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Approximation of Immersed Surfaces Into a Tetrahedral Mesh Generated by the Meccano Method

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Abstract— In this paper, we present a new method to insert open surfaces into an existing tetrahedral mesh generated by the meccano method. The surfaces must be totally immersed in the mesh and must not intersect between them. The strategy includes a mesh refinement to obtain an initial approximation of each surface capturing its geometric features, the projection of the nodes from the approximation onto the actual surface, and the mesh optimization. The proposed method provides a high-quality conformal mesh with interpolations of the inserted surfaces. These approximations are suitable for operations where roughness is a major problem and a smoother solution is required, such as the estimation of normal vectors or the imposition of Neumann conditions.

Keywords: Meccano mesh, Kossaczký refinement, surface parameterization, simultaneous untangling and smoothing, element quality.

1 Introduction

In a wide range of application in numerical simulation, the insertion of surfaces in the geometric model of the problem is required. Such applications comprise those related with dynamic fronts that depends on the simulation results, interfaces between subdomains, moving surfaces or the inclusion of new features in an already existing geometry.

A first approach to address the approximation of surfaces is to generate a new mesh for each scenario or time step but this solution is unaffordable because of its computational cost. An alternative is to locally remesh ([9]) the tetrahedral mesh to include the changes in the geometric model. This approach is only suitable for changes restricted to a small region in the domain. Finally, mesh moving techniques ([7]) allow to track moving surfaces and insert new ones in the mesh.

In this work we propose a novel approach to insert immersed surfaces in a tetahedral mesh generated by the meccano method. It combines mesh refinement, node projection and mesh optimization to capture the immersed surfaces and improve the quality of the resulting mesh. An academic example is presented to illustrate the steps and analyse the performance of the procedure.

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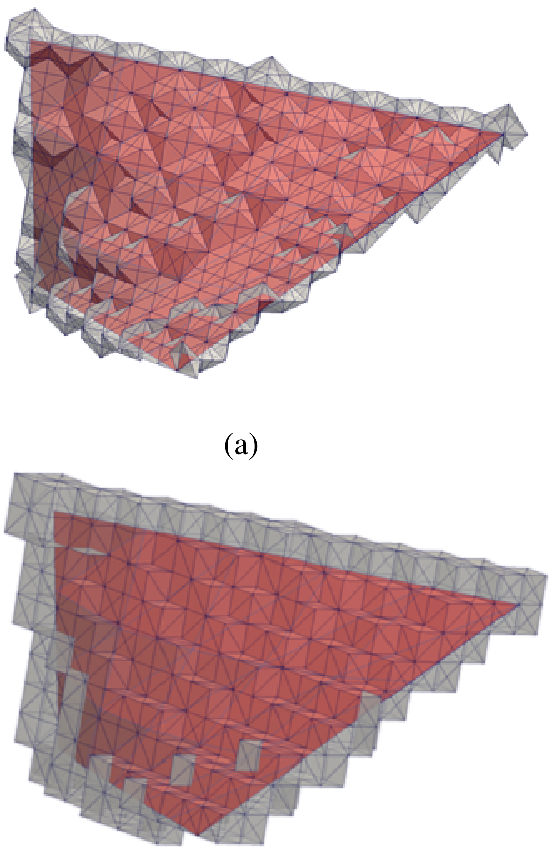
2 Overview of the Meccano method

The Meccano method ([6]) is an automatic tetrahedral mesh generator for complex solids. The input data is the definition of the boundary of the solid. From the geometry of this boundary, the method creates a computational domain that coarsely approximates the solid as a juxtaposition of simple pieces.

The solid boundary is parameterized to the boundary of the meccano, establishing a bijective transformation between both surfaces. The mesh generation implies a Kossaczký refinement process performed in the computational domain, and the projection of the boundary nodes onto their actual location in the real domain through the transformation. The locations for inner nodes are obtained by means of a simultaneous untangling and smoothing procedure that resolves inverted elements in the mesh and improves the quality of tetrahedra. This quality is related to the similarity between cell shapes in both, the real and computational domains.



(a)

(b)

Figure 1: Volumetric approximation: (a) before extension, as a set of tetrahedra (gray) that intersect with the immersed surface (red); and (b) after extension.

Note that the meccano method not only generates a valid mesh with high-quality elements for the solid, but also provides a piecewise parameterization of the volume of the solid to the meccano.

3 Problem statement

This work is focused in the insertion of a simply connected, single oriented, immersed surfaces, $\mathcal{S}_{\mathcal{M}}$, in a tetrahedral mesh, $\mathcal{M}$, generated by the meccano method. In order to capture geometric features of the surface, the refinement of the mesh and the relocation of nodes are allowed. Our goal is to obtain a surface, $\mathcal{S}_{\mathcal{M}}$, composed of triangles of $\mathcal{M}$ that approximates the immersed surface with a given tolerance.

Under the hypothesis of nonintersecting surfaces, the insertion of several surfaces is achieved iterating the strategy for each one.

4 Surface insertion algorithm

The strategy for surface insertion is composed of four steps. In the first step, the cells intersected by the immersed surface are refined using the Kossaczký algorithm ([5]) to capture its geometrical features, providing a coarse volumetric approximation. In the second step, a set of faces in the boundary of the volumetric approximation is selected to approximate the surface, verifying some required topological and geometric properties. Third, both, the original surface and the set of faces in the approximation, are parameterized to the same parametric space, allowing the projection of the nodes of the approximation onto the corresponding surface. Finally, we optimize a regularized distortion measure for tetrahedra using the Simultaneous Untangling and Smoothing technique, resolving inverted elements and improving the overall quality of mesh cells.

4.1 Volumetric approximation

As a first stage, we obtain a volumetric approximation of the surface $\mathcal{M}$, composed of the set $\mathcal{V}_{\mathcal{M}}$ of tetrahedra of $\mathcal{M}$ that intersect $\mathcal{S}$. To capture the surface, we apply the Kossaczký method and refine those tetrahedra on $\mathcal{V}_{\mathcal{M}}$ such that

$$l_T \geq l_r, \qquad T \in \mathcal{M}, \tag{1}$$

where l_T is the longest edge of the tetrahedron T, and l_r is a threshold that determines the capturing resolution. The refined elements that still intersect $\mathcal{S}$ are preserved in $\mathcal{V}_{\mathcal{M}}$. Otherwise, the tetrahedron is removed from the volumetric approximation. The process iterates until all the tetrahedra in $\mathcal{V}_{\mathcal{M}}$ verify condition (1)

Once the refinement process is finished, the approximation $\mathcal{V}_{\mathcal{M}}$ is extended to obtain a volume topologically equivalent to a ball in $\mathbb{R}^3$ that encloses the immersed surface. For each node N in the boundary of $\mathcal{V}_{\mathcal{M}}$, its adjacent tetrahedra are added to the volumetric approximation if the Euler characteristic, χ, of N in $\mathcal{V}_{\mathcal{M}}$ does not verify:

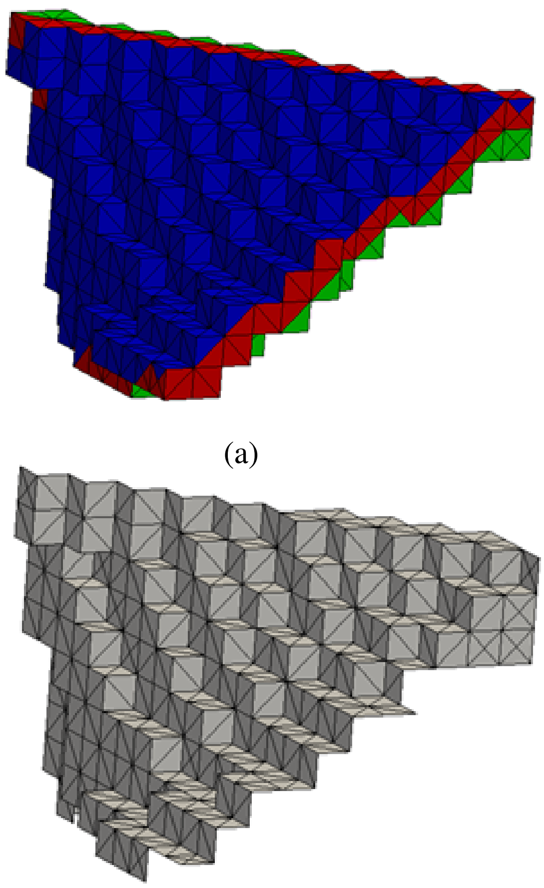
(2) $$\chi := n_E - n_F + n_T = 1,$$

where n_E, n_F and n_C are, respectively, the number of edges, faces and tetrahedra adjacent to the node.

Finally, we perform additional extensions to impose that boundary faces of $\mathcal{V}_{\mathcal{M}}$ are parallel to any coordinate surfaces, as shown in Figure 1.

4.2 Surface approximation

We obtain a surface approximation of $\mathcal{S}$ as a set of faces, $\mathcal{S}_{\mathcal{M}}$, in the boundary of the volumetric approximation $\mathcal{V}_{\mathcal{M}}$. To construct $\mathcal{S}_{\mathcal{M}}$, we label the nodes of $\partial\mathcal{V}_{\mathcal{M}}$ attending to the side of the immersed surface they are located at. Thus, a single side is selected and we consider all the triangles in $\mathcal{S}_{\mathcal{M}}$ with the three nodes to that side of $\mathcal{S}$, see Figure 2.



(a)

(b)

Figure 2: Surface approximation: (a) labelling of nodes in $\mathcal{V}_{\mathcal{M}}$ attending to the sides of the surface; and (b) healed version of the surface approximation corresponding to the blue side.

In order to ensure that $\mathcal{S}_{\mathcal{M}}$ is topologically equivalent to a 2D ball defined over $\partial\mathcal{V}_{\mathcal{M}}$, and to avoid unresolvable element degenerations in the projection stage, a surface healing has to be performed. First, $\mathcal{S}_{\mathcal{M}}$ is extended to ensure that the edges on $\partial\mathcal{S}_{\mathcal{M}}$ are parallel to any coordinate axis. Then, the healing iterates the following actions until no changes is introduced:

- Remove corbels, i. e., pairs of connected triangles in the surface approximation with three nodes at $\partial\mathcal{S}_{\mathcal{M}}$;
- Remove nodes on $\partial\mathcal{S}_{\mathcal{M}}$ with high connectivity, i. e., adjacent to five or more triangles in $\mathcal{S}_{\mathcal{M}}$;
- Refine tetrahedra in $\mathcal{M}$ with more than one triangle on $\mathcal{S}_{\mathcal{M}}$;
- Refine triangles of $\mathcal{M}$ with more than one edge in $\partial\mathcal{S}_{\mathcal{M}}$;
- Refine edges with nodes on different surfaces (immersed or boundary ones); and
- Refine dividing edges, i. e., edges not included in $\partial\mathcal{S}_{\mathcal{M}}$ with nodes on $\partial\mathcal{S}_{\mathcal{M}}$.

4.3 Surface projection

To project the nodes of the $\mathcal{S}_{\mathcal{M}}$ onto $\mathcal{S}$, we compute the Floater ([2, 3]) parameterization of $\mathcal{S}$ and $\mathcal{S}_{\mathcal{M}}$, denoted respectively by φ and $\varphi_{\mathcal{M}}$, to the same parametric space $\mathcal{P} = [0,1] \times [0,1]$

(3) $$\varphi : \mathcal{P} \to \mathcal{S}, \qquad \varphi_{\mathcal{M}} : \mathcal{P} \to \mathcal{S}_{\mathcal{M}},$$

presented in Figure 3. Thus, the projection is performed by composing the two previous parameterizations. Specifically, we obtain the new location of the node, $\boldsymbol{x}_i$, from its previous location in the surface approximation step, $\boldsymbol{x}_{\mathcal{M}i}$, according to

(4) $$\boldsymbol{x}_i = \varphi \circ \varphi_{\mathcal{M}}^{-1}(\boldsymbol{x}_{\mathcal{M}i}).$$

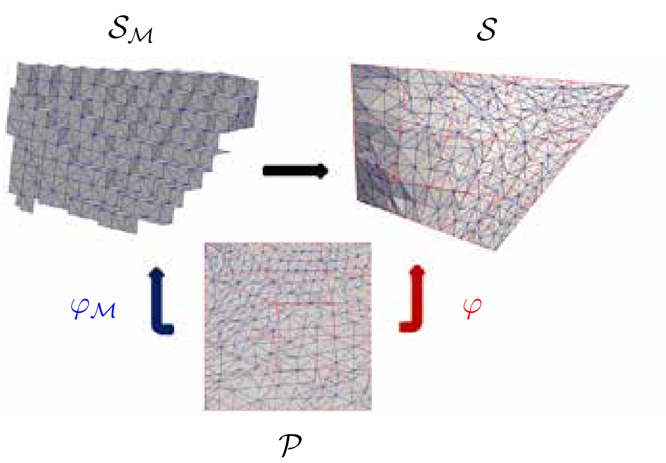


Figure 3: Surface projection by means of simultaneous parameterization of immersed surface $\mathcal{S}$ (red) and surface approximation $\mathcal{S}_{\mathcal{M}}$ (blue) with the same parametric space $\mathcal{P}$.

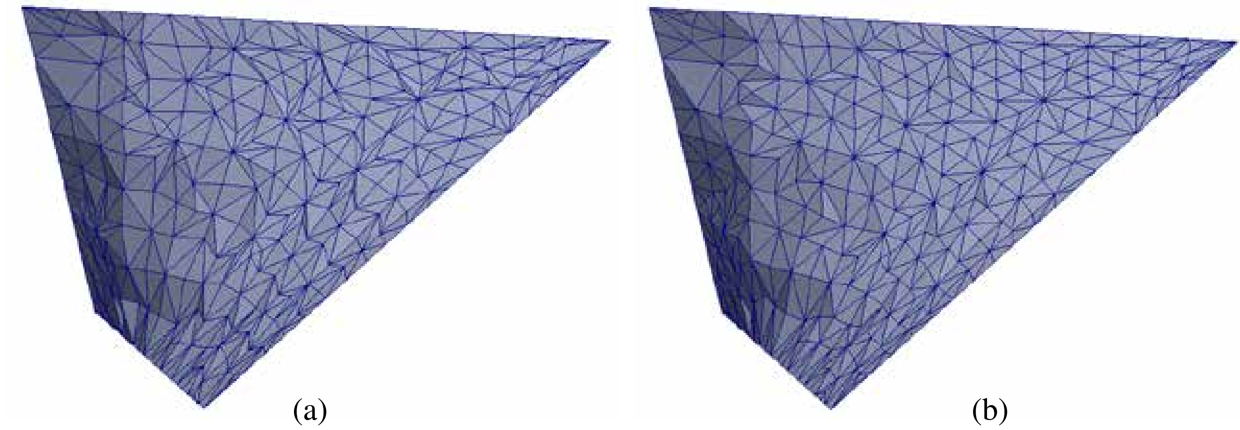


Figure 4: Mesh optimization. Improvement in the quality of the triangles in the surface approximation $\mathcal{S}_{\mathcal{M}}$: (a) mesh with nodes in the approximation projected onto the immersed surface; and (b) after optimization.

4.4 Mesh optimization

The projection process may introduce degenerated elements in the mesh $\mathcal{M}$. To resolve the element inversions and obtain a valid mesh with high-quality tetrahedra, we apply the simultaneous untangling and smoothing technique ([1, 8]).

Specifically, the objective function to optimize is

$$f(\boldsymbol{x}) = \sum_e {\eta^*}_e(\boldsymbol{J}\phi), \tag{5}$$

defined in terms of the regularized version, η^*, of the shape distortion measure, η ([4]),

$$\eta(\boldsymbol{J}\phi) = \frac{\|\boldsymbol{J}\phi\|^2}{3\,|\sigma(\boldsymbol{J}\phi)|^{\frac{2}{3}}}, \tag{6}$$

where $\boldsymbol{J}\phi$ is the Jacobian matrix of the affine mapping of elements from its counterpart in the computational domain, $\|\cdot\|$ is the Frobenius norm, σ is the determinant, and $h(\cdot)$ is the regularization function

$$h(\sigma) = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right), \tag{7}$$

proposed in [1] to replace σ in the shape distortion measure (6).

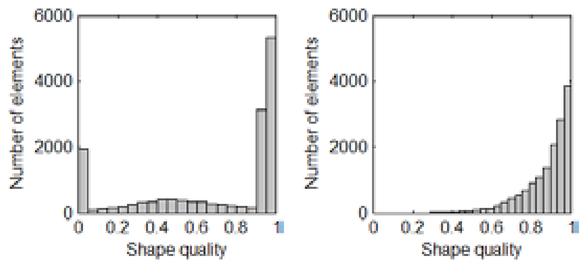
For inner nodes of $\mathcal{M}$, optimization is performed in the real domain. For mesh boundary nodes and those projected to the immersed surface, the optimization domain is the parametric one whereas locations of nodes in the real domain are taken into account to measure the element distortion.

5 Example: simple cube

In this section we present the insertion of the curved surface $\mathcal{S}$ used in Section 4 inside the tetrahedral mesh, $\mathcal{M}$, for a cube. Figure 5 (a) shows the outline of the mesh generated by the Meccano method, as well as the immersed surface to be inserted. The Kossaczký refinement allows to capture the geometrical features of the surface and then, applying the strategy described in Sections 4.1 and 4.2, we obtain the surface approximation, $\mathcal{S}_{\mathcal{M}}$, shown in Figure 5 (b).

In the next step, the nodes of $\mathcal{S}_{\mathcal{M}}$ are projected onto the immersed surface $\mathcal{S}$ and inverted and low-quality elements may appear, as shown in Figure 5 (c). The simultaneous untangling and smoothing of $\mathcal{M}$ provides a valid mesh and optimizes the quality of its elements, see Figure 5 (d). The optimization relocates inner nodes and those that lie on surfaces (boundary of $\mathcal{M}$ or immersed surface).

Figure 5 presents the histograms of qualities for the elements of the mesh $\mathcal{M}$ before and after optimization. It can be observed that the optimization resolves all the degenerated elements, increasing the lowest qualities to achieve a minimum value of 0.29 and a mean value of 0.88. The standard deviation of qualities is reduced from 0.36 to 0.11 in the smoothed mesh.



(a) (b)

Figure 6: Mesh quality histograms for: (a) mesh with nodes in the approximation projected onto the immersed surfaces; and (b) mesh after optimization.

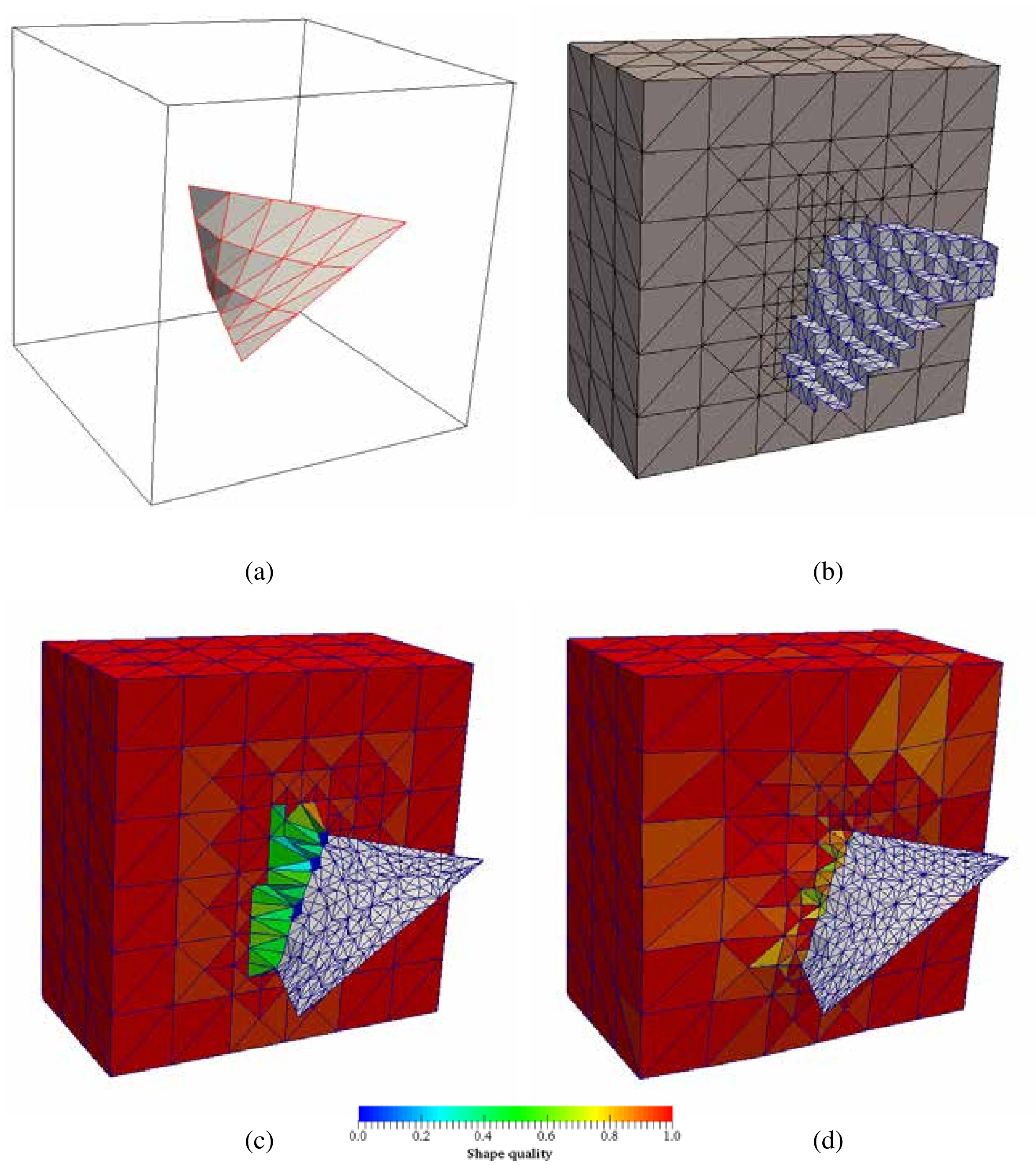


Figure 5: Steps in the insertion of an immersed curved surface in the mesh of a cube: (a) outline of the cube mesh, $\mathcal{M}$, and triangulation of the immersed surface, $\mathcal{S}$; (b) refined tetrahedral mesh and surface approximation $\mathcal{S}_{\mathcal{M}}$ as a set of faces of $\mathcal{M}$; (c) projection of nodes in the approximation onto the immersed surface; and (d) smoothed final mesh.

6 Conclusions

We have presented a novel approach to insert surfaces in a mesh generated by the meccano method. The strategy includes a mesh refinement step to capture the geometric features of each surface. Then, a set of triangles of the mesh is considered to coarsely approximate the surface. We impose to this approximation a set of topological and geometric properties that makes it suitable for projection onto the actual surface, avoiding unresolvable tangles in the mesh. The nodes in the approximation are then projected onto the correspondig surface by means of a parameterization of both, surface and approximation, with the same parametric space. Finally, a simultaneous untangling and smoothing technique provides a valid mesh and improves the quality of the elements, as shown in the quality histograms for the cube example.

Although we have described the insertion for a single surface, the strategy can be applied to insert several totally immersed surfaces that do not intersect between them. Furthermore, the method deal with triangulated surfaces and this approach can be considered to handle any alternative definition of the immersed surface. In addition, the proposed strategy is computed efficiently, taking advantage of parallelism in steps of the method such as calculation of intersections, parameterization, projection and mesh optimization.

Additional future work will focus on handling the insertion of intersecting surfaces and the extension of the strategy to surfaces with more complex topologies.

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