid X \in V\} \subset T_\xi N, \quad \xi \in N, \quad (5.4)$$

i.e., the function $r^V(\xi) = \dim \mathcal{D}_\xi^V$, is locally constant around ξ_0 . We call *generic domain* or simply *domain* of V the set of generic points of V .

Example 2. Consider the Lie algebra $I_4 = \langle X_1, X_2, X_3 \rangle$ of vector fields on \mathbb{R}^2 detailed in Table 1. By using the expressions of X_1 , X_2 and X_3 in coordinates, we see that $r^{I_4}(x, y)$ equals the rank of the matrix

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \end{pmatrix}, \quad (5.5)$$

which is two for every $(x, y) \in \mathbb{R}^2$ except for points with $y - x = 0$, where the rank is one. So, the domain of I_4 is $\mathbb{R}_{x \neq y}^2 = \{(x, y) \mid x \neq y\} \subset \mathbb{R}^2$.

Lemma 3. *Let V be a finite-dimensional real Lie algebra of Hamiltonian vector fields on \mathbb{R}^2 with respect to a Poisson structure and let $\xi_0 \in \mathbb{R}^2$ be a generic point of V . There exists a $U \ni \xi_0$ such that $V|_U$ consists of Hamiltonian vector fields relative to a symplectic structure.*

Proof. If $\dim \mathcal{D}_{\xi_0}^V = 0$, then $\dim \mathcal{D}_\xi^V = 0$ for every ξ in a $U \ni \xi_0$ because the rank of \mathcal{D}^V is locally constant around generic points. Consequently, $V|_U = 0$ and its unique element become Hamiltonian relative to the restriction of $\omega = dx \wedge dy$ to U . Let us assume now $\dim \mathcal{D}_{\xi_0}^V \neq 0$. By assumption, the elements of V are Hamiltonian vector

fields with respect to a Poisson bivector $\Lambda \in \Gamma(\Lambda^2 T\mathbb{R}^2)$. Hence, $\mathcal{D}_\xi^V \subset \mathcal{D}_\xi^\Lambda$ for every $\xi \in \mathbb{R}^2$, with \mathcal{D}^Λ being the *characteristic distribution* of Λ [401]. Since $\dim \mathcal{D}_{\xi_0}^V \neq 0$ and r^V is locally constant at ξ_0 , then $\dim \mathcal{D}_\xi^V > 0$ for every ξ in a $U \ni \xi_0$. Since the rank of \mathcal{D}^Λ is even at every point of \mathbb{R}^2 and $\mathcal{D}_\xi^V \subset \mathcal{D}_\xi^\Lambda$ for every $\xi \in U$, the rank of \mathcal{D}^Λ is two on U . So, Λ comes from a symplectic structure on U and $V|_U$ is a Lie algebra of Hamiltonian vector fields relative to it. \square

Roughly speaking, the previous lemma establishes that any Lie–Hamilton system X on \mathbb{R}^2 can be considered around each generic point of V^X as a Lie–Hamilton system admitting a minimal Lie algebra of Hamiltonian vector fields with respect to a symplectic structure. As our study of such systems is local, we hereafter focus on analysing minimal Lie algebras of this type.

A *volume form* Ω on an n -dimensional manifold N is a non-vanishing n -form on N . The divergence of a vector field X on N with respect to Ω is the unique function $\operatorname{div} X : N \rightarrow \mathbb{R}$ satisfying $\mathcal{L}_X \Omega = (\operatorname{div} X) \Omega$. An *integrating factor* for X on $U \subset N$ is a function $f : U \rightarrow \mathbb{R}$ such that $\mathcal{L}_{fX} \Omega = 0$ on U . Next we have the following result [305].

Lemma 4. *Consider the volume form $\Omega = dx \wedge dy$ on a $U \subset \mathbb{R}^2$ and a vector field X on U . Then, X is Hamiltonian with respect to a symplectic form $\omega = f\Omega$ on U if and only if $f : U \rightarrow \mathbb{R}$ is a non-vanishing integrating factor of X with respect to Ω , i.e., $Xf = -f \operatorname{div} X$ on U .*

Proof. Since ω is a symplectic form on U , then f must be non-vanishing. As

$$\mathcal{L}_X \omega = \mathcal{L}_X (f\Omega) = (Xf)\Omega + f\mathcal{L}_X \Omega = (Xf + f \operatorname{div} X)\Omega = \mathcal{L}_{fX} \Omega, \quad (5.6)$$

then X is locally Hamiltonian with respect to ω , i.e., $\mathcal{L}_X \omega = 0$, if and only if f is a non-vanishing integrating factor for X on U . As U is a contractible open subset, the Poincaré Lemma ensures that X is a local Hamiltonian vector field if and only if it is a Hamiltonian vector field. Consequently, the lemma follows. \square

Definition 5. Given a vector space V of vector fields on U , we say that V admits a *modular generating system* (U_1, X_1, \dots, X_p) if U_1 is a dense open subset of U such that every $X \in V|_{U_1}$ can be brought into the form $X|_{U_1} = \sum_{i=1}^p g_i X_i|_{U_1}$ for certain functions $g_1, \dots, g_p \in C^\infty(U_1)$ and vector fields $X_1, \dots, X_p \in V$.

Example 3. Given the Lie algebra $\mathfrak{P}_3 \simeq \mathfrak{so}(3)$ on \mathbb{R}^2 of Table 1 of Appendix 1, the vector fields

$$X_1 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad X_2 = (1 + x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \quad (5.7)$$

of P_3 satisfy that $X_3 = g_1X_1 + g_2X_2$ on $U_1 = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ for the functions $g_1, g_2 \in C^\infty(U_1)$

$$g_1 = \frac{x^2 + y^2 - 1}{x}, \quad g_2 = \frac{y}{x}. \quad (5.8)$$

Obviously, U_1 is an open dense subset of \mathbb{R}^2 . As every element of V is a linear combination of X_1, X_2 and $X_3 = g_1X_1 + g_2X_2$, then every $X \in V|_{U_1}$ can be written as a linear combination with smooth functions on U_1 of X_1 and X_2 . So, (U_1, X_1, X_2) form a generating modular system for P_3 .

In Table 1 in Appendix 1 we detail a modular generating system, which is indicated by the first one or two vector fields written between brackets in the list of the X_i 's, for every finite-dimensional Lie algebra of vector fields of the GKO classification.

Theorem 6. *Let V be a Lie algebra of vector fields on $U \subset \mathbb{R}^2$ admitting a modular generating system (U_1, X_1, \dots, X_p) . We have that: 1) The space V consists of Hamiltonian vector fields relative to a symplectic form on U if and only if*

- *i) Let g_1, \dots, g_p be certain smooth functions on $U_1 \subset U$. Then,*

$$X|_{U_1} = \sum_{i=1}^p g_i X_i|_{U_1} \in V|_{U_1} \implies \operatorname{div} X|_{U_1} = \sum_{i=1}^p g_i \operatorname{div} X_i|_{U_1}. \quad (5.9)$$

- *ii) The elements X_1, \dots, X_p admit a common non-vanishing integrating factor on U .*

2) *If the rank of \mathcal{D}^V is two on U , the symplectic form is unique up to a multiplicative non-zero constant.*

Proof. Let us prove the direct part of 1). Since (U_1, X_1, \dots, X_p) form a modular generating system for V , we have that every $X|_{U_1} \in V|_{U_1}$ can be brought into the form $X|_{U_1} = \sum_{i=1}^p g_i X_i|_{U_1}$ for certain $g_1, \dots, g_p \in C^\infty(U_1)$. As V is a Lie algebra of Hamiltonian vector fields with respect to a symplectic structure on U , let us say

$$\omega = f(x, y)dx \wedge dy, \quad (5.10)$$

then Lemma 4 ensures that $Yf = -f \operatorname{div} Y$ for every $Y \in V$. Then,

$$\begin{aligned} f \operatorname{div} X &= -Xf = -\sum_{i=1}^p g_i X_i f = f \sum_{i=1}^p g_i \operatorname{div} X_i \\ &\iff f \left(\operatorname{div} X - \sum_{i=1}^p g_i \operatorname{div} X_i \right) = 0 \end{aligned} \quad (5.11)$$

on U_1 . As ω is non-degenerate, then f is non-vanishing and $i)$ follows. Since all the vector fields of V are Hamiltonian with respect to ω , they share a common non-vanishing integrating factor, namely f . From this, $ii)$ holds.

Conversely, if $ii)$ is satisfied, then Lemma 4 ensures that X_1, \dots, X_p are Hamiltonian with respect to (5.10) on U , with f being a non-vanishing integrating factor. As (U_1, X_1, \dots, X_p) form a generating modular system for V , every $X \in V$ can be written as $\sum_{i=1}^p g_i X_i$ on U_1 for certain functions $g_1, \dots, g_p \in C^\infty(U_1)$. From $i)$ we obtain $\operatorname{div} X = \sum_{i=1}^p g_i \operatorname{div} X_i$ on U_1 . Then,

$$Xf = \sum_{i=1}^p g_i X_i f = -f \sum_{i=1}^p g_i \operatorname{div} X_i = -f \operatorname{div} X \quad (5.12)$$

on U_1 and, since the elements of V are smooth and U_1 is dense on U , the above expression also holds on U . Hence, f is a non-vanishing integrating factor for X , which becomes a Hamiltonian vector field with respect to ω on U in virtue of Lemma 4. Hence, part 1) is proven.

As far as part 2) of the theorem is concerned, if the vector fields of V are Hamiltonian with respect to two different symplectic structures on U , they admit two different non-vanishing integrating factors f_1 and f_2 . Hence,

$$X(f_1/f_2) = (f_2 X f_1 - f_1 X f_2)/f_2^2 = (f_2 f_1 \operatorname{div} X - f_1 f_2 \operatorname{div} X)/f_2^2 = 0 \quad (5.13)$$

and f_1/f_2 is a common constant of motion for all the elements of V . Hence, it is a constant of motion for every vector field taking values in the distribution \mathcal{D}^V . Then rank of \mathcal{D}^V on U is two by assumption. So, \mathcal{D}^V is generated by the vector fields ∂_x and ∂_y on U . Thus, the only constants of motion on U common to all the vector fields taking values in \mathcal{D}^V , and consequently common to the elements of V , are constants. Since f_1 and f_2 are non-vanishing, then $f_1 = \lambda f_2$ for a $\lambda \in \mathbb{R} \setminus \{0\}$ and the associated symplectic structures are the same up to an irrelevant non-zero proportionality constant. \square

Using Theorem 6, we can immediately prove the following result.

Corollary 7. *If a Lie algebra of vector fields V on a $U \subset \mathbb{R}^2$ consists of Hamiltonian vector fields with respect to a symplectic form and admits a modular generating system whose elements are divergence free, then every element of V is divergence free.*

5.2 Lie–Hamilton algebras

It is known that given a Lie–Hamilton system X , its Lie–Hamilton algebras are not uniquely defined in general. Moreover, the existence of different types of Lie–Hamilton algebras for the same Lie–Hamilton system is important in its linearization

and the use of certain methods [109]. For instance, if a Lie–Hamilton system X on N admits a Lie–Hamilton algebra isomorphic to V^X and $\dim V^X = \dim N$, then X can be linearized together with its associated Poisson structure [109].

Example 4. Consider again the Lie–Hamilton system X given by the complex Riccati equations with t -dependent real coefficients (2.7) and assume $V^X \simeq \mathfrak{sl}(2)$. Recall that X admits a Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega) \simeq \mathfrak{sl}(2)$ spanned by the Hamiltonian functions h_1, h_2, h_3 given by (3.3) relative to the symplectic structure ω detailed in (3.2). We can also construct a second (non-isomorphic) Lie–Hamilton algebra for X with respect to ω . The vector fields X_i , with $i = 1, 2, 3$, spanning V^X (see (2.10)) have also Hamiltonian functions $\bar{h}_i = h_i + 1$, for $i = 1, 2, 3$, respectively. Hence, $(\mathbb{R}_{y \neq 0}^2, \omega, h = a_0(t)\bar{h}_1 + a_1(t)\bar{h}_2 + a_2(t)\bar{h}_3)$ is a Lie–Hamiltonian structure for X giving rise to a Lie–Hamilton algebra $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega) \equiv (\langle \bar{h}_1, \bar{h}_2, \bar{h}_3, 1 \rangle, \{\cdot, \cdot\}_\omega) \simeq \mathfrak{sl}(2) \oplus \mathbb{R}$ for X .

Proposition 8. *A Lie–Hamilton system X on a symplectic connected manifold (N, ω) possesses an associated Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ isomorphic to V^X if and only if every Lie–Hamilton algebra non-isomorphic to V^X is isomorphic to $V^X \oplus \mathbb{R}$.*

Proof. Let $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ be an arbitrary Lie–Hamilton algebra for X . As X is defined on a connected manifold, the sequence of Lie algebras

$$0 \hookrightarrow (\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega) \cap \langle 1 \rangle \hookrightarrow (\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega) \xrightarrow{\varphi} V^X \rightarrow 0, \quad (5.14)$$

where $\varphi : \bar{\mathcal{H}}_\Lambda \rightarrow V^X$ maps every function of $\bar{\mathcal{H}}_\Lambda$ to minus its Hamiltonian vector field, is always exact (cf. [109]). Hence, $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ can be isomorphic either to V^X or to a Lie algebra extension of V^X of dimension $\dim V^X + 1$.

If $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ is isomorphic to V^X and there exists a second Lie–Hamilton algebra $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ for X non-isomorphic to V^X , we see from (5.14) that $1 \in \bar{\mathcal{H}}_\Lambda$ and $1 \notin \mathcal{H}_\Lambda$. Given a basis X_1, \dots, X_r of V^X , each element X_i , with $i = 1, \dots, r$, has a Hamiltonian function $\bar{h}_i \in \bar{\mathcal{H}}_\Lambda$ and another $h_i \in \mathcal{H}_\Lambda$. As V^X is defined on a connected manifold, then $h_i = \bar{h}_i - \lambda_i \in \bar{\mathcal{H}}_\Lambda$ with $\lambda_i \in \mathbb{R}$ for every $i = 1, \dots, r$. From this and using again that $1 \in \bar{\mathcal{H}}_\Lambda \setminus \mathcal{H}_\Lambda$, we obtain that $\{h_1, \dots, h_r, 1\}$ is a basis for $\bar{\mathcal{H}}_\Lambda$ and $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega) \simeq (\mathcal{H}_\Lambda \oplus \mathbb{R}, \{\cdot, \cdot\}_\omega)$.

Let us assume now that every Lie–Hamilton algebra $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ non-isomorphic to V^X is isomorphic to $V^X \oplus \mathbb{R}$. We can define a Lie algebra anti-isomorphism $\mu : V^X \rightarrow \bar{\mathcal{H}}_\Lambda$ mapping each element of V^X to a Hamiltonian function belonging to a Lie subalgebra of $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ isomorphic to V^X . Hence, $(N, \omega, h = \mu(X))$, where $h_t = \mu(X_t)$ for each $t \in \mathbb{R}$, is a Lie–Hamiltonian structure for X and $(\mu(V^X), \{\cdot, \cdot\}_\omega)$ is a Lie–Hamilton algebra for X isomorphic to V^X . \square

Proposition 9. *If a Lie–Hamilton system X on a symplectic connected manifold (N, ω) admits an associated Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ isomorphic to V^X , then it admits a Lie–Hamilton algebra isomorphic to $V^X \oplus \mathbb{R}$.*

Proof. Let (N, ω, h) be a Lie–Hamiltonian structure for X giving rise to the Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$. Consider the linear space L_h spanned by linear combinations of the functions $\{h_t\}_{t \in \mathbb{R}}$. Since we assume $\mathcal{H}_\Lambda \simeq V^X$, the exact sequence (5.14) involves that $1 \notin L_h$. Moreover, we can write $h = \sum_{i=1}^p b_i(t)h_{t_i}$, where h_{t_i} are the values of h at certain times t_1, \dots, t_p such that $\{h_{t_1}, \dots, h_{t_p}\}$ are linearly independent and b_1, \dots, b_p are certain t -dependent functions. Observe that the vector fields $(b_1(t), \dots, b_p(t))$, with $t \in \mathbb{R}$, span a p -dimensional linear space. If we choose a t -dependent Hamiltonian $\bar{h} = \sum_{i=1}^p b_i(t)h_{t_i} + b_{p+1}(t)$, where $b_{p+1}(t)$ is not a linear combination of $b_1(t), \dots, b_p(t)$, and we recall that $1, h_{t_1}, \dots, h_{t_p}$ are linearly independent over \mathbb{R} , we obtain that the linear hull of the functions $\{\bar{h}_t\}_{t \in \mathbb{R}}$ has dimension $\dim L_h + 1$. Moreover, $(N, \{\cdot, \cdot\}_\omega, \bar{h})$ is a Lie–Hamiltonian structure for X . Hence, they span, along with their successive Lie brackets, a Lie–Hamilton algebra isomorphic to $\mathcal{H}_\Lambda \oplus \mathbb{R}$. \square

Corollary 10. *If X is a Lie–Hamilton system with respect to a symplectic connected manifold (N, ω) admitting a Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ satisfying that $1 \in \{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$, then X does not possess any Lie–Hamilton algebra isomorphic to V^X .*

Proof. If $1 \in \{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$, then \mathcal{H}_Λ cannot be isomorphic to $V^X \oplus \mathbb{R}$ because the derived Lie algebra of \mathcal{H}_Λ , i.e. $\{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$, contains the constant function 1 and the derived Lie algebra of a \mathcal{H}_Λ isomorphic to $V^X \oplus \mathbb{R}$ does not. In view of Proposition 8, system X does not admit any Lie–Hamilton algebra isomorphic to V^X . \square

Proposition 11. *If X is a Lie–Hamilton system on a connected manifold N whose V^X consists of Hamiltonian vector fields with respect to a symplectic structure ω that does not possess any Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ isomorphic to V^X , then all its Lie–Hamilton algebras (with respect to $\{\cdot, \cdot\}_\omega$) are isomorphic.*

Proof. Let $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ and $(\bar{\mathcal{H}}_\Lambda, \{\cdot, \cdot\}_\omega)$ be two Lie–Hamilton algebras for X . Since they are not isomorphic to V^X and in view of the exact sequence (5.14), then $1 \in \mathcal{H}_\Lambda \cap \bar{\mathcal{H}}_\Lambda$. Let X_1, \dots, X_r be a basis of V^X . Every vector field X_i admits a Hamiltonian function $h_i \in \mathcal{H}_\Lambda$ and another $\bar{h}_i \in \bar{\mathcal{H}}_\Lambda$. The functions h_1, \dots, h_r are linearly independent and the same applies to $\bar{h}_1, \dots, \bar{h}_r$. Then, $\{h_1, \dots, h_r, 1\}$ is a basis for \mathcal{H}_Λ and $\{\bar{h}_1, \dots, \bar{h}_r, 1\}$ is a basis for $\bar{\mathcal{H}}_\Lambda$. As N is connected, then $h_i = \bar{h}_i - \lambda_i$ with $\lambda_i \in \mathbb{R}$ for each $i \in \mathbb{R}$. Hence, the functions h_i belong to $\bar{\mathcal{H}}_\Lambda$ and the functions \bar{h}_i belong to \mathcal{H}_Λ . Thus $\mathcal{H}_\Lambda = \bar{\mathcal{H}}_\Lambda$. \square

For instance, the case P₁ from Table 1 in Appendix 1 corresponds to the two-dimensional Euclidean algebra $\mathfrak{iso}(2) \simeq \langle X_1, X_2, X_3 \rangle$, but the Hamiltonian functions $\langle h_1, h_2, h_3, h_0 = 1 \rangle$ span the centrally extended Euclidean algebra $\overline{\mathfrak{iso}}(2)$. A similar fact arises in classes P₃ $\simeq \mathfrak{so}(3)$, P₅ $\simeq \mathfrak{sl}(2) \times \mathbb{R}^2$, I₈ $\simeq \mathfrak{iso}(1, 1)$ (the (1 + 1)-dimensional Poincaré algebra), I_{14B} $\simeq \mathbb{R} \times \mathbb{R}^r$ and I₁₆ $\simeq \mathfrak{h}_2 \times \mathbb{R}^{r+1}$. Among them,

only the family $P_3 \simeq \mathfrak{so}(3)$ is a simple Lie algebra in such a manner that $h_0 = 1$ gives rise to a trivial central extension which means that the LH algebra is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}$; otherwise the central extension is a non-trivial one and it cannot be ‘removed’.

In this respect, notice that the appearance of a non-trivial central extension is the difference between the family I_{14B} with respect to I_{14A} . We also recall that the LH algebra corresponding to the class P_5 , that is $\overline{\mathfrak{sl}(2)} \times \mathbb{R}^2$, is isomorphic to the two-photon Lie algebra \mathfrak{h}_6 (see [31, 428] and references therein) and, therefore, also to the $(1 + 1)$ -dimensional centrally extended Schrödinger Lie algebra [38].

We stress that the Lie algebra $\mathfrak{sl}(2)$ appears four times (classes P_2, I_3, I_4 and I_5 in the GKO classification) which means that there may be different LH systems sharing isomorphic Vessiot–Guldberg Lie algebras that are non-diffeomorphic, namely there exists no diffeomorphism mapping the elements of one into the other. In other words, only LH systems belonging to each class can be related through a change of variables. We shall explicitly apply this property throughout the paper.

5.3 Local classification

In this section we describe the local structure of Lie–Hamilton systems on the plane, i.e., given the minimal Lie algebra of a Lie–Hamilton system X on the plane, we prove that V^X is locally diffeomorphic around a generic point of V^X to one of the Lie algebras given in Table 2 in Appendix 1. We also prove that, around a generic point of V^X , the Lie–Hamilton algebras of X must have one of the algebraic structures described in Table 2, Appendix 1.

If X is a Lie–Hamilton system, its minimal Lie algebra must be locally diffeomorphic to one of the Lie algebras of the GKO classification that consists of Hamiltonian vector fields with respect to a Poisson structure. As we are concerned with generic points of minimal Lie algebras, Lemma 3 ensures that V^X is locally diffeomorphic around generic points to a Lie algebra of Hamiltonian vector fields with respect to a symplectic structure. So, its minimal Lie algebra is locally diffeomorphic to one of the Lie algebras of the GKO classification consisting of Hamiltonian vector fields with respect to a symplectic structure on a certain open contractible subset of its domain. By determining which of the Lie algebras of the GKO classification admit such a property, we can classify the local structure of all Lie–Hamilton systems on the plane.

Proposition 12. *The primitive Lie algebras $P_1^{\alpha \neq 0}, P_4, P_6$ – P_8 and the imprimitive ones $I_2, I_3, I_6, I_7, I_8^{(\alpha \neq -1)}, I_9$ – $I_{11}, I_{13}, I_{15}, I_{16}^{(\alpha \neq -1)}, I_{17}$ – I_{20} are not Lie algebras of Hamiltonian vector fields on any $U \subset \mathbb{R}^2$.*

Proof. Apart from I_{15} , the remaining Lie algebras detailed in this statement admit a modular generating system whose elements are divergence free on the whole \mathbb{R}^2 (see

the elements written between brackets in Table 1). At the same time, we also observe in Table 1 that these Lie algebras admit a vector field with non-zero divergence on any U . In view of Corollary 7, they cannot be Lie algebras of Hamiltonian vector fields with respect to any symplectic structure on any $U \subset \mathbb{R}^2$.

In the case of the Lie algebra I_{15} , we have that $(\mathbb{R}_{y \neq 0}^2, X_1 = \partial_x, X_2 = y\partial_y)$ form a generating modular system of I_{15} . Observe that $X_2 = y\partial_y$ and $X_3 = \eta_1(x)\partial_y$, where η_1 is a non-null function—it forms with $\eta_2(x), \dots, \eta_r(x)$ a basis of solutions of a system of r first-order linear homogeneous differential equations in normal form with constant coefficients (cf. [210, 295])—satisfy $\operatorname{div} X_2 = 1$ and $\operatorname{div} X_3 = 0$. Obviously, $\operatorname{div} X_3 \neq \eta_1 \operatorname{div} X_2 / y$ on any U . So, I_{15} does not satisfy Theorem 6 on any U and it is not a Lie algebra of Hamiltonian vector fields on any $U \subset \mathbb{R}^2$. \square

To simplify the notation, we assume in this Chapter that all objects are defined on a contractible $U \subset \mathbb{R}^2$ of the domain of the Lie algebra under study. Additionally, U_1 stands for a dense open subset of U . In the following two subsections, we explicitly show that *all* of the Lie algebras of the GKO classification *not* listed in Proposition 12 consist of Hamiltonian vector fields on any U of their domains. For each Lie algebra, we compute the integrating factor f of ω given by (5.10) turning the elements of a basis of the Lie algebra into Hamiltonian vector fields and we work out their Hamiltonian functions. Additionally, we obtain the algebraic structure of all the Lie–Hamilton algebras of the Lie–Hamilton systems admitting such minimal Lie algebras.

We stress that the main results covering the resulting Hamiltonian functions h_i , the symplectic form ω and the Lie–Hamilton algebra are summarized in Table 2 accordingly to the GKO classification of Table 1. We point out that the Lie algebras of the class I_{14} give rise to two non-isomorphic Lie–Hamilton algebras: I_{14A} whenever $1 \notin \langle \eta_1, \dots, \eta_r \rangle$ and I_{14B} otherwise. Consequently, we obtain twelve finite-dimensional real Lie algebras of Hamiltonian vector fields on the plane.

In order to shorten the presentation of the following results, we remark that for some of such Lie–Hamilton algebras their symplectic structure is just the standard one

Proposition 13. *The Lie algebras $P_1^{(\alpha=0)}$, P_5 , $I_8^{(\alpha=-1)}$, I_{14B} and $I_{16}^{(\alpha=-1)}$ are Lie algebras of Hamiltonian vector fields with respect to the standard symplectic form $\omega = dx \wedge dy$, that is, $f \equiv 1$.*

Proof. We see in Table 1 that all the aforementioned Lie algebras admit a modular generating system $(U, X_1 = \partial_x, X_2 = \partial_y)$ and all their elements have zero divergence. So, they satisfy condition (5.9). The vector fields X_1, X_2 are Hamiltonian with respect to the symplectic structure $\omega = dx \wedge dy$. In view of Theorem 6, the whole Lie algebra consists of Hamiltonian vector fields relative to ω . \square

Primitive Lie algebras

Lie algebra $\mathbf{P}_1^{(\alpha=0)}$: $A_0 \simeq \mathfrak{iso}(2)$

Proposition 13 states that A_0 is a Lie algebra of Hamiltonian vector fields with respect to the symplectic form $\omega = dx \wedge dy$. The basis of vector fields X_1, X_2, X_3 of A_0 given in Table 1 satisfy the commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = X_1. \quad (5.15)$$

So, A_0 is isomorphic to the two-dimensional Euclidean algebra $\mathfrak{iso}(2)$. Using the relation $\iota_X \omega = dh$ between a Hamiltonian vector field and one of its Hamiltonian functions, we get that the Hamiltonian functions for X_1, X_2, X_3 read

$$h_1 = y, \quad h_2 = -x, \quad h_3 = \frac{1}{2}(x^2 + y^2), \quad (5.16)$$

correspondingly. Along with $h_0 = 1$, these functions fulfil

$$\{h_1, h_2\}_\omega = h_0, \quad \{h_1, h_3\}_\omega = h_2, \quad \{h_2, h_3\}_\omega = -h_1, \quad \{h_0, \cdot\}_\omega = 0. \quad (5.17)$$

Consequently, if X is a Lie–Hamilton system admitting a minimal Lie algebra A_0 , i.e. $X = \sum_{i=1}^3 b_i(t)X_i$ for certain t -dependent functions b_1, b_2, b_3 such that $V^X \simeq A_0$, then it admits a Lie–Hamiltonian structure $(U, \omega, h = \sum_{i=1}^3 b_i(t)h_i)$ and a Lie–Hamilton algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ generated by the functions $\langle h_1, h_2, h_3, h_0 \rangle$. Hence, $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega)$ is a finite-dimensional real Lie algebra of Hamiltonian functions isomorphic to the *centrally extended* Euclidean algebra $\overline{\mathfrak{iso}(2)}$ [29]. Indeed, note that $1 \in \{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$. In virtue of Corollary 10, system X does not admit any Lie–Hamilton algebra isomorphic to V^X . Moreover, Proposition 11 ensures that all Lie–Hamilton algebras for X are isomorphic to $\overline{\mathfrak{iso}(2)}$.

Lie algebra \mathbf{P}_2 : $\mathfrak{sl}(2)$

We have already proved that the Lie algebra of vector fields \mathbf{P}_2 , which is spanned by the vector fields (2.10), is a Lie algebra of Hamiltonian vector fields with respect to the symplectic structure (3.2). The Hamiltonian functions h_1, h_2, h_3 for X_1, X_2 and X_3 were found to be (3.3), correspondingly. Then, a Lie system X with minimal Lie algebra \mathbf{P}_2 , i.e. a system of the form $X = \sum_{i=1}^3 b_i(t)X_i$ for certain t -dependent functions b_1, b_2, b_3 such that $V^X = \mathbf{P}_2$, is a Lie–Hamilton system with respect to the Poisson bracket induced by (3.2). Then, X admits a Lie–Hamiltonian structure $(U, \omega, h = \sum_{i=1}^3 b_i(t)h_i)$ and a Lie–Hamilton algebra isomorphic to $\mathfrak{sl}(2)$ with commutation relations (3.3). In view of Proposition 9, any Lie–Hamilton system associated to \mathbf{P}_2 also admits a Lie–Hamilton algebra isomorphic to $\mathfrak{sl}(2) \oplus \mathbb{R}$. In view of Proposition 8, these are the only algebraic structures of the Lie–Hamilton algebras for such Lie–Hamilton systems.

Lie algebra \mathbf{P}_3 : $\mathfrak{so}(3)$

In this case, we must determine a symplectic structure ω turning the elements of the modular generating system (U_1, X_1, X_2) of \mathbf{P}_3 into locally Hamiltonian vector fields with respect to a symplectic structure ω (5.10). In view of Theorem 6, this ensures that every element of \mathbf{P}_3 is Hamiltonian with respect to ω . The condition $\mathcal{L}_{X_1}\omega = 0$ gives

$$y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} = 0. \quad (5.18)$$

Applying the characteristics method, we find that f must be constant along the integral curves of the system $x dx + y dy = 0$, namely curves with $x^2 + y^2 = k \in \mathbb{R}$. So, $f = f(x^2 + y^2)$. If we now require $\mathcal{L}_{X_2}\omega = 0$, we obtain that

$$(1 + x^2 - y^2) \frac{\partial f}{\partial x} + 2xy \frac{\partial f}{\partial y} + 4xf = 0. \quad (5.19)$$

Using that $f = f(x^2 + y^2)$, we have

$$\frac{f'}{f} = -\frac{2}{1 + x^2 + y^2} \Rightarrow f(x^2 + y^2) = (1 + x^2 + y^2)^{-2}. \quad (5.20)$$

Then,

$$\omega = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}. \quad (5.21)$$

So, \mathbf{P}_3 becomes a Lie algebra of Hamiltonian vector fields relative to ω . The vector fields X_1 , X_2 and X_3 admit the Hamiltonian functions

$$h_1 = -\frac{1}{2(1 + x^2 + y^2)}, \quad h_2 = \frac{y}{1 + x^2 + y^2}, \quad h_3 = -\frac{x}{1 + x^2 + y^2}, \quad (5.22)$$

which along $h_0 = 1$ satisfy the commutation relations

$$\begin{aligned} \{h_1, h_2\}_\omega &= -h_3, & \{h_1, h_3\}_\omega &= h_2, \\ \{h_2, h_3\}_\omega &= -4h_1 - h_0, & \{h_0, \cdot\}_\omega &= 0, \end{aligned} \quad (5.23)$$

with respect to the Poisson bracket induced by ω . Then, $\langle h_1, h_2, h_3, h_0 \rangle$ span a Lie algebra of Hamiltonian functions isomorphic to a central extension of $\mathfrak{so}(3)$, denoted $\bar{\mathfrak{so}}(3)$. It is well known [29] that the central extension associated with h_0 is a trivial one; if we define $\bar{h}_1 = h_1 + h_0/4$, then $\langle \bar{h}_1, h_2, h_3 \rangle$ span a Lie algebra isomorphic to $\mathfrak{so}(3)$ and $\bar{\mathfrak{so}}(3) \simeq \mathfrak{so}(3) \oplus \mathbb{R}$. In view of this and using Propositions 8 and 9, a Lie system admitting a minimal Lie algebra \mathbf{P}_3 admits Lie–Hamilton structures isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}$ and $\mathfrak{so}(3)$.

Lie algebra \mathbf{P}_5 : $\mathfrak{sl}(2) \ltimes \mathbb{R}^2$

From Proposition 13, this Lie algebra consists of Hamiltonian vector fields with respect to the symplectic form $\omega = dx \wedge dy$. The vector fields of the basis X_1, \dots, X_5 for \mathbf{P}_5 given in Table 1 are Hamiltonian vector fields relative to ω with Hamiltonian functions

$$h_1 = y, \quad h_2 = -x, \quad h_3 = xy, \quad h_4 = \frac{1}{2}y^2, \quad h_5 = -\frac{1}{2}x^2, \quad (5.24)$$

correspondingly. These functions together with $h_0 = 1$ satisfy the relations

$$\begin{aligned} \{h_1, h_2\}_\omega &= h_0, & \{h_1, h_3\}_\omega &= -h_1, & \{h_1, h_4\}_\omega &= 0, \\ \{h_1, h_5\}_\omega &= -h_2, & \{h_2, h_3\}_\omega &= h_2, & \{h_2, h_4\}_\omega &= -h_1, \\ \{h_2, h_5\}_\omega &= 0, & \{h_3, h_4\}_\omega &= 2h_4, & \{h_3, h_5\}_\omega &= -2h_5, \\ \{h_4, h_5\}_\omega &= h_3, & \{h_0, \cdot\}_\omega &= 0. \end{aligned} \quad (5.25)$$

Hence $\langle h_1, \dots, h_5, h_0 \rangle$ span a Lie algebra $\overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2}$ which is a non-trivial central extension of \mathbf{P}_5 , i.e., it is not isomorphic to $\mathbf{P}_5 \oplus \mathbb{R}$. In fact, it is isomorphic to the so-called two-photon Lie algebra \mathfrak{h}_6 (see [31] and references therein); this can be proven to be $\mathfrak{h}_6 \simeq \mathfrak{sl}(2) \oplus_s \mathfrak{h}_3$, where $\mathfrak{sl}(2) \simeq \langle h_3, h_4, h_5 \rangle$, $\mathfrak{h}_3 \simeq \langle h_1, h_2, h_0 \rangle$ is the Heisenberg–Weyl Lie algebra, and \oplus_s stands for a semidirect sum. Furthermore, \mathfrak{h}_6 is also isomorphic to the $(1+1)$ -dimensional centrally extended Schrödinger Lie algebra [38].

In view of Corollary 10, Proposition 11 and following the same line of reasoning than in previous cases, a Lie system admitting a minimal Lie algebra \mathbf{P}_5 only possesses Lie–Hamilton algebras isomorphic to \mathfrak{h}_6 .

Imprimitive Lie algebras

Lie algebra \mathbf{I}_1 : \mathbb{R}

Note that $X_1 = \partial_x$ is a modular generating basis of \mathbf{I}_1 . By solving the PDE $\mathcal{L}_{X_1}\omega = 0$ with ω written in the form (5.10), we obtain that $\omega = f(y)dx \wedge dy$ with $f(y)$ being any non-vanishing function of y . In view of Theorem 6, the Lie algebra \mathbf{I}_1 becomes a Lie algebra of Hamiltonian vector fields with respect to ω . Observe that X_1 , a basis of \mathbf{I}_1 , has a Hamiltonian function, $h_1 = \int^y f(y')dy'$. As h_1 spans a Lie algebra isomorphic to \mathbb{R} , it is obvious that a system X with $V^X \simeq \mathbf{I}_1$ admits a Lie–Hamilton algebra isomorphic to \mathbf{I}_1 . Proposition 9 yields that X possesses a Lie–Hamilton algebra isomorphic to \mathbb{R}^2 . In view of Proposition 8, these are the only algebraic structures for the Lie–Hamilton algebras for X_1 .

Lie algebra \mathbf{I}_4 : $\mathfrak{sl}(2)$ of type II

This Lie algebra admits a modular generating system $(\mathbb{R}_{x \neq y}^2, X_1 = \partial_x + \partial_y, X_2 = x\partial_x + y\partial_y)$. Let us search for a symplectic form ω (5.10) turning X_1 and X_2 into local

Hamiltonian vector fields with respect to it. In rigour of Theorem 6, if X_1, X_2 become locally Hamiltonian, then we can ensure that every element of I_4 is Hamiltonian with respect to ω . By imposing $\mathcal{L}_{X_i}\omega = 0$ ($i = 1, 2$), we find that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0, \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0. \quad (5.26)$$

Applying the method of characteristics to the first equation, we have that $dx = dy$. Then $f = f(x-y)$. Using this in the second equation, we obtain a particular solution $f = (x-y)^{-2}$ which gives rise to a closed and non-degenerate two-form, namely

$$\omega = \frac{dx \wedge dy}{(x-y)^2}. \quad (5.27)$$

Hence,

$$h_1 = \frac{1}{x-y}, \quad h_2 = \frac{x+y}{2(x-y)}, \quad h_3 = \frac{xy}{x-y} \quad (5.28)$$

are Hamiltonian functions of the vector fields X_1, X_2, X_3 of the basis for I_4 given in Table 1, respectively. Using the Poisson bracket $\{\cdot, \cdot\}_\omega$ induced by (5.27), we obtain that h_1, h_2 and h_3 satisfy

$$\{h_1, h_2\}_\omega = -h_1, \quad \{h_1, h_3\}_\omega = -2h_2, \quad \{h_2, h_3\}_\omega = -h_3. \quad (5.29)$$

Then, $(\langle h_1, h_2, h_3 \rangle, \{\cdot, \cdot\}_\omega) \simeq \mathfrak{sl}(2)$. Consequently, if X is a Lie–Hamilton system admitting a minimal Lie algebra I_4 , it admits a Lie–Hamilton algebra that is isomorphic to $\mathfrak{sl}(2)$ or, from Proposition 9, to $\mathfrak{sl}(2) \oplus \mathbb{R}$. From Proposition 8, these are the only algebraic structures for its Lie–Hamilton algebras.

Lie algebra I_5 : $\mathfrak{sl}(2)$ of type III

Observe that $(U, X_1 = \partial_x, X_2 = 2x\partial_x + y\partial_y)$ form a modular generating system of I_5 . The conditions $\mathcal{L}_{X_1}\omega = \mathcal{L}_{X_2}\omega = 0$ ensuring that X_1 and X_2 are locally Hamiltonian with respect to ω give rise to the equations

$$\frac{\partial f}{\partial x} = 0, \quad 2x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 3f = 0, \quad (5.30)$$

so that $f(x, y) = y^{-3}$ is a particular solution of the second equation right above, and X_1, X_2 become locally Hamiltonian vector fields relative to the symplectic form

$$\omega = \frac{dx \wedge dy}{y^3}. \quad (5.31)$$

In view of Theorem 6, this implies that every element of I_5 is Hamiltonian with respect to ω . Hamiltonian functions for the elements of the basis X_1, X_2, X_3 for I_5 given in Table 1 read

$$h_1 = -\frac{1}{2y^2}, \quad h_2 = -\frac{x}{y^2}, \quad h_3 = -\frac{x^2}{2y^2}. \quad (5.32)$$

They span a Lie algebra isomorphic to $\mathfrak{sl}(2)$

$$\{h_1, h_2\}_\omega = -2h_1, \quad \{h_1, h_3\}_\omega = -h_2, \quad \{h_2, h_3\}_\omega = -2h_3. \quad (5.33)$$

Therefore, a Lie system possessing a minimal Lie algebra I_5 possesses a Lie–Hamilton algebra isomorphic to $\mathfrak{sl}(2)$ and, in view of Proposition 9, to $\mathfrak{sl}(2) \oplus \mathbb{R}$. In view of Proposition 8, these are the only possible algebraic structures for the Lie–Hamilton algebras for X .

Lie algebra $I_8^{(\alpha=-1)}$: $B_{-1} \simeq \mathfrak{iso}(1, 1)$

In view of Proposition 13, this Lie algebra consists of Hamiltonian vector fields with respect to the standard symplectic structure $\omega = dx \wedge dy$. The elements of the basis for B_{-1} detailed in Table 1 satisfy the commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2. \quad (5.34)$$

Hence, these vector fields span a Lie algebra isomorphic to the (1+1)-dimensional Poincaré algebra $\mathfrak{iso}(1, 1)$. Their corresponding Hamiltonian functions turn out to be

$$h_1 = y, \quad h_2 = -x, \quad h_3 = xy, \quad (5.35)$$

which together with a central generator $h_0 = 1$ fulfil the commutation relations

$$\{h_1, h_2\}_\omega = h_0, \quad \{h_1, h_3\}_\omega = -h_1, \quad \{h_2, h_3\}_\omega = h_2, \quad \{h_0, \cdot\}_\omega = 0. \quad (5.36)$$

Thus, a Lie system X admitting a minimal Lie algebra B_{-1} possesses a Lie–Hamilton algebra isomorphic to the centrally extended Poincaré algebra $\overline{\mathfrak{iso}}(1, 1)$ which, in turn, is also isomorphic to the harmonic oscillator algebra \mathfrak{h}_4 . As is well known [29], this Lie algebra is not of the form $\mathfrak{iso}(1, 1) \oplus \mathbb{R}$, then Proposition 8 ensures that X does not admit any Lie–Hamilton algebra isomorphic to $\mathfrak{iso}(1, 1)$. Moreover, Proposition 11 states that all Lie–Hamilton algebras of X must be isomorphic to $\overline{\mathfrak{iso}}(1, 1)$.

Lie algebra I_{12} : \mathbb{R}^{r+1}

The vector field $X_1 = \partial_y$ is a modular generating system for I_{12} and all the elements of this Lie algebra have zero divergence. By solving the PDE $\mathcal{L}_{X_1}\omega = 0$, where we recall that ω has the form (5.10), we see that $f = f(x)$ and X_1 becomes Hamiltonian for any non-vanishing function $f(x)$. In view of Theorem 6, the remaining elements of I_{12} become automatically Hamiltonian with respect to ω . Then, we obtain that X_1, \dots, X_{r+1} are Hamiltonian vector fields relative to the symplectic structure $\omega = f(x)dx \wedge dy$ with Hamiltonian functions

$$h_1 = - \int^x f(x')dx', \quad h_{j+1} = - \int^x f(x')\xi_j(x')dx', \quad (5.37)$$

with $j = 1, \dots, r$, and $r \geq 1$, which span the Abelian Lie algebra \mathbb{R}^{r+1} . In consequence, a Lie–Hamilton system X related to a minimal Lie algebra I_{12} possesses a Lie–Hamilton algebra isomorphic to \mathbb{R}^{r+1} . From Propositions 8 and 9, it only admits an additional Lie–Hamilton algebra isomorphic to \mathbb{R}^{r+2} .

Lie algebra I_{14} : $\mathbb{R} \ltimes \mathbb{R}^r$

The functions $\eta_1(x), \dots, \eta_r(x)$ form a fundamental system of solutions of a system of r differential equations with constant coefficients [14, 210]. Hence, none of them vanishes in an open interval of \mathbb{R} and I_{14} is such that (U_1, X_1, X_2) , where X_1 and X_2 are given in Table 1, form a modular generating system. Since all the elements of I_{14} have divergence zero and using Theorem 6, we infer that I_{14} consists of Hamiltonian vector fields relative to a symplectic structure if and only if X_1 and X_2 are locally Hamiltonian vector fields with respect to a symplectic structure. By requiring $\mathcal{L}_{X_i}\omega = 0$, with $i = 1, 2$ and ω of the form (5.10), we obtain that

$$\frac{\partial f}{\partial x} = 0, \quad \eta_j(x) \frac{\partial f}{\partial y} = 0, \quad j = 1, \dots, r. \quad (5.38)$$

So, I_{14} is only compatible, up to nonzero multiplicative factor, with $\omega = dx \wedge dy$. The Hamiltonian functions corresponding to X_1, \dots, X_{r+1} turn out to be

$$h_1 = y, \quad h_{j+1} = - \int^x \eta_j(x') dx', \quad (5.39)$$

with $j = 1, \dots, r$, and $r \geq 1$. We remark that different Lie–Hamilton algebras, corresponding to the Lie algebras hereafter called I_{14A} and I_{14B} , are spanned by the above Hamiltonian functions:

- Case I_{14A} : If $1 \notin \langle \eta_1, \dots, \eta_r \rangle$, then the functions (5.39) span a Lie algebra $\mathbb{R} \ltimes \mathbb{R}^r$ and, by considering Propositions 8 and 9, this case only admits an additional Lie–Hamilton algebra isomorphic to $(\mathbb{R} \ltimes \mathbb{R}^r) \oplus \mathbb{R}$.
- Case I_{14B} : If $1 \in \langle \eta_1, \dots, \eta_r \rangle$, we can choose a basis of I_{14} in such a way that there exists a function, let us say η_1 , equal to 1. Then the Hamiltonian functions (5.39) turn out to be

$$h_1 = y, \quad h_2 = -x, \quad h_{j+1} = - \int^x \eta_j(x') dx', \quad (5.40)$$

with $j = 2, \dots, r$, and $r \geq 1$ which require a central generator $h_0 = 1$ in order to close a centrally extended Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\omega) \simeq (\overline{\mathbb{R} \ltimes \mathbb{R}^r})$.

In view of the above, a Lie system X with a minimal Lie algebra I_{14} is a Lie–Hamilton system. Its Lie–Hamilton algebras can be isomorphic to I_{14} or $I_{14} \oplus \mathbb{R}$ when $1 \notin \langle \eta_1, \dots, \eta_r \rangle$ (class I_{14A}). If $1 \in \langle \eta_1, \dots, \eta_r \rangle$ (class I_{14B}), a Lie–Hamilton algebra is isomorphic to $\overline{\mathbb{R} \ltimes \mathbb{R}^r}$ and since $1 \in \{\mathcal{H}_\Lambda, \mathcal{H}_\Lambda\}_\omega$, we obtain from Corollary 10 and Proposition 11 that every Lie–Hamilton algebra for X is isomorphic to it.

Lie algebra $\mathbf{I}_{16}^{(\alpha=-1)}$: $C_{-1}^r \simeq \mathfrak{h}_2 \times \mathbb{R}^{r+1}$

In view of Proposition 13, this Lie algebra consists of Hamiltonian vector fields with respect to the standard symplectic structure. The resulting Hamiltonian functions for X_1, \dots, X_{r+3} are given by

$$h_1 = y, \quad h_2 = -x, \quad h_3 = xy, \quad h_{j+3} = -\frac{x^{j+1}}{j+1}, \quad (5.41)$$

with $j = 1, \dots, r$, and $r \geq 1$, which again require an additional central generator $h_0 = 1$ to close on a finite-dimensional Lie algebra. The commutation relations for this Lie algebra are given by

$$\begin{aligned} \{h_1, h_2\}_\omega &= h_0, & \{h_1, h_3\}_\omega &= -h_1, \\ \{h_2, h_3\}_\omega &= h_2, & \{h_1, h_4\}_\omega &= -h_2, \\ \{h_1, h_{k+4}\}_\omega &= -(k+1)h_{k+3}, & \{h_2, h_{j+3}\}_\omega &= 0, \\ \{h_3, h_{j+3}\}_\omega &= -(j+1)h_{j+3}, & \{h_{j+3}, h_{k+4}\}_\omega &= 0, \\ \{h_0, \cdot\}_\omega &= 0, \end{aligned} \quad (5.42)$$

with $j = 1, \dots, r$ and $k = 1, \dots, r-1$, which define the centrally extended Lie algebra $\overline{\mathfrak{h}_2 \times \mathbb{R}^{r+1}}$. This Lie algebra is not a trivial extension of $\mathfrak{h}_2 \times \mathbb{R}^{r+1}$.

Then, given a Lie system X with a minimal Lie algebra C_{-1}^r , the system is a Lie–Hamilton one which admits a Lie–Hamilton algebra isomorphic to $\overline{\mathfrak{h}_2 \times \mathbb{R}^{r+1}}$. As $1 \in \left\{ \overline{\mathfrak{h}_2 \times \mathbb{R}^{r+1}}, \mathfrak{h}_2 \times \mathbb{R}^{r+1} \right\}_\omega$, Corollary 10 and Proposition 11 ensure that every Lie–Hamilton algebra for X is isomorphic to $\overline{\mathfrak{h}_2 \times \mathbb{R}^{r+1}}$.

6 Applications of Lie–Hamilton systems on the plane

Here we show Lie–Hamilton systems of relevance in Science. Given the recurrent appearance of the $\mathfrak{sl}(2, \mathbb{R})$ -Lie algebra, we show a few instances of Lie–Hamilton systems with a $\mathfrak{sl}(2, \mathbb{R})$ Vessiot–Guldberg Lie algebra. These are: the Milne–Pinney and second-order Kummer–Schwarz equations, the complex Riccati equations with t -dependent real coefficients, the coupled Riccati equations, planar diffusion Riccati equations, etc. We will establish certain equivalences among some of the mentioned systems.

The importance of Lie–Hamilton systems in biological applications is undoubtable. Some examples will be displayed: the generalized Buchdahl equations, certain Lotka–Volterra systems, quadratic polynomial systems and models for viral infections. Another two Lie algebras worth of mention in the realm of Lie–Hamilton systems are the two-photon and \mathfrak{h}_2 -Lie algebras. Their respective applications are: the dissipative harmonic oscillator and the second-order Riccati equation for the first, and the

complex Bernoulli and generalized Buchdahl equations and certain Lotka–Volterra systems for the second. To conclude, we will add some other reputable Lie–Hamilton systems as the Cayley–Klein Riccati equation and double–Clifford or split complex Riccati equations. The forthcoming subsections show the abovementioned systems in detail.

6.1 $\mathfrak{sl}(2, \mathbb{R})$ -Lie–Hamilton systems

Let us employ our techniques to study $\mathfrak{sl}(2)$ -Lie–Hamilton systems [308, 353]. More specifically, we analyse Lie systems used to describe Milne–Pinney equations [97], Kummer–Schwarz equations [87] and complex Riccati equations with real t -dependent coefficients [154]. As a byproduct, our results also cover the t -dependent frequency harmonic oscillator.

Example 1. The **Milne–Pinney equation**, which is well known for its multiple properties and applications in physics (see [287] and references therein). For example, it is useful to modelize the propagation of laser beams in nonlinear media, plasma dynamics, Bose-Einstein condensates through the so-called Gross-Pitaevskii equation, etc.

It takes the form

$$\frac{d^2x}{dt^2} = -\omega^2(t)x + \frac{c}{x^3}, \quad (6.1)$$

where $\omega(t)$ is any t -dependent function and c is a real constant. It was first introduced by Ermakov as a way to find first-integrals of its corresponding t -dependent harmonic oscillator when $c = 0$. By adding a new variable $y \equiv dx/dt$, we can study these equations through the first-order system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\omega^2(t)x + \frac{c}{x^3}, \end{cases} \quad (6.2)$$

which is a Lie system [97, 419]. We recall that (6.2) can be regarded as the equations of motion of the one-dimensional Smorodinsky–Winternitz system [35, 195]; moreover, when the parameter c vanishes, this reduces to the harmonic oscillator (both with a t -dependent frequency). Explicitly, (6.2) is the associated system to the t -dependent vector field $X_t = X_3 + \omega^2(t)X_1$, where

$$X_1 = -x \frac{\partial}{\partial y}, \quad X_2 = \frac{1}{2} \left(y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \right), \quad X_3 = y \frac{\partial}{\partial x} + \frac{c}{x^3} \frac{\partial}{\partial y}, \quad (6.3)$$

span a finite-dimensional real Lie algebra V of vector fields isomorphic to $\mathfrak{sl}(2)$ with commutation relations given by

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \quad (6.4)$$

There are *four* classes of finite-dimensional Lie algebras of vector fields isomorphic to $\mathfrak{sl}(2)$ in the GKO classification: P_2 and I_3 – I_5 . To determine which one is locally diffeomorphic to V , we first find out whether V is imprimitive or not. In this respect, recall that V is *imprimitive* if there exists a one-dimensional distribution \mathcal{D} invariant under the action (by Lie brackets) of the elements of V . Hence, \mathcal{D} is spanned by a non-vanishing vector field

$$Y = g_x(x, y) \frac{\partial}{\partial x} + g_y(x, y) \frac{\partial}{\partial y}, \quad (6.5)$$

which must be invariant under the action of X_1 , X_2 and X_3 . As g_x and g_y cannot vanish simultaneously, Y can be taken either of the following local forms

$$Y = \frac{\partial}{\partial x} + g_y \frac{\partial}{\partial y}, \quad Y = g_x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \quad (6.6)$$

Let us assume that \mathcal{D} is spanned by the first one and search for Y . Now, if \mathcal{D} is invariant under the Lie brackets of the elements of V , we have that

$$\mathcal{L}_{X_1} Y = \left(1 - x \frac{\partial g_y}{\partial y}\right) \frac{\partial}{\partial y} = \gamma_1 Y, \quad (6.7a)$$

$$\mathcal{L}_{X_2} Y = \frac{1}{2} \left[\frac{\partial}{\partial x} + \left(y \frac{\partial g_y}{\partial y} - x \frac{\partial g_y}{\partial x} - g_y\right) \frac{\partial}{\partial y} \right] = \gamma_2 Y, \quad (6.7b)$$

$$\mathcal{L}_{X_3} Y = -g_y \frac{\partial}{\partial x} + \left(\frac{3c}{x^4} + y \frac{\partial g_y}{\partial x} + \frac{c}{x^3} \frac{\partial g_y}{\partial y}\right) \frac{\partial}{\partial y} = \gamma_3 Y, \quad (6.7c)$$

for certain functions $\gamma_1, \gamma_2, \gamma_3$ locally defined on \mathbb{R}^2 . The left-hand side of (6.7a) has no term ∂_x but the right-hand one has it provided $\gamma_1 \neq 0$. Therefore, $\gamma_1 = 0$ and $g_y = y/x + G$ for a certain $G = G(x)$. Next by introducing this result in (6.7b), we find that $\gamma_2 = 1/2$ and $2G + xG' = 0$, so that $G(x) = \mu/x^2$ for $\mu \in \mathbb{R}$. Substituting this into (6.7c), we obtain that $\gamma_3 = -(\mu + xy)/x^2$ and $\mu^2 = -4c$. Consequently, when $c > 0$ it does not exist any non-zero Y spanning locally \mathcal{D} satisfying (6.7a)–(6.7c) and V is therefore primitive, whilst if $c \leq 0$ there exists a vector field

$$Y = \frac{\partial}{\partial x} + \left(\frac{y}{x} + \frac{\mu}{x^2}\right) \frac{\partial}{\partial y}, \quad \mu^2 = -4c, \quad (6.8)$$

which spans \mathcal{D} , so that V is imprimitive. The case of \mathcal{D} being spanned by the second form of Y (6.6) can be analysed analogously and drives to the same conclusion.

Therefore the system (6.2) belongs to different classes within the GKO classification according to the value of the parameter c . The final result is established in the following statement.

Proposition 1. *The Vessiot–Guldberg Lie algebra for system (6.2), corresponding to the the Milne–Pinney equations, is locally diffeomorphic to P_2 for $c > 0$, I_4 for $c < 0$ and I_5 for $c = 0$.*

Proof. Since V is primitive when $c > 0$ and this is isomorphic to $\mathfrak{sl}(2)$, the GKO classification given in Table 1 implies that V is locally diffeomorphic to the primitive class P_2 .

Let us now consider that $c < 0$ and prove that the system is diffeomorphic to the class I_4 . We do this by showing that there exists a local diffeomorphism $\phi : (x, y) \in U \subset \mathbb{R}_{x \neq y}^2 \mapsto \bar{U} \subset (u, v) \in \mathbb{R}_{u \neq 0}^2$, satisfying that ϕ_* maps the basis for I_4 listed in Table 1 into (6.3). Due to the Lie bracket $[X_1, X_3] = 2X_2$, verified in both bases, it is only necessary to search the map for the generators X_1 and X_3 (so for X_2 this will be automatically fulfilled). By writing in coordinates

$$\phi_*(\partial_x + \partial_y) = -x\partial_y, \quad \phi_*(x^2\partial_x + y^2\partial_y) = y\partial_x + c/x^3\partial_y, \quad (6.9)$$

we obtain

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -u \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} v \\ c/u^3 \end{pmatrix}.$$

Hence, $\partial_x u + \partial_y u = 0 \Rightarrow u = g(x - y)$ for a certain $g : z \in \mathbb{R} \mapsto g(z) \in \mathbb{R}$. Since $x^2\partial_x u + y^2\partial_y u = v$, then $v = (x^2 - y^2)g'$, where g' is the derivative of $g(z)$ in terms of z . Using now that $\partial_x v + \partial_y v = -u$ we get $2(x - y)g' = -g$ so that $g = \lambda/|x - y|^{1/2}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. Substituting this into the remaining equation $x^2\partial_x v + y^2\partial_y v = c/u^3$, we find that $\lambda^4 = -4c$. Since $c < 0$, we consistently find that

$$u = \frac{\lambda}{|x - y|^{1/2}}, \quad v = -\frac{\lambda(x + y)}{2|x - y|^{1/2}}, \quad \lambda^4 = -4c. \quad (6.10)$$

Finally, let us set $c = 0$ and search for a local diffeomorphism $\phi : (x, y) \in U \subset \mathbb{R}_{y \neq 0}^2 \mapsto \bar{U} \subset (u, v) \in \mathbb{R}^2$ such that ϕ_* maps the basis corresponding to I_5 into (6.3); namely

$$\phi_*(\partial_x) = -x\partial_y, \quad \phi_*(x^2\partial_x + xy\partial_y) = y\partial_x, \quad (6.11)$$

yielding

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -u \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

Hence, $\partial_x u = 0 \Rightarrow u = g_1(y)$ for a certain $g_1 : \mathbb{R} \rightarrow \mathbb{R}$. Since $\partial_x v = -u$, then $v = -g_1(y)x + g_2(y)$ for another $g_2 : \mathbb{R} \rightarrow \mathbb{R}$. Using now the PDEs of the second matrix, we see that $xy\partial_y u = xyg_1' = v = -g_1x + g_2$, so that $g_2 = 0$ and $g_1 = \lambda/y$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\lambda \neq 0$ to avoid ϕ not to be a diffeomorphism. It can be checked that the remaining equation is so fulfilled. Therefore $u = \lambda/y$ and $v = -\lambda x/y$. \square

We remark that, since the three classes P_2 , I_4 and I_5 appear in Table 2 in Appendix 1, system (6.2) can always be associated to a symplectic form turning

their vector fields Hamiltonian. In this respect, recall that it was recently proved, that the system (6.2) is a Lie–Hamilton one for any value of c [35]. However, we shall show that identifying it to one of the classes of the GKO classification will be useful to study the relation of this system to other ones.

Example 2. Let us turn now to reconsider the **second-order Kummer–Schwarz equation** (2.38) written as a first-order system (2.40)

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = \frac{3}{2} \frac{v^2}{x} - 2c_0x^3 + 2b_1(t)x, \end{cases} \quad (6.12)$$

on $\mathbb{T}\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$, obtained by adding the new variable $v \equiv dx/dt$ to the KS-2 equation (2.38).

It is well known that (6.12) is a Lie system [87, 308]. In fact, it describes the integral curves of the t -dependent vector field $X_t = X_3 + b_1(t)X_1$ where the vector fields X_1, X_2, X_3 correspond to those in (2.42). Since their corresponding commutation relations are the same as in (2.42), spanning a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2)$. Thus, V can be isomorphic to one of the four $\mathfrak{sl}(2)$ -Lie algebras of vector fields in the GKO classification.

As in the previous subsection, we analyse if there exists a distribution \mathcal{D} stable under V and locally generated by a vector field Y of the first form given in (6.6) (the same results can be obtained by assuming the second one). Notice that in this section we change the notation in the variable v in (6.12), as $v \leftrightarrow y$. So, imposing \mathcal{D} to be stable under V yields

$$\mathcal{L}_{X_1}Y = 2 \left(x \frac{\partial g_y}{\partial y} - 1 \right) \frac{\partial}{\partial y} = \gamma_1 Y, \quad (6.13a)$$

$$\mathcal{L}_{X_2}Y = -\frac{\partial}{\partial x} + \left(x \frac{\partial g_y}{\partial x} + 2y \frac{\partial g_y}{\partial y} - 2g_y \right) \frac{\partial}{\partial y} = \gamma_2 Y, \quad (6.13b)$$

$$\mathcal{L}_{X_3}Y = -g_y \frac{\partial}{\partial x} + \left[X_3 g_y + \frac{3y^2}{2x^2} + 6cx^2 - \frac{3y}{x} g_y \right] \frac{\partial}{\partial y} = \gamma_3 Y, \quad (6.13c)$$

for certain functions $\gamma_1, \gamma_2, \gamma_3$ locally defined on \mathbb{R}^2 . The left-hand side of (6.13a) has no term ∂_x and the right-hand one does not have it provided $\gamma_1 = 0$. Hence, $\gamma_1 = 0$ and $g_y = y/x + F$ for a $F = F(x)$. In view of (6.13b), we then obtain $\gamma_2 = -1$ and $F - xF' = 0$, that is, $F(x) = \mu x$ for $\mu \in \mathbb{R}$. Substituting g_y in (6.13c), we obtain that $\gamma_3 = -\mu x - y/x$ and $\mu^2 = -4c$. Hence, as in the Milne–Pinney equations, we find that if $c > 0$ it does not exist any Y spanning locally \mathcal{D} satisfying (6.13a)–(6.13c) and V is primitive, meanwhile if $c \leq 0$, then \mathcal{D} is spanned by the vector field

$$Y = \frac{\partial}{\partial x} + \left(\frac{y}{x} + \mu x \right) \frac{\partial}{\partial y}, \quad \mu^2 = -4c, \quad (6.14)$$

and V is imprimitive.

The precise classes of the GKO classification corresponding to the system (2.40) are summarized in the following proposition.

Proposition 2. *The Vessiot–Guldberg Lie algebra for system (6.12), associated with the second-order Kummer–Schwarz equations, is locally diffeomorphic to P_2 for $c > 0$, I_4 for $c < 0$ and I_5 for $c = 0$.*

Proof. The case with $c > 0$ provides the primitive class P_2 since $Y = 0$. If $c < 0$ we look for a local diffeomorphism $\phi : (x, y) \in U \subset \mathbb{R}_{x \neq y}^2 \mapsto \bar{U} \subset (u, v) \in \mathbb{R}_{u \neq 0}^2$, such that ϕ_* maps the basis of I_4 into (2.38), that is,

$$\phi_*(\partial_x + \partial_y) = 2x\partial_y, \quad \phi_*(x^2\partial_x + y^2\partial_y) = y\partial_x + \left(\frac{3}{2}y^2/x - 2cx^3\right)\partial_y. \quad (6.15)$$

Then

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2u \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} v \\ \frac{3}{2}v^2/u - 2cu^3 \end{pmatrix}.$$

Proceeding as in the proof of Proposition 1 we find that $u = g(x - y)$ and $v = (x^2 - y^2)g'$ for $g : \mathbb{R} \rightarrow \mathbb{R}$. As now $\partial_x v + \partial_y v = 2u$ we obtain $2(x - y)g' = 2g$, so that $g = \lambda(x - y)$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and assume $\lambda \neq 0$ to avoid ϕ not being a diffeomorphism. The remaining equation $x^2\partial_x v + y^2\partial_y v = \frac{3}{2}v^2/u - 2cu^3$ implies that $4\lambda^2 = -1/c$, which is consistent with the value $c < 0$. Then

$$u = \lambda(x - y), \quad v = \lambda(x^2 - y^2), \quad 4\lambda^2 = -1/c. \quad (6.16)$$

In the third possibility with $c = 0$ we require that ϕ_* maps the basis of I_5 into (2.38) so fulfilling

$$\phi_*(\partial_x) = 2x\partial_y, \quad \phi_*(x^2\partial_x + xy\partial_y) = y\partial_x + \frac{3}{2}y^2/x\partial_y, \quad (6.17)$$

that is,

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2u \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} v \\ \frac{3}{2}v^2/u \end{pmatrix}.$$

By taking into account the proof of Proposition 1, it is straightforward to check that the four PDEs are satisfied for $u = \lambda y^2$ and $v = 2\lambda xy^2$ with $\lambda \in \mathbb{R} \setminus \{0\}$. \square

Example 3. Let us return to **complex Riccati equations with t -dependent real coefficients** in the form (2.7). We already showed that this system has a Vessiot–Guldberg Lie algebra $P_2 \simeq \mathfrak{sl}(2)$. Therefore, it is locally diffeomorphic to the Vessiot–Guldberg Lie algebra appearing in the above Milne–Pinney and Kummer–Schwarz equations whenever the parameter $c > 0$. In view of the GKO classification,

there exist local diffeomorphisms relating the three first-order systems associated with these equations. For instance, we can search for a local diffeomorphism $\phi : (x, y) \in U \subset \mathbb{R}_{y \neq 0}^2 \mapsto \bar{U} \subset (u, v) \in \mathbb{R}_{u \neq 0}^2$ mapping every system (2.7) into one of the form (6.2), e.g., satisfying that ϕ_* maps the basis of P_2 in Table 1, related to the planar Riccati equation, into the basis (6.3) associated with the Milne–Pinney one. By writing in coordinates

$$\phi_*(\partial_x) = -x\partial_y, \quad \phi_*[(x^2 - y^2)\partial_x + 2xy\partial_y] = y\partial_x + c/x^3\partial_y, \quad (6.18)$$

we obtain

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -u \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} v \\ c/u^3 \end{pmatrix}.$$

Similar computations to those performed in the proof of Proposition 1 for the three PDEs $\partial_x u = 0$, $\partial_x v = -u$ and $(x^2 - y^2)\partial_x u + 2xy\partial_y u = v$ gives $u = \lambda/|y|^{1/2}$ and $v = -\lambda x/|y|^{1/2}$ with $\lambda \in \mathbb{R} \setminus \{0\}$. Substituting these results into the remaining equation we find that $\lambda^4 = c$ which is consistent with the positive value of c . Consequently, this maps the system (2.7) into (6.2) and the solution of the first one is locally equivalent to solutions of the second one.

Summing up, we have explicitly proven that the three $\mathfrak{sl}(2)$ Lie algebras of the classes P_2 , I_4 and I_5 given in Table 2 in Appendix 1 cover the following $\mathfrak{sl}(2)$ -Lie systems

- P_2 : Milne–Pinney and Kummer–Schwarz equations for $c > 0$ as well as complex Riccati equations with t -dependent coefficients,
- I_4 : Milne–Pinney and Kummer–Schwarz equations for $c < 0$,
- I_5 : Milne–Pinney and Kummer–Schwarz equations for $c = 0$ and the harmonic oscillator with t -dependent frequency.

This means that, only within each class, they are locally diffeomorphic and, therefore, there exists a local change of variables mapping one into another. Thus, for instance, there does not exist any diffeomorphism mapping the Milne–Pinney and Kummer–Schwarz equations with $c \neq 0$ to the harmonic oscillator. These results also explain from an algebraic point of view the existence of the known diffeomorphism mapping Kummer–Schwarz equations to Milne–Pinney equations [287] provided that the sign of c is the same in both systems.

6.2 Lie–Hamilton biological models

In this section we focus on new applications of the Lie–Hamilton approach to Lotka–Volterra-type systems and to a viral infection model. We also consider here the

analysis of Buchdahl equations which can be studied through a Lie–Hamilton system diffeomorphic to a precise t -dependent Lotka–Volterra system.

Example 4. We call **generalized Buchdahl equations** [73, 118, 122] the second-order differential equations

$$\frac{d^2x}{dt^2} = a(x) \left(\frac{dx}{dt} \right)^2 + b(t) \frac{dx}{dt}, \quad (6.19)$$

where $a(x)$ and $b(t)$ are arbitrary functions of their respective arguments. In order to analyse whether these equations can be studied through a Lie system, we add the variable $y \equiv dx/dt$ and consider the first-order system of differential equations

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = a(x)y^2 + b(t)y. \end{cases} \quad (6.20)$$

Note that if $(x(t), y(t))$ is a particular solution of this system with $y(t_0) = 0$ for a particular $t_0 \in \mathbb{R}$, then $y(t) = 0$ for every $t \in \mathbb{R}$ and $x(t) = \lambda \in \mathbb{R}$. Moreover, if $a(x) = 0$ then the solution of the above system is also trivial. As a consequence, we can restrict ourselves to studying particular solutions on $\mathbb{R}_{y \neq 0}^2$ with $a(x) \neq 0$.

Next let us prove that (6.20) is a Lie system. Explicitly, (6.20) is associated with the t -dependent vector field $X_t = X_1 + b(t)X_2$, where

$$X_1 = y \frac{\partial}{\partial x} + a(x)y^2 \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}. \quad (6.21)$$

Since

$$[X_1, X_2] = -X_1, \quad (6.22)$$

these vector fields span a Lie algebra V isomorphic to \mathfrak{h}_2 leaving invariant the distribution \mathcal{D} spanned by $Y \equiv X_1$. Since the rank of \mathcal{D}^V is two, V is locally diffeomorphic to the imprimitive class I_{14A} with $r = 1$ and $\eta_1(x) = e^x$ given in Table 2. This proves for the first time that generalized Buchdahl equations written as the system (6.20) are, in fact, not only a Lie system [87] but a Lie–Hamilton one.

Next by determining a symplectic form obeying $\mathcal{L}_{X_i}\omega = 0$, with $i = 1, 2$ for the vector fields (6.21) and the generic ω (5.10), it can be shown that this reads

$$\omega = \frac{\exp\left(-\int a(x)dx\right)}{y} dx \wedge dy, \quad (6.23)$$

which turns X_1 and X_2 into Hamiltonian vector fields with Hamiltonian functions

$$h_1 = y \exp\left(-\int^x a(x')dx'\right), \quad h_2 = -\int^x \exp\left(-\int^{\bar{x}} a(\bar{x})d\bar{x}\right) dx', \quad (6.24)$$

respectively. Note that all these structures are properly defined on $\mathbb{R}_{y \neq 0}^2$ and hold $\{h_1, h_2\} = h_1$. Consequently, the system (6.20) has a t -dependent Hamiltonian given by $h_t = h_1 + b(t)h_2$.

Example 5. Consider the specific t -dependent Lotka–Volterra system [263, 400] of the form

$$\begin{cases} \frac{dx}{dt} = ax - g(t)(x - ay)x, \\ \frac{dy}{dt} = ay - g(t)(bx - y)y, \end{cases} \quad (6.25)$$

where $g(t)$ is a t -dependent function representing the variation of the seasons and a, b are certain real parameters describing the interactions among the species. We hereafter focus on the case $a \neq 0$, as otherwise the above equation becomes, up to a t -reparametrization, an autonomous differential equation that can easily be integrated. We also assume $g(t)$ to be a non-constant function and we restrict ourselves to particular solutions on $\mathbb{R}_{x,y \neq 0} = \{(x, y) | x \neq 0, y \neq 0\}$ (the remaining ones can be trivially obtained).

Let us prove that (6.25) is a Lie system and that for some values of the real parameters $a \neq 0$ and b this is a Lie–Hamilton system as well. This system describes the integral curves of the t -dependent vector field $X_t = X_1 + g(t)X_2$ where

$$X_1 = ax \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}, \quad X_2 = -(x - ay)x \frac{\partial}{\partial x} - (bx - y)y \frac{\partial}{\partial y}, \quad (6.26)$$

satisfy

$$[X_1, X_2] = aX_2, \quad a \neq 0. \quad (6.27)$$

Hence, X_1 and X_2 are the generators of a Lie algebra V of vector fields isomorphic to \mathfrak{h}_2 leaving invariant the distribution \mathcal{D} on $\mathbb{R}_{x,y \neq 0}$ spanned by $Y \equiv X_2$. According to the values of the parameters $a \neq 0$ and b we find that

- When $a = b = 1$, the rank of \mathcal{D}^V on the domain of V is one. In view of Table 1 the Lie algebra V is thus isomorphic to \mathfrak{I}_2 and, by taking into account Table 2, we conclude that X is a Lie system, but not a Lie–Hamilton one.
- Otherwise, the rank of \mathcal{D}^V is two, so that this Lie algebra is locally diffeomorphic to \mathfrak{I}_{14A} with $r = 1$ and $\eta_1 = e^{ax}$ given in Table 2 and, consequently, X is a Lie–Hamilton system. As a straightforward consequence, when $a = 1$ and $b \neq 1$ the system (6.25) is locally diffeomorphic to the generalized Buchdahl equations (6.20).

Let us now derive a symplectic structure (5.10) turning the elements of V into Hamiltonian vector fields by solving the system of PDEs $\mathcal{L}_{X_1}\omega = \mathcal{L}_{X_2}\omega = 0$. The first condition reads in local coordinates

$$\mathcal{L}_{X_1}\omega = (X_1f + 2af)dx \wedge dy = 0. \quad (6.28)$$

So we obtain that $f = F(x/y)/y^2$ for any function $F : \mathbb{R} \rightarrow \mathbb{R}$. By imposing that $\mathcal{L}_{X_2}\omega = 0$, we find

$$\mathcal{L}_{X_2}\omega = [(b-1)x^2 + (a-1)yx] \frac{\partial f}{\partial x} + f[(b-2)x + ay] = 0. \quad (6.29)$$

Notice that, as expected, f vanishes when $a = b = 1$. We study separately the remaining cases: i) $a \neq 1$ and $b \neq 1$; ii) $a = 1$ and $b \neq 1$; and iii) $a \neq 1$ and $b = 1$.

When both $a, b \neq 1$ we write $f = F(x/y)/y^2$, thus obtaining that ω reads, up to a non-zero multiplicative constant, as

$$\omega = \frac{1}{y^2} \left(\frac{x}{y}\right)^{\frac{a}{1-a}} \left[1 - a + (1-b)\frac{x}{y}\right]^{\frac{1}{a-1} + \frac{1}{b-1}} dx \wedge dy, \quad a, b \neq 1. \quad (6.30)$$

From this, we obtain the following Hamiltonian functions for X_1 and X_2

$$\begin{aligned} h_1 &= a(1-b)^{1+\frac{1}{a-1}+\frac{1}{b-1}} \left(\frac{x}{y}\right)^{\frac{1}{b-1}} \times \\ &\quad \times {}_2F_1\left(\frac{1}{1-b}, \frac{1}{1-a} + \frac{1}{1-b}; \frac{b-2}{b-1}, \frac{y(1-a)}{x(b-1)}\right), \\ h_2 &= -y \left(\frac{x}{y}\right)^{\frac{1}{1-a}} \left[(1-a) + (1-b)\frac{x}{y}\right]^{\frac{1}{a-1} + \frac{1}{b-1} + 1}, \end{aligned} \quad (6.31)$$

where ${}_2F_1(\alpha, \beta, \gamma, \zeta)$ stands for the hypergeometric function

$${}_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} [(\alpha)_n (\beta)_n / (\gamma)_n] z^n / n! \quad (6.32)$$

with $(\delta)_n = \Gamma(\delta + n)/\Gamma(\delta)$ being the rising Pochhammer symbol. As expected, $\{h_1, h_2\}_\omega = -ah_2$.

When $a = 1$ and $b \neq 1$, the symplectic form for X becomes

$$\omega = \frac{1}{y^2} \exp\left(\frac{y - (b-2)x \ln|x/y|}{(b-1)x}\right) dx \wedge dy, \quad b \neq 1, \quad (6.33)$$

and the Hamiltonian functions for X_1 and X_2 read

$$\begin{aligned} h_1 &= -\left(\frac{1}{1-b}\right)^{\frac{1}{b-1}} \Gamma\left(\frac{1}{1-b}, \frac{y}{x(1-b)}\right), \\ h_2 &= (b-1)x \left(\frac{x}{y}\right)^{\frac{1}{b-1}} \exp\left(\frac{y}{(b-1)x}\right), \end{aligned} \quad (6.34)$$

with $\Gamma(u, v)$ being the incomplete Gamma function, which satisfy $\{h_1, h_2\}_\omega = -h_2$.

Finally, when $b = 1$ and $a \neq 1$, we have

$$\omega = \frac{1}{y^2} \left(\frac{x}{y} \right)^{\frac{a}{1-a}} \exp \left(\frac{x}{y(a-1)} \right) dx \wedge dy, \quad a \neq 1. \quad (6.35)$$

Then, the Hamiltonian functions for X_1 and X_2 are, in this order,

$$\begin{aligned} h_1 &= a(1-a)^{\frac{1}{1-a}} \Gamma \left(\frac{1}{1-a}, \frac{x}{y(1-a)} \right), \\ h_2 &= (a-1)y \exp \left(\frac{x}{y(a-1)} \right) \left(\frac{x}{y} \right)^{\frac{1}{1-a}}. \end{aligned} \quad (6.36)$$

Indeed, $\{h_1, h_2\}_\omega = -ah_2$.

Example 6. The system of differential equations [306], is known as a **class of quadratic polynomial Lie systems**

$$\begin{cases} \frac{dx}{dt} = b(t)x + c(t)y + d(t)x^2 + e(t)xy + f(t)y^2, \\ \frac{dy}{dt} = y, \end{cases} \quad (6.37)$$

where $b(t), c(t), d(t), e(t)$ and $f(t)$ are arbitrary t -dependent functions, is an interacting species model of Lotka–Volterra type that belongs to the class of quadratic-linear polynomial systems with a unique singular point at the origin [306].

In general, this system is not a Lie system. For instance, consider the particular system associated to the t -dependent vector field $X_t = d(t)X_1 + e(t)X_2 + X_3$,

$$X_1 = x^2 \frac{\partial}{\partial x}, \quad X_2 = xy \frac{\partial}{\partial x}, \quad X_3 = y \frac{\partial}{\partial y}, \quad (6.38)$$

where $d(t)$ and $e(t)$ are non-constant and non-proportional functions. Notice that V^X contains X_1, X_2 and their successive Lie brackets, i.e. the vector fields

$$\overbrace{[X_2, \dots [X_2, X_1] \dots]}^{n\text{-times}} = x^2 y^n \frac{\partial}{\partial x} \equiv Y_n. \quad (6.39)$$

Hence, $[X_2, Y_n] = Y_{n+1}$ and the family of vector fields $X_1, X_2, X_3, Y_1, Y_2, \dots$ span an infinite-dimensional family of linearly independent vector fields over \mathbb{R} so that X is not a Lie system.

Hereafter we analyse the cases of (6.37) with $d(t) = e(t) = 0$ which provides quadratic polynomial systems that are Lie systems. We call them *quadratic polynomial Lie systems*; these are related to the system of differential equations [306]

$$\begin{cases} \frac{dx}{dt} = b(t)x + c(t)y + f(t)y^2, \\ \frac{dy}{dt} = y. \end{cases} \quad (6.40)$$

Note that if a solution $(x(t), y(t))$ of the above system is such that $y(t_0) = 0$ for a certain t_0 , then $y(t) = 0$ for all $t \in \mathbb{R}$ and the corresponding $x(t)$ can be then easily obtained. In view of this, we focus on those particular solutions within $\mathbb{R}_{y \neq 0}^2$. The system (6.40) is associated with the t -dependent vector field on $\mathbb{R}_{y \neq 0}^2$ of the form $X_t = X_1 + b(t)X_2 + c(t)X_3 + f(t)X_4$, where

$$X_1 = y \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x}, \quad X_3 = y \frac{\partial}{\partial x}, \quad X_4 = y^2 \frac{\partial}{\partial x}, \quad (6.41)$$

satisfy the commutation rules

$$\begin{aligned} [X_1, X_2] &= 0, & [X_1, X_3] &= X_3, & [X_1, X_4] &= 2X_4, \\ [X_2, X_3] &= -X_3, & [X_2, X_4] &= -X_4, & [X_3, X_4] &= 0. \end{aligned} \quad (6.42)$$

Note that $V \simeq V_1 \times V_2$ where $V_1 = \langle X_1, X_2 \rangle \simeq \mathbb{R}^2$ and $V_2 = \langle X_3, X_4 \rangle \simeq \mathbb{R}^2$. In addition, the distribution \mathcal{D} spanned by $Y \equiv \partial_x$ is invariant under the action of the above vector fields so, V is imprimitive. In view of Table 1, we find that (6.40) is a Lie system corresponding to the imprimitive class I_{15} with $V \simeq \mathbb{R}^2 \times \mathbb{R}^2$. By taking into account our classification given in 2 in Appendix 1, we know that this is not a Lie algebra of vector fields with respect to any symplectic structure.

Example 7. We now consider a subcase of (6.40) that provides a Lie–Hamilton system. We will refer to such subcase as **quadratic polynomial Lie–Hamilton systems**. In view of Table 2, the Lie subalgebra $\mathbb{R} \times \mathbb{R}^2$ of V is a Lie algebra of Hamiltonian vector fields with respect to a symplectic structure, that is, $I_{14} \subset I_{15}$. So, it is natural consider the restriction of (6.40) to

$$\begin{cases} \frac{dx}{dt} = bx + c(t)y + f(t)y^2, \\ \frac{dy}{dt} = y, \end{cases} \quad (6.43)$$

where $b \in \mathbb{R} \setminus \{1, 2\}$ and $c(t), f(t)$ are still t -dependent functions. The system (6.43) is associated to the t -dependent vector field $X_t = X_1 + c(t)X_2 + f(t)X_3$ on $\mathbb{R}_{y \neq 0}^2 = \{(x, y) \in \mathbb{R} \mid y \neq 0\}$, where

$$X_1 = bx \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial x}, \quad X_3 = y^2 \frac{\partial}{\partial x} \quad (6.44)$$

satisfy the commutation relations

$$[X_1, X_2] = (1 - b)X_2, \quad [X_1, X_3] = (2 - b)X_3, \quad [X_2, X_3] = 0. \quad (6.45)$$

Therefore, the vector fields (6.44) generate a Lie algebra $V \simeq V_1 \times V_2$, where $V_1 = \langle X_1 \rangle \simeq \mathbb{R}$ and $V_2 = \langle X_2, X_3 \rangle \simeq \mathbb{R}^2$. The domain of V is $\mathbb{R}_{y \neq 0}^2$ and the rank of

\mathcal{D}^V is two. Moreover, the distribution \mathcal{D} spanned by the vector field $Y \equiv \partial_x$ is stable under the action of the elements of V , which turns V into an imprimitive Lie algebra. So, V must be locally diffeomorphic to the imprimitive Lie algebra I_{14} displayed in Table 1 for $r = 2$. We already know that the class I_{14} is a Lie algebra of Hamiltonian vector fields with respect to a symplectic structure.

By imposing $\mathcal{L}_{X_i}\omega = 0$ for the vector fields (6.44) and the generic symplectic form (5.10), it can be shown that ω reads

$$\omega = \frac{dx \wedge dy}{y^{b+1}}, \quad (6.46)$$

which turns (6.44) into Hamiltonian vector fields with Hamiltonian functions

$$h_1 = -\frac{x}{y^b}, \quad h_2 = \frac{y^{1-b}}{1-b}, \quad h_3 = \frac{y^{2-b}}{2-b}, \quad b \in \mathbb{R} \setminus \{1, 2\}. \quad (6.47)$$

Note that all the above structures are properly defined on $\mathbb{R}_{y \neq 0}^2$. The above Hamiltonian functions span a three-dimensional Lie algebra with commutation relations

$$\{h_1, h_2\}_\omega = (b-1)h_2, \quad \{h_1, h_3\}_\omega = (b-2)h_3, \quad \{h_2, h_3\}_\omega = 0. \quad (6.48)$$

Consequently, V is locally diffeomorphic to the imprimitive Lie algebra I_{14A} of Table 2 such that the Lie–Hamilton algebra is $\mathbb{R} \times \mathbb{R}^2$ (also $(\mathbb{R} \times \mathbb{R}^2) \oplus \mathbb{R}$). The system (6.43) has a t -dependent Hamiltonian

$$h_t = b h_1 + c(t)h_2 + d(t)h_3 = -b\frac{x}{y^b} + c(t)\frac{y^{1-b}}{1-b} + d(t)\frac{y^{2-b}}{2-b}. \quad (6.49)$$

We point out that the cases of (6.43) with either $b = 1$ or $b = 2$ also lead to Lie–Hamilton systems, but now belonging, both of them, to the class I_{14B} of Table 2 as a central generator is required. For instance if $b = 1$, the commutation relations (6.45) reduce to

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_3, \quad [X_2, X_3] = 0, \quad (6.50)$$

while the symplectic form and the Hamiltonian functions are found to be

$$\omega = \frac{dx \wedge dy}{y^2}, \quad h_1 = -\frac{x}{y}, \quad h_2 = \ln y, \quad h_3 = y, \quad (6.51)$$

which together with $h_0 = 1$ close the (centrally extended) Lie–Hamilton algebra $\mathbb{R} \times \mathbb{R}^2$, that is,

$$\{h_1, h_2\}_\omega = -h_0, \quad \{h_1, h_3\}_\omega = -h_3, \quad \{h_2, h_3\}_\omega = 0, \quad \{h_0, \cdot\}_\omega = 0. \quad (6.52)$$

A similar result can be found for $b = 2$.

Example 8. Finally, let us consider a **simple viral infection model** given by [153]

$$\begin{cases} \frac{dx}{dt} = (\alpha(t) - g(y))x, \\ \frac{dy}{dt} = \beta(t)xy - \gamma(t)y, \end{cases} \quad (6.53)$$

where $g(y)$ is a real positive function taking into account the power of the infection. Note that if a particular solution satisfies $x(t_0) = 0$ or $y(t_0) = 0$ for a $t_0 \in \mathbb{R}$, then $x(t) = 0$ or $y(t) = 0$, respectively, for all $t \in \mathbb{R}$. As these cases are trivial, we restrict ourselves to studying particular solutions within $\mathbb{R}_{x,y \neq 0}^2 = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$.

The simplest possibility consists in setting $g(y) = \delta$, where δ is a constant. Then, (6.53) describes the integral curves of the t -dependent vector field $X_t = (\alpha(t) - \delta)X_1 + \gamma(t)X_2 + \beta(t)X_3$, on $\mathbb{R}_{x,y \neq 0}^2$, where the vector fields

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = -y \frac{\partial}{\partial y}, \quad X_3 = xy \frac{\partial}{\partial y}, \quad (6.54)$$

satisfy the relations (6.50). So, X is a Lie system related to a Vessiot–Guldberg Lie algebra $V \simeq \mathbb{R} \ltimes \mathbb{R}^2$ where $\langle X_1 \rangle \simeq \mathbb{R}$ and $\langle X_2, X_3 \rangle \simeq \mathbb{R}^2$. The distribution \mathcal{D}^V has rank two on $\mathbb{R}_{x,y \neq 0}^2$. Moreover, V is imprimitive, as the distribution \mathcal{D} spanned by $Y \equiv \partial_y$ is invariant under the action of vector fields of V . Thus V is locally diffeomorphic to the imprimitive Lie algebra I_{14B} for $r = 2$ and, in view of Table 2, the system X is a Lie–Hamilton one.

Next we obtain that V is a Lie algebra of Hamiltonian vector fields with respect to the symplectic form

$$\omega = \frac{dx \wedge dy}{xy}. \quad (6.55)$$

Then, the vector fields X_1 , X_2 and X_3 have Hamiltonian functions: $h_1 = \ln|y|$, $h_2 = \ln|x|$, $h_3 = -x$, which along $h_0 = 1$ close the relations (6.52). If we assume $V^X = V$, the t -dependent Hamiltonian $h_t = (\alpha(t) - \delta)h_1 + \gamma(t)h_2 + \beta(t)h_3$ gives rise to a Lie–Hamiltonian structure $(\mathbb{R}_{x,y \neq 0}^2, \omega, h)$ for X defining the Lie–Hamilton algebra $(\overline{\mathbb{R} \ltimes \mathbb{R}^2})$.

6.3 Other Lie–Hamilton systems on the plane

Example 9. Let us reconsider the **Cayley–Klein Riccati equation** reviewed in (2.7),

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)(x^2 - y^2), \quad \frac{dy}{dt} = a_1(t)y + a_2(t)2xy, \quad (6.56)$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary t -dependent real functions with t -dependent vector field

$$X_t = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3, \quad (6.57)$$

where X_1, X_2, X_3 have commutation relations (2.11). According to Table 1 in Appendix 1, system (2.7) is a LH system possessing a Vessiot–Guldberg Lie algebra diffeomorphic to the primitive Lie algebra P_2 and the vector fields X_1, X_2, X_3 are Hamiltonian vector fields with respect to the symplectic form

$$\omega = \frac{dx \wedge dy}{y^2}. \quad (6.58)$$

Then, some Hamiltonian functions h_i of X_i (2.10) can be chosen to be

$$h_1 = -\frac{1}{y}, \quad h_2 = -\frac{x}{y}, \quad h_3 = -\frac{x^2 + y^2}{y}, \quad (6.59)$$

which satisfy

$$\{h_1, h_2\}_\omega = -h_1, \quad \{h_1, h_3\}_\omega = -2h_2, \quad \{h_2, h_3\}_\omega = -h_3. \quad (6.60)$$

Hence, $(\langle h_1, h_2, h_3 \rangle, \{\cdot, \cdot\}_\omega)$ is a Lie–Hamilton algebra for X isomorphic to $\mathfrak{sl}(2)$ and

$$h = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3 \quad (6.61)$$

is a t -dependent Hamiltonian function.

The above result can be generalized by making use of analytic continuation and contractions which can also be understood as a Cayley–Klein approach [37, 219, 377, 423], which underlies the structure of the referred to as two-dimensional Cayley–Klein geometries.

Consider the real plane with coordinates $\{x, y\}$ and an ‘additional’ unit j such that

$$j^2 \in \{-1, +1, 0\}. \quad (6.62)$$

Next, we define

$$z := x + jy, \quad \bar{z} := x - jy, \quad (x, y) \in \mathbb{R}^2. \quad (6.63)$$

Assuming that j commutes with real numbers, we can write

$$|z|^2 := z\bar{z} = x^2 - j^2y^2, \quad z^2 = x^2 + j^2y^2 + 2jxy. \quad (6.64)$$

In this way we find that the number z in (6.63) comprises three possibilities

- $j^2 = -1$. In this case we are dealing with the usual *complex numbers* $j := i$ and $z \in \mathbb{C}$. Hence

$$|z|^2 = z\bar{z} = x^2 + y^2, \quad z^2 = x^2 - y^2 + 2ixy, \quad z \in \mathbb{C}. \quad (6.65)$$

- $j^2 = +1$. Now we are dealing with the so-called *split-complex numbers* $z \in \mathbb{C}'$. The additional unit is usually known as the *double* or *Clifford* unit $j := e$ [423]. Thus

$$|z|^2 = z\bar{z} = x^2 - y^2, \quad z^2 = x^2 + y^2 + 2exy, \quad z \in \mathbb{C}'. \quad (6.66)$$

- $j^2 = 0$. In this last possibility z is known as a *dual* or *Study number* [423], which can be regarded as a *contracted* case since

$$|z|^2 = z\bar{z} = x^2, \quad z^2 = x^2 + 2\epsilon xy, \quad z \in \mathbb{D}. \quad (6.67)$$

With these ingredients we shall call the *Cayley–Klein Riccati equation* [166] the generalization of the complex Riccati equation (2.7) to $z \in \{\mathbb{C}, \mathbb{C}', \mathbb{D}\}$, that is,

$$\frac{dz}{dt} = a_0(t) + a_1(t)z + a_2(t)z^2, \quad z := x + jy, \quad (6.68)$$

which, for real t -dependent coefficients $a_0(t), a_1(t), a_2(t)$, gives rise to a system of two differential equations

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)(x^2 + j^2y^2), \quad \frac{dy}{dt} = a_1(t)y + a_2(t)2xy, \quad (6.69)$$

that generalizes the planar Riccati equation (2.7).

We now prove separately that the two remaining cases with $j^2 = +1$ and $j^2 = 0$ are also Lie–Hamilton systems.

Example 10. If we set $j^2 := +1$, the system (6.69) is known as a **Double-Clifford or split-complex Riccati equation**

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)(x^2 + y^2), \quad \frac{dy}{dt} = a_1(t)y + a_2(t)2xy. \quad (6.70)$$

The point is to analyze whether this is also a Lie system and, in affirmative case, whether it is furthermore a Lie–Hamilton one with a Vessiot–Guldberg Lie algebra diffeomorphic to a class given in Table 1.

Indeed, system (6.70) is related to the t -dependent vector field of the form (6.57) with

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = (x^2 + y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}. \quad (6.71)$$

These again span a Vessiot–Guldberg real Lie algebra $V \simeq \mathfrak{sl}(2)$ with the same commutation rules given by (2.11). Hence (6.70) is a Lie system.

Vector fields (6.71) do not arise exactly in the Lie algebras isomorphic to $\mathfrak{sl}(2)$ in Table 1, namely, P_2 , I_4 and I_5 . Nevertheless, if we introduce in (6.71) the new variables $\{u, v\}$ defined by

$$u = x + y, \quad v = x - y, \quad x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v), \quad (6.72)$$

we find that

$$X_1 = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad X_2 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \quad X_3 = u^2 \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}, \quad (6.73)$$

which are, exactly, the vector fields appearing in the imprimitive Lie algebra $I_4 \simeq \mathfrak{sl}(2)$ of Table 1 and Table 2. Thus these are endowed with a closed and non-degenerate two-form; the latter and some associated Hamiltonian functions are given by

$$\omega = \frac{du \wedge dv}{(u - v)^2}, \quad h_1 = \frac{1}{u - v}, \quad h_2 = \frac{u + v}{2(u - v)}, \quad h_3 = \frac{uv}{u - v}, \quad (6.74)$$

which satisfy the same commutation relations (6.60).

In terms of the initial variables $\{x, y\}$, the expressions (6.74) turn out to be

$$\omega = -\frac{dx \wedge dy}{2y^2}, \quad h_1 = \frac{1}{2y}, \quad h_2 = \frac{x}{2y}, \quad h_3 = \frac{x^2 - y^2}{2y}, \quad (6.75)$$

to be compared with (6.58) and (6.59).

Example 11. When $j^2 = 0$ the system (6.69) reduces to a **Dual-Study Riccati equation**

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad \frac{dy}{dt} = a_1(t)y + a_2(t)2xy, \quad (6.76)$$

which can be regarded as a ‘contracted’ system from either (2.7) or (6.70). We stress that this system appears as a part of the Riccati system employed in the resolution of diffusion-type equations [389, 390]. Its relevance is due to the fact that its general solution allows us to solve the whole Riccati system and to map diffusion-type equations into an easily integrable PDE.

Let us prove that (6.76) is a Lie–Hamilton system. It is clear that this is a Lie system as this is related to the t -dependent vector field $X_t = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}, \quad (6.77)$$

span a Vessiot–Guldberg real Lie algebra $V \simeq \mathfrak{sl}(2)$ with commutation rules given by (2.11). These vector fields do not appear again in Table 1. Nevertheless, we

can map them into a basis given in Table 1, by choosing an appropriate change of variables.

We assume $y > 0$, while the case $y < 0$ can be studied analogously giving a similar result. Consider the new variables $\{u, v\}$ defined by

$$u = x, \quad v = \sqrt{y}, \quad x = u, \quad y = v^2. \quad (6.78)$$

By writing vector fields (6.77) in the new variables, we obtain

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = u \frac{\partial}{\partial u} + \frac{1}{2}v \frac{\partial}{\partial v}, \quad X_3 = u^2 \frac{\partial}{\partial u} + uv \frac{\partial}{\partial v}, \quad (6.79)$$

which are those appearing in the imprimitive Lie algebra $I_5 \simeq \mathfrak{sl}(2)$ of Table 1 and Table 2 and so we get

$$\omega = \frac{du \wedge dv}{v^3}, \quad h_1 = -\frac{1}{2v^2}, \quad h_2 = -\frac{u}{2v^2}, \quad h_3 = -\frac{u^2}{2v^2}. \quad (6.80)$$

These Hamiltonian functions satisfy the commutation relations (6.60). Consequently, h_1, h_2, h_3 span a Lie algebra isomorphic to $\mathfrak{sl}(2)$.

Finally we write the expressions (6.80) in terms of the initial variables $\{x, y\}$

$$\omega = \frac{dx \wedge dy}{2y^2}, \quad h_1 = -\frac{1}{2y}, \quad h_2 = -\frac{x}{2y}, \quad h_3 = -\frac{x^2}{2y}, \quad (6.81)$$

which fulfill the commutation relations (6.60). Therefore, the system (6.76) is, once more, an $\mathfrak{sl}(2)$ -Lie-Hamilton system possessing a Vessiot–Guldberg Lie algebra diffeomorphic to I_5 and a LH algebra $(\langle h_1, h_2, h_3 \rangle, \{\cdot, \cdot\}_\omega) \simeq \mathfrak{sl}(2)$.

We conclude that the Cayley–Klein Riccati equation (6.68) comprises in a unified way the *three* non-diffeomorphic Vessiot–Guldberg Lie algebras of Hamiltonian vector fields isomorphic to $\mathfrak{sl}(2)$: P_2, I_4 and I_5 .

Other $\mathfrak{sl}(2)$ -Lie–Hamilton systems and ‘equivalence’

In this section we present some $\mathfrak{sl}(2)$ -Lie–Hamilton systems of mathematical and physical interest. To keep notation simple, we say that a second-order differential equation is a Lie system (resp. LH system) when the first-order system obtained from it by adding a new variable $y := dx/dt$, is a Lie system (resp. Lie–Hamilton system).

We here study the coupled Riccati equation, Milne–Pinney and second-order Kummer–Schwarz equations, certain Lie systems appearing in the study of diffusion equations, the Smorodinsky–Winternitz system and the harmonic oscillator, both with a t -dependent frequency. Next we establish, according to Table 1, the equivalence among them and also among the three cases covered by the Cayley–Klein Riccati equation. More precisely, we establish which of all of the above systems are locally diffeomorphic.

Example 12. Consider the system of **coupled Riccati equations** given in (4.10) for $n = 2$ [318], with $x_1 = x$ and $x_2 = y$.

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad \frac{dy}{dt} = a_0(t) + a_1(t)y + a_2(t)y^2, \quad (6.82)$$

This system can be expressed as a t -dependent vector field (6.57) where

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, \quad (6.83)$$

so that these vector fields exactly reproduce those given in Table 1 for the class $I_4 \simeq \mathfrak{sl}(2)$ which, in turn, means that this system is locally diffeomorphic to the split-complex Riccati equation (6.70).

Example 13. The **Milne–Pinney equation** [287] was shown in (6.2) as a first-order system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\omega^2(t)x + \frac{c}{x^3}, \end{cases} \quad (6.84)$$

It has an associated t -dependent vector field $X = X_3 + \omega^2(t)X_1$, with vector fields

$$X_1 = -x \frac{\partial}{\partial y}, \quad X_2 = \frac{1}{2} \left(y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \right), \quad X_3 = y \frac{\partial}{\partial x} + \frac{c}{x^3} \frac{\partial}{\partial y}, \quad (6.85)$$

The vector fields X_1, X_2, X_3 span a Lie algebra isomorphic to $\mathfrak{sl}(2)$ with Lie brackets given by (6.4).

Proposition 3. *The system (6.2) is a Lie–Hamilton system of class P_2 for $c > 0$, I_4 for $c < 0$ and I_5 for $c = 0$ given in Table 1.*

Therefore, as the Cayley–Klein Riccati equation (6.69), the Milne–Pinney equations include the three possibilities of Vessiot–Guldberg Lie algebras isomorphic to $\mathfrak{sl}(2)$ of Hamiltonian vector fields.

Example 14. Reconsider the **second-order Kummer–Schwarz equation** studied in [87] and it was earlier shown in (2.38), which was rewritten in terms of a first-order system (2.40),

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = \frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 + 2b_1(t)x, \end{cases} \quad (6.86)$$

on $T\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$, obtained by adding the new variable $v \equiv dx/dt$ to the KS-2 equation (2.38).

This system has an associated t -dependent vector field $M = M_3 + \eta(t)M_1$, where the vector fields correspond with

$$M_1 = 2x \frac{\partial}{\partial v}, \quad M_2 = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad M_3 = v \frac{\partial}{\partial x} + \left(\frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 \right) \frac{\partial}{\partial v} \quad (6.87)$$

satisfy the commutation relations

$$[M_1, M_3] = 2M_2, \quad [M_1, M_2] = M_1, \quad [M_2, M_3] = M_3. \quad (6.88)$$

that span a Lie algebra isomorphic to $\mathfrak{sl}(2)$. It can be proven that (6.86) comprises, once more, the three Vessiot–Guldberg Lie algebras of Hamiltonian vector fields isomorphic to $\mathfrak{sl}(2)$ given in Table 1 according to the value of the parameter c [32].

Proposition 4. *The system (6.86) is a LH system of class P_2 for $c > 0$, I_4 for $c < 0$ and I_5 for $c = 0$ given in Table 1 in Appendix 1.*

Example 15. A diffusion equation can be transformed into a simpler PDE by solving a system of seven first-order ordinary differential equations (see [389] and [390, p. 104] for details). This system can be easily solved by integrating its projection to \mathbb{R}^2 , known as the **planar diffusion Riccati system** given by

$$\frac{dx}{dt} = -b(t) + 2c(t)x + 4a(t)x^2 + a(t)c_0y^4, \quad \frac{dy}{dt} = (c(t) + 4a(t)x)y, \quad (6.89)$$

where $a(t), b(t)$ and $c(t)$ are arbitrary t -dependent functions and $c_0 \in \{0, 1\}$. We call this system planar diffusion Riccati system. This system is related to the t -dependent vector field

$$X = a(t)X_3 - b(t)X_1 + c(t)X_2, \quad (6.90)$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = (4x^2 + c_0y^4) \frac{\partial}{\partial x} + 4xy \frac{\partial}{\partial y}, \quad (6.91)$$

satisfy the commutation relations

$$[X_1, X_2] = 2X_1, \quad [X_1, X_3] = 4X_2, \quad [X_2, X_3] = 2X_3. \quad (6.92)$$

Consequently, they span a Lie algebra isomorphic to $\mathfrak{sl}(2)$. For $c_0 = 1$ the change of variables

$$u = 2x + y^2, \quad v = 2x - y^2, \quad x = \frac{1}{4}(u + v), \quad y = \sqrt{(u - v)/2} \quad (6.93)$$

maps these vector fields to a basis of I_4 (see Table 1). Writing the symplectic structure and the Hamiltonian functions in the coordinate system $\{x, y\}$, we obtain

$$\omega = \frac{dx \wedge dy}{y^3}, \quad h_1 = -\frac{1}{2y^2}, \quad h_2 = -\frac{x}{y^2}, \quad h_3 = \frac{-2x^2}{y^2} + \frac{1}{2}y^2, \quad (6.94)$$

that satisfy

$$\{h_1, h_2\}_\omega = -2h_1, \quad \{h_1, h_3\}_\omega = -4h_2, \quad \{h_2, h_3\}_\omega = -2h_3. \quad (6.95)$$

For the case $c_0 = 0$, we have that the vector fields (6.91) form a basis of I_5 (see (6.79)). Hence, their associated symplectic form and some corresponding Hamiltonian functions can easily be obtained from Table 1.

Equivalence among $\mathfrak{sl}(2)$ -Lie-Hamilton systems

Consequently, by taking into account all the above results, we are led to the following statement.

Theorem 5. *The $\mathfrak{sl}(2)$ -Lie-Hamilton systems (2.40), (4.10), (6.2), (6.69) and (6.89) are equivalent through local diffeomorphisms whenever their Vessiot–Guldberg Lie algebras belong to the same class in Table 1, that is,*

- P_2 : Milne–Pinney and Kummer–Schwarz equations for $c > 0$ as well as complex Riccati equations.
- I_4 : Milne–Pinney and Kummer–Schwarz equations for $c < 0$, coupled Riccati equations, split-complex Riccati equations and the planar diffusion Riccati system with $c_0 = 1$.
- I_5 : Milne–Pinney and Kummer–Schwarz equations for $c = 0$ as well as dual-Study Riccati equations, planar diffusion Riccati system with $c_0 = 0$ and the harmonic oscillator with t -dependent frequency.

All of the above systems are considered to have t -dependent coefficients.

Only within each class, these systems are locally diffeomorphic and, therefore, there exists a local t -independent change of variables mapping one into another. For instance, there does not exist any diffeomorphism on \mathbb{R}^2 mapping the Milne–Pinney and Kummer–Schwarz equations with $c \neq 0$ to the harmonic oscillator with a t -dependent frequency as the latter correspond to set $c = 0$ and belong to class I_5 . Moreover, these results also explain the existence of the known diffeomorphism mapping Kummer–Schwarz equations to Milne–Pinney equations, which from our approach, should be understood in a ‘unified map’ [287], that is, the value of the parameter c should be considered and everything works whenever the same sign of c is preserved.

6.4 Two-photon Lie–Hamilton systems

In this section, we study two different Lie–Hamilton systems that belong to the same class P_5 in Table 1; these are a dissipative harmonic oscillator and the second-order Riccati equation in Hamiltonian form. As a consequence, there exists a diffeomorphism mapping one into the other when they are written as first-order systems.

The five generators X_1, \dots, X_5 written in Table 1 satisfy the Lie brackets

$$\begin{aligned} [X_1, X_2] &= 0, & [X_1, X_3] &= X_1, & [X_1, X_4] &= 0, & [X_1, X_5] &= X_2, \\ [X_2, X_3] &= -X_2, & [X_2, X_4] &= X_1, & [X_2, X_5] &= 0, & [X_3, X_4] &= -2X_4, \\ [X_3, X_5] &= 2X_5, & [X_4, X_5] &= -X_3, \end{aligned} \quad (6.96)$$

in such a manner that they span a Lie algebra isomorphic to $\mathfrak{sl}(2) \times \mathbb{R}^2$ where $\mathbb{R}^2 = \langle X_1, X_2 \rangle$ and $\mathfrak{sl}(2) = \langle X_3, X_4, X_5 \rangle$. Such vector fields are Hamiltonian ones with respect to the canonical symplectic form $\omega = dx \wedge dy$. Nevertheless, the corresponding Hamiltonian functions must be enlarged with a central generator $h_0 = 1$ giving rise to the centrally extended Lie algebra $\overline{\mathfrak{sl}(2) \times \mathbb{R}^2}$ which is, in fact, isomorphic to the two-photon Lie algebra $\mathfrak{h}_6 = \langle h_1, h_2, h_3, h_4, h_5, h_0 \rangle$ [31, 428]. That is why we shall call these systems *two-photon Lie–Hamilton systems*. The Lie brackets between these functions read

$$\begin{aligned} \{h_1, h_2\}_\omega &= h_0, & \{h_1, h_3\}_\omega &= -h_1, & \{h_1, h_4\}_\omega &= 0, & \{h_1, h_5\}_\omega &= -h_2, \\ \{h_2, h_3\}_\omega &= h_2, & \{h_2, h_4\}_\omega &= -h_1, & \{h_2, h_5\}_\omega &= 0, & \{h_3, h_4\}_\omega &= 2h_4, \\ \{h_3, h_5\}_\omega &= -2h_5, & \{h_4, h_5\}_\omega &= h_3, & \{h_0, \cdot\}_\omega &= 0. \end{aligned} \quad (6.97)$$

Notice that $\mathfrak{h}_6 \simeq \mathfrak{sl}(2) \times \mathfrak{h}_3$, where $\mathfrak{h}_3 \simeq \langle h_0, h_1, h_2 \rangle$ is the Heisenberg–Weyl Lie algebra and $\mathfrak{sl}(2) \simeq \langle h_3, h_4, h_5 \rangle$. Since $\mathfrak{h}_4 \simeq \langle h_0, h_1, h_2, h_3 \rangle$ is the harmonic oscillator algebra (isomorphic to $\overline{\mathfrak{iso}(1, 1)}$ in the class I_8), we have the embeddings $\mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \mathfrak{h}_6$.

Example 16. The t -dependent Hamiltonian for the **dissipative harmonic oscillator** studied in [113] is given by

$$h(t, q, p) = \alpha(t) \frac{p^2}{2} + \beta(t) \frac{pq}{2} + \gamma(t) \frac{q^2}{2} + \delta(t)p + \epsilon(t)q + \phi(t), \quad (6.98)$$

where $\alpha(t), \beta(t), \gamma(t), \delta(t), \epsilon(t), \phi(t)$ are real t -dependent functions. The corresponding Hamilton equations read

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial h}{\partial p} = \alpha(t)p + \frac{\beta(t)}{2}q + \delta(t), \\ \frac{dp}{dt} &= -\frac{\partial h}{\partial q} = -\left(\beta(t)\frac{p}{2} + \gamma(t)q + \epsilon(t)\right). \end{aligned} \quad (6.99)$$

This system has an associated t -dependent vector field

$$X = \delta(t)X_1 - \epsilon(t)X_2 + \frac{\beta(t)}{2}X_3 + \alpha(t)X_4 - \gamma(t)X_5 \quad (6.100)$$

where

$$\begin{aligned} X_1 &= \frac{\partial}{\partial q}, & X_2 &= \frac{\partial}{\partial p}, & X_3 &= q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}, \\ X_4 &= p\frac{\partial}{\partial q}, & X_5 &= q\frac{\partial}{\partial p}, \end{aligned} \quad (6.101)$$

are, up to a trivial change of variables $x = q$ and $y = p$, the vector fields of the basis of \mathbb{P}_5 given in Table 1. Hence, their Hamiltonian functions with respect to the symplectic structure $\omega = dq \wedge dp$ are indicated in Table 1.

Example 17. We reconsider the family of **second-order Riccati equation** (2.23) which arose by reducing third-order linear differential equations through a dilation symmetry and a t -reparametrization [115]. The crucial point is that a quite general family of second-order Riccati equations (2.23) admits a t -dependent Hamiltonian (see [115, 308] for details) given by

$$h(t, x, p) = -2\sqrt{-p} - p(a_0(t) + a_1(t)x + a_2(t)x^2), \quad p < 0, \quad (6.102)$$

where $a_0(t), a_1(t), a_2(t)$ are certain functions related to the t -dependent coefficients of (2.23). The corresponding Hamilton equations are (2.28) so that the associated t -dependent vector field has the expression

$$X = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4, \quad (6.103)$$

such these vector fields correspond with (2.29) and close the commutation relations (2.30), where

$$\begin{aligned} Y_1 &= \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, & Y_2 &= \frac{\partial}{\partial x}, & Y_3 &= x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \\ Y_4 &= x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p}. \end{aligned} \quad (6.104)$$

Another vector field

$$Y_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p} \quad (6.105)$$

If we set

$$\begin{aligned} Y_1 &= -\frac{1}{\sqrt{2}}X_5, & Y_2 &= \frac{1}{\sqrt{2}}X_1, & Y_3 &= -2X_3, \\ Y_4 &= -X_4, & Y_5 &= X_2, \end{aligned} \quad (6.106)$$

we find that the commutation relations for Y_1, \dots, Y_5 coincide with (6.96) and therefore span a Lie algebra isomorphic to $\mathfrak{sl}(2) \times \mathbb{R}^2$. Indeed, it is a Lie–Hamilton system

endowed with a canonical symplectic form $\omega = dx \wedge dp$. The Hamiltonian functions corresponding to the vector fields (2.29) turn out to be

$$\begin{aligned}\tilde{h}_1 &= -2\sqrt{-p}, & \tilde{h}_2 &= p, & \tilde{h}_3 &= xp, \\ \tilde{h}_4 &= x^2p, & \tilde{h}_5 &= -2x\sqrt{-p},\end{aligned}\quad (6.107)$$

which span along with $\tilde{h}_0 = 1$ a Lie algebra of functions isomorphic to the two-photon Lie algebra \mathfrak{h}_6 [35, 109] with non-vanishing Poisson brackets given by

$$\begin{aligned}\{\tilde{h}_1, \tilde{h}_3\}_\omega &= -\frac{1}{2}\tilde{h}_1, & \{\tilde{h}_1, \tilde{h}_4\}_\omega &= -\tilde{h}_5, & \{\tilde{h}_1, \tilde{h}_5\}_\omega &= 2\tilde{h}_0, \\ \{\tilde{h}_2, \tilde{h}_3\}_\omega &= -\tilde{h}_2, & \{\tilde{h}_2, \tilde{h}_4\}_\omega &= -2\tilde{h}_3, & \{\tilde{h}_2, \tilde{h}_5\}_\omega &= -\tilde{h}_1, \\ \{\tilde{h}_3, \tilde{h}_4\}_\omega &= -\tilde{h}_4, & \{\tilde{h}_3, \tilde{h}_5\}_\omega &= -\frac{1}{2}\tilde{h}_5.\end{aligned}\quad (6.108)$$

Under the identification

$$\begin{aligned}h_1 &= -\frac{1}{\sqrt{2}}\tilde{h}_5, & h_2 &= \frac{1}{\sqrt{2}}\tilde{h}_1, & h_3 &= -2\tilde{h}_3, & h_4 &= -\tilde{h}_4, \\ h_5 &= \tilde{h}_2, & h_0 &= \tilde{h}_0,\end{aligned}\quad (6.109)$$

we recover the Poisson brackets (6.97). Furthermore, the expressions for the vector fields and their Hamiltonian functions written in Table 1 for the class P_5 , in variables $\{u, v\}$, can be obtained from X_1, \dots, X_5 by applying the change of variables defined by

$$u = \sqrt{-2p}, \quad v = x\sqrt{-2p}, \quad x = \frac{v}{u}, \quad p = -\frac{1}{2}u^2, \quad (6.110)$$

such that $\omega = dx \wedge dp := du \wedge dv$. Notice that $p/\sqrt{-p} = -\sqrt{-p}$ since $p < 0$.

The main results of this section are then summarized as follows.

Proposition 6. *The dissipative harmonic oscillator (6.98) and the Hamilton equations (2.28) corresponding to the second-order Riccati equation (2.23) are P_5 -Lie-Hamilton systems with Lie-Hamilton algebra isomorphic to the two-photon one \mathfrak{h}_6 . Consequently, all are locally diffeomorphic.*

6.5 \mathfrak{h}_2 -Lie-Hamilton systems

Let us study class $I_{14A} \simeq \mathbb{R} \times \mathbb{R}^r$ with $r = 1$ of Table 1. It admits a basis of vector fields $X_1 = \partial_x$ and $X_2 = \eta_1(x)\partial_y$. If we require that these two vector fields close a non-abelian Lie algebra, we choose $[X_1, X_2] = X_2$, with no loss of generality, then $\eta_1(x) = e^x$, that is:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = e^x \frac{\partial}{\partial y}, \quad [X_1, X_2] = X_2, \quad (6.111)$$

that we denote by $\mathfrak{h}_2 \simeq \mathbb{R} \times \mathbb{R}$. This is a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields when endowed with the canonical symplectic form $\omega = dx \wedge dy$. Hence, we can choose:

$$h_1 = y, \quad h_2 = -e^x, \quad \{h_1, h_2\}_\omega = -h_2. \quad (6.112)$$

In the following, we prove that the complex Bernoulli equation with t -dependent real coefficients has an underlying \mathfrak{h}_2 -Lie algebra. Furthermore, we relate this fact with the generalized Buchdahl equations and t -dependent Lotka–Volterra systems. It is remarkable that all particular cases of Cayley–Klein Riccati equations (6.69) with $a_2(t) = 0$ are \mathfrak{h}_2 -Lie–Hamilton systems. Finally, we prove that all of these systems are locally diffeomorphic and belong to the same class $I_{14A}^{r=1} \simeq \mathfrak{h}_2$.

Example 18. The **complex Bernoulli equation** [317] with t -dependent real coefficients takes the form

$$\frac{dz}{dt} = a_1(t)z + a_2(t)z^n, \quad n \notin \{0, 1\}, \quad (6.113)$$

where $z \in \mathbb{C}$ and $a_1(t), a_2(t)$ are arbitrary t -dependent real functions. This is a generalization to the complex numbers of the usual Bernoulli equation. By writing $z := x + iy$, equation (6.113) reads

$$\frac{dx}{dt} = a_1(t)x + a_2(t) \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} x^{n-k} i^k y^k, \quad (6.114)$$

$$\frac{dy}{dt} = a_1(t)y + a_2(t) \sum_{\substack{k=1 \\ k \text{ odd}}}^n \binom{n}{k} x^{n-k} i^{k-1} y^k. \quad (6.115)$$

This can be studied in terms of the t -dependent vector field $X = a_1(t)X_1 + a_2(t)X_2$, where

$$\begin{aligned} X_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \\ X_2 &= \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} x^{n-k} i^k y^k \frac{\partial}{\partial x} + \sum_{\substack{k=1 \\ k \text{ odd}}}^n \binom{n}{k} x^{n-k} i^{k-1} y^k \frac{\partial}{\partial y} \end{aligned} \quad (6.116)$$

and their commutation relation is

$$[X_1, X_2] = (n-1)X_2, \quad (6.117)$$

which is isomorphic to \mathfrak{h}_2 . In the GKO classification [32, 210] there is just one Lie algebra isomorphic to \mathfrak{h}_2 whose vector fields are not proportional at each point: I_{14A} with $r = 1$. So, $\langle X_1, X_2 \rangle$ is a Lie algebra of Hamiltonian vector fields in view of the results of Table 2.

Let us inspect the case $n = 2$. So, the vector fields in (6.116) read $X_1 = x\partial_x + y\partial_y$ and $X_2 = (x^2 - y^2)\partial_x + 2xy\partial_y$, and become Hamiltonian with respect to the symplectic form and Hamiltonian functions

$$\omega = \frac{dx \wedge dy}{y^2}, \quad h_1 = -\frac{x}{y}, \quad h_2 = -\frac{x^2 + y^2}{y}. \quad (6.118)$$

Additionally, $\{h_1, h_2\}_\omega = -h_2$. Notice that, in this very particular case, the Hamiltonian functions become in the form given in Table 2 for the positive Borel subalgebra of $P_2 \simeq \mathfrak{sl}(2)$. In fact, for $n = 2$ the equation (6.113) is a particular case of the complex Riccati equation (2.7) with $a_0 = 0$.

Let us now take a closer look at $n = 3$. So, the vector field X_2 read

$$X_2 = (x^3 - 3xy^2) \frac{\partial}{\partial x} + (3x^2y - y^3) \frac{\partial}{\partial y} \quad (6.119)$$

and it can be shown that X_1 and X_2 become Hamiltonian vector fields with respect to the symplectic form

$$\omega = \frac{x^2 + y^2}{x^2y^2} dx \wedge dy. \quad (6.120)$$

Moreover, X_1 and X_2 admit Hamiltonian functions

$$h_1 = \frac{y^2 - x^2}{xy}, \quad h_2 = -\frac{(x^2 + y^2)^2}{xy}, \quad (6.121)$$

satisfying $\{h_1, h_2\}_\omega = -2h_2$. Other new results come out by setting $n > 3$.

Example 19. The **generalized Buchdahl equations** [73, 118, 122] are the second-order differential equations given by (6.19) which can be rewritten in terms of a first-order system (6.20),

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = a(x)y^2 + b(t)y. \end{cases} \quad (6.122)$$

such that $X = X_1 + b(t)X_2$, where the vector fields corresponded with (6.21).

These vector fields span a Lie algebra diffeomorphic to $I_{14A}^{r=1} \simeq \mathfrak{h}_2$ so that (6.122) is a Lie–Hamilton system. The corresponding symplectic form and Hamiltonian functions can be found in [32].

Example 20. Finally, consider the **particular Lotka–Volterra systems** [263, 400] given by

$$\frac{dx}{dt} = ax - g(t)(x - ay)x, \quad \frac{dy}{dt} = ay - g(t)(bx - y)y, \quad a \neq 0, \quad (6.123)$$

where $g(t)$ determines the variation of the seasons, while a and b are constants describing the interactions among the species. System (6.123) is associated with the t -dependent vector field $X = X_1 + g(t)X_2$ where

$$X_1 = ax \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}, \quad X_2 = -(x - ay)x \frac{\partial}{\partial x} - (bx - y)y \frac{\partial}{\partial y}, \quad (6.124)$$

satisfy

$$[X_1, X_2] = aX_2, \quad a \neq 0. \quad (6.125)$$

Hence, (6.123) is a Lie system. Moreover, it has been proven in [32] that, except for the case with $a = b = 1$, this is also a LH system belonging to the family $I_{14A}^{r=1} \simeq \mathfrak{h}_2$.

Hence, we conclude this section with the following statement.

Proposition 7. *The complex Bernoulli equation (6.115), the generalized Buchdahl equations (6.122) and the t -dependent Lotka–Volterra systems (6.123) (with the exception of $a = b = 1$) are LH systems with a Vessiot–Guldberg Lie algebra diffeomorphic to $I_{14A}^{r=1} \simeq \mathfrak{h}_2$ in Table 1. Thus all of these systems are locally diffeomorphic.*

7 Dirac–Lie systems

A step further on the study of structure-Lie systems is the consideration of Dirac–Lie systems. The introduction of this type of system presents some advantages over different approaches to certain physical systems. In the following paragraph, the importance of Dirac–Lie systems will be clearly stated.

7.1 Motivation for Dirac–Lie systems

Dirac–Lie systems arise due to the existence of certain Lie systems which do not have a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Poisson structure, but they do with respect to a presymplectic one. A possible structure incorporating the presymplectic and Poisson structure is the Dirac structure. We will illustrate this fact with the following example.

Example 1. Let us reconsider the **third-order Kummer–Schwarz equation** [52, 216],

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = a, \\ \frac{da}{dt} = \frac{3}{2} \frac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v, \end{cases} \quad (7.1)$$

on the open submanifold $\mathcal{O}_2 = \{(x, v, a) \in T^2\mathbb{R} \mid v \neq 0\}$ of $T^2\mathbb{R} \simeq \mathbb{R}^3$. Its associated t -dependent vector field is $X^{3KS} = N_3 + b_1(t)N_1$.

If $b_1(t)$ is not a constant, the Vessiot–Guldberg Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ on $\mathcal{O}_2 = \{(x, v, a) \in T^2\mathbb{R} \mid v \neq 0\}$ made up with vector fields (2.65). Hence, X^{3KS} is a Lie system.

However, X^{3KS} is not a Lie–Hamilton system when $b_1(t)$ is not a constant. Indeed, in this case $\mathcal{D}^{X^{3KS}}$ coincides with $T\mathcal{O}_2$ on \mathcal{O}_2 . If X^{3KS} were also a Lie–Hamilton system with respect to (N, Λ) , then $V^{X^{3KS}}$ would consist of Hamiltonian vector fields and the characteristic distribution associated to Λ would have odd-dimensional rank on \mathcal{O}_2 . This is impossible, as the local Hamiltonian vector fields of a Poisson manifold span a generalized distribution of even rank at each point. Our previous argument can easily be generalized to formulate the following ‘no-go’ theorem.

Theorem 1. (Lie–Hamilton no-go Theorem) *If X is a Lie system on an odd-dimensional manifold N satisfying that $\mathcal{D}_{x_0}^X = T_{x_0}N$ for a point x_0 in N , then X is not a Lie–Hamilton system on N .*

Note that from the properties of r^X it follows that, if $\mathcal{D}_{x_0}^X = T_{x_0}N$ for a point x_0 , then $\mathcal{D}_x^X = T_xN$ for x in an open neighborhood $U_{x_0} \ni x_0$. Hence, we can merely consider whether X is a Lie–Hamilton system on $N \setminus U_{x_0}$.

Despite the previous negative results, system (2.64) admits another interesting property: we can endow the manifold \mathcal{O}_2 with a presymplectic form ω_{3KS} in such a way that $V^{X^{3KS}}$ consists of Hamiltonian vector fields with respect to it. Indeed, by considering the equations $\mathcal{L}_{Y_1}\omega_{3KS} = \mathcal{L}_{Y_2}\omega_{3KS} = \mathcal{L}_{Y_3}\omega_{3KS} = 0$ and $d\omega_{3KS} = 0$, we can readily find the presymplectic form

$$\omega_{3KS} = \frac{dv \wedge da}{v^3} \quad (7.2)$$

on \mathcal{O}_2 . Additionally, we see that

$$\iota_{Y_1}\omega_{3KS} = d\left(\frac{2}{v}\right), \quad \iota_{Y_2}\omega_{3KS} = d\left(\frac{a}{v^2}\right), \quad \iota_{Y_3}\omega_{3KS} = d\left(\frac{a^2}{2v^3} + 2c_0v\right). \quad (7.3)$$

So, the system X^{3KS} becomes a Lie system with a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to ω_{3KS} . As seen later on, systems of this type can be studied through appropriate generalizations of the methods employed to investigate Lie–Hamilton systems.

Example 2. Another example of a Lie system which is not a Lie–Hamilton system but admits a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect

to a presymplectic form is the **Riccati system**

$$\left\{ \begin{array}{ll} \frac{ds}{dt} = -4a(t)us - 2d(t)s, & \frac{dx}{dt} = (c(t) + 4a(t)u)x + f(t) - 2ug(t), \\ \frac{du}{dt} = -b(t) + 2c(t)u + 4a(t)u^2, & \frac{dy}{dt} = (2a(t)x - g(t))v, \\ \frac{dv}{dt} = (c(t) + 4a(t)u)v, & \frac{dz}{dt} = a(t)x^2 - g(t)x, \\ \frac{dw}{dt} = a(t)v^2, & \end{array} \right. \quad (7.4)$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$ and $g(t)$ are arbitrary t -dependent functions. The interest of this system is due to its use in solving diffusion-type equations, Burger's equations, and other PDEs [389].

Since every particular solution $(s(t), u(t), v(t), w(t), x(t), y(t), z(t))$ of (7.4), with $v(t_0) = 0$ ($s(t_0) = 0$) for a certain $t_0 \in \mathbb{R}$, satisfies $v(t) = 0$ ($s(t) = 0$) for every t , we can restrict ourselves to analyzing system (7.4) on the submanifold $M = \{(s, u, v, w, x, y, z) \in \mathbb{R}^7 \mid v \neq 0, s \neq 0\}$. This will simplify the application of our techniques without omitting any relevant detail.

System (7.4) describes integral curves of the t -dependent vector field

$$X_t^{RS} = a(t)X_1 - b(t)X_2 + c(t)X_3 - 2d(t)X_4 + f(t)X_5 + g(t)X_6, \quad (7.5)$$

where

$$\begin{aligned} X_1 &= -4us \frac{\partial}{\partial s} + 4u^2 \frac{\partial}{\partial u} + 4uv \frac{\partial}{\partial v} + v^2 \frac{\partial}{\partial w} + 4ux \frac{\partial}{\partial x} + 2xv \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z}, \\ X_2 &= \frac{\partial}{\partial u}, \quad X_3 = 2u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + x \frac{\partial}{\partial x}, \quad X_4 = s \frac{\partial}{\partial s}, \quad X_5 = \frac{\partial}{\partial x}, \\ X_6 &= -2u \frac{\partial}{\partial x} - v \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad X_7 = \frac{\partial}{\partial z}. \end{aligned} \quad (7.6)$$

Their commutation relations are

$$\begin{aligned} [X_1, X_2] &= 4(X_4 - X_3), \quad [X_1, X_3] = -2X_1, \quad [X_1, X_5] = 2X_6, \quad [X_1, X_6] = 0, \\ [X_2, X_3] &= 2X_2, \quad [X_2, X_5] = 0, \quad [X_2, X_6] = -2X_5, \\ [X_3, X_5] &= -X_5, \quad [X_3, X_6] = X_6, \\ [X_5, X_6] &= -X_7, \end{aligned} \quad (7.7)$$

and X_4 and X_7 commute with all the vector fields. Hence, system (7.4) is a Lie system associated to a Vessiot–Guldberg Lie algebra V isomorphic to $(\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{h}_2) \oplus \mathbb{R}$, where $\mathfrak{sl}(2, \mathbb{R}) \simeq \langle X_1, X_2, X_4 - X_3 \rangle$, $\mathfrak{h}_2 \simeq \langle X_5, X_6, X_7 \rangle$ and $\mathbb{R} \simeq \langle X_4 \rangle$. It is worth noting that this new example of Lie system is one of the few Lie systems related to remarkable PDEs until now [91].

Observe that (7.4) is not a Lie–Hamilton system when $V^{X^{RS}} = V$. In this case $\mathcal{D}_p^{X^{RS}} = T_pM$ for any $p \in M$ and, in view of Theorem 1 and the fact that $\dim T_pM = 7$, the system X^{RS} is not a Lie–Hamilton system on M .

Nevertheless, we can look for a presymplectic form turning X^{RS} into a Lie system with a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields. Looking for a non-trivial solution of the system of equations $\mathcal{L}_{X_k}\omega_{RS} = 0$, with $k = 1, \dots, 7$, and $d\omega_{RS} = 0$, one can find the presymplectic two-form

$$\omega_{RS} = -\frac{4wdu \wedge dw}{v^2} + \frac{dv \wedge dw}{v} + \frac{4w^2du \wedge dv}{v^3}. \quad (7.8)$$

In addition, we can readily see that $d\omega_{RS} = 0$ and X_1, \dots, X_7 are Hamiltonian vector fields

$$\begin{aligned} \iota_{X_1}\omega_{RS} &= d\left(4uw - \frac{8u^2w^2}{v^2} - \frac{v^2}{2}\right), & \iota_{X_2}\omega_{RS} &= -d\left(\frac{2w^2}{v^2}\right), \\ \iota_{X_3}\omega_{RS} &= d\left(w - \frac{4w^2u}{v^2}\right), \end{aligned} \quad (7.9)$$

and $\iota_{X_k}\omega_{RS} = 0$ for $k = 4, \dots, 7$.

Apart from the above examples, other non Lie–Hamilton systems that admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a presymplectic form can be found in the study of certain reduced Ermakov systems [226], Wei–Norman equations for dissipative quantum oscillators [102], and $\mathfrak{sl}(2, \mathbb{R})$ –Lie systems [353].

A straightforward generalization of the concept of a Lie–Hamilton system to Dirac manifolds would be a Lie system admitting a Vessiot–Guldberg Lie algebra V of vector fields for which there exists a Dirac structure L such that V consists of L –Hamiltonian vector fields.

7.2 Dirac–Lie Hamiltonians

In view of Theorem 7, every Lie–Hamilton system admits a Lie–Hamiltonian. Since Dirac–Lie systems are generalizations of these systems, it is natural to investigate whether Dirac–Lie systems admit an analogous property.

Definition 2. A *Dirac–Lie Hamiltonian structure* is a triple (N, L, h) , where (N, L) stands for a Dirac manifold and h represents a t –parametrized family of admissible functions $h_t : N \rightarrow \mathbb{R}$ such that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L)$ is a finite-dimensional real Lie algebra. A t –dependent vector field X is said to admit, to have or to possess a Dirac–Lie Hamiltonian (N, L, h) if $X_t + dh_t \in \Gamma(L)$ for all $t \in \mathbb{R}$.

Note 3. For simplicity, we hereafter call Dirac–Lie Hamiltonian structures Dirac–Lie Hamiltonians.

From the above definition, we see that system (2.64) related to the third-order Kummer–Schwarz equations possesses a Dirac–Lie Hamiltonian $(N, L^{\omega_{3KS}}, h^{3KS})$ and system (7.4), used to analyze diffusion equations, admits a Dirac–Lie Hamiltonian $(N, L^{\omega_{RS}}, h^{RS})$.

Consider the third-order Kummer–Schwarz equation in first-order form (2.64). Remind that N_1 , N_2 , and N_3 are Hamiltonian vector fields with respect to the presymplectic manifold $(\mathcal{O}_2, \omega_{3KS})$. It follows from relations (7.3) that the vector fields N_1 , N_2 , and N_3 have Hamiltonian functions

$$h_1 = -\frac{2}{v}, \quad h_2 = -\frac{a}{v^2}, \quad h_3 = -\frac{a^2}{2v^3} - 2c_0v, \quad (7.10)$$

respectively. Moreover,

$$\{h_1, h_3\} = 2h_2, \quad \{h_1, h_2\} = h_1, \quad \{h_2, h_3\} = h_3, \quad (7.11)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on $\text{Adm}(\mathcal{O}_2, \omega_{3KS})$ induced by ω_{3KS} . In consequence, h_1, h_2 , and h_3 span a finite-dimensional real Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Thus, every X_t^{3KS} is a Hamiltonian vector field with Hamiltonian function $h_t^{3KS} = h_3 + b_1(t)h_1$ and the space $\text{Lie}(\{h_t^{3KS}\}_{t \in \mathbb{R}}, \{\cdot, \cdot\})$ becomes a finite-dimensional real Lie algebra. This enables us to associate X^{3KS} to a curve in $\text{Lie}(\{h_t^{3KS}\}_{t \in \mathbb{R}}, \{\cdot, \cdot\})$. The similarity of $(\mathcal{O}_2, \omega_{3KS}, h^{3KS})$ with Lie–Hamiltonians are immediate.

If we now turn to the Riccati system (7.4), we will see that we can obtain a similar result. More specifically, relations (7.9) imply that X_1, \dots, X_7 have Hamiltonian functions

$$h_1 = \frac{(v^2 - 4uw)^2}{2v^2}, \quad h_2 = \frac{2\omega^2}{v^2}, \quad h_3 = \frac{4w^2u}{v^2} - w, \quad (7.12)$$

and $h_4 = h_5 = h_6 = h_7 = 0$. Moreover, given the Poisson bracket on admissible functions induced by ω_{3KS} , we see that

$$\{h_1, h_2\} = -4h_3, \quad \{h_1, h_3\} = -2h_1, \quad \{h_2, h_3\} = 2h_2. \quad (7.13)$$

Hence, h_1, \dots, h_7 span a real Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and, as in the previous case, the t -dependent vector fields X_t^{RS} possess Hamiltonian functions $h_t^{RS} = a(t)h_1 - b(t)h_2 + c(t)h_3$. Again, we can associate X^{RS} to a curve $t \mapsto h_t^{RS}$ in the finite-dimensional real Lie algebra $(\text{Lie}(\{h_t^{RS}\}_{t \in \mathbb{R}}), \{\cdot, \cdot\})$.

Let us analyze the properties of Dirac–Lie structures. Observe first that there may be several systems associated with the same Dirac–Lie Hamiltonian. For instance, the systems X^{RS} and

$$\begin{aligned} X_2^{RS} &= a(t)X_1 - b(t)X_2 + c(t)X_3 - 2d(t)X_4 \\ &\quad + f(t)z^3X_5 + g(t)X_6 + h(t)z^2X_7 \end{aligned} \quad (7.14)$$

admit the same Dirac–Lie Hamiltonian $(N, L^{\omega_{RS}}, h^{RS})$. It is remarkable that X_2^{RS} is not even a Lie system in general. Indeed, in view of

$$[z^2 X_7, z^n X_5] = n z^{n+1} X_5, \quad n = 3, 4, \dots, \quad (7.15)$$

we easily see that the successive Lie brackets of $z^n X_5$ and $z^2 X_7$ span an infinite set of vector fields which are linearly independent over \mathbb{R} . So, in those cases in which X_5 and X_7 belong to $V^{X_2^{RS}}$, this Lie algebra becomes infinite-dimensional.

In the case of a Dirac–Lie system, Proposition 5 in Chapter 2 shows easily the following.

Corollary 4. *Let (N, L, X) be a Dirac–Lie system admitting a Dirac–Lie Hamiltonian (N, L, h) . Then, we have the exact sequence of Lie algebras*

$$0 \hookrightarrow \text{Cas}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L) \hookrightarrow \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L) \xrightarrow{BL} \pi(V^X) \rightarrow 0, \quad (7.16)$$

where $\text{Cas}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L) = \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L) \cap \text{Cas}(N, L)$. In other words, we have that $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L)$ is a Lie algebra extension of the space $\pi(V^X)$ by $\text{Cas}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L)$.

Theorem 5. *Each Dirac–Lie system (N, L, X) admits a Dirac–Lie Hamiltonian (N, L, h) .*

Proof. Since $V^X \subset \text{Ham}(N, L)$ is a finite-dimensional Lie algebra, we can define a linear map $T : X_f \in V^X \mapsto f \in C^\infty(N)$ associating each L -Hamiltonian vector field in V^X with an associated L -Hamiltonian function, e.g., given a basis X_1, \dots, X_r of V^X we define $T(X_i) = h_i$, with $i = 1, \dots, r$, and extend T to V^X by linearity. Note that the functions h_1, \dots, h_r need not be linearly independent over \mathbb{R} , as a function can be Hamiltonian for two different L -Hamiltonian vector fields X_1 and X_2 when $X_1 - X_2 \in \mathfrak{G}(N, L)$. Given the system X , there exists a smooth curve $h_t = T(X_t)$ in $\mathfrak{W}_0 \equiv \text{Im } T$ such that $X_t + dh_t \in \Gamma(L)$. To ensure that h_t gives rise to a Dirac–Lie Hamiltonian, we need to demonstrate that $\dim \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L) < \infty$. This will be done by constructing a finite-dimensional Lie algebra of functions containing the curve h_t .

Consider two elements $Y_1, Y_2 \in V^X$. Note that the functions $\{T(Y_1), T(Y_2)\}_L$ and $T([Y_1, Y_2])$ have the same L -Hamiltonian vector field. So, $\{T(Y_1), T(Y_2)\}_L - T([Y_1, Y_2]) \in \text{Cas}(N, L)$ and, in view of Proposition 5 from Chapter 2, it Poisson commutes with all other admissible functions. Let us define $\Upsilon : V^X \times V^X \rightarrow C^\infty(N)$ of the form

$$\Upsilon(X_1, X_2) = \{T(X_1), T(X_2)\}_L - T[X_1, X_2]. \quad (7.17)$$

The image of Υ is contained in a finite-dimensional real Abelian Lie subalgebra of $\text{Cas}(N, L)$ of the form

$$\mathfrak{W}_c \equiv \langle \Upsilon(X_i, X_j) \rangle, \quad (7.18)$$

for $i, j = 1, \dots, r$, and being X_1, \dots, X_r a basis for V^X . From here, it follows that

$$\{\mathfrak{W}_C, \mathfrak{W}_C\}_L = 0, \quad \{\mathfrak{W}_C, \mathfrak{W}_0\}_L = 0, \quad \{\mathfrak{W}_0, \mathfrak{W}_0\}_L \subset \mathfrak{W}_C + \mathfrak{W}_0. \quad (7.19)$$

Hence, $(\mathfrak{W} \equiv \mathfrak{W}_0 + \mathfrak{W}_C, \{\cdot, \cdot\}_L)$ is a finite-dimensional real Lie algebra containing the curve h_t , and X admits a Dirac–Lie Hamiltonian $(N, L, T(X_t))$. \square

The following proposition is easy to verify.

Proposition 6. *Let (N, L, X) be a Dirac–Lie system. If (N, L, h) and (N, L, \bar{h}) are two Dirac–Lie Hamiltonians for (N, L, X) , then*

$$h = \bar{h} + f^X, \quad (7.20)$$

where $f^X \in C^\infty(\mathbb{R} \times N)$ is a t -dependent function such that each $f_t^X : x \in N \mapsto f^X(x, t) \in \mathbb{R}$ is a Casimir function that is constant on every integral manifold \mathcal{O} of \mathcal{D}^X .

Note that if we have a Dirac–Lie Hamiltonian (N, L, h) and we define a linear map $\hat{T} : h \in \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}) \mapsto X_h \in \text{Ham}(N, L)$, the space $\hat{T}(\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}))$ may span an infinite-dimensional Lie algebra of vector fields. For instance, consider again the Lie–Hamiltonian $(\mathcal{O}_2, \omega_{3KS}, h_t^{3KS} = h_3 + b_1(t)h_1)$ for the system (2.64). The functions h_1, h_2 , and h_3 are also Hamiltonian for the vector fields

$$\begin{aligned} Y_1 &= 2v \frac{\partial}{\partial a} + e^{v^2} \frac{\partial}{\partial x}, & Y_2 &= v \frac{\partial}{\partial v} + 2a \frac{\partial}{\partial a}, \\ Y_3 &= a \frac{\partial}{\partial v} + \left(\frac{3}{2} \frac{a^2}{v} - 2c_0 v^3 \right) \frac{\partial}{\partial a}, \end{aligned} \quad (7.21)$$

which satisfy

$$\overbrace{[Y_2, [\dots, [Y_2, Y_1] \dots]]}^{j\text{-times}} = f_j(v) \frac{\partial}{\partial x} + 2(-1)^j v \frac{\partial}{\partial a}, \quad f_j(v) \equiv \overbrace{v \frac{\partial}{\partial v} \dots v \frac{\partial}{\partial v}}^{j\text{-times}}(e^{v^2}). \quad (7.22)$$

In consequence, $\text{Lie}(\hat{T}(\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\})), [\cdot, \cdot])$ contains an infinite-dimensional Lie algebra of vector fields because the functions $\{f_j\}_{j \in \mathbb{R}}$ form an infinite family of linearly independent functions over \mathbb{R} . So, we need to impose additional conditions to ensure that the image of \hat{T} is finite-dimensional.

The following theorem yields an alternative definition of a Dirac–Lie system.

Theorem 7. *Given a Dirac manifold (N, L) , the triple (N, L, X) is a Dirac–Lie system if and only if there exists a curve $\gamma : t \in \mathbb{R} \rightarrow \gamma_t \in \Gamma(L)$ satisfying that $\rho(\gamma_t) = X_t \in \text{Ham}(N, L)$ for every $t \in \mathbb{R}$ and $\text{Lie}(\{\gamma_t\}_{t \in \mathbb{R}}, [[\cdot, \cdot]]_C)$ is a finite-dimensional real Lie algebra.*

Proof. Let us prove the direct part of the theorem. Assume that (N, L, X) is a Dirac–Lie system. In virtue of Theorem 5, it admits a Dirac–Lie Hamiltonian (N, L, h) , with $h_t = T(X_t)$ and $T : V^X \rightarrow \text{Adm}(N, L)$ a linear morphism associating each element of V^X with one of its L -Hamiltonian functions. We aim to prove that the curve in $\Gamma(L)$ of the form $\gamma_t = X_t + d(T(X_t))$ satisfies that $\dim \text{Lie}(\{\gamma_t\}_{t \in \mathbb{R}}, [[\cdot, \cdot]]_C) < \infty$.

The sections of $\Gamma(L)$ of the form

$$X_1 + dT(X_1), \dots, X_r + dT(X_r), d\Upsilon(X_i, X_j), \quad (7.23)$$

with $i, j = 1, \dots, r$, X_1, \dots, X_r is a basis of V^X and $\Upsilon : V^X \times V^X \rightarrow \text{Cas}(N, L)$ is the map (7.17), span a finite-dimensional Lie algebra $(E, [[\cdot, \cdot]]_C)$. Indeed,

$$[[X_i + dT(X_i), X_j + dT(X_j)]]_C = [X_i, X_j] + d\{T(X_i), T(X_j)\}_L, \quad (7.24)$$

for $i, j = 1, \dots, r$. Taking into account that $\{T(X_i), T(X_j)\}_L - T([X_i, X_j]) = \Upsilon(X_i, X_j)$, we see that the above is a linear combination of the generators (7.23). Additionally, we have that

$$[[X_i + dT(X_i), d\Upsilon(X_j, X_k)]]_C = d\{T(X_i), \Upsilon(X_j, X_k)\}_L = 0. \quad (7.25)$$

So, sections (7.23) span a finite-dimensional subspace E of $(\Gamma(L), [[\cdot, \cdot]]_C)$. As $\gamma_t \in E$, for all $t \in \mathbb{R}$, we conclude the direct part of the proof.

The converse is straightforward from the fact that $(L, [[\cdot, \cdot]]_C, \rho)$ is a Lie algebroid. Indeed, given the curve γ_t within a finite-dimensional real Lie algebra of sections E satisfying that $X_t = \rho(\gamma_t) \in \text{Ham}(N, L)$, we have that $\{X_t\}_{t \in \mathbb{R}} \subset \rho(E)$ are L -Hamiltonian vector fields. As E is a finite-dimensional Lie algebra and ρ is a Lie algebra morphism, $\rho(E)$ is a finite-dimensional Lie algebra of vector fields and (N, L, X) becomes a Dirac–Lie system. \square

The above theorem shows the interest of defining a class of Lie systems related to general Lie algebroids.

7.3 Diagonal prolongations

We analyze the properties of diagonal prolongations of Dirac–Lie systems. As a result, we discover new features that can be applied to study their superposition rules. The process will be similar to the one described in Section 3 for prolongations and the coalgebra method. Let $\tau : E \rightarrow N$ be a vector bundle. Its *diagonal prolongation* to N^m is the Cartesian product bundle $E^{[m]} = E \times \dots \times E$ of m copies of E , viewed as a vector bundle over N^m in a natural way

$$E_{(x_{(1)}, \dots, x_{(m)})}^{[m]} = E_{x_{(1)}} \oplus \dots \oplus E_{x_{(m)}}. \quad (7.26)$$

Every section $X : N \rightarrow E$ of E has a natural *diagonal prolongation* to a section $X^{[m]}$ of $E^{[m]}$

$$X^{[m]}(x_{(1)}, \dots, x_{(m)}) = X(x_{(1)}) + \dots + X(x_{(m)}). \quad (7.27)$$

Given a function $f : N \rightarrow \mathbb{R}$, we call *diagonal prolongation* of f to N^m the function $\tilde{f}^{[m]}$ on N^m of the form $\tilde{f}^{[m]}(x_{(1)}, \dots, x_{(m)}) = f(x_{(1)}) + \dots + f(x_{(m)})$.

We can consider also sections $X^{(j)}$ of $E^{[m]}$ given by

$$X^{(j)}(x_{(1)}, \dots, x_{(m)}) = 0 + \dots + X(x_{(j)}) + \dots + 0. \quad (7.28)$$

It is clear that, if $\{X_i \mid i = 1, \dots, p\}$ is a basis of local sections of E , then $\{X_i^{(j)} \mid i = 1, \dots, p, j = 1, \dots, m\}$ is a basis of local sections of $E^{[m]}$. Note that all this can be repeated also for *generalized vector bundles*, like generalized distributions.

Since there are obvious canonical isomorphisms

$$(TN)^{[m]} \simeq TN^m \quad \text{and} \quad (T^*N)^{[m]} \simeq T^*N^m, \quad (7.29)$$

we can interpret the diagonal prolongation $X^{[m]}$ of a vector field on N as a vector field $\tilde{X}^{[m]}$ on N^m , and the diagonal prolongation $\alpha^{[m]}$ of a 1-form on N as a 1-form $\tilde{\alpha}^{[m]}$ on N^m . In the case when m is fixed, we will simply write \tilde{X} and $\tilde{\alpha}$. The proof of the following properties of diagonal prolongations is straightforward.

Proposition 8. *The diagonal prolongation to N^m of a vector field X on N is the unique vector field $\tilde{X}^{[m]}$ on N^m , projectable under the map $\pi : (x_{(1)}, \dots, x_{(m)}) \in N^m \mapsto x_{(1)} \in N$ onto X and invariant under the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 1, \dots, m$. The diagonal prolongation to N^m of a 1-form α on N is the unique 1-form $\tilde{\alpha}^{[m]}$ on N^m such that $\tilde{\alpha}^{[m]}(\tilde{X}^{[m]}) = \widetilde{\alpha(X)}^{[m]}$ for every vector field $X \in \Gamma(TN)$. We have $d\tilde{\alpha} = \widetilde{d\alpha}$ and $\mathcal{L}_{\tilde{X}^{[m]}}\tilde{\alpha}^{[m]} = \widetilde{\mathcal{L}_X\alpha}^{[m]}$. In particular, if α is closed (exact), so is its diagonal prolongation $\tilde{\alpha}^{[m]}$ to N^m .*

Using local coordinates (x^a) in N and the induced system $(x_{(i)}^a)$ of coordinates in N^m , we can write, for $X = \sum_a X^a(x)\partial_{x^a}$ and $\alpha = \sum_a \alpha_a(x)dx^a$,

$$\tilde{X}^{[m]} = \sum_{a,i} X^a(x_{(i)})\partial_{x_{(i)}^a} \quad \text{and} \quad \tilde{\alpha}^{[m]} = \sum_{a,i} \alpha_a(x_{(i)})dx_{(i)}^a. \quad (7.30)$$

Let us fix m . Obviously, given two vector fields X_1 and X_2 on N , we have $[\widetilde{X_1}, \widetilde{X_2}] = [\tilde{X}_1, \tilde{X}_2]$. In consequence, the prolongations to N^m of the elements of a finite-dimensional real Lie algebra V of vector fields on N form a real Lie algebra \tilde{V} isomorphic to V . Similarly to standard vector fields, we can define the diagonal prolongation of a t -dependent vector field X on N to N^m as the only t -dependent vector field \tilde{X} on N^m satisfying that \tilde{X}_t is the prolongation of X_t to N^m for each $t \in \mathbb{R}$.

When X is a Lie–Hamilton system, its diagonal prolongations are also Lie–Hamilton systems in a natural way [35]. Let us now focus on proving an analogue of this result for Dirac–Lie systems.

Definition 9. Given two Dirac manifolds (N, L_N) and (M, L_M) , we say that $\varphi : N \rightarrow M$ is a *forward Dirac map* between them if $(L_M)_{\varphi(x)} = \mathfrak{F}_\varphi(L_N)_x$, where

$$\mathfrak{F}_\varphi(L_N)_x = \{\varphi_* X_x + \omega_{\varphi(x)} \in T_{\varphi(x)}M \oplus T_{\varphi(x)}^*M \mid X_x + (\varphi^* \omega_{\varphi(x)})_x \in (L_N)_x\}, \quad (7.31)$$

for all $x \in N$.

Proposition 10. *Given a Dirac structure (N, L) and the natural isomorphism*

$$(TN^m \oplus_{N^m} T^*N^m)_{(x_{(1)}, \dots, x_{(m)})} \simeq (T_{x_{(1)}}N \oplus T_{x_{(1)}}^*N) \oplus \dots \oplus (T_{x_{(m)}}N \oplus T_{x_{(m)}}^*N), \quad (7.32)$$

the diagonal prolongation $L^{[m]}$, viewed as a vector subbundle in $TN^m \oplus_{N^m} T^*N^m = \mathcal{P}N^{[m]}$, is a Dirac structure on N^m .

The forward image of $L^{[m]}$ through each $\pi_i : (x_{(1)}, \dots, x_{(m)}) \in N^m \rightarrow x_{(i)} \in N$, with $i = 1, \dots, m$, equals L . Additionally, $L^{[m]}$ is invariant under the permutations $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 1, \dots, m$.

Proof. Being a diagonal prolongation of L , the subbundle $L^{[m]}$ is invariant under permutations $x_{(i)} \leftrightarrow x_{(j)}$ and each element of a basis $X_i + \alpha_i$ of L , with $i = 1, \dots, n$, can naturally be considered as an element $X_i^{(j)} + \alpha_i^{(j)}$ of the j th-copy of L within $L^{[m]}$. This gives rise to a basis of $L^{[m]}$, which naturally becomes a smooth mn -dimensional subbundle of $\mathcal{P}N^m$. Considering the natural pairing $\langle \cdot, \cdot \rangle_+$ of $\mathcal{P}N^m$ and using $\langle \alpha_j^{(i)}, X_l^{(k)} \rangle = 0$ for $i \neq k$, we have

$$\begin{aligned} \left\langle \left(X_j^{(i)} + \alpha_j^{(i)} \right) (x_{(1)}, \dots, x_{(m)}), \left(X_l^{(k)} + \alpha_l^{(k)} \right) (x_{(1)}, \dots, x_{(m)}) \right\rangle_+ = \\ \delta_k^i \langle (X_j + \alpha_j)(x_{(i)}), (X_l + \alpha_l)(x_{(i)}) \rangle_+ = 0, \end{aligned} \quad (7.33)$$

for every $p = (x_{(1)}, \dots, x_{(m)}) \in N^m$. As the pairing is bilinear and vanishes on a basis of $L^{[m]}$, it does so on the whole $L^{[m]}$, which is therefore isotropic. Since $L^{[m]}$ has rank mn , it is maximally isotropic.

Using that

$$[X_j^{(i)}, X_l^{(k)}] = 0, \quad \iota_{X_j^{(i)}} d\alpha_l^{(k)} = 0, \quad \text{and} \quad \mathcal{L}_{X_j^{(i)}} \omega_l^{(k)} = 0 \quad (7.34)$$

for $i \neq k = 1, \dots, m$ and $j, l = 1, \dots, \dim N$, we obtain

$$[[X_j^{(i)} + \alpha_j^{(i)}, X_l^{(k)} + \alpha_l^{(k)}]]_C = \delta_k^i [[X_j^{(i)} + \alpha_j^{(i)}, X_l^{(i)} + \alpha_l^{(i)}]]_C \in \Gamma(L^{[m]}). \quad (7.35)$$

So, $L^{[m]}$ is involutive. Since it is also maximally isotropic, it is a Dirac structure.

Let us prove that $\mathfrak{P}_{\pi_a}(L^{[m]}) = L$ for every π_a . Note that $(X_j^{(a)} + \alpha_j^{(a)})_p \in L_p^{[m]}$ is such that $\pi_{a*}(X_j^{(a)})_p = (X_j)_{x_{(a)}}$ and $(\alpha_j)_{x_{(a)}} \circ (\pi_{*a})_p = (\alpha_j^{(a)})_p$ for every $p \in \pi_a^{-1}(x_{(a)})$. So, $(X_j + \alpha_j)_{x_{(a)}} \in (\mathfrak{P}_{\pi_a}(L^{[m]}))_{x_{(a)}} \subset L_{x_{(a)}}$ for $j = 1, \dots, n$ and every $x_{(a)} \in N$. Using that $X_j + \alpha_j$ is a basis for L and the previous results, we obtain $L \subset \mathfrak{P}_{\pi_a}(L^{[m]})$. Conversely, $\mathfrak{P}_{\pi_a}(L^{[m]}) \subset L$. Indeed, if $(X + \alpha)_{x_{(a)}} \in \mathfrak{P}_a(L^{[m]})$, then there exists an element $(Y + \beta)_p \in L_p^{[m]}$, with $p \in \pi^{-1}(x_{(a)})$, such that $\pi_{a*}Y_p = X_{x_{(a)}}$ and $(\alpha)_{x_{(a)}} \circ (\pi_{*a})_p = \beta_p$. Using that $(Y + \beta)_p = \sum_{ij} c_{ij}(X_j^{(i)} + \alpha_j^{(i)})_p$ for a unique set of constants c_{ij} , with $i = 1, \dots, m$ and $j = 1, \dots, n$, we have $\pi_{a*}(\sum_{ij} c_{ij}(X_j^{(i)})_p) = \sum_j c_{aj}(X_j)_{x_{(a)}} = X_{x_{(a)}}$. Meanwhile, $\beta_p = \alpha_{x_{(a)}} \circ (\pi_{*a})_p$ means that $\sum_j c_{aj}(\alpha_j)_{x_{(a)}} = \alpha_{x_{(a)}}$. So, $(X + \alpha)_{x_{(a)}} = \sum_j c_{aj}(X_j + \alpha_j)_{x_{(a)}} \in L_{x_{(a)}}$. \square

Corollary 11. *Given a Dirac structure (N, L) , we have $\rho_m(L^{[m]}) = \rho(L)^{[m]}$, where ρ_m is the projection $\rho_m : \mathcal{P}N^m \rightarrow TN^m$. Then, if X is an L -Hamiltonian vector field with respect to L , its diagonal prolongation $\tilde{X}^{[m]}$ to N^m is an L -Hamiltonian vector field with respect to $L^{[m]}$. Moreover, $\rho_m^*(L^{[m]}) = \rho^*(L)^{[m]}$, where ρ_m^* is the canonical projection $\rho_m^* : \mathcal{P}N^m \rightarrow T^*N^m$.*

Corollary 12. *If (N, L, X) is a Dirac–Lie system, then $(N^m, L^{[m]}, \tilde{X}^{[m]})$ is also a Dirac–Lie system.*

Proof. If X admits a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields with respect to (N, L) , then \tilde{X} possesses a Vessiot–Guldberg Lie algebra \tilde{V} given by the diagonal prolongations of the elements of V , which are $L^{[m]}$ -Hamiltonian vector fields, by construction of $L^{[m]}$ and Corollary 11. \square

Similarly to the prolongations of vector fields, one can define prolongations of functions and 1-forms in an obvious way.

Proposition 13. *Let X be a vector field and f be a function on N . Then:*

- (a) *If f is an L -Hamiltonian function for X , its diagonal prolongation \tilde{f} to N^m is an $L^{[m]}$ -Hamiltonian function of the diagonal prolongation \tilde{X} to N^m .*
- (b) *If $f \in \text{Cas}(N, L)$, then $\tilde{f} \in \text{Cas}(N^m, L^{[m]})$.*
- (c) *The map $\lambda : (\text{Adm}(N, L), \{\cdot, \cdot\}_L) \ni f \mapsto \tilde{f} \in (\text{Adm}(N^m, L^{[m]}), \{\cdot, \cdot\}_{L^{[m]}})$ is an injective Lie algebra morphism.*

Proof. Let f be an L -Hamiltonian function for X . Then, $X + df \in \Gamma(L)$ and $\tilde{X} + d\tilde{f} = \tilde{X} + d\tilde{f}$ is as an element of $\Gamma(L^{[m]})$. By a similar argument, if $f \in \text{Cas}(N, L)$, then $\tilde{f} \in \text{Cas}(N^m, L^{[m]})$. Given $f, g \in \text{Adm}(N, L)$, we have $\widetilde{\{f, g\}_L} = \widetilde{Xfg} = \widetilde{Xf}g = X_{\tilde{f}}\tilde{g} = \{\tilde{f}, \tilde{g}\}_{L^{[m]}}$, i.e., $\lambda(\{f, g\}_L) = \{\lambda(f), \lambda(g)\}_{L^{[m]}}$. Additionally, as λ is linear, it becomes a Lie algebra morphism. Moreover, it is easy to see that $\tilde{f} = 0$ if and only if $f = 0$. Hence, λ is injective. \square

Note, however, that in the above we cannot ensure that λ is a Poisson algebra morphism, as in general $\widetilde{fg} \neq \widetilde{f}\widetilde{g}$.

Using the above proposition, we can easily prove the following corollaries.

Corollary 14. *If $h_1, \dots, h_r : N \rightarrow \mathbb{R}$ is a family of functions on a Dirac manifold (N, L) spanning a finite-dimensional real Lie algebra of functions with respect to the Lie bracket $\{\cdot, \cdot\}_L$, then their diagonal prolongations $\widetilde{h}_1, \dots, \widetilde{h}_r$ to N^m close an isomorphic Lie algebra of functions with respect to the Lie bracket $\{\cdot, \cdot\}_{L^{[m]}}$ induced by the Dirac structure $(N^m, L^{[m]})$.*

Corollary 15. *If (N, L, X) is a Dirac–Lie system admitting a Lie–Hamiltonian (N, L, h) , then $(N^m, L^{[m]}, \widetilde{X}^{[m]})$ is a Dirac–Lie system with a Dirac–Lie Hamiltonian $(N^m, L^{[m]}, h^{[m]})$, where $h_t^{[m]} = \widetilde{h}_t^{[m]}$ is the diagonal prolongation of h_t to N^m .*

7.4 Superposition rules and t -independent constants of motion

Let us give a first straightforward application of Dirac–Lie systems to obtain constants of the motion.

Proposition 16. *Given a Dirac–Lie system (N, L, X) , the elements of $\text{Cas}(N, L)$ are constants of the motion for X . Moreover, the set \mathcal{I}_L^X of its admissible t -independent constants of the motion form a Poisson algebra $(\mathcal{I}_L^X, \cdot, \{\cdot, \cdot\}_L)$.*

Proof. Two admissible functions f and g are t -independent constants of the motion for X if and only if $X_t f = X_t g = 0$ for every $t \in \mathbb{R}$. Using that every X_t is a derivation of the associative algebra $C^\infty(N)$, we see that given $f, g \in \mathcal{I}_L^X$, then $f + g$, λf , and $f \cdot g$ are also constants of the motion for X for every $\lambda \in \mathbb{R}$. Since the sum and product of admissible functions are admissible functions, then \mathcal{I}_L^X is closed under the sum and product of elements and real constants. So (\mathcal{I}_L^X, \cdot) is an associative subalgebra of $(C^\infty(N), \cdot)$.

As (N, L, X) is a Dirac–Lie system, the vector fields $\{X_t\}_{t \in \mathbb{R}}$ are L -Hamiltonian. Therefore,

$$X_t \{f, g\}_L = \{X_t f, g\}_L + \{f, X_t g\}_L. \quad (7.36)$$

As f and g are constants of the motion for X , then $\{f, g\}_L$ is so also. Using that $\{f, g\}_L$ is also an admissible function, we finish the proof. \square

The following can easily be proved.

Proposition 17. *Let (N, L, X) be a Dirac–Lie system possessing a Dirac–Lie Hamiltonian (N, L, h) . An admissible function $f : N \rightarrow \mathbb{R}$ is a constant of the motion for X if and only if it Poisson commutes with all the elements of $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_L)$.*

Consider a Dirac–Lie system (N, L^ω, X) with ω being a symplectic structure and X being an autonomous system. Consequently, $\text{Adm}(N, L) = C^\infty(N)$ and the above proposition entails that $f \in C^\infty(N)$ is a constant of the motion for X if and only if it Poisson commutes with a Hamiltonian function h associated to X . This shows that Proposition 17 recovers as a particular case this well-known result [13]. Additionally, Proposition 17 suggests us that the rôle played by autonomous Hamiltonians for autonomous Hamiltonian systems is performed by finite-dimensional Lie algebras of admissible functions associated with a Dirac–Lie Hamiltonian for Dirac–Lie systems. This fact can be employed, for instance, to study t -independent first-integrals of Dirac–Lie systems, e.g., the maximal number of such first-integrals in involution, which would lead to the interesting analysis of integrability/superintegrability and action/angle variables for Dirac–Lie systems [430].

Another reason to study t -independent constants of the motion of Lie systems is their use in deriving superposition rules [89]. More explicitly, a superposition rule for a Lie system can be obtained through the t -independent constants of the motion of one of its diagonal prolongations [91]. The following proposition provides some ways of obtaining such constants.

Proposition 18. *If X be a system possessing a t -independent constant of the motion f , then:*

1. *The diagonal prolongation $\tilde{f}^{[m]}$ is a t -independent constant of the motion for $\tilde{X}^{[m]}$.*
2. *If Y is a t -independent Lie symmetry of X , then $\tilde{Y}^{[m]}$ is a t -independent Lie symmetry of $\tilde{X}^{[m]}$.*
3. *If h is a t -independent constant of the motion for $\tilde{X}^{[m]}$, then $\tilde{Y}^{[m]}h$ is another t -independent constant of the motion for $\tilde{X}^{[m]}$.*

Proof. This result is a straightforward application of Proposition 8 and the properties of the diagonal prolongations of t -dependent vector fields. \square

Proposition 19. *Given a Dirac–Lie system (N, L, X) that admits a Dirac–Lie Hamiltonian (N, L, h) such that $\{h_t\}_{t \in \mathbb{R}}$ is contained in a finite-dimensional Lie algebra of admissible functions $(\mathfrak{M}, \{\cdot, \cdot\}_L)$. Given the momentum map $J : N^m \rightarrow \mathfrak{W}^*$ associated with the Lie algebra morphism $\phi : f \in \mathfrak{W} \mapsto \tilde{f} \in \text{Adm}(N^m, L^{[m]})$, the pull-back $J^*(C)$ of any Casimir function C on \mathfrak{W}^* is a constant of the motion for the diagonal prolongation $\tilde{X}^{[m]}$. If $\mathfrak{W} \simeq \text{Lie}(\{\tilde{h}_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{L^{[m]}})$, the function $J^*(C)$ Poisson commutes with all $L^{[m]}$ -admissible constants of the motion of $\tilde{X}^{[m]}$.*

Example 3. Let us use the above results to devise a superposition rule for the **third-order Kummer–Schwarz equation** in first-order form (2.64) with $c_0 = 0$, the so-called *Schwarzian equations* [381, 396]. To simplify the presentation, we will

always assume $c_0 = 0$ in this section. It is known (cf. [87]) that the derivation of a superposition rule for this system can be reduced to obtaining certain three t -independent constants of the motion for the diagonal prolongation \widetilde{X}^{3KS} of X^{3KS} to \mathcal{O}_2^2 . In [87] such constants were worked out through the method of characteristics which consists in solving a series of systems of ODEs. Nevertheless, we can determine such constants more easily through Dirac–Lie systems. System (2.64) is represented by the t -dependent vector field

$$X_t^{3KS} = v \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + \left(\frac{3}{2} \frac{a^2}{v} - 2c_0 v^3 + 2b_1(t)v \right) \frac{\partial}{\partial a} = Y_3 + b_1(t)Y_1, \quad (7.37)$$

where the vector fields on $\mathcal{O}_2 = \{(x, v, a) \in T^2\mathbb{R} \mid v \neq 0\}$ given by

$$\begin{aligned} N_1 &= 2v \frac{\partial}{\partial a}, & N_2 &= v \frac{\partial}{\partial v} + 2a \frac{\partial}{\partial a}, \\ N_3 &= v \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + \left(\frac{3}{2} \frac{a^2}{v} - 2c_0 v^3 \right) \frac{\partial}{\partial a}, \end{aligned} \quad (7.38)$$

If so, t -dependent vector field \widetilde{X}^{3KS} is spanned by a linear combination of the diagonal prolongations of N_1 , N_2 , and N_3 to \mathcal{O}_2^2 . From (7.38), we have

$$\begin{aligned} \widetilde{N}_1 &= \sum_{i=1}^2 v_i \frac{\partial}{\partial a_i}, & \widetilde{N}_2 &= \sum_{i=1}^2 \left(v_i \frac{\partial}{\partial v_i} + 2a_i \frac{\partial}{\partial a_i} \right), \\ \widetilde{N}_3 &= \sum_{i=1}^2 \left(v_i \frac{\partial}{\partial x_i} + a_i \frac{\partial}{\partial v_i} + \frac{3}{2} \frac{a_i^2}{v_i} \frac{\partial}{\partial a_i} \right). \end{aligned} \quad (7.39)$$

From Proposition 13 and functions (7.10), the vector fields $\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3$ are $L^{[2]}$ -Hamiltonian with $L^{[2]}$ -Hamiltonian functions

$$\widetilde{h}_1 = -\frac{2}{v_1} - \frac{2}{v_2}, \quad \widetilde{h}_2 = -\frac{a_1}{v_1^2} - \frac{a_2}{v_2^2}, \quad \widetilde{h}_3 = -\frac{a_1^2}{2v_1^3} - \frac{a_2^2}{2v_2^3}. \quad (7.40)$$

Indeed, these are the diagonal prolongations to \mathcal{O}_2^2 of the L -Hamiltonian functions of N_1, N_2 , and N_3 . Moreover, they span a real Lie algebra of functions isomorphic to that one spanned by h_1, h_2, h_3 and to $\mathfrak{sl}(2, \mathbb{R})$. We can then define a Lie algebra morphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow C^\infty(N^2)$ of the form $\phi(e_1) = \widetilde{h}_1$, $\phi(e_2) = \widetilde{h}_2$ and $\phi(e_3) = \widetilde{h}_3$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathfrak{sl}(2, \mathbb{R})$. Using that $\mathfrak{sl}(2, \mathbb{R})$ is a simple Lie algebra, we can compute the Casimir invariant on $\mathfrak{sl}(2, \mathbb{R})^*$ as $e_1 e_3 - e_2^2$ (where e_1, e_2, e_3 can be considered as functions on $\mathfrak{sl}(2, \mathbb{R})$). Proposition 19 ensures then that $\widetilde{h}_1 \widetilde{h}_3 - \widetilde{h}_2^2$ Poisson commutes with $\widetilde{h}_1, \widetilde{h}_2$ and \widetilde{h}_3 . In this way, we obtain a constant of the motion for \widetilde{X}^{3KS} given by

$$I = \widetilde{h}_1 \widetilde{h}_3 - \widetilde{h}_2^2 = \frac{(a_2 v_1 - a_1 v_2)^2}{v_1^3 v_2^3}. \quad (7.41)$$

Schwarzian equations admit a Lie symmetry $Z = x^2\partial/\partial x$ [346]. Its prolongation to $T^2\mathbb{R}$, i.e.,

$$Z_P = x^2 \frac{\partial}{\partial x} + 2vx \frac{\partial}{\partial v} + 2(ax + v^2) \frac{\partial}{\partial a}, \quad (7.42)$$

is a Lie symmetry of X^{3KS} . From Proposition 13, we get that \tilde{Z}_P is a Lie symmetry of \tilde{X}^{3KS} . So, we can construct constants of the motion for \tilde{X}^{3KS} by applying \tilde{Z}_P to any of its t -independent constants of the motion. In particular,

$$F_2 \equiv -\tilde{Z}_P \log |I| = x_1 + x_2 + \frac{2v_1v_2(v_1 - v_2)}{a_2v_1 - a_1v_2} \quad (7.43)$$

is constant on particular solutions $(x_{(1)}(t), v_{(1)}(t), a_{(1)}(t), x_{(2)}(t), v_{(2)}(t), a_{(2)}(t))$ of \tilde{X}^{3KS} . If $(x_{(2)}(t), v_{(2)}(t), a_{(2)}(t))$ is a particular solution for X^{3KS} , its opposite is also. So, the function

$$F_3 \equiv x_1 - x_2 + \frac{2v_1v_2(v_1 + v_2)}{a_2v_1 - a_1v_2} \quad (7.44)$$

is also constant along solutions of \tilde{X}^{3KS} , i.e., it is a new constant of the motion. In consequence, we get three t -independent constants of the motion: $\Upsilon_1 = I$ and

$$\Upsilon_2 = \frac{F_2 + F_3}{2} = x_1 + \frac{2v_1^2v_2}{a_2v_1 - a_1v_2}, \quad \Upsilon_3 = \frac{F_2 - F_3}{2} = x_2 - \frac{2v_1v_2^2}{a_2v_1 - a_1v_2}. \quad (7.45)$$

This gives rise to three t -independent constants of the motion for \tilde{X}^{3KS} . Taking into account that $\partial(\Upsilon_1, \Upsilon_2, \Upsilon_3)/\partial(x_1, v_1, a_1) \neq 0$, the expressions $\Upsilon_1 = \lambda_1$, $\Upsilon_2 = \lambda_2$, and $\Upsilon_3 = \lambda_3$ allow us to obtain the expressions of x_1, v_1, a_1 in terms of the remaining variables and $\lambda_1, \lambda_2, \lambda_3$. More specifically,

$$x_1 = \frac{4}{\lambda_1(\lambda_3 - x_2)} + \lambda_2, \quad v_1 = \frac{4v_2}{\lambda_1(\lambda_3 - x_2)^2}, \quad a_1 = \frac{8v_2^2 + 4a_2(\lambda_3 - x_2)}{\lambda_1(\lambda_3 - x_2)^3}. \quad (7.46)$$

According to the theory of Lie systems [91], the map $\Phi : (x_2, v_2, a_2; \lambda_1, \lambda_2, \lambda_3) \in \mathcal{O}_2^2 \times \mathbb{R}^3 \mapsto (x_1, v_1, a_1) \in \mathcal{O}_2^2$ enables us to write the general solution of (2.64) into the form

$$(x(t), v(t), a(t)) = \Phi(x_2(t), v_2(t), a_2(t); \lambda_1, \lambda_2, \lambda_3). \quad (7.47)$$

This is the known superposition rule for Schwarzian equations (in first-order form) derived in [308] by solving a system of PDEs. Meanwhile, our present techniques enable us to obtain the same result without any integration. Note that $x(t)$, the general solution of Schwarzian equations, can be written as $x(t) = \tau \circ \Phi(x_2(t), \lambda_1, \lambda_2, \lambda_3)$, with τ the projection $\tau : (x_2, v_2, a_2) \in T^2\mathbb{R} \mapsto x_2 \in \mathbb{R}$, from a unique particular solution of (2.39), recovering a known feature of these equations [346].

7.5 Bi-Dirac-Lie systems

It can happen that a Lie system X on a manifold N possesses Vessiot-Guldberg Lie algebras of vector fields with respect to two different Dirac structures. This results in defining two Dirac-Lie systems. For instance, the system of coupled Riccati equations (4.10) admits two Dirac-Lie structures [32]: the one previously given, $(\mathcal{O}, L^\omega, X)$, where ω is given by (4.12), and a second one, $(\mathcal{O}, L^{\bar{\omega}}, X)$, with

$$\bar{\omega} = \sum_{i < j=1}^4 \frac{dx_i \wedge x_j}{(x_i - x_j)^2}. \quad (7.48)$$

In the following sections, several similar examples will be shown. This suggests us to define the following notion.

Definition 20. A *bi-Dirac-Lie system* is a four-tuple (N, L_1, L_2, X) , where (N, L_1) and (N, L_2) are two different Dirac manifolds and X is a Lie system on N such that $V^X \subset \text{Ham}(N, L_1) \cap \text{Ham}(N, L_2)$.

Given a bi-Dirac-Lie system (N, L_1, L_2, X) , we can apply indistinctly the methods of the previous sections to (N, L_1, X) and (N, L_2, X) to obtain superposition rules, constants of the motion, and other properties of X . This motivates studies on constructions of this type of structures.

Let us depict a new procedure to build up bi-Dirac-Lie systems from (N, L^ω, X) whose X possesses a t -independent Lie symmetry Z . This method is a generalization to nonautonomous systems, associated to presymplectic manifolds, of the method for autonomous Hamiltonian systems devised in [111].

Consider a Dirac-Lie system (N, L^ω, X) , where ω is a presymplectic structure, and a t -independent Lie symmetry Z of X , i.e. $[Z, X_t] = 0$ for all $t \in \mathbb{R}$. Under the above assumptions, $\omega_Z = \mathcal{L}_Z \omega$ satisfies $d\omega_Z = d\mathcal{L}_Z \omega = \mathcal{L}_Z d\omega = 0$, so (N, ω_Z) is a presymplectic manifold. The vector fields of V^X are still Hamiltonian with respect to (N, ω_Z) . Indeed, we can see that Theorem 5 ensures that X admits a Dirac-Lie Hamiltonian (N, L^ω, h) and

$$\begin{aligned} [Z, X_t] = 0 &\implies \iota_{X_t} \circ \mathcal{L}_Z = \mathcal{L}_Z \circ \iota_{X_t} \\ &\Downarrow \\ \iota_{X_t} \omega_Z = \iota_{X_t} \mathcal{L}_Z \omega &= \mathcal{L}_Z \iota_{X_t} \omega = -\mathcal{L}_Z dh_t = -d(Zh_t), \end{aligned} \quad (7.49)$$

$\forall t \in \mathbb{R}$. So, the vector fields $\{X_t\}_{t \in \mathbb{R}}$ are L^{ω_Z} -Hamiltonian. Since the successive Lie brackets and linear combinations of L -Hamiltonian vector fields and elements of V^X are L -Hamiltonian vector fields, the whole Lie algebra V^X is Hamiltonian with respect to ω_Z . Consequently, (N, L^{ω_Z}, X) is also a Dirac-Lie system. In view of (7.6) and since $\dim \text{Cas}(N, L^{\omega_Z}) = 1$, we see that $(B^{\omega_Z})^{-1}(V^X)$ is a finite-dimensional Lie algebra. As the curve $\bar{h} : t \in \mathbb{R} \mapsto Zh_t \in \text{Adm}(N, L^{\omega_Z})$ is included within

$(B^{\omega_Z})^{-1}(V^X)$, the Lie algebra $\text{Lie}(\{Zh_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{L^{\omega_Z}})$, where $\{\cdot, \cdot\}_{L^{\omega_Z}}$ is the Poisson bracket induced by L^{ω_Z} , becomes finite-dimensional. In other words, (N, L^{ω_Z}, Zh_t) is also a Lie–Hamiltonian for X . Moreover,

$$\{\bar{h}_t, \bar{h}_{t'}\}_{L^{\omega_Z}} = X_t(\bar{h}_{t'}) = X_t(Zh_{t'}) = Z(X_t h_{t'}) = Z\{h_t, h_{t'}\}_{L^{\omega}}, \forall t \in \mathbb{R}. \quad (7.50)$$

Summarizing, we have the following proposition.

Proposition 21. *If (N, L^{ω}, X) is a Dirac–Lie system for which X admits a t -independent Lie symmetry Z , then $(N, L^{\omega}, L^{\mathcal{L}Z\omega}, X)$ is a bi–Dirac–Lie system. If (N, L^{ω}, h) is a Dirac–Lie Hamiltonian for X , then $(N, L^{\mathcal{L}Z\omega}, Zh)$ is a Dirac–Lie Hamiltonian for X and there exists an exact sequence of Lie algebras*

$$(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{L^{\omega}}) \xrightarrow{Z} (\{Zh_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{L^{\omega_Z}}) \rightarrow 0. \quad (7.51)$$

Note that, given a Lie–Hamilton system (N, L^{ω}, X) , the triple (N, L^{ω_Z}, X) need not be a Lie–Hamilton system: ω_Z may fail to be a symplectic two-form (cf. [111]). This causes that the theory of Lie–Hamilton systems cannot be applied to study (N, L^{ω_Z}, X) , while the methods of our work do.

Let us illustrate the above theory with an example. Recall that Schwarzian equations admit a Lie symmetry $Z = x^2\partial/\partial x$. As a consequence, system (2.64), with $c_0 = 0$, possesses a t -independent Lie symmetry Z_P given by (7.42) and

$$\omega_{Z_P} \equiv \mathcal{L}_{Z_P}\omega_{3KS} = -\frac{2}{v^3}(xdv \wedge da + vda \wedge dx + adx \wedge dv). \quad (7.52)$$

Moreover,

$$\begin{aligned} \iota_{Y_1}\omega_{Z_P} &= -d(Z_P h_1) = -d\left(\frac{4x}{v}\right), & \iota_{Y_2}\omega_{Z_P} &= -d(Z_P h_2) = d\left(2 - \frac{2ax}{v^2}\right), \\ \iota_{Y_3}\omega_{Z_P} &= -d(Z_P h_3) = d\left(\frac{2a}{v} - \frac{a^2x}{v^3}\right). \end{aligned} \quad (7.53)$$

So, Y_1, Y_2 , and Y_3 are Hamiltonian vector fields with respect to ω_{Z_P} . Moreover, since

$$\begin{aligned} \{Z_P h_1, Z_P h_2\}_{L^{\omega_{Z_P}}} &= Z_P h_1, \\ \{Z_P h_2, Z_P h_3\}_{L^{\omega_{Z_P}}} &= Z_P h_3, \\ \{Z_P h_1, Z_P h_3\}_{L^{\omega_{Z_P}}} &= 2Z_P h_2, \end{aligned} \quad (7.54)$$

we see that $Z_P h_1, Z_P h_2$, and $Z_P h_3$ span a new finite-dimensional real Lie algebra. So, if $(\mathcal{O}_2, L^{\omega}, h)$ is a Lie–Hamiltonian for X , then $(\mathcal{O}_2, L^{\omega_{Z_P}}, Z_P h)$ is a Dirac–Lie Hamiltonian for X .

Let us devise a more general method to construct bi–Dirac–Lie systems. Given a Dirac manifold (N, L) and a closed two-form ω on N , the sections on $TN \oplus_N T^*N$ of the form

$$X + \alpha - \iota_X \omega, \quad (7.55)$$

where $X + \alpha \in \Gamma(L)$, span a new Dirac structure (N, L^ω) [75]. When two Dirac structures are connected by a transformation of this type, it is said that they are *gauge equivalent*. Using this, we can prove the following propositions.

Proposition 22. *Let Z be a vector field on N . Then, the Dirac structures L^ω and L^{ω_Z} , with $\omega_Z = \mathcal{L}_Z\omega$, are gauge equivalent.*

Proof. The Dirac structure L^ω is spanned by sections of the form $X - \iota_X\omega$, with $X \in \Gamma(N)$, and the Dirac structure L^{ω_Z} is spanned by sections of the form $X - \iota_X\omega_Z$. Recall that $d\omega = d\omega_Z = 0$. So, L^{ω_Z} is of the form

$$X - \iota_X\omega - \iota_X(\omega_Z - \omega), \quad X - \iota_X\omega \in \Gamma(L^\omega). \quad (7.56)$$

As $d(\omega_Z - \omega) = 0$, then L^ω and L^{ω_Z} are connected by a gauge transformation. \square

This result gives us a hint to construct a more general method to create bi-Dirac-Lie systems.

Proposition 23. *Let (N, L, X) be a Dirac-Lie system and ω be a closed two-form such that $\widehat{\omega}(V^X) \subset \mathcal{B}^1(N)$, where $\mathcal{B}^1(N)$ is the space of exact one-forms on N . Then, $(N, L, {}^\omega L, X)$ is a bi-Dirac-Lie system.*

Proof. If $Y \in V^X$, then it is L -Hamiltonian and $Y + df \in \Gamma(L)$ for a certain function $f \in C^\infty(N)$. By definition of ${}^\omega L$, we have that $Y + df - \iota_Y\omega \in \Gamma({}^\omega L)$. Since $\widehat{\omega}(V^X) \subset \mathcal{B}^1(N)$ by assumption, then $\widehat{\omega}(Y) = -dg$ for a certain $g \in C^\infty(N)$. So, $Y + d(f + g) \in \Gamma({}^\omega L)$ and Y is ${}^\omega L$ -Hamiltonian. Hence, V^X is a finite-dimensional real Lie algebra of ${}^\omega L$ -Hamiltonian vector fields, $(N, {}^\omega L, X)$ is a Dirac-Lie system and $(N, L, {}^\omega L, X)$ is a bi-Dirac-Lie system. \square

Note 24. Note that two gauge equivalent Dirac structures may have different spaces of admissible functions. This causes that they can be used to obtain different admissible constants of the motion and other properties of X . In brief, gauge equivalent Dirac structures are not equivalent from the point of view of their associated Dirac-Lie systems.

7.6 Dirac-Lie systems and Schwarzian-KdV equations

Let us give some final relevant applications of our methods. In particular, we devise a procedure to construct traveling wave solutions for some relevant nonlinear PDEs by means of Dirac-Lie systems. For simplicity, we hereafter denote the partial derivatives of a function $f : (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto f(x_1, \dots, x_n) \in \mathbb{R}$ by $\partial_{x_i}f$.

Example 4. Consider the so-called **Schwarzian Korteweg de Vries equation** (SKdV equation)[27]

$$\{\Phi, x\}\partial_x\Phi = \partial_t\Phi, \quad (7.57)$$

where $\Phi : (t, x) \in \mathbb{R}^2 \rightarrow \Phi(t, x) \in \mathbb{R}$ and

$$\{\Phi, x\} \equiv \frac{\partial_x^3 \Phi}{\partial_x \Phi} - \frac{3}{2} \left(\frac{\partial_x^2 \Phi}{\partial_x \Phi} \right)^2, \quad (7.58)$$

which is equivalent to that introduced in (2.79). This PDE has been attracting some attention due to its many interesting properties [27, 217, 362]. For instance, Dorfman established a bi-symplectic structure for this equation [148], and many others have been studying its solutions and generalizations [27, 362]. As a relevant result, we can mention that, given a solution Φ of the SKdV equation, the function $\{\Phi, x\}$ is a particular solution of the Korteweg-de Vries equation (KdV equation) [46]

$$\partial_t u = \partial_x^3 u + 3u \partial_x u. \quad (7.59)$$

We now look for traveling wave solutions of (7.57) of the type $\Phi(t, x) = g(x - f(t))$ for a certain fixed t -dependent function f with $df/dt = v_0 \in \mathbb{R}$. Substituting $\Phi = g(x - f(t))$ within (7.57), we obtain that g is a particular solution of the Schwarzian equation

$$\frac{d^3 g}{dz^3} = \frac{3}{2} \frac{(d^2 g/dz^2)^2}{dg/dz} - v_0 \frac{dg}{dz}, \quad (7.60)$$

where $z \equiv x - f(t)$. We already know that the Schwarzian equations can be studied through the superposition rule (7.46), which can better be obtained by using that Schwarz equations can be studied through a Dirac–Lie system, as seen in this section. More specifically, we can generate all their solutions from a known one as

$$g_2(z) = \frac{\alpha g_1(z) + \beta}{\gamma g_1(z) + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \quad (7.61)$$

In addition, (7.60) is a HODE Lie system, i.e., when written as a first-order system by adding the variables $v = dx/dz$ and $a = dv/dz$, it becomes a Lie system X , namely one of the form (7.37). It can be proved that (7.60) can be integrated for any $v_0 = df/dt$. For instance, particular solutions of this equation read

$$\bar{g}_1(z) = \operatorname{th} \left[\sqrt{v_0/2z} \right] \quad (v_0 > 0), \quad g_1(z) = \frac{1}{z+1} \quad (v_0 = 0). \quad (7.62)$$

Note that $g_1(z)$ has the shape of a solitary stationary solution, i.e., $\lim_{x \rightarrow \pm\infty} g_1(x - \lambda_0) = 0$ for every $\lambda_0 \in \mathbb{R}$. Meanwhile, \bar{g}_1 is a traveling wave solution. Moreover, the general solution of (7.60) in both cases can be obtained from (7.61).

8 Jacobi–Lie systems

Following the research line of this first part of the thesis, we here study Lie systems with Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect

to Jacobi manifolds. Roughly speaking, a *Jacobi manifold* is a manifold N endowed with a local Lie algebra $(C^\infty(N), \{\cdot, \cdot\})$ [136, 147, 270, 293, 378]. Using that Poisson manifolds are a particular case of Jacobi manifolds, we can consider Jacobi–Lie systems as a generalization of Lie–Hamilton systems. Although each Jacobi manifold gives rise to an associated Dirac manifold, not all Hamiltonian vector fields with respect to the Jacobi manifold become Hamiltonian with respect to its associated Dirac manifold (cf. [136]). Hence, not every Jacobi–Lie system can be straightforwardly understood as a Dirac–Lie system. Even in that case, the Jacobi manifold allows us to construct a Dirac manifold to study the system. The main difference between Jacobi–Lie systems and Lie–Hamilton systems is that Jacobi manifolds do not naturally give rise to Poisson brackets on a space of smooth functions on the manifold, which makes difficult to prove analogues and/or extensions of the results for Lie–Hamilton systems.

In this section, we extend to Jacobi–Lie systems some of the main structures found for Lie–Hamilton systems, e.g., Lie–Hamiltonians [109], and we classify all Jacobi–Lie systems on \mathbb{R} and \mathbb{R}^2 by determining all Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to Jacobi manifolds on \mathbb{R} and \mathbb{R}^2 . This is achieved by using the local classification of Lie algebras of vector fields on \mathbb{R} and \mathbb{R}^2 derived by Lie [295] and improved by González-López, Kamran and Olver (GKO) [210] (see also [32]). Summarizing, we obtain that every Lie system on the real line is a Jacobi–Lie system and we show that every Jacobi–Lie system on the plane admits a Vessiot–Guldberg Lie algebra diffeomorphic to one of the 14 classes indicated in Table 5 in the Appendix 1.

8.1 Jacobi–Lie systems

We now introduce Jacobi–Lie systems as Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Jacobi manifold.

Definition 1. A *Jacobi–Lie system* (N, Λ, R, X) consists of the Jacobi manifold (N, Λ, R) and a Lie system X admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to (N, Λ, R) .

Example 1. We reconsider the **continuous Heisenberg group** [413] given in Chapter 2, which describes the space of matrices

$$\mathbb{H} = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\}, \quad (8.1)$$

endowed with the standard matrix multiplication, where $\{x, y, z\}$ is the natural coordinate system on \mathbb{H} induced by (8.1).

A straightforward calculation shows that the Lie algebra \mathfrak{h} of left-invariant vector fields on \mathbb{H} is spanned by

$$X_1^L = \frac{\partial}{\partial x}, \quad X_2^L = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad X_3^L = \frac{\partial}{\partial z}. \quad (8.2)$$

Consider now the system on \mathbb{H} given by

$$\frac{dh}{dt} = \sum_{\alpha=1}^3 b_\alpha(t) X_\alpha^L(h), \quad h \in \mathbb{H}, \quad (8.3)$$

for arbitrary t -dependent functions $b_1(t), b_2(t)$ and $b_3(t)$. Since the associated t -dependent vector field $X_t^{\mathbb{H}} = b_1(t)X_1^L + b_2(t)X_2^L + b_3(t)X_3^L$ takes values in a finite-dimensional Lie algebra of vector fields, the system $X^{\mathbb{H}}$ is a Lie system. The interest of $X^{\mathbb{H}}$ is due to its appearance in the solution of the so-called *quantum Lie systems* as well as Lie systems admitting a Vessiot–Guldberg Lie algebra isomorphic to \mathfrak{h} [102].

Let us show that system (8.3) gives rise to a Jacobi–Lie system. Consider the bivector $\Lambda_{\mathbb{H}}$ given by

$$\Lambda_{\mathbb{H}} \equiv -y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \quad (8.4)$$

and the vector field $R_{\mathbb{H}} \equiv \partial/\partial z$. Then,

$$\begin{aligned} X_1^L &= [\Lambda_{\mathbb{H}}, -y]_{SN} - yR_{\mathbb{H}}, & X_2^L &= [\Lambda_{\mathbb{H}}, x]_{SN} + xR_{\mathbb{H}}, \\ X_3^L &= [\Lambda_{\mathbb{H}}, 1]_{SN} + R_{\mathbb{H}}. \end{aligned} \quad (8.5)$$

That is, X_1^L, X_2^L and X_3^L are Hamiltonian vector fields with Hamiltonian functions $h_1 = -y, h_2 = x$ and $h_3 = 1$, respectively. Hence, we obtain that $(\mathbb{H}, \Lambda_{\mathbb{H}}, R_{\mathbb{H}}, X^{\mathbb{H}})$ is a Jacobi–Lie system. It is remarkable that each Hamiltonian function h_i is a first-integral of X_i^L and $R_{\mathbb{H}}$ for $i = 1, 2, 3$, respectively.

Lemma 2. *The space of good Hamiltonian functions $G(N, \Lambda, R)$ with respect to a Jacobi structure (N, Λ, R) is a Poisson algebra with respect to the Jacobi bracket $\{\cdot, \cdot\}_{\Lambda, R}$. Additionally, $\star_g : f \in C^\infty(N) \mapsto \{g, f\}_{\Lambda, R} \in C^\infty(N)$, for any $g \in G(N, \Lambda, R)$, is a derivation on $(C^\infty(N), \cdot)$.*

Proof. First, we prove that the Jacobi bracket of two good Hamiltonian functions is a good Hamiltonian function. For general functions $u_1, u_2 \in C^\infty(N)$, we have

$$R\{u_1, u_2\}_{\Lambda, R} = R(\Lambda(du_1, du_2) + u_1Ru_2 - u_2Ru_1). \quad (8.6)$$

If $u_1, u_2 \in G(N, \Lambda, R)$, then $Ru_1 = Ru_2 = 0$. Using this and $[\Lambda, R]_{SN} = 0$, we obtain

$$R\{u_1, u_2\}_{\Lambda, R} = R(\Lambda(du_1, du_2)) = [R, [[\Lambda, u_1]_{SN}, u_2]_{SN}]_{SN} = 0. \quad (8.7)$$

Hence, $\{u_1, u_2\}_{\Lambda, R} \in G(N, \Lambda, R)$, which becomes a Lie algebra relative to $\{\cdot, \cdot\}_{\Lambda, R}$. Note also that $R(u_1 \cdot u_2) = 0$ and $u_1 \cdot u_2 \in G(N, \Lambda, R)$.

Given an arbitrary $u_1 \in G(N, \Lambda, R)$ and any $u_2, u_3 \in C^\infty(N)$, we have that

$$\begin{aligned} \star_{u_1}(u_2 \cdot u_3) &= \Lambda(\mathrm{d}u_1, \mathrm{d}(u_2 u_3)) + u_1 R(u_2 u_3) - u_2 u_3 R u_1 \\ &= X_{u_1}(u_2 \cdot u_3) = u_3 X_{u_1} u_2 + u_2 X_{u_1} u_3 = u_3 \star_{u_1} u_2 + u_2 \star_{u_1} u_3. \end{aligned} \quad (8.8)$$

So, \star_{u_1} is a derivation on $(C^\infty(N), \cdot)$ and also on $(G(N, \Lambda, R), \{\cdot, \cdot\}_{\Lambda, R})$. From this it trivially follows that $(G(N, \Lambda, R), \cdot, \{\cdot, \cdot\}_{\Lambda, R})$ is a Poisson algebra. \square

8.2 Jacobi–Lie Hamiltonians

Definition 3. We call *Jacobi–Lie Hamiltonian* a quadruple (N, Λ, R, h) , where (N, Λ, R) is a Jacobi manifold and $h : (t, x) \in \mathbb{R} \times N \mapsto h_t(x) \in N$ is a t -dependent function such that $(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{\Lambda, R})$ is a finite-dimensional real Lie algebra. Given a system X on N , we say that X admits a *Jacobi–Lie Hamiltonian* (N, Λ, R, h) if X_t is a Hamiltonian vector field with Hamiltonian function h_t (with respect to (N, Λ, R)) for each $t \in \mathbb{R}$.

Example 2. So, we find that $h_t = b_1(t)h_1 + b_2(t)h_2 + b_3(t)h_3 = -b_1(t)y + b_2(t)x + b_3(t)$ is a Hamiltonian function for each $X_t^{\mathbb{H}}$ in (8.3), with $t \in \mathbb{R}$. In addition,

$$\{h_1, h_2\}_{\Lambda_{\mathbb{H}}, R_{\mathbb{H}}} = h_3, \quad \{h_1, h_3\}_{\Lambda_{\mathbb{H}}, R_{\mathbb{H}}} = 0, \quad \{h_2, h_3\}_{\Lambda_{\mathbb{H}}, R_{\mathbb{H}}} = 0. \quad (8.9)$$

In other words, the functions $\{h_t\}_{t \in \mathbb{R}}$ span a finite-dimensional real Lie algebra of functions with respect to the Poisson bracket (8.3) induced by the Jacobi manifold. Thus, $X^{\mathbb{H}}$ admits a Jacobi–Lie Hamiltonian $(N, \Lambda_{\mathbb{H}}, R_{\mathbb{H}}, h)$.

Theorem 4. *Given a Jacobi–Lie Hamiltonian (N, Λ, R, h) , the system X of the form $X_t = X_{h_t}$, $\forall t \in \mathbb{R}$, is a Jacobi–Lie system. If X is a Lie system whose $\{X_t\}_{t \in \mathbb{R}}$ are good Hamiltonian vector fields, then it admits a Jacobi–Lie Hamiltonian.*

Proof. Let us prove the direct part. By assumption, the Hamiltonian functions $\{h_t\}_{t \in \mathbb{R}}$ are contained in a finite-dimensional Lie algebra $\mathrm{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{\Lambda, R})$. The Lie algebra morphism $\phi_{\Lambda, R} : f \in C^\infty(N) \mapsto X_f \in \mathrm{Ham}(N, \Lambda, R)$ maps the curve h_t into a curve X_t within $\phi_{\Lambda, R}(\mathrm{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{\Lambda, R}))$.

Since $\mathrm{Lie}(\{h_t\}_t, \{\cdot, \cdot\}_{\Lambda, R})$ is finite-dimensional and $\phi_{\Lambda, R}$ is a Lie algebra morphism,

$$\phi_{\Lambda, R}(\mathrm{Lie}(\{h_t\}_t, \{\cdot, \cdot\}_{\Lambda, R}))$$

is a finite-dimensional Lie algebra. Hence, X takes values in a finite-dimensional Lie algebra of Hamiltonian vector fields and it is a Jacobi–Lie system.

Let us prove the converse. Since the elements of $\{X_t\}_{t \in \mathbb{R}}$ are good Hamiltonian vector fields by assumption and $\mathrm{Lie}(\{X_t\}_{t \in \mathbb{R}}) = V^X$, every element of V^X is a

good Hamiltonian vector field and we can choose a basis X_1, \dots, X_r of V^X with good Hamiltonian functions h_1, \dots, h_r . The Jacobi bracket $\{h_i, h_j\}_{\Lambda, R}$ is a good Hamiltonian function for $[X_i, X_j]$.

Since $[X_i, X_j] = \sum_{k=1}^r c_{ijk} X_k$ for certain constants c_{ijk} , we obtain that each function:

$$s_{ij} = \{h_i, h_j\}_{\Lambda, R} - \sum_{k=1}^r c_{ijk} h_k, \quad i < j, \quad (8.10)$$

is the difference of two good Hamiltonian functions with the same Hamiltonian vector field. Hence, $\{s_{ij}, h\}_{\Lambda, R} = 0$ for all $h \in C^\infty(N)$. Using this, we obtain that the linear space generated by h_1, \dots, h_r, s_{ij} , with $1 \leq i < j \leq r$, is a finite-dimensional Lie algebra relative to $\{\cdot, \cdot\}_{\Lambda, R}$. If $X = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha$, then $(N, \Lambda, R, h = \sum_{\alpha=1}^r b_\alpha(t) h_\alpha)$ is a Jacobi–Lie Hamiltonian for X . \square

One of the relevant properties of the Jacobi–Lie Hamiltonians is given by the following proposition, whose proof is straightforward.

Proposition 5. *Let (N, Λ, R, X) be a Jacobi–Lie system admitting a Jacobi–Lie Hamiltonian (N, Λ, R, h) of good Hamiltonian functions $\{h_t\}_{t \in \mathbb{R}}$. A function $f : N \rightarrow \mathbb{R}$ is a t -independent constant of motion for X if and only if f commutes with all the elements of $\text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_{\Lambda, R})$ relative to $\{\cdot, \cdot\}_{\Lambda, R}$.*

8.3 Jacobi–Lie systems on low dimensional manifolds

Let us prove that every Lie system on the real line gives rise to a Jacobi–Lie system. In the case of two-dimensional manifolds, we display, with the aid of the GKO classification [32, 210] of Lie algebras of vector fields on the plane given in Table 1, all the possible Vessiot–Guldberg Lie algebras related to Jacobi–Lie systems on the plane given in Table 5.

Example 3. Let us show that the **coupled Riccati equation**

$$\frac{dx_i}{dt} = a_0(t) + a_1(t)x_i + a_2(t)x_i^2, \quad i = 1, \dots, n, \quad (8.11)$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary t -dependent functions,

can be associated with a Jacobi–Lie system for $n = 1$, which proves that every Lie system on the real line can be considered as a Jacobi–Lie system. Recall that (2.12) is a Lie system with a Vessiot–Guldberg Lie algebra V spanned by (2.13). Observe that V consists of Hamiltonian vector fields with respect to a Jacobi manifold on \mathbb{R} given by $\Lambda = 0$ and $R = \frac{\partial}{\partial x_1}$. Indeed, the vector fields $X_1, X_2, X_3 \in V$ admit Hamiltonian functions

$$h_1 = 1, \quad h_2 = x_1, \quad h_3 = x_1^2. \quad (8.12)$$

We now classify Jacobi–Lie systems $(\mathbb{R}^2, \Lambda, R, X)$, where we may assume Λ and R to be locally equal or different from zero. There exists just one Jacobi–Lie system with $\Lambda = 0$ and $R = 0$: $(\mathbb{R}^2, \Lambda = 0, R = 0, X = 0)$. Jacobi–Lie systems of the form $(\mathbb{R}^2, \Lambda \neq 0, R = 0)$ are Lie–Hamilton systems, whose Vessiot–Guldberg Lie algebras were obtained in [32]. In Table 5 we indicate these cases by writing *Poisson*. A Jacobi–Lie system $(\mathbb{R}^2, \Lambda = 0, R \neq 0, X)$ is such that if $Y \in V^X$, then $Y = fR$ for certain $f \in C^\infty(\mathbb{R}^2)$. All cases of this type can be easily obtained out of the bases given in Table 1. We describe them by writing $(0, R)$ at the last column.

Propositions 9 and 10, show that the Vessiot–Guldberg Lie algebras of Table 5 in Chapter 2 that do not fall into the mentioned categories are not Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to Jacobi manifolds $(\mathbb{R}^2, \Lambda \neq 0, R \neq 0)$. This means that every $(\mathbb{R}^2, \Lambda, R, X)$ admits a Vessiot–Guldberg Lie algebra belonging to one of the previously mentioned classes¹.

Lemma 6. *Every Jacobi manifold on the plane with $R \neq 0$ and $\Lambda \neq 0$ admits a local coordinate system $\{s, t\}$ where $R = \partial_s$ and $\Lambda = \partial_s \wedge \partial_t$.*

Proof. Since it is assumed $R \neq 0$, there exist local coordinates $\{s, t_0\}$ on which $R = \partial_s$. Meanwhile, we have that $\Lambda = \Lambda(s, t_0)\partial_s \wedge \partial_{t_0}$. Since $[R, \Lambda]_{SN} = 0$, we get that $\partial_s \Lambda = 0$ and $\Lambda = \Lambda(t_0)$. Hence, $\Lambda = \Lambda(t_0)\partial_s \wedge \partial_{t_0}$. As we consider $\Lambda \neq 0$, we can define a new variable $t = t(t_0)$ such that $dt/dt_0 \equiv \Lambda^{-1}(t_0)$. Finally, $\Lambda = \partial_s \wedge \partial_t$. \square

Definition 7. We call the local coordinate variables $\{s, t\}$ of the above lemma *local rectifying coordinates* of the Jacobi manifold on the plane.

Lemma 8. *Let $(\mathbb{R}^2, \Lambda, R)$ be a Jacobi manifold with $R_\xi \neq 0$ and $\Lambda_\xi \neq 0$ at every $\xi \in \mathbb{R}^2$. The Lie algebra morphism $\phi : C^\infty(\mathbb{R}^2) \rightarrow \text{Ham}(\mathbb{R}^2, \Lambda, R)$ has non-trivial kernel. On local rectifying coordinates, we have $\ker \phi = \langle e^t \rangle$.*

Proof. If $f \in C^\infty(\mathbb{R}^2)$ belongs to $\ker \phi$, then $\widehat{\Lambda}(df) + fR = 0$. In local rectifying coordinates, we get

$$\partial_s f \partial_t - \partial_t f \partial_s + f \partial_s = 0 \Rightarrow \begin{cases} \partial_s f = 0 \\ \partial_t f = f \end{cases} \Rightarrow f = \lambda e^t, \quad \lambda \in \mathbb{R}. \quad \square$$

Proposition 9. *If a Lie system on the plane is related to a Vessiot–Guldberg Lie algebra V of vector fields containing two non-zero vector fields X_1, X_2 satisfying $[X_1, X_2] = X_1$ and $X_1 \wedge X_2 = 0$, then V does not consist of Hamiltonian vector fields relative to any Jacobi manifold with $R \neq 0$ and $\Lambda \neq 0$.*

Proof. Let us take a rectifying coordinate system for $(\mathbb{R}^2, \Lambda, R)$. Since ϕ is a morphism of Lie algebras, we get that X_1, X_2 amount to the existence of non-zero functions h_1 and h_2 such that $\{h_1, h_2\}_{\Lambda, R} = h_1 + g$, where $g \in \ker \phi$. So,

$$\{h_1, h_2\}_{\Lambda, R} = \Lambda(dh_1, dh_2) + h_1 R h_2 - h_2 R h_1 = h_1 + g. \quad (8.13)$$

¹To exclude P_1 with $\alpha \neq 0$, we need a trivial modification of Proposition 10 using exactly the same line of thought.

Meanwhile, $X_1 \wedge X_2 = 0$ implies that

$$\widehat{\Lambda}(dh_1) \wedge \widehat{\Lambda}(dh_2) + R \wedge [h_1 \widehat{\Lambda}(dh_2) - h_2 \widehat{\Lambda}(dh_1)] = 0. \quad (8.14)$$

Using local rectifying coordinates, we see that $\widehat{\Lambda}(dh_i) = (Rh_i)\partial_t - \partial_t h_i R$ and $R \wedge \widehat{\Lambda}(dh_i) = (Rh_i)\Lambda$ for $i = 1, 2$. Hence,

$$[(Rh_1)\partial_t - \partial_t h_1 R] \wedge [(Rh_2)\partial_t - \partial_t h_2 R] + [h_1(Rh_2) - h_2(Rh_1)]\Lambda = 0 \quad (8.15)$$

and

$$\begin{aligned} 0 &= (Rh_1\partial_t h_2 - Rh_2\partial_t h_1)\Lambda + (h_1Rh_2 - h_2Rh_1)\Lambda \\ &\quad \Downarrow \\ &= \Lambda(dh_1, dh_2) + h_1Rh_2 - h_2Rh_1 = 0. \end{aligned} \quad (8.16)$$

This amounts to $\{h_1, h_2\}_{\Lambda, R} = 0$, which implies that $0 = h_1 + g$ and $X_1 = 0$. This is impossible by assumption and X_1 and X_2 cannot be Hamiltonian. \square

Proposition 10. *There exists no Jacobi manifold on the plane with $\Lambda \neq 0$ and $R \neq 0$ turning the elements of a Lie algebra diffeomorphic to $\langle \partial_x, \partial_y, x\partial_x + \alpha y\partial_y \rangle$, with $\alpha \notin \{0, -1\}$, into Hamiltonian vector fields.*

Proof. Let us proceed by reduction to absurd. Assume (N, Λ, R) to be a Jacobi manifold turning the above mentioned vector fields into Hamiltonian ones. Taking local rectifying coordinates for the Jacobi manifold, it turns out that every Lie algebra diffeomorphic to the previous one can be written in terms of three vector fields X_1, X_2, X_3 diffeomorphic to $\partial_x, \partial_y, x\partial_x + \alpha y\partial_y$. Then, they satisfy $[X_1, X_2] = 0, [X_1, X_3] = X_1, [X_2, X_3] = \alpha X_2, X_1 \wedge X_2 \neq 0$ and

$$X_3 = \mu_2 X_1 + \alpha \mu_1 X_2 \quad (8.17)$$

for certain first-integrals μ_1 and μ_2 for X_1 and X_2 , respectively. From (8.17) and using that X_1, X_2, X_3 are Hamiltonian for certain Hamiltonian functions h_1, h_2, h_3 , correspondingly, we obtain

$$\widehat{\Lambda}(dh_3) + h_3 R = \mu_2 \widehat{\Lambda}(dh_1) + \mu_2 h_1 R + \alpha \mu_1 \widehat{\Lambda}(dh_2) + \alpha \mu_1 h_2 R \quad (8.18)$$

and, by means of the rectified expression for Λ and R , we get

$$\begin{aligned} \frac{\partial h_3}{\partial s} &= \mu_2 \frac{\partial h_1}{\partial s} + \alpha \mu_1 \frac{\partial h_2}{\partial s}, \\ \frac{\partial h_3}{\partial t} &= \mu_2 \frac{\partial h_1}{\partial t} + \alpha \mu_1 \frac{\partial h_2}{\partial t} - \mu_2 h_1 - \alpha \mu_1 h_2 + h_3. \end{aligned} \quad (8.19)$$

Since $[X_1, X_3] = X_1$, then $\{h_1, h_3\}_{\Lambda, R} = h_1 + \lambda_1 e^t$, where e^t is a function with zero Hamiltonian vector field and $\lambda_1 \in \mathbb{R}$. Hence,

$$h_1 + \lambda_1 e^t = \{h_1, h_3\}_{\Lambda, R} = \Lambda(dh_1, dh_3) + h_1(Rh_3) - h_3(Rh_1). \quad (8.20)$$

Simplifying and using previous expressions (8.19), we find that

$$h_1 + \lambda_1 e^t = \mu_1 \left(\frac{\partial h_2}{\partial t} \frac{\partial h_1}{\partial s} - \frac{\partial h_1}{\partial t} \frac{\partial h_2}{\partial s} + h_1 \frac{\partial h_2}{\partial s} - h_2 \frac{\partial h_1}{\partial s} \right) = \alpha \mu_1 \{h_1, h_2\}_{\Lambda, R}. \quad (8.21)$$

As $[X_1, X_2] = 0$, then $\{h_1, h_2\}_{\Lambda, R} = \lambda e^t$ for a certain constant $\lambda \in \mathbb{R}$. Hence, $h_1 = (\alpha \mu_1 \lambda - \lambda_1) e^t$ and $\lambda \neq 0$. Proceeding analogously for $\{h_2, h_3\}_{\Lambda, R} = \alpha(h_2 + \lambda_2 e^t)$, we obtain

$$h_2 + \lambda_2 e^t = \mu_2 \{h_2, h_1\}_{\Lambda, R} \Rightarrow h_2 = (-\mu_2 \lambda / \alpha - \lambda_2) e^t. \quad (8.22)$$

Writing the compatibility condition for the system (8.19), we reach to

$$(\alpha + 1) \lambda \left(\frac{\partial \mu_1}{\partial s} \frac{\partial \mu_2}{\partial t} - \frac{\partial \mu_2}{\partial s} \frac{\partial \mu_1}{\partial t} \right) = 0. \quad (8.23)$$

This implies that $d\mu_1 \wedge d\mu_2 = 0$. Since μ_1 is a first-integral for X_1 and μ_2 is a first-integral for X_2 , this means that $X_1 \wedge X_2 = 0$, which is impossible by assumption. This finishes the proof. \square

Lie symmetry for differential equations

1 What can a symmetry transformation achieve?

Invariance of a differential equation under a transformation is synonymous of *existence of symmetry* and, consequently, of *conserved quantities*. The invariance of a differential equation under a transformation [331] helps us to achieve partial or complete integration of such an equation. Many of the existing solutions to physical phenomena described by differential equations have been obtained through symmetry arguments [339, 358, 388].

A conserved quantity for a first-order differential equation can lead to its integration by quadrature, whilst for higher-order ones, it leads to a reduction of their order. A number of conserved quantities equals to the number of the degrees of freedom of the ordinary differential equation leads to the complete integration of such an equation [339, 388]. Nevertheless, finding as many conserved quantities as the number of degrees of freedom of the differential equation, is a nontrivial task.

The most famous and established method for finding point symmetries is the *classical Lie symmetry method* (CLS) developed by Lie from 1881 [296, 298, 299, 339, 388]. Although the classical Lie symmetry analysis represents a very powerful tool, it yields cumbersome calculations to be solved by hand. Notwithstanding, the increased number of available software packages for symbolic calculus has made of generalizations of the CLS analysis very tractable approaches to find conservation laws, explicit solutions, etc.

If we impose that symmetries leave certain submanifold invariant, we find the class of *conditional or nonclassical symmetries method* (NSM) introduced in 1969 by Bluman and Cole [64]. The origin of this proposal rooted in the obtention of exact solutions for the linear *heat equation* [181] that were not deducible from the former CLS. The NSM method has been successfully applied by many authors [26,

127, 332, 334] and the author of this manuscript in [167, 169]. A few examples will be displayed in forthcoming sections. A remarkable difference between the classical and the nonclassical Lie symmetry analysis is that the latter provides us with no longer linear systems of differential equations to calculate the symmetries.

During the last decades of the XX century, attention has been focused on Lie symmetry computation methods for nonlinear evolution equations of hydrodynamic type in Plasma Physics, cosmological models and other fields [5, 66, 102]. These equations have solutions in form of solitons. In order to study and to derive solutions of this kind of equations, the classical and nonclassical Lie symmetry approaches have proven their efficiency. Given their recurrent appearance in the Physics and Mathematics literature, it is important to settle an algorithmic method for the calculation of their symmetries, eventual reduction and finding their solutions. For this reason, we will dedicate this part of this thesis to calculating symmetries and reducing differential equations, ranging from ODEs to hierarchies of PDEs and Lax pairs. In this way, the plan of the chapter goes as follows.

Section 1: The CLS analysis for ODEs and applications: We explain the CLS for ODEs. We consider Lie transformations of the most general type and we give theorems which facilitate the calculation of Lie symmetries. A number of steps will summarize the quasi-algorithmical symmetry search and it will be applied to some particular ODEs: the second- and third-order Kummer–Schwarz equations, the first- and second-order Riccati equations and the Milne–Pinney equations, revised in Chapter 3.

Section 2: Lie symmetry for Lie systems: We inspect certain types of Lie symmetries for Lie systems. We prove that every Lie system admits a Lie algebra of Lie symmetries whose geometric properties are determined by the Vessiot–Guldberg Lie algebra of the Lie system. To illustrate our results, we analyze a particular type of Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems and we apply our findings to inspect Riccati equations, the Cayley–Klein Riccati equations (defined for the first time in this thesis), quaternionic Riccati equations with real t -dependent coefficients, generalized Darboux–Brioschi–Halphen systems and second-order Kummer–Schwarz equations. Next, we analyze a particular type of Lie symmetries for $\text{Aff}(\mathbb{R})$ -Lie systems and we apply our results to Buchdahl and Painlevé–Ince equations. To conclude, we generalize the search of this type of Lie symmetries to PDE Lie systems. In a similar fashion, we search for a particular type of Lie symmetries and illustrate the interest of our theory by studying partial Riccati equations.

It is remarkable that the application of our methods to Lie systems, leads to obtaining Lie symmetries for all Lie systems sharing isomorphic Vessiot–Guldberg Lie algebras. In this way, we are concluding results for several different Lie systems, simultaneously.

Section 3: The CLS and NSM for PDEs: This section contains the theory for Lie point symmetry search applied to PDEs. We explain in a number of steps

how to apply the method and to implement the difference between the CLS and the NSM. For PDEs, we can reduce the system to lower dimension by getting rid of one of the independent variables through symmetry arguments. We will explain the process of reduction with the aid of the method of characteristics. We will illustrate the calculation of classical Lie symmetries for PDEs with the example of the inviscid Burgers equation.

Section 4: Application to PDEs and spectral problems in Physics: We start by applying both the CLM and NSM analysis to nonlinear PDEs of hydrodynamical type, and their corresponding Lax pairs. We start with the example of the Bogoyanlevski–Kadomtsev–Petviashvili equation in $2 + 1$ dimensions ($2 + 1$ -BKP, henceforth) and its corresponding Lax pair, a two component, nonisospectral problem defined over the complex field, in $2 + 1$. The author of this thesis and her collaborators have obtained both the classical and nonclassical symmetries and have compared them. Through the obtained symmetries, we have reduced both the equation and Lax pair and we have achieved several reductions. Two of them are of special interest. The first corresponds with the KdV equation in $1 + 1$ dimensions, what implies that $2 + 1$ -BKP is a generalization of the $1 + 1$ KdV equation. The second interesting reduction can be interpreted from a physical point of view. We proceed by studying this reduction and its associated Lax pair in $1 + 1$ dimensions, with a second iteration through the Lie symmetry calculus (both classical and nonclassical) and later reduction to an ODE (considering the nonisospectral parameter as another independent variable). Specifically, we prove that the nonclassical symmetries are more global and contain the classical case. A second list of reductions is shown for the $1 + 1$ dimensional case.

Section 5: Application to spectral problems associated with hierarchies of PDEs: Next, we apply the CLS and NSM to complete hierarchies of nonlinear partial differential equations and their corresponding Lax pairs. A *hierarchy* is a set of differential equations that are related through a *recursion operator* [125]. The n -th iteration of the recursion operator provides us with the n -th equation member of the hierarchy. In particular, we take a look at two examples of hierarchies of partial differential equations of a similar form: the so called hierarchy of Camassa–Holm in $2 + 1$ dimensions [79] and the Qiao hierarchy or modified Camassa–Holm hierarchy in $2 + 1$ dimensions [359, 360, 361] (they are correspondingly denoted as CHH($2 + 1$) and Qiao $2 + 1$ or mCHH($2 + 1$)). We will make an exhaustive description of the nonclassical symmetries attending to different values of the functions and constants of integration present in the symmetries.

A number of reductions arise for both hierarchies. We will show the reductions of their corresponding Lax pairs and we check whether the nonisospectral character is inherited within lower dimensions. For CHH($2 + 1$), among its reductions we find the positive and negative Camassa–Holm hierarchies in $1 + 1$ dimensions and their corresponding Lax pairs [167]. Some of the reductions of the Lax pair will maintain

their nonisospectral character, what is quite unusual for 1 + 1 dimensions. For the Qiao hierarchy, we obtained reductions which can be physically interpreted as well [168].

2 The CLS for ODEs

Consider a p -th order system of q ordinary differential equations, with an independent variable t and k dependent fields $u_1(t), \dots, u_k(t)$, given by

$$\Psi^l(t, u_j^r(t)) = 0, \quad l = 1, 2, \dots, q, \quad (2.1)$$

where

$$u_j^r = \frac{\partial^r u_j}{\partial t^r}, \quad r = 1, \dots, p, \quad j = 1, \dots, k.$$

and occasionally we will use $u = (u_1, \dots, u_k)$. Eventually, we will also write $(u_j)_{\underbrace{t \dots t}_{r\text{-times}}}$ for $\partial^r u_j / \partial t^r$. A solution to this system of p -th order ODEs is a set of functions $u_j(t)$ such that

$$\Psi^l(t, u_j(t), u_j^1(t), \dots, u_j^p(t)) = 0, \quad l = 1, \dots, q, \quad j = 1, \dots, k. \quad (2.2)$$

Geometrically, the variables $\{t, u_j^r(t) \mid j = 1, \dots, k, r = 1, \dots, p\}$ can be understood as a coordinate system for $J^p\pi \simeq \mathbb{R} \times T^pN$ with $\pi : (t, u_1, \dots, u_k) \in N_{\mathbb{R}} \equiv \mathbb{R} \times N \mapsto t \in \mathbb{R}$. Recall that we write

$$j_i^p u \equiv (t, u_j, u_j^1, \dots, u_j^p), \quad \forall j \quad (2.3)$$

for a general point of $J^p\pi$. Moreover, the differential equation (2.1) can be understood as a submanifold $\mathcal{E} = \Psi^{-1}(0) \subset J^p\pi$, where $\Psi = (\Psi^1, \dots, \Psi^q)$. Meanwhile, the particular solutions of (2.1) are can be understood sections $u : t \in \mathbb{R} \rightarrow (t, u(t)) \in N_{\mathbb{R}}$ of the bundle $(N_{\mathbb{R}}, \mathbb{R}, \pi)$ whose prolongation to $J^p\pi$ satisfies (2.1), i.e. we have (2.2).

Let us consider the ϵ -parametric group of transformations

$$\begin{cases} \bar{t} \rightarrow t + \epsilon \xi(t, u_j) + O(\epsilon^2), \\ \bar{u}_j \rightarrow u_j + \epsilon \eta_j(t, u_j) + O(\epsilon^2), \end{cases} \quad j = 1, \dots, k. \quad (2.4)$$

The associated vector field with the transformation reads

$$X = \xi(t, u_j) \frac{\partial}{\partial t} + \sum_{j=1}^k \eta_j(t, u_j) \frac{\partial}{\partial u_j} \quad (2.5)$$

Definition 1. We call a transformation (2.4) a *point Lie transformation* if the coefficients $\xi(t, u_j)$ and $\eta_j(t, u_j)$ do not depend on higher-order derivatives u_j^r for $r = 1, \dots, p$.

Up to p -order derivatives, transformation (2.4) reads

$$\begin{cases} t \rightarrow t + \epsilon\xi + O(\epsilon^2), \\ u_j \rightarrow u_j + \epsilon\eta_j(t, u_j) + O(\epsilon^2), \\ u_j^{(1)} \rightarrow u_j^{(1)} + \epsilon\eta_j^{(1)}(t, u_j) + O(\epsilon^2), \\ u_j^{(2)} \rightarrow u_j^{(2)} + \epsilon\eta_j^{(2)}(t, u_j) + O(\epsilon^2), \\ \dots \\ u_j^{(p)} \rightarrow u_j^{(p)} + \epsilon\eta_j^{(p)}(t, u_j) + O(\epsilon^2), \end{cases} \quad i = 1, \dots, k. \quad (2.6)$$

on $J^p\pi$. We shall drop the dependency (t, u_j) in $\xi(t, u_j), \eta_j(t, u_j), \eta_j^{(1)}(t, u_j), \dots$ for convenience and express them simply as $\xi, \eta_j, \eta_j^{(1)}, \dots$ in the forthcoming expressions. Geometrically, this amounts to a Lie group action $\Phi : \mathbb{R} \times J^p\pi \rightarrow J^p\pi$.

The infinitesimal generator of this ϵ -parametric group of transformation reads

$$X^p = \xi \frac{\partial}{\partial t} + \sum_{j=1}^k \left(\eta_j \frac{\partial}{\partial u_j} + \eta_j^{(1)} \frac{\partial}{\partial u_j^{(1)}} + \eta_j^{(2)} \frac{\partial}{\partial u_j^{(2)}} + \dots + \eta_j^{(p)} \frac{\partial}{\partial u_j^{(p)}} \right), \quad (2.7)$$

where we have not specified the dependency of the coefficients of the infinitesimal generator, due to simplicity.

Recall that we gave in (4.20) in Chapter 2 how to obtain the coefficients $\eta_j^{(p)}$. From a geometric viewpoint, transformation (2.4) can be viewed as the infinitesimal part of a Lie group action $\Phi : (\epsilon; t; u) \in \mathbb{R} \times N_{\mathbb{R}} \mapsto (\bar{t}, \bar{u}) \equiv \Phi(\epsilon; t, u) \in N_{\mathbb{R}}$. Meanwhile, transformation (2.6) can be understood as a Lie group action $\widehat{\Phi} : (\epsilon; j_t^p u) \in \mathbb{R} \times J^p\pi \mapsto j_t^p \bar{u} \equiv \widehat{\Phi}(j_t^p u) \equiv \widehat{\Phi}(\epsilon; j_t^p u) \in J^p\pi$.

Definition 2. We say that transformation (2.6) is a *Lie point symmetry* of (2.1) if it maps points of \mathcal{E} into points of \mathcal{E} , namely

$$\Psi^l(t, u_j(t), u_j^{(r)}(t)) \Rightarrow \Psi^l(\bar{t}, \bar{u}_j(\bar{t}), \bar{u}_j^{(r)}(\bar{t})), \quad \forall j, r, l. \quad (2.8)$$

Notice that the invariance under transformations is independent of the choice of the coordinate system in which the system of ODEs and its solutions are given. Only the explicit form of the symmetry will change.

Lemma 3. Given a Lie point symmetry (2.6), we have that if $u_j(t)$ is a solution of (2.1), then $\bar{u}_j(\bar{t})$ for all $j = 1, \dots, k$, is a new solution of the same equation.

Theorem 4. We say that the infinitesimal generator X^p is a Lie point symmetry of the system of ODEs of p -order given by (2.1) if

$$X^p(\Psi^l) = 0, \quad l = 1, \dots, q, \quad (2.9)$$

on the submanifold $\mathcal{E} = \Psi^{-1}(0)$, with $\Psi = (\Psi^1, \dots, \Psi^q)$.

Proof. According to the Definition 2 of Lie point symmetry, if $\Psi(j_t^p u) = 0$, then $\Psi(\widehat{\Phi}(j_t^p u)) = 0$ for every $\epsilon \in \mathbb{R}$. Hence,

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Psi^l(\widehat{\Phi}(j_t^p u)) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Psi^l(j_t^p u) = 0, \quad (2.10)$$

$\forall j_t^p u \in \Psi^{-1}(0)$, and $l = 1, \dots, q$. Meanwhile,

$$\begin{aligned} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Psi^l(\widehat{\Phi}(j_t^p u)) &= d\Psi^l \Big|_{j_t^p u} \left(\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \widehat{\Phi}(j_t^p u) \right) \\ &= d\Psi^l \Big|_{j_t^p u_j} (X^p(j_t^p u)) = X^p(\Psi^l)(j_t^p \bar{u}), \end{aligned} \quad (2.11)$$

for all $j_t^p u \in \Psi^{-1}(0)$ and $l = 1, \dots, q$. Therefore, $X^p \Psi^l = 0$ on $\Psi^{-1}(0)$. \square

Definition 5. We call *invariant surface condition* of a system of ODEs the relation

$$\eta_j - (u_j)_t \xi = 0, \quad j = 1, \dots, k, \quad (2.12)$$

2.1 Algorithmic computation of Lie point symmetry for systems of ODEs

It is possible to follow an algorithmic procedure to compute the Lie point symmetries of p -order systems of ODEs. The process goes as follows.

1. Introduce the Lie point transformation (2.6) in (2.1).
2. Select the terms in the zero-order in ϵ , which retrieves the initial equations. Isolate the higher-order derivative, $u_j^{(p)}$, appearing in each of the untransformed equations in system (2.1).
3. Select the terms in the first-order of ϵ . Introduce the prolongations computed according to (4.20) given in Chapter 2, together with the higher-order derivatives $u_j^{(p)}$ obtained in the previous step. Take the coefficients of the remaining derivatives of the dependent variables in different orders, from the $u_j^{(p-1)}$ to the first $u_j^{(1)}$ for $1 \leq j \leq k$. It is legit to set each of these conditions equal to zero, given the definition of Lie point symmetries in which ξ and η_j do not depend on derivatives of any order.
4. At this point, we decide for a classical or nonclassical approach. For the nonclassical, we implement the invariant surface condition (2.12), namely $u_j^{(1)} = \eta_j / \xi$, with $j = 1, \dots, k$. This leads to an overdetermined system of differential equations (which is no longer linear) for the coefficients ξ and η_j of the infinitesimal generator.

The more possible conditions we impose, the more general the symmetries will be. *This implies that nonclassical symmetries are more general than the classical ones and they must include them as a particular case.*

5. Solving the resulting system is not an easy task, specially in the nonclassical approach, but it leads to the desired symmetries.

We aim to apply the theory exposed for the CLS to some ODEs with physical relevance. The equations we inspect have been devised in Chapter 3 of this manuscript.

From now on, we will use the following notation in the forthcoming examples and sections. We will denote the successive derivatives with respect to t as $u_j^{(r)} = \underbrace{(u_j)_{t, \dots, t}}_{r\text{-times}}$.

Example 1. Let us study the **second-order Kummer–Schwarz equation** [52, 308, 320, 393]

$$u_{tt} - \frac{3}{2} \frac{u_t^2}{u} + 2c_0 u^3 - 2c_1(t)u = 0, \quad (2.13)$$

with c_0 being a real constant and $c_1(t)$ being any t -dependent function. It is a particular case of the Gambier equations [87, 224] and appears in cosmological models [329]. We search for a general Lie point symmetry for (2.13). As usual, we denote by ξ the coefficient of the infinitesimal generator associated with t and the one associated with u is denoted by η . Hence, the infinitesimal ϵ -parametric group of diffeomorphisms of the symmetry reads

$$\begin{cases} \bar{t} \rightarrow t + \epsilon \xi(t, u), \\ \bar{u} \rightarrow u + \epsilon \eta(t, u). \end{cases} \quad (2.14)$$

In a similar fashion, the functions present in the transformation must be expanded in a Taylor expansion up to first-order in ϵ

$$c_1(\bar{t}) \rightarrow c_1(t) + \epsilon \frac{dc_1(t)}{dt} \xi(t, u). \quad (2.15)$$

Since (2.13) is a second-order equation, we prolong this transformation to first and second-order derivatives as well,

$$\begin{cases} \bar{u}_{\bar{t}} \rightarrow u_t + \epsilon \eta_t(t, u), \\ \bar{u}_{\bar{t}\bar{t}} \rightarrow u_{tt} + \epsilon \eta_{tt}(t, u) \end{cases} \quad (2.16)$$

and we calculate the prolongations η_t and η_{tt} according to (4.20) in Chapter 2. Introducing (2.14) into (2.13) leads us to an overdetermined system which can only be solved in case of considering particular cases of symmetry. Let us for example suppose that $\xi(t, u)$ is not a function of u , i.e., $\xi = \xi(t)$. Hence,

$$\xi(t) = -G_1(t) - K_1, \quad \eta(t, u) = \frac{dG_1}{dt} u, \quad (2.17)$$

for a certain function $G_1 = G_1(t)$.

Example 2. The **third-order Kummer-Schwarz** equation reads

$$u_{ttt} - \frac{3}{2} \frac{u_{tt}}{u} + 2C_0(u)u_t^3 - 2b_0(t)u_t = 0. \quad (2.18)$$

We propose the same Lie transformation as in (2.14). This time, we need to expand the functions $C_0(u)$ and $b_0(t)$

$$C_0(\bar{u}) \rightarrow C_0(u) + \epsilon \frac{\partial C_0(u)}{\partial u} \eta(t, u), \quad b_0(\bar{u}) \rightarrow b_0(u) + \epsilon \frac{\partial b_0(t)}{\partial t} \xi(t, u). \quad (2.19)$$

Since (2.18) is a third-order equation, transformation (2.14) has to be extended up to the third-order derivative apart from considering (2.16),

$$\bar{u}_{ttt} \rightarrow u_{ttt} + \epsilon \eta_{ttt}(t, u), \quad (2.20)$$

where η_{ttt} is calculated according to (4.20) in Chapter 2. Invariability of (2.18) under such a transformation arises in an overdetermined system of differential equations whose unique possibility reads

$$\xi = K_1, \quad \eta(u) = K_2 u, \quad (2.21)$$

with K_1 and K_2 being two constants and the two following conditions being satisfied

$$\frac{db_0(t)}{dt} = 0, \quad C_0(u) = \frac{C_1}{u^2}, \quad (2.22)$$

where C_1 is another constant of integration. Conditions (2.22) had been known previously [308], but here we have retrived them by our Lie symmetry approach which strictly needs (2.22) being satisfied in order to (2.16), (2.19) and (2.20) be a Lie symmetry transformation. As a conclusion, we see that the constants K_1, K_2 appearing in (2.21) are arbitrary functions of time, $G_1(t)$ in (2.17).

Example 3. Consider now the **first-order Riccati equation**,

$$u_t = a_0(t) + a_1(t)u + a_2(t)u^2. \quad (2.23)$$

This equation is very important in Theoretical Physics, in the introduction of Witten's Supersymmetric Quantum Mechanics [421]. A general Lie point symmetry transformation for these equations reads

$$\begin{cases} \bar{t} \rightarrow t + \epsilon \xi(t, u), \\ \bar{u} \rightarrow u + \epsilon \eta(t, u). \end{cases} \quad (2.24)$$

Applying this transformation to the time-dependent functions of (2.23),

$$a_i(\bar{t}) \rightarrow a_i(t) + \epsilon \frac{\partial a_i(t)}{\partial t} \xi(t, u), \quad i = 0, 1, 2. \quad (2.25)$$

Introducing this transformation into (2.24), we need to compute the first-order prolongation

$$\bar{u}_{\bar{t}} \rightarrow u_t + \epsilon \eta_t(t, u) \quad (2.26)$$

with the computation of η_t according to (4.20) in Chapter 2. An overdetermined system for the symmetries suggests the consideration of an Ansatz $\xi = \xi(t)$. This leads to

$$\xi(t) = \frac{1}{\sqrt{a_2(t)}\sqrt{a_0(t)}}, \quad \eta(t, u) = \left(-\frac{1}{2} \frac{\frac{da_2(t)}{dt}}{a_2^{3/2}\sqrt{a_0(t)}} + \frac{1}{2} \frac{\frac{da_0(t)}{dt}}{\sqrt{a_2(t)}a_0^{3/2}(t)} \right) u. \quad (2.27)$$

Example 4. The **second-order Riccati equation** accounts with multiple applications in scientific fields (see Chapter 3 and [120, 223]). It takes the form

$$u_{tt} + (f_0(t) + f_1(t)u)u_t + C_0(t) + C_1(t)u + C_2(t)u^2 + C_3(t)u^3 = 0, \quad (2.28)$$

with

$$f_1(t) = 3\sqrt{C_3(t)}, \quad f_0(t) = \frac{C_2(t)}{\sqrt{C_3(t)}} - \frac{1}{2C_3(t)} \frac{dC_3}{dt}(t), \quad (2.29)$$

where $C_1(t), C_2(t)$ are arbitrary t -dependent functions and $C_3(t) > 0$ [419]. These conditions are retrieved from the property of integrability of the equation in the Painlevé sense. We search for a transformation as in (2.24), which needs to be prolonged up to the second derivative as

$$\bar{u}_{\bar{t}} \rightarrow u_t + \epsilon \eta_t(t, u), \quad \bar{u}_{\bar{t}\bar{t}} \rightarrow u_{tt} + \epsilon \eta_{tt}(t, u), \quad (2.30)$$

with η_t and η_{tt} being calculated with the aid of (4.20) in Chapter 2. The functions appearing in (2.28) are Taylor expanded up to first order as

$$C_j(\bar{t}) \rightarrow C_j(t) + \epsilon \frac{dC_j(t)}{dt} \xi(t, u), \quad j = 1, 2, 3. \quad (2.31)$$

and

$$f_0(\bar{t}) \rightarrow f_0(t) + \epsilon \frac{df_0(t)}{dt} \xi(t, u), \quad f_1(\bar{t}) \rightarrow f_1(t) + \epsilon \frac{df_1(t)}{dt} \xi(t, u). \quad (2.32)$$

We consider the Ansatz $\xi = \xi(t)$. Under such a circumstance, we find

$$\xi(t) = F(t), \quad \eta(t, u) = g_1(t)u + g_2(t), \quad (2.33)$$

where

$$F(t) = -\frac{\int \sqrt{C_3(t)} g_1(t) dt}{\sqrt{C_3(t)}}. \quad (2.34)$$

The function $g_2(t)$ is not arbitrary, it satisfies that

$$g_2(t) = -\frac{\frac{dg_1(t)}{dt}}{\sqrt{C_3(t)}} + \frac{1}{3} \frac{\frac{dC_2(t)}{dt} \int \sqrt{C_3(t)} g_1(t) dt}{C_3(t)^{3/2}} + \frac{1}{3} \frac{C_2(t) g_1(t)}{C_3(t)} - \frac{1}{3} \frac{C_2(t) \frac{dC_3(t)}{dt} \int \sqrt{C_3(t)} g_1(t) dt}{C_3(t)^{5/2}}. \quad (2.35)$$

Nevertheless, the nonarbitrary function $g_1(t)$ has to satisfy a cumbersome equation whose explicit expression is omitted but can be straightforwardly obtained after a long but easy calculation. Given this situation, we try to simplify our result by imposing further additional conditions. For example, let us suppose that $g_2(t) = 0$. Now, the symmetries have a more pleasant expression

$$\xi(t) = F(t), \quad \eta(t, u) = g_1(t)u, \quad (2.36)$$

with the additional relations

$$F(t) = -\frac{\int \sqrt{C_3(t)} g_1(t) dt}{\sqrt{C_3(t)}}, \quad g_1(t) = \frac{C_2(t) e^{\frac{1}{3} \int \frac{C_2(t)}{\sqrt{C_3(t)}} dt}}{C_3(t)}. \quad (2.37)$$

The process of symmetry calculation recovers some well-known relations among the coefficients of the equation (2.28) in order to (2.24) and (2.30) to (2.32) be a symmetry of (2.28). Such relations are

$$f_1(t) = 3\sqrt{C_3(t)}, \quad f_0(t) = \frac{C_2(t)}{\sqrt{C_3(t)}} - \frac{1}{2} \frac{dC_3(t)/dt}{C_3(t)} \quad (2.38)$$

which had previously been devised in other papers [108].

Example 5. The Milne–Pinney equation, namely

$$u_{tt} + \omega(t)^2 u - \frac{K}{u^3} = 0. \quad (2.39)$$

where $\omega(t)$ is the frequency of oscillation of the system and $K > 0$ is a constant, was studied in Chapter 3 [97, 109, 354]. It is defined in $\mathbb{R} - \{0\}$ and it is invariant under parity. That is, for a particular solution $x(t)$, we have that $\bar{x}(t) = -x(t)$ is another solution.

From the point of view of the Lie analysis, we search for a Lie point symmetry

$$\begin{cases} \bar{t} \rightarrow t + \epsilon \xi(t, u), \\ \bar{u} \rightarrow u + \epsilon \eta(t, u). \end{cases} \quad (2.40)$$

Similarly, we expand in Taylor series up to first-order the function $w(t)$

$$w(\bar{t}) \rightarrow w(t) + \epsilon \frac{dw(t)}{dt} \xi(t, u). \quad (2.41)$$

Extending the transformation to the second derivative, we have $\bar{u}_{\bar{t}\bar{t}} = u_{tt} + \epsilon \eta_{tt}(t, u)$, where η_{tt} is calculated according to (4.20) in Chapter 2. The resulting symmetries read

$$\xi(t, u) = -4\omega(t) \frac{d\omega(t)}{dt} F(t) - 4\omega(t)^2 \frac{dF(t)}{dt}, \quad \eta(t, u) = \frac{dF(t)}{dt} u, \quad (2.42)$$

where $F(t)$ reads

$$F(t) = \exp \left(-2 \int \frac{10\omega_t^3 \omega + \omega_t^2 + \omega \omega_{tt} + 4\omega_t \omega_{tt} \omega^2 - 2\omega^3 \omega_{ttt} - 16\omega^6 \omega_t^2}{16\omega \omega_t - 16\omega^3 \omega_{tt} + 44\omega_t^2 \omega^2 + 16\omega^6 - 32\omega^7 \omega_t + 1} dt \right). \quad (2.43)$$

3 Lie symmetries for Lie systems

Given the interest of Lie symmetries for the study of differential equations, we devote this section to the study of Lie symmetries for Lie systems. We pioneer the study and application of certain types of Lie symmetries for higher-order Lie systems and PDE Lie systems, i.e., the generalizations of Lie systems to the realms of higher-order and partial differential equations [87, 91, 97, 166]. As a byproduct, we develop one of the few applications of the theory of Lie systems in the investigation of systems of partial differential equations. Additionally, only a few particular results about Lie symmetries for Lie systems had appeared before in [109]. We here accomplish a careful and exhaustive study and calculation of such Lie symmetries [166].

3.1 On certain Lie symmetries for Lie systems

We are now concerned with the study of certain Lie symmetries for Lie systems. We prove that the features of these Lie symmetries are determined by the algebraic structure of a Vessiot–Guldberg Lie algebra of the Lie system.

Consider a general the Lie system X given by

$$X = \sum_{\alpha=1}^r b_{\alpha}(t) X_{\alpha}, \quad (3.1)$$

where b_1, \dots, b_r are some t -dependent functions. Then, (3.1) admits a Vessiot–Guldberg Lie algebra V with basis X_1, \dots, X_r and structure constants $c_{\alpha\beta\gamma}$. Let us study the Lie symmetries of X of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_{\alpha}(t) X_{\alpha}, \quad (3.2)$$

where f_0, \dots, f_r are certain t -dependent functions. We denote by \mathcal{S}_X^V the space of such Lie symmetries. Recall that $Y \in \mathcal{S}_X^V$ if and only if

$$[Y, \bar{X}] = h\bar{X} \quad (3.3)$$

for a function $h \in C^\infty(\mathbb{R} \times N)$, with \bar{X} being the autonomization of X [339]. From this, it immediately follows that \mathcal{S}_X^V is a real Lie algebra of vector fields.

Using the properties of \mathcal{S}_X^V , we now characterize the elements of \mathcal{S}_X^V as particular solutions of a family of Lie systems.

Lemma 1. *The vector field Y of the form (3.2) is a Lie symmetry for the Lie system (3.1) if and only if the t -dependent functions f_0, \dots, f_r satisfy the system of differential equations*

$$\frac{df_0}{dt} = b_0(t), \quad \frac{df_\alpha}{dt} = f_0 \frac{db_\alpha}{dt}(t) + b_\alpha(t)b_0(t) + \sum_{\beta,\gamma=1}^r b_\beta(t)f_\gamma c_{\gamma\beta\alpha}, \quad (3.4)$$

for a certain t -dependent function b_0 and $\alpha = 1, \dots, r$.

Proof. From (3.1), (3.2) and (3.3), we have that

$$\begin{aligned} [Y, \bar{X}] &= \left[f_0 \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha X_\alpha, \frac{\partial}{\partial t} + \sum_{\beta=1}^r b_\beta X_\beta \right] \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(\left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} \right) X_\alpha + \sum_{\beta=1}^r b_\beta f_\alpha [X_\alpha, X_\beta] \right) \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(\left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} \right) X_\alpha + \sum_{\beta,\gamma=1}^r b_\beta f_\alpha c_{\alpha\beta\gamma} X_\gamma \right) \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} + \sum_{\beta,\gamma=1}^r b_\beta f_\gamma c_{\gamma\beta\alpha} \right) X_\alpha \\ &= h \left(\frac{\partial}{\partial t} + \sum_{\alpha=1}^r b_\alpha X_\alpha \right). \end{aligned} \quad (3.5)$$

Thus, $[Y, \bar{X}] = h\bar{X}$ is equivalent to

$$\left(-\frac{df_0}{dt} - h \right) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left[\sum_{\beta,\gamma=1}^r b_\beta f_\gamma c_{\gamma\beta\alpha} + f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} - hb_\alpha \right] X_\alpha = 0. \quad (3.6)$$

Since $\partial/\partial t, X_1, \dots, X_r$, are linearly independent over \mathbb{R} , we obtain that $[Y, \bar{X}] = h\bar{X}$ if and only if (3.4) is fulfilled with h being an arbitrary t -dependent function such that $b_0(t) = -h(t)$ for every $t \in \mathbb{R}$. Hence, the elements of \mathcal{S}_X^V are the particular solutions of (3.4) for arbitrary t -dependent functions b_0 . \square

Definition 2. We call (3.4) the *symmetry system* of the Lie system (3.1) with respect to its Vessiot–Guldberg Lie algebra V . We write Γ_X^V for the t -dependent vector field associated with (3.4), that is,

$$\Gamma_X^V = b_0(t) \frac{\partial}{\partial f_0} + \sum_{\alpha=1}^r \left(f_0 \frac{db_\alpha}{dt}(t) + b_0(t)b_\alpha(t) + \sum_{\gamma,\beta=1}^r b_\beta(t)f_\gamma c_{\gamma\beta\alpha} \right) \frac{\partial}{\partial f_\alpha}. \quad (3.7)$$

Note that (3.1) may have different Vessiot–Guldberg Lie algebras (see [102] for details). When V is understood from context, we simply call (3.4) the symmetry system of X . Let us prove that (3.4) is a Lie system.

Theorem 3. *The system Γ_X^V is a Lie system possessing a Vessiot–Guldberg Lie algebra*

$$(A_1 \oplus_S A_2) \oplus_S V_L \simeq (\mathbb{R}^{r+1} \oplus_S \mathbb{R}^r) \oplus_S V/Z(V), \quad (3.8)$$

where

$$\begin{aligned} A_1 &= \langle Z_0, \dots, Z_r \rangle \simeq \mathbb{R}^{r+1}, & A_2 &= \langle W_1, \dots, W_r \rangle \simeq \mathbb{R}^r, \\ V_L &= \langle Y_1, \dots, Y_r \rangle \simeq V/Z(V), \end{aligned} \quad (3.9)$$

with

$$Y_\alpha = \sum_{\beta,\gamma=1}^r f_\beta c_{\beta\alpha\gamma} \frac{\partial}{\partial f_\gamma}, \quad W_\alpha = f_0 \frac{\partial}{\partial f_\alpha}, \quad Z_0 = \frac{\partial}{\partial f_0}, \quad Z_\alpha = \frac{\partial}{\partial f_\alpha}, \quad (3.10)$$

with $\alpha = 1, \dots, r$, we write $A \oplus_S B$ for the semi-direct sum of the ideal A of $A + B$ with B , and $Z(V)$ is the center of the Lie algebra V .

Proof. By defining t -dependent functions

$$c_0 \equiv b_0, \quad \bar{b}_0 \equiv 0, \quad c_\alpha \equiv b_0 b_\alpha, \quad \bar{b}_\alpha \equiv \frac{db_\alpha}{dt}, \quad \alpha = 1, \dots, r, \quad (3.11)$$

we can write

$$\Gamma_X^V(t, f) = \sum_{\alpha=0}^r c_\alpha(t) Z_\alpha(f) + \sum_{\alpha=1}^r [\bar{b}_\alpha(t) W_\alpha(f) + b_\alpha(t) Y_\alpha(f)], \quad f \in \mathbb{R}^{n+1}. \quad (3.12)$$

Hence, Γ_X^V is a t -dependent vector field taking values in the linear space $V^S \equiv A_1 + A_2 + V_L$. Let us show that V^S is also a Lie algebra of vector fields. To do

so, let us first prove that $V_L \simeq V/Z(V)$. Consider $Y_\alpha, Y_\beta \in V_L$. Recalling that $[X_\alpha, X_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_\gamma$, we obtain

$$\begin{aligned}
[Y_\alpha, Y_\beta] &= \sum_{i,j,m,n=1}^r \left[c_{i\alpha j} f_i \frac{\partial}{\partial f_j}, c_{m\beta n} f_m \frac{\partial}{\partial f_n} \right] \\
&= \sum_{i,j,m,n=1}^r c_{i\alpha j} c_{m\beta n} \left(f_i \delta_j^m \frac{\partial}{\partial x_n} - f_m \delta_n^i \frac{\partial}{\partial f_j} \right) \\
&= \sum_{i,m,n=1}^r c_{i\alpha m} c_{m\beta n} f_i \frac{\partial}{\partial f_n} - \sum_{n,j,m=1}^r c_{n\alpha j} c_{m\beta n} f_m \frac{\partial}{\partial f_j} \\
&= \sum_{i,m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{m\alpha n} c_{\beta i m}) f_i \frac{\partial}{\partial f_n}. \tag{3.13}
\end{aligned}$$

Using the Jacobi identity for the structure constants $c_{\alpha\beta\gamma}$, we see that

$$\sum_{m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{\alpha\beta m} c_{m i n} + c_{\beta i m} c_{m \alpha n}) = 0, \quad \forall i, \alpha, \beta = 1, \dots, r. \tag{3.14}$$

From this,

$$\begin{aligned}
[Y_\alpha, Y_\beta] &= \sum_{i,m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{m\alpha n} c_{\beta i m}) f_i \frac{\partial}{\partial f_n} = - \sum_{i,m,n=1}^r c_{\alpha\beta m} c_{m i n} f_i \frac{\partial}{\partial f_n} \\
&= \sum_{m=1}^r c_{\alpha\beta m} Y_m. \tag{3.15}
\end{aligned}$$

So, Y_1, \dots, Y_r span a Lie algebra. We can define a Lie algebra morphism $\phi : V \rightarrow V_L$ of the form $\phi(X_\alpha) = Y_\alpha$ for $\alpha = 1, \dots, r$. The vector fields Y_1, \dots, Y_r do not need to be linearly independent. Let us show this. We can assume with no loss of generality that X_1, \dots, X_s , with $s \leq r$, form a basis for $\ker \phi$. Since $\phi(X_\alpha) = 0$ for $\alpha = 1, \dots, s$, we have $f_\beta c_{\beta\alpha\gamma} = 0$ for $\alpha = 1, \dots, s$ and $\beta, \gamma = 1, \dots, r$. Thus, we see that $[X_\alpha, X_\beta] = 0$ for $\alpha = 1, \dots, s$ and $\beta = 1, \dots, r$. This means that $X_\alpha \in Z(V)$. Conversely, we get by similar arguments that if $X \in Z(V)$, then $X \in \ker \phi$. Hence, $X \in \ker \phi$ if and only if $X \in Z(V)$. In consequence, $\ker \phi = Z(V)$ and Y_1, \dots, Y_r span a Lie algebra isomorphic to $V/Z(V)$.

It is obvious that A_1 is an ideal of $A_1 + A_2$. Moreover, as $[A_1, V_L] \subset A_1$ and $[A_2, V_L] \subset A_2$, then $A_1 \oplus_S A_2$ is an ideal of V^S . Consequently, V^S is a Lie algebra of the form (3.8). \square

Definition 4. We say that the *Lie systems* X_1 and X_2 are *isomorphic* when they take values in two isomorphic Vessiot–Guldberg Lie algebras V_1, V_2 and there exists a Lie algebra isomorphism $\phi : V_1 \rightarrow V_2$ such that $(X_2)_t = \phi((X_1)_t)$ for each $t \in \mathbb{R}$.

Proposition 5. *Given two isomorphic Lie systems X_1 and X_2 related to Vessiot–Guldberg Lie algebras V_1, V_2 , their symmetry systems relative to such Lie algebras are, up to a change of basis in V_1 and/or V_2 , the same.*

3.2 Lie algebras of Lie symmetries for Lie systems

In this section we study different Lie subalgebras of \mathcal{S}_X^V . Their interest resides in the fact that, when finite-dimensional, they can be integrated to form Lie group actions of symmetries for X . In turn, they can be employed to simplify the Lie system they are referred to.

Lemma 6. *The space of functions $C^\infty(\mathbb{R})$ can be endowed with a Lie bracket given by*

$$\{f, \bar{f}\}_{\mathbb{R}} = f \frac{d\bar{f}}{dt} - \bar{f} \frac{df}{dt}, \quad \forall f, \bar{f} \in C^\infty(\mathbb{R}). \quad (3.16)$$

Proof. In order to prove that (3.16) is a Lie bracket, we must show that (3.16) is bilinear, antisymmetric and satisfies the Jacobi identity. From its definition (3.16) is clearly bilinear and antisymmetric. To see that (3.16) holds the Jacobi identity, we consider the map $\phi : f \in C^\infty(\mathbb{R}) \mapsto f\partial/\partial t \in \Gamma(\mathbb{T}\mathbb{R})$. Observe that ϕ is a linear isomorphism.

Moreover, it follows that

$$\begin{aligned} \phi(\{f, \bar{f}\}_{\mathbb{R}}) &= \{f, \bar{f}\}_{\mathbb{R}} \frac{\partial}{\partial t} = \left(f \frac{d\bar{f}}{dt} - \bar{f} \frac{df}{dt} \right) \frac{\partial}{\partial t} = \left[f \frac{\partial}{\partial t}, \bar{f} \frac{\partial}{\partial t} \right] \\ &= [\phi(f), \phi(\bar{f})], \end{aligned} \quad (3.17)$$

for arbitrary $f, \bar{f} \in C^\infty(\mathbb{R})$. By using the Jacobi identity for vector fields on \mathbb{R} with respect to the Lie bracket $[\cdot, \cdot]$, we obtain

$$\begin{aligned} \phi(\{\{f, \bar{f}\}_{\mathbb{R}}, \bar{\bar{f}}\}_{\mathbb{R}} + \{\{\bar{f}, \bar{\bar{f}}\}_{\mathbb{R}}, f\}_{\mathbb{R}} + \{\{\bar{\bar{f}}, f\}_{\mathbb{R}}, \bar{f}\}_{\mathbb{R}}) &= \\ &= [[\phi(f), \phi(\bar{f})], \phi(\bar{\bar{f}})] + [[\phi(\bar{f}), \phi(\bar{\bar{f}})], \phi(f)] + [[\phi(\bar{\bar{f}}), \phi(f)], \phi(\bar{f})] \\ &= 0, \end{aligned} \quad (3.18)$$

$\forall f, \bar{f}, \bar{\bar{f}} \in C^\infty(\mathbb{R})$. Since ϕ is a linear isomorphism, then

$$\{\{f, \bar{f}\}_{\mathbb{R}}, \bar{\bar{f}}\}_{\mathbb{R}} + \{\{\bar{f}, \bar{\bar{f}}\}_{\mathbb{R}}, f\}_{\mathbb{R}} + \{\{\bar{\bar{f}}, f\}_{\mathbb{R}}, \bar{f}\}_{\mathbb{R}} = 0 \quad (3.19)$$

and (3.16) satisfies the Jacobi identity giving rise to a Lie bracket on $C^\infty(\mathbb{R})$. Moreover, ϕ becomes a Lie algebra isomorphism. \square

Definition 7. Let X be a Lie system on N with a Vessiot–Guldberg Lie algebra V and let \mathfrak{W} be a nonempty set of t -dependent functions that form a Lie algebra with

respect to the Lie bracket defined in (3.16). We call $\mathcal{S}_{X, \mathfrak{W}}^V$ the space

$$\mathcal{S}_{X, \mathfrak{W}}^V = \left\{ Y \in \mathcal{S}_X^V \mid Y = f_0 \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha X_\alpha, f_0 \in \mathfrak{W} \right\}, \quad (3.20)$$

where X_1, \dots, X_r is a basis for V .

Proposition 8. *The space of symmetries $\mathcal{S}_{X, \mathfrak{W}}^V$ is a Lie algebra of symmetries of X .*

Proof. Since \mathfrak{W} and \mathcal{S}_X^V are linear spaces, the linear combinations of elements of $\mathcal{S}_{X, \mathfrak{W}}^V$ belong to $\mathcal{S}_{X, \mathfrak{W}}^V$. So, this space becomes a vector space. Moreover, given two elements, $Y, Y^* \in \mathcal{S}_{X, \mathfrak{W}}^V$, their Lie bracket reads

$$\begin{aligned} [Y, Y^*] &= \left[f_0(t) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha(t) X_\alpha, f_0^*(t) \frac{\partial}{\partial t} + \sum_{\beta=1}^r f_\beta^*(t) X_\beta \right] \\ &= \{f_0, f_0^*\}_{\mathbb{R}} \frac{\partial}{\partial t} + \sum_{\beta=1}^r \left[\left(f_0 \frac{df_\beta^*}{dt} - f_0^* \frac{df_\beta}{dt} \right) X_\beta + \sum_{\alpha, \gamma=1}^r f_\alpha f_\beta^* c_{\alpha\beta\gamma} X_\gamma \right] \\ &= \{f_0, f_0^*\}_{\mathbb{R}} \frac{\partial}{\partial t} + \sum_{\gamma=1}^r \left[\left(f_0 \frac{df_\gamma^*}{dt} - f_0^* \frac{df_\gamma}{dt} \right) + \sum_{\alpha, \beta=1}^r f_\alpha f_\beta^* c_{\alpha\beta\gamma} \right] X_\gamma. \end{aligned} \quad (3.21)$$

Since \mathcal{S}_X^V is a Lie algebra and $Y, Y^* \in \mathcal{S}_X^V$, then $[Y, Y^*] \in \mathcal{S}_X^V$. As additionally $\{f_0, f_0^*\}_{\mathbb{R}} \in \mathfrak{W}$, then $[Y, Y^*] \in \mathcal{S}_{X, \mathfrak{W}}^V$. Hence, the Lie bracket of elements of $\mathcal{S}_{X, \mathfrak{W}}^V$ belongs to $\mathcal{S}_{X, \mathfrak{W}}^V$, which becomes a Lie algebra. \square

Corollary 9. *Given a Lie system X on N related to a Vessiot–Guldberg Lie algebra V , the elements of $\mathcal{S}_{X, \mathfrak{W}}^V$ with*

1. $\mathfrak{W} = \{f_0 \in C^\infty(\mathbb{R}) \mid df_0/dt = 0\}$,
2. $\mathfrak{W} = \{f_0 \in C^\infty(\mathbb{R}) \mid f_0 = 0\}$,

are finite-dimensional Lie algebras of vector fields. In the second case, $\mathcal{S}_{X, \mathfrak{W}}^V$ is isomorphic to V .

Proof. In both cases, \mathfrak{W} is nonempty. In the first case, the functions with $df_0/dt = 0$ are constant. These functions form an Abelian Lie algebra with respect to the Lie bracket $\{\cdot, \cdot\}_{\mathbb{R}}$. In view of Proposition 8, the space $\mathcal{S}_{X, \mathfrak{W}}^V$ is a Lie algebra.

In the second case, the function zero is also a zero-dimensional Lie algebra relative to $\{\cdot, \cdot\}_{\mathbb{R}}$. Since $f_0 = 0$ and using Proposition 8, we obtain that $\mathcal{S}_{X, \mathfrak{W}}^V$ can be understood as a Lie algebra of t -dependent vector fields taking values in V . To

prove that $\mathcal{S}_{X, \mathfrak{W}}^V \simeq V$, let us consider the morphism which maps each t -dependent vector field with its value at $t = 0$, namely

$$\begin{aligned} \phi : \mathcal{S}_{X, \mathfrak{W}}^V &\longrightarrow V \\ Z &\mapsto Z_0. \end{aligned}$$

Let X_1, \dots, X_r be a basis for V . From Lemma (1), we have that $(f_1(t), \dots, f_r(t))$ is a particular solution of the system

$$\frac{df_\alpha}{dt} = \sum_{\delta, \beta=1}^r b_\beta(t) f_\delta c_{\delta\beta\alpha}, \quad \alpha = 1, \dots, r. \quad (3.22)$$

For each initial condition $f_\alpha(0) = c_\alpha \in \mathbb{R}$, with $\alpha = 1, \dots, r$, i.e., by fixing Z_0 , there exists a unique solution of the above system. Hence, there exists a unique t -dependent vector field Z of $\mathcal{S}_{X, \mathfrak{W}}^V$ with $Z_0 = \sum_{\alpha=1}^r c_\alpha X_\alpha$. Thus, ϕ is a bijection. Using that for two vector fields $Z_1, Z_2 \in \mathcal{S}_{X, \mathfrak{W}}^V$ we have $[Z_1, Z_2] \in \mathcal{S}_{X, \mathfrak{W}}^V$ and $[Z_1, Z_2]_t = [(Z_1)_t, (Z_2)_t]$, we see that ϕ is a Lie algebra morphism and $\mathcal{S}_{X, \mathfrak{W}}^V \simeq V$. \square

3.3 Applications to systems of ODEs and HODEs

Let us work out the symmetry systems and related Lie symmetries for some Lie systems of interest. In particular, we will illustrate that Proposition 5 enables us to determine simultaneously Lie symmetries for different Lie systems with isomorphic Vessiot–Guldberg Lie algebras.

Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems

Let us obtain the symmetry systems and related Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems. This shall be used in following subsections to obtain simultaneously Lie symmetries of isomorphic $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems appearing in the physics and/or mathematical literature.

Let us choose a basis of vector fields $\{X_1, X_2, X_3\}$ of $V \simeq \mathfrak{sl}(2, \mathbb{R})$ with commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \quad (3.23)$$

Every Lie system with Vessiot–Guldberg Lie algebra V can be brought into the form

$$X = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3 \quad (3.24)$$

for certain t -dependent functions b_1, b_2 and b_3 . The Lie symmetries of \mathcal{S}_X^V take the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2 + f_3(t)X_3, \quad (3.25)$$

where f_0, f_1, f_2, f_3 are some t -dependent functions to be determined. In view of (3.4) and the commutation relations (3.23), the symmetry system for X relative to V reads

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{db_1}{dt}(t) + f_1 b_2(t) - f_2 b_1(t) + b_0(t) b_1(t), \\ \frac{df_2}{dt} = f_0 \frac{db_2}{dt}(t) + 2f_1 b_3(t) - 2f_3 b_1(t) + b_0(t) b_2(t), \\ \frac{df_3}{dt} = f_0 \frac{db_3}{dt}(t) + f_2 b_3(t) - f_3 b_2(t) + b_0(t) b_3(t). \end{cases} \quad (3.26)$$

As stated in Theorem 3, this is a Lie system. Indeed, system (3.26) is related to the t -dependent vector field

$$\begin{aligned} \Gamma_X^{\mathfrak{sl}(2, \mathbb{R})} = & \frac{db_1(t)}{dt} W_1 + \frac{db_2(t)}{dt} W_2 + \frac{db_3(t)}{dt} W_3 + b_0(t) Z_0 + b_0(t) b_1(t) Z_1 + b_0(t) b_2(t) Z_2 \\ & + b_0(t) b_3(t) Z_3 + b_1(t) Y_1 + b_2(t) Y_2 + b_3(t) Y_3, \end{aligned} \quad (3.27)$$

where

$$Z_\alpha = \frac{\partial}{\partial f_\alpha}, \quad \alpha = 0, 1, 2, 3, \quad W_\beta = f_0 \frac{\partial}{\partial f_\beta}, \quad \beta = 1, 2, 3 \quad (3.28)$$

and

$$Y_1 = -f_2 \frac{\partial}{\partial f_1} - 2f_3 \frac{\partial}{\partial f_2}, \quad Y_2 = f_1 \frac{\partial}{\partial f_1} - f_3 \frac{\partial}{\partial f_3}, \quad Y_3 = 2f_1 \frac{\partial}{\partial f_2} + f_2 \frac{\partial}{\partial f_3}. \quad (3.29)$$

These vector fields hold

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = 2Y_2, \quad [Y_2, Y_3] = Y_3. \quad (3.30)$$

Since $Z(V) = \{0\}$, then $V_L = \langle Y_1, Y_2, Y_3 \rangle$ is a Lie algebra isomorphic to $V/Z(V) \simeq \mathfrak{sl}(2, \mathbb{R})$ as stated in Theorem 3. The rest of commutation relations read

$$\begin{aligned} [Y_1, Z_0] &= 0, & [Y_1, Z_1] &= 0, & [Y_1, Z_2] &= Z_1, & [Y_1, Z_3] &= 2Z_2, \\ [Y_2, Z_0] &= 0, & [Y_2, Z_1] &= -Z_1, & [Y_2, Z_2] &= 0, & [Y_2, Z_3] &= Z_3, \\ [Y_3, Z_0] &= 0, & [Y_3, Z_1] &= -2Z_2, & [Y_3, Z_2] &= -Z_3, & [Y_3, Z_3] &= 0. \end{aligned} \quad (3.31)$$

Moreover,

$$\begin{aligned} [Y_1, W_1] &= 0, & [Y_1, W_2] &= W_1, & [Y_1, W_3] &= 2W_2 \\ [Y_2, W_1] &= -W_1, & [Y_2, W_2] &= 0, & [Y_2, W_3] &= W_3, \\ [Y_3, W_1] &= -2W_2, & [Y_3, W_2] &= -W_3, & [Y_3, W_3] &= 0 \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} [Z_0, W_j] &= Z_j, & [Z_i, W_j] &= 0, & [W_i, W_j] &= 0, & i, j &= 1, 2, 3, \\ [Z_\alpha, Z_\beta] &= 0, & \alpha, \beta &= 0, \dots, 3. \end{aligned} \quad (3.33)$$

Hence, $A_1 = \langle Z_0, Z_1, Z_2, Z_3 \rangle$ is an ideal of $A_1 + A_2$. And $A_1 + A_2$ is an ideal of $A_1 + A_2 + V_L$, with $A_2 = \langle W_1, W_2, W_3 \rangle$.

Example 1. We study a particular type of **first-order Riccati equation** [257, 364]

$$\frac{dx}{dt} = \eta(t) + x^2, \quad (3.34)$$

where η is an arbitrary t -dependent function.

It is well known that the Riccati equation is a Lie system with a Vessiot–Guldberg Lie algebra isomorphic to $V^{\text{Ric}} \simeq \mathfrak{sl}(2, \mathbb{R})$ [109]. Indeed, equation (3.34) has the associated t -dependent vector field $X^{\text{Ric}} = X_3^{\text{Ric}} + \eta(t)X_1^{\text{Ric}}$, where

$$X_1^{\text{Ric}} = \frac{\partial}{\partial x}, \quad X_2^{\text{Ric}} = x \frac{\partial}{\partial x}, \quad X_3^{\text{Ric}} = x^2 \frac{\partial}{\partial x} \quad (3.35)$$

satisfy the same commutation relations as X_1, X_2, X_3 in (3.23). In view of this, (3.34) is related to a t -dependent vector field taking values in a finite-dimensional Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, (3.34) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Moreover, we can consider X^{Ric} as a particular case of system (3.24). By applying the results of the previous sections to generic $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems, we find that the symmetry system for X^{Ric} is of the form (3.26) with $b_1 = \eta, b_2 = 0$ and $b_3 = 1$, namely

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{d\eta}{dt}(t) - \eta(t)f_2 + b_0(t)\eta(t), \\ \frac{df_2}{dt} = 2f_1 - 2\eta(t)f_3, \\ \frac{df_3}{dt} = f_2 + b_0(t). \end{cases} \quad (3.36)$$

We can recover and generalize the results given in [209] by means of our approach. From their expressions in (3.36), we can differentiate df_3/dt twice and df_2/dt once. By substituting d^2f_2/dt^2 in d^3f_3/dt^3 and using the remaining equations in (3.36), we obtain that

$$\frac{d^3f_3}{dt^3} = \frac{d^3f_0}{dt^3} + 2\frac{df_1}{dt} - 2\frac{d\eta}{dt}f_3 - 2\eta(t)\frac{df_3}{dt}. \quad (3.37)$$

By substituting the value of df_1/dt from (3.36) and using that $f_2 = df_3/dt - b_0(t)$, we obtain the following equation for f_3 in terms of the coefficients of (3.34) and f_0

$$\frac{d^3f_3}{dt^3} = \frac{d^3f_0}{dt^3} + 4b_0\eta(t) + 2\frac{d\eta}{dt}f_0 - 2\frac{d\eta}{dt}f_3 - 4\eta(t)\frac{df_3}{dt}. \quad (3.38)$$

From this, we can retrieve the following corollary given in [209].

Corollary 10. *The Riccati equation (3.34) admits the Lie symmetry*

$$Y = f_0 \frac{\partial}{\partial t} - \frac{1}{2} \frac{d^2 f_0}{dt^2} \frac{\partial}{\partial x} - \frac{df_0}{dt} x \frac{\partial}{\partial x}, \quad (3.39)$$

where

$$\frac{d^3 f_0}{dt^3} + 4b_0(t)\eta(t) + 2 \frac{d\eta}{dt} f_0 = 0. \quad (3.40)$$

Proof. Since we are looking for Lie symmetries with $f_3 = 0$, equation (3.38) reduces to (3.40). Moreover, by substituting $f_3 = 0$ in (3.36), we obtain that

$$\frac{df_3}{dt} = b_0(t) + f_2 = 0, \quad \frac{df_2}{dt} = 2f_1, \quad (3.41)$$

which yields $f_2 = -df_0/dt$ and $2f_1 = -d^2 f_0/dt^2$. Hence, the corollary follows. \square

Going back to general symmetries of (3.34), we can obtain some of its Lie symmetries by solving (3.36) for certain values of $\eta(t)$ summarized in Table 6 in Appendix 1.

Example 2. Reconsider the **Cayley–Klein Riccati equation** appeared in (6.68) within Chapter 3. Its associated first-order system (6.69) was described in Chapter 3 by means of the t -dependent vector field

$$X^{\text{CK}} = b_1(t)X_1^{\text{CK}} + b_2(t)X_2^{\text{CK}} + b_3(t)X_3^{\text{CK}}, \quad (3.42)$$

where

$$X_1^{\text{CK}} = \frac{\partial}{\partial x}, \quad X_2^{\text{CK}} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3^{\text{CK}} = (x^2 + \iota^2 y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \quad (3.43)$$

satisfy the same commutation relations as X_1, X_2, X_3 in (3.23). Since X_1, X_2, X_3 span a Lie algebra $V \simeq \mathfrak{sl}(2, \mathbb{R})$, the vector fields $X_1^{\text{CK}}, X_2^{\text{CK}}$ and X_3^{CK} span a real Lie algebra, V^{CK} , which is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ independently of the square of ι . Then, (3.42) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system.

Moreover, we can define a the Lie algebra isomorphism $\phi : V^{\text{CK}} \rightarrow V$ mapping

$$\phi(X_\alpha^{\text{CK}}) = X_\alpha, \quad \alpha = 1, 2, 3, \quad (3.44)$$

where X_α , with $\alpha = 1, 2, 3$, are the generic vector fields in (3.23). Hence, we have that $\phi(X_t^{\text{CK}}) = X_t$ for every $t \in \mathbb{R}$ with X being the generic $\mathfrak{sl}(2, \mathbb{R})$ -Lie system (3.24). Then, (3.42) is isomorphic to (3.24) and, in view of Proposition 5, the symmetry system of X^{CK} becomes (3.26).

Example 3. Let us consider the **quaternionic Riccati equation** with t -dependent real coefficients. Quaternions are the elements of the real vector space $\mathbb{H} \simeq \mathbb{R}^4$ of elements of the form $q = q_0 + q_1i + q_2j + q_3k$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and with the standard sum of elements and multiplication by real numbers. We can define a multiplication of quaternions (see [133] for details) by assuming that real numbers commute with all quaternions and that the following operations are fulfilled

$$\begin{aligned} i^2 = -1, \quad j^2 = -1, \quad k^2 = -1, \\ i \cdot j = -j \cdot i = k, \quad k \cdot i = -i \cdot k = j, \quad j \cdot k = -k \cdot j = i. \end{aligned} \quad (3.45)$$

The quaternionic Riccati equation [418] takes the form

$$\frac{dq}{dt} = b_1(t) + a_1(t)q + qa_2(t) + qb_3(t)q, \quad (3.46)$$

where q and the t -dependent functions $a_1, a_2, b_1, b_3 : \mathbb{R} \rightarrow \mathbb{H}$ take values in \mathbb{H} [133]. The existence of periodic solutions for particular cases of (3.46) has been studied in [84, 418] and, for real t -dependent coefficients, in [418]. In this work, we focus on the latter case.

Writing q in coordinates, we obtain that (3.46) reads

$$\begin{cases} \frac{dq_0}{dt} = b_1(t) + b_2(t)q_0 + b_3(t)(q_0^2 - q_1^2 - q_2^2 - q_3^2), \\ \frac{dq_1}{dt} = b_2(t)q_1 + 2b_3(t)q_0q_1, \\ \frac{dq_2}{dt} = b_2(t)q_2 + 2b_3(t)q_0q_2, \\ \frac{dq_3}{dt} = b_2(t)q_3 + 2b_3(t)q_0q_3, \end{cases} \quad (3.47)$$

where $b_2(t) \equiv a_1(t) + a_2(t)$. This system is related to the t -dependent vector field

$$X^{\mathbb{H}} = b_1(t)X_1^{\mathbb{H}} + b_2(t)X_2^{\mathbb{H}} + b_3(t)X_3^{\mathbb{H}}, \quad (3.48)$$

with

$$\begin{aligned} X_1^{\mathbb{H}} &= \frac{\partial}{\partial q_0}, & X_2^{\mathbb{H}} &= q_0 \frac{\partial}{\partial q_0} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3}, \\ X_3^{\mathbb{H}} &= 2q_0 \left(q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3} \right) + (q_0^2 - q_1^2 - q_2^2 - q_3^2) \frac{\partial}{\partial q_0}, \end{aligned} \quad (3.49)$$

which satisfy the same commutation relations as X_1, X_2 and X_3 in (3.23). So, $X_1^{\mathbb{H}}, X_2^{\mathbb{H}}, X_3^{\mathbb{H}}$ span a real Lie algebra $V^{\mathbb{H}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Hence, the quaternionic Riccati equation (3.46) with t -dependent real coefficients is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie

system isomorphic to (3.24) with respect to the Lie algebra isomorphism $\phi : V^{\mathbb{H}} \rightarrow V$ being given by

$$\phi(X_{\alpha}^{\mathbb{H}}) = X_{\alpha}, \quad \alpha = 1, 2, 3. \quad (3.50)$$

Moreover, we have that $\phi(X_t^{\mathbb{H}}) = X_t$ for every $t \in \mathbb{R}$. Hence, the symmetry system for (3.46) is (3.26). If we assume for instance $f_0 = k \in \mathbb{R}$, and $b_1(t) = \eta(t)$, $b_2(t) = 0$ and $b_3(t) = 1$, we obtain that (3.46) is isomorphic to the Lie system (3.36). Hence, for certain values of $\eta(t)$ given in Table 6 in Appendix 1, we can derive several Lie symmetries for quaternionic Riccati equations.

Example 4. We now show that our theory in particular, and the whole theory of Lie systems in general, can be used to study autonomous systems of first-order ODEs. We consider the **generalized Darboux–Brioschi–Halphen system** (DBH system)[141]

$$\begin{cases} \frac{dw_1}{dt} = w_3w_2 - w_1w_3 - w_1w_2 + \tau^2, \\ \frac{dw_2}{dt} = w_1w_3 - w_2w_1 - w_2w_3 + \tau^2, \\ \frac{dw_3}{dt} = w_2w_1 - w_3w_2 - w_3w_1 + \tau^2, \end{cases} \quad (3.51)$$

where

$$\begin{aligned} \tau^2 \equiv & \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) \\ & + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3) \end{aligned} \quad (3.52)$$

and $\alpha_1, \alpha_2, \alpha_3$ are real constants.

When $\tau = 0$, system (3.51) retrieves the classical DBH system solved by Halphen [117, 141, 227] which appears in the study of triply orthogonal surfaces and the vacuum Einstein equations for hyper-Kähler Bianchi-IX metrics. For $\tau \neq 0$, the generalized DBH system can be considered as a reduction of the self-dual Yang–Mills equations corresponding to an infinite-dimensional gauge group of diffeomorphisms of a three-dimensional sphere [117].

It can be proved that (3.51) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Indeed, it is the associated system to the t -dependent vector field

$$\begin{aligned} X_t^{\text{DBH}} = & (w_3w_2 - w_1(w_3 + w_2) + \tau^2) \frac{\partial}{\partial w_1} + (w_1w_3 - w_2(w_1 + w_3) + \tau^2) \frac{\partial}{\partial w_2} \\ & + (w_2w_1 - w_3(w_2 + w_1) + \tau^2) \frac{\partial}{\partial w_3} = -X_3^{\text{DBH}}. \end{aligned} \quad (3.53)$$

This vector field span a Lie algebra V^{DBH} of vector fields along with

$$X_1^{\text{DBH}} = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} + \frac{\partial}{\partial w_3}, \quad X_2^{\text{DBH}} = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} + w_3 \frac{\partial}{\partial w_3} \quad (3.54)$$

satisfying the commutation relations (3.23). In consequence, $X_1^{\text{DBH}}, X_2^{\text{DBH}}$ and X_3^{DBH} span a three-dimensional Lie algebra of vector fields V^{DBH} isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and then X^{DBH} is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Since X_1, X_2, X_3 admit the same structure constants as (3.23), the symmetry system for X^{DBH} becomes (3.26) with $b_1(t) = b_2(t) = 0$ and $b_3(t) = -1$, namely

$$\frac{df_0}{dt} = b_0(t), \quad \frac{df_1}{dt} = 0, \quad \frac{df_2}{dt} = -2f_1, \quad \frac{df_3}{dt} = -f_2 - b_0(t). \quad (3.55)$$

Hence, for $b_0(t) = 0$, we obtain

$$Y = t_0 \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + (\lambda_1 t^2 - \lambda_2 t + \lambda_3) X_3, \quad (3.56)$$

with $\lambda_1, \lambda_2, \lambda_3, t_0 \in \mathbb{R}$. Evidently, these vector fields span a Lie algebra of Lie symmetries isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for $t_0 = 0$. For $b_0(t) = c_0$, we obtain

$$Y = (c_0 t + t_0) \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + (\lambda_1 t^2 - (\lambda_2 + c_0)t + \lambda_3) X_3, \quad (3.57)$$

with $\lambda_1, \lambda_2, \lambda_3, t_0, c_0 \in \mathbb{R}$. Finally, for $b_0(t) = c_0 t$, we get

$$Y = \left(t_0 + \frac{c_0 t^2}{2} \right) \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + \left[\lambda_1 t^2 - \left(\lambda_2 + \frac{c_0 t}{2} \right) t + \lambda_3 \right] X_3, \quad (3.58)$$

with $\lambda_1, \lambda_2, \lambda_3, t_0, c_0 \in \mathbb{R}$. Since $\mathfrak{W} = \langle 1, t, t^2 \rangle$ is a Lie algebra with respect to the Lie bracket, $\{\cdot, \cdot\}_{\mathbb{R}}$, we obtain in view of Proposition 8 that

$$\mathcal{S}_{X, \mathfrak{W}}^V = \{Y \in \mathcal{S}_X^V \mid f_0 \in \langle 1, t, t^2 \rangle\} \quad (3.59)$$

is a Lie algebra of Lie symmetries. By choosing appropriately the constant coefficients of the above vector fields and setting $\tau = 0$, we recover the Lie algebra of symmetries isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for classical DBH systems [333].

The study of systems of HODEs through Lie systems implies the addition of extra variables in order to express a higher-order system as a first-order one. The introduction of these extra variables frequently results in the obtainment of non-point Lie symmetries, as we shall exemplify in forthcoming examples. Certain non-local Lie symmetries can be identified with the prolongations of certain vector fields.

Lemma 11. *Given a Lie algebra of Lie point symmetries V , its prolongations form a Lie algebra \widehat{V} isomorphic to the former.*

Proof. Consider the map

$$\begin{aligned} \Phi : \Gamma(\mathbb{T}(\mathbb{R} \times N)) &\rightarrow \Gamma(\mathbb{T}[\mathbb{R} \times \mathbb{T}^p N]) \\ X &\mapsto \widehat{X}, \end{aligned} \quad (3.60)$$

mapping sections of $\mathbb{T}(\mathbb{R} \times N)$, i.e., vector fields on $\mathbb{R} \times N$, onto sections of $\mathbb{T}(\mathbb{R} \times \mathbb{T}^p N)$, i.e., vector fields on the manifold $\mathbb{R} \times \mathbb{T}^p N$, where $\mathbb{T}^p N$ is the p -order tangent space. Recall that, roughly speaking, $\mathbb{T}^p N$ is the space of equivalence classes of curves in N with the same Taylor expansion up to order p . It admits a differentiable structure induced by the variables x_1, \dots, x_n of the coordinate systems of N and the induced variables $x_i^{(k)}$ describing the derivatives in terms of t up to order p of the coordinates of a curve within N . Given the so called *contact one-forms* $\theta_i^k = dx_i^{(k)} - x_i^{(k+1)} dt$ with $i = 1, \dots, n$ and $k = 0, \dots, p-1$ on $\mathbb{R} \times \mathbb{T}^p N$, we say that \widehat{X} is the prolongation of $X \in \Gamma(\mathbb{T}(\mathbb{R} \times N))$ to $\mathbb{R} \times \mathbb{T}^p N$ if and only if every $\mathcal{L}_{\widehat{X}} \theta_i^k$ belongs to the contact ideal spanned by all the contact forms and $J_{\pi_*}^p \widehat{X} = X$, with $J_{\pi_*}^p$ being the tangent map to the projection $J_{\pi}^p : (t, x, x^{(k)}) \in \mathbb{R} \times \mathbb{T}^p N \mapsto (t, x) \in \mathbb{R} \times N$ with $k = 1, \dots, p$. This implies that Φ is \mathbb{R} -linear and injective. Additionally,

$$\mathcal{L}_{[\widehat{X}_1, \widehat{X}_2]} \theta_i^k = \left(\mathcal{L}_{\widehat{X}_1} \mathcal{L}_{\widehat{X}_2} - \mathcal{L}_{\widehat{X}_2} \mathcal{L}_{\widehat{X}_1} \right) \theta_i^k \quad (3.61)$$

belongs to the ideal of the contact forms because $\mathcal{L}_{\widehat{X}_1}$ and $\mathcal{L}_{\widehat{X}_2}$ do so. Moreover,

$$J_{\pi_*}^p [\widehat{X}_1, \widehat{X}_2] = [J_{\pi_*}^p \widehat{X}_1, J_{\pi_*}^p \widehat{X}_2] = [X_1, X_2]. \quad (3.62)$$

Hence, $[\widehat{X}_1, \widehat{X}_2]$ must be the prolongation of $[X_1, X_2]$, i.e., $[\widehat{X}_1, \widehat{X}_2] = [\widehat{X}_1, \widehat{X}_2]$. In this way, Φ is a Lie algebra morphism. Obviously, given a Lie algebra of Lie point symmetries V , its prolongations form a Lie algebra $\widehat{V} = \Phi(V)$ isomorphic to V . \square

Definition 12. We call \widehat{V} the Lie algebra whose elements are the prolongations of the elements in V .

Example 5. As an example of a second-order Lie system possessing a Vessiot–Gulberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, we recall the **second-order Kummer–Schwarz equation** given in (2.38) in Chapter 3. In form of a first order system, it took the expression

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = \frac{3}{2} \frac{v^2}{x} - 2c_0 x^3 + 2b_1(t)x, \end{cases} \quad (3.63)$$

on $\mathbb{T}\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Such system is associated with the t -dependent vector

field $M^{\text{KS}} = M_3^{\text{KS}} + \eta(t)M_1^{\text{KS}}$, with vector fields

$$\begin{aligned} M_1^{\text{KS}} &= 2x \frac{\partial}{\partial v}, & M_2^{\text{KS}} &= x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \\ M_3^{\text{KS}} &= v \frac{\partial}{\partial x} + \left(\frac{3v^2}{2x} - 2c_0x^3 \right) \frac{\partial}{\partial v} \end{aligned} \quad (3.64)$$

satisfy the same commutation relations as the vector fields X_1, X_2, X_3 in (3.23) and they therefore span a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since the basis $M_1^{\text{KS}}, M_2^{\text{KS}}, M_3^{\text{KS}}$ have the same structure constants as the $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems analyzed in Subsection 3.3, we can define, for instance, a Lie algebra morphism ϕ mapping M_1, M_2, M_3 to the basis $X_1^{\text{Ric}}, X_2^{\text{Ric}}, X_3^{\text{Ric}}$ for Riccati equations (3.34). In such a case, (2.40) maps to X^{Ric} and in view of Proposition 5, the symmetry system for (2.40) is of the form of the symmetry system (3.36). As a consequence, the particular solutions f_0, f_1, f_2 and f_3 for Riccati equations, detailed in Table 6, are valid for (2.38) as well.

3.4 Lie symmetries for $\text{Aff}(\mathbb{R})$ -Lie systems

In this section, we aim to obtain elements of \mathcal{S}_X^V for $\text{Aff}(\mathbb{R})$ -Lie systems. We choose a basis of vector fields $\{X_1, X_2\}$ of $V \simeq \text{Aff}(\mathbb{R})$ with $[X_1, X_2] = X_1$, and express the most general Lie system with Vessiot–Guldberg Lie algebra V as $X = a(t)X_1 + b(t)X_2$. Let us now look for its Lie symmetries of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2. \quad (3.65)$$

The symmetry condition gives rise to a symmetry system

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{da}{dt}(t) + a(t)b_0(t) + b(t)f_1 - a(t)f_2, \\ \frac{df_2}{dt} = f_0 \frac{db}{dt}(t) + b(t)b_0(t), \end{cases} \quad (3.66)$$

associated with the t -dependent vector field

$$\begin{aligned} \Gamma_X^{\text{Aff}(\mathbb{R})} &= b_0(t)Z_0 + \frac{da}{dt}W_1 + \frac{db}{dt}W_2 + b_0(t)a(t)Z_1 \\ &\quad + b_0(t)b(t)Z_2 + a(t)Y_1 + b(t)Y_2 \end{aligned} \quad (3.67)$$

where

$$\begin{aligned} Y_1 &= -f_2 \frac{\partial}{\partial f_1}, & Y_2 &= f_1 \frac{\partial}{\partial f_1}, & W_1 &= f_0 \frac{\partial}{\partial f_1}, \\ W_2 &= f_0 \frac{\partial}{\partial f_2}, & Z_\alpha &= \frac{\partial}{\partial f_\alpha}, \end{aligned} \quad (3.68)$$

with $\alpha = 0, 1, 2$. Since, $[Y_1, Y_2] = Y_1$, then $V_L = \langle Y_1, Y_2 \rangle$ gives rise to a Lie algebra isomorphic to $\text{Aff}(\mathbb{R})$. Moreover,

$$\begin{aligned} [Z_0, W_1] &= Z_1, & [Z_0, W_2] &= Z_2, & [Z_i, W_j] &= 0, & [Z_\alpha, Z_\beta] &= 0, \\ [Y_1, W_1] &= 0, & [Y_1, W_2] &= W_1, & [Y_1, Z_0] &= 0, & [Y_1, Z_1] &= 0, \\ [Y_1, Z_2] &= Z_1, & [Y_2, W_1] &= -W_1, & [Y_2, W_2] &= 0, & [Y_2, Z_0] &= 0, \\ [Y_2, Z_1] &= -Z_1, & [Y_2, Z_2] &= 0. \end{aligned} \quad (3.69)$$

with $i, j = 1, 2$ and $\alpha, \beta = 0, 1, 2$. In this way, $A_1 = \langle Z_0, Z_1, Z_2 \rangle$ is an ideal of $A_1 + A_2$, with $A_2 = \langle W_1, W_2 \rangle$ and $A_1 \oplus_S A_2$ is an ideal of $A_1 + A_2 + V_L$. Hence, system (3.66) possesses a Vessiot–Guldberg Lie algebra

$$(A_1 \oplus_S A_2) \oplus_S V_L \simeq (\mathbb{R}^3 \oplus_S \mathbb{R}^2) \oplus_S \text{Aff}(\mathbb{R}). \quad (3.70)$$

We can solve (3.66) when $b_0(t) = 0$. In this case, $f_0 = k \in \mathbb{R}$ and (3.66) reduces to a trivial equation for f_2 and a linear one for f_1 . The general solution reads

$$\begin{aligned} f_1(t) &= \left[\int_0^t \left[k \frac{da}{dt'}(t') - a(t')(kb(t') + c_1) \right] e^{-\int_0^{t'} b(t'') dt''} dt' + c_2 \right] e^{\int_0^t b(t') dt'}, \\ f_2(t) &= kb(t) + c_1, \end{aligned} \quad (3.71)$$

where c_1 and c_2 are integration constants.

Example 6. In order to illustrate $\text{Aff}(\mathbb{R})$ -Lie systems through a physical example, we reconsider the **Buchdahl equation** [73, 118, 122], which was analyzed in Chapter 3 as a first-order system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = a(x)y^2 + b(t)y. \end{cases} \quad (3.72)$$

Recall that we proved there that (3.72) is a Lie system [32]. Indeed, (3.72) describes the integral curves of the t -dependent vector field

$$X^{BD} = v \frac{\partial}{\partial x} + (f(x)v^2 + a_2(t)v) \frac{\partial}{\partial v} = X_1 - a_2(t)X_2, \quad (3.73)$$

where

$$X_1^{BD} = v \frac{\partial}{\partial x} + f(x)v^2 \frac{\partial}{\partial v}, \quad X_2^{BD} = -v \frac{\partial}{\partial v}, \quad (3.74)$$

satisfy $[X_1, X_2] = X_1$. That is, X is an $\text{Aff}(\mathbb{R})$ -Lie system. By applying Theorem 3, we see that the Lie symmetries of (3.72) of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2 \quad (3.75)$$

are determined by the first-order system (3.66) with $a(t) = 1$ and $b(t) = -a_2(t)$, i.e.,

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = b_0(t) - a_2(t)f_1 - f_2, \\ \frac{df_2}{dt} = -f_0 \frac{da_2}{dt} - b_0(t)a_2(t), \end{cases} \quad (3.76)$$

which is associated to the t -dependent vector field

$$\Gamma_X^{\text{BD}} = b_0(t)Z_0 - \frac{da_2}{dt}W_2 + b_0(t)Z_1 - a_2(t)b_0(t)Z_2 + Y_1 - a_2(t)Y_2, \quad (3.77)$$

where the vector fields $Z_0, Z_1, Z_2, Y_1, Y_2, W_2$ are those detailed in (3.68) and have the commutation relations (3.69). Hence, these vector fields span a Lie algebra (3.70). Therefore, we can obtain the Lie symmetries for this system of the form (3.65) by substituting $a(t) = 1$ and $b(t) = -a_2(t)$ in (3.71).

Example 7. Let us study the Lie symmetries of the **Painlevé–Ince equation** [12, 257, 266]

$$\frac{d^2x}{dt^2} = -3x \frac{dx}{dt} - x^3. \quad (3.78)$$

This differential equation has been widely studied in many works (see [104] and references therein) due to its interesting properties and applications. From the point of view of the theory of Lie systems, it is a HODE Lie system, i.e., by adding the new variable $v \equiv dx/dt$, we obtain a system

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = -3xv - x^3, \end{cases} \quad (3.79)$$

which can be viewed as a Lie system. Indeed, it admits a Vessiot–Guldberg Lie algebra $V \simeq \mathfrak{sl}(3, \mathbb{R})$ spanned by the vector fields (see [101])

$$\begin{aligned} X_1 &= v \frac{\partial}{\partial x} - (3xv + x^3) \frac{\partial}{\partial v}, & X_2 &= \frac{\partial}{\partial v}, \\ X_3 &= -\frac{\partial}{\partial x} + 3x \frac{\partial}{\partial v}, & X_4 &= x \frac{\partial}{\partial x} - 2x^2 \frac{\partial}{\partial v}, \\ X_5 &= (v + 2x^2) \frac{\partial}{\partial x} - x(v + 3x^2) \frac{\partial}{\partial v}, & X_6 &= 2x(v + x^2) \frac{\partial}{\partial x} + 2(v^2 - x^4) \frac{\partial}{\partial v}, \\ X_7 &= \frac{\partial}{\partial x} - x \frac{\partial}{\partial v}, & X_8 &= 2x \frac{\partial}{\partial x} + 4v \frac{\partial}{\partial v}. \end{aligned} \quad (3.80)$$

As a consequence, we can apply our theory to obtain Lie symmetries of this HODE Lie system. Moreover, as (3.79) determines the integral curves of X_1 , we can search for a vector field within V commuting with X_1 . Since $[X_1, X_6] = 0$ (see [104]), we obtain that X_6 is a Lie symmetry for (3.79) and a non-local Lie symmetry for (3.78).

3.5 Lie symmetries for PDE Lie systems

Let us consider a system of PDEs

$$\frac{\partial x_i}{\partial t_l} = X_{li}(t, x), \quad i = 1, \dots, n, \quad l = 1, \dots, s, \quad (3.81)$$

where the $X_{li} : \mathbb{R}^s \times \mathbb{R}^n \mapsto \mathbb{R} \in X(t, x)$ are arbitrary functions. Its particular solutions are sections $x : t \equiv (t_1, \dots, t_s) \in \mathbb{R}^s \mapsto x(t) \in N$ of the bundle $(\mathbb{R}^s \times N, N, \pi_s : (t, x) \in \mathbb{R}^s \times N \mapsto x \in N)$. In particular, we recover the simple case of ODEs when $s = 1$.

Let us assume $t \in \mathbb{R}^s$. We call *t-dependent vector field* X on N a map $X : \mathbb{R}^s \times N \rightarrow TN$ such that $\tau \circ X = \pi_s$. We call \bar{X}^l , with $l = 1, \dots, s$, the *autonomization* of X with respect to t_l , i.e.,

$$\bar{X}^l(t, x) \equiv \frac{\partial}{\partial t_l} + X(t, x). \quad (3.82)$$

Definition 13. A system of PDEs of the form (3.81)

$$\frac{\partial x_i}{\partial t_l} = \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x), \quad l = 1, \dots, s, \quad i = 1, \dots, n, \quad (3.83)$$

is a *PDE Lie system* if

- There exist vector fields X_1, \dots, X_r on N spanning an r -dimensional real Lie algebra V satisfying that each t -dependent vector field $X_l(t, x) = \sum_{i=1}^n X_{li}(t, x) \times \partial/\partial x_i$, where $l = 1, \dots, s$, can be written in the form

$$X_l(t, x) = \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x), \quad l = 1, \dots, s \quad (3.84)$$

for certain t -dependent functions $b_{\alpha l}$.

- Let $c_{\alpha\beta\gamma}$ be the structure constants of X_1, \dots, X_r , i.e., $[X_{\alpha}, X_{\beta}] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_{\gamma}$. Then,

$$\frac{\partial b_{\gamma k}}{\partial t_l}(t) - \frac{\partial b_{\gamma l}}{\partial t_k}(t) + \sum_{\alpha, \beta=1}^r b_{\alpha l}(t) b_{\beta k}(t) c_{\alpha\beta\gamma} = 0, \quad (3.85)$$

with $l \neq k = 1, \dots, s, \quad \gamma = 1, \dots, r$.

If these conditions are satisfied, we call V a *Vessiot–Guldberg Lie algebra for the PDE Lie system (3.81)*.

Theorem 14. *Let V be a finite-dimensional real Lie algebra of vector fields on \mathbb{R}^n with a basis X_1, \dots, X_r . Given a PDE Lie system as (3.83), then*

$$Y(t, x) = \sum_{\beta=1}^r f_{\beta}(t) X_{\beta}(x) \quad (3.86)$$

is a Lie symmetry of (3.83) if and only if $[\bar{X}^l, Y] = 0$ for every l , where \bar{X}^l is the l -autonomization given by (3.82) of the t -dependent vector field

$$X_l(t, x) \equiv \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x), \quad l = 1, \dots, s. \quad (3.87)$$

Proof. The coordinate systems $\{t_1, \dots, t_s\}$ on \mathbb{R}^s and $\{x_1, \dots, x_n\}$ on \mathbb{R}^n induce a coordinate system on the fiber bundle $J^1\pi \simeq \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^{ns}$ with respect to the projection $\pi : (t_l, x_j) \in \mathbb{R}^s \times \mathbb{R}^n \mapsto (t_l) \in \mathbb{R}^s$ of the form $t_l, x_j, x_{j,l}$, with $1 \leq j \leq n$ and $1 \leq l \leq s$. A vector field $Y = \sum_{k=1}^n \eta_k(t, x) \partial / \partial x_k$, $t \in \mathbb{R}^s$ and $x \in \mathbb{R}^n$, is a Lie symmetry of (3.83) if and only if

$$\widehat{Y} F_l^i = 0, \quad i = 1, \dots, n, \quad l = 1, \dots, s \quad (3.88)$$

on the submanifold $\mathcal{S} = \bigcap_{i=1}^n \bigcap_{l=1}^s (F_l^i)^{-1}(0)$, with $\widehat{Y} : J^1\pi \rightarrow T(J^1\pi)$ being the prolongation of Y , namely

$$\widehat{Y} = \sum_{k=1}^n \left[\eta_k \frac{\partial}{\partial x_k} + \sum_{q=1}^s \left(\frac{\partial \eta_k}{\partial t_q} + \sum_{j=1}^n \frac{\partial \eta_k}{\partial x_j} x_{j,q} \right) \frac{\partial}{\partial x_{k,q}} \right], \quad (3.89)$$

and

$$F_l^i = x_{i,l} - \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}^i(x), \quad i = 1, \dots, n, \quad l = 1, \dots, s, \quad (3.90)$$

$$X_{\alpha} = \sum_{i=1}^n X_{\alpha}^i(x) \frac{\partial}{\partial x_i}, \quad \alpha = 1, \dots, r.$$

By assumption, $\eta_k(t, x) = \sum_{\beta=1}^r f_{\beta}(t) X_{\beta}^k(x)$ for $k = 1, \dots, n$. So,

$$\widehat{Y} = \sum_{k=1}^n \sum_{\beta=1}^r \left[f_{\beta}(t) X_{\beta}^k \frac{\partial}{\partial x_k} + \sum_{q=1}^s \left(\frac{\partial f_{\beta}(t)}{\partial t_q} X_{\beta}^k + \sum_{j=1}^n f_{\beta}(t) \frac{\partial X_{\beta}^k}{\partial x_j} x_{j,q} \right) \frac{\partial}{\partial x_{k,q}} \right]. \quad (3.91)$$

Substituting this in $\widehat{Y} F_l^i = 0$, we obtain

$$\sum_{k=1}^n \sum_{\beta=1}^r \left[f_{\beta}(t) X_{\beta}^k \left(- \sum_{\alpha=1}^r b_{\alpha l}(t) \frac{\partial X_{\alpha}^i}{\partial x_k} \right) \right]$$

$$\begin{aligned}
& + \sum_{q=1}^s \left(\frac{\partial f_\beta(t)}{\partial t_q} X_\beta^k + \sum_{j=1}^n f_\beta(t) \frac{\partial X_\beta^k}{\partial x_j} x_{j,q} \right) \delta_q^l \delta_k^i \Big] = \\
& = \sum_{\beta=1}^r \left[- \sum_{k=1}^n \sum_{\alpha=1}^r f_\beta(t) b_{\alpha l}(t) X_\beta^k \frac{\partial X_\alpha^i}{\partial x_k} + \frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{j=1}^n f_\beta(t) \frac{\partial X_\beta^i}{\partial x_j} x_{j,l} \right] \\
& = 0. \tag{3.92}
\end{aligned}$$

Restricting the above expression to the submanifold $\mathcal{S} = \cap_{i=1}^n \cap_{l=1}^s (F_l^i)^{-1}(0)$ and renaming indexes appropriately, we obtain

$$\sum_{\beta=1}^r \left[\frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{k=1}^n \sum_{\alpha=1}^r \left(f_\beta(t) \frac{\partial X_\beta^i}{\partial x_k} b_{\alpha l}(t) X_\alpha^k - f_\beta(t) b_{\alpha l}(t) X_\beta^k \frac{\partial X_\alpha^i}{\partial x_k} \right) \right] = 0. \tag{3.93}$$

Hence,

$$\sum_{\beta=1}^r \left[\frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{\gamma=1}^r \sum_{k=1}^n f_\beta(t) b_{\gamma l}(t) \left(\frac{\partial X_\beta^i}{\partial x_k} X_\gamma^k - X_\beta^k \frac{\partial X_\gamma^i}{\partial x_k} \right) \right] = 0, \tag{3.94}$$

whose right-hand side becomes, for each fixed l , the coefficients in the basis $\partial/\partial x^i$, with $i = 1, \dots, n$, of $[\bar{X}^l, Y]$. So, the above amounts to

$$\begin{aligned}
[\bar{X}^l, Y] & = \left[\frac{\partial}{\partial t_l} + \sum_{k=1}^n \sum_{\gamma=1}^r b_{\gamma l}(t) X_\gamma^k \frac{\partial}{\partial x_k}, \sum_{i=1}^n \sum_{\beta=1}^r f_\beta(t) X_\beta^i \frac{\partial}{\partial x_i} \right] \\
& = \left[\frac{\partial}{\partial t_l} + \sum_{\gamma=1}^r b_{\gamma l}(t) X_\gamma, \sum_{\beta=1}^r f_\beta(t) X_\beta \right] = 0. \tag{3.95}
\end{aligned}$$

Then, Y is a Lie symmetry of (3.83) if and only if the condition

$$[\bar{X}^l, Y] = 0 \tag{3.96}$$

is satisfied for $l = 1, \dots, s$. \square

Theorem 15. *Given a Lie symmetry of the form (3.86) for the system (3.83), the coefficients $f_1(t), \dots, f_r(t)$ satisfy a PDE Lie system admitting a Vessiot–Guldberg Lie algebra $V^S \simeq V/Z(V)$, where we recall that V is a Vessiot–Guldberg Lie algebra for (3.83).*

Proof. Let X_1, \dots, X_r be a basis for V with structure constants $c_{\alpha\beta\gamma}$. From Theorem 14, we have

$$[\bar{X}_l^l, Y] = \left[\frac{\partial}{\partial t_l} + \sum_{\alpha=1}^r b_{\alpha l}(t) X_\alpha, \sum_{\delta=1}^r f_\delta(t) X_\delta \right] = \sum_{\delta=1}^r \left(\frac{\partial f_\delta}{\partial t_l} X_\delta + \sum_{\alpha=1}^r b_{\alpha l}(t) f_\delta [X_\alpha, X_\delta] \right)$$

$$= \sum_{\pi=1}^r \left(\frac{\partial f_{\pi}}{\partial t_l} + \sum_{\alpha=1}^r \sum_{\delta=1}^r b_{\alpha l}(t) f_{\delta} c_{\alpha \delta \pi} \right) X_{\pi} = 0. \quad (3.97)$$

Since X_1, \dots, X_r are linearly independent over \mathbb{R} and the coefficients of the above expression are only t -dependent, we get that the above amounts to

$$\frac{\partial f_{\pi}}{\partial t_l} = \sum_{\alpha, \delta=1}^r b_{\alpha l}(t) f_{\delta} c_{\delta \alpha \pi}, \quad \pi = 1, \dots, r, \quad l = 1, \dots, s. \quad (3.98)$$

To prove that this is a PDE Lie system, we define the vector fields

$$Y_{\alpha} = \sum_{\delta, \pi=1}^r c_{\delta \alpha \pi} f_{\delta} \frac{\partial}{\partial f_{\pi}}, \quad \alpha = 1, \dots, r. \quad (3.99)$$

We have already proved that $[Y_{\alpha}, Y_{\beta}] = \sum_{\delta=1}^r c_{\alpha \beta \delta} Y_{\delta}$ in Theorem 3. So, these vector fields span a Lie algebra isomorphic to $V/Z(V)$ (for a proof of this fact follow the same line of reasoning as in Theorem 3). In terms of these vector fields, we see that (3.98) is related to the t -dependent vector fields $X_l(t, x) = \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x)$, with $l = 1, \dots, s$. Additionally, to be a PDE Lie system, the above system (3.98) must satisfy the condition

$$\sum_{\alpha, \mu=1}^r \left(\frac{\partial b_{\alpha \pi}}{\partial \kappa} - \frac{\partial b_{\alpha \kappa}}{\partial \pi} + \sum_{\delta, \epsilon=1}^r b_{\delta \kappa} b_{\epsilon \pi} c_{\delta \epsilon \alpha} \right) f_{\mu} c_{\mu \alpha \sigma} = 0, \quad (3.100)$$

with $\kappa \neq \pi = 1, \dots, r$, $\sigma = 1, \dots, r$. The expression in brackets vanishes due to the integrability condition for system (3.83). Hence, (3.98) is a PDE Lie system. We call (3.98) the *symmetry system* for (3.83) relative to V . \square

Definition 16. Given a PDE Lie system X with a Vessiot–Guldberg Lie algebra V , we call \mathcal{S}_X^V the space of Lie symmetries of X that are also t -dependent vector fields taking values in V .

We can straightforwardly prove that the space \mathcal{S}_X^V is a Lie algebra.

3.6 Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -PDE Lie systems

An $\mathfrak{sl}(2, \mathbb{R})$ -PDE Lie system is a PDE Lie system admitting a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let us obtain the elements of \mathcal{S}_X^V for this case. Let us choose a basis of vector fields $\{X_1, X_2, X_3\}$ for V satisfying the same commutation relations as in (3.23). Let us write a general PDE Lie system whose autonomization for a fixed value l is

$$\bar{X}^l = \frac{\partial}{\partial t_l} + b_{1l}(t)X_1 + b_{2l}(t)X_2 + b_{3l}(t)X_3, \quad (3.101)$$

with $t = (t_1, \dots, t_s) \in \mathbb{R}^s$, $1 \leq l \leq s$ and a certain type of possible Lie symmetry $Y = f_1(t)X_1 + f_2(t)X_2 + f_3(t)X_3$, where $f_1(t), f_2(t), f_3(t)$ are t -dependent functions to be determined by the symmetry condition (3.96). This leads us to the system of s first-order PDEs

$$\begin{cases} \frac{\partial f_1}{\partial t_l} = b_{2l}(t)f_1 - b_{1l}(t)f_2, \\ \frac{\partial f_2}{\partial t_l} = 2(b_{3l}(t)f_1 - b_{1l}(t)f_3), \\ \frac{\partial f_3}{\partial t_l} = b_{3l}(t)f_2 - b_{2l}(t)f_3, \end{cases} \quad (3.102)$$

with $l = 1, \dots, s$. Expressed in terms of t -dependent vector fields, for a fixed value of l

$$\Gamma_l^{\mathfrak{sl}(2, \mathbb{R})} = b_{1l}(t)Y_1 + b_{2l}(t)Y_2 + b_{3l}(t)Y_3, \quad l = 1, \dots, s, \quad (3.103)$$

where

$$\begin{aligned} Y_1 &= -f_2 \frac{\partial}{\partial f_1} - 2f_3 \frac{\partial}{\partial f_2}, & Y_2 &= f_1 \frac{\partial}{\partial f_1} - f_3 \frac{\partial}{\partial f_3}, \\ Y_3 &= 2f_1 \frac{\partial}{\partial f_2} + f_2 \frac{\partial}{\partial f_3} \end{aligned} \quad (3.104)$$

close the commutation relations in (3.23). So, they span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Example 8. Let us consider the **partial Riccati equation**, i.e., the PDE system

$$\frac{\partial x}{\partial t_1} = b_{11}(t) + b_{21}(t)x + b_{31}(t)x^2, \quad \frac{\partial x}{\partial t_2} = b_{12}(t) + b_{22}(t)x + b_{32}(t)x^2, \quad (3.105)$$

with the t -dependent coefficients satisfying the appropriate integrability condition (3.100). Such systems appear in the study of WZNW equations and multidimensional Toda systems [188]. Observe that the partials $\partial x / \partial t_1$ and $\partial x / \partial t_2$ are related to the t -dependent vector fields $X_{t_1}^{\text{pRic}} = b_{11}(t)X_1^{\text{pRic}} + b_{21}(t)X_2^{\text{pRic}} + b_{31}(t)X_3^{\text{pRic}}$ and $X_{t_2}^{\text{pRic}} = b_{12}(t)X_1^{\text{pRic}} + b_{22}(t)X_2^{\text{pRic}} + b_{32}(t)X_3^{\text{pRic}}$, with

$$X_1^{\text{pRic}} = \frac{\partial}{\partial x}, \quad X_2^{\text{pRic}} = x \frac{\partial}{\partial x}, \quad X_3^{\text{pRic}} = x^2 \frac{\partial}{\partial x} \quad (3.106)$$

satisfying the commutation relations (3.23). That is, the vector fields $\langle X_1^{\text{pRic}}, X_2^{\text{pRic}}, X_3^{\text{pRic}} \rangle$ span a Vessiot–Guldberg Lie algebra $V^{\text{pRic}} \simeq \mathfrak{sl}(2, \mathbb{R})$. Since we assume that the functions $b_{ij}(t)$ with $i = 1, 2, 3$ and $j = 1, 2$ satisfy (3.100), we get that (3.105) is a PDE Lie system.

Let us look for Lie symmetries of the form $Y = f_1(t)X_1^{\text{pRic}} + f_2(t)X_2^{\text{pRic}} + f_3(t)X_3^{\text{pRic}}$ for (3.105). In view of Theorem 15, such Lie symmetries are solutions of

the system of PDEs

$$\begin{cases} \frac{\partial f_1}{\partial t_j} = b_{2j}(t)f_1 - b_{1j}(t)f_2, \\ \frac{\partial f_2}{\partial t_j} = 2(b_{3j}(t)f_1 - b_{1j}(t)f_3), \\ \frac{\partial f_3}{\partial t_j} = b_{3j}(t)f_2 - b_{2j}(t)f_3, \end{cases} \quad (3.107)$$

with $j = 1, 2$. This resulting system can be interpreted in terms of the t -dependent vector fields $\Gamma_j^{\text{pRic}} = b_{1j}(t)Y_1 + b_{2j}(t)Y_2 + b_{3j}(t)Y_3$, with $j = 1, 2$ and (3.104). These vector fields have the same structure constants as the $X_1^{\text{pRic}}, X_2^{\text{pRic}}, X_3^{\text{pRic}}$. Therefore, (3.105) is a PDE Lie system with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

4 The CLS and NSM for PDEs

Let us now consider a manifold $J^p\pi$ locally isomorphic to $\mathbb{R}^n \times T^pN$ where we now assume the bundle induced by $\pi : (x, u) \in \mathbb{R}^n \times \mathbb{R}^k \simeq N_{\mathbb{R}^n} \mapsto x \in \mathbb{R}^n$, where $x = (x_1, \dots, x_n)$ and $u = (u_1, \dots, u_k)$. The manifold $J^p\pi$ admits a coordinate system $\{x_1, \dots, x_n, u_1, \dots, u_k, u_J\}_{|J|=p}$, where J is a multi-index $J \equiv (j_1, \dots, j_n)$ and $|J| = p$ means that $|J| = j_1 + \dots + j_n$. When necessary, we will write

$$u_J = (u_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}} \equiv \frac{\partial^{|J|} u}{\partial x_{i_1}^{j_1} \partial x_{i_2}^{j_2} \dots \partial x_{i_n}^{j_n}}, \quad j_1 + \dots + j_n = p. \quad (4.1)$$

We study systems of PDEs of p -order of the form

$$\Psi^l = \Psi^l \left(x_i, u_j, (u_j)_{x_i}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}} \right), \quad l = 1, \dots, q. \quad (4.2)$$

such that $j_1 + \dots + j_n = p$, $1 \leq i_1, \dots, i_n \leq n$ and $x_1 \leq x_i \leq x_n$, $j = 1, \dots, k$. Geometrically, we understand this expression as a submanifold of $\mathcal{E} \subset J^p\pi$.

We search for an ϵ -parametric group of transformations on $N_{\mathbb{R}^n}$, namely a Lie group action $\Phi : \mathbb{R} \times N_{\mathbb{R}^n} \rightarrow N_{\mathbb{R}^n}$. Infinitesimally, this ϵ -parametric group reads

$$\begin{cases} \bar{x}_i \rightarrow x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \\ \bar{u}_j \rightarrow u_j + \epsilon \eta_j(x, u) + O(\epsilon^2), \end{cases} \quad (4.3)$$

for all $i = 1, \dots, n$ and $j = 1, \dots, k$.

Definition 1. Given a system of PDEs differential equations (4.2), we say that Φ is a *Lie point symmetry* for (4.2) if for every particular solution $(x, u(x))$ is mapped into a new particular solution $(\bar{x}, \bar{u}(\bar{x})) \equiv \Phi \circ (x, u(x))$.

Definition 2. We call *invariant surface conditions* the set of expressions

$$\eta_j - \sum_{i=1}^n (u_j)_{x_i} \xi_i = 0, \quad j = 1, \dots, k. \quad (4.4)$$

Definition 3. A *nonclassical Lie point symmetry* is a transformation that do not only leave the PDE problem (4.2) invariant, but the set surface conditions (4.4) as well.

For systems of PDEs with derivatives that up to order p , we propose a general transformation

$$\left\{ \begin{array}{l} \bar{x}_i \rightarrow x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \\ \bar{u}_j \rightarrow u_j + \epsilon \eta_j(x, u) + O(\epsilon^2), \\ (\bar{u}_j)_{\bar{x}_i} \rightarrow (u_j)_{x_i} + \epsilon (\eta_j)_{x_i} + O(\epsilon^2), \\ (\bar{u}_j)_{\bar{x}_i^{j_1}, \bar{x}_i^{j_2}} \rightarrow (u_j)_{x_i^{j_1}, x_i^{j_2}} + \epsilon (\eta_j)_{x_i^{j_1}, x_i^{j_2}} + O(\epsilon^2), \\ \dots \\ (\bar{u}_j)_{\bar{x}_i^{j_1}, \bar{x}_i^{j_2}, \dots, \bar{x}_i^{j_n}} \rightarrow (u_j)_{x_i^{j_1}, x_i^{j_2}, \dots, x_i^{j_n}} + \epsilon (\eta_j)_{x_i^{j_1}, x_i^{j_2}, \dots, x_i^{j_n}} + O(\epsilon^2), \end{array} \right. \quad (4.5)$$

such that $j_1 + \dots + j_n \leq p$, $x_1 \leq x_i \leq x_n$ and where $(\eta_{u_j})_{x_i}, \dots, (\eta_{u_j})_{x_i^{j_1}, x_i^{j_2}, \dots, x_i^{j_n}}$ are the prolongations of η_j from first, up to p -order. We have not specified the dependency on (x_i, u_j) for a matter of simplicity.

We can write the p -order *infinitesimal generator* X^p on N for a transformation (4.5) of the system of p -order differential equations (4.2), as

$$\begin{aligned} X^p = & \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k \eta_j \frac{\partial}{\partial u_j} + \sum_{i=1}^n \sum_{j=1}^n (\eta_j)_{x_i} \frac{\partial}{\partial (u_j)_{x_i}} \\ & + \sum_{i_1, i_2=1}^n \sum_{j=1}^k (\eta_j)_{x_i^{j_1}, x_i^{j_2}} \frac{\partial}{\partial (u_j)_{x_i^{j_1}, x_i^{j_2}}} \\ & + \dots + \sum_{i_1, i_2, \dots, i_n=1}^n \dots \sum_{j=1}^k (\eta_j)_{x_i^{j_1}, \dots, x_i^{j_n}} \frac{\partial^{|J|}}{\partial (u_j)_{x_i^{j_1}, \dots, x_i^{j_n}}}. \end{aligned} \quad (4.6)$$

such that $j_1 + \dots + j_n \leq p$

4.1 Algorithmic computation of Lie point symmetries for systems of PDEs

It is possible to follow an algorithmic procedure to compute Lie point symmetries of PDEs. We resume the process in the following number of steps

1. Introduce the ϵ -parametric group of transformations (4.5) in the PDE problem (4.2).
2. Take the zero-order coefficient in ϵ . We recover the initial, untransformed equations. Isolate the higher-order derivative $(u_j)_{x_{i_1}^{j_1} \dots x_{i_n}^{j_n}}$ in terms of the remaining ones by using the relations (4.2).
3. Take the coefficient of first-order in ϵ . Introduce all the prolongations $(\eta_{u_j})_{x_{i_1}^{j_1} \dots x_{i_n}^{j_n}}, \dots, (\eta_{u_j})_{x_i}$, computed according to (4.20) given in Chapter 2. Introduce the correspondent expressions of the higher-order derivatives $(u_j)_{x_{i_1}^{j_1} \dots x_{i_n}^{j_n}}$ of the untransformed equations retrieved in the previous step.

Select the coefficients of the different order derivatives, from the $p - 1$ order $(u_j)_{x_{i_1}^{j_1} \dots x_{i_{n-1}}^{j_{n-1}}}$, (since the highest-order has already been substituted in the previous step) down to the first-order $(u_j)_{x_i}$ and such that $j_1 + \dots + j_{n-1} = p - 1$. It is legit to set such coefficients equal to zero, given the definition of point symmetries, in which ξ_i and η_j do not depend on derivatives of any order. We obtain an overdetermined linear system of differential equations for the coefficients ξ_i and η_j of the infinitesimal generator (4.6).

4. At this point, we have to decide whether we search for *classical or nonclassical Lie point symmetries*. We implement the invariant surface conditions (4.4) in case of searching for nonclassical ones.

If $\xi_{\hat{i}} \neq 0$, where $1 \leq \hat{i} \leq n$, for a particular fixed value \hat{i} of the index i ,

$$(u_j)_{x_i} = \frac{\eta_j - \sum_{i \neq \hat{i}=1}^n (u_j)_{x_i} \xi_i}{\xi_{\hat{i}}}, \quad 1 \leq j \leq k. \quad (4.7)$$

In forthcoming examples we shall see that there exists a plethora of cases depending on whether certain ξ_i are nonzero or null.

4.2 Reduction method of systems of PDEs

Given a p -order system of PDEs (4.2) and an ϵ -parametric group of transformations (4.5) for such a system, there exists an associated infinitesimal generator X^p given in (4.6) for such a transformation which is tangent to the submanifold \mathcal{E} given by

$$0 = \Psi^l \left(x_i, u_j, (u_j)_{x_i}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1} \dots x_{i_n}^{j_n}} \right), \quad l = 1, \dots, q, \quad (4.8)$$

on $\mathbb{J}^p \pi$, such that $j_1 + \dots + j_n = p$, namely X^p satisfies that $X^p \Psi^l = 0$ for $l = 1, \dots, q$ on \mathcal{E} . Observe that

$$\begin{aligned} \xi_i \Psi_{x_i}^l + \eta_j \Psi_{u_j}^l + (\eta_j)_{x_i} \Psi_{(u_j)_{x_i}}^l + (\eta_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}} \Psi_{(u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}}^l \\ + \dots + (\eta_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}} \Psi_{(u_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}}}^l = 0, \end{aligned} \quad (4.9)$$

where we denote by Ψ_y^l the partial derivative $\partial\Psi^l/\partial y$. Geometrically, this means that a tangent vector to \mathcal{E} has coordinates

$$\left(\xi_i, \eta_j, (\eta_j)_{x_i}, (\eta_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (\eta_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}} \right) \quad (4.10)$$

such that $j_1 + \dots + j_n = p$, $i_1, \dots, i_n = 1, \dots, n$, $j = 1, \dots, k$ and $x_1 \leq x_i \leq x_n$ in the natural basis given by

$$\frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial u_j}, \quad \frac{\partial}{\partial (u_j)_{x_i}}, \quad \frac{\partial}{\partial (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}}, \quad \dots \quad \frac{\partial}{\partial (u_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}}}. \quad (4.11)$$

Then, (2.1) is equivalent to saying that the vector

$$\left(\Psi_{x_i}^l, \Psi_{u_j}^l, \Psi_{(u_j)_{x_i}}^l, \Psi_{(u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}}^l, \dots, \Psi_{(u_j)_{x_{i_1}^{j_1}, \dots, x_{i_n}^{j_n}}}^l \right) \quad (4.12)$$

in the basis (4.11) is normal to the submanifold $\mathcal{E} \subset J^p\pi$ with respect to the standard Euclidean product. Observe that this is so for every vector tangent to \mathcal{E} . In other words, the graph of the solution must be a union of integral curves of this vector field. These integral curves are called the characteristics curves of the original PDE.

The integral curves of X^p are given by the particular solutions of

$$\frac{dx_1}{\xi_1} = \dots = \frac{dx_n}{\xi_n} = \frac{du_1}{\eta_1} = \dots = \frac{du_k}{\eta_k}, \quad (4.13)$$

the so called *Lagrange-Charpit system* [388]. The recommended procedure to solve the above system is to fix one of the variables, for example x_1 , and integrate by pairs

$$\begin{aligned} \frac{dx_1}{\xi_1} - \frac{dx_2}{\xi_2} = 0, \quad \dots \quad \frac{dx_1}{\xi_1} - \frac{dx_n}{\xi_n} = 0, \\ \frac{dx_1}{\xi_1} - \frac{du_1}{\eta_1} = 0, \quad \dots \quad \frac{dx_1}{\xi_1} - \frac{du_k}{\eta_k} = 0. \end{aligned} \quad (4.14)$$

Each of these one-form differential equations gives rise to a constant of integration, a total of $n + k - 1$, respectively $z_2, \dots, z_n, U_1, \dots, U_k$.

Definition 4. We call z_2, \dots, z_n *reduced variables* and U_1, \dots, U_k *reduced scalar fields* arising from the integration of the above mentioned system.

One of variables from the $n + k$ -tuple $\{x_1, \dots, x_n, u_1, \dots, u_k\}$ can be written in terms of the remaining $n + k - 1$ functions $\{z_2, \dots, z_n, U_1, \dots, U_k\}$.

$$\begin{cases} z_{\hat{i}} = z_{\hat{i}}(x_1, x_{\hat{i}}, \xi_1(x_1, x_{\hat{i}}), \xi_{\hat{i}}(x_1, x_{\hat{i}})), & 2 \leq \hat{i} \leq n, \\ U_{\hat{j}} = U_{\hat{j}}(x_1, u_{\hat{j}}, \xi_1(x_1, u_{\hat{j}}), \eta_{\hat{j}}(x_1, u_{\hat{j}})), & 1 \leq \hat{j} \leq k. \end{cases} \quad (4.15)$$

To obtain the reduction, we introduce the new variables by inverting (4.15) as

$$\begin{cases} x_{\hat{i}} = F_{\hat{i}}(x_1, z_{\hat{i}}, \xi_1(x_1, z_{\hat{i}}), \xi_{\hat{i}}(x_1, z_{\hat{i}})), & 2 \leq \hat{i} \leq n, \\ u_{\hat{j}} = G_{\hat{j}}(x_1, U_{\hat{j}}, \xi_1(x_1, U_{\hat{j}}), \eta_{\hat{j}}(x_1, U_{\hat{j}})), & 1 \leq \hat{j} \leq k. \end{cases} \quad (4.16)$$

The transformation has to be taken to the derivatives as well. To reduce the initial problem, we make use of the change of coordinates in the tangent space

$$\begin{aligned} x_1 &= \frac{\partial}{\partial x_1} + \sum_{l=2}^n \frac{\partial z_l}{\partial x_1} \frac{\partial}{\partial z_l} + \sum_{m=1}^k \frac{\partial U_m}{\partial x_1} \frac{\partial}{\partial U_m}, \\ x_i &= \sum_{l=2}^n \frac{\partial z_l}{\partial x_i} \frac{\partial}{\partial z_l} + \sum_{m=1}^k \frac{\partial U_m}{\partial x_i} \frac{\partial}{\partial U_m}, \\ u_j &= \sum_{l=2}^n \frac{\partial z_l}{\partial u_j} \frac{\partial}{\partial z_l} + \sum_{m=1}^k \frac{\partial U_m}{\partial u_j} \frac{\partial}{\partial U_m}, \end{aligned} \quad (4.17)$$

and extended for every $2 \leq i \leq n$ and $1 \leq j \leq k$, where (4.15) gives us the coefficients for (4.17). For higher-order derivatives we only have to apply these relations recursively.

Example 1. Consider the simplest case of a nonlinear equation, that is the particular case of the **Burgers equation** [42, 247]

$$u_t + uu_x = \mu u_{xx}. \quad (4.18)$$

This model contributes to the case of gas dynamics and traffic flow. It is equivalent to the Navier-Stokes equation for incompressible flow with the pressure term removed. The coefficient μ is known as the viscosity. For a zero value of μ we have

$$u_t + uu_x = 0, \quad (4.19)$$

which is known as the inviscid Burgers equation, which is the prototype equation developing solutions in form of discontinuities (shock waves) [42]. Equation (4.19) is a first-order partial differential equation with one dependent variable and two independent variables, i.e. a submanifold in $J^1(\mathbb{R}^2, \mathbb{R})$. This equation, although

seemingly trivial at a first glance, shows the phenomena of narrowing and distortion of waves. Let us solve it by the method of the characteristics. We search for curves $x = x(s), t = t(s)$, such that $u = u(x(s), t(s))$ is constant along such curves. Then,

$$0 = \frac{du}{ds} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} = u_x \frac{dx}{dt} - uu_x \frac{dt}{ds} = u_x \frac{d}{ds}(x - ut). \quad (4.20)$$

This means that $x - ut = cte$. Then, the general solution can be written explicitly as $u = f(x - ut)$, with f an arbitrary function $f(x) = u(x, 0)$. This implies that every point of the curve $f(x)$ travels with a velocity equal to its height. The higher points travel at a faster speed than the lower ones. As a consequence, the curve suffers distortion and gets narrower.

We propose a general transformation according to the general theory exposed in this section

$$\begin{cases} \bar{x} \rightarrow x + \epsilon \xi_1(x, t, u) + O(\epsilon^2), \\ \bar{t} \rightarrow t + \epsilon \xi_2(x, t, u) + O(\epsilon^2), \\ \bar{u} \rightarrow u + \epsilon \eta(x, t, u) + O(\epsilon^2) \end{cases} \quad (4.21)$$

By direct prolongation of the transformation to the first derivative, we have the prolonged transformation

$$\begin{cases} \bar{u}_x \rightarrow u_x + \epsilon \eta_x(x, t, u) + O(\epsilon^2), \\ \bar{u}_t \rightarrow u_t + \epsilon \eta_t(x, t, u) + O(\epsilon^2). \end{cases} \quad (4.22)$$

Making use of the formula (4.20) for first order prolongations and considering the invariant surface condition

$$\eta - u_x \xi_1 - u_t \xi_2 = 0, \quad (4.23)$$

we arrive at

$$\begin{cases} \eta_x = u_x \frac{\partial \eta_u}{\partial u} + \frac{\partial \eta}{\partial x} - u_x \frac{\partial \xi_1}{\partial x} - u_t \frac{\partial \xi_2}{\partial x} - u_x^2 \frac{\partial \xi_1}{\partial u} - \frac{\partial \xi_2}{\partial u} u_x u_t, \\ \eta_t = u_t \frac{\partial \eta_u}{\partial u} + \frac{\partial \eta}{\partial t} - u_x \frac{\partial \xi_1}{\partial t} - u_t \frac{\partial \xi_2}{\partial t} - u_x u_t \frac{\partial \xi_1}{\partial u} - \frac{\partial \xi_2}{\partial u} u_t^2. \end{cases} \quad (4.24)$$

The introduction of the complete transformation (4.21) and (4.24) into (4.19) when $\mu = 0$ provides us with a system in different orders in ϵ . The first-order in ϵ is system of differential equations

$$\frac{\partial \eta}{\partial t} - \xi_{1t} u_x + \xi_{2t} u u_x + u \frac{\partial \eta_u}{\partial x} - u \xi_{1x} u_x + \xi_{2x} u^2 u_x + \eta_u u_x = 0. \quad (4.25)$$

where we have substituted the prolongations η_x and $(\eta_u)_t$ and the relation retrieved from the zero-order term in ϵ which is $u_t = -u u_x$. The remaining coefficients of

derivatives, as for u_x can be set equal to zero, because of the definition of Lie point symmetries. In this way, we obtain the following result

$$\xi_1 = a_1 + a_2x + a_3t, \quad (4.26)$$

$$\xi_2 = b_1 + b_2x + b_3t, \quad (4.27)$$

$$\eta_u = a_3 + (a_2 - b_3)u - b_2u^2, \quad (4.28)$$

with $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. These symmetries are classical Lie symmetries.

5 Applications to PDEs in Physics

In this section we aim to calculate Lie symmetries of certain PDEs of different nature. In particular, we study evolution equations appearing in the literature of hydrodynamic systems. Some of these equations belong to certain hierarchies of PDE from which they are one of their lowest-order members. We will study the complete hierarchies from the point of view of Lie point symmetry and we will obtain their reductions as well. Associated with such equations we will present their corresponding Lax pairs. We find really interesting to calculate the Lie point symmetries of a Lax pair. It is innovative in a way that we can see *how the spectral parameter and the eigenfunction reduce under the symmetry*. The interest resides in the fact that Lax pairs in higher-order dimensions, when reduced to a lower dimension by the symmetry, they can lose their nonisospectral character giving rise to an isospectral problem, which could be tractable with the aid of the inverse scattering method, for example. In this section we aim to review the 1 + 1-BKP and the 2 + 1-BKP equations and their corresponding Lax pairs by calculating their symmetries and reductions. The first is an isospectral Lax pair, while the second is nonisospectral.

Generally, equations with nonisospectral Lax pairs in 2 + 1 dimensions may provide more realistic models in the propagation of small amplitude surface waves in straits or large channels with no dramatic varying depth and width and vorticity [142, 312, 395].

From our experience, it is easier to deal with 2 + 1 dimensional Lax pairs rather than with 1 + 1 dimensional versions. Therefore, a good way of proceeding in order to study 1 + 1 versions is to start by 2 + 1 nonisospectral LP and obtain their nontrivial reductions to 1 + 1 through classical and non-classical methods.

The nonisospectral Lax pairs are interesting because the compatibility condition ($\psi_{xxt} = \psi_{ttx}$) imply that the spectral parameter λ is a function, instead of a constant as in the isospectral cases, and shall be included as an scalar field in the Lie point transformation. If we wish to know the 1 + 1 reductions of the spectral problem, it is specially important to establish how the spectral parameter reduces, simultaneously, their compatibility reductions yield the reduction of the nonlinear problem [158].

This means that the spectral parameter λ has to satisfy an equation, which is the *nonisospectral condition*, and that the symmetries are not only symmetries of the LP and the nonlinear problem, but of the equation for the spectral parameter as well. We will illustrate these facts in the following subsections.

5.1 The 2 + 1 dimensional BKP equation and corresponding Lax pair

We consider the 2 + 1 dimensional **Bogoyanlevski–Kadomtsev–Petviashvili equation** (2 + 1-BKP equation henceforth) [158, 165], studied for the first time here from the point of view of the classical and nonclassical Lie point symmetry symmetry analysis [169]. The equation can be expressed as follows

$$(u_{xt} + u_{xxx}y + 8u_xu_{xy} + 4u_{xx}u_y)_x + \sigma u_{yyy} = 0, \quad \sigma = \pm 1, \quad (5.1)$$

i.e. a submanifold of $J^4(\mathbb{R}^3, \mathbb{R})$. It has been proved that this equation is the 2 + 1 dimensional reduction of a 3 + 1 dimensional Kadomtsev–Petviashili (KP) equation [150, 265], that it has the Painlevé property and admits a Lax pair representation. It resembles the Calogero–Bogoyanlevski–Schiff (CBS) equations by similarity in terms [69, 77, 380]. The solutions of (5.1) have solitonic behavior. In particular, this equation contains *lump solutions*, or solitons that decay polynomially in all directions and have non-trivial interactions [158, 174, 325]. Expression (5.1) can be presented in the non-local form as a submanifold of $J^4(\mathbb{R}^3, \mathbb{R}^2)$ of the form

$$\begin{aligned} \omega_{yy} &= u_{xt} + u_{xxx}y + 8u_xu_{xy} + 4u_{xx}u_y, \\ u_y &= \omega_x, \end{aligned} \quad (5.2)$$

where we have chosen $\sigma = -1$ and have introduced the auxiliary scalar field $\omega(x, y, t)$. If $u(x, y, t)$ is a solution for $\sigma = -1$, then $u(x, iy, it)$ is a solution for $\sigma = 1$. This implies that the sign of σ makes no real distinction in the problem. The 2 + 1-BKP equation in non-local form (5.2), has an associated spectral problem which is complex and has two component (ψ, χ) giving rise to a submanifold of $J^2(\mathbb{C}^3, \mathbb{C}^4)$ [165]

$$\begin{aligned} \psi_{xx} &= -i\psi_y - 2u_x\psi, \\ \psi_t &= 2i\psi_{yy} - 4u_y\psi_x + (2u_{xy} + 2i\omega_y)\psi, \end{aligned} \quad (5.3)$$

and its complex conjugate

$$\begin{aligned} \chi_{xx} &= i\chi_y - 2u_x\chi, \\ \chi_t &= -2i\chi_{yy} - 4u_y\chi_x + (2u_{xy} - 2i\omega_y)\chi, \end{aligned} \quad (5.4)$$

with $\psi = \psi(x, y, t)$ and $\chi = \chi(x, y, t)$ over \mathbb{C}^3 . The compatibility condition of the cross derivatives ($\psi_{xxt} = \psi_{txx}$) in (5.3) retrieves the equations in (5.2). So does

the compatibility condition ($\chi_{xxt} = \chi_{txx}$). Since (5.3) and (5.4) are essentially the same, we shall perform calculations on (5.3) and the results should be equivalently extrapolated to the eigenfunction χ in (5.4). We can denote by

$$\Psi = \begin{bmatrix} \psi \\ \chi \end{bmatrix} \quad (5.5)$$

the two component wave function.

We propose the Lie transformation

$$\begin{cases} x \rightarrow x + \epsilon \xi_1(x, y, t, u, \omega) + O(\epsilon^2), \\ y \rightarrow y + \epsilon \xi_2(x, y, t, u, \omega) + O(\epsilon^2), \\ t \rightarrow t + \epsilon \xi_3(x, y, t, u, \omega) + O(\epsilon^2), \\ u \rightarrow u + \epsilon \eta_u(x, y, t, u, \omega) + O(\epsilon^2), \\ \omega \rightarrow \omega + \epsilon \eta_\omega(x, y, t, u, \omega) + O(\epsilon^2), \\ \psi \rightarrow \psi + \epsilon \eta_\psi(x, y, t, u, \omega, \lambda) + O(\epsilon^2) \end{cases} \quad (5.6)$$

where we have dropped the bar notation \bar{x}_i and \bar{u}_j for simplicity. Associated with this transformation, there exists an infinitesimal generator

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_u \frac{\partial}{\partial u} + \eta_\omega \frac{\partial}{\partial \omega} + \eta_\psi \frac{\partial}{\partial \psi}. \quad (5.7)$$

where the subscripts in η have been added according to the field to which each η it is associated. This transformation must leave (5.2) invariant. We compute the first- and second-order derivatives appearing in the equations

$$\begin{cases} u_x \rightarrow u_x + \epsilon(\eta_u)_x + O(\epsilon^2), & \omega_x \rightarrow \omega_x + \epsilon(\eta_\omega)_x + O(\epsilon^2), \\ \omega_y \rightarrow \omega_y + \epsilon(\eta_\omega)_y + O(\epsilon^2), & u_{xx} \rightarrow u_{xx} + \epsilon(\eta_u)_{xx} + O(\epsilon^2), \\ u_{xy} \rightarrow u_{xy} + \epsilon(\eta_u)_{xy} + O(\epsilon^2), & u_{xt} \rightarrow u_{xt} + \epsilon(\eta_u)_{xt} + O(\epsilon^2), \\ u_{xxy} \rightarrow u_{xxy} + \epsilon(\eta_u)_{xxy} + O(\epsilon^2). \end{cases} \quad (5.8)$$

The Lax pair must be simultaneously invariant (5.3) and (5.4).

$$\begin{cases} \psi_{xx} \rightarrow \psi_{xx} + \epsilon(\eta_\psi)_{xx} + O(\epsilon^2), & \psi_{yy} \rightarrow \psi_{yy} + \epsilon(\eta_\psi)_{yy} + O(\epsilon^2), \\ \psi_x \rightarrow \psi_x + \epsilon(\eta_\psi)_x + O(\epsilon^2), & \psi_y \rightarrow \psi_y + \epsilon(\eta_\psi)_y + O(\epsilon^2), \\ \psi_t \rightarrow \psi_t + \epsilon(\eta_\psi)_t + O(\epsilon^2). \end{cases} \quad (5.9)$$

The prolongations $(\eta_u)_{xxy}, (\eta_u)_{xt}, (\eta_u)_{xy}, (\eta_u)_{xx}, (\eta_u)_x, (\eta_u)_y, (\eta_\omega)_x, (\eta_\omega)_y, (\eta_\psi)_{yy}, (\eta_\psi)_{xx}, (\eta_\psi)_t, (\eta_\psi)_y, (\eta_\psi)_x$ can be calculated according to (4.20) given in Chapter 2.

In the case of nonclassical Lie symmetries, we have to include the invariant surface conditions by extracting u_t , ω_t and ψ_t from

$$\begin{aligned}\xi_1 u_x + \xi_2 u_y + \xi_3 u_t &= \eta_u, \\ \xi_1 \omega_x + \xi_2 \omega_y + \xi_3 \omega_t &= \eta_\omega, \\ \xi_1 \psi_x + \xi_2 \psi_y + \xi_3 \psi_t &= \eta_\psi\end{aligned}\quad (5.10)$$

and from the original, untransformed equations, we make use of

$$\begin{aligned}\omega_{yy} &= u_{xt} + u_{xxx}y + 8u_x u_{xy} + 4u_{xx}u_y, \\ u_y &= \omega_x,\end{aligned}\quad (5.11)$$

and the untransformed equations of the Lax pair

$$\begin{aligned}\psi_{xx} &= -i\psi_y - 2u_x\psi, \\ \psi_t &= 2i\psi_{yy} - 4u_y\psi_x + (2u_{xy} + 2i\omega_y)\psi.\end{aligned}\quad (5.12)$$

Introducing such relations we arrive at the nonclassical symmetries:

$$\xi_1 = \frac{\dot{A}_3(t)}{4}x + A_1(t), \quad (5.13)$$

$$\xi_2 = \frac{\dot{A}_3(t)}{2}y + A_2(t), \quad (5.14)$$

$$\xi_3 = A_3(t), \quad (5.15)$$

$$\eta_u = -\frac{\dot{A}_3(t)}{4}u + \frac{\dot{A}_2(t)}{8}x + \frac{\ddot{A}_3(t)}{16}xy + \frac{\dot{A}_1(t)y}{4} + B_1(t), \quad (5.16)$$

$$\eta_\omega = -\frac{\dot{A}_3(t)}{2}w + \frac{\dot{A}_1(t)}{4}x + \frac{\ddot{A}_3(t)}{32}x^2 + B_3(t)y + \frac{\ddot{A}_2(t)}{16}y^2 \quad (5.17)$$

$$+ \frac{\ddot{A}_3(t)}{96}y^3 + B_2(t), \quad (5.18)$$

$$\eta_\psi = \left[-2\lambda - \frac{\dot{A}_3(t)}{8} + i \left(\frac{\dot{A}_2(t)}{4}y + \frac{\ddot{A}_3(t)}{16}y^2 + 2 \int B_3(t)dt \right) \right] \psi. \quad (5.19)$$

These symmetries depend on six arbitrary functions of time, $A_1(t)$, $A_2(t)$, $A_3(t)$ and $B_1(t)$, $B_2(t)$, $B_3(t)$, which shall serve us as a way to classify the possible reductions, and a constant λ . The classical symmetries are a particular case of the nonclassical and lead to less general reductions. For this matter, we will only specify the nonclassical ones.

To reduce the problem, we have to solve the Lagrange-Charpit system

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{du}{\eta_u} = \frac{d\omega}{\eta_\omega} = \frac{d\psi}{\eta_\psi}. \quad (5.20)$$

The most relevant possible reductions agree with the cases

Case I : $A_3(t) \neq 0$		Case II : $A_3(t) = 0$	
1.	$A_1(t) \neq 0 \quad A_2(t) \neq 0$	1.	$A_1(t) \neq 0 \quad A_2(t) \neq 0$
2.	$A_1(t) \neq 0 \quad A_2(t) = 0$	2.	$A_1(t) \neq 0 \quad A_2(t) = 0$
3.	$A_1(t) = 0 \quad A_2(t) \neq 0$	3.	$A_1(t) = 0 \quad A_2(t) \neq 0$

Table 4.1. Reductions for BKP $2 + 1$

We shall use the next notation for the reduced variables

$$x, y, t \rightarrow x_1, x_2 \quad (5.21)$$

and for the reduced fields:

$$\begin{aligned} \omega(x, y, t) &\rightarrow \Omega(x_1, x_2), \\ u(x, y, t) &\rightarrow U(x_1, x_2), \\ \psi(x, y, t) &\rightarrow \Phi(x_1, x_2). \end{aligned} \quad (5.22)$$

As a matter of simplification, we shall drop the dependency x_1, x_2 of all the fields in the forthcoming expressions.

- *Case I.1.* $A_3(t) \neq 0, A_1 \neq 0, A_2 \neq 0$

– Reduced variables

$$x_1 = \frac{x}{A_3(t)^{1/4}} - \int \frac{A_1(t)}{A_3(t)^{5/4}} dt, \quad x_2 = \frac{y}{A_3(t)^{1/2}} - \int \frac{A_2(t)}{A_3(t)^{3/2}} dt. \quad (5.23)$$

– Reduced fields

$$\begin{aligned} u(x, y, t) &= \frac{U(x_1, x_2)}{A_3(t)^{1/4}} + \frac{x_1 x_2}{16} \frac{\dot{A}_3(t)}{A_3(t)^{1/4}} + \frac{x_1}{16} \left(2 \frac{A_2(t)}{A_3(t)^{3/4}} + B(t) \frac{\dot{A}_3(t)}{A_3(t)^{1/4}} \right) \\ &\quad + \frac{x_2}{16} \left(4 \frac{A_1(t)}{A_3(t)^{1/2}} + A(t) \frac{\dot{A}_3(t)}{A_3(t)^{1/4}} \right), \end{aligned} \quad (5.24)$$

$$\begin{aligned} \omega(x, y, t) &= \frac{\Omega(x_1, x_2)}{A_3(t)^{1/2}} + \frac{x_2^3}{192} \left(2A_3(t)^{1/2} \ddot{A}_3(t) - \frac{\dot{A}_3(t)^2}{A_3(t)^{1/2}} \right) + \frac{x_1^2}{32} \frac{\dot{A}_3(t)}{A_3(t)^{1/2}} \\ &\quad + \frac{x_1}{16} \left(4 \frac{A_1(t)}{A_3(t)^{3/4}} + A(t) \frac{\dot{A}_3(t)}{A_3(t)^{1/2}} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{x_2^2}{64} \left[4\dot{A}_2(t) - 2\frac{A_2(t)\dot{A}_3(t)}{A_3(t)} + \frac{B(t)}{A_3(t)^{1/2}} \left(2A_3(t)\ddot{A}_3(t) - \dot{A}_3(t)^2 \right) \right] \\
& + x_2 \left[\frac{\int B_3(t)dt}{A_3(t)^{1/2}} - \frac{1}{16} \frac{A_2(t)^2}{A_3(t)^{3/2}} + \frac{B(t)}{16} \left(2\dot{A}_2(t) - \frac{A_2(t)\dot{A}_3(t)}{A_3(t)} \right) \right. \\
& \left. + \frac{B(t)^2}{64} \left(2A_3(t)^{1/2}\ddot{A}_3(t) - \frac{\dot{A}_3(t)^2}{A_3(t)^{1/2}} \right) \right], \quad (5.25)
\end{aligned}$$

where we have used the definitions

$$A(t) = \int \frac{A_1(t)}{A_3(t)^{5/4}} dt, \quad B(t) = \int \frac{A_2(t)}{A_3(t)^{3/2}} dt. \quad (5.26)$$

– Reduced equation

$$\begin{aligned}
U_{x_2} &= \Omega_{x_1}, \\
\Omega_{x_2x_2} &= U_{x_2x_1x_1x_1} + 8U_{x_2x_1}U_{x_1} + 4U_{x_1x_1}U_{x_2}.
\end{aligned} \quad (5.27)$$

These two equations can be summarized in

$$U_{x_2x_1x_1x_1x_1} + 4\left((U_{x_2}U_{x_1})_{x_1} + U_{x_1}U_{x_1x_2}\right)_{x_1} - U_{x_2x_2x_2} = 0, \quad (5.28)$$

which appears in [411] and has *multiple soliton solutions*.

– Reduced eigenfunction

$$\begin{aligned}
\psi(x, y, t) &= \frac{\Phi(x_1, x_2)}{A_3(t)^{1/8}} \exp \left[i\frac{x_2^2}{16}\dot{A}_3(t) + i\frac{x_2}{8} \left(2\frac{A_2(t)}{A_3(t)^{1/2}} + B(t)\dot{A}_3(t) \right) \right. \\
& - 2\lambda \int \frac{dt}{A_3(t)} \\
& \left. + i \int \left(2\frac{\int B_3(t)dt}{A_3(t)} + \frac{B(t)}{4} \frac{\dot{A}_2(t)}{A_3(t)^{1/2}} + \frac{B(t)^2}{16}\ddot{A}_3(t) \right) dt \right]. \quad (5.29)
\end{aligned}$$

– Reduced Lax pair

$$\begin{aligned}
\Phi_{x_1x_1} + i\Phi_{x_2} + 2U_{x_1}\Phi &= 0, \\
i\Phi_{x_2x_2} - 2U_{x_2}\Phi_{x_1} + (U_{x_1x_2} + i\Omega_{x_2} + \lambda)\Phi &= 0.
\end{aligned} \quad (5.30)$$

- The cases I.2. and I.3. will be omitted for being equivalent to I.1.
- Case II.1. $A_3 = 0$, $A_1 \neq 0$, $A_2 \neq 0$

– Reduced variables

$$x_1 = A_2(t)x - A_1(t)y \quad x_2 = \int A_1(t)A_2(t)^2 dt. \quad (5.31)$$

– Reduced fields

$$u(x, y, t) = A_2(t)U(x_1, x_2) - \frac{A_1(t)^3 y}{A_2(t)^3 4} + \left(\frac{\dot{A}_1(t)}{A_2(t)} - 2A_1(t) \frac{\dot{A}_2(t)}{A_2(t)^2} \right) \frac{y^2}{8} - \frac{\dot{A}_2(t) xy}{A_2(t) 4}, \quad (5.32)$$

$$\omega(x, y, t) = -A_1(t)\Omega(x_1, x_2) + \left(\frac{\dot{A}_1(t)}{A_2(t)} - 2A_1(t) \frac{\dot{A}_2(t)}{A_2(t)^2} \right) \frac{xy}{4} - \frac{\dot{A}_2(t) x^2}{A_2(t) 8} - \frac{A_1(t)^3 x}{A_2(t)^3 4} - \left(4\lambda^2 A_2(t)^4 + 3i\lambda A_1(t)^2 + \frac{3A_1(t)^4}{16A_2(t)^4} - i \frac{\dot{A}_2(t)}{4A_2(t)} \right) y + \left(3 \frac{\dot{A}_2(t)^2}{A_2(t)^2} - \frac{\ddot{A}_2(t)}{A_2(t)} \right) \frac{y^3}{24}. \quad (5.33)$$

– Reduced equation

$$\begin{aligned} \Omega_{x_1} &= U_{x_1}, \\ U_{x_1 x_1 x_1 x_1} &= U_{x_1 x_2} - 12U_{x_1} U_{x_1 x_1}, \end{aligned} \quad (5.34)$$

which can be equivalently rewritten as

$$U_{x_1 x_1 x_1 x_1} + 12U_{x_1 x_1}^2 + 12U_{x_1} U_{x_1 x_1 x_1} - U_{x_1 x_1 x_2} = 0. \quad (5.35)$$

This reduced equation corresponds with the Korteweg de Vries equation (KdV) in 1 + 1 dimensions [149]. *Therefore, we can conclude that the equation BKP is a generalization of KdV to 2 + 1 dimensions.*

– Reduced Eigenfunction

$$\begin{aligned} \psi(x, y, t) &= \Phi(x_1, x_2) e^{-2\lambda A_2(t)^2 y} \times \\ &\times \exp \left[i \left(\frac{A_1(t) x}{A_2(t) 2} - \frac{A_1(t)^2 y}{A_2(t)^2 4} - \frac{\dot{A}_2(t) y^2}{A_2(t) 4} \right) \right]. \end{aligned} \quad (5.36)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} &= 2i\lambda \Phi - 2U_{x_1} \Phi, \\ \Phi_{x_2} &= 8i\lambda \Phi_{x_1} + 4U_{x_1} \Phi_{x_1} - 2\Phi U_{x_1 x_1}, \end{aligned} \quad (5.37)$$

which is the Lax pair corresponding with the KdV equation.

- *Case II.2.* $A_3(t) = 0$, $A_1(t) \neq 0$, $A_2(t) = 0$

– Reduced variables

$$x_1 = y, \quad x_2 = t. \quad (5.38)$$

– Reduced fields

$$\begin{aligned} u(x, y, t) &= U(x_1, x_2) + \frac{B_2(t)}{A_1(t)}x_1 + \frac{B_3(t)}{2A_1(t)}x_1^2 + \frac{B_1(t)}{A_1(t)}x \\ &\quad + \frac{A_1'(t)}{4A_1(t)}xy, \end{aligned} \quad (5.39)$$

$$\begin{aligned} \omega(x, y, t) &= \frac{1}{2i}\Omega(x_1, x_2) + \left(\frac{B_1(t)A_1'(t)}{2A_1(t)^2} + \frac{B_1'(t)}{2A_1(t)} \right) x_1^2 \\ &\quad + \left(\frac{A_1'(t)^2 + A_1(t)A_1''(t)}{24A_1^2} \right) x_1^3 + \frac{B_2(t)}{A_1(t)}x + \frac{A_1'(t)}{8A_1(t)}x^2 \\ &\quad + \frac{B_3(t)}{A_1(t)}xy. \end{aligned} \quad (5.40)$$

– Reduced equation

$$\begin{aligned} U_{x_1} &= 0, \\ U_{x_1x_1x_1} &= 0, \\ \Omega_{x_1x_1} &= 0. \end{aligned} \quad (5.41)$$

– Reduced Eigenfunction

$$\begin{aligned} \psi(x, y, t) &= \frac{\Phi(x_1, x_2)}{\sqrt{A_1(t)}} \exp \left[\int 8 \frac{B_2(t)}{A_1(t)^2} \left(\lambda - i \int B_2(t) dt \right) \right. \\ &\quad \left. - i \frac{8}{A_1(t)^2} \left(B_1(t) + 2 \frac{(\lambda - i \int B_3(t) dt)^2}{A_1(t)} \right)^2 \right. \\ &\quad \left. + \frac{2i}{A_1(t)} \left(B_1(t) + \frac{2}{A_1(t)} \left(\lambda - i \int B_3(t) dt \right)^2 \right) x_1 \right. \\ &\quad \left. + i \frac{A_1'(t)}{4A_1(t)} x_1^2 - \frac{2}{A_1(t)} \left(\lambda - i \int B_3(t) dt \right) x \right]. \end{aligned} \quad (5.42)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1} &= 0, \\ \Phi_{x_2} &= \Omega_{x_1} \Phi. \end{aligned} \quad (5.43)$$

- *Case II.3.* $A_3(t) = 0$, $A_1(t) = 0$, $A_2(t) \neq 0$

– Reduced variables

$$x_1 = \frac{x}{\sqrt{A_2(t)}} - 4 \int \frac{B_1(t)}{A_2(t)^{3/2}} dt, \quad x_2 = t. \quad (5.44)$$

– Reduced fields

$$u(x, y, t) = \frac{1}{2\sqrt{A_2(t)}} \left(U(x_1, x_2) + 2x_1 \int B_3(t) dt \right) + \frac{B_1(t)}{A_2(t)} y + \frac{A_2'(t)}{8A_2(t)} xy, \quad (5.45)$$

$$\omega(x, y, t) = \Omega(x_1, x_2) + \left(\frac{B_1}{\sqrt{A_2}} + \frac{1}{2} A_2'(t) \int \frac{B_1(t)}{A_2(t)^{3/2}} dt \right) x_1 + \frac{A_2'(t)}{16} x_1^2 + \frac{B_2(t)}{A_2(t)} y + \frac{B_3(t)}{2A_2(t)} y^2 + \frac{A_2''(t)}{48A_2(t)} y^3. \quad (5.46)$$

– Reduced equation

$$\begin{aligned} \Omega_{x_1} &= 0, \\ U_{x_1 x_2} &= 0, \\ U_{x_1 x_1 x_2} &= 0. \end{aligned} \quad (5.47)$$

– Reduced eigenfunction

$$\begin{aligned} \psi(x, y, t) &= \frac{\Phi(x_1, x_2)}{A_2(t)^{1/4}} \times \\ &\times e^{2i \int \left(\frac{B_2(t)}{A_2(t)} + \frac{4(\lambda - i \int B_3(t) dt)^2}{A_2(t)^2} \right) dt} e^{-\frac{2(\lambda - i \int B_3(t) dt)}{A_2(t)} y + i \frac{A_2(t)'}{8A_2(t)} y^2}. \end{aligned} \quad (5.48)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_2} &= 0, \\ \Phi_{x_1 x_1} &= (2i\lambda - U_{x_1})\Phi. \end{aligned} \quad (5.49)$$

5.2 A reduction of the 2 + 1-BKP equation and corresponding Lax pair to 1 + 1 dimensions

Let us now study the nontrivial reduction I.1. obtained in the past subsection 5.1. of the initial 2 + 1-BKP equation to 1 + 1 dimensions [169]. We consider this reduction of interest from a possible physical interpretation viewpoint. Reduction II.1. is another nontrivial reduction corresponding to the very celebrated Korteweg

de Vries equation which has been studied in multiple occasions and already accounts with a wide physical literature dedicated to it [71, 260, 382]. This is why in this subsection we shall focus on reduction I.1. Let us consider the differential equation in $J^4(\mathbb{R}^2, \mathbb{R}^2)$.

$$\begin{aligned} u_{x_2x_2} &= \Omega_{x_1x_2}, \\ u_{x_1x_1x_1x_2} + 8u_{x_1x_2}u_{x_1} + 4u_{x_1x_1}u_{x_2} - \Omega_{x_2x_2} &= 0. \end{aligned} \quad (5.50)$$

This equation is also integrable in the Painlevé sense and possesses an associated linear spectral problem or Lax pair in $J^2(\mathbb{C}^3, \mathbb{C}^3)$

$$\begin{aligned} \Phi_{x_1x_1} + i\Phi_{x_2} + 2U_{x_1}\Phi &= 0, \\ i\Phi_{x_2x_2} - 2U_{x_2}\Phi_{x_1} + (U_{x_1x_2} + i\Omega_{x_2} + \lambda)\Phi &= 0. \end{aligned} \quad (5.51)$$

whose compatibility condition recovers (5.50). We propose a general transformation for the independent and dependent variables as

$$\left\{ \begin{array}{l} x_1 \rightarrow x_1 + \epsilon\xi_1(x_1, x_2, u, \Omega) + O(\epsilon^2), \\ x_2 \rightarrow x_2 + \epsilon\xi_2(x_1, x_2, u, \Omega) + O(\epsilon^2), \\ \lambda \rightarrow \lambda + \epsilon\eta_\lambda(x_1, x_2, \lambda, \Phi) + O(\epsilon^2), \\ u \rightarrow u + \epsilon\eta_u(x_1, x_2, u, \Omega) + O(\epsilon^2), \\ \Omega \rightarrow \Omega + \epsilon\eta_\Omega(x_1, x_2, u, \Omega) + O(\epsilon^2), \\ \Phi \rightarrow \Phi + \epsilon\eta_\Phi(x_1, x_2, u, \Omega, \lambda, \Phi) + O(\epsilon^2). \end{array} \right. \quad (5.52)$$

Here we can see that we have considered λ as an independent variable in order to make the reductions properly.

By definition of symmetry of the equation and its associated Lax pair, this ϵ -parametric group of transformations must leave the equations in (5.50) and (5.51) invariant. Equivalently, the associated symmetry vector field has the expression

$$X = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \eta_\lambda \frac{\partial}{\partial \lambda} + \eta_u \frac{\partial}{\partial u} + \eta_\Omega \frac{\partial}{\partial \Omega} + \eta_\Phi \frac{\partial}{\partial \Phi}. \quad (5.53)$$

From the terms in $\epsilon = 0$ we retrieve the original, untransformed equations. We retrieve the conditions

$$\begin{aligned} u_{x_2x_2} &= \Omega_{x_1x_2}, \\ u_{x_1x_1x_1x_2} &= \Omega_{x_2x_2} - 8u_{x_1x_2}u_{x_1} - 4u_{x_1x_1}u_{x_2}, \\ \Phi_{x_1x_1} &= -i\Phi_{x_2} - 2U_{x_1}\Phi = 0, \\ i\Phi_{x_2x_2} &= 2U_{x_2}\Phi_{x_1} - (U_{x_1x_2} - i\Omega_{x_2} - \lambda)\Phi = 0 \end{aligned} \quad (5.54)$$

that shall be used in the forthcoming steps. First introduce (5.52) into the system of differential equations in (5.54) and set the linear term in ϵ equal to zero. Introduce

the prolongations $(\eta_u)_{x_1x_1x_1x_2}$, $(\eta_u)_{x_1x_2}$, $(\eta_u)_{x_1x_1}$, $(\eta_u)_{x_2x_2}$, $(\eta_u)_{x_1}$, $(\eta_u)_{x_2}$, $(\eta_\Omega)_{x_2x_2}$, $(\eta_\Omega)_{x_1x_2}$, $(\eta_\Omega)_{x_1}$, $(\eta_\Phi)_{x_1x_1}$, $(\eta_\Phi)_{x_2x_2}$, $(\eta_\Phi)_{x_1}$, $(\eta_\Phi)_{x_2}$, calculated following formula (4.20) in Chapter 2 and $u_{x_2x_2}$, $u_{x_1x_1x_1x_2}$, $\Phi_{z_1z_1}$, $\Phi_{z_2z_2}$ from (5.54). The difference between classical and nonclassical symmetries in this case is the implementation of the invariant surface conditions. Depending on the case we choose, we have the classification

Values of infinitesimal generators		
Case I.	$\xi_1 \neq 0$	$\xi_2 \neq 0$
Case II	$\xi_1 \neq 0$	$\xi_2 = 0$
Case III.	$\xi_1 = 0$	$\xi_2 \neq 0$

Table 4.2. Iterative reduction of the BKP

Nontrivial results are led by $\xi_1 \neq 0, \xi_2 \neq 0$, which is case I., we shall consider. The invariant surface conditions provide us with

$$\begin{aligned} u_{x_2} &= \frac{\eta_u - \xi_1 U_{x_2}}{\xi_2}, \\ \Omega_{x_2} &= \frac{\eta_\Omega - \xi_1 \Omega_{x_2}}{\xi_2}, \\ \Phi_{x_2} &= \frac{\eta_\Phi - \xi_1 \Phi_{x_1} - \eta_\lambda \Phi_\lambda}{\xi_2}, \end{aligned} \quad (5.55)$$

that we substitute and their multiple higher-order derivatives. We now pick the remaining coefficients in first or higher-order derivatives of the fields $(u_{x_1}, u_{x_1x_1}, \Omega_{x_2x_2}, \dots)$ and set them equal to zero. It is an overdetermined system of equations whose solutions are the symmetries. If we divide by η_λ all the expressions, we resulting reductions will be invariant. This results in

$$\xi_1(x_1, x_2, u, \Omega) = \frac{1}{\eta_\lambda} \left(\frac{1}{2} K_1 x_1 + K_2 \right), \quad (5.56)$$

$$\xi_2(x_1, x_2, u, \Omega) = \frac{1}{\eta_\lambda} (K_1 x_2 + K_3), \quad (5.57)$$

$$\eta_\lambda = 1, \quad (5.58)$$

$$\eta_u(x_1, x_2, u, \Omega) = \frac{1}{\eta_\lambda} \left[-\frac{1}{2} K_1 u + K_5 + C_0 (K_3 x_1 + K_1 x_1 x_2 + 2K_2 x_2) \right], \quad (5.59)$$

$$\eta_\Omega(x_1, x_2, u, \Omega) = \frac{1}{\eta_\lambda} \left[-K_1 \Omega + K_4 x_2 + K_6(x_1) + C_0^2 \left(K_3 x_2^2 + \frac{1}{3} K_1 x_2^3 \right) \right], \quad (5.60)$$

$$\eta_\Phi(x_1, x_2, \lambda, U, W, \Phi, \lambda) = \frac{B(x_1, x_2, \lambda)}{\eta_\lambda} \Phi. \quad (5.61)$$

These symmetries depend on 6 arbitrary constants of integration $K_1, K_2, K_3, K_4, K_5, C_0 \in \mathbb{R}$ and two arbitrary functions $K_6(x_1)$ and $B(x_1, x_2)$.

In the classical case, the symmetries read

$$\xi_1(x_1, x_2, u, \Omega) = \frac{1}{2}k_1x_1 + k_2, \quad (5.62)$$

$$\xi_2(x_1, x_2, u, \Omega) = k_1x_2 + k_3 \quad (5.63)$$

$$\eta_\lambda(x_1, x_2, u, \Omega, \Phi, \lambda) = -2k_1\lambda - ik_4, \quad (5.64)$$

$$\eta_u(x_1, x_2, u, \Omega) = -\frac{1}{2}k_1u + k_5, \quad (5.65)$$

$$\eta_\Omega(x_1, x_2, u, \Omega) = -k_1\Omega + k_4x_2, \quad (5.66)$$

$$\eta_\Phi(x_1, x_2, U, \Omega, \Phi, \lambda) = G(\lambda)\Phi. \quad (5.67)$$

These symmetries depend on 5 arbitrary constants of integration $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$. It is obvious that the classical case is included within the nonclassical one, as expected. *From here we see that there exist big differences between the classical and nonclassical symmetries for this example. It is interesting, since for many other examples, the classical and nonclassical symmetries have coincided.* Nevertheless, the difference between the classical and nonclassical case is not substantial in the case of reductions. Here we show a list of reductions for the nonclassical case which do not differ from the reduced versions obtained in the classical approach.

For the nonclassical case in which $\eta_\lambda = 1$, we have the subclassification

Case I: $k_1 \neq 0$		Case II: $k_1 = 0$	
1.	$A_1(t) \neq 0 \quad A_2(t) \neq 0$	1.	$A_1(t) \neq 0 \quad A_2(t) \neq 0$
		2.	$A_1(t) \neq 0 \quad A_2(t) = 0$

Table 4.3. Subclassification of reductions for BKP in 1 + 1

We introduce the following notation for the reduced variables

$$x_1, x_2, \lambda \rightarrow z, \Lambda \quad (5.68)$$

and the reduced fields and eigenfunctions

$$\begin{aligned} u(x_1, x_2) &\rightarrow V(z, \Lambda), \\ \Omega(x_1, x_2) &\rightarrow W(z, \Lambda), \\ \Phi(x_1, x_2) &\rightarrow \varphi(z, \Lambda). \end{aligned} \quad (5.69)$$

We find the following reductions

- *Case I.1.* $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0$

– Reduced variables

$$z(x_1, x_2, \lambda) = \frac{k_1(k_1x_2 + k_3)}{(k_1x_1 + 2k_2)^2}, \quad (5.70)$$

$$\Lambda(x_1, x_2, \lambda) = k_1^{-5}(2k_1\lambda + ik_4)(k_1x_1 + 2k_2)^4. \quad (5.71)$$

– Reduced fields

$$U(x_1, x_2, \lambda) = k_1 \frac{V(z, \Lambda)}{(k_1x_1 + 2k_2)}, \quad (5.72)$$

$$\begin{aligned} \Omega(x_1, x_2, \lambda) &= \frac{k_4}{2k_1^3}(k_1x_1 + 2k_2)^2 z(x_1, x_2, \lambda) \\ &\quad - \frac{k_1^2}{2(k_1x_1 + 2k_2)^2} \frac{W(z(x_1, x_2, \lambda), \Lambda(x_1, x_2, \lambda))}{z(x_1, x_2, \lambda)}. \end{aligned} \quad (5.73)$$

– Reduced Eigenfunction

$$\Phi(x_1, x_2, \lambda) = \varphi(z, \Lambda) e^{i \int \frac{G(\lambda)}{k_4 - 2ik_1\lambda} d\lambda}. \quad (5.74)$$

– Reduced Lax pair

$$\begin{aligned} &2z^2\varphi_{zz} + 8iz^2V_z(2\Lambda\varphi_\Lambda - z\varphi_z) \\ &\quad + \left[W - z(1 - 6iz)V_z + iz^2(4zV_{zz} - \Lambda) \right] \varphi = 0, \\ &16\Lambda^2\varphi_{\Lambda\Lambda} - 16z\Lambda\varphi_{z\Lambda} + 4\Lambda(3 - 8iz^2V_z)\varphi_\Lambda + (i + 6z + 16iz^3V_z)\varphi_z \\ &\quad - 2 \left[W + V + z(1 + 6iz)V_z + iz^2(4zV_{zz} - \Lambda) \right] \varphi = 0. \end{aligned} \quad (5.75)$$

– Reduced equations

$$\begin{aligned} &W_z = U_z, \\ &16z^6V_{zzzz} + 144z^5V_{zzz} - (1 - 300z^2 + 32z^2V + 96z^3V_z)z^2V_{zz} \\ &\quad - 176z^4V_z^2 + 2(1 + 60z^2 - 32z^2V)zV_z - 2W = 0. \end{aligned} \quad (5.76)$$

- *Case II.1.* $k_1 = 0, k_2 \neq 0, k_3 = 0$

– Reduced variables

$$z = -\frac{k_2}{k_3}x_1 + \frac{k_2^2}{k_3^2}x_2, \quad (5.77)$$

$$\Lambda = \frac{k_3^4}{k_2^4}\lambda + i\frac{k_2}{k_3}x_1. \quad (5.78)$$

– Reduced fields

$$U(x_1, x_2) = -\frac{k_2}{k_3}V(z, \Lambda) + \frac{k_2^2}{8k_3^2}x_1, \quad (5.79)$$

$$\Omega(x_1, x_2) = \frac{k_2^2}{k_3^2} \left(W, \Lambda \right) + \frac{1}{2}z^2 + \frac{k_2}{k_3}zx_1 + \frac{1}{2} \frac{k_2^2}{k_3^2}x_1^2. \quad (5.80)$$

– Reduced Eigenfunction

$$\Phi(x_1, x_2, \lambda) = \varphi(z, \Lambda). \quad (5.81)$$

– Reduced Lax pair

$$\begin{aligned} \varphi_{\Lambda\Lambda} - \frac{1}{4}\varphi - 2V_z\varphi + 2i\varphi_{z\Lambda} - \varphi_{zz} - i\varphi_z &= 0, \\ (z - i\Lambda)\varphi + (\varphi + 2\varphi_\Lambda + 2i\varphi_z)V_z - iV_{zz}\varphi + \varphi_{zz} &= 0. \end{aligned} \quad (5.82)$$

– Reduced equation

$$\begin{aligned} W_z &= V_z, \\ V_{zzzz} + 12V_zV_{zz} - 1 &= 0. \end{aligned} \quad (5.83)$$

• *Case II.2.* $k_1 = 0, k_2 = 0, k_3 \neq 0$

– Reduced variables

$$z = z_2 \left(-\frac{k_4}{k_2} \right)^{2/5}, \quad (5.84)$$

$$\Lambda = \frac{-k_2\lambda - ik_4z_1}{(-k_2k_4)^{1/5}} + z_2 \left(-\frac{k_4}{k_2} \right)^{2/5}. \quad (5.85)$$

– Reduced fields

$$u(z_1, z_2) = i \left(\frac{k_4}{-k_2} \right)^{1/5} \left(U(z) + \frac{i}{2}z^2 \right), \quad (5.86)$$

$$\omega(z_1, z_2) = \left(\frac{k_4}{-k_2} \right)^{2/5} \left(\Omega(z) - \frac{1}{4}z \right) + \frac{k_4}{k_2}z_1z_2. \quad (5.87)$$

– Reduced Eigenfunction

$$\psi(z_1, z_2) = \varphi(z, \Lambda). \quad (5.88)$$

– Reduced Lax pair

$$\begin{aligned} \varphi_{\Lambda\Lambda} - i\varphi_\Lambda - i\varphi_z &= 0, \\ (i - 2z)\varphi_\Lambda + i\varphi_z + \left[-\frac{1}{4} + i(z - \Lambda) + \Omega_z \right] \varphi + 2\varphi_{z\Lambda} + \varphi_{zz} &= 0. \end{aligned} \quad (5.89)$$

– Reduced equation

$$\begin{aligned} V_z &= 0, \\ W_{zz} &= 0. \end{aligned} \tag{5.90}$$

6 Explicit calculation for hierarchies of PDEs

A *hierarchy* is a set of differential equations that are related via a *recursion operator*. The recursive application of such operator gives members of different orders of the hierarchy. For instance, let us consider the well-known **Camassa-Holm equation** [79] as a submanifold of $J^3(\mathbb{R}^2, \mathbb{R})$ (CH1 + 1, henceforth)

$$u_t + Cu_x - u_{xxt} = 2u_x u_{xx} + u u_{xxx}. \tag{6.1}$$

We can interpret u as the fluid velocity and t, x as the temporal and spatial coordinates, respectively. It is an integrable, nonlinear PDE. Though not integrable in the strict defined Painlevé sense, there exists a change of variables (action-angle variables) such that the evolution equation in the new variables is equivalent to a linear flow at constant speed. This change of variables is achieved by studying its associated spectral problem and it is reminiscent of the fact that integrable classical hamiltonian systems are equivalent to linear flows on tori. Indeed, (6.1) is a bi-Hamiltonian model for shallow water waves propagation. When the parameter $C > 0 \in \mathbb{R}$ the solutions are smooth solitary waves or solitons. The case $C = 0$ that mostly concerns us, has peakon solutions. That is, solitons with sharp peak, discontinuity in the slope.

$$u = ce^{-|x-ct|} + O(k \log k). \tag{6.2}$$

Example 1. Equation (6.1) can be understood as the first member of the n -dimensional **negative Camassa-Holm hierarchy** (NCHH 1 + 1) that reads

$$u_t = R^{-n} u_x \tag{6.3}$$

where n is an integer and $R = JK^{-1}$ is the recursion operator formed by the two operators $K = u\partial_x + \partial_x u$ and $J = \partial_x - \partial_{xxx}$. Since calculating the inverse of an operator is a difficult computation, we can introduce auxiliary scalar fields $v_1(x, t), \dots, v_n(x, t)$ such that for the n -iteration of the operator, we can expand (6.3) on $J^3(\mathbb{R}^2, \mathbb{R}^{n+1})$ as

$$\begin{aligned} u_t &= Jv_{[n]}, \\ Jv_{[j-1]} &= Kv_{[j]}, \quad j = 2, \dots, n, \\ u_x &= Kv_{[1]}. \end{aligned} \tag{6.4}$$

It is easy to prove that when $n = 1$ we recover (6.1). Then we can say that (6.1) is the first member of the negative Camassa-Holm hierarchy. There exists a positive

Camassa-Holm hierarchy (PCHH1+1) when $n = -1$ in (6.3) and has been considered in [124, 262].

6.1 The negative Camassa-Holm hierarchy in 2 + 1 dimensions and corresponding Lax pair

This equation was generalized to 2+1 in [173]. It is a simple integrable generalization

$$u_t = R^{-n}u_y, \quad (6.5)$$

where R is the same as defined for the 1 + 1 case. This hierarchy has been proven to be equivalent to n copies of the AKNS equation in 2 + 1 variables [170, 277] by a reciprocal transformation. It is well known that the AKNS has the Painlevé property and a nonisospectral Lax pair. The NCHH correspondent LP can be obtained by means of the singular manifold method [170] or by applying the inverse reciprocal transform on the known AKNS associated Lax pair. Similarly, we can introduce additional scalar fields and express it onto $J^3(\mathbb{R}^3, \mathbb{R}^{n+1})$ as

$$\begin{aligned} u_y &= Jv_{[n]}, \\ Jv_{[j-1]} &= Kv_{[j]}, \quad j = 2, \dots, n \\ u_t &= Kv_{[1]}. \end{aligned} \quad (6.6)$$

where $u = u(x, y, t)$ and $v_{[j]} = v_{[j]}(x, y, t)$ are defined over \mathbb{R}^3 . If we write explicitly the form of the operator, the hierarchy reads

$$\begin{aligned} u_y &= (v_{[n]})_x - (v_{[n]})_{xxx}, \\ (v_{[j-1]})_x - (v_{[j-1]})_{xxx} &= u_x v_{[j]} + 2u(v_{[j]})_x, \quad j = 2, \dots, n \\ u_t &= u_x v_{[1]} + 2u(v_{[1]})_x. \end{aligned} \quad (6.7)$$

Indeed, (6.4) can be recovered if we set $\partial/\partial y = \partial/\partial t$. The PCHH1+1 is recovered by $\partial/\partial y = \partial/\partial x$. The first member of (6.7) is the (6.1) equivalent in 2 + 1 dimensions. Lie point symmetries of the (6.7) hierarchy were computed in [167].

The Lax pair associated with the NCHH(2+1) (6.6) is a nonisospectral Lax pair on $J^2(\mathbb{C}^3, \mathbb{C}^{n+3})$.

$$\begin{aligned} \psi_{xx} - \left(\frac{1}{4} - \frac{\lambda}{2}u \right) \psi &= 0, \\ \psi_y - \lambda^n \psi_t + \hat{\mathcal{A}}\psi_x - \frac{\hat{\mathcal{A}}_x}{2}\psi &= 0, \end{aligned} \quad (6.8)$$

where $\hat{\mathcal{A}} = \sum_{i=1}^n \lambda^{(n-j+1)}v^j$ such that the compatibility condition ($\psi_{xxt} = \psi_{txx}$) recovers (6.7) and imposes the additional condition

$$\lambda_y - \lambda^n \lambda_t = 0, \quad (6.9)$$

which is the *nonisospectrality condition*. As a consequence, we can interpret physically the spectrum of eigenvalues as a distorted spectrum whose eigenvalues are no longer constant and their values get narrower.

In order to obtain symmetries of this Lax pair, together with those of (6.7), we propose the following transformation

$$\begin{cases} x \rightarrow x + \epsilon \xi_1(x, y, t, u, v_{[j]}) + O(\epsilon^2), \\ y \rightarrow y + \epsilon \xi_2(x, y, t, u, v_{[j]}) + O(\epsilon^2), \\ t \rightarrow t + \epsilon \xi_3(x, y, t, u, v_{[j]}) + O(\epsilon^2), \\ \lambda \rightarrow \lambda + \epsilon \eta_\lambda(x, y, t, u, v_{[j]}, \lambda, \psi), \\ u \rightarrow u + \epsilon \eta_u(x, y, t, u, v_{[j]}) + O(\epsilon^2), \\ v_j \rightarrow v_j + \epsilon \eta_{v_j}(x, y, t, u, v_{[j]}) + O(\epsilon^2), \\ \psi \rightarrow \psi + \epsilon \eta_\psi(x, y, t, u, v_{[j]}, \lambda, \psi), \end{cases} \quad j = 1, \dots, n, \quad (6.10)$$

where λ has to be considered as a scalar field due to the nonisospectrality condition (6.9). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_\lambda \frac{\partial}{\partial \lambda} + \eta_u \frac{\partial}{\partial u} + \sum_{j=1}^n \eta_{v_{[j]}} \frac{\partial}{\partial v^{(j)}} + \eta_\psi \frac{\partial}{\partial \psi}. \quad (6.11)$$

We need to know how the derivatives of the fields transform under this symmetry transformation. We introduce the prolongations of the action of the group to the different derivatives that appear in (6.7). To calculate the prolongations $(\eta_{v_{[j-1]}})_{xxx}$, $(\eta_{v_{[j-1]}})_x$, $(\eta_{v_{[j]}})_x$, $(\eta_{v_{[1]}})_x$, $(\eta_{v_{[n]}})_{xxx}$, $(\eta_{v_{[n]}})_x$, $(\eta_u)_x$, $(\eta_u)_y$, $(\eta_u)_t$, ψ_{xx} , ψ_y , ψ_t , ψ_x , we follow the process explained in (4.20) given in Chapter 2. It is therefore necessary that (6.10) leaves the hierarchy (6.7) invariant. We obtain an overdetermined system for the infinitesimals ξ_{x_1} , ξ_{x_2} , ξ_{x_3} , η_λ , η_u , $\eta_{v_{[j]}}$, η_ψ . According to the procedure for symmetry calculation step by step explained above, from the zero-order in ϵ we can isolate the higher-order derivatives appearing in

$$\begin{aligned} (v_{[n]})_{xxx} &= (v_{[n]})_x - u_y, \\ (v_{[j-1]})_{xxx} &= (v_{[j-1]})_x - u_x v_{[j]} - 2u(v_{[j]})_x, \quad j = 2, \dots, n, \\ u_t &= u_x v_{[1]} + 2u(v_{[1]})_x, \end{aligned} \quad (6.12)$$

and

$$\begin{aligned} \psi_{xx} &= \left(\frac{1}{4} + \frac{\lambda}{2} u \right) \psi, \\ \psi_y &= \lambda^n \psi_t - \hat{A} \psi_x + \frac{\hat{A}_x}{2} \psi. \end{aligned} \quad (6.13)$$

with $\hat{\mathcal{A}} = \sum_{i=1}^n \lambda^{(n-j+1)} v_{[j]}$, in order to be introduced in the system of differential equations from which to derive the symmetries.

The generalization to nonclassical symmetries [65, 339] imposes the following invariant surface conditions

$$\begin{aligned}\eta_u &= \xi_1 u_x + \xi_2 u_y + \xi_3 u_t, \\ \eta_{v_{[j]}} &= \xi_1 (v_{[j]})_x + \xi_2 (v_{[j]})_y + \xi_3 (v_{[j]})_t, \quad j = 1, \dots, n. \\ \eta_\lambda &= \xi_2 \lambda_y + \xi_3 \lambda_t, \\ \eta_\psi &= \xi_1 \psi_x + \xi_2 \psi_y + \xi_3 \psi_t.\end{aligned}\tag{6.14}$$

which shall be used as in the addressed method for nonclassical Lie point symmetry calculation.

We distinguish between three different types of nonclassical symmetry according to the values of the infinitesimal generators

Owing to (6.14) there is no restriction in selecting $\xi_i = 1$ when $\xi_i \neq 0$ [339].

Values of infinitesimal generators			
Case I.	$\xi_3 = 1$	any	any
Case II	$\xi_3 = 0$	$\xi_2 = 1$	any
Case III.	$\xi_3 = 0$	$\xi_2 = 0$	$\xi_1 = 1$

Table 4.4. First classification of reductions of CHH in $2 + 1$

In the case of nonclassical symmetries for $\xi_3 = 1$, we eliminate derivatives with respect to t in

$$\begin{aligned}u_t &= \eta_u - \xi_1 u_x - \xi_2 u_y, \\ (v_{[j]})_t &= \eta_{v_{[j]}} - \xi_1 (v_{[j]})_x - \xi_2 (v_{[j]})_y, \quad j = 1, \dots, n. \\ \lambda_t &= \eta_\lambda - \xi_2 \lambda_y, \\ \psi_t &= \eta_\psi - \xi_1 \psi_x - \xi_2 \psi_y.\end{aligned}\tag{6.15}$$

Under these conditions, we obtain the following symmetries

$$\xi_1 = S_1/S_3,\tag{6.16}$$

$$\xi_2 = S_2/S_3,\tag{6.17}$$

$$\xi_3 = 1,\tag{6.18}$$

$$\eta_\lambda = \frac{1}{S_3} \left(\frac{a_3 - a_2}{n} \right) \lambda,\tag{6.19}$$

$$\eta_u = \frac{1}{S_3} \left(\frac{1}{2} \frac{\partial S_1}{\partial x} + a_0 \right) \psi,\tag{6.20}$$

$$\eta_{v_{[1]}} = \frac{1}{S_3} \left[v_{[1]} \left(\frac{\partial S_1}{\partial x} - a_3 \right) - \frac{\partial S_1}{\partial t} \right], \quad (6.21)$$

$$\eta_{v_{[j]}} = \frac{1}{S_3} \left(\frac{\partial S_1}{\partial x} - a_2 \frac{j-1}{n} - a_3 \frac{n-j+1}{n} \right) v_{[j]}, \quad j = 2, \dots, n. \quad (6.22)$$

$$\eta_\psi = \frac{1}{S_3} \left(\frac{1}{2} \frac{\partial S_1}{\partial x} + a_0 \right) \psi. \quad (6.23)$$

where

$$\begin{aligned} S_1 &= S_1(x, t) = A_1(t) + B_1(t)e^x + C_1(t)e^{-x}, \\ S_2 &= S_2(y) = a_2 y + b_2, \\ S_3 &= S_3(t) = a_3 t + b_3. \end{aligned} \quad (6.24)$$

with $A_1(t), A_2(t), A_3(t)$ arbitrary functions of t . Furthermore, a_0, a_2, b_2, a_3, b_3 are arbitrary constants, such that a_3 and b_3 cannot be simultaneously 0.

For the second type of symmetry, $\xi_3 = 0, \xi_2 = 1$ we can now write

$$u_y = \eta_u - \xi_1 u_x, \quad (6.25)$$

$$(v_{[j]})_y = \eta_{v_{[j]}} - \xi_1 (v_{[j]})_x, \quad (6.26)$$

$$\lambda_y = \eta_\lambda, \quad (6.27)$$

$$\psi_y = \eta_\psi - \xi_1 \psi_x. \quad (6.28)$$

which combined with (6.10) leads us to

$$\xi_1 = S_1/S_2, \quad (6.29)$$

$$\xi_2 = 1, \quad (6.30)$$

$$\xi_3 = 0, \quad (6.31)$$

$$\eta_\lambda = \frac{1}{S_2} \left(-\frac{a_2}{n} \right) \lambda, \quad (6.32)$$

$$\eta_u = \frac{1}{S_2} \left(-2 \frac{\partial S_1}{\partial x} + \frac{a_2}{n} \right) u, \quad (6.33)$$

$$\eta_{v_{[1]}} = \frac{1}{S_2} \left(\frac{\partial S_1}{\partial x} v_{[1]} - \frac{\partial S_1}{\partial t} \right), \quad (6.34)$$

$$\eta_{v_{[j]}} = \frac{1}{S_2} \left(\frac{\partial S_1}{\partial x} - a_2 \frac{j-1}{n} \right) v_{[j]}, \quad j = 2, \dots, n. \quad (6.35)$$

$$\eta_\psi = \frac{1}{S_2} \left(-2 \frac{\partial S_1}{\partial x} + \frac{a_2}{n} \right) \psi, \quad (6.36)$$

where S_1 and S_2 are those given in (6.24). Evidently, a_2 and b_2 cannot be 0 at the same time. For the last case $\xi_3 = \xi_2 = 0, \xi_1 = 1$, the invariant surface conditions

turn into

$$\begin{aligned} u_x &= \eta_u, \\ v_{[j]} &= \eta_{v_{[j]}}, \quad j = 1, \dots, n. \\ \psi_x &= \eta_\psi. \end{aligned} \tag{6.37}$$

And the symmetries take a very trivial form

$$\xi_1 = 1, \tag{6.38}$$

$$\xi_2 = 0, \tag{6.39}$$

$$\xi_3 = 0, \tag{6.40}$$

$$\eta_\lambda = 0, \tag{6.41}$$

$$\eta_u = -2u, \tag{6.42}$$

$$\eta_{v_{[j]}} = v_{[j]}, \quad j = 1, \dots, n. \tag{6.43}$$

$$\eta_\psi = \frac{1}{2} \left(1 \pm i\sqrt{2\lambda u} \right) \psi. \tag{6.44}$$

For any of the cases, we have to solve the following Lagrange-Charpit system with their corresponding values of infinitesimal generators

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\lambda}{\eta_\lambda} = \frac{du}{\eta_u} = \frac{dv_{[j]}}{\eta_{v_{[j]}}} = \frac{d\psi}{\eta_\psi}. \tag{6.45}$$

For the first case in which $\xi_3 = 1$, the classification of reductions can be performed attending to Tables 4.5 and 4.6

Possible values of the t-dependent functions			
Case I.	$A_1(t) \neq 0$	$B_1(t) = 0$	$C_1(t) = 0$
Case II	$A_1(t) = 0$	$B_1(t) \neq 0$	$C_1(t) = 0$
Case III.	$A_1(t) = 0$	$B_1(t) = 0$	$C_1(t) \neq 0$

Table 4.5. First subclassification of reductions for CHH 2 + 1

Case II yields the same reduced spectral problems as the obtained for I. Case III is easy to prove that it is equivalent to II owing to the invariance of the Lax pair and equations under the transformation $x \rightarrow -x$, $y \rightarrow -y$ and $t \rightarrow -t$. Below we only consider cases I and II.

For each of these cases we have subclasses depending on the values of the constants a_j and b_j , which are the 5 subclasses exposed. We introduce the notation for the reduced variables as

$$x, y, t \rightarrow x_1, x_2 \tag{6.46}$$

Values of the constants			
1.	$a_2 = 0$	$a_3 = 0$	$b_2 = 0$
2.	$a_2 = 0$	$a_3 = 0$	$b_2 \neq 0$
3.	$a_2 = 0$	$a_3 \neq 0$	$b_2 = 0$
4.	$a_2 = 0$	$a_3 \neq 0$	$b_2 \neq 0$
5.	$a_2 \neq 0$	any	any

Table 4.6. Second subclassification of reductions for CHH 2 + 1

and the reduced fields and reduced eigenfunction

$$\begin{aligned}
u(x, y, t) &\rightarrow U(x_1, x_2), \\
v_{[1]}(x, y, t) &\rightarrow V_{[1]}(x_1, x_2), \\
v_{[j]}(x, y, t) &\rightarrow V_{[j]}(x_1, x_2), \\
\lambda(y, t) &\rightarrow \Lambda(x_2), \\
\psi(x, y, t) &\rightarrow \Phi(x_1, x_2).
\end{aligned} \tag{6.47}$$

We obtain 5 different nontrivial reductions. We obtain each of them separately with their correspondent reduced variables, reduced fields and transformations. In the next subsection, the reduced spectral problem will be obtained.

- *Reduction I.1.* $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 = 0, a_3 = 0, b_2 = 0$

By solving the characteristic equation for these values, we obtain

– Reduced variables

$$x_1 = x - \frac{1}{b_3} \int A_1(t) dt, \quad x_2 = y. \tag{6.48}$$

– Reduced spectral parameter

$$\lambda = \lambda_0. \tag{6.49}$$

– Reduced vector fields

$$u(x, y, t) \rightarrow U(x_1, x_2), \tag{6.50}$$

$$v_{[1]}(x, y, t) \rightarrow V_{[1]}(x_1, x_2) - \frac{A_1}{b_3}, \tag{6.51}$$

$$v_{[j]}(x, y, t) = V_{[j]}(x_1, x_2), \tag{6.52}$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.53)$$

$$\frac{2U \partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} = 0, \quad (6.54)$$

$$\frac{2U \partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.55)$$

which is *the positive Camassa-Holm hierarchy* whose first component $n = 1$ is a *modified Dym equation* [23].

– Reduced eigenfunction

$$\psi = e^{\frac{a_0 t}{b_3}} e^{\frac{\lambda_0^n a_0 x_2}{b_3}} \Phi(x_1, x_2). \quad (6.56)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2} U \right) \Phi &= 0, \\ \Phi_{x_2} + \hat{\mathcal{B}} \Phi_{x_1} - \frac{\hat{\mathcal{B}}}{2} \Phi &= 0, \end{aligned} \quad (6.57)$$

with $\hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2)$.

- *Reduction I.2.* $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 = 0, a_3 = 0, b_2 \neq 0$

– Reduced variables

$$x_1 = x - \frac{1}{b_3} \int A_1(t) dt, \quad x_2 = \frac{y}{b_2} - \frac{t}{b_3}. \quad (6.58)$$

– Reduced spectral parameter

$$\lambda = \left(\frac{b_3}{b_2} \right)^{1/n} \lambda_0. \quad (6.59)$$

– Reduced fields

$$u(x, y, t) = \left(\frac{b_2}{b_3} \right)^{1/n} U(x_1, x_2), \quad (6.60)$$

$$v_{[1]}(x, y, t) = \left(\frac{1}{b_3} \right) V_{[1]}(x_1, x_2) - \frac{A_1}{b_3}, \quad (6.61)$$

$$v_{[j]}(x, y, t) = \left(\frac{1}{b_3} \right) \left(\frac{b_2}{b_3} \right)^{\frac{1-j}{n}} V_{[j]}(x_1, x_2). \quad (6.62)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.63)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.64)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.65)$$

– Reduced eigenfunction

$$\psi = e^{\frac{a_0 t}{b_3}} e^{\frac{\lambda_0^n a_0 x_2}{1 + \lambda_0^n}} \Phi(x_1, x_2). \quad (6.66)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2} U \right) \Phi &= 0, \\ \Phi_{x_2} (1 + \lambda_0^n) + \hat{\mathcal{B}} \Phi_{x_1} - \frac{\hat{\mathcal{B}}}{2} \Phi &= 0, \end{aligned} \quad (6.67)$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2).$$

- *Reduction I.3.* $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 = 0, a_3 \neq 0, b_2 = 0$

– Reduced variables

$$x_1 = x - \int \frac{A_1(t)}{S_3} dt, \quad x_2 = a_3 y. \quad (6.68)$$

– Reduced spectral parameter

$$\lambda(y, t) = S_3^{1/n} \Lambda(x_2). \quad (6.69)$$

– Reduced fields

$$u(x, y, t) = S_3^{-1/n} U(x_1, x_2), \quad (6.70)$$

$$v_{[1]} = \frac{a_3}{S_3} V_{[1]}(x_1, x_2) - \frac{A_1}{S_3}, \quad (6.71)$$

$$v_{[j]} = a_3 S_3^{\frac{j-1}{n}-1} V_{[j]}(x_1, x_2). \quad (6.72)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.73)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} + \frac{U}{n} = 0, \quad (6.74)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.75)$$

– Reduced eigenfunction

$$\psi = \Lambda(x_2) \frac{a_0^n}{a_3} S_3^{\frac{a_0}{a_3}} \Phi(x_1, x_2) \quad (6.76)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} &= \left(\frac{1}{4} + \frac{\Lambda(x_2)}{2} U \right) \Phi, \\ \Phi_{x_2} &= -\hat{\mathcal{B}} \Phi_{x_1} + \frac{\hat{\mathcal{B}}}{2} \Phi, \end{aligned} \quad (6.77)$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2).$$

- *Reduction I.4.* $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 = 0, a_3 \neq 0, b_2 \neq 0$

– Reduced variables

$$x_1 = x - \int \frac{A_1(t)}{S_3} dt, \quad x_2 = \frac{a_3 y}{b_2} - \ln S_3. \quad (6.78)$$

– Reduced spectral parameter

$$\lambda(y, t) = \left(\frac{S_3}{b_2} \right)^{1/n} \Lambda(x_2). \quad (6.79)$$

– Reduced fields

$$u(x, y, t) = \left(\frac{b_2}{S_3} \right)^{1/n} U(x_1, x_2), \quad (6.80)$$

$$v_{[1]} = \frac{a_3}{S_3} V_{[1]}(x_1, x_2) - \frac{A_1}{S_3}, \quad (6.81)$$

$$v_{[j]} = \frac{a_3}{S_3} \left(\frac{S_3}{b_2} \right)^{\frac{j-1}{n}} V_{[j]}(x_1, x_2). \quad (6.82)$$

– The reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.83)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} + \frac{U}{n} = 0, \quad (6.84)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.85)$$

– Reduced eigenfunction

$$\psi = \Lambda(x_2) \frac{a_0^n}{a_3} S_3^{a_3} \Phi(x_1, x_2). \quad (6.86)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} - \left(\frac{1}{4} - \frac{\Lambda(x_2)}{2} U \right) \Phi &= 0, \\ \Phi_{x_2} (1 + \Lambda(x_2)^n) + \hat{\mathcal{B}} \Phi_{x_1} - \frac{\hat{\mathcal{B}}}{2} \Phi &= 0, \end{aligned} \quad (6.87)$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2).$$

- *Reduction I.5.* $B_1(t) = C_1(t) = 0, A_1(t) \neq 0, a_2 \neq 0$

– The reduced variables

$$x_1 = x - \int \frac{A_1(t)}{S_3(t)} dt, \quad x_2 = S_2 S_3^{-\frac{a_2}{a_3}}. \quad (6.88)$$

– Reduced spectral parameter

$$\lambda(y, t) = S_3^{\frac{a_3 - a_2}{a_3^n}} \Lambda(x_2). \quad (6.89)$$

– Reduced fields

$$u(x, y, t) = S_3^{\frac{a_2 - a_3}{a_3^n}} U(x_1, x_2), \quad (6.90)$$

$$v_{[1]}(x, y, t) = \left(\frac{a_2}{S_3} \right) V_{[1]}(x_1, x_2) - \frac{A_1}{S_3}, \quad (6.91)$$

$$v_{[j]}(x, y, t) = \left(\frac{a_2}{S_3} \right) V_{[j]}(x_1, x_2) S_3^{\frac{(a_3 - a_2)(j-1)}{a_3^n}}. \quad (6.92)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \quad (6.93)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} + x_2 \frac{\partial U}{\partial x_2} + \frac{a_3 - a_2}{a_2} \frac{U}{n} = 0, \quad (6.94)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.95)$$

– Reduced eigenfunction

$$\psi = \Lambda(x_2)^{\frac{na_0}{a_3 - a_2}} S_3^{\frac{a_0}{a_3}} \Phi(x_1, x_2). \quad (6.96)$$

– Reduced Lax pair

$$\Phi_{x_1 x_1} = \left(\frac{1}{4} + \frac{\Lambda(x_2)}{2} U \right) \Phi, \quad (6.97)$$

$$\Phi_{x_2} (1 + x_2 \Lambda(x_2)^2) + \hat{\mathcal{B}} \Phi_{x_1} - \frac{\hat{\mathcal{B}}_{x_1}}{2} \Phi = 0,$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2).$$

For the type of symmetry in which $\xi_3 = 0$, $\xi_2 = 1$, one of the reduced variables is t . This means that the integrals that involve S_1 can be performed without any restrictions for the functions $A_1(t)$, $B_1(t)$, $C_1(t)$. We have four different cases (see Table 4.7).

Values of infinitesimal generators		
1.	$a_2 = 0$	$E = \sqrt{A_1^2 - 4B_1C_1} = 0$
2.	$a_2 = 0$	$E = \sqrt{A_1^2 - 4B_1C_1} \neq 0$
3.	$a_2 \neq 0$	$E = \sqrt{A_1^2 - 4B_1C_1} = 0$
4.	$a_2 \neq 0$	$E = \sqrt{A_1^2 - 4B_1C_1} \neq 0$

Table 4.7. Second classification of reductions for CHH2 + 1

- *Reduction 1.* $a_2 = 0$, $E = \sqrt{A_1^2 - 4B_1C_1} = 0$

– Reduced variables

$$x_1 = \int \frac{dx}{S_1(x, t)} - \frac{y}{b_2}, \quad x_2 = \frac{t}{b_2}. \quad (6.98)$$

– Reduced spectral parameter

$$\lambda(y, t) = \lambda_0. \quad (6.99)$$

– Reduced fields

$$u(x, y, t) = \frac{U(x_1, x_2)}{S_1^2}, \quad (6.100)$$

$$v_{[1]}(x, y, t) = \frac{S_1}{b_2} V_{[1]}(x_1, x_2) + S_1 \frac{d}{dt} \left(\int \frac{dx}{S_1(x, t)} \right), \quad (6.101)$$

$$v_{[j]}(x, y, t) = \frac{S_1}{b_2} V_{[j]}(x_1, x_2). \quad (6.102)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial U}{\partial x_1} = 0, \quad (6.103)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_2} = 0, \quad (6.104)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} = 0. \quad (6.105)$$

– Reduced eigenfunction

$$\psi(x, y, t) = \sqrt{S_1} e^{\frac{a_0 y}{b_2}} e^{\frac{a_0 t}{b_2 \lambda_0^2}} \Phi(x_1, x_2). \quad (6.106)$$

– Reduced Lax pair

$$\Phi_{x_1 x_1} + \frac{\lambda_0}{2} U \Phi = 0, \quad (6.107)$$

$$\lambda_0^n \Phi_{x_2} = (\hat{\mathcal{B}} - 1) \Phi_{x_1} - \frac{\hat{\mathcal{B}}}{2} \Phi,$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \lambda_0^{n-j+1} V_{[j]}(x_1, x_2).$$

• *Reduction 2.* $a_2 = 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} \neq 0$

– Reduced variables

$$x_1 = E \left(\int \frac{dx}{S_1(x, t)} - \frac{y}{b_2} \right), \quad x_2 = \frac{1}{b_2} \int E(t) dt. \quad (6.108)$$

– Reduced spectral parameter

$$\lambda(y, t) = \lambda_0. \quad (6.109)$$

– Reduced fields

$$u(x, y, t) = \frac{E^2}{S_1^2} U(x_1, x_2), \quad (6.110)$$

$$v_{[1]}(x, y, t) = \frac{S_1}{b_2} V_{[1]}(x_1, x_2) + S_1 \frac{d}{dt} \left(\int \frac{dx}{S_1(x, t)} \right) + S_1 \frac{E_t}{E^2} x_1, \quad (6.111)$$

$$v_{[j]}(x, y, t) = \frac{S_1}{b_2} V_{[j]}(x_1, x_2). \quad (6.112)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} - \frac{\partial U}{\partial x_1} = 0, \quad (6.113)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_2} = 0, \quad (6.114)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.115)$$

– Reduced eigenfunction

$$\psi = \sqrt{\frac{S_1}{E}} e^{\frac{a_0 y}{b_2}} e^{\frac{a_0 t}{b_2 \lambda_0^n}} \Phi(x_1, x_2). \quad (6.116)$$

– Reduced Lax pair

$$\Phi_{x_1 x_1} + \left(\frac{\lambda_0}{2} U - \frac{1}{4} \right) \Phi = 0, \quad (6.117)$$

$$\lambda_0^n \Phi_{x_2} = (\hat{\mathcal{B}} - 1) \Phi_{x_1} - \frac{\hat{\mathcal{B}}_{x_1}}{2} \Phi,$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \Lambda(x_2)^{n-j+1} V_{[j]}(x_1, x_2).$$

- *Reduction 3.* $a_2 \neq 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} = 0$

– Reduced variables

$$x_1 = \int \frac{dx}{S_1(x, t)} - \frac{\ln S_2}{a_2}, \quad x_2 = t. \quad (6.118)$$

– Reduced spectral parameter

$$\lambda(y, t) = S_2^{-1/n} \Lambda(x_2), \quad (6.119)$$

where $\Lambda(x_2)$ satisfies the nonisospectral condition

$$n \frac{d\Lambda(x_2)}{dx_2} + a_2 \lambda(x_2)^{1-n} = 0. \quad (6.120)$$

– Reduced fields

$$u(x, y, t) = \frac{S_2^{1/n}}{S_1^2} U(x_1, x_2), \quad (6.121)$$

$$v_{[1]}(x, y, t) = S_1 V_{[1]}(x_1, x_2) + S_1 \frac{d}{dt} \left(\int \frac{dx}{S_1(x, t)} \right), \quad (6.122)$$

$$v_{[j]}(x, y, t) = S_1 S_2^{\frac{1-j}{n}} V_{[j]}(x_1, x_2). \quad (6.123)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial U}{\partial x_1} + \frac{a_2}{n} U = 0, \quad (6.124)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_2} = 0, \quad (6.125)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} = 0. \quad (6.126)$$

– Reduced eigenfunction

$$\psi = \sqrt{S_1} \lambda^{-\frac{a_0 n}{a_2}} \Phi(x_1, x_2). \quad (6.127)$$

– Reduced Lax pair

$$\begin{aligned} \Phi_{x_1 x_1} + \frac{\Lambda(x_2)}{2} U \Phi &= 0, \\ \Lambda(x_2)^n \Phi_{x_2} &= (\hat{\mathcal{B}} - 1) \Phi_{x_1} - \frac{\hat{\mathcal{B}}_{x_1}}{2} \Phi, \end{aligned} \quad (6.128)$$

with $\hat{\mathcal{B}} = \sum_{j=1}^n \Lambda(x_2)^{n-j+1} V_{[j]}(x_1, x_2)$.

• *Reduction 4.* $a_2 \neq 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} \neq 0$

– Reduced variables

$$x_1 = E \left(\int \frac{dx}{S_1(x, t)} - \frac{\ln S_2}{a_2} \right), \quad x_2 = \int E(t) dt. \quad (6.129)$$

– Reduced spectral parameter

$$\lambda(y, t) = S_2^{-1/n} \Lambda(x_2), \quad (6.130)$$

where $\Lambda(x_2)$ satisfies the nonisospectral condition

$$n \frac{d\Lambda(x_2)}{dx_2} + \frac{a_2}{E(x_2)} \Lambda(x_2)^{n-1} = 0. \quad (6.131)$$

– Reduced fields

$$u(x, y, t) = \frac{E^2}{S_1^2} S_2^{1/n} U(x_1, x_2), \quad (6.132)$$

$$v_{[1]}(x, y, t) = S_1 V_{[1]}(x_1, x_2) + S_1 \frac{d}{dt} \left(\int \frac{dx}{S_1(x, t)} \right) + S_1 \frac{E_t}{E^2} x_1, \quad (6.133)$$

$$v_{[j]}(x, y, t) = S_1 S_2^{\frac{1-j}{n}} V_{[j]}(x_1, x_2). \quad (6.134)$$

– Reduced hierarchy

$$\frac{\partial^3 V_{[n]}}{\partial x_1^3} - \frac{\partial V_{[n]}}{\partial x_1} - \frac{\partial U}{\partial x_1} + \frac{a_2}{nE(x_2)} U = 0, \quad (6.135)$$

$$2U \frac{\partial V_{[1]}}{\partial x_1} + V_{[1]} \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_2} = 0, \quad (6.136)$$

$$2U \frac{\partial V_{[j+1]}}{\partial x_1} + V_{[j+1]} \frac{\partial U}{\partial x_1} + \frac{\partial^3 V_{[j]}}{\partial x_1^3} - \frac{\partial V_{[j]}}{\partial x_1} = 0. \quad (6.137)$$

– Reduced eigenfunction

$$\psi(x, y, t) = \sqrt{\frac{S_1}{E}} \lambda^{-\frac{a_0 n}{a_2}} \phi(x_1, x_2). \quad (6.138)$$

– Reduced Lax pair

$$\Phi_{x_1 x_1} + \left(\frac{\Lambda(x_2)}{2} U - \frac{1}{4} \right) \Phi = 0, \quad (6.139)$$

$$\Lambda(x_2)^n \Phi_{x_2} = (\hat{\mathcal{B}} - 1) \phi_{x_1} - \frac{\hat{\mathcal{B}}}{2} \Phi,$$

$$\text{with } \hat{\mathcal{B}} = \sum_{j=1}^n \Lambda(x_2)^{n-j+1} V_{[j]}(x_1, x_2).$$

6.2 The Qiao hierarchy in 2 + 1 dimensions and corresponding Lax pair

Another important hierarchy in relation to the (6.7) is the **Qiao hierarchy** (Qiao2+1). It was first introduced as an integrable generalization of the Qiao hierarchy in 1 + 1 [361] to 2 + 1 dimensions. This hierarchy can be proven to be connected with the NCHH(2 + 1) by reciprocal transformations [160]. We will refer to it as mCHH2 + 1. In the reduced form, it is formally equivalent to (6.3)

$$u_t = R^{-n} u_y, \quad (6.140)$$

but the operators are substantially different. In this case, the operator $K = \partial_{xxx} - \partial_x$ and $J = -\partial_x u \partial_x^{(-1)} u \partial_x$ that form the recursion operator $R = JK^{-1}$. In compact form as a submanifold of $J^3\mathbb{R}^3, \mathbb{R}^{n+1}$) the mCHH2 + 1 reads

$$\begin{aligned} u_y &= Jv_{[n]}, \\ Jv_{[j+1]} &= K\bar{v}_{[j]}, \\ u_t &= K\bar{v}_{[n]}. \end{aligned} \quad (6.141)$$

It is necessary to introduce auxiliary scalar fields $v_{[1]}(x, y, t), \dots, v_{[n]}(x, y, t)$ as in (6.6), (6.7) for utilizing the inverse of the operator J . Also, due to the difficulty of operating with the term $\delta^{-1}u$ in J it is necessary to include another n -tuple of scalar fields $\omega_{[1]}, \dots, \omega_{[n]}$. Introducing the explicit form of the operators and making use of the additional scalar fields, we expanded expressions to $J^3(\mathbb{R}^3, \mathbb{R}^{2n+1})$ as

$$\begin{aligned} u_y &= -(u\omega_{[1]})_x, \\ (v_{[j]})_{xx} - v_{[j]} &= -u\omega_{[j+1]}, \quad j = 1, \dots, n-1, \\ (\omega_{[j]})_x &= u(v_{[j]})_x, \\ u_t &= ((v_{[n]})_{xx} - v_{[n]})_x, \end{aligned} \quad (6.142)$$

The Lax pair for mNCHH(2 + 1) was first introduced in [160] as an integrable generalization of the Qiao(1 + 1)'s Lax pair to 2 + 1 dimensions [359, 360, 361]. It is a two component Lax pair given geometrically as a submanifold of $J^2(\mathbb{C}^3, \mathbb{C}^{2n+4})$ whose temporal and spatial parts read correspondingly

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_t = \lambda^n \begin{pmatrix} \phi \\ \psi \end{pmatrix}_y + \lambda p \begin{pmatrix} \phi \\ \psi \end{pmatrix}_x + \frac{i\sqrt{\lambda}}{2} \begin{bmatrix} 0 & q_x - q \\ q_x + q & 0 \end{bmatrix}_x \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad (6.143)$$

where $p = \sum_{j=1}^n \lambda^{n-j}(y, t)\omega_{[j]}(x, y, t)$ and $q = \sum_{j=1}^n \lambda^{n-j}(y, t)v_{[j]}(x, y, t)$ and $i = \sqrt{-1}$, and

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_x = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\lambda}u \\ i\sqrt{\lambda}u & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (6.144)$$

The compatibility condition of $(\phi_{xt} = \phi_{tx})$ or $(\psi_{xt} = \psi_{tx})$ recovers (6.142). Additionally, from the compatibility condition we obtain the nonisospectrality condition

$$\lambda_t - \lambda^n \lambda_y = 0, \quad (6.145)$$

which is the same as for the NCHH2 + 1.

In order to calculate the Lie point symmetries of the mCHH(2 + 1) and its associated Lax pair, we propose a uniparametric Lie point symmetry transformation

of the form

$$\left\{ \begin{array}{l} x \rightarrow x + \epsilon \xi_1(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ y \rightarrow y + \epsilon \xi_2(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ t \rightarrow t + \epsilon \xi_3(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ \lambda \rightarrow \lambda + \epsilon \eta_\lambda(x, y, t, \lambda, u, \omega_{[j]}, v_{[j]}, \phi, \psi), \\ u \rightarrow u + \epsilon \eta_u(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ \omega_{[j]} \rightarrow \omega_{[j]} + \epsilon \eta_{\omega_{[j]}}(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ v_{[j]} \rightarrow v_{[j]} + \epsilon \eta_{v_{[j]}}(x, y, t, u, \omega_{[j]}, v_{[j]}, \phi, \psi) + O(\epsilon^2), \\ \phi \rightarrow \phi + \epsilon \eta_\phi(x, y, t, \lambda, u, \omega_{[j]}, v_{[j]}, \phi, \psi), \\ \psi \rightarrow \psi + \epsilon \eta_\psi(x, y, t, \lambda, u, \omega_{[j]}, v_{[j]}, \phi, \psi), \end{array} \right. \quad (6.146)$$

with $j = 1, \dots, n$. The Lie algebra corresponds with the set of vector fields

$$\begin{aligned} X = & \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_\lambda \frac{\partial}{\partial \lambda} + \\ & \eta_u \frac{\partial}{\partial u} + \sum_{j=1}^n \omega_{[j]} \frac{\partial}{\partial \omega_{[j]}} + \sum_{j=1}^n v_{[j]} \frac{\partial}{\partial v_{[j]}} + \eta_\psi \frac{\partial}{\partial \psi} + \eta_\phi \frac{\partial}{\partial \phi}. \end{aligned} \quad (6.147)$$

Symmetries of this hierarchy have been particularly difficult to obtain. For this matter, we will restrict ourselves to the classical realm. In general, an overdetermined system of nonlinear PDEs is very difficult to solve, as for the nonclassical case. The classical case has also been tough, due to the number of auxiliary fields that have been introduced. Following a similar procedure as explained in the algorithmic steps, we need the prolongations corresponding with $(\eta_{v_{[n]}})_{xxx}$, $(\eta_{v_{[j]}})_{xx}$, $(\eta_{v_{[j]}})_x$, $(\eta_{v_{[n]}})_x$, $(\eta_{\omega_{[j]}})_x$, $(\eta_u)_t$, $(\eta_u)_y$, $(\eta_u)_x$, $(\eta_{\omega_{[1]}})_x$, $(\eta_\psi)_t$, $(\eta_\psi)_y$, $(\eta_\psi)_x$, $(\eta_\phi)_t$, $(\eta_\phi)_y$, $(\eta_\phi)_x$, according to (4.20) in Chapter 2. Also we make use of the zero-order in ϵ retrieved initial, untransformed equations whose higher-order derivative has to be

introduced into the system of PDEs for the symmetry search

$$\begin{aligned}
u_y &= -(u\omega_{[1]})_x, \\
(v_{[j]})_{xx} &= v_{[j]} - u\omega_{[j+1]}, \\
(\omega_{[j]})_x &= u(v_{[j]})_x, \\
(v_{[n]})_{xxx} &= u_t + (v_{[n]})_x, \\
\lambda_t &= \lambda\lambda_y, \\
\phi_t &= \lambda^n \phi_y + \lambda p \phi_x + \frac{i\sqrt{\lambda}}{2}(q_x - q)\psi, \\
\psi_t &= \lambda^n \psi_y + \lambda p \psi_x + \frac{i\sqrt{\lambda}}{2}(q_x + q)\phi, \\
\phi_x &= -\frac{1}{2}\phi + \frac{i\sqrt{\lambda}u}{2}\psi, \\
\psi_x &= \frac{1}{2}\psi + \frac{i\sqrt{\lambda}u}{2}\phi,
\end{aligned} \tag{6.148}$$

with $j = 1, \dots, n - 1$.

From here, we obtain the results

$$\xi_1 = A_1(t), \tag{6.149}$$

$$\xi_2 = a_2 y + b_2, \tag{6.150}$$

$$\xi_3 = a_3 t + b_3, \tag{6.151}$$

$$\eta_\lambda(y, t, \lambda) = \frac{a_2 - a_3}{n} \lambda, \tag{6.152}$$

$$\eta_u(x, y, t, u) = \frac{a_3 - a_2}{2n} u, \tag{6.153}$$

$$\eta_{\omega_{[j]}}(x, y, t, \omega_{[j]}) = \delta^{(j,1)} \frac{dA_1(y)}{dy} - \frac{(n-j+1)a_2 + (j-1)a_3}{n} \omega_{[j]}, \tag{6.154}$$

$$\eta_{v_{[j]}}(x, y, t, v_{[j]}) = \delta^{(j,n)} A_n(y, t) - \frac{(2(n-j)+1)a_2 + (2j-1)a_3}{2n} v_{[j]} \tag{6.155}$$

$$\eta_\phi(x, y, t, \lambda, \psi, \phi) = \gamma(y, t, \lambda) \phi, \tag{6.156}$$

$$\eta_\psi(x, y, t, \lambda, \psi, \phi) = \gamma(y, t, \lambda) \psi, \tag{6.157}$$

for all $j = 1, \dots, n$, and where the function γ satisfies the following equation

$$\frac{\partial \gamma(y, t, \lambda)}{\partial t} = \lambda^n \frac{\partial \gamma(y, t, \lambda)}{\partial t}, \tag{6.158}$$

which is equivalent to the equation (4.19). The functions are $A_1(y)$, $A_n(y, t)$ and constants a_2, a_3, b_2, b_3 are arbitrary and $\delta^{(j,1)}$ and $\delta^{(j,n)}$ are Kronecker deltas.

The reductions can be achieved by the following characteristic system

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\lambda}{\eta_\lambda} = \frac{du}{\eta_u} = \frac{d\omega_{[j]}}{\eta_{\omega_{[j]}}} = \frac{dv_{[j]}}{\eta_{v_{[j]}}} = \frac{d\psi}{\eta_\psi} = \frac{d\phi}{\eta_\phi} \quad (6.159)$$

and attending to the values for the arbitrary constants

	Case I: $a_2 \neq 0, b_2 = 0$	Case II: $a_2 = 0, b_2 \neq 0$	Case III: $a_2 = 0, b_2 = 0$
1.	$a_3 \neq 0$	$b_3 = 0$	$a_3 \neq 0$
2.	$a_3 = 0$	$b_3 \neq 0$	$a_3 = 0$
3.	$a_3 = 0$	$b_3 = 0$	$a_3 = 0$

Table 4.8. Classification of reductions for mCHH2 + 1

If we introduce the next notation for the reduced variables and vector fields

$$\begin{aligned} x, y, t, & \rightarrow x_1, x_2, \\ \lambda & \rightarrow \Lambda(x_2), \\ U(x, y, t) & \rightarrow U(z_1, z_2), \\ \omega_{[1]}(x, y, t) & \rightarrow \Omega_{[1]}(x_1, x_2), \\ \omega_{[j]}(x, y, t) & \rightarrow \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \\ v_{[j]}(x, y, t) & \rightarrow v_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \\ v_{[n]}(x, y, t) & \rightarrow v_{[n]}(x_1, x_2), \\ \phi(x, y, t) & \rightarrow \Phi(x_1, x_2), \\ \psi(x, y, t) & \rightarrow \Psi(x_1, x_2), \\ p(x, y, t) & \rightarrow P(x_1, x_2), \\ q(x, y, t) & \rightarrow Q(x_1, x_2). \end{aligned} \quad (6.160)$$

according to the exposed tables, we have

- *Reduction I.1.* $a_2 \neq 0, b_2 = 0, a_3 \neq 0, b_3 = 0$

Solving the characteristic system (6.159) for the values of constants in I.1.,

– Reduced variables

$$x_1 = x - \frac{1}{a_2} \int \frac{A_1(y)}{y} dy, \quad x_2 = \frac{t}{y^r}, \quad (6.161)$$

where $r = a_3/a_2$.

– The reduced spectral parameter reads

$$\lambda(y, t) = y^{\frac{1-r}{n}} \Lambda(x_2), \quad (6.162)$$

where $\Lambda(x_2)$ satisfies the *nonisospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = \left(\frac{1-r}{n} \right) \frac{\Lambda(x_2)^{n+1}}{1 + rx_2 \Lambda(x_2)^n}. \quad (6.163)$$

– Reduced fields

$$u(x, y, t) = y^{\frac{r-1}{2n}} U(x_1, x_2), \quad (6.164)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{a_2 y} + \frac{\Omega_{[1]}(x_1, x_2)}{y}, \quad (6.165)$$

$$\omega_{[j]}(x, y, t) = y^{\frac{(r-1)(1-j)}{n}} \left(\frac{\Omega_{[j]}}{y} \right), \quad j = 2, \dots, n, \quad (6.166)$$

$$v_{[j]}(x, y, t) = y^{\frac{(r-1)(1-2j)}{2n}} \left(\frac{V_{[j]}(x_1, x_2)}{y} \right), \quad j = 1, \dots, n-1, \quad (6.167)$$

$$v_{[n]}(x, y, t) = y^{\frac{(r-1)(1-2n)}{2n}} \left(\frac{1}{a_2 y} \int \frac{A_n(y, t) y^{\frac{2rn-r+1}{2n}}}{y} dy + \frac{V_{[n]}(x_1, x_2)}{y} \right). \quad (6.168)$$

– Reduced hierarchy

$$\left((V_{[n]})_{x_1 x_1} - V_{[n]} \right)_{x_1} - U_{x_2} = 0, \quad (6.169)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.170)$$

$$\left(\Omega_{[1]} U \right)_{x_1} + \frac{r-1}{2n} U - rx_2 U_{x_2} = 0, \quad (6.171)$$

$$\left(\Omega_{[j]} \right)_{x_1} = U (V_{[j]})_{x_1}. \quad (6.172)$$

– Reduced eigenfunctions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Phi(x_1, x_2), \quad (6.173)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Psi(x_1, x_2), \quad (6.174)$$

$$p(x, y, t) = \frac{y^{\frac{r-1}{n}}}{y^r} \left(\frac{A_1(y) \Lambda(x_2)^{n-1}}{a_2} + P(x_1, x_2) \right), \quad (6.175)$$

$$q(x, y, t) = \frac{y^{\frac{r-1}{2n}}}{y^r} \left(\frac{1}{a_2} \int \frac{A_n(y, t) y^{\frac{2rn-r+1}{2n}}}{y} dy + Q(x_1, x_2) \right) \quad (6.176)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.177)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.178)$$

– The reduced spectral problem is

$$(1 + rx_2\Lambda^n) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad (6.179)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda}U \\ i\sqrt{\Lambda}U & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (6.180)$$

- *Reduction I.2.* $a_2 \neq 0, b_2 = 0, a_3 = 0, b_3 \neq 0$

– Reduced variables

$$x_1 = x - \frac{1}{a_2} \int \frac{A_1(y)}{y} dy, \quad x_2 = \frac{a_2 t}{b_3} - \ln y. \quad (6.181)$$

– Reduced spectral parameter

$$\lambda(y, t) = \left(\frac{a_2 y}{b_3} \right)^{\frac{1}{n}} \Lambda(x_2), \quad (6.182)$$

with $\Lambda(x_2)$ satisfying the nonisospectral condition

$$\frac{\partial \Lambda(x_2)}{dx_2} = \frac{\Lambda(x_2)^{n+1}}{n(1 + \Lambda(x_2)^n)}. \quad (6.183)$$

– Reduced fields

$$u(x, y, t) = \left(\frac{a_2 y}{b_3} \right)^{-\frac{1}{2n}} U(x_1, x_2), \quad (6.184)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{a_2 y} + \frac{\Omega_{[1]}(x_1, x_2)}{y}, \quad (6.185)$$

$$\omega_{[j]}(x, y, t) = \left(\frac{a_2 y}{b_3} \right)^{\frac{j-1}{n}} \frac{\Omega_{[j]}(x_1, x_2)}{y}, \quad (6.186)$$

$$v_{[j]}(x, y, t) = \left(\frac{a_2 y}{b_3} \right)^{\frac{2j-1}{2n}} \frac{V_{[j]}(x_1, x_2)}{y}, \quad j = 1, \dots, n-1, \quad (6.187)$$

$$v_{[n]}(x, y, t) = y^{-\frac{1}{2n}} \int \frac{A_n(y, t) y^{\frac{1}{2n}}}{a_2 y} dy + \left(\frac{a_2 y}{b_3} \right)^{\frac{2n-1}{2n}} \frac{V_{[n]}}{y}. \quad (6.188)$$

– Reduced hierarchy

$$((V_{[n]})_{x_1 x_1} - V_{[n]})_{x_1} - U_{x_2} = 0, \quad (6.189)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.190)$$

$$(\Omega_{[1]} U)_{x_1} - U/2n - U_{x_2} = 0, \quad (6.191)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.192)$$

– The reduced eigenfunctions read

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Phi, \quad (6.193)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Psi, \quad (6.194)$$

$$p(x, y, t) = \left(\frac{a_2 y}{b_3} \right)^{\frac{n-1}{n}} \left(\frac{A_1(y) \Lambda(x_2)^{n-1}}{a_2 y} + \frac{P(x_1, x_2)}{y} \right), \quad (6.195)$$

$$q(x, y, t) = y^{-\frac{1}{2n}} \int \frac{A_n(y, t) y^{\frac{1-2n}{2n}}}{a_2} dy + \left(\frac{a_2 y}{b_3} \right)^{\frac{2n-1}{2n}} \frac{Q(x_1, x_2)}{y}, \quad (6.196)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.197)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.198)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(y, x_2) = \Gamma \left[y, t = \frac{b_3}{a_2} (x_2 + \ln y) \right] \quad (6.199)$$

that yields

$$\gamma_t = \frac{a_2}{b_3} \Gamma_{x_2}, \quad \gamma_y = \Gamma_y - \frac{1}{y} \Gamma_{x_2}. \quad (6.200)$$

Therefore, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$(1 + \Lambda^n) \Gamma_{x_2} = y \Lambda^y \Gamma_y. \quad (6.201)$$

– Reduced Lax pair

$$(1 + \Lambda^n) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad (6.202)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda}U \\ i\sqrt{\Lambda}U & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (6.203)$$

- *Reduction I.3.* $a_2 \neq 0, b_2 = 0, a_3 = 0, b_3 = 0$

– Reduced variables

$$x_1 = x - \frac{1}{a_2} \int \frac{A_1(y)}{y} dy, \quad x_2 = t. \quad (6.204)$$

– Reduced spectral parameter

$$\lambda(y, t) = y^{\frac{1}{n}} \Lambda(x_2), \quad (6.205)$$

where $\Lambda(x_2)$ satisfies the *non-isospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = \frac{\Lambda(x_2)^{n+1}}{n}. \quad (6.206)$$

– Reduced fields

$$u(x, y, t) = y^{-\frac{1}{2n}} U(x_1, x_2), \quad (6.207)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{a_2 y} + \frac{\Omega_{[1]}(x_1, x_2)}{y}, \quad (6.208)$$

$$\omega_{[j]}(x, y, t) = y^{\frac{j-1}{n}} \frac{\Omega_{[j]}(x_1, x_2)}{y}, \quad j = 2, \dots, n, \quad (6.209)$$

$$v_{[j]}(x, y, t) = y^{\frac{2j-1}{2n}} \frac{V_{[j]}(x_1, x_2)}{y}, \quad j = 1, \dots, n-1, \quad (6.210)$$

$$v_{[n]}(x, y, t) = y^{-\frac{1}{2n}} \left(\int \frac{A_n(y, t) y^{\frac{1}{2n}}}{a_2 y} dy + V_{[n]}(x_1, x_2) \right). \quad (6.211)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} - U_{x_2} = 0, \quad (6.212)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.213)$$

$$(\Omega_{[1]} U)_{x_1} - \frac{U}{2n} = 0, \quad (6.214)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.215)$$

– The reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Phi(x_1, x_2), \quad (6.216)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{a_2 y} dy} \Psi(x_1, x_2), \quad (6.217)$$

$$p(x, y, t) = y^{-\frac{1}{n}} \left(\frac{A_1(y) \Lambda(x_2)^{n-1}}{a_2} + P(x_1, x_2) \right), \quad (6.218)$$

$$q(x, y, t) = y^{-\frac{1}{2n}} \left(\frac{1}{a_2} \int \frac{A_n(y, t) y^{\frac{1}{2n}}}{y} dy + Q(x_1, x_2) \right), \quad (6.219)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.220)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.221)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(y, x_2) = \gamma(y, t = x_2) \quad (6.222)$$

that yields

$$\gamma_t = \Gamma_{x_2}, \quad \gamma_y = \Gamma_y. \quad (6.223)$$

Therefore, according to (6.158), the field $\Gamma(y, x_2)$ satisfies the equation

$$\Gamma_{x_2} = y \Lambda^n \Gamma_y. \quad (6.224)$$

– Reduced Lax pair

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \quad (6.225)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.226)$$

- *Reduction II.1.* $a_2 = 0, b_2 \neq 0, a_3 \neq 0, b_3 = 0$

– Reduced variables

$$x_1 = x - \frac{1}{b_2} \int A_1(y) dy, \quad x_2 = \frac{a_3 t}{b_2} e^{-\frac{a_3 y}{b_2}}. \quad (6.227)$$

– Reduced fields

$$u(x, y, t) = e^{\frac{a_3 y}{2n b_2}} U(x_1, x_2), \quad (6.228)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{b_2} + \frac{a_3}{b_2} \Omega_{[1]}(x_1, x_2), \quad (6.229)$$

$$\omega_{[j]}(x, y, t) = \frac{a_3}{b_2} e^{-\frac{a_3(j-1)y}{n b_2}} \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \quad (6.230)$$

$$v_{[j]}(x, y, t) = \frac{a_3}{b_2} e^{-\frac{a_3(2j-1)y}{2n b_2}} V_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \quad (6.231)$$

$$v_{[n]}(x, y, t) = e^{-\frac{a_3(2n-1)y}{2n b_2}} \left(\int \frac{A_n(y, t) e^{\frac{a_3(2n-1)y}{2n b_2}}}{b_2} dy + \frac{a_3}{b_2} V_{[n]}(x_1, x_2) \right). \quad (6.232)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} - U_{x_2} = 0, \quad (6.233)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.234)$$

$$(\Omega_{[1]} U)_{x_1} + \frac{U}{2n} - x_2 U_{x_2} = 0, \quad (6.235)$$

$$(\Omega_{[j]})_{x_1} = U (V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.236)$$

– Reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Phi(x_1, x_2), \quad (6.237)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Psi(x_1, x_2), \quad (6.238)$$

$$p(x, y, t) = e^{-\frac{a_3(n-1)y}{n b_2}} \left(\frac{A_1(y) \Lambda(x_2)^{n-1}}{b_2} + \frac{a_3}{b_2} P(x_1, x_2) \right), \quad (6.239)$$

$$q(x, y, t) = e^{-\frac{a_3(2n-1)y}{2n b_2}} \left[\int \frac{A_n(y, t) e^{\frac{a_3(2n-1)y}{2n b_2}}}{b_2} dy + \frac{a_3}{b_2} Q(x_1, x_2) \right], \quad (6.240)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.241)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.242)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(y, x_2) = \gamma \left(y, t = \frac{b_2}{a_3} e^{\frac{a_3}{b_2} y} \right) \quad (6.243)$$

that yields

$$\gamma_t = \frac{a_3}{b_2} e^{\frac{-a_3 y}{b_2}} \Gamma_{x_2}, \quad \gamma_y = \Gamma_y - \frac{a_3}{b_2} x_2 \Gamma_{x_2}. \quad (6.244)$$

Therefore, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$a_3(1 + x_2 \Lambda^n) \Gamma_{x_2} = b_2 \Lambda^n \Gamma_y. \quad (6.245)$$

– Reduced spectral parameter

$$\lambda(y, t) = e^{\frac{-a_3 y}{n b_2}} \Lambda(x_2), \quad (6.246)$$

where $\Lambda(x_2)$ satisfies the *non-isospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = -\frac{\Lambda(x_2)^{n+1}}{n(1 + x_2 \Lambda(x_2)^n)}. \quad (6.247)$$

– Reduced Lax pair

$$(1 + x_2 \Lambda^n) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \quad (6.248)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.249)$$

- *Reduction II.2.* $a_2 = 0, b_2 \neq 0, a_3 = 0, b_3 \neq 0$.

– Reduced variables

$$x_1 = x - \int \frac{A_1(y)}{b_2} dy, \quad x_2 = y - \frac{b_2}{b_3} t. \quad (6.250)$$

– Reduced spectral parameter

$$\lambda(y, t) = \left(\frac{b_2}{b_3}\right)^{\frac{1}{n}} \Lambda(x_2), \quad (6.251)$$

where $\Lambda(x_2)$ satisfies the *isospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = 0. \quad (6.252)$$

– Reduced fields

$$u(x, y, t) = \left(\frac{b_3}{b_2}\right)^{\frac{1}{2n}} U(x_1, x_2), \quad (6.253)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{b_2} + \Omega_{[1]}(x_1, x_2), \quad (6.254)$$

$$\omega_{[j]}(x, y, t) = \left(\frac{b_2}{b_3}\right)^{\frac{j-1}{n}} \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \quad (6.255)$$

$$v_{[j]}(x, y, t) = \left(\frac{b_2}{b_3}\right)^{\frac{2j-1}{2n}} V_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \quad (6.256)$$

$$v_{[n]}(x, y, t) = \frac{1}{b_2} \int A_n(y, t) dy + \left(\frac{b_2}{b_3}\right)^{\frac{2n-1}{2n}} V_{[n]}(x_1, x_2). \quad (6.257)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} + U_{x_2} = 0, \quad (6.258)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.259)$$

$$(\Omega_{[1]} U)_{x_1} + U_{x_2} = 0, \quad (6.260)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.261)$$

– Reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Phi(x_1, x_2), \quad (6.262)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Psi(x_1, x_2), \quad (6.263)$$

$$p(x, y, t) = \left(\frac{b_2}{b_3}\right)^{\frac{n-1}{n}} \left(\frac{A_1(y) \Lambda(x_2)^{n-1}}{b_2} + P(x_1, x_2) \right), \quad (6.264)$$

$$q(x, y, t) = \frac{1}{b_2} \int A_n(y, t) dy + \left(\frac{b_2}{b_3}\right)^{\frac{2n-1}{2n}} Q(x_1, x_2), \quad (6.265)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.266)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.267)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(y, x_2) = \gamma \left(y, t = \frac{b_3}{b_2}(y - x_2) \right) \quad (6.268)$$

that yields

$$\gamma_t = -\frac{b_2}{b_3} \Gamma_{x_2}, \quad \gamma_y = \Gamma_y + \Gamma_{x_2}. \quad (6.269)$$

Thus, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$(1 + \Lambda^n) \Gamma_{x_2} = -\Lambda^n \Gamma_y. \quad (6.270)$$

– Reduced Lax pair

$$(1 + \Lambda^n) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad (6.271)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.272)$$

- *Reduction II.3.* $a_2 = 0, b_2 \neq 0, a_3 = 0, b_3 = 0$

– Reduced variables

$$x_1 = x - \int \frac{A_1(y)}{b_2} dy, \quad x_2 = t. \quad (6.273)$$

– Reduced spectral parameter

$$\lambda(y, t) = \Lambda(x_2), \quad (6.274)$$

where $\Lambda(x_2)$ satisfies the *isospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = 0. \quad (6.275)$$

– Reduced fields

$$u(x, y, t) = U(x_1, x_2), \quad (6.276)$$

$$\omega_{[1]}(x, y, t) = \frac{A_1(y)}{b_2} + \Omega_{[1]}(x_1, x_2), \quad (6.277)$$

$$\omega_{[j]}(x, y, t) = \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \quad (6.278)$$

$$v_{[j]}(x, y, t) = V_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \quad (6.279)$$

$$v_{[n]}(x, y, t) = \frac{1}{b_2} \int A_n(y, t) dy + V_{[n]}(x_1, x_2). \quad (6.280)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} - U_{x_2} = 0, \quad (6.281)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.282)$$

$$(\Omega_{[1]} U)_{x_1} = 0, \quad (6.283)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.284)$$

– Reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Phi(x_1, x_2), \quad (6.285)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(y, x_2)}{b_2} dy} \Psi(x_1, x_2), \quad (6.286)$$

$$p(x, y, t) = \frac{A_1(y) \Lambda(x_2)^{n-1}}{b_2} + P(x_1, x_2), \quad (6.287)$$

$$q(x, y, t) = \frac{1}{b_2} \int A_n(y, t) dy + Q(x_1, x_2), \quad (6.288)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.289)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.290)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(y, x_2) = \gamma(y, t = x_2) \quad (6.291)$$

that yields

$$\gamma_t = \Gamma_{x_2}, \quad \gamma_y = \Gamma_y. \quad (6.292)$$

Therefore, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$\Gamma_{x_2} = \Lambda^n \Gamma_y. \quad (6.293)$$

– Reduced Lax pair

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = \Lambda P \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad (6.294)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda}U \\ i\sqrt{\Lambda}U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.295)$$

- *Reduction III.1.* $a_2 = 0, b_2 = 0, a_3 \neq 0, b_3 = 0$

– Reduced variables

$$x_1 = x - \frac{A_1(y)}{a_3} \ln t, \quad x_2 = y. \quad (6.296)$$

– Reduced spectral parameter

$$\lambda(y, t) = t^{\frac{-1}{n}} \Lambda(x_2), \quad (6.297)$$

where $\Lambda(x_2)$ satisfies the *nonisospectral condition*

$$n \frac{d\Lambda(x_2)}{dx_2} + \Lambda(x_2)^{1-n} = 0. \quad (6.298)$$

– Reduced fields

$$u(x, y, t) = t^{\frac{1}{2n}} U(x_1, x_2), \quad (6.299)$$

$$\omega_{[1]}(x, y, t) = \frac{\ln(t)}{a_3} \frac{dA_1(y)}{dy} + \Omega_{[1]}(x_1, x_2), \quad (6.300)$$

$$\omega_{[j]}(x, y, t) = t^{\frac{1-j}{n}} \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \quad (6.301)$$

$$v_{[j]}(x, y, t) = t^{\frac{1-2j}{2n}} V_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \quad (6.302)$$

$$v_{[n]}(x, y, t) = t^{\frac{1-2n}{2n}} \left(\frac{1}{a_3} \int t^{\frac{-1}{2n}} A_n(y, t) dt + V_{[n]}(x_1, x_2) \right). \quad (6.303)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} - \frac{U}{2n} + \frac{\hat{A}_1(x_2)}{a_3} U_{x_1} = 0, \quad (6.304)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.305)$$

$$(\Omega_{[1]} U)_{x_1} + U_{x_2} = 0, \quad (6.306)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.307)$$

– Reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(t, x_2)}{a_3 t} dt} \Phi(x_1, x_2), \quad (6.308)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(t, x_2)}{a_3 t} dt} \Psi(x_1, x_2), \quad (6.309)$$

$$p(x, y, t) = t^{\frac{1-n}{n}} \left(\frac{\ln(t) \Lambda(x_2)^{n-1}}{a_3} \frac{dA_1(y)}{dy} + P(x_1, x_2) \right), \quad (6.310)$$

$$q(x, y, t) = t^{\frac{1-2n}{2n}} \left(\frac{1}{a_3} \int t^{\frac{-1}{2n}} A_n(y, t) dt + Q(x_1, x_2) \right), \quad (6.311)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.312)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.313)$$

and the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(t, x_2) = \gamma(y = x_2, t) \quad (6.314)$$

that yields

$$\gamma_t = \Gamma_t, \quad \gamma_y = \Gamma_{x_2}. \quad (6.315)$$

Therefore, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$\Lambda^n \Gamma_{x_2} = t \Gamma_t. \quad (6.316)$$

– Reduced Lax pair

$$\begin{aligned} -\Lambda^n \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} &= \left(\Lambda P + \frac{\hat{A}_1}{a_3} \right) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} \\ &+ \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \end{aligned} \quad (6.317)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.318)$$

and $\hat{A}_1 = \hat{A}_1(x_2) = A_1(y = y(x_2))$.

- *Reduction III.2.* $a_2 = 0, b_2 = 0, a_3 = 0, b_3 \neq 0$.

– Reduced variables

$$x_1 = x - \frac{A_1(y)t}{b_3}, \quad x_2 = \frac{1}{b_3} \int A_1(y) dy. \quad (6.319)$$

– Reduced spectral parameter

$$\lambda(y, t) = \Lambda(x_2), \quad (6.320)$$

where $\Lambda(x_2)$ satisfies the *isospectral condition*

$$\frac{d\Lambda(x_2)}{dx_2} = 0. \quad (6.321)$$

– Reduced fields

$$u(x, y, t) = U(x_1, x_2), \quad (6.322)$$

$$\omega_{[1]}(x, y, t) = \frac{t}{b_3} \frac{dA_1(y)}{dy} + \frac{A_1(y)}{b_3} \Omega_{[1]}(x_1, x_2), \quad (6.323)$$

$$\omega_{[j]}(x, y, t) = \frac{A_1(y)}{b_3} \Omega_{[j]}(x_1, x_2), \quad j = 2, \dots, n, \quad (6.324)$$

$$v_{[j]}(x, y, t) = \frac{A_1(y)}{b_3} V_{[j]}(x_1, x_2), \quad j = 1, \dots, n-1, \quad (6.325)$$

$$v_{[n]}(x, y, t) = \frac{1}{b_3} \int A_n(y, t) dt + \frac{A_1(y)}{b_3} V_{[n]}(x_1, x_2), \quad (6.326)$$

– Reduced hierarchy

$$(V_{[n]})_{x_1 x_1 x_1} - (V_{[n]})_{x_1} + U_{x_1} = 0, \quad (6.327)$$

$$(V_{[j]})_{x_1 x_1} - V_{[j]} + U \Omega_{[j+1]} = 0, \quad j = 1, \dots, n-1, \quad (6.328)$$

$$(\Omega_{[1]} U)_{x_1} + U_{x_2} = 0, \quad (6.329)$$

$$(\Omega_{[j]})_{x_1} = U(V_{[j]})_{x_1}, \quad j = 1, \dots, n. \quad (6.330)$$

– Reduced spectral functions

$$\phi(x, y, t) = e^{\int \frac{\Gamma(t, x_2)}{b_3} dt} \Phi(x_1, x_2), \quad (6.331)$$

$$\psi(x, y, t) = e^{\int \frac{\Gamma(t, x_2)}{b_3} dt} \Psi(x_1, x_2), \quad (6.332)$$

$$p(x, y, t) = \frac{t}{b_3} \frac{dA_1(y)}{dy} \Lambda^{n-1} + \frac{A_1(y)}{b_3} P(x_1, x_2), \quad (6.333)$$

$$q(x, y, t) = \frac{1}{b_3} \int A_n(y, t) dt + \frac{A_1(y)}{b_3} Q(x_1, x_2). \quad (6.334)$$

$$P(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} \Omega_{[j]}(x_1, x_2), \quad (6.335)$$

$$Q(x_1, x_2) = \sum_{j=1}^n \Lambda(x_2)^{n-j} V_{[j]}(x_1, x_2), \quad (6.336)$$

where the function $\Gamma(y, x_2)$ is obtained through the following identification

$$\Gamma(t, x_2) = \gamma(y = y(x_2), t) \quad (6.337)$$

that yields

$$\gamma_t = \Gamma_t, \quad \gamma_y = \frac{A_1}{a_3} \Gamma_{x_2}. \quad (6.338)$$

Therefore, according to (6.158), $\Gamma(y, x_2)$ satisfies the equation

$$\Lambda^n \Gamma_{x_2} = \frac{b_3}{\hat{A}_1(x_2)} \Gamma_t, \quad \hat{A}_1(x_2) = A_1(y = y(x_2)). \quad (6.339)$$

– Reduced spectral problem

$$-\Lambda^n \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_2} = (\Lambda P + 1) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} + \frac{i\sqrt{\Lambda}}{2} \begin{bmatrix} 0 & Q_{x_1} - Q \\ Q_{x_1} + Q & 0 \end{bmatrix}_{x_1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad (6.340)$$

and

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{x_1} = \frac{1}{2} \begin{bmatrix} -1 & i\sqrt{\Lambda} U \\ i\sqrt{\Lambda} U & 1 \end{bmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (6.341)$$

Reciprocal transformations

1 What is a reciprocal transformation?

This chapter is dedicated to the study of reciprocal transformations. Reciprocal transformations are a particular type of hodograph transformations in which the use of conserved quantities is fundamental [367, 368, 369, 370, 372].

On a first approximation, *hodograph transformations* are transformations involving the interchange of dependent and independent variables [123, 172]. When the variables are switched, the space of independent variables is called the *reciprocal space*. In particular case of two variables, we refer to it as the *reciprocal plane*. As a physical interpretation, whereas the independent variables play the role of positions in the reciprocal space, this number is increased by turning certain fields or dependent variables into independent variables and viceversa [132]. For example, for the hydrodynamical case, usually fields that represent the height of the wave or its velocity, are turned into a new set of independent variables. *Reciprocal transformations* share this definition but require the employment of conservative forms and their properties, as we shall see in forthcoming paragraphs [159, 160, 175, 176].

One of the biggest advantages of dealing with hodograph and reciprocal transformations is that many of the equations reported integrable, which are considered seemingly different from one another, happen to be related via reciprocal transformations. If this were the case, two apparently unrelated equations, even two complete hierarchies of PDEs that are linked via reciprocal transformation are tantamount versions of a unique problem. In this way, hodograph and reciprocal transformations give rise to a procedure of relating allegedly new equations to the rest of their equivalent integrable sisters.

Let us consider a manifold $N_{\mathbb{R}^n}$ explained in Chapter 4. We call a p -order

quasilinear PDE to a submanifold of $J^p(\mathbb{R}^n, \mathbb{R}^n)$ as

$$(u_j)_{x_i} = g(u)(u_j)_{x_1^{j_1}, x_2^{j_2}, x_3^{j_3}, \dots, x_n^{j_n}} + f\left(u_j, (u_j)_{x_i}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}}\right), \quad (1.1)$$

with $\frac{\partial g}{\partial u} \neq 0$, the subindices i_1, \dots, i_n can take any value between $1, \dots, n$, the sum $j_1 + j_2 + j_3 + \dots + j_n \leq p$ and $i = 1, \dots, n$, $j = 1, \dots, k$, where u denotes $u = (u_1, \dots, u_k)$ and $x = (x_1, \dots, x_n)$. In [123], the most general quasilinear equation above is found to be mapped via hodograph transformation into linearizable equations, that is, equations solvable in terms of either a linear PDE or a linear integral equation. Indeed, Gardner associated the solution of the KdV equation and the time-independent Schrödinger equation and showed, using ideas of the direct and inverse scattering, that the Cauchy problem for the KdV equation could be solved in terms of a linear equation [198, 200]. This novelty is today the well-known IST, leading to numerous solutions in branches of water waves, stratified fluids, Plasma Physics, statistical Mechanics and Quantum Field theory [5, 6, 8, 9]. In this way, hodograph transformations facilitate the solvability of certain PDEs that can either be treated by the IST scheme or by transformation to a linear PDE, (those said to be linearizable). In [143, 245], hodograph transformations were proven to be a useful instruments to transform equations with peakon solutions into equations that pass the Painlevé test.

During the past decades, more attention has been laid upon reciprocal transformations due to their manageability and quasi-algorithmical way of approach [159, 175, 176]. The increasing number of works in the Physics literature manifests their importance [2, 4, 159, 175, 176, 185, 187, 245, 367]. For dealing with reciprocal transformations, the concept of conservation laws is fundamental. Conservation laws are characteristic of real physical processes.

By *conservation law* we understand an expression on of the form

$$\frac{\partial \psi_1}{\partial x_{i_1}} + \frac{\partial \psi_2}{\partial x_{i_2}} = 0, \quad (1.2)$$

for certain two values of the indices in between $1 \leq i_1, i_2 \leq n$ and two scalar functions $\psi_1, \psi_2 \in C^\infty(J^p N_{\mathbb{R}^n})$. One can find a more formal definition of conservation law in [67].

A second significant role of reciprocal transformations is their utility in the identification of hierarchies which do not pass the Painlevé test and in which the SMM is not applicable [160, 172]. Precisely, our motivation for the study of reciprocal transformations roots in the study of the Camassa-Holm hierarchy [167]. Such hierarchy, has been known to be integrable for some time and has an associated linear problem. Nevertheless, in its most commonly expressed form, given in (6.6) in Chapter 4, it is not integrable in the sense of the Painlevé property, nor the SMM is constructive.

Our conjecture is that if an equation is integrable, there must be a transformation that will let us turn the initial equation into a new one in which the Painlevé test is successful and the SMM can be applied. Hence, a Lax pair could be derived and other methods mentioned in the introduction could be worked out. Therefore, discarding any pathology, one should be able to write down a transformation that brings the equation into a suitable form, in which the test is applicable.

From now on, this chapter will be dedicated to reciprocal transformations exclusively. Reciprocal transformations have a long story. Alongside with the IST, the two procedures gave rise to the discovery of other integrable nonlinear evolution equations similar to the KdV equations. For example, Zakharov and Shabat [427] presented the now famous nonlinear Schrödinger (NLS) equation, which presents an infinite number of integrals of motion and possesses n -soliton solutions with purely elastic interaction. A third important equation, worth of mention, that arose from the IST, is the modified Korteweg de Vries (mKdV) equation, found by Wadati [407] in 1972, which also attracted a lot of attention in fields of Plasma Physics, nonlinear Optics or superconductivity. In 1971, the study of the blow-up stability in the theory of solitons lead to the full understanding of the special properties of the KdV [275].

In 1928, the invariance of nonlinear gas dynamics, magnetogas dynamics and general hydrodynamic systems under reciprocal transformations was extensively studied [185, 371]. Stationary and moving boundary problems in soil mechanics and nonlinear heat conduction has likewise been a subject of much research [187, 367].

Applications of the reciprocal transformation in continuum mechanics are to be found in the monographs by Rogers and Shadwick [375]. These transformations have also played an important role in the soliton theory and between hierarchies of PDEs [143, 245]. Indeed, the invariance of certain integrable hierarchies under reciprocal transformations induces auto-Bäcklund transformations [175, 176, 336, 370, 372].

The most representative and differing properties of reciprocal transformations from the hodograph ones, are: they map conservation laws to conservation laws and diagonalizable systems to diagonalizable systems, but act nontrivially on metrics and on Hamiltonian structures. For instance, the flatness property or the locality of the Hamiltonian structure are not preserved, in general [3, 4].

Finding a proper reciprocal transformation is usually a very complicated task. Notwithstanding, in the case that concerns us, Fluid Mechanics, etc., a change of this type is usually reliable, specially for systems of hydrodynamic type with time evolution defined on manifolds $N_{\mathbb{R}^n}$ as a submanifold on $J^p(\mathbb{R}^{n+1}, \mathbb{R}^{2k})$

$$(u_j)_t = \sum_{l=1}^k v_l^j(u)(u_l)_{x_i}, \quad \forall 1 \leq i \leq n, \quad j = 1, \dots, n \quad (1.3)$$

and $v_l^j(u)$ are $C^\infty(J^p N_{\mathbb{R}^n})$, appearing Gas Dynamics, Hydrodynamics, Chemical Kinetics, Differential Geometry and Topological Field Theory [151, 398].

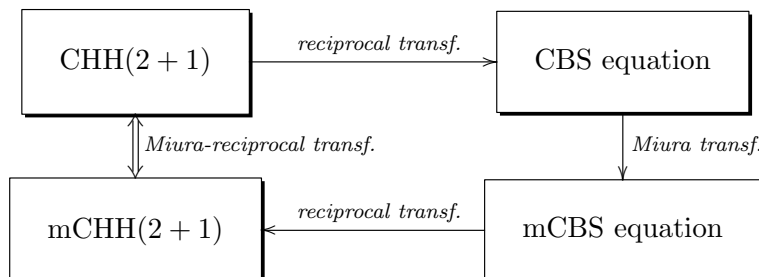


Figure 5.1. Miura-reciprocal transformation.

Lately, we have been focusing on differing versions of reciprocal transformations. Specifically, on the composition of two or more reciprocal transformations or composition with transformations of other nature. For example, the composition of a Miura [6, 379] and a reciprocal transformation, the so called *Miura-reciprocal transformations* helps us relate two different hierarchies, which is the purpose of this chapter. In particular, we show the example of the n -component Camassa–Holm hierarchy $\text{CHH}(2 + 1)$ which does not possess the PP in the variables expressed in (6.7), in Chapter 4. Nevertheless, it can be proven [175, 176] that there exists a reciprocal transformation that turns the $\text{CHH}(2 + 1)$ into n copies of the Calogero–Bogoyavlenskii–Schiff (CBS) equation [70, 78, 277]. This latter equation possesses the PP and the SMM can be applied to obtain its LP and other relevant properties. On the other hand, the n -component Qiao hierarchy $\text{mCHH}(2 + 1)$ (6.142), given in Chapter 4, has been reported as an integrable hierarchy for which the PP is not applicable, nor constructive [160, 359, 360, 361]. Notwithstanding, one can prove [175, 176] that there exists a reciprocal transformation which turns $\text{mCHH}(2 + 1)$ into n copies of the modified Calogero–Bogoyavlenskii–Schiff (mCBS) equation, which is known to have the PP, etc [170].

Summarizing, $\text{CHH}(2 + 1)$ and $\text{mCHH}(2 + 1)$ are related to the CBS and mCBS, correspondingly. Aside from this property, there exists a Miura transformation [116, 161] relating the CBS and the mCBS equations. One then wonders if $\text{mCHH}(2 + 1)$ is related to $\text{CHH}(2 + 1)$ in any way. It seems clear that the relationship between $\text{mCHH}(2 + 1)$ and $\text{CHH}(2 + 1)$ necessarily includes a composition of a Miura and a reciprocal transformation. Figure 5.1 clarifies this idea.

In this way, the plan of this chapter goes as follows

Section 1: Quasi-algorithmical construction of reciprocal transformations: We introduce the theoretical framework of a reciprocal transformation on a general manifold of type $N_{\mathbb{R}^n}$ and give a quasi-algorithmical method for the construction of such a transformation. It is our intention to give some (generally

common) steps in the construction of the transformation, hence the name “quasi-algorithmical”. Nevertheless, the study of this theory will become much simpler by applying it to low dimensional cases, that are the cases that concern us.

Section 2: CHH(2+1) versus CBS: We explicitly construct the reciprocal transformation that turns the n -component CHH(2 + 1) into n copies of the CBS equation.

Section 3: mCHH(2+1) versus mCBS: We explicitly construct the reciprocal transformation that turns the n -component mCHH(2 + 1), or initially called Qiao(2 + 1), into n copies of the mCBS equation. Here, given the Lax pair for the mCBS equation, we will invert the reciprocal transformation and we will achieve a Lax pair for the former n -component mCHH(2 + 1). Here is where the role of reciprocal transformations as a tool for deriving Lax pairs will be explicit.

Section 4: Reciprocal-Miura transformation between CHH(2+1) and mCHH(2+1): Here we give a Miura transformation between the CBS and mCBS equations. This Miura transformation has been well known in the literature for some years. By means of a composition of a Miura and reciprocal transformations it is possible to link the CHH(2+1) and mCHH(2+1) and relate their dependent variables U and u , correspondingly, as well as the triple of independent variables in which they are written (X, Y, T) and (x, y, t) , respectively. We will take a closer look at two particular cases: one is the Qiao equation arisen from the first component of the hierarchy mCHH(2 + 1), (that is when $n = 1$) and U is independent of Y and u is independent of y . The second particular case is: the Camassa–Holm equation arising from the first-component of the hierarchy CHH(2 + 1), ($n = 1$) if $T = X$ and $t = x$.

2 Quasi-algorithmical construction of reciprocal transformations

Let us consider the most general nonlinear system on $J^p(\mathbb{R}^n, \mathbb{R}^k)$, and we propose a system of q partial differential equations

$$\Psi^l = \Psi^l \left(x_i, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}} \right), \quad (2.1)$$

for all $l = 1, \dots, q$ and $i = 1, \dots, n$, $j = 1, \dots, k$, $j_1 + \dots + j_n \leq p$. Suppose that we know a number n of conserved quantities that are expressible in the following

manner

$$\begin{aligned}
 A_{x_{i_1}}^{(j)} \left(x_{i_1}, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots \right) &= A_{x_{i_2}}^{(j')} \left(x_i, u_j, (u_j)_{x_{i_1}^{j_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots \right), \\
 x_{i_1} &\neq x_{i_2}, \quad A^{(j)} \neq A^{(j')}, \\
 1 \leq i_1, \dots, i_n \leq n, \quad j_1 + \dots + j_n &\leq p, \quad 1 \leq j, j' \leq 2n.
 \end{aligned} \tag{2.2}$$

where $A_{x_{i_1}}^{(j)} = \partial A^{(j)} / \partial x_{i_1}$, $A_{x_{i_2}}^{(j')} = \partial A^{(j')} / \partial x_{i_2}$ and $A^{(j)}, A^{(j')} \in C^\infty(\mathbb{J}^p N_{\mathbb{R}^n})$, which in principle do not need to be equal.

If the number of equations in (2.2) is equal to the number of independent variables, then we propose a transformation for each of the coordinates $\{x_1, \dots, x_n\}$ to a new set of coordinates $\{z_1, \dots, z_n\}$ as

$$dz_i = A^{(j)} dx_{i_2} + A^{(j')} dx_{i_1}, \quad \forall 1 \leq i_1, i_2 \leq n, \quad \forall 1 \leq j, j' \leq 2n. \tag{2.3}$$

such that if the property of closeness is satisfied, $d^2 z_i = 0$ for all $z_i, i = 1, \dots, n$ and we recover the conserved quantities (2.2).

Concerning examples of physical nature that we shall deal with, we will focus on the particular case in which only one equation with conserved quantities is used. This implies that only one independent variable will be transformed. The transformed independent variable is a particular one. Let us denote it by $x_{\hat{i}}$. We now search for a function $X(z_1, \dots, z_n)$ such that

$$x_{\hat{i}} = X(z_1, \dots, z_n) \tag{2.4}$$

is turned into a dependent variable. Simultaneously, we use the conserved quantity to propose the transformation

$$\begin{aligned}
 dz_{\hat{i}} &= A^{(j)} dx_{i_2} + A^{(j')} dx_{i_1}, \\
 dz_i &= dx_i,
 \end{aligned} \tag{2.5}$$

$\forall 1 \leq i_1 \neq i_2 \leq n$ and for a fixed value in between $1 \leq \hat{i} \leq n$, where we see that independent variables $x_i, \forall i \neq \hat{i} = 1, \dots, n$ have been left untransformed but renamed as z_i .

Deriving relation (2.4),

$$dx_{\hat{i}} = \sum_{i=1}^n X_{z_i} dz_i, \quad X_{z_i} = \frac{\partial X}{\partial z_i} \tag{2.6}$$

and by isolating $dz_{\hat{i}}$ in (2.6), we have

$$dz_{\hat{i}} = \frac{dx_{\hat{i}}}{X_{z_{\hat{i}}}} - \sum_{i \neq \hat{i}=1}^n \frac{X_{z_i}}{X_{z_{\hat{i}}}} dx_i, \tag{2.7}$$

where we have used that $dz_i = dx_i$ for all $i \neq \hat{i}$ according to (2.5).

Here, by direct comparison of coefficients in (2.5) and (2.7), we have that

$$\begin{aligned} A^{(j')} &= \frac{1}{X_{z_{\hat{i}}}}, \\ A^{(j)} &= -\frac{X_{z_i}}{X_{z_{\hat{i}}}}. \end{aligned} \quad (2.8)$$

It is important that we show the pass to the tangent space. For example, in the case of first-order derivatives, we have

$$\begin{aligned} u_{x_{\hat{i}}} &= \frac{\partial u}{\partial z_{\hat{i}}} \frac{\partial z_{\hat{i}}}{\partial x_{\hat{i}}} + \sum_{i \neq \hat{i}=1}^n \frac{\partial u}{\partial z_i} \frac{\partial z_i}{\partial x_{\hat{i}}} = \frac{1}{X_{z_{\hat{i}}}}, \\ u_{x_i} &= \frac{\partial u}{\partial z_{\hat{i}}} \frac{\partial z_{\hat{i}}}{\partial x_i} + \sum_{i \neq \hat{i}=1}^n \frac{\partial u}{\partial z_i} \frac{\partial z_i}{\partial x_i} = -\frac{X_{z_i}}{X_{z_{\hat{i}}}} u_{z_i} + u_{z_i}, \quad \forall i \neq \hat{i} \end{aligned} \quad (2.9)$$

This process can be applied recursively for higher-order derivatives in the jet space $J^p N$.

In this way, using expressions in (2.8), (2.9), etc., we transform a system (2.1) with initial variables $\{x_1, \dots, x_n\}$ into a new system in variables $\{z_1, \dots, z_n\}$ and fields $u_j(z_1, \dots, z_n)$, $\forall j = 1, \dots, k$, depending on the new variables. From (2.8), we can extract expressions for

$$x_i, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, \dots, x_{i_n}^{j_n}} \quad (2.10)$$

for $j_1 + \dots + j_n \leq p$ if possible, given the functional form of $A^{(j)}$, $A^{(j')} \in C^\infty J^p N_{\mathbb{R}^n}$. Bearing in mind expression (2.4), the transformation of the initial system (2.1) will then read

$$\Psi^l = \bar{\Psi}^l \left(z_i, X, X_{z_i}, X_{z_{i_1}^{j_1}, z_{i_2}^{j_2}}, \dots, X_{z_{i_1}^{j_1}, z_{i_2}^{j_2}, z_{i_3}^{j_3}, \dots, z_{i_n}^{j_n}} \right) \quad (2.11)$$

for all $l = 1, \dots, q$ such that $j_1 + \dots + j_n \leq p$ and $z_i = z_1, \dots, z_n$.

The forthcoming examples will show the application of this procedure to the particular case in which the number of initial independent variables is 3, with the identification $x_1 = x, x_2 = y, x_3 = t$. We deal with hierarchies of PDEs in which a great number of dependent variables is present, as CHH(2 + 1) and mCHH(2 + 1). *Here the necessity of introducing auxiliary independent variables to which such auxiliary fields are reciprocally-transformed is obvious. Hence, the introduction of intermediate variables z_4, z_5, \dots will account for the dependent variables.* We depict this fact by showing a clever choice of variables in the next two examples. In order to make things clearer, some slight changes in the notation of the theory or the dimension of $N_{\mathbb{R}^n}$ shall be altered to fit our concrete examples.

3 CHH(2+1) versus CBS

We shall briefly summarize and improve the results of [160] and [172] in order to establish the reciprocal transformation that connects the CHH(2+1) and mCHH(2+1) hierarchies with CBS and mCBS equations, respectively. Many details (especially those referring to the detailed calculation) are omitted and can be obtained in the above cited references.

It could be useful from the beginning to say that we shall use capital letters for the dependent and independent variables connected with CHH(2+1), and lower case letters will be used to refer to mCHH(2+1).

The CHH(2+1) hierarchy can be written in a compact form as

$$U_T = R^{-n}U_Y, \quad (3.1)$$

where R is the recursion operator defined as

$$R = JK^{-1}, \quad K = \partial_{XXX} - \partial_X, \quad J = -\frac{1}{2}(\partial_X U + U\partial_X). \quad (3.2)$$

Note that the factor $-\frac{1}{2}$ in the definition of J is not essential and it has been introduced to make the later identification between the time variables easier. This is the difference between the CHH(2+1) introduced in (6.6) in Chapter 4 and the present.

The n component of this hierarchy can also be rewritten as a set of PDEs by introducing n dependent fields $\Omega_{[i]}$, ($i = 1, \dots, n$) in the following way

$$\begin{aligned} U_Y &= J\Omega_{[1]} \\ J\Omega_{[i+1]} &= K\Omega_{[i]}, \quad i = 1, \dots, n-1, \\ U_T &= K\Omega_{[n]}, \end{aligned} \quad (3.3)$$

and by introducing two new fields, P and Δ , related to U as

$$U = P^2, \quad P_T = \Delta_X, \quad (3.4)$$

we can write the hierarchy in the form of the following set of equations

$$\begin{aligned} P_Y &= -\frac{1}{2}(P\Omega_{[1]})_X, \\ (\Omega_{[i]})_{XXX} - (\Omega_{[i]})_X &= -P(P\Omega_{[i+1]})_X, \quad i = 1, \dots, n-1, \\ P_T &= \frac{(\Omega_{[n]})_{XXX} - (\Omega_{[n]})_X}{2P} = \Delta_X. \end{aligned} \quad (3.5)$$

It was shown in [172] that (2.12) can be reduced to the the negative Camassa-Holm hierarchy under the reduction $\frac{\partial}{\partial t} = 0$. The positive flow can be obtained under the reduction $\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$.

The conservative form of the first two equations allows us to define the exact derivative

$$dz_0 = P dX - \frac{1}{2} P \Omega_{[1]} dY + \Delta dT. \quad (3.6)$$

A reciprocal transformation [245], [368], [369] can be introduced by considering the former independent variable X as a field depending on z_0 , $z_1 = Y$ and $z_{n+1} = T$. From (3.6) we have

$$\begin{aligned} dX &= \frac{1}{P} dz_0 + \frac{\Omega_{[1]}}{2} dz_1 - \frac{\Delta}{P} dz_{n+1}, \\ Y &= z_1, \\ T &= z_{n+1}, \end{aligned} \quad (3.7)$$

and therefore

$$\begin{aligned} X_{z_0} &= \frac{1}{P}, \\ X_{z_1} &= \frac{\Omega_{[1]}}{2}, \\ X_{z_{n+1}} &= -\frac{\Delta}{P}, \end{aligned} \quad (3.8)$$

where $X_{z_i} = \frac{\partial X}{\partial z_i}$. We can now extend the transformation by introducing a new independent variable z_i for each field $\Omega_{[i]}$ by generalizing (3.8) as

$$X_i = \frac{\Omega_{[i]}}{2}, \quad i = 1, \dots, n. \quad (3.9)$$

Hence, the new field $X = X(z_0, z_1, \dots, z_n, z_{n+1})$ depends on $n+2$ independent variables, where each of the former dependent fields $\Omega_{[i]}$, ($i = 1, \dots, n$) allows us to define a new dependent variable z_i through definition (3.9). It requires some calculation (see [172] for details) but it can be proved that the reciprocal transformation (3.7)-(3.9) transforms (3.5) to the following set of n PDEs on $J^4(\mathbb{R}^{n+2}, \mathbb{R})$

$$-\left(\frac{X_{z_{i+1}}}{X_{z_0}}\right)_{z_0} = \left[\left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)_{z_0} - \frac{1}{2} \left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)^2 \right]_{z_i}, \quad (3.10)$$

with $i = 1, \dots, n$. Note that each equation depends on only three variables z_0, z_i, z_{i+1} . This result generalizes the one found in [245] for the first component of the hierarchy. The conservative form of (3.10) allows us to define a field $M(z_0, z_1, \dots, z_{n+1})$ such that

$$\begin{aligned} M_{z_i} &= -\frac{1}{4} \left(\frac{X_{z_{i+1}}}{X_{z_0}}\right), \quad i = 1, \dots, n, \\ M_{z_0} &= \frac{1}{4} \left[\left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)_{z_0} - \frac{1}{2} \left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)^2 \right]. \end{aligned} \quad (3.11)$$

It is easy to prove that each M_i should satisfy the following CBS equation [70, 78] on $J^4(\mathbb{R}^{n+2}, \mathbb{R})$

$$M_{z_0, z_{i+1}} + M_{z_0, z_0, z_0, z_i} + 4M_{z_i} M_{z_0, z_0} + 8M_{z_0} M_{z_0, z_i} = 0, \quad (3.12)$$

with $i = 1, \dots, n$.

4 mCHH(2+1) versus mCBS

In [160], the mCHH(2+1) was introduced

$$u_t = r^{-n} u_y, \quad (4.1)$$

where r is the recursion operator, defined as

$$r = jk^{-1}, \quad k = \partial_{xxx} - \partial_x, \quad j = -\partial_x u (\partial_x)^{-1} u \partial_x \quad (4.2)$$

This hierarchy generalizes the one introduced by Qiao in [361], whose second positive member was studied in [359, 360, 361]. We shall briefly summarize the results of [160] when a procedure similar to that described above for CHH(2+1) is applied to mCHH(2+1).

If we introduce $2n$ auxiliary fields $v_{[i]}$, $\omega_{[i]}$ defined through

$$\begin{aligned} u_y &= jv_{[1]}, \\ jv_{[i+1]} &= kv_{[i]}, \\ (\omega_{[i]})_x &= u(v_{[i]})_x, \\ u_t &= kv_{[n]}, \end{aligned} \quad i = 1, \dots, n-1, \quad (4.3)$$

the hierarchy can be written as the system

$$\begin{aligned} u_y &= -(u\omega_{[1]})_x, \\ (v_{[i]})_{xxx} - (v_{[i]})_x &= -(u\omega_{[i+1]})_x, \\ u_t &= (v_{[n]})_{xxx} - (v_{[n]})_x = \delta_x, \end{aligned} \quad i = 1, \dots, n-1, \quad (4.4)$$

which allows to define the exact derivative

$$dz_0 = u dx - u\omega_{[1]} dy + \delta dt \quad (4.5)$$

and $z_1 = y, z_{n+1} = t$. We can define a reciprocal transformation such that the former independent variable x is a new field $x = x(z_0, z_1, \dots, z_{n+1})$ depending on $n+2$ variables in the form

$$\begin{aligned} x_{z_0} &= \frac{1}{u}, \\ x_{z_1} &= \omega_{[1]}, \\ x_{z_{n+1}} &= -\frac{\delta}{u}. \end{aligned} \quad (4.6)$$

If we introduce the auxiliary variables for the auxiliary fields, $x_i = \omega_{(i)}$ for $i = 2, \dots, n$, the transformation of the equations (4.4) yields the system of equations on $J^4(\mathbb{R}^{n+2}, \mathbb{R})$

$$\left(\frac{x_{z_{i+1}}}{x_{z_0}} + \frac{x_{z_i, z_0, z_0}}{x_{z_0}} \right)_{z_0} = \left(\frac{x_{z_0}^2}{2} \right)_{z_i}, \quad i = 1, \dots, n. \quad (4.7)$$

Note that each equation depends on only three variables: z_0, z_i, z_{i+1} .

The conservative form of (4.7) allows us to define a field $m = m(z_0, z_1, \dots, z_{n+1})$ such that on $J^3(\mathbb{R}^{n+2}, \mathbb{R}^2)$ we have

$$m_{z_0} = \frac{x_{z_0}^2}{2}, \quad m_{z_i} = \frac{x_{z_{i+1}}}{x_{z_0}} + \frac{x_{z_i, z_0, z_0}}{x_{z_0}}, \quad i = 1, \dots, n. \quad (4.8)$$

Equation (4.7) has been extensively studied from the point of view of Painlevé analysis [170] and it can be considered as the modified version of the CBS equation (3.12). Actually, in [170] it was proved that the Miura transformation that relates (3.12) and (4.8) is

$$4M = x_{z_0} - m. \quad (4.9)$$

A non-isospectral Lax pair was obtained for (4.8) in [170]. By inverting this Lax pair through the reciprocal transformation (4.6) the following spectral problem was obtained for mCHH(2+1). This Lax pair [170] reads

$$\begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_x = \frac{1}{2} \begin{bmatrix} -1 & I\sqrt{\lambda}u \\ I\sqrt{\lambda}u & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}, \quad (4.10)$$

$$\begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_t = \lambda^n \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_y + \lambda a \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_x + I \frac{\sqrt{\lambda}}{2} \begin{bmatrix} 0 & b_{xx} - b_x \\ b_{xx} + b_x & 0 \end{bmatrix} \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}, \quad (4.11)$$

where

$$a = \sum_{i=1}^n \lambda^{n-i} \omega_{[i]}, \quad b = \sum_{i=1}^n \lambda^{n-i} v_{[i]}, \quad I = \sqrt{-1}, \quad (4.12)$$

and $\lambda(y, t)$ is a non-isospectral parameter that satisfies

$$\lambda_x = 0, \quad \lambda_t - \lambda^n \lambda_y = 0. \quad (4.13)$$

Although the Painlevé test cannot be applied to mCHH(2+1), reciprocal transformations are a tool that can be used to write the hierarchy as a set of mCBS equation to which the Painlevé analysis (the SMM in particular) can be successfully applied.

5 Reciprocal-Miura transformations

As stated in the previous section, there are two reciprocal transformations (3.8) and (4.6) that relate CHH(2+1) and mCHH(2+1) hierarchies with CBS (3.12) and mCBS (4.8) respectively. Furthermore, it is known that a Miura transformation (4.9) relates CBS and mCBS. The natural question that arises is whether the mCHH(2+1) hierarchy can be considered as the modified version of CHH(2+1). Evidently, the relationship between both hierarchies cannot be a simple Miura transformation because they are written in different variables (X, Y, T) and (x, y, t) . The answer is provided by the relationship of both sets of variables with the same set z_0, z_1, z_{n+1} . By combining (3.7) and (4.5), we have

$$\begin{aligned} P dX - \frac{1}{2} P \Omega_{[1]} dY + \Delta dT &= u dx - u \omega_{[1]} dy + \delta dt, \\ Y &= y, \\ T &= t, \end{aligned} \tag{5.1}$$

which yields the required relationship between the independent variables of CHH(2+1) and those of mCHH(2+1). The Miura transformation (4.9), also provides the following results

$$\begin{aligned} 4M_{z_0} = x_{z_0, z_0} - m_{z_0} &\implies \frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0} = x_{z_0}, \\ 4M_{z_i} = x_{z_0, z_i} - m_{z_i} &\implies -\frac{X_{z_i+1}}{X_{z_0}} = x_{z_0, z_i} - \frac{x_{z_0, z_0, z_i}}{x_{z_0}} - \frac{x_{z_i+1}}{x_{z_0}}, \end{aligned} \tag{5.2}$$

with $i = 1, \dots, n$, where (3.11) and (4.8) have been used. With the aid of (3.8), (3.9) and (4.6), the following results arise from (5.2) (see Appendix 2)

$$\begin{aligned} \frac{1}{u} &= \left(\frac{1}{P} \right)_X + \frac{1}{P}, \\ P \Omega_{[i+1]} &= 2(v_{[i]} - (v_{[i]})_x) \implies \omega_{[i+1]} = \frac{(\Omega_{[i+1]})_X + \Omega_{[i+1]}}{2}, \\ \Delta &= (v_{[n]})_x - v_{[n]}. \end{aligned} \tag{5.3}$$

with $i = 1, \dots, n-1$. Furthermore, (5.1) can be written as

$$dx = \left[1 - \frac{P_X}{P} \right] dX + \left[\omega_{[1]} - \frac{\Omega_{[1]}}{2} \left(1 - \frac{P_X}{P} \right) \right] dY + \frac{\Delta - \delta}{u} dT. \tag{5.4}$$

The cross derivatives of (5.4) imply (see Appendix 2) that

$$\begin{aligned} \left[1 - \frac{P_X}{P} \right]_Y &= \left[\omega_{[1]} - \frac{\Omega_{[1]}}{2} \left(1 - \frac{P_X}{P} \right) \right]_X \implies \omega_{[1]} = \frac{(\Omega_{[1]})_X + \Omega_{[1]}}{2}, \\ \left[1 - \frac{P_X}{P} \right]_T &= \left[\frac{\Delta - \delta}{P} \left(1 - \frac{P_X}{P} \right) \right]_X \implies \frac{\delta}{u} = \left(\frac{\Delta}{P} \right)_X + \frac{\Delta}{P} \end{aligned} \tag{5.5}$$

and therefore with the aid of (5.5), (5.4) reads

$$dx = \left[1 - \frac{P_X}{P}\right] dX - \frac{P_Y}{P} dY - \frac{P_T}{P} dT. \quad (5.6)$$

This exact derivative can be integrated as

$$x = X - \ln P. \quad (5.7)$$

By summarizing the above conclusions, we have proved that the mCHH(2+1) hierarchy

$$u_t = r^{-n} u_y, \quad u = u(x, y, t), \quad (5.8)$$

can be considered as the modified version of the Camassa-Holm hierarchy

$$U_T = R^{-n} U_Y, \quad U = U(X, Y, T). \quad (5.9)$$

The transformation that connects the two hierarchies involves the reciprocal transformation

$$x = X - \frac{1}{2} \ln U \quad (5.10)$$

as well as the following transformation between the fields

$$\begin{aligned} \frac{1}{u} &= \frac{1}{\sqrt{U}} \left(1 - \frac{U_X}{2U}\right), \quad \implies \quad \omega_{[i]} = \frac{(\Omega_{[i]})_X + \Omega_{[i]}}{2}, \quad i = 1, \dots, n, \\ \frac{\delta}{u} &= \left(\frac{\Delta}{\sqrt{U}}\right)_X + \frac{\Delta}{\sqrt{U}}. \end{aligned} \quad (5.11)$$

5.1 Particular case 1: The Qiao equation

We are now restricted to the first component of the hierarchies $n = 1$ in the case in which the field u is independent of y and U is independent of Y .

- From (2.13) and (2.14), for the restriction of CHH(2+1) we have

$$\begin{aligned} U &= P^2, \\ U_T &= (\Omega_{[1]})_{XXX} - (\Omega_{[1]})_X, \\ (P\Omega_{[1]})_X &= 0. \end{aligned} \quad (5.12)$$

which can be summarized as

$$\begin{aligned} \Omega_{[1]} &= \frac{k_1}{P} = \frac{k_1}{\sqrt{U}}, \\ U_T &= k_1 \left[\left(\frac{1}{\sqrt{U}}\right)_{XXX} - \left(\frac{1}{\sqrt{U}}\right)_X \right], \end{aligned} \quad (5.13)$$

that is the *Dym equation*.

- The reduction of mCHH(2+1) can be achieved from (3.4) in the form

$$\begin{aligned}(\omega_{[1]})_x &= u(v_{[1]})_x, \\ u_t &= (v_{[1]})_{xxx} - (v_{[1]})_x, \\ (u\omega_{[1]})_x &= 0,\end{aligned}\tag{5.14}$$

which can be written as

$$\begin{aligned}\omega_{[1]} = \frac{k_2}{u} &\implies v_{[1]} = \frac{k_2}{2u^2}, \\ u_t &= k_2 \left[\left(\frac{1}{2u^2} \right)_{xx} - \left(\frac{1}{2u^2} \right)_x \right].\end{aligned}\tag{5.15}$$

that is the Qiao equation.

- From (5.13) and (5.15) it is easy to see that $k_1 = 2k_2$. By setting $k_2 = 1$, we can conclude that the Qiao equation

$$u_t = \left(\frac{1}{2u^2} \right)_{xxx} - \left(\frac{1}{2u^2} \right)_x\tag{5.16}$$

is the modified version of the Dym equation

$$U_T = \left(\frac{2}{\sqrt{U}} \right)_{XXX} - \left(\frac{2}{\sqrt{U}} \right)_X.\tag{5.17}$$

- From (2.17) and (3.27), it is easy to see that the independence from y implies that $X_1 = X_0$ and $x_1 = x_0$, which means that the CBS and modified CBS equations (2.21) and (3.29) reduce to the following potential versions of the KdV and modified KdV equations

$$\begin{aligned}(M_{z_2} + M_{z_0, z_0, z_0} + 6M_{z_0}^2)_{z_0} &= 0, \\ x_{z_2} + x_{z_0, z_0, z_0} - \frac{1}{2}x_{z_0}^3 &= 0.\end{aligned}\tag{5.18}$$

5.2 Particular case 2: The Camassa-Holm equation

If we are restricted to the $n = 1$ component when $T = X$ and $t = x$, the following results hold

- From (2.13) and (2.14), for the restriction of CHH(2+1) we have

$$\begin{aligned}\Delta &= P = \sqrt{U}, \\ U &= (\Omega_{[1]})_{XX} - \Omega_{[1]}, \\ U_Y + U(\Omega_{[1]})_X + \frac{1}{2}\Omega_{[1]}U_X &= 0\end{aligned}\tag{5.19}$$

which is the Camassa-Holm equation.

- The reduction of mCHH(2+1) can be obtained from From (3.24) and (3.25) in the form

$$\begin{aligned}\delta &= u = (v_{[1]})_{xx} - v_{[1]}, \\ u_y + (u\omega_{[1]})_x &= 0, \\ (\omega_{[1]})_x - u(v_{[1]})_x &= 0,\end{aligned}\tag{5.20}$$

which can be considered as a modified Camassa-Holm equation.

- From (2.17) and (3.27), it is easy to see that $X_2 = x_2 = -1$. Therefore, the reductions of (2.21) and (3.18) are

$$M_{z_0, z_0, z_0, z_1} + 4M_{z_1} M_{z_0, z_0} + 8M_{z_0} M_{z_0, z_1} = 0,\tag{5.21}$$

which is the AKNS equation and

$$\left(\frac{x_{z_1, z_0, z_0} - 1}{x_{z_0}}\right)_{z_0} = \left(\frac{x_{z_0}^2}{2}\right)_{z_1},\tag{5.22}$$

that is the modified AKNS equation.

1 Lie systems

- There remains a lot of work to continue on Lie systems. Apart from the obtained results, e.g., linearizability conditions, constants of motion, Lie symmetries, superposition rules for such Lie systems, etc., further properties need to be analyzed: the existence of several Lie–Hamiltonian structures for a Lie system, the study of conditions for the existence of Lie–Hamilton systems, other methods to derive superposition rules, the analysis of integrable and superintegrable Lie–Hamilton systems, etc.
- Our work on the second-order Riccati equations has introduced many advantages with respect to previous methods. We directly transform a quite general family of second-order Riccati equations into Lie–Hamilton systems by Legendre transforms. When possible, this is a much simpler approach than the previous, based on more elaborated geometric theories which map second-order Riccati equations into Lie systems associated with a $\mathfrak{sl}(3, \mathbb{R})$ Lie algebra. Our method reduces the explicit integration of second-order Riccati equations to solving Lie systems related to $\mathfrak{sl}(2, \mathbb{R})$, e.g., Riccati equations. We have also provided a new approach to Kummer–Schwarz equations (second and third-order) that retrieves results in a simpler way.
- In this thesis we have detailed numerous new applications of Lie systems. This has much enlarged the potential applications of these systems. We have shown for the first time that Lie systems appear in the study of viral models and certain types of Lotka–Volterra systems. We have also proved that Riccati equations over several composition algebras, e.g., quaternions or dual-complex numbers, are also Lie systems. Other new applications concern Darboux–Brioschi–Halphen systems, certain systems with quadratic nonlinear-

ities, planar systems with trigonometric nonlinearities appearing in the theory of integrable systems, complex Bernoulli equations, etcetera. Concerning PDEs, we have studied partial Riccati equations and we have analyzed certain systems appearing in the study of flat connection forms.

- We have classified Lie–Hamilton systems on the plane attending to their underlying Lie algebras. We found that only twelve out of the initial 28 classes of finite-dimensional real Lie algebras of vector fields on the plane detailed in the GKO classification, are of Lie–Hamilton type. In turn, these classes give rise to twelve families of Lie algebras of Hamiltonian vector fields. This led to classifying Lie–Hamilton systems on the plane. In particular, we have given examples belonging to the classes P_2 , I_2 , I_4 , I_5 , I_{14A} , I_{14B} and I_{15} of our classification of finite-dimensional Lie algebras of Hamiltonian vector fields. We have shown that Kummer–Schwarz, Milne–Pinney equations (both with $c > 0$) and complex Riccati equations with t -dependent coefficients are related to the same Lie algebra P_2 , a fact which was used to explain the existence of a local diffeomorphism that maps each one of these systems into another. We also showed that the t -dependent harmonic oscillator, arising from Milne–Pinney equations when $c = 0$, corresponds to class I_5 and this can only be related through diffeomorphisms to the Kummer–Schwarz equations with $c = 0$, but not with complex Riccati equations. We plan to study the existence and the maximal number of their cyclic limits for planar Lie systems so as to investigate the so called second Hilbert’s number $H(2)$ for these systems. This could be a first step to analyze the XVI Hilbert’s problem through our Lie techniques. Work on these lines is currently in progress.
- We have proven the efficiency of the coalgebra method for obtaining superposition rules in the case of $\mathfrak{sl}(2)$ - and $\mathfrak{so}(3)$ -Lie–Hamilton systems. Obtaining superposition rules for all of the two-photon Lie–Hamilton systems remains another open problem. As a byproduct of this further construction, Lie–Hamilton systems defined on subalgebras of P_5 would be also obtained as particular cases. Recall that P_5 has as relevant Lie–Hamilton subalgebras: $\mathfrak{sl}(2)$, $P_1 \simeq \overline{\mathfrak{iso}}(2)$ and $I_8 \simeq \overline{\mathfrak{iso}}(1, 1) \simeq \mathfrak{h}_4$. Work on these lines is currently in progress.
- For structure-Lie systems, we have proposed a method for generating bi–Dirac–Lie systems. Superposition rules for Dirac–Lie systems have been achieved for the first time. We have gone a step further by classifying Lie algebras of Hamiltonian vector fields with respect to a Jacobi structure. Based on the previous GKO classification, we have established which Lie algebras in the GKO classification table are Hamiltonian in the Jacobi sense. Therefore, we are classifying Jacobi–Lie systems on low dimensional manifolds, since the

GKO is a planar classification. Another classification table in terms of the Jacobi structure has arisen and has been displayed. It is not shocking to find a smaller number of Jacobi–Lie systems with physical or mathematical applications, given the restrictive nature of the Jacobi structure.

- For future research, we aim at finding new types of Lie systems related to other geometric structures. For instance, it would be interesting to study the existence of Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to almost or twisted Poisson structures. We are also interested in developing a generalization of the theory of Lie systems to the framework of Lie algebroids. The latter has shown to be very fruitful in Geometric Mechanics and Control Theory. Moreover, a further analysis of the properties of Dirac–Lie systems is being performed. The theory of partial superposition rules for partial differential equations is still an open question, specifically from the point of view of the coalgebra method. We would like to develop a solid geometrical framework and expand the applications of Lie systems to partial differential equations. The author has plans of continuing a similar procedure as the one presented for Lie–Hamilton systems, applied to the realm of k -symplectic Lie systems. Also, she is particularly interested in research in other types of geometries: as Kähler or Calabi–Yau with more applications in Theoretical Physics, given her Physics training background.

2 Lie symmetries

- The wide applicability of Lie symmetries has suggested the inspection of Lie symmetries of Lie systems. We have shown that these symmetries can be obtained by solving a system of differential equations that is another Lie system possessing a Vessiot–Guldberg Lie algebra related to the Vessiot–Guldberg Lie algebra of the Lie system. Our results have been generalized to the realm of higher-order differential equations and partial differential Lie systems. Our procedures enable us to find Lie symmetries of isomorphic partial differential Lie systems simultaneously.
- We have calculated the point Lie symmetries of (nonlinear) partial differential equations with strong applications in the field of Fluid Dynamics. An important example is that of the BKP equation and its corresponding two-component Lax pair in $2 + 1$ dimensions. We have reduced both the equation and its Lax pair attending to different choices of the arbitrary functions present in the symmetries. Out of 6 reductions, 2 of them happen to be knowledgeable equations in the literature: one is the Korteweg de Vries equation and another is an equation showing multisoliton solutions. In this way, we can say that BKP in $2 + 1$ is a generalization of the KdV equation in $1 + 1$.

- Next, we perform an iterative process of symmetry search on another nonlinear equation (obtained as one of the reductions of the previous step) and its corresponding two component Lax pair in $1 + 1$. The spectral parameter has been conveniently introduced in the reduction and it needs to be considered as an independent variable for further reduction. The nonclassical approach has resulted in 6 constants of integration and 2 arbitrary functions, whilst the classical approach gives rise to symmetries only containing 5 constants of integration. Another 4 interesting reductions have been contemplated out of this second reduction.
- We have calculated point Lie symmetries for higher-order hierarchies of differential equations and their corresponding Lax pairs. It seems natural to expect the same Lie symmetries for both the hierarchy and the Lax pair. Nevertheless, the results obtained do not match our expected criteria, as we have proven for the Camassa–Holm hierarchy and its Lax pair. The difference between the symmetries of the hierarchy and the Lax pair is one arbitrary function of time in the Lax pair but a constant appearing for the equation. It is our future duty to identify the reasons for these differing results. In the case of the Camassa–Holm hierarchy, we have started with the non-isospectral Lax pair in $2 + 1$ dimensions. We have searched for non-classical Lie symmetries of this non-isospectral problem and each possibility has led us to reduced spectral problems in $1 + 1$ whose compatibility conditions give rise to $1 + 1$ dimensional reduced hierarchies. Each of these possibilities comes from the choice of the 5 arbitrary constants and 3 arbitrary time dependent functions that appear in the symmetries. The most outstanding result of this research is that our procedure also provides the reduction of the eigenfunction, as well as that of the spectral parameter. In many cases, the spectral parameter also proves to be nonisospectral, even in the $1 + 1$ reductions. We achieved several reductions that can be summarized in 9 nontrivial cases. 5 of the arisen hierarchies have a nonisospectral Lax pair, 2 of them are the positive and negative Camassa–Holm hierarchies. The rest of reductions happen to be isospectral, as most problems found in $1 + 1$ in the literature. It is interesting then to remark that we have found nonisospectral problems in $1 + 1$, which are not abundant.

A similar problem to this one has constituted another section of this second part. We have dealt with the Qiao hierarchy and its associated nonisospectral Lax pair. The interest of this problem resides in the spinorial or two component Lax pair. We have followed a very similar procedure as for the Camassa–Holm hierarchy, with the difference that this time we have only pursued classical symmetries, instead of nonclassical. The reason for contemplating this change is the difficulty in dealing with the arising nonlinear equations for the obtainance of the symmetry. Solving the classical case has appeared

more reliable to us in this particular case. The symmetries depend on 4 arbitrary constants and 2 arbitrary functions whereas there exists a function that has to satisfy an equation identical to the nonisospectrality condition. Each possibility contemplated leads to a possible reduction, we are counting a total of 8. Then again, we find nonisospectral reductions in $1 + 1$ dimensions, which are not frequent. These results confirm the group interpretation of the spectral parameter proposed by other authors.

- A future idea of research is identifying the differences between the classical and nonclassical symmetries obtained for our examples. It is noticeable that several differences occur, in particular, in the character of the arbitrary functions and constants arising out of the integration. It is our idea to guess where their differences root. It is also interesting to compare these symmetries with contact symmetries and obtain more through the contact symmetry method.

3 Reciprocal transformations

- We have proposed a reciprocal transformation between two hierarchies of interest: the Camassa–Holm and Qiao hierarchies. We have presented both hierarchies in $2 + 1$ dimensions and we have discussed some general properties of the Camassa–Holm hierarchy for a field $U(X, Y, T)$ and the Qiao hierarchy for $u(x, y, t)$. We have constructed reciprocal transformations that connect both hierarchies with the CBS and mCBS equations, respectively. The Lax pair for both of the hierarchies can be retrieved through those of the CBS and mCBS. If we consider the Lax pair of the CBS and mCBS and undo the reciprocal transformation, we achieve the Lax pair of the hierarchies. There exists a Miura transformation between the CBS and mCBS equations. So, it is possible to achieve a relation between the fields U and u and the variables (X, Y, T) and (x, y, t) . This carefully constructed relation proves that the Qiao hierarchy is a transformed version of that of Camassa–Holm. Two particular cases of interest have been contemplated: one is the relation between the two hierarchies when they are independent of Y and y . This shows that the Qiao equation is a modified Dym equation. Another is the modified Camassa–Holm equation when the reduction $T = X$ is applied to the first component of the Camassa–Holm hierarchy.
- We can say that somehow reciprocal transformations help us reduce the number of available nonlinear equations in the literature, as two seemingly different equations can be turned from one into another by reciprocal transformation, which is the case of the reviewed example. A question for future work is whether there exists a canonical description for differential equations. Intuitively, we expect that if two equations are essentially the same, although

apparently different in disguised versions, they must share the same singular manifold equations. But this is just an initial guess worth of further research in the future. Also, reciprocal transformations have proven their utility in the derivation of Lax pairs, as one initial equation whose Lax pair is unknown, can be interpreted as the reciprocally-transformed equation whose Lax pair is well-known. In this way, undoing the transformation on the latter Lax pair, the achieve the Lax pair of the former. As we know, obtaining Lax pairs is a nontrivial subject. The common way is to impose ad hoc forms for such linear problems and make them fit according to the compatibility condition. Accounting for reciprocal transformations, we do not face the problem of imposing ad hoc Ansätze. To finish, we mention that reciprocal transformations permit us to obtain (sometimes) equations integrable in the Painlevé sense if they did not have this property before the change. This is due to the non invariability of the Painlevé test under changes of variables.

- Some possible future research on this topic would consist of understanding whether the singular manifold equations can constitute a canonical representation of a partial differential equation and designing techniques to derive Lax pairs in a more unified way. Also, the trial of composition of reciprocal transformations with transformations of other nature, as performed in the case of Miura-reciprocal transformations, can lead to more unexpected but desirable results, as in the example exposed.

Appendix 1: Tables

Table 1. The GKO classification of the 8 + 20 finite-dimensional real Lie algebras of vector fields on the plane and their most relevant characteristics. The vector fields which are written between brackets form a modular generating system. The functions $\xi_1(x), \dots, \xi_r(x)$ and 1 are linearly independent over \mathbb{R} and the functions $\eta_1(x), \dots, \eta_r(x)$ form a basis of solutions for an r -order differential equation with constant coefficients [14, pp. 470–471]. Finally, $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ stands for the semi-direct sum (as Lie algebras) of \mathfrak{g}_1 by \mathfrak{g}_2 . Dom stands for the domain of the Lie algebra of vector fields.

#	Primitive	Basis of vector fields X_i	Dom
P ₁	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	\mathbb{R}^2
P ₂	$\mathfrak{sl}(2)$	$\{\partial_x, x\partial_x + y\partial_y\}, (x^2 - y^2)\partial_x + 2xy\partial_y$	$\mathbb{R}_{y \neq 0}^2$
P ₃	$\mathfrak{so}(3)$	$\{y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y\}, 2xy\partial_x + (1 + y^2 - x^2)\partial_y$	\mathbb{R}^2
P ₄	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	\mathbb{R}^2
P ₅	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	\mathbb{R}^2
P ₆	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	\mathbb{R}^2
P ₇	$\mathfrak{so}(3, 1)$	$\{\partial_x, \partial_y\}, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (y^2 - x^2)\partial_y$	\mathbb{R}^2
P ₈	$\mathfrak{sl}(3)$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	\mathbb{R}^2
#	Imprimitive	Basis of vector fields X_i	Dom
I ₁	\mathbb{R}	$\{\partial_x\}$	\mathbb{R}^2
I ₂	\mathfrak{h}_2	$\{\partial_x\}, x\partial_x$	\mathbb{R}^2
I ₃	$\mathfrak{sl}(2)$ (type I)	$\{\partial_x\}, x\partial_x, x^2\partial_x$	\mathbb{R}^2
I ₄	$\mathfrak{sl}(2)$ (type II)	$\{\partial_x + \partial_y, x\partial_x + y\partial_y\}, x^2\partial_x + y^2\partial_y$	$\mathbb{R}_{x \neq y}^2$
I ₅	$\mathfrak{sl}(2)$ (type III)	$\{\partial_x, 2x\partial_x + y\partial_y\}, x^2\partial_x + xy\partial_y$	$\mathbb{R}_{y \neq 0}^2$
I ₆	$\mathfrak{gl}(2)$ (type I)	$\{\partial_x, \partial_y\}, x\partial_x, x^2\partial_x$	\mathbb{R}^2
I ₇	$\mathfrak{gl}(2)$ (type II)	$\{\partial_x, y\partial_y\}, x\partial_x, x^2\partial_x + xy\partial_y$	$\mathbb{R}_{y \neq 0}^2$
I ₈	$B_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\{\partial_x, \partial_y\}, x\partial_x + \alpha y\partial_y, \quad 0 < \alpha \leq 1$	\mathbb{R}^2
I ₉	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y$	\mathbb{R}^2
I ₁₀	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y, x^2\partial_x$	\mathbb{R}^2
I ₁₁	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\{\partial_x, \partial_y\}, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$	\mathbb{R}^2
I ₁₂	\mathbb{R}^{r+1}	$\{\partial_y\}, \xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₃	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$	$\{\partial_y\}, y\partial_y, \xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₄	$\mathbb{R} \ltimes \mathbb{R}^r$	$\{\partial_x, \eta_1(x)\partial_y\}, \eta_2(x)\partial_y, \dots, \eta_r(x)\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₅	$\mathbb{R}^2 \ltimes \mathbb{R}^r$	$\{\partial_x, y\partial_y\}, \eta_1(x)\partial_y, \dots, \eta_r(x)\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₆	$C_\alpha^r \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x + \alpha y\partial_y, x\partial_y, \dots, x^r\partial_y, \quad r \geq 1, \quad \alpha \in \mathbb{R}$	\mathbb{R}^2
I ₁₇	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$	$\{\partial_x, \partial_y\}, x\partial_x + (ry + x^r)\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₈	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \dots, x^r\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₁₉	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_y, 2x\partial_x + ry\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \dots, x^r\partial_y, \quad r \geq 1$	\mathbb{R}^2
I ₂₀	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$	$\{\partial_x, \partial_y\}, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \dots, x^r\partial_y, \quad r \geq 1$	\mathbb{R}^2

Table 2. The classification of the 4 + 8 finite-dimensional real Lie algebras of Hamiltonian vector fields on \mathbb{R}^2 . For I_{12} , I_{14A} and I_{16} , we have $j = 1, \dots, r$ and $r \geq 1$; in I_{14B} the index $j = 2, \dots, r$.

#	Primitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
P ₁	$A_0 \simeq \mathfrak{iso}(2)$	$y, -x, \frac{1}{2}(x^2 + y^2), 1$	$dx \wedge dy$	$\overline{\mathfrak{iso}(2)}$
P ₂	$\mathfrak{sl}(2)$	$-\frac{1}{y}, -\frac{x}{y}, -\frac{x^2 + y^2}{y}$	$\frac{dx \wedge dy}{y^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
P ₃	$\mathfrak{so}(3)$	$\frac{-1}{2(1 + x^2 + y^2)}, \frac{y}{1 + x^2 + y^2},$ $-\frac{x}{1 + x^2 + y^2}, 1$	$\frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$	$\mathfrak{so}(3)$ or $\mathfrak{so}(3) \oplus \mathbb{R}$
P ₅	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$y, -x, xy, \frac{1}{2}y^2, -\frac{1}{2}x^2, 1$	$dx \wedge dy$	$\overline{\mathfrak{sl}(2) \ltimes \mathbb{R}^2} \simeq \mathfrak{h}_6$
#	Imprimitive	Hamiltonian functions h_i	ω	Lie–Hamilton algebra
I ₁	\mathbb{R}	$\int^y f(y')dy'$	$f(y)dx \wedge dy$	\mathbb{R} or \mathbb{R}^2
I ₄	$\mathfrak{sl}(2)$ (type II)	$\frac{1}{x - y}, \frac{x + y}{2(x - y)}, \frac{xy}{x - y}$	$\frac{dx \wedge dy}{(x - y)^2}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I ₅	$\mathfrak{sl}(2)$ (type III)	$-\frac{1}{2y^2}, -\frac{x}{y^2}, -\frac{x^2}{2y^2}$	$\frac{dx \wedge dy}{y^3}$	$\mathfrak{sl}(2)$ or $\mathfrak{sl}(2) \oplus \mathbb{R}$
I ₈	$B_{-1} \simeq \mathfrak{iso}(1, 1)$	$y, -x, xy, 1$	$dx \wedge dy$	$\overline{\mathfrak{iso}(1, 1)} \simeq \mathfrak{h}_4$
I ₁₂	\mathbb{R}^{r+1}	$-\int^x f(x')dx', -\int^x f(x')\xi_j(x')dx'$	$f(x)dx \wedge dy$	\mathbb{R}^{r+1} or \mathbb{R}^{r+2}
I _{14A}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type I)	$y, -\int^x \eta_j(x')dx', 1 \notin \langle \eta_j \rangle$	$dx \wedge dy$	$\mathbb{R} \ltimes \mathbb{R}^r$ or $(\mathbb{R} \ltimes \mathbb{R}^r) \oplus \mathbb{R}$
I _{14B}	$\mathbb{R} \ltimes \mathbb{R}^r$ (type II)	$y, -x, -\int^x \eta_j(x')dx', 1$	$dx \wedge dy$	$\overline{(\mathbb{R} \ltimes \mathbb{R}^r)}$
I ₁₆	$C_{-1}^r \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$y, -x, xy, -\frac{x^{j+1}}{j+1}, 1$	$dx \wedge dy$	$\overline{\mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}}$

Table 3. Specific Lie–Hamilton systems according to the family given in the classification of Table 1. All of these systems have t -dependent real coefficients.

LH algebra	#	LH systems
$\mathfrak{sl}(2)$	P_2	Milne–Pinney and Kummer–Schwarz equations with $c > 0$ Complex Riccati equation
$\mathfrak{sl}(2)$	I_4	Milne–Pinney and Kummer–Schwarz equations with $c < 0$ Split-complex Riccati equation Coupled Riccati equations Planar diffusion Riccati system for $c_0 = 1$
$\mathfrak{sl}(2)$	I_5	Milne–Pinney and Kummer–Schwarz equations with $c = 0$ Dual-Study Riccati equation Harmonic oscillator Planar diffusion Riccati system for $c_0 = 0$
$\mathfrak{h}_6 \simeq \overline{\mathfrak{sl}(2) \times \mathbb{R}^2}$	P_5	Dissipative harmonic oscillator Second-order Riccati equation in Hamiltonian form
$\mathfrak{h}_2 \simeq \mathbb{R} \times \mathbb{R}$	$I_{14A}^{r=1}$	Complex Bernoulli equation Generalized Buchdahl equations Lotka–Volterra systems

Table 4. Non-exhaustive tree of inclusion relations between classes of the GKO classification. We write $A \dashv\vdash B$ when a subclass of A is diffeomorphic to a Lie subalgebra in B . Every Lie algebra includes I_1 . In bold and italics are classes with Hamiltonian Lie algebras and rank one associated distribution, respectively. Colors help distinguishing the arrows.

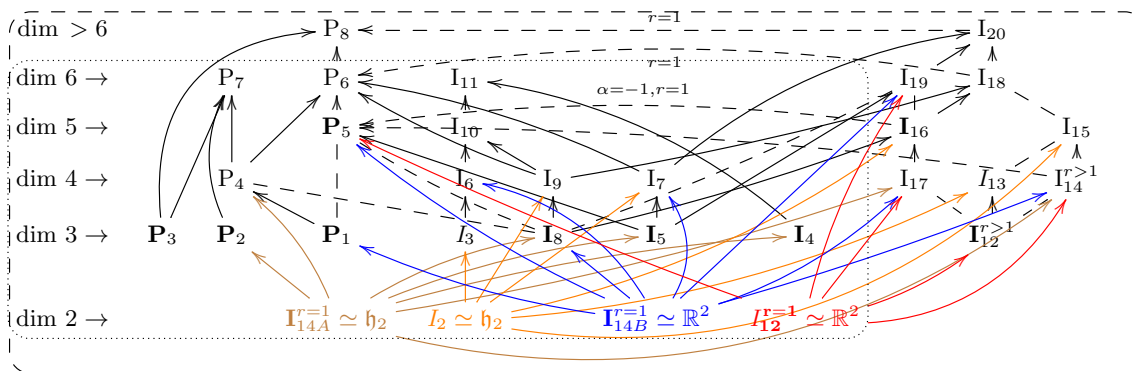


Table 5. Functions $1, \xi_1(x), \dots, \xi_r(x)$ are linearly independent and $\eta_1(x), \dots, \eta_r(x)$ form a basis of solutions for a system of r -order linear differential equations with constant coefficients. Notice that $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$ stands for the semi-direct sum of \mathfrak{g}_1 by \mathfrak{g}_2 , i.e., \mathfrak{g}_2 is an ideal of $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$.

#	Lie algebra	Basis of vector fields X_i	Jacobi
P ₁	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$	$(\alpha = 0)$ Pois.
P ₂	$\mathfrak{sl}(2)$	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	Poisson
P ₃	$\mathfrak{so}(3)$	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1 + y^2 - x^2)\partial_y$	Poisson
P ₄	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	No
P ₅	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$	Poisson
P ₆	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y$	No
P ₇	$\mathfrak{so}(3, 1)$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (y^2 - x^2)\partial_y$	No
P ₈	$\mathfrak{sl}(3)$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$	No
I ₁	\mathbb{R}	∂_x	$(0, \partial_x)$, Pois.
I ₂	\mathfrak{h}_2	$\partial_x, x\partial_x$	$(0, \partial_x)$
I ₃	$\mathfrak{sl}(2)$ (type I)	$\partial_x, x\partial_x, x^2\partial_x$	$(0, \partial_x)$
I ₄	$\mathfrak{sl}(2)$ (type II)	$\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y$	Poisson
I ₅	$\mathfrak{sl}(2)$ (type III)	$\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y$	Poisson
I ₆	$\mathfrak{gl}(2)$ (type I)	$\partial_x, \partial_y, x\partial_x, x^2\partial_x$	No
I ₇	$\mathfrak{gl}(2)$ (type II)	$\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y$	No
I ₈	$B_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, \quad 0 < \alpha \leq 1$	$(\alpha = -1)$ Pois.
I ₉	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y$	No
I ₁₀	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x$	No
I ₁₁	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$	No
I ₁₂	\mathbb{R}^{r+1}	$\partial_y, \xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y$	$(0, \partial_y)$, Pois.
I ₁₃	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$	$\partial_y, y\partial_y, \xi_1(x)\partial_y, \dots, \xi_r(x)\partial_y$	$(0, \partial_y)$
I ₁₄	$\mathbb{R} \ltimes \mathbb{R}^r$	$\partial_x, \eta_1(x)\partial_y, \eta_2(x)\partial_y, \dots, \eta_r(x)\partial_y$	Poisson
I ₁₅	$\mathbb{R}^2 \ltimes \mathbb{R}^r$	$\partial_x, y\partial_y, \eta_1(x)\partial_y, \dots, \eta_r(x)\partial_y$	No
I ₁₆	$C_\alpha^r \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, x\partial_y, \dots, x^r\partial_y, \quad \alpha \in \mathbb{R}$	$(\alpha = -1)$ Pois.
I ₁₇	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$	$\partial_x, \partial_y, x\partial_x + (ry + x^r)\partial_y, x\partial_y, \dots, x^{r-1}\partial_y$	No
I ₁₈	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \dots, x^r\partial_y$	No
I ₁₉	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_y, 2x\partial_x + rxy\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \dots, x^r\partial_y$	No
I ₂₀	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \dots, x^r\partial_y$	No

Table 6. Lie symmetries for Riccati equations (3.34) for different $\eta(t)$. We assume $f_0 = k \in \mathbb{R}$. We have $f_2 = df_3/dt$ and $f_1 = k\eta(t) - \int \eta(t)f_3 dt$. $Airy_A$ and $Airy_B$ denote the Airy and Bairy functions. J, Y are the Bessel functions of first and second kind. Function f_3 follows from (3.40).

$\eta(t)$	$f_3(t)$
$\frac{k}{at+b}$	$k + c_1(at+b)J^2\left(1, 2\sqrt{(at+b)/a^2}\right) + c_2(at+b)Y^2\left(1, 2\sqrt{at+b/a^2}\right) + c_3(at+b)J\left(1, 2\sqrt{t+b/a}\right)Y\left(1, 2\sqrt{t+b/a}\right)$
$\frac{k}{(at+b)^2}$	$-\frac{kat}{b} + c_1\left(\frac{at+b}{a}\right)^{\frac{a+\sqrt{a^2-4k}}{a}} + c_2\left(\frac{at+b}{a}\right)^{-\frac{-a+\sqrt{a^2-4k}}{a}} + c_3(at+b),$
$at+b$	$k + c_1 Airy_A\left(-\frac{at+b}{a^{2/3}}\right)^2 + c_2 Airy_B\left(-\frac{at+b}{a^{2/3}}\right)^2 + c_3 Airy_A\left(-\frac{at+b}{a^{2/3}}\right) Airy_B\left(-\frac{at+b}{a^{2/3}}\right)$

Appendix 2: Auxiliar calculations for Chapter 5

This appendix contains intermediate calculations for the obtainance of some equations displayed in Chapter 5. All numerical labels of the equations are in relation with those of Chapter 5.

Obtaining 5.4

Equation (5.2) provides

$$x_0 = X_0 + \partial_0(\ln X_0).$$

If we use the fact that $X_0 = \frac{1}{P}$ and $x_0 = \frac{1}{u}$, we obtain

$$\frac{1}{u} = \frac{1}{P} - \partial_0(\ln P),$$

and by using (3.6) we have

$$\frac{1}{u} = \frac{1}{P} - \frac{1}{P}(\ln P)_X$$

which yields (5.4).

Obtaining 5.5

By taking $i = 1, \dots, n - 1$ in (5.2), we have

$$-\frac{X_{i+1}}{X_0} = x_{0,i} - \frac{x_{0,0,i}}{x_0} - \frac{x_{i+1}}{x_0}, \quad i = 1, \dots, n - 1$$

If we use (3.7) and (4.5) the result is

$$\frac{P\Omega^{[i+1]}}{2} = \partial_0(\omega^{[i]}) - u\partial_{00}(\omega^{[i]}) - u\omega^{[i+1]}, \quad i = 1, \dots, n - 1$$

And now (4.5) gives us

$$-\frac{P\Omega^{[i+1]}}{2} = \frac{\omega_x^{[i]}}{u} - \left(\frac{\omega_x^{[i]}}{u}\right)_x - u\omega^{[i+1]}, \quad i = 1, \dots, n-1$$

If we use the following expressions arising from (4.4)

$$\omega_x^{[i]} = uv_x^{[i]}, \quad u\omega^{[i+1]} = v^{[i]} - v_{xx}^{[i]}, \quad i = 1, \dots, n-1,$$

the result is

$$-\frac{P\Omega^{[i+1]}}{2} = v_x^{[i]} - v_i, \quad i = 1, \dots, n-1$$

as is required in (5.3), and $u\omega^{[i+1]} = v^{[i]} - v_{xx}^{[i]}$, $i = 1, \dots, n-1$ can be written as

$$u\omega^{[i+1]} = (v^{[i]} - v_x^{[i]}) + (v_x^{[i]} - v_{xx}^{[i]}) = \left(\frac{P\Omega^{[i+1]}}{2}\right) + \left(\frac{P\Omega^{[i+1]}}{2}\right)_x, \quad i = 1, \dots, n-1$$

We have $\partial_x = \frac{u}{P}\partial_X$. Therefore,

$$u\omega^{[i+1]} = \left(\frac{P\Omega^{[i+1]}}{2}\right) + \frac{u}{P} \left(\frac{P\Omega^{[i+1]}}{2}\right)_X, \quad i = 1, \dots, n-1$$

$$\omega^{[i+1]} = \left(\frac{P\Omega^{[i+1]}}{2u}\right) + \frac{1}{2P} (P_X\Omega^{[i+1]} + P\Omega_X^{[i+1]}), \quad i = 1, \dots, n-1$$

We can eliminate u with the aid of the first equation in (5.3). The result is

$$\omega^{[i+1]} = \frac{\Omega_X^{[i+1]} + \Omega^{[i+1]}}{2}, \quad i = 1, \dots, n-1.$$

Obtaining 5.6

By taking $i = n$ in (5.2) we have

$$-\frac{X_{n+1}}{X_0} = x_{0,n} - \frac{x_{0,0,n}}{x_0} - \frac{x_{n+1}}{x_0}.$$

Equations (3.8) and (4.6) allow us to write the above equation as

$$\Delta = \partial_0(\omega^{[n]}) - u\partial_{00}(\omega^{[n]}) + \delta$$

With the aid of (4.5), it reads

$$\Delta = \frac{\omega_x^{[n]}}{u} - \left(\frac{\omega_x^{[n]}}{u}\right)_x + \delta.$$

If $\omega_x^{[n]} = uv_x^{[n]}$ and $\delta = v_{xx}^{[n]} - v^{[n]}$ are used, we obtain (5.5).

Bibliography

- [1] L. Abellanas, L. Martínez Alonso, A general setting for Casimir invariants, *J. Math. Phys.* **16**, 1580–1584 (1975).
- [2] S. Abenda, Reciprocal transformations and local hamiltonian structures of hydrodynamic type systems, *J. Phys. A* **42**, 095298 (2009).
- [3] S. Abenda, T. Grava, Modulation of Camassa–Holm equation and reciprocal transformations, *Ann. Inst. Fourier* **55**, 1803–1834 (2005).
- [4] S. Abenda, T. Grava, Reciprocal transformations and flat metrics on Hurwitz spaces, *J. Phys. A* **40**, 10769–10790 (2007).
- [5] M.J. Ablowitz, P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society, Lecture Notes Series **149**, Cambridge University Press, Cambridge, (1991).
- [6] M.J. Ablowitz, M. Kruskal, H. Segur, A note on Miura’s transformation, *J. Math. Phys.* **20**, 999–1003 (1979).
- [7] M.J. Ablowitz, A. Ramani, H. Segur, Nonlinear evolution equations and ordinary differential equations of Painlevé type, *Lett. Nuov. Cim.* **23**, 333–338 (1978).
- [8] M.J. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P -type I, *J. Math. Phys.* **21**, 715–721 (1980).
- [9] M.J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, (1981).
- [10] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equation in control and systems theory*, Birkhäuser Verlag, Berlin, (2003).

-
- [11] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, Benjamin/Cummings publishing, Reading, (1978).
- [12] B. Abraham–Shrauner, Hidden symmetries and linearization of the modified Painlevé–Ince equation, *J. Math. Phys.* **34**, 4809–4816 (1993).
- [13] B. Abramovich, *A history of non-euclidean geometry: evolution of the concept of a geometric space*, Springer–Verlag, New York, (1988).
- [14] M. Ackerman, R. Hermann, *Sophus Lie’s transformation group paper*, Math. Sci. Press., Brookline, (1975).
- [15] J.M. Ancochea, R. Campoamor–Stursberg, L. Garcia Vergnolle, Solvable Lie algebras with naturally graded nilradicals and their invariants, *J. Phys. A* **39**, 1339–1355 (2006).
- [16] R.L. Anderson, V.A. Baikov *et al.*, *CRC handbook of Lie group analysis of differential equations. Vol. 1. Symmetries, exact solutions and conservation laws*, CRC Press, Boca Raton, (1994).
- [17] R.L. Anderson, V.A. Baikov *et al.*, *CRC handbook of Lie group analysis of differential equations. vol 3. New trends in theoretical developments and computational methods*, CRC Press, Boca Raton, (1996).
- [18] R.L. Anderson, J. Harnard, P. Winternitz, Group theoretical approach to superposition rules for systems of Riccati equations, *Lett. Math. Phys.* **5**, 143–148 (1981).
- [19] R.L. Anderson, J. Harnard, P. Winternitz, Systems of ordinary differential equations with nonlinear superposition principles, *Physica D* **4**, 164–182 (1982).
- [20] I. Anderson, N. Kamran, P.J. Olver, Internal, external and generalized symmetries, *Adv. in Math.* **100**, 53–100 (1993).
- [21] R.M. Angelo, E.I. Duzzioni, A.D. Ribeiro, Integrability in t -dependent systems with one degree of freedom, *J. Phys. A* **45**, 055101 (2012).
- [22] R.M. Angelo, W.F. Wreszinski, Two-level quantum dynamics, integrability and unitary not gates, *Phys. Rev. A* **72**, 034105 (2005).
- [23] M. Antonowitz, A. Fordy, Coupled Harry–Dym equations with multiHamiltonian structures, *J. Phys. A* **21**, L269–L275 (1988).
- [24] C. Arnold, Formal continued fractions solutions of the generalized second order Riccati equations, applications, *Numer. Algorithms* **15**, 111–134 (1997).

-
- [25] V.I. Arnold, *Mathematical Methods of Classical Mechanics. Second Edition*, Springer–Verlag, New York, (1989).
- [26] D. Arrigo, J.R. Beckham, Nonclassical symmetries of evolutionary partial differential equations and compatibility conditions, *J. Math. Anal. Appl.* **289**, 55–65 (2004).
- [27] Í. Aslan, The Exp-function approach to the Schwarzian Korteweg de Vries equation, *Computers & Mathematics with Applications* **59**, 2896–2900 (2010).
- [28] F. Avram, J.F. Cariñena, J. de Lucas, *A Lie systems approach for the first passage-time of piecewise deterministic processes*, in: *Modern Trends of Controlled Stochastic Processes: Theory and Applications*, Luniver Press, Frome, 144–160 (2010).
- [29] J.A. de Azcárraga, F.J. Herranz, J.C. Pérez Bueno, M. Santander, Central extensions of the quasi-orthogonal Lie algebras, *J. Phys. A* **31**, 1373–1394 (1998).
- [30] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, (2003).
- [31] A. Ballesteros, A. Blasco, F.J. Herranz, N -dimensional integrability from two-photon coalgebra symmetry, *J. Phys. A: Math. Theor.* **42**, 265205 (2009).
- [32] A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, C. Sardón, Lie–Hamilton systems on the plane: theory, classification and applications, *J. Differential Equations* **258**, 2873–2907 (2015).
- [33] A. Ballesteros, A. Blasco, F.J. Herranz, F. Musso, O. Ragnisco, (Super) integrability from coalgebra symmetry: formalism and applications, *J. Phys.: Conf. Ser.* **175**, 012004 (2009).
- [34] A. Ballesteros, A. Blasco, O. Ragnisco, *Poisson coalgebras, symplectic realizations and integrable systems*, in: *Monogr. Real Acad. Cienc. Exact. Fis. Quím. Nat. Zaragoza* **29**, Zaragoza: Acad. Cienc. Exact. Fís. Quím. Nat. Zaragoza, 27–36, (2006).
- [35] A. Ballesteros, J.F. Cariñena, F.J. Herranz J. de Lucas, C. Sardón, From constants of motion to superposition rules of Lie–Hamilton systems, *J. Phys. A* **46**, 285203 (2013).
- [36] A. Ballesteros, M. Corsetti, O. Ragnisco, N -dimensional classical integrable systems from Hopf algebras, *Czech. J. Phys.* **46**, 1153–1165 (1996).

- [37] A. Ballesteros, F.J. Herranz, M.A. del Olmo, M. Santander, Quantum structure of the motion groups of the two-dimensional Cayley–Klein geometries, *J. Phys. A* **26**, 5801–5823 (1993).
- [38] A. Ballesteros, F.J. Herranz, P. Parashar, $(1 + 1)$ Schrödinger Lie bialgebras and their Poisson–Lie groups, *J. Phys. A* **33**, 3445–3465 (2000).
- [39] Á. Ballesteros, F. J. Herranz, O. Ragnisco, Superintegrability on $\mathfrak{sl}(2)$ -coalgebra spaces, *Physics of Atomic Nuclei* **71**, 812–818 (2008).
- [40] A. Ballesteros, F. Musso, O. Ragnisco, Classical and Quantum integrable systems: the coalgebra approach, *Regular and chaotic dynamics* **7**, 4 (2002).
- [41] A. Ballesteros, O. Ragnisco, A systematic construction of completely integrable Hamiltonian from coalgebras, *J. Phys. A* **31**, 3791–3813 (1998).
- [42] T. Beatus, T. Tlustý, R. Bai-Ziu, Burgers shock waves and sound in a 2D microfluid droplets ensemble, *Phys. Rev. Lett.* **103**, 114502 (2009).
- [43] J. Beckers, L. Gagnon, V. Hussin, P. Winternitz, Superposition formulas for nonlinear superequations, *J. Math. Phys.* **31**, 2528–2534 (1990).
- [44] J. Beckers, V. Hussin, P. Winternitz, Complex parabolic subgroups of G_2 and nonlinear differential equations, *Lett. Math. Phys.* **11**, 81–86 (1986).
- [45] J. Beckers, V. Hussin, P. Winternitz, Nonlinear equations with superposition formulas and the exceptional group G_2 . I. Complex and real forms of g_2 and their maximal subalgebras, *J. Math. Phys.* **27**, 2217–2227 (1986).
- [46] G.M. Beffa, Moving frames, geometric Poisson brackets and the KdV–Schwarzian evolution of pure spinors, *Ann. Inst. Fourier (Grenoble)* **61**, 2405–2434 (2011).
- [47] A.A. Bekov, Integrable cases and trajectories in the Gylden–Meshcherskivi problem, *Soviet Astronom.* **33**, 71–78 (1989).
- [48] I.M. Benn, R.W. Tucker, *An introduction to spinors and geometry with applications in Physics*, Adam Hilger, Ltd., Bristol, (1987).
- [49] F. Berdjis, A criterion for completeness of Casimir operators, *J. Math. Phys.* **22**, 1851–56 (1981).
- [50] L.M. Berkovich, Gylden–Mevsvcerskivi problem, *Celestian Mech.* **24**, 407–429 (1981).
- [51] L.M. Berkovich, Transformation of Sturm–Liouville differential equations, *Funct. Anal. Appl.* **16**, 190–192 (1982).

- [52] L.M. Berkovich, Canonical forms of ordinary linear differential equations, *Arch. Math. (Brno)* **24**, 25–42 (1988).
- [53] L.M. Berkovich, The generalized Emden-Fowler equation, *Symmetry in nonlinear mathematical physics*, Vol. 1, 2 Natl. Acad. Sci. Ukraine, Inst. Math., Kiev, 155–163 (1997).
- [54] L.M. Berkovich, Discussion of Lie’s nonlinear superposition theory, in: *Proceedings of the International Conference of Modern Group, Analysis for the New Millennium (MOGRAN '00)*, Ufa, (2000).
- [55] L.M. Berkovich, Method of factorization of ordinary differential operators and some of its applications, *Appl. Anal. Discret. Math.* **1**, 122–149 (2007).
- [56] L.M. Berkovich, F.L. Berkovich, Transformation and factorization of second order linear ordinary differential equations and its implementation in REDUCE, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **6**, 11–24 (1995).
- [57] L.M. Berkovich, N.H. Rozov, Transformations of linear differential equations of second order and adjoined nonlinear equations, *Arch. Math. (Brno)* **33**, 75–98 (1997).
- [58] M.V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. R. Soc. Lond. Ser. A* **392**, 45–57 (1984).
- [59] G. Birkhoff, *Hydrodynamics: a study in logic, fact and similitude*, Princeton University Press, Princeton, (1950).
- [60] G. Birkhoff, Sound waves in fluids, *Appl. num. Math.* **3**, 3–24 (1987).
- [61] D. Blázquez-Sanz, *Differential Galois theory and Lie–Vessiot systems*, VDM Verlag, (2008).
- [62] D. Blázquez-Sanz, J.J. Morales-Ruiz, Local and Global Aspects of Lie’s Superposition Theorem, *J. Lie Theory* **20**, 483–517 (2010).
- [63] D. Blázquez-Sanz, J.J. Morales-Ruiz, Lie’s reduction method and differential Galois theory in the complex analytic context, *Discrete Contin. Dyn. Syst. - Series A* **32**, 353–379 (2012).
- [64] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.* **18**, 1025–1042 (1969).
- [65] G.W. Bluman, J. D. Cole, *Similarity methods for differential equations*, Springer-Verlag, (1974).

-
- [66] G.W. Bluman, S. Kumei, *Symmetries and differential equations*, Springer NY, (1989).
- [67] A.V. Bocharov, V.N. Chetverikov *et al.*, *Symmetries and conservation laws for differential equations of mathematical physics*, Translations of mathematical monographs **182**, American Mathematical Society, Providence, R.I. (1999).
- [68] O.I. Bogoyavlenskii, Euler equations on finite-dimensional Lie algebras arising in physical problems, *Comm. Math. Phys.* **95**, 307–315 (1984).
- [69] O.I. Bogoyavlenskii, Overturning solitons in new two-dimensional integrable equations, *Math. URSS. Izv.*, **34**, 245 (1990).
- [70] O.I. Bogoyavlenskii, Breaking solitons in 2+1 dimensional integrable equations, *Russian Math. Surv.* **45**, 1–86 (1990).
- [71] J.L. Bona, Forced oscillations of a damped Korteweg de Vries equation in a quarter plane, *Commun. Contemp. Math.* **5**, 369 (2003).
- [72] M. Boutin, *On invariants of Lie group actions and their application to some equivalence problems*, Thesis (Ph.D.)—University of Minnesota, ProQuest LLC, Ann Arbor, (2001).
- [73] H.A. Buchdahl, A relativistic fluid spheres resembling the Emden polytrope of index 5, *Astrophys. J.* **140**, 1512–1516 (1964).
- [74] H. Bursztyn, *A brief introduction to Dirac manifolds*, in: *Geometric and Topological Methods for Quantum Field Theory. Proceedings of the 2009 Villa de Leyva Summer School*, Cambridge University Press, Cambridge, 4–38 (2013).
- [75] H. Bursztyn, O. Radko, Gauge equivalence of Dirac structures and symplectic groupoids, *Ann. Inst. Fourier (Grenoble)* **53**, 309–337 (2003).
- [76] S. Callot, D. Hulin, J. Lafontane, *Riemannian Geometry*, Springer, Berlin, (1987).
- [77] F. Calogero, A method to generate solvable non-linear evolution equations, *Lett. Nuovo Cimento*, **14**, 443–448 (1975).
- [78] F. Calogero, Generalized Wronskian relations, one-dimensional Schrödinger equation and non-linear partial differential equations solvable by the inverse-scattering method, *Nuovo Cimento B* **31**, 229–249 (1976).
- [79] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71**, 1661–1664 (1993).

-
- [80] R. Campoamor-Stursberg, Invariants of solvable rigid Lie algebras up to dimension 8, *J. Phys. A* **35**, 6293–6306 (2002).
- [81] R. Campoamor-Stursberg, The structure of the invariants of perfect Lie algebras, *J. Phys. A* **36**, 6709–6723 (2003).
- [82] R. Campoamor-Stursberg, Solvable Lie algebras with an \mathbb{N} -graded nilradical of maximal nilpotency degree and their invariants, *J. Phys. A* **43**, 145202 (2010).
- [83] J. Campos, Möbius transformations and periodic solutions of complex Riccati equations, *Bull. London Math. Soc.* **29**, 205–215 (1997).
- [84] J. Campos, J. Mawhin, Periodic solutions of quaternionic-valued ordinary differential equations, *Ann. Mat. Pura Appl.* **185**, suppl., S109–S127 (2006).
- [85] J.F. Cariñena, J. Grabowski, J. de Lucas, Quasi-Lie schemes: theory and applications, *J. Phys. A* **42**, 335206 (2009).
- [86] J.F. Cariñena, J. Grabowski, J. de Lucas, Lie families: theory and applications, *J. Phys. A* **43**, 305201 (2010).
- [87] J.F. Cariñena, J. Grabowski, J. de Lucas, Superposition rules for higher-order differential equations, and their applications, *J. Phys. A* **45**, 185202 (2012).
- [88] J.F. Cariñena, J. Grabowski, J. de Lucas, C. Sardón, Dirac–Lie systems and Schwarzian equations, *J. Differential Equations* **257**, 2303–2340 (2014).
- [89] J.F. Cariñena, J. Grabowski, G. Marmo, *Lie–Scheffers systems: a geometric approach*, *Napoli Series in Physics and Astrophysics*, Bibliopolis, Napoli, (2000).
- [90] J.F. Cariñena, J. Grabowski, G. Marmo, *Some physical applications of systems of differential equations admitting a superposition rule. Proceedings of the XXXII Symposium on Mathematical Physics (Toruń, 2000)* *Rep. Math. Phys.* **48**, 47–58 (2001).
- [91] J.F. Cariñena, J. Grabowski, G. Marmo, Superposition rules, Lie theorem and partial differential equations, *Rep. Math. Phys.* **60**, 237–258 (2007).
- [92] J.F. Cariñena, J. Grabowski, A. Ramos, Reduction of time-dependent systems admitting a superposition principle, *Acta Appl. Math.* **66**, 67–87 (2001).
- [93] J.F. Cariñena, P. Guha, M.F. Rañada, A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion, *J. Phys.: Conf. Ser.* **175**, 012009 (2009).
- [94] J.F. Cariñena, A. Ibort, G. Marmo and G. Morandi, *Geometry from dynamics, Classical and Quantum* Springer, Dordrecht, (2015).

- [95] J.F. Cariñena, P.G.L. Leach, J. de Lucas, Quasi-Lie schemes and Emden-Fowler equations, *J. Math. Phys.* **50**, 103515 (2009).
- [96] J.F. Cariñena, C. López, Group theoretical perturbative treatment of nonlinear Hamiltonians on the dual of a Lie algebra, *Rep. Math. Phys.* **43**, 43–51 (1999).
- [97] J.F. Cariñena, J. de Lucas, Recent applications of the theory of Lie systems in Ermakov systems, *SIGMA* **4**, 031 (2008).
- [98] J.F. Cariñena, J. de Lucas, A nonlinear superposition rule for solutions of the Milne-Pinney equation, *Phys. Lett. A* **372**, 5385–5389 (2008).
- [99] J.F. Cariñena, J. de Lucas, Applications of Lie systems in dissipative Milne-Pinney equations, *Int. J. Geom. Methods Mod. Phys.* **6**, 683 (2009).
- [100] J.F. Cariñena, J. de Lucas, Quantum Lie systems and integrability conditions, *Int. Geom. Methods Mod. Phys.* **6**, 1235–1252 (2009).
- [101] J.F. Cariñena, J. de Lucas, Superposition rules and second-order Riccati equations, *J. Geom. Mech.* **3**, 1–22 (2011).
- [102] J.F. Cariñena, J. de Lucas, Lie systems: theory, generalisations, and applications, *Dissertationes Math. (Rozprawy Mat.)* **479**, 1–162 (2011).
- [103] J.F. Cariñena, J. de Lucas, *Superposition rules and second-order differential equations*, in: *XIX International Fall Workshop on Geometry and Physics*, AIP Conf. Proc. **360**. Amer. Inst. Phys., New York, 127–132 (2011).
- [104] J.F. Cariñena, J. de Lucas, Quasi-Lie schemes and second-order Riccati equations, *J. Geom. Mech.* **3**, 1–22 (2011).
- [105] J.F. Cariñena, J. de Lucas, M.F. Rañada, Recent applications of the theory of Lie systems in Ermakov systems, *SIGMA Symmetry Integrability Geom. Methods Appl.* **4**, 031 (2008).
- [106] J.F. Cariñena, J. de Lucas, M.F. Rañada, Lie systems and integrability conditions for t-dependent frequency harmonic oscillators, *Int. J. Geom. Methods Mod. Phys.* **7**, 289–310 (2010).
- [107] J.F. Cariñena, J. de Lucas, M.F. Rañada, A geometric approach to integrability of Abel differential equations, *Int. J. Theor. Phys.* **50**, 2114–2124 (2011).
- [108] J.F. Cariñena, J. de Lucas, C. Sardón, A new Lie system’s approach to second-order Riccati equations, *Int. J. Geom. Methods Mod. Phys.* **9**, 1260007 (2012).
- [109] J.F. Cariñena, J. de Lucas, C. Sardón, Lie-Hamilton systems: theory and applications, *Int. Geom. Methods Mod. Phys.* **10**, 0912982 (2013).

- [110] J.F. Cariñena, G. Marmo, J. Nasarre, The nonlinear superposition principle and the Wei–Norman method, *Int. J. Mod. Phys. A* **13**, 3601 (1998).
- [111] J.F. Cariñena, G. Marmo, M. Rañada, Non-symplectic symmetries and bi-Hamiltonian structures of the rational harmonic oscillator, *J. Phys. A* **35**, L679–L686 (2002).
- [112] J.F. Cariñena, A. Ramos, Integrability of the Riccati equation from a group-theoretical viewpoint, *Internat. J. Modern Phys. A* **14**, 1935–1951 (1999).
- [113] J.F. Cariñena, A. Ramos, *Applications of Lie systems in Quantum Mechanics and Control Theory*, in: *Classical and Quantum integrability*, Banach Center, Publ. **59**, Warsaw, 143–162 (2003).
- [114] J.F. Cariñena, A. Ramos, *Lie systems and connections in fiber bundles: applications in Quantum Mechanics*, in: *Differential Geometry and its applications*, Matfyzpress, Prague, 437–452 (2005).
- [115] J.F. Cariñena, M.F. Rañada, M. Santander, Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability, *J. Math. Phys.* **46**, 062703 (2005).
- [116] J.M. Cerveró, P.G. Estévez, Miura transformation between two non-linear equations in $2 + 1$ dimensions, *J. Math. Phys.* **39**, 2800–2807 (1998).
- [117] S. Chakravartya, R. Halburd, First integrals of a generalized Darboux–Halphen system, *J. Math. Phys.* **44**, 1751–1762 (2003).
- [118] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461**, 2451–2476 (2005).
- [119] V. Chari, A. Pressley, *A Guide to Quantum Groups* Cambridge: Cambridge University Press, (1994).
- [120] J.S.R. Chisholm, A.K. Common, A class of second-order differential equations and related first-order systems, *J. Phys. A* **20**, 5459–5472 (1987).
- [121] A. Choudhuri, B. Talukdar, U. Das, Modified KdV hierarchy: Lax pair representation and bi-hamiltonian structure, *Zeitschrift für Naturforschung* **64**, 171–179 (2009).
- [122] J.L. Cieśliński, T. Nikiciuk, A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients, *J. Phys. A: Math. Theor.* **43**, 175205 (2010).

- [123] P.A. Clarkson, A.S. Fokas, M.J. Ablowitz, Hodograph transformations of linearizable partial differential equations, *SIAM J. of Appl. Math.* **49**, 1188–1209 (1989).
- [124] P.A. Clarkson, P.R. Gordoa, A. Pickering, Multicomponent equations associated to nonisospectral scattering problems, *Inverse Problems* **13**, 1463–1476 (1997).
- [125] P.A. Clarkson, A.N.W. Hone, N. Joshi, Hierarchies of differential equations and Bäcklund transformations, *J. Nonlin. Math. Phys.* **10**, 13–26 (2003).
- [126] P.A. Clarkson, M.D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.* **30**, 2201–2213 (1989).
- [127] P.A. Clarkson, P. Winternitz, Nonclassical symmetry reductions for the Kadomtsev-Petviashvili equation, *Physica D* **49**, 257–272 (1991).
- [128] J.N. Clellanda, P.J. Vassiliou, A solvable string on a Lorentzian surface, *Differential Geometry and its Applications* **33**, 177–198 (2014).
- [129] J. Clemente-Gallardo, On the relations between control systems and Lie systems. In: *Groups, geometry and physics. Monogr. Real Acad. Ci. Exact. Fís.-Quím. Nat. Zaragoza* **29**, Acad. Cienc. Exact. Fís. Quím. Nat. Zaragoza, Zaragoza, 65–68 (2006).
- [130] R. Conte, Singularities of differential equations and integrability, in: *An Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Non-linear Waves*, Frontières, Gif-sur-Yvette, 49–143 (1994).
- [131] R. Conte, *The Painlevé property. One century later*, Springer-Verlag, New York, (1999).
- [132] R. Conte, M. Musette, *The Painlevé Handbook*, Springer & Canopus Publishing Limited, Bristol, (2008).
- [133] J.H. Conway, D.A. Smith, *On quaternions and octonions: their geometry, arithmetic and symmetry*, A.K. Peters, Ltd., Natick, (2003).
- [134] R. Cooke, *The mathematics of Sonya Kovalevskaya*, Springer-Verlag, New York, (1984).
- [135] J. Cortés, E. Martínez, Mechanical control systems on Lie algebroids, *IMA J. Math. Control. Inform.* **21**, 457–492 (2004).
- [136] T. Courant, *Dirac Manifolds*, Ph.D. Thesis, University of California, Berkeley, (1987).

- [137] T. Courant, Dirac manifolds, *Trans. Amer. Math. Soc.* **319**, 631–661 (1990).
- [138] R. Courant, K.O. Friedrichs, *Supersonic flows and shock waves. Applied Mathematical Sciences* **21**, Springer, New York, (1991).
- [139] T. Courant, A. Weinstein, *Beyond Poisson structures*, in: *Action Hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), Travaux en Cours Vol. 27*. Hermann, Paris, 39–49 (1988).
- [140] M. Crampin, Homogeneous systems of higher-order ordinary differential equations, *Commun. Math.* **18**, 37–50 (2010).
- [141] G. Darboux, Sur la théorie des coordonnées curvilignes et les systèmes orthogonaux, *Ann. Ec. Norm. Supér.* **7**, 101–150 (1878).
- [142] D. David, D. Levi, P. Winternitz, Integrable nonlinear equations for water waves in straits of varying width and depth, *Stud. Appl. Math.* **76**, 133–168 (1987).
- [143] A. Degasperis, D.D. Holm, A.N.W. Hone, A new integral equation with peakon solutions, *Theor. Math. Phys.* **133**, 1463–1474 (2002).
- [144] M. Delgado, The Lagrange–Charpit Method, *SIAM* **39**, 298–304 (1997).
- [145] M.H. Dida, F. Hathout, M. Djaa, On the geometry of the second order tangent bundle with the diagonal lift metric, *Int. Journal of Math. Analysis* **3**, 443–456 (2009).
- [146] P.A.M. Dirac, Generalized Hamiltonian dynamics, *Canadian J. Math.* **2**, 129–148 (1950).
- [147] I. Dorfman, Dirac structures of integrable evolution equations, *Phys. Lett. A* **125**, 240–246 (1987).
- [148] I. Dorfman, Krichever–Novikov equations and local symplectic structures, *Sov. Math. Dokl.* **38**, 340–343 (1989).
- [149] P.G. Drazin, *Solitons, London Mathematical Society Lecture Note Series* **85**, Cambridge University Press, Cambridge, (1983).
- [150] V.S. Druyma, On the analytical solution of the two dimensional Korteweg de Vries equation, *Sov. Phys. JETP Lett.* **19**, 753–757 (1974).
- [151] B.A. Dubrovin, S.P. Novikov, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, *Russ. Math. Surv.* **44**, 35–124 (1989).

- [152] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, Reduction of presymplectic manifolds with symmetry, *Rev. Math. Phys.* **11**, 1209–1247 (1999).
- [153] L. Edelstein-Keshet, *Mathematical Models in Biology*. SIAM Classics in Applied Mathematics **46**, SIAM, Philadelphia, (2005).
- [154] A.I. Egorov, *Riccati equations*, Russian Academic Monographs 5, Pensoft Publ., Sofia-Moscow, (2007).
- [155] L. Erbe, Comparison theorems for second order Riccati equations with applications, *SIAM J. Math. Anal.* **8**, 1032–1037 (1997).
- [156] V.P. Ermakov, Second-order differential equations. Conditions of complete integrability, *Univ. Izv. Kiev Series* **9**, 1–25 (1880).
- [157] V.P. Ermakov, Second-order differential equations: conditions of complete integrability, *Appl. Anal. Discrete Math.* **2**, 123–145 (2008).
- [158] P.G. Estévez, A nonisospectral problem in $2 + 1$ dimensions derived from KP, *Inverse problems* **17**, 1043–1052 (2001).
- [159] P.G. Estévez, Reciprocal transformations for a spectral problem in $2 + 1$ dimensions, *Theor. and Math. Phys.* **159**, 763–769 (2009).
- [160] P.G. Estévez, Generalized Qiao hierarchy in $2+1$ dimensions: reciprocal transformations, spectral problem and non-isospectrality, *Phys. Lett. A* **375**, 537–540 (2011).
- [161] P.G. Estévez, E. Conde, P.R. Gordoá, Unified approach to Miura, Bäcklund and Darboux transformations for nonlinear partial differential equations, *J. Nonlin. Math. Phys.* **5**, 82–114 (1998).
- [162] P.G. Estévez, M.L. Gandarias, J. de Lucas, Classical Lie symmetries and reductions of a nonisospectral Lax pair, *J. Nonlin. Math. Phys.* **18**, 51–60 (2011).
- [163] P.G. Estévez, M.L. Gandarias, J. Prada, Symmetries reductions of a Lax Pair, *Phys. Lett. A* **343**, 40–47 (2005).
- [164] P.G. Estévez, P.R. Gordoá, Singular manifold method: Darboux transformations and nonclassical symmetries, *J. Nonlin. Math. Phys.* **2**, 334–355 (1995).
- [165] P.G. Estévez, G.A. Hernáez, Non-isospectral problem in $2 + 1$ dimensions, *J. Phys. A: Math. and Gen.* **33**, 1405–1413 (2000).
- [166] P.G. Estevez, F.J. Herranz, J. de Lucas, C. Sardón, Lie symmetries for Lie systems: applications to systems of ODEs and PDEs, arXiv:1404.2740 (2014).

-
- [167] P.G. Estévez, J.D. Lejarreta, C. Sardón, Non isospectral $1 + 1$ hierarchies arising from a Camassa–Holm hierarchy in $2 + 1$ dimensions, *J. Nonlin. Math. Phys.* **8**, 9–28 (2011).
- [168] P.G. Estévez, J.D. Lejarreta, C. Sardón, Integrable $1 + 1$ dimensional hierarchies arising from reduction of a non-isospectral problem in $2 + 1$ dimensions, *Appl. Math. Comput.* **224**, 311–324 (2013).
- [169] P.G. Estévez, J.D. Lejarreta, C. Sardón, Classical and nonclassical approach for a wave model in $2 + 1$ dimensions, In preparation.
- [170] P.G. Estévez, J. Prada, A generalization of the sine-Gordon equation to $2 + 1$ dimensions, *J. Nonlin. Math. Phys.* **11**, 164–179 (2004).
- [171] P.G. Estévez, J. Prada, Singular manifold method for an equation in $2 + 1$ dimensions, *J. Nonlin. Math. Phys.* **12**, 266–279 (2005).
- [172] P.G. Estévez, J. Prada, Hodograph transformations for a Camassa–Holm hierarchy in $2 + 1$ dimensions, *J. Phys. A* **38**, 1–11 (2005).
- [173] P.G. Estévez, J. Prada, Solutions of a Camassa-Holm hierarchy in $2 + 1$ dimensions, *Theor. Math. Phys.* **144**, 1132–1137 (2005).
- [174] P.G. Estévez, J. Prada, Lump solutions for PDE's: Algorithmic construction and classification, *J. Nonlin. Math. Phys.* **15**, 166–175 (2008).
- [175] P.G. Estévez, C. Sardón, Miura reciprocal Transformations for hierarchies in $2+1$ dimensions, *J. Nonlin. Math. Phys.* **20**, 552–564 (2013).
- [176] P.G. Estévez, C. Sardón, Miura reciprocal transformations for two integrable hierarchies in $1 + 1$ dimensions, *Proceedings GADEIS (2012)*, Protaras, Cyprus, (2012).
- [177] N. Euler, M. Euler, A tree of linearizable second-order evolution equation: Generalized hodograph transformations, *J. Nonlin. Math. Phys.* **8**, 770–778 (2011).
- [178] M. Euler, N. Euler, P. Leach, The Riccati and Ermakov-Pinney hierarchies, *J. Nonlin. Math. Phys.* **14**, 290–310 (2007).
- [179] N. Euler, M. Euler, S. Lundberg, Reciprocal Bäcklund transformations of autonomous evolution equations, *Theor. Math. Phys.* **159**, 770–778 (2009).
- [180] N. Euler, M.L. Gandarias, M. Euler, O. Lindblom, Auto-hodograph transformations for a hierarchy of nonlinear evolution equations, *J. Math. Anal. Appl.* **257**, 21–28 (2001).

- [181] L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, (1998).
- [182] W. Fair, Y.L. Luke, Rational approximations to the solution of the second order Riccati equation, *Math. Comp.* **20**, 602–606 (1966).
- [183] M.U. Farooq, F.M. Mahomed, M.A. Rashid, Integration of systems of ODEs via nonlocal symmetry-like operators, *Math. and Comp. Appl.* **15**, 585–600 (2010).
- [184] F. Fassò, Superintegrable Hamiltonian systems: geometry and applications, *Acta Appl. Math.* **87**, 93–121 (2005).
- [185] E.V. Ferapontov, Reciprocal transformations and their invariants, *Differential equations* **25**, 898–905 (1989).
- [186] E.V. Ferapontov, M.V. Pavlov, Reciprocal transformations of Hamiltonian operators of hydrodynamic type: nonlocal Hamiltonian formalism for nonlinearly degenerate systems, *J. Math. Phys.* **44**, 1150–72 (2003).
- [187] E.V. Ferapontov, C. Rogers, W.K. Schief, Reciprocal transformations of two component hyperbolic system and their invariants, *J. Math. Anal. Appl.* **228**, 365–376 (1998).
- [188] L.A. Ferreira, F.J. Gomes, A.V. Razumov, M.V. Saveliev, A.H. Zimerman, Riccati-type equations, generalized WZNW equations and multidimensional Toda systems, *Commun. Math. Phys.* **203**, 649–666 (1999).
- [189] Z. Fiala, Evolution equation of Lie-type for finite deformations, time-discrete integration, and incremental methods, *Acta Mech.* **226** 17–35 (2015).
- [190] R. Flores-Espinoza, *Monodromy factorization for periodic Lie systems and reconstruction phases*, in: *Geometric methods in Physics*. AIP Conf. Proc. **1079**, Amer. Inst. Phys., New York, 189–195 (2008).
- [191] R. Flores-Espinoza, Periodic first integrals for Hamiltonian systems of Lie type, *Int. J. Geom. Methods Mod. Phys.* **8**, 1169–1177 (2011).
- [192] R. Florez-Espinoza, J. de Lucas, Y.M. Vorobiev, Phase splitting for periodic Lie systems, *J. Phys. A* **43**, 205208 (2010).
- [193] R. Florez-Espinoza, Y.M. Vorobiev, On dynamical and geometric phases of time-periodic linear Euler equations, *Russ. J. Math. Phys.* **12**, 326–349 (2005).
- [194] A.S. Fokas, A.R. Its, A. Kapaev, V.Y. Novokshenov, *Painlevé transcendents, the Riemann-Hilbert approach*. *Mathematical Surveys and Monographs* **128**, American Mathematical Society, Rhode Island, (2006).

- [195] J. Fris, V. Mandrosov, Y.A. Smorodinsky, M. Uhler, P. Winternitz, On higher symmetries in quantum mechanics, *Phys. Lett.* **16**, 354–356 (1965).
- [196] G. Fröbenius, Über das Pfaffsche probleme, *J. für Reine und Angew. Math.* **82**, 230–315 (1877).
- [197] I.A. García, J. Giné, J. Llibre, Liénard and Riccati differential equations related via Lie algebras, *Discrete Contin. Dyn. Syst. Ser. B* **10**, 485–494 (2008).
- [198] C.S. Gardner, Korteweg de Vries equation and generalizations, IV. The Korteweg de Vries equation as a bihamiltonian system, *J. Math. Phys.* **12**, 1548 (2003).
- [199] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg de Vries equation, *Phys. Rev. Lett.* **19**, 1095–1097 (1967).
- [200] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Korteweg de Vries equation and generalization. IV Method for exact solution, *Comm. Pure Appl. Math.* **27**, 97–133 (1974).
- [201] C.F. Gauss, *Disquisitiones generales circa superficies curvas*, (1827).
- [202] C.F. Gauss, *General investigations of curves surfaces 1827-1825*, The Princeton University Library, (1902).
- [203] C.F. Gauss, *General investigations of curved surfaces unabridged*, Wexford College Press, (2007).
- [204] H. Geiges, *Contact Geometry, in Handbook of Differential Geometry. Vol. II*, Elsevier/North-Holland, Amsterdam, 315–382, (2006).
- [205] A. Gheorghie, Quantum and classical Lie systems for extended symplectic groups, *Rom. Journ. Phys.* **58**, 1436–1445 (2013).
- [206] A. Ghose Choudhury, P. Guha, B. Khanra, Solutions of some second order ODEs by the extended Prolle–Singer method and symmetries, *J. Nonlin. Math. Phys.* **15**, 365–382 (2008).
- [207] C.R. Gilson, A. Pickering, Factorization and Painlevé analysis of a class of nonlinear third-order partial differential equations, *J. Phys. A* **28**, 2871–2878 (1995).
- [208] H. Goldstein, *Classical Mechanics. Second Edition. Addison-Wesley series in Physics*, Addison-Wesley Publishing Co., Reading, (1980).
- [209] C.A. Gómez, A. Salas, Special symmetries to standard Riccati equations and applications, *Appl. Math. Comp.* **216**, 3089–3096 (2010).

- [210] A. González-López, N. Kamran, P.J. Olver, Lie algebras of vector fields in the real plane, *Proc. London Math. Soc.* **64**, 339–368 (1992).
- [211] K. Grabowska, J. Grabowski, Variational calculus with constraints on general algebroids, *J. Phys. A* **41**, 175204 (2008).
- [212] K. Grabowska, J. Grabowski, Dirac algebroids in Lagrangian and Hamiltonian mechanics, *J. Geom. Phys.* **61**, 2233–2253 (2011).
- [213] K. Grabowska, J. Grabowski, P. Urbański, Geometrical mechanics on algebroids, *Int. J. Geom. Meth. Mod. Phys.* **3**, 559–575 (2006).
- [214] J. Grabowski, M. Jóźwikowski, Pontryagin maximum principle on almost Lie algebroids, *SIAM J. Control Optim.* **49**, 1306–1357 (2011).
- [215] J. Grabowski, M. de León, J.C. Marrero, D. Martín de Diego, Nonholonomic constraints: a new viewpoint, *J. Math. Phys.* **50**, 013520 (2009).
- [216] J. Grabowski, J. de Lucas, Mixed superposition rules and the Riccati hierarchy, *J. Differential Equations* **254**, 179–198 (2013).
- [217] B. Grammaticos, A. Ramani, S. Lafortune, The Gambier mapping, revisited, *Phys. A* **253**, 260–270 (1998).
- [218] V. Gritsev, P. Barmettler, E. Demler, Scaling approach to quantum non-equilibrium dynamics of many-body systems, *New J. Phys.* **12**, 113005 (2010).
- [219] N.A. Gromov, *Contractions and Analytical Continuations of the Classical Groups. Unified Approach*, (Syktyvkar: Komi Scientific Center) (in russian), (1992).
- [220] C. Grosche, G.S. Pogosyan, A.N. Sissakian, Path integral discussion for Smorodinsky-Winternitz potentials: I. Two- and three dimensional Euclidean space, *Fortschr. Phys.* **43**, 453–521 (1995).
- [221] H. Grosse, Solitons of the modified KdV equation, *Lett. in Math. Phys.* **8**, 313–319 (1984).
- [222] B. Gruber, L. O’Raifeartaigh, *S* theorem and construction of the invariants of the semisimple compact Lie algebras, *J. Math. Phys.* **5**, 1796–1804 (1964).
- [223] M.A. Grundland, D. Levi, On higher-order Riccati equations as Bäcklund transformations, *J. Phys. A: Math. Gen.* **32**, 3931–3937 (1999).
- [224] P. Guha, A.G. Choudhury, B. Grammaticos, Dynamical studies of equations from the Gambier family, *SIGMA Symmetry Integrability Geom. Methods Appl.* **7**, 028 (2011).

- [225] A. Guldberg, Sur les équations différentielles ordinaires qui possèdent un système fondamental d'intégrales, *C.R. Math. Acad. Sci. Paris* **116**, 964–965 (1893).
- [226] F. Haas, J. Goedert, Lie point symmetries for reduced Ermakov systems, *Phys. Lett. A* **332**, 25–34 (2004).
- [227] G. Halphen, Sur un système d'équations différentielles, *C.R. Acad. Sci. Paris* **92**, 1101–1103 (1881).
- [228] J. Harnad, P. Winternitz, R.L. Anderson, Superposition principles for matrix Riccati equations, *J. Math. Phys.* **24**, 1062–1072 (1983).
- [229] S.E. Harris, P.A. Clarkson, Painlevé analysis and similarity reductions for the magma equation, *SIGMA* **2**, 068 (2006).
- [230] M. Havlíček, S. Post, P. Winternitz, Nonlinear superposition formulas based on imprimitive group action, *J. Math. Phys.* **40**, 3104–3122 (1999).
- [231] M. Hazewinkel, *Hyperbolic partial differential equations, numerical methods* Encyclopedia of mathematics, Springer, (2001).
- [232] R. Hermann, Cartanian Geometry, nonlinear waves and Control theory, Part A: *Math. Sci. Press*, Brookline, Mass (1979).
- [233] R. Hermann, Cartanian Geometry, nonlinear waves and Control theory, Part B: *Math. Sci. Press*, Brookline, (1980).
- [234] F.J. Herranz, A. Ballesteros, Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature, *SIGMA* **2**, 010 (2006).
- [235] F.J. Herranz, A. Ballesteros, M. Santander, T. Sanz-Gil, *Maximally superintegrable Smorodinsky-Winternitz systems on the N-dimensional sphere and hyperbolic spaces*, in: *Superintegrability in classical and quantum systems*, CRM Proc. Lect. Notes **37**, Amer. Math. Soc., Providence, 75–89, (2004).
- [236] A. Blasco, F.J. Herranz, J. de Lucas, C. Sardón, Lie–Hamilton systems on the plane: applications and superposition rules, arXiv:1410.7336, (2015).
- [237] F.J. Herranz, J. de Lucas, C. Sardón, Jacobi–Lie systems: fundamentals and low-dimensional classification, Proceedings AIMS, 2014, Madrid. arXiv:1412.0300, (2014).
- [238] F.J. Herranz, M. Santander, Casimir invariants for the complete family of quasisimple orthogonal algebras, *J. Phys. A: Math. Gen.* **30**, 5411–5426 (1997).

- [239] J. Hietarinta, *Recent results from the search for bilinear equations having three soliton solutions*, in: *Nonlinear evolution equations: integrability and spectral methods*, Manchester University Press, Manchester, 307–318 (1990).
- [240] J. Hietarinta, *Hirota's bilinear method and partial integrability*, in: *Partially integrable evolution equations in Physics. NATO ASI Series* **310**, 459–478 (1990).
- [241] E. Hille, *Ordinary differential equations in the complex domain*, Dover Publication, Inc., New York, (1997).
- [242] P. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.* **27**, 1192 (1971).
- [243] R. Hirota, *Direct methods in soliton theory*, in: *Solitons. Topics in current physics* **17**, Springer–Verlag, Berlin, 157–176 (1980).
- [244] R. Hirota, *The direct method in soliton theory*, Cambridge University Press, Cambridge, (2004).
- [245] A.N.W. Hone, Reciprocal link for 2 + 1 dimensions extensions of shallow water equations, *Appl. Math. Lett.* **13**, 37–42 (2000).
- [246] A.N.W. Hone, J.P. Wang, Prolongation algebras and Hamiltonian operators for peakon equations, *Inverse Problems* **19**, 129–145 (2003).
- [247] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure and Appl. Math.* **3**, 201–230 (1950).
- [248] J. Hoppe, *Lectures on integrable systems, Lect. Notes Phys.* **10**, New Series m: Monographs, Springer–Verlag, Berlin Heidelberg, (1992).
- [249] V. Hussin, J. Beckers, L. Gagnon, P. Winternitz, Nonlinear differential equations and Lie superalgebras, *Lett. Math. Phys.* **13**, 113–120 (1987).
- [250] V. Hussin, J. Beckers, L. Gagnon, P. Winternitz, Superposition formulas for nonlinear superequations, *J. Math. Phys.* **31**, 2528–2534 (1990).
- [251] P. E. Hydon, *Symmetry methods for differential equations: A beginner's guide*, Cambridge University Press, (2000).
- [252] N.H. Ibragimov, Discussion of Lies nonlinear superposition theory, in: *MOGRAM 2000, Modern group analysis for the new millenium, USATU Publishers, Ufa*, 116–119, (2000).
- [253] N.H. Ibragimov, Sur l'équivalence des équations d'évolution, qui admettent une algèbre de Lie–Bäcklund infinie, *C. R. Acad. Sci. Paris Ser. I* **293**, 657–660 (1981).

- [254] N.H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*, J. Wiley & Sons, Chichester, (1999).
- [255] N.H. Ibragimov, Utilization of canonical variables for integration of systems of first-order differential equations, *ALGA* **6**, (2009).
- [256] N.H. Ibragimov, A.V. Aksenov, V.A. Baikov, V.A. Chugunov, R.K. Azizov, A.G. Meshkov, *CRC handbook of Lie group analysis of differential equations. Vol. 2. Applications in engineering and physical sciences*, CRC Press, Boca Raton, (1995).
- [257] E.L. Ince, *Ordinary differential equations*, Dover Publications, New York, (1956).
- [258] A. Inselberg, Superpositions for nonlinear operators: I. Strong superpositions and linearizability, *J. Math. Anal. Appl.* **40**, 494–508 (1972).
- [259] M.V. Ioffe, H.J. Korsch, Nonlinear supersymmetric (Darboux) covariance of the Ermakov–Milne–Pinney equation, *Phys. Lett. A* **311**, 200–205 (2003).
- [260] H.N.A. Ismail, K. Raslam, G. Salem, Solitary wave solution for the general Korteweg de Vries equation by Adomian decomposition method, *Appl. Math. and Comput.* **154**, 17–29 (2004).
- [261] K. Itô, *Encyclopedic dictionary of mathematics. Vol. I-IV. Second edition*, MIT Press, (1987).
- [262] R.I. Ivanov, Equations of the Camassa Holm hierarchy, *Theor. Math. Phys.* **160**, 952–959 (2009).
- [263] Z. Jin, H. Maoan, L. Guihua, The persistence in a Lotka–Volterra competition systems with impulsive, *Chaos, solitons and Fractals* **24**, 1105–1117 (2005).
- [264] M. Jotz, T. Ratiu, Induced Dirac structures on isotropy-type manifolds, *Transform. Groups* **16**, 175–191 (2011).
- [265] B.B. Kadomtsev, V.I. Petviashvili, On stability of solitary waves in weakly dispersive media, *Sov. Phys. Dokl.* **15**, 539–541 (1970).
- [266] A. Karasu, P.G.L. Leach, Nonlocal symmetries and integrable ordinary differential equations: $\ddot{x} + 3x\dot{x} + x^3 = 0$ and its generalisations, *J. Math. Phys.* **50**, 073509 (2009).
- [267] R.S. Kaushal, Construction of Exact Invariants for Time Dependent Classical Dynamical Systems, *Int. J. Theor. Phys.* **37**, 1793–1856 (1998).

- [268] R.S. Kaushal, H.J. Korsch, Dynamical Noether invariants for time-dependent nonlinear systems, *J. Math. Phys.* **22**, 1904–1908 (1981).
- [269] J.G. Kingston, C. Rogers, Reciprocal Bäcklund transformations of conservation laws, *Phys. Lett. A* **92**, 261–264 (1982).
- [270] A. Kirillov, Local Lie algebras, *Uspekhi Mat. Nauk.* **31**, 57–76 (1976).
- [271] L. Königsberger, Über die einer beliebigen differentialgleichung erster ordnung angehörigen selbständigen transcendenten, *Acta Math.* **3**, 1–48 (1883).
- [272] B.G. Konopelchenko, *Introduction to multidimensional integrable equations*, the inverse spectral transform in $2 + 1$ dimensions, Plenum Press, New York and London, (1992).
- [273] D.J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a New Type of Long Stationary Waves, *Philosophical Magazine* **39**, 42–443 (1895).
- [274] V. Kravchenko, V. Kravchenko, B. Williams, A quaternionic generalization of the Riccati differential equation, in: *Clifford analysis and its applications*, Kluwer Acad. Publ. 143–154 (2001).
- [275] M.D. Kruskal, N.J. Zabusky, Progress on the Fermi-Pasta-Ulam nonlinear string problem, Princeton Plasma Physics laboratory Annual Rep. Technical report, MATT-Q-21, Princeton, (1963).
- [276] E.S. Kryachko, Few Sketches on Connections Between the Riccati and Ermakov–Milne–Pinney Equations, *Int. J. Quantum Chem.* **109**, 2897–2902 (2009).
- [277] N. Kudryashov, A. Pickering, Rational solutions for Schwarzian integrable hierarchies, *J. Phys. A: Math. Gen.* **31**, 9505–9518 (1998).
- [278] S. Lafortune, P. Winternitz, Superposition formulas for pseudounitary matrix Riccati equations, *J. Math. Phys.* **37**, 1539–1550 (1996).
- [279] L.D. Landau, E.M. Lifshitz, *The classical theory of fields*, **2** Pergamin Press, Butterworth-Heinemann (1975).
- [280] L.D. Landau, E.M. Lifshitz, *Mechanics. Course of Theoretical Physics* **1**, Pergamon Press, Oxford-London-New York-Paris, (1976).
- [281] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21**, 467–490 (1968).

- [282] P.D. Lax, R.S. Phillips, *Scattering Theory for Automorphic Functions*, Annals of Mathematics Studies 87, Princeton University Press, Princeton, (1976).
- [283] J.A. Lázaro-Camí, J.P. Ortega, Superposition rules and stochastic Lie-Scheffers systems, *Ann. Inst. H. Poincaré Probab. Stat.* **45**, 910–931 (2009).
- [284] P.G.L. Leach, First integrals for the modified Emden equation $\ddot{q} + \alpha(t)q + q^n = 0$, *J. Math. Phys.* **26**, 2510–2514 (1985).
- [285] P.G.L. Leach, Symmetry and singularity properties of the generalized Kummer–Schwarz and related equations, *J. Math. Anal. Appl.* **348**, 487–493 (2008).
- [286] P.G.L. Leach, Generalized Ermakov systems, *Phys. Lett. A* **158**, 102–106 (1991).
- [287] P.G.L. Leach, K. Andriopoulos, The Ermakov equation: A commentary, *Appl. Anal. Discrete Math.* **2**, 146–157 (2008).
- [288] P.G.L. Leach, K.G. Govinder, On the uniqueness of the Schwarzian and linearization by nonlocal contact transformation, *J. Math. Anal. Appl.* **235**, 84–107 (1999).
- [289] M. Legare, Symmetry reductions of the Lax Pair of the four dimensional Euclidean Self Dual Yang Mills equation, *J. Nonlin. Math. Phys.* **3**, 266–285 (1996).
- [290] M. de León, J.C. Marrero, E. Martínez, Lagrangian submanifolds and dynamics on Lie algebroids, *J. Phys. A* **38**, R241–R308 (2005).
- [291] D. Levi, M.C. Nucci, C. Rogers, P. Winternitz, Group theoretical analysis of a rotating shallow liquid in a rigid container, *J. Phys. A: Math. Gen.* **22**, 4743–4767 (1989).
- [292] J.J. Levin, On the matrix Riccati equation, *Proc. Amer. Math. Soc.* **10**, 519–524 (1959).
- [293] P. Libermann, C.M. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publishing Co., Dordrecht, (1987).
- [294] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry* **12**, 253–300 (1977).
- [295] S. Lie, Theorie der Transformationsgruppen I, *Math. Ann.* **16**, 441–528 (1880).
- [296] S. Lie, Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen, *Arch. for Math.* **6**, 328–368 (1881).

- [297] S. Lie, Allgemeine Untersuchungen über Differentialgleichungen, die eine kontinuierliche endliche Gruppe gestatten, *Math. Ann.* **25**, 71–151 (1885).
- [298] S. Lie, *Theorie der Transformationgruppen* **2**, Teubner, Leipzig, (1890).
- [299] S. Lie, *Theorie der Transformationgruppen* **3**, Teubner, Leipzig, (1893).
- [300] S. Lie, Sur une classe d'équations différentielles qui possèdent des systèmes fondamentaux d'intégrales, *C.R. Math. Acad. Sci. Paris* **116**, 1233–1236 (1893).
- [301] S. Lie, *On differential equations possessing fundamental integrals*, Leipziger, Berichte, (1893).
- [302] S. Lie, *Theorie der Transformationsgruppen Dritter Abschnitt*, Teubner, Leipzig, (1893).
- [303] S. Lie, G. Scheffers, *Vorlesungen über kontinuierliche Gruppen mit geometrischen und anderen Anwendungen*, Edited and revised by G. Scheffers, Teubner, Leipzig, (1893).
- [304] J. Liouville, Note sur les équations de la dynamique, *J. Math. Pures Appl.* **20**, 137–138 (1855).
- [305] J. Llibre, D. Peralta-Salas, A note on the first integrals of vector fields with integrating factors and normalizers, *SIGMA Symmetry Integrability Geom. Methods Appl.* **8**, 035 (2012).
- [306] J. Llibre, C.Valls, Liouvillian first integrals of quadratic-linear polynomial differential systems, *J. Math. Anal. Appl.* **379**, 188–199 (2011).
- [307] P.G. Luan, C.S. Tang, Lewis-Riesenfeld approach to the solutions of the Schrödinger equation in the presence of a time-dependent linear potential, *Phys. Rev. A* **71**, 014101 (2005).
- [308] J. de Lucas, C. Sardón, On Lie systems and Kummer–Schwarz equations, *J. Math. Phys.* **54**, 033505 (2013).
- [309] J. de Lucas, C. Sardón, A Lie systems approach to Kummer–Schwarz equations, *J. Math. Phys.* **54**, 033505 (2013).
- [310] J. de Lucas, M. Tobolski, S. Vilariño, A new application of k -symplectic Lie systems, to appear in *Int. J. Geom. Methods Mod. Physics*, arXiv:1404.1596 (2015).
- [311] J. de Lucas, S. Vilariño, k -symplectic Lie systems: theory and applications, *J. of Differential Equations* **256**, 2221–2255 (2015).

- [312] W.X. Ma, Lax representations and Lax operator algebras of isospectral and nonisospectral hierarchies of evolution equations, *J. Math. Phys.* **33**, 2464–2476 (1992).
- [313] M. Maamache, Ermakov systems, exact solution, and geometrical angles and phases, *Phys. Rev. A* **52**, 936–940 (1995).
- [314] A.A. Makarov, Ya.A. Smorodinsky, Kh.Valiev, P.Winternitz, A systematic search for nonrelativistic systems with dynamical symmetries, *Nuovo Cimento A* **52**, 1061 (1967).
- [315] S.V. Manakov, P.M. Santini, A hierarchy of integrable partial differential equations in $2 + 1$ dimensions associated with one parameter families of one dimensional vector fields, *Theor. Math. Phys.* **152**, 1004–1011 (2007).
- [316] V.A. Marchenko, *Sturm–Liouville operators and applications. Second Edition*, American Mathematical Society, Providence, (2011).
- [317] A. Marino, Topological methods, variational inequalities and elastic bounce trajectories, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **22**, 269–290 (2011).
- [318] M. Mariton, P. Bertrand, A homotopy algorithm for solving coupled Riccati equations, *Potim. Contr. Appl. Met.* **6**, 351–357 (1985).
- [319] C.M. Marle, Calculus on Lie algebroids, Lie groupoids and Poisson manifolds, *Diss. Math. (Rozprawy Mat.)* **457**. Warsaw: Institute of Mathematics of the Polish Academy of Science, (2008).
- [320] A.J. Marshall, *A new, harder proof that continuous functions with Schwarz derivative 0 are lines*, in: *Fourier analysis*, Lecture Notes in Pure and Appl. Math. **157**, 35–46 (1994).
- [321] E. Martínez, Lagrangians mechanics on Lie algebroids, *Acta Appl. Math.* **67**, 295–320 (2001).
- [322] J.B. McLeod, P.J. Olver, The connection between partial differential equations soluble by the inverse scattering and ordinary differential equations of Painlevé type, *SIAM J. Math. Anal.* **14**, 488–506 (1983).
- [323] G. Meng, Y. Gao, X. Yu, Y. Shen, Y. Qin, Multi-soliton solutions for the coupled nonlinear Schrödinger type equation, *Nonlin. Dynamics* **70**, 609–617 (2012).
- [324] A.V. Mikhailov, *Integrability. Lect. Notes Phys.* **767**, Springer, Berlin–Heidelberg, (2009).

- [325] A.A. Minzoni, N.F. Smyth, Evolution of lump solutions for the KP equation, *Wave Motion* **24**, 291–305 (1996).
- [326] R.M. Miura, Korteweg de Vries equation and generalizations. A remarkable explicit non-linear transformation, *J. Math. Phys.* **9**, 1202–1204 (1968).
- [327] Z. Nehari, The Schwarzian derivative and Schlicht functions, *Bull. Amer. Math. Soc.* **55**, 545–551 (1949).
- [328] F. Nijhoff, On some "Schwarzian" equations and their discrete analogues, in: Algebraic aspects of integrable systems, Progr. Nonlinear Differential Equations Appl. **26**, Birkhäuser Boston, Boston, MA, 237–260 (1997).
- [329] M. Nowakowski, H.C. Rosu, Newton's laws of motion in form of Riccati equation, *Phys. Rev. E* **65**, 047602 (2002).
- [330] M.C. Nucci, The complete Kepler group can be derived by Lie group analysis, *J. Math. Phys.* **37**, 1772–1775 (1996).
- [331] M.C. Nucci, The role of symmetries in solving differential equations, *Mathl. Comput. Modelling* **25**, 181–193 (1997).
- [332] M.C. Nucci, Nonclassical symmetries as special solutions of heir-equations, *Math. Anal. Appl.* **279**, 168–179 (2003).
- [333] M.C. Nucci, Jacobi last multiplier and Lie symmetries: A novel application of an old relationship, *J. Nonlin. Math. Phys.* **12**, 284–304 (2005).
- [334] M.C. Nucci, W.F. Ames, Classical and nonclassical symmetries of the Helmholtz equation, *J. Math. An. Appl.* **178**, 584–591 (1993).
- [335] A. Odziejewicz, A.M. Grundland, The Superposition Principle for the Lie Type First Order PDEs, *Rep. Math. Phys.* **45**, 293–306 (2000).
- [336] W. Oevel, C. Rogers, Gauge transformations and reciprocal links in $2 + 1$ dimensions, *Rev. Math. Phys.* **5**, 299–330 (1993).
- [337] M.A. del Olmo, M.A. Rodríguez, P. Winternitz, Simple subgroups of simple Lie groups and nonlinear differential equations with superposition principles, *J. Math. Phys.* **27**, 14–23 (1986).
- [338] M.A. del Olmo, M.A. Rodríguez, P. Winternitz, Superposition formulas for the rectangular matrix Riccati equations, *J. Math. Phys.* **28**, 530–535 (1987).
- [339] P.J. Olver, *Applications of Lie Groups to Differential equations. Second Edition. Graduate Texts in Mathematics* **107**, Springer–Verlag, New York, (1993).

- [340] P.J. Olver, J. Peter, *Applications of Lie groups to differential equations*, Graduate Texts in Mathematics **107**, Springer-Verlag, (1993).
- [341] P.J. Olver, P. Rosenau, The construction of special solutions to partial differential equations, *Phys. Lett. A* **114**, 107–112 (1986).
- [342] P.J. Olver, P. Rosenau, Group invariant solutions of differential equations, *SIAM J. Appl. Math* **47**, 263–278 (1987).
- [343] R. Ortega, The complex periodic problem for a Riccati equation, *Ann. Univ. Buchar. Math. Ser. 3*, 219–226 (2012).
- [344] J.P. Ortega, T.S. Ratiu, *Momentum maps and Hamiltonian reduction*. *Prog. Math.* **222**, Boston: Birkhäuser Boston, Inc. (2004).
- [345] V. Ovsienko, S. Tabachnikov, *Projective differential geometry old and new: from the Schwarzian derivative to cohomology of diffeomorphism groups*, Cambridge University Press, Cambridge, (2005).
- [346] V. Ovsienko, S. Tabachnikov, What is the Schwarzian derivative, *Notices of the AMS* **56**, 34–36 (2009).
- [347] P. Painlevé, Memoire sur les équations différentielles dont l'intégrale générale est uniforme, *Bull. Soc. Math. Phys. France* **28**, 201–261 (1900).
- [348] P. Painlevé, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math* **25**, 1–85 (1902).
- [349] R.S. Palais, Global formulation of the Lie theory of transformation groups, *Mem. Amer. Math. Soc.* **22**, 1–123 (1957).
- [350] J. Patera, R.T. Sharp, P. Winternitz, Invariants of real low-dimensional Lie algebras, *J. Math. Phys.* **17**, 986–994 (1976).
- [351] A.M. Perelomov, V.S. Popov, Casimir operators for semisimple Lie groups, *Math. USSR-Izvestija* **2**, 1313–1335 (1968).
- [352] N.P. Petrov, The dynamical Casimir effect in a periodically changing domain: a dynamical systems approach, *J. Opt. B: Quantum Semiclass. Opt.* **7**, S89 (2005).
- [353] G. Pietrzkowski, Explicit solutions of the \mathfrak{a}_1 -type Lie-Scheffers system and a general Riccati equation, *J. Dyn. Control Syst.* **18**, 551–571 (2012).
- [354] E. Pinney, The nonlinear differential equation $y'' + p(x)y + cy - \vartheta = 0$, *Proc. Amer. Math. Soc.* **1**, 681 (1950).

-
- [355] A.D. Polyanin, *Handbook of linear partial differential equations for engineers and scientists*, Chapman and hall/CRC Press, Boca Raton, (2002).
- [356] M. Postnikov, *Lie Groups and Lie Algebras. Lectures in Geometry. Semester V* Moscow: Mir (1986).
- [357] E. Pucci, Similarity reductions of partial differential equations, *J. Phys. A* **25**, 2631–2640 (1992).
- [358] E. Pucci, G. Saccomandi, On the reduction methods for ordinary differential equations, *J. Phys. A* **35**, 6145–6155 (2002).
- [359] Z. Qiao, New integrable hierarchy, parametric solutions, cuspons, one peak solutions, M/W shape peak solution, *J. Math. Phys.* **48**, 0822701-20 (2010).
- [360] Z. Qiao, X. Li, An integrable equation with nonsmooth solitons, *Theor. Math. Phys.* **167**, 584–589 (2011).
- [361] Z. Qiao, L. Liu, A new integrable equation with no smooth solitons, *Chaos Solitons Fract.* **41**, 587–593 (2009).
- [362] J. Ramírez, M.S. Bruzón, C. Muriel, M.L. Gandarias, The Schwarzian Korteweg–de Vries equation in $(2 + 1)$ dimensions, *J. Phys. A* **36**, 1467–1484 (2003).
- [363] A. Ramos, *A connection approach to Lie systems*, in: Proceedings of the XI Fall Workshop on Geometry and Physics, *Publ. R. Soc. Mat. Esp.* **6**, 235–239 (2004).
- [364] J. Riccati, Animadversiones in aequationes differentiales secundi gradus, Actorum Eruditorum, quae Lipsiae publicantur, *Supplementa* **8**, 66–73 (1724).
- [365] B. Riemann, *Gesammelte mathematische Werke und wissenschaftlicher Nachlass, Sändig Reprint*, Grundlagen für eine allgemeine Theorie der Functionen einer veränderlich complexen Grösse, (1851).
- [366] B. Riemann, *Gesammelte mathematische Werke und wissenschaftlicher Nachlass, Sändig Reprint*, Über die Hypothesen, welche der Geometrie zu Grunde liegen, (1854).
- [367] C. Rogers, Application of a reciprocal transformation to a two-phase Stefan Problem, *J. Phys. A* **18**, L105–L109 (1985).
- [368] C. Rogers, Reciprocal transformations in $(2 + 1)$ dimensions, *J. Phys. A* **19**, L491–L496 (1986).

- [369] C. Rogers, The Harry Dym equation in $2 + 1$ dimensions: A reciprocal link with the Kadomtsev-Petviashvili equation, *Phys. Lett. A* **120**, 15–18 (1987).
- [370] C. Rogers, S. Carillo, On reciprocal properties of the Caudrey, Dodd–Gibbon and Kaup–Kuppersmidt hierarchies, *Phys. Scripta* **36**, 865–869 (1987).
- [371] C. Rogers, J.G. Kingston, W.F. Shadwick, On reciprocal type invariant transformations in magneto-gas dynamics, *J. Math. Phys.* **21**, 395–397 (1980).
- [372] C. Rogers, M.C. Nucci, On reciprocal Bäcklund transformations and the Korteweg de Vries hierarchy, *Phys. Scripta* **33**, 289–292 (1986).
- [373] C. Rogers, W.K. Schief, Multi-component Ermakov Systems: Structure and Linearization, *J. Math. Anal. Appl.* **198**, 194–220 (1996).
- [374] C. Rogers, W.K. Schief, P. Winternitz, Lie-theoretical generalization and discretization of the Pinney equation, *J. Math. Anal. Appl.* **216**, 246–264 (1997).
- [375] C. Rogers, W.F. Shadwick, *Bäcklund transformations and their applications*, Academic Press, New York, (1982).
- [376] D.E. Rourke, M.P. Augustine, Exact Linearization of the Radiation-Damped Spin System, *Phys. Rev. Lett.* **84**, 1685–1688 (2000).
- [377] B.A. Rozenfeld, *A history of non-euclidean geometry*, Springer, New York, (1988).
- [378] T. Rybicki, On automorphisms of a Jacobi manifold, *Univ. Iagel. Acta Math.* **38**, 89–98 (2000).
- [379] S.Y. Sakovich, On Miura transformations of evolution equations, *J. Phys. A: Math. Gen.* **26**, L369–L373 (1993).
- [380] C. Sardón, P.G. Estévez, Miura-Reciprocal Transformations for two Integrable Hierarchies in $1+1$ Dimensions, pp. 203–211, *Proc. 6th Workshop “Group Analysis of Differential Equations & Integrable Systems” (Protaras, Cyprus, 2012)*, University of Cyprus, Nicosia, (2013).
- [381] A. Sebbar, A. Sebbar, Eisenstein series and modular differential equations, *Canad. Math. Bull.* **55**, 400–409 (2012).
- [382] G. Segal, The geometry of the Korteweg de Vries equation, *Int. J. Mod. Phys. A* **6**, 2859 (1991).
- [383] S.I. Senashov, A. Yakhno, Conservation laws, hodograph transformation and boundary value problems of plane plasticity, *SIGMA* **071**, (16 pages) (2012).

- [384] S. Shnider, P. Winternitz, Classification of systems of nonlinear ordinary differential equations with superposition principles, *J. Math. Phys.* **25**, 3155–3165 (1984).
- [385] S. Shnider, P. Winternitz, Nonlinear equations with superposition principles and the theory of transitive primitive Lie algebras, *Lett. Math. Phys.* **8**, 69–78 (1984).
- [386] M. Sorine, P. Winternitz, Superposition rules for solutions of differential matrix Riccati equations arising in Control theory, *IEEE Trans. Automat. Control* **30**, 266–272 (1985).
- [387] W.H. Steeb, N. Euler, *Nonlinear Evolution Equations and Painlevé test. Computational and applied mathematics, II*, North-Holland, Amsterdam, (1992).
- [388] H. Stephani, *Differential equations: their solution using symmetries*, Cambridge University Press, Cambridge, (1990).
- [389] E. Suazo, S. Suslov, K. Sergei, M. Vega-Guzmán, The Riccati differential equation and a diffusion-type equation, *New York J. Math.* **17A**, 225–244 (2011).
- [390] E. Suazo, K.S. Suslov, J.M. Vega-Guzmán, The Riccati system and a diffusion-type equation, *Mathematics 2014* **2**, 96–118 (2014).
- [391] H. Sussmann, Orbits of families of vector fields and integrability of systems with singularities, *Bull. Amer. Math. Soc.* **79**, 197–199 (1973).
- [392] S. Tabachnikov, *On zeros of the Schwarzian derivative*, in: *Topics in singularity theory*, *Amer. Math. Soc. Transl. Ser. 2* **180**, Amer. Math. Soc., 229–239 (1997).
- [393] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics. An Introduction*, John–Wiley & Sons, New York, (1989).
- [394] S. Tanaka, Some remarks on the modified Korteweg de Vries equation, *Publ. RIMS, Kyoto Univ.* **8**, 429–437 (1972).
- [395] S. Tian, T. Zhang, H. Zhang, Binary Darboux–Bäcklund transformation and new singular soliton solutions for the nonisospectral Kadomtsev–Petviashvili equation, *Int. J. Nonlin. Sci.* **9**, 422–429 (2010).
- [396] A. Tongas, D. Tsoubelis, P. Xenitidis, Integrability aspects of a Schwarzian PDE, *Phys. Lett. A* **284**, 266–274 (2011).
- [397] V. Torrisi, M.C. Nucci, *Application of Lie group analysis to a mathematical model which describes HIV transmission*, in: *The Geometrical Study of Differential Equations*, A.M.S., Providence, 11–20 (2001).

- [398] S.P. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type: the generalized hodograph method, *Math. USSR Izv.* **37**, 397–419 (1991).
- [399] A. Tsitskishvili, Solution of the Schwarz differential equation, *Mem. Differential Equations Math. Phys.* **11**, 129–156 (1997).
- [400] D.P. Tsvetkov, A periodic Lotka–Volterra system, *Serdica Math. J.* **22**, 109–116 (1996).
- [401] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, *Progress in Mathematics* **118**, Birkhäuser Verlag, Basel, (1994).
- [402] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, *Graduate Texts in Mathematics* **102**, Springer-Verlag, New York, (1984).
- [403] M.E. Vessiot, Sur une classe d'équations différentielles, *Ann. Sci. École Norm. Sup.* **10**, 53–64 (1893).
- [404] M.E. Vessiot, Sur une classe d'équations différentielles, *C.R. Math. Acad. Sci. Paris* **116**, 959–961 (1893).
- [405] M.E. Vessiot, Sur les systèmes d'équations différentielles du premier ordre qui ont des systèmes fondamentaux d'intégrales, *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.* **8**, H1–H33 (1894).
- [406] Y. Vorobiev, On Poisson realizations of transitive Lie algebroids, *J. Nonlin. Math. Phys.* **11**, 43–48 (2004).
- [407] M. Wadati, The exact solution of the modified Korteweg de Vries equation, *J. Phys. Soc. Japan* **32**, 1681 (1972).
- [408] M. Wadati, K. Sawada, New representations of the soliton solution for the Korteweg de Vries equation, *J. Phys. Soc Jpn* **52**, 749–760 (1983).
- [409] R.M. Wald, *General Relativity*, Chicago University Press, Chicago, (1984).
- [410] Y. Wang-Chuan, L. Biao, W. Jia, Similarity reductions of nonispectral BKP equation, *Commun. Theor. Phys.* **52**, 565 (2009).
- [411] A.M. Wazwaz, Analytic study on the one and two spatial dimensional potential KdV equations, *Chaos, Solitons and Fractals* **36**, 175–181 (2008).
- [412] A.M. Wazwaz, New higher dimensional fifth-order non-linear equations with multiple soliton solutions, *Phys. Scr.* **84**, 025007 (2011).
- [413] N. Weaver, Sub-Riemannian metrics for quantum Heisenberg manifolds, *J. Operator Theory* **43**, 223–242 (2000).

- [414] A. Weinstein, The local structure of Poisson manifolds, *J. Differential Geom.* **18**, 523–557 (1983).
- [415] J. Weiss, The Painlevé property for partial differential equations II, Bäcklund transformation, Lax pairs and Schwarzian derivative, *J. Math. Phys.* **24**, 1405–1413 (1983).
- [416] J. Weiss, *Solitons in Physics, Mathematics and nonlinear Optics. IMA Series* **17**, Springer-Verlag, New York, (1990).
- [417] J. Weiss, M.J. Tabor, G. Carnevale, The Painlevé Property for partial differential equations, *J. Math. Phys.* **24**, 522–526 (1983).
- [418] P. Wilczyński, Quaternionic-valued ordinary differential equations. The Riccati equation, *J. Differential Equations* **247**, 2163–2187 (2007).
- [419] P. Winternitz, Lie groups and solutions of nonlinear differential equations. In: *Nonlinear phenomena. Lect. Notes Phys.* **189**, Springer-Verlag, Oaxtepec, 263–331 (1983).
- [420] P. Winternitz, A. Smorodinsky, M. Uhler, J. Fris, Symmetry groups in classical and quantum mechanics, *Soviet J. Nuclear Phys.* **4**, 444–50 (1967).
- [421] E. Witten, Dynamical breaking of supersymmetry, *Nucl. Phys. B* **188**, 513–554 (1981).
- [422] T. Xue, Y. Zhang, Bihamiltonian systems of hydrodynamic type and reciprocal transformations, *Lett. Math. Phys.* **75**, 79–92 (2006).
- [423] I.M. Yaglom, A simple non-euclidean geometry and its physical basis. An elementary account of Galilean geometry and the Galilean principle of relativity, Springer-Verlag, New York, (1979).
- [424] L. You-Ning, H. Hua-Jun, Nonlinear dynamical symmetries of Smorodinsky–Winternitz and Fokas–Lagerstorm systems, *Chinese Phys. B* **20**, 010302 (2011).
- [425] S.J. Yu, K. Toda, Lax Pairs, Painlevé properties and exact solutions of the Calogero Korteweg de Vries equation and a new (2+1)-dimensional equation, *J. Nonlinear Math. Phys.* **7**, 1–13 (2000).
- [426] V.E. Zakharov, S.V. Manakov, On the complete integrability of a nonlinear Schrodinger equation, *J. Theor. Math. Phys.* **19**, 551–559 (1974).
- [427] V.E. Zakharov, A.B. Shabat, Exact theory of two dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. -JETP*, **34**, 62–69 (1972).

- [428] W.M. Zhang, D.H. Feng, R. Gilmore, Coherent states: theory and some applications, *Rev. Mod. Phys.* **62**, 867–927 (1990).
- [429] V.V. Zharinov, *Lecture notes of geometrical aspects of partial differential equations*, Series on Soviet and East European Mathematics **9**, (1992).
- [430] N.T. Zung, Action-angle variables on Dirac manifolds, arXiv.org:1204.3865 (2013).