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TRABAJO DE FIN DE GRADO:

SOME RESULTS ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Chapter 1. Introduction

The subject of this work are the Navier-Stokes equations which are used to model the behaviour of fluids and are of immense importance in subjects as diverse as geophysics, the modelling of climate and climate change, oceanography and the design of aircrafts and cars.

The study of fluid mechanics can be traced back to antiquity with, among others, the studies of Archimedes (287-212 B.C.). Of great importance to the future development of fluid mechanics was the publication of Newton's foundational work *Philosophiæ Naturalis Principia Mathematica* (1687) which laid the groundwork for classical mechanics, from which the Navier-Stokes equations would be derived. In the year 1755 Euler established the equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p$$
$$\nabla \cdot u = 0$$

where u and p are respectively the velocity and pressure of the fluid in question. These equations are now known as the Euler equations. The first of these equations is derived from the use of Newton's second law, $F = ma$ and his second equation corresponds to the principle of conservation of mass and the requirement that the fluid be incompressible.

To derive these equations Euler assumed an ideal fluid, that is, one without friction in addition to being incompressible. As a result of this, one may formally derive from his equation that the kinetic energy of the fluid is conserved. Nonetheless as D'Alambert (one of Euler's compeers who had also been studying fluid mechanics) noticed, these equations have relatively simple solutions which do not allow for many naturally occurring phenomena such as the flight of birds.

D'Alambert was not able to resolve this issue, and it remained unsolved until the later work of the French physicist Navier, who was one of Fourier's students, and the Irish mathematician Stokes, who is also known for, among many things, the Stokes Theorem. These two thinkers introduced to Euler's equations in the years 1820 and 1845 respectively the so called viscosity term $\nu\Delta u$, where $\nu > 0$ is known as the kinematic viscosity, to obtain the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u = \nu\Delta u - \nabla p$$
$$\nabla \cdot u = 0$$

which no longer verify a conservation of kinetic energy. These are the equations which we will study in this work.

During the 19th century many thinkers attempted to obtain analytic and explicit solutions without much success. However, in the 20th century many advances were made thanks to, among others, the work of Leray [10] (1934), Ladyzhenskaya [9] (1963) and Fujita-Kato [6] (1964).

Despite decades of work the Navier-Stokes equations remains one of the most challenging and open problems in all of mathematics and physics. To sum up the current understanding of these equations we mention the following: in the two dimensional case the Navier-Stokes equations are well posed (i.e. they have a unique global solution that obeys a stability condition) and global existence and uniqueness of weak solutions to the Navier-Stokes equations has also been established. In the three dimensional case it is known that weak solutions exist globally, however, it is not known whether these weak solutions are unique. The existence of strong solutions in the three dimensional case is also established, however only locally, that is, for a finite amount of time. Another unknown is whether in a bounded domain solutions to the Navier-Stokes equations converge to solutions to the Euler equations as ν converges to zero.

A large part of the complexity of tackling the Navier-Stokes equations is due to the great difficulty in choosing a correct function space in which to situate the initial data. While a solution to a typical ordinary differential equation takes for each value of time a value in a finite dimensional vector space, in the case of a evolution partial differential equation for each value of time said solution takes values in a space of functions, which is infinite dimensional and determined by what space one supposes the initial data to be in. The importance of choice of said space is reflected in the estimates that one may obtain for solutions to the Navier-Stokes equations, which are the tools one uses in proving (ideally global) existence, uniqueness and regularity of said solutions. In our case the main estimate we will use will be the one established in Lemma 5.9, lemma which will be of crucial importance in the use of the fixed point theorem (much as in the classical Cauchy existence proof for solutions to ODE) to prove existence of “mild” solutions and many further properties of said solutions.

Our study will follow the notes [16] of the Fields Medallist Terence Tao who worked for many years at the forefront of the Navier-Stokes equations, see for example [15], taking us through a brief overview of the heat equation and the Leray equations up to the theory of well posedness for the Navier-Stokes equations.

Our work is structured as follows: the second chapter is dedicated to some preliminary results. In the third chapter we study the heat equations, to which we shall prove existence and uniqueness of solutions for sufficiently regular initial data. The heat equations are closely related to the Navier-Stokes equations and they will provide a key tool for our future study of the Navier-Stokes equations. In the fourth chapter we study the Leray equations and define the Leray projection, which will allow us to obtain an equivalent formulation to the Navier-Stokes equations. This formulation will be developed in the fifth and final chapter which is devoted to the study of the Navier-Stokes equations in any dimension. In this last chapter we shall first prove the uniqueness of smooth solutions to the Navier-Stokes equations. We will then define the concept of mild solutions (which are also weak solutions) to the Navier-Stokes equations, prove that they obey a stability condition, exist on some maximal time interval $[0, T_*)$ and that if T_* is finite then the Navier Stokes equations “blow-up” at time T_*. We continue by proving that: mild solutions are unique, are smooth if the initial data is smooth and are globally defined for small initial data. Finally, we conclude our work by discussing some generalizations of the preceding results. Such as showing how one may enlarge the class of mild functions given in [16] while still obtaining the previously discussed results.

Chapter 2. Preliminaries

We commence our work with a physical derivation of the incompressible Euler and Navier-Stokes equations. In our deduction we will suppose all functions to be smooth and will consider spatial volumes with smooth boundary. Let us suppose that the fluid is located in some region of space $W \subset \mathbb{R}^d$. Let us denote the value of the velocity field of said fluid at some time t and some point x by $u(t,x)$, its pressure by $p(t,x)$ and its density by $\rho(t,x)$. The first of these quantities being a vector in $\mathbb{R}^d$ and the other two being scalars. Furthermore, let us suppose that the density of said fluid is constant, say $\rho(t,x) = \rho$, and hence that the fluid is in particular incompressible. We now restrict ourselves to a volume of fluid $V \subset W$. Since the density of the fluid is constant so is its mass

$$m(t) = \int_V \rho(t,x)dx = \rho V.$$

Let us denote by ∂V the boundary of V and by n the unit normal which, by convention, we take to be outward pointing. Then we have that the amount of mass entering V through its surface per unit time at the instant t is

$$-\rho \int_{\partial V} u(t,x) \cdot n dx = -\rho \int_V \nabla \cdot u(t,x) dx,$$

where the minus sign appears as the normal vector is outward pointing and the equality is due to Gauss's formula. Since the mas of fluid in V is constant we have that this quantity is 0 and hence, since $S \subset V$ was any, we deduce the so called continuity equation

$$\nabla \cdot u = 0 \tag{2.1}$$

To further proceed we will suppose that the fluid is an ideal fluid. That is, that the stress force acting on the total volume of the fluid is due exclusively to the pressure p and is

$$-\int_{\partial V} p(t,x) n dx = -\int_V \nabla p(t,x) dx,$$

where we have once again used a version of Gauss's theorem. That is, the force acting per unit volume on a point x is $\nabla p(t,x)$. By Newton's second law we deduce that, if we denote the acceleration filed of the fluid by $a(t,x)$, the force per unit volume is also

$$\rho a(t,x) = -\nabla p(t,x) \tag{2.2}$$

We now wish to calculate the acceleration field in terms of the velocity field. To do so consider a fluid particle b and denote its position at time t by $x_b(t)$. We then have that, by the chain rule

$$a(t, x_b(t)) = \partial_t(u(t, x_b(t))) = (\partial_t u)(t, x_b(t)) + \partial_t x_b(t) \cdot \nabla u(t, x_b(t)).$$

By now using that the term $\partial_t x_b(t) = u(t, x_b(t))$ is just the velocity of the particle at the point $x_b(t)$ we have that $a(t, x_b(t)) = (\partial_t u + u \cdot \nabla u)(t, x_b(t)$. Finally since the particle b was any we deduce that $a = \partial_t u + u \cdot \nabla u$.

By (2.2) we deduce that

$$\partial_t u + u \cdot \nabla u = -\frac{1}{\rho}\nabla p \tag{2.3}$$

This last equation together with equation (2.1) are the previously mentioned Euler equations. Note that (2.3) may also be rewritten as

$$\rho \partial_t u_i = -\rho u_k \partial_k u_i - \partial_i p = -\rho \partial_k (u_k u_i) - \partial_i p = -\partial_k(\rho u_k u_i + p\delta_{i,k}) =: -\partial_k(\Pi_{i,k}),$$

where throughout we use Einstein's summation convention, in the second equality we used the continuity equation and where $\Pi_{i,k}$ is known as the momentum flux density tensor.

We now drop the hypothesis that our fluid is ideal, supposing that it has some constant viscosity $\nu \in \mathbb{R}^+$ and therefore has interior frictional forces not due to the pressure. This supposition leads us to hypothesize that the tensor $\Pi_{i,k}$ should be rewritten as something of the form

$$\Pi_{i,k} = \rho u_k u_i + p\delta_{i,k} - \sigma_{i,k}.$$

We will call $\sigma_{i,k}$ the viscous stress tensor. By reasoning that viscous forces occur only when there is a difference of velocity in different particles of the fluid, we hypothesize that $\sigma_{i,k}$ must depend only on the gradient of the velocity. We further suppose that $\sigma_{i,k}$ depends only of the first derivatives of the velocity and does so linearly. Additionally, by observing that $\sigma_{i,k}$ must vanish when the fluid is subject to a uniform rotation (as no internal friction would occur), one may show that $\sigma_{i,k}$ only contains symmetric terms of the form $\partial_k u_i + \partial_i u_k$. The most general tensor that verifies all these properties is of the form

$$\sigma_{i,k} = \eta\left(\partial_k u_i + \partial_i u_k - \frac{2}{3}\frac{\partial_l}{u_l}\sigma_{i,k}\right) + \xi \partial_l u_l$$

where $\eta, \xi > 0$ are called the coefficients of viscosity, see for example [11]. By further supposing these coefficients to be constant (instead of functions of time and space) we deduce from substituting said tensor back into (2.3) and using the continuity equation that

$$\partial_t u + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu \Delta u \tag{2.4}$$

where $\nu := \eta/\rho$ is the "kinematic viscosity". This equation, together with the continuity equation (2.1), are the Navier-Stokes equations.

Notation

We now give a brief aside on notation: throughout the text we shall use Einstein summation convention that indices are summed over their range when repeated in an expression. We will write $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d \to \mathbb{R}^m)$ to denote the space of equivalence classes of p integrable functions from $\mathbb{R}^d$ to $\mathbb{R}$ and $\mathbb{R}^m$ respectively.

Given a function defined on a factor space $u : X \times Y \to Z$ we shall frequently write $u(x)$ to stand for the slightly more cumbersome $u(x, \cdot) : Y \to Z$. Given $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}^d$ we shall write as is standard

$$x^\alpha := x^{\alpha_1} \cdots x^{\alpha_d}; \quad D^\alpha := \partial_1^{\alpha_1} \cdots \partial_1^{\alpha_d}$$

and whenever the expression D^α appears we will assume implicitly that $\alpha \in \mathbb{Z}^d$. Given $x \in \mathbb{R}^d$ we will write $|x|$ to denote its norm and the Japanese bracket $\langle x \rangle$ will signify

$$\langle x \rangle := (1 + |x|^2)^{1/2}.$$

Finally we will employ the notation $f \lesssim g$ to mean that there exists some constant C such that $f \leq Cg$. If the value of C depends on some other parameter such as the dimension d we shall make this explicit by writing $f \lesssim_d g$.

Throughout our work we will often be working with functions of the form:

$$u : I \times \mathbb{R}^d \to \mathbb{R}^m$$

with I a (possibly unbounded) interval in $\mathbb{R}$. Due to the physical interpretation we will denote the first variable by t and call it the time variable. The second variable we will denote by x and call the space variable. We shall often call $I \times \mathbb{R}^d$ the space-time region.

Ordinary Differential Equations

In this section we give a brief discussion of the theory of ODE and touch on some of the motivating ideas behind our study of the Navier-Stokes equations. The principal result of ODE's is Picard's theorem which we now discuss. Consider the ODE Cauchy problem

$$u' = F(u); \quad u(0) = u_0 \tag{2.5}$$

with $u_0 \in \mathbb{R}^d$, and with $F : \mathbb{R}^d \to \mathbb{R}^d$ Lipschitz on compact sets. Then by defining for continuous u

$$\Phi(f)(t) := u_0 + \int_0^t F(f)(t')dt'$$

one has that for $T(|u_0|)$ (where T is an increasing function of $|u_0|$, F being understood to be fixed) small enough the restriction of Φ to $C([-T(|u_0|), T(|u_0|)] \to B(0, 2|u_0|)$ is a contractive endomorphism and thus has a fixed point u which solves (2.5) on $[-\epsilon, \epsilon]$.

One may show without much difficulty that this solution is unique and, by translating the ODE and iterating said argument, one further obtains that u may be extended to a maximal interval (T_-, T_+) (in the sense that no larger time interval admits a solution to (2.5)). Furthermore, the method of proof shows that the only way that T_+ (respectively T_-) is finite is if u "blows-up" at T_+ (respectively T_-) by leaving every compact set of $\mathbb{R}^d$. We end this section with two results that will be used later in our work.

Lemma 2.1 (**Gronwall's inequality**). *Let $u \in C^1(I \to \mathbb{R})$, $A \in C(I \to \mathbb{R})$ and $x_0 \in I$. Then*

$$\partial_t u \leq A(t)u(t) \quad \forall t \in I \implies u(t) \leq e^{\int_0^t A(s)ds} \quad \forall t \in I.$$

For a short proof based on the fundamental theorem of calculus see [16] page 4.

Lemma 2.2 (**Duhamel's formula**). *Let $\lambda, T, \in \mathbb{R}$, $u_0 \in \mathbb{R}^d$, $F \in C([0,T] \to \mathbb{R}^d)$, then the linear ODE*

$$u'(t) = \lambda u(t) + F(t); \quad u(0) = u_0$$

has unique solution

$$u(t) = e^{\lambda t}u_0 + \int_0^t e^{\lambda(t-t')}F(t')dt'.$$

See for example [18] page 52.

Some Useful Results on Integration

Here we state some results on integration which we shall use throughout our work. The first two propositions will be (along with the dominated and monotone convergence theorem) our main tool to commute differentials with integrals.

Proposition 2.3. *Let X be a measure space, T a first countable metric space and*

$$f : T \times X \to \mathbb{R}$$

such that $f(t)$ is integrable $\forall t \in T$ and such that

$$|f(t)| \leq g \quad \forall t \in T$$

for some integrable function $g : X \to \mathbb{R}$. Then given $t_0 \in T$ we have that

$$\lim_{t \to t_0} \int_X f(t,x) = \int_X \lim_{t \to t_0} f(t,x).$$

In consequence if $f(x)$ is continuous so is $\int_X f(t,x)dx$.

Proof. The proof is an application of the dominated convergence theorem to the sequence of functions $f_n := f(t_n)$ where $\{t_n\}_{n=1}^{\infty}$ is some sequence converging to t. □

Proposition 2.4 (differentiation under the integral sign). *Let X be a measure space and U be an open interval of $\mathbb{R}$ and*

$$f : U \times X \to \mathbb{R}$$

such that:
a) $f(t)$ is measurable for every $t \in U$ and integrable for some $t_0 \in U$
b) For each $x \in X$ $f(x)$ is differentiable and there exists an integrable function $g : X \to \mathbb{R}$ such that

$$|\partial_t f(t,x)| \leq g(x) \quad \forall (t,x) \in U \times X$$

then

$$\partial_t \int_X f(t)dx = \int_X \partial_t f(t)dx.$$

For a proof see for example [3] page 108. As a consequence by a simple limiting argument we derive a special case known as Leibnitz's integration rule which will be useful in our study of the heat equation.

Proposition 2.5. *Let $f : U \times V \to \mathbb{R}$ be a continuous function where U, V are open intervals of $\mathbb{R}$ containing a, t. Then if f verifies a) and b) of the previous proposition:*

$$\partial_t \int_a^t f(t,x)dx = f(t,t) + \int_a^t \partial_t f(t,x)dx.$$

We have thus far studied the interchange of derivatives with integrals. We shall also have occasion to interchange two integrals with themselves:

Theorem 2.6 (**Fubini-Tonelli**)**.** *Let $f : X \times Y \to \mathbb{R}^d$ be a measurable function where $(X, \mu), (Y, \sigma)$ are two sigma-finite measure spaces. Then if one of the iterated integrals*

$$\int_X \left(\int_Y |f(x,y)| d\sigma \right) d\mu \quad or \quad \int_Y \left(\int_X |f(x,y)| d\mu \right) d\sigma \tag{2.6}$$

is finite then f is integrable. On the other hand if f is integrable both iterated integrals in (2.6) exist and are equal to

$$\int_{X \times Y} |f(x,y)| d(\sigma \times \mu)$$

where $\sigma \times \mu$ is the product measure on $X \times Y$.

For a proof for the euclidean case (which is all we will need) we refer the reader once more to [3] pages 144-146.

It will also be useful to have access to the following result:

Lemma 2.7. *Let $s \in \mathbb{N}$ then*

$$\int_{\mathbb{R}^d} \langle x \rangle^{-s} dx < \infty \iff s > d \iff \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-s} < \infty \tag{2.7}$$

The equivalence of these two statements is a consequence of fact that $\langle x \rangle^{-s}$ is monotone decreasing. To prove (2.7) for example it suffices to apply Fubini's theorem and some basic inequalities. The proof of our next theorem may be found in [13] page 22.

Theorem 2.8 (**Young's inequality**)**.** *Let $1 \leq p, q, r \leq \infty$ verify*

$$\frac{1}{r} + \frac{1}{p} - 1 = \frac{1}{q}$$

and consider $f \in L^p(\mathbb{R}^n)$, $K : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ such that for some constant C

$$\|K(x)\|_{L^r(\mathbb{R}^n)} \leq C \qquad \|K(y)\|_{L^r(\mathbb{R}^m)} \leq C$$

for almost all $x \in \mathbb{R}^m$ and for almost all $y \in \mathbb{R}^n$. Then if we set

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y) f(x,y) dy$$

we have that Tf is in $L^q(\mathbb{R}^n)$ and verifies the bound

$$\|Tf\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

Finally, we will also have use of a more general form of integration known as Bochner integration, whose theory deals with the integration of functions

$$f : (X, \mu) \to Z$$

where (X, μ) is a measure space and where Z is now a Banach space. So as to give a brief idea of this theory we mention that such a function f is said to be strongly measurable if it may be written as the limit of step functions f_n and in this case one defines its integral as

$$\int_X f d\mu := \lim_{n \to \infty} \int_X f_n d\mu$$

where the definition of the integrals of the step functions is analogous to the real case. If one has that the integral of $\|f\|_Z$ is finite it is said that f is strongly integrable. As in the real case if f is strongly integrable it is also strongly measurable and one has the inequality

$$\left\|\int_X f d\mu\right\|_Z \leq \int_X \|f\|_Z d\mu$$

which, for the Banach space $Z = L^p(Y)$ (where (Y, σ) is another measure space), gives Minkowski's integral inequality

$$\left(\int_Y \left|\int_X f d\mu\right|^p d\sigma\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f|^p d\sigma\right)^{\frac{1}{p}} d\mu \tag{2.8}$$

For a more complete discussion than the one given see [20] pages 132-135. We will also have use of the following proposition which may be found in [8] page 21.

Proposition 2.9. *Let (X, μ) be a measure space, Z a Banach space and let us define $L^p(X, Z)$ to be the set of equivalence classes of functions $f : X \to Z$ such that*

$$\|f\|_{L^p(X\to Z)} := \left(\int_X \|f\|_Z^p d\mu\right)^{\frac{1}{p}} < \infty$$

where $1 \leq p < \infty$. Then $L^p(X, Z)$ is a Banach Space. Furthermore, given a sequence of function $\{f_n\}_{n=1}^{\infty} \subset L_p(X, Z)$ converging to f in $L_p(X, Z)$, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} \|f(x) - f_{n_k}(x)\|_Z = 0 \qquad \mu\text{-almost everywhere.}$$

For completeness we observe that this also holds for $L^\infty(X, Z)$ whose definition is analogous to the real case.

Fourier Transform and Fourier Series

Due to its important and pervasive use in this work in this section we study he Fourier transform and Fourier series. For a full account see for example [17] pages 222-226.

We begin by introducing the Fourier transform.

Definition 2.10. *Given $f \in L^1(\mathbb{R}^d \to \mathbb{C}^m)$ the* Fourier transform *of f is the linear operator defined by:*

$$\mathcal{F}_1 : L^1(\mathbb{R}^d \to \mathbb{C}^m) \longrightarrow L^\infty(\mathbb{R}^d \to \mathbb{C}^m); \quad f \mapsto \hat{f}$$

where

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx \qquad \forall \xi \in \mathbb{R}^d.$$

Note that the definition is correct as indeed

$$\left|\hat{f}(\xi)\right| \leq \int_{\mathbb{R}^d} |f(x)| dx = \|f\|_{L^1(\mathbb{R}^d\to\mathbb{C}^m)} \qquad \forall \xi \in \mathbb{R}^d, \tag{2.9}$$

the linearity of $\mathcal{F}_1$ also being clear by the linearity of the integral. We also define the inverse Fourier transform for integrable functions:

Definition 2.11. *Given $f \in L^1(\mathbb{R}^d \to \mathbb{C}^m)$ the* inverse Fourier transform *of f is the linear operator defined by:*

$$\mathcal{F}_1^* : L^1(\mathbb{R}^d \to \mathbb{C}^m) \longrightarrow L^\infty(\mathbb{R}^d \to \mathbb{C}^m); \quad f \mapsto \check{f}$$

where: $\check{f}(\xi) := \hat{f}(-\xi)$.

It is immediately possible to prove that $\hat{f}$ and $\check{f}$ are not only bounded but also continuous.

Proposition 2.12. *Let $f \in L^1(\mathbb{R}^d \to \mathbb{C}^m)$. Then $\hat{f}, \check{f}$ are continuous.*

Proof. The proof is an immediate application of Proposition 2.3. □

We now state two basic properties of the Fourier transform:

Proposition 2.13. *Given $f \in L^1(\mathbb{R}^d \to \mathbb{C}^m)$:*

a) If $x^\alpha f(x) \in L^1(\mathbb{R}^d \to \mathbb{C}^m)$ then:

$$D^\alpha \hat{f}(\xi) = (-2\pi i)^{|\alpha|} \widehat{x^\alpha f}(\xi) \qquad \forall \xi \in \mathbb{R}^d.$$

b) If f is absolutely continuous in x_i for almost every $x_1, ...x_{i-1}, x_{i+1}, ..., x_d$ then

$$\widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \hat{f}(\xi) \qquad \forall \xi \in \mathbb{R}^d.$$

The proof is a differentiation under the integral sign for a) and an integration by parts for b). See for example [14] page 181. Analogous properties also hold for the inverse Fourier transform where it is only necessary to introduce a factor of $(-1)^\alpha$ in a) and a change of the sign in b).

The importance of these two properties is that they give us information on the Fourier transform of a function f based on its regularity and decay. They can be stated informally as: decay gives regularity and regularity gives decay (as the Fourier transform of any $f \in L^1(\mathbb{R}^d \to \mathbb{R}^d)$ decays to 0 at infinity by the Riemann-Lebesgue lemma).

In general the Fourier transform of an integrable function is not itself integrable. That is, $L^1(\mathbb{R}^d \to \mathbb{C}^m)$ is not closed under the Fourier transform. We now introduce a space that is closed under the Fourier transform; the Schwartz space, which can be thought of as the space of infinitely regular functions with infinite decay:

Definition 2.14. *The* Schwartz space $\mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$ *is:*

$$\mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d \to \mathbb{C}^m) : x^\alpha D^\beta f \in L^\infty(\mathbb{R}^d \to \mathbb{C}^m) \quad \forall \alpha, \beta \in \mathbb{Z}^d\}.$$

As is clear from properties a) and b) of Proposition 2.13 given $f \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$

$$\xi^\alpha D^\beta \hat{f}(\xi) = (-2\pi i)^{|\beta|} \xi^\alpha \widehat{x^\beta f} = \frac{(-2\pi i)^{|\beta|}}{(2\pi i)^{|\alpha|}} \widehat{D^\alpha x^\beta f} \in L^\infty(\mathbb{R}^d \to \mathbb{C}^m) \tag{2.10}$$

$$\xi^\alpha D^\beta \check{f}(\xi) = \frac{(2\pi i)^{|\beta|}}{(-2\pi i)^{|\alpha|}} \widehat{D^\alpha x^\beta f} \in L^\infty(\mathbb{R}^d \to \mathbb{C}^m) \tag{2.11}$$

and hence the Fourier transform and the inverse Fourier transform restrict to endomorphisms of the Schwartz space which we will denote respectively by

$$\mathcal{F}_\mathcal{S}, \mathcal{F}_\mathcal{S}^* : \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m) \to \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m).$$

The next item on the agenda is Plancherel's theorem which is proven via the following lemma, in which we shall use the notation

$$\langle f, g \rangle_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} := \int_{\mathbb{R}^d} f(x) \cdot \overline{g(x)} dx$$

for the inner product on $L^2(\mathbb{R}^d \to \mathbb{C}^m)$.

Lemma 2.15. *Given $f, g \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$*

(i) $\langle \mathcal{F}_\mathcal{S} f, g \rangle_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} = \langle f, \mathcal{F}_\mathcal{S}^* g \rangle_{L^2(\mathbb{R}^d \to \mathbb{C}^m)}$; $\quad \langle \mathcal{F}_\mathcal{S}^* f, g \rangle_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} = \langle f, \mathcal{F}_\mathcal{S} g \rangle_{L^2(\mathbb{R}^d \to \mathbb{C}^m)}$

(ii) $\mathcal{F}_\mathcal{S}^* \mathcal{F}_\mathcal{S} f = \mathcal{F}_\mathcal{S} \mathcal{F}_\mathcal{S}^* f = f$

For a detailed proof see [17] pages 222-226. From (i) and (ii) we deduce immediately that, given a Schwartz function f,

$$\|f\|_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} = \|\mathcal{F}_\mathcal{S} f\|_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} = \|\mathcal{F}_\mathcal{S}^* f\|_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} \tag{2.12}$$

That is, the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_1^*$

$$\mathcal{F}_\mathcal{S}, \mathcal{F}_\mathcal{S}^* : \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m) \to \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$$

are linear unitary operators which are the inverse the one to the other. We obtain the following result.

Proposition 2.16 (Plancherel's theorem). *$\mathcal{F}_\mathcal{S}$ and $\mathcal{F}_\mathcal{S}^*$ may be extended to unitary operators:*

$$\mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}^d \to \mathbb{C}^m) \to L^2(\mathbb{R}^d \to \mathbb{C}^m)$$

With $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = Id$.

Proof. This is a immediate result of the density of $\mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$ in $L^2(\mathbb{R}^d \to \mathbb{C}^m)$ and the completeness of $L^2(\mathbb{R}^d \to \mathbb{C}^m)$ together with (ii) and (2.12). As by a simple limiting argument it is easy to see that continuous extensions of continuous linear operators preserve the norm and the inverses of the operators in question. □

Finally we note that given f in $L^1(\mathbb{R}^d \to \mathbb{C}^m) \cap L^2(\mathbb{R}^d \to \mathbb{C}^m)$ we have that, as one would desire:

$$\mathcal{F}f(\xi) = \mathcal{F}_1 f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \qquad \forall \xi \in \mathbb{R}^d.$$

This can be seen by taking a sequence of functions $\{f_n\}_{n=1}^\infty \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$ converging to f in $L^1(\mathbb{R}^d \to \mathbb{C}^m)$ and in $L^2(\mathbb{R}^d \to \mathbb{C}^m)$. As then $\mathcal{F}_1 f_n$ converges uniformly to $\mathcal{F}_1 f$ by (2.9) and in L^2 to $\mathcal{F} f$ by Proposition 2.16, from where we deduce that $\mathcal{F}_1 f = \mathcal{F} f$.

We now extend our previous results to $\mathbb{Z}^d$ periodic functions, though, for any period, analogous results may be achieved. By identifying $\mathbb{Z}^d$ periodic functions with functions on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ we will work with functions $f : \mathbb{T}^d \to \mathbb{C}^m$.

Definition 2.17. *Given $f \in L^1(\mathbb{T}^d \to \mathbb{C}^m)$ we define the k-th* Fourier coefficient *of f as:*

$$\hat{f}(k) := \int_{\mathbb{T}^d} f(x)e^{-2\pi ik\cdot x}dx.$$

We thus obtain a function $\hat{f}$ on $\mathbb{Z}^d$ which we shall call the Fourier series of f and a continuous linear function which we shall denote as in the euclidean case:

$$\mathcal{F}_1 : L^1(\mathbb{T}^d \to \mathbb{C}^m) \to l^\infty(\mathbb{Z}^d \to \mathbb{C}^m)$$

where $l^\infty(\mathbb{Z}^d \to \mathbb{C}^m)$ is the set of bounded sequences from $\mathbb{Z}^d$ to $\mathbb{C}^m$. As before we have the following result:

Proposition 2.18. *Given $f \in L^1(\mathbb{T}^d \to \mathbb{C}^m)$ if f is absolutely continuous in x_i for almost every $x_1, ...x_{i-1}, x_{i+1}, ..., x_d$ then*

$$\widehat{\partial_{x_j} f}(k) = 2\pi ik_j\hat{f}(k) \qquad \forall k \in \mathbb{Z}^d.$$

In particular if $f \in C^\infty(\mathbb{T}^d \to \mathbb{C}^m)$ we have that:

$$\widehat{D^\alpha f}(k) = (2\pi ik)^\alpha \hat{f}(k) \tag{2.13}$$

and hence we have that $\hat{f}$ is of rapid decrease (i.e. $\hat{f}$ decreases faster than the inverse of any polynomial). We thus have, as before, an induced map:

$$\mathcal{F}_{C^\infty} : C^\infty(\mathbb{T}^d \to \mathbb{C}^m) \to s(\mathbb{Z}^d \to \mathbb{C}^m)$$
$$f \mapsto \hat{f}$$

where $s(\mathbb{Z}^d \to \mathbb{C}^m)$ are the sequences from $\mathbb{Z}^d$ to $\mathbb{C}^m$ that are of rapid decrease.

Similarly to the non-periodic case we now define

$$\mathcal{F}^*_{C^\infty} : s(\mathbb{Z}^d \to \mathbb{C}^m) \to C^\infty(\mathbb{T}^d \to \mathbb{C}^m); \quad a \mapsto \check{a}$$

with:

$$\check{a}(x) := \sum_{k\in\mathbb{Z}^d} a(k)e^{2\pi ik\cdot x}.$$

Which is indeed smooth as, by the rapid decay of a, we may by Proposition 2.4 commute all derivatives with the above sum to obtain that

$$D^\alpha\check{a}(x) = \sum_{k\in\mathbb{Z}^d} (2\pi i)^{|\alpha|}k^\alpha a(k)e^{2\pi ik\cdot x} \quad \forall a \in s(\mathbb{Z}^d \to \mathbb{C}^m) \tag{2.14}$$

It is now possible to prove as with the euclidean case that

$$\mathcal{F}_{C^\infty}\mathcal{F}^*_{C^\infty} = \mathcal{F}^*_{C^\infty}\mathcal{F}_{C^\infty} = Id$$

and that analogously given f smooth, a of rapid decrease

$$\langle \mathcal{F}_{c^\infty} f, a\rangle_{l^2(\mathbb{Z}^d\to\mathbb{C}^m)} = \langle f, \mathcal{F}^*_{C^\infty} a\rangle_{L^2(\mathbb{R}^d\to\mathbb{C}^m)}; \quad \langle \mathcal{F}^*_{C^\infty} a, f\rangle_{L^2(\mathbb{R}^d\to\mathbb{C}^m)} = \langle a, \mathcal{F}f\rangle_{l^2(\mathbb{Z}^d\to\mathbb{C}^m)}$$

see for example [17] pages 197-206. We conclude that $\mathcal{F}_{C^\infty}$ are unitary linear functions and that hence:

Proposition 2.19 (Plancherel's (periodic) theorem). *$\mathcal{F}_{C^\infty}$ and $\mathcal{F}^*_{C^\infty}$ may be extended to unitary operators:*

$$\mathcal{F} : L^2(\mathbb{R}^d \to \mathbb{C}^m) \to l^2(\mathbb{Z}^d \to \mathbb{C}^m); \quad \mathcal{F}^* : l^2(\mathbb{Z}^d \to \mathbb{C}^m) \to L^2(\mathbb{R}^d \to \mathbb{C}^m)$$

with $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = Id$.

We note that, as for the Euclidean case, given $f \in L^2(\mathbb{T}^d \to \mathbb{C}^m) \cap L^1(\mathbb{T}^d \to \mathbb{C}^m)$ by an identical argument it holds that

$$\mathcal{F}f(k) = \mathcal{F}_1 f(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x}$$

and where now Plancherel's theorem gives that:

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{2\pi i k \cdot x}. \tag{2.15}$$

Distributions and Sobolev Spaces

Here we will quickly recall the concepts of tempered distributions and Sobolev spaces, which are concepts of utmost importance in the field of PDEs and Fourier analysis. We begin by introducing the space of tempered distributions. In general given a real vector space X together with a countable family of semi-norms $\{p_j\}_{j=0}^{\infty}$ with the property that: given $x \neq 0$, there exists j such that $p_j(x) \neq 0$. Then

$$d(x, y) := \sum_{j=0}^{\infty} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)} \quad \forall x, y \in X \tag{2.16}$$

is a translation invariant metric on X. We shall denote the dual of X by

$$X' := \{w : X \to \mathbb{C} \; : w \, \text{continuous}\}.$$

Due to the metric we set on X it is simple to verify that $w \in X'$ iff there exist $C \in \mathbb{R}$, $N \in \mathbb{N}$ such that

$$|w(u)| \leq C \sum_{j=1}^{N} p_j(u) \quad \forall u \in X \tag{2.17}$$

In the case of the Schwartz space $\mathcal{S}(\mathbb{R}^d \to \mathbb{R})$ we give it the topology induced by

$$p_k(u) := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} \langle x \rangle^k |D^\alpha u(x)| \tag{2.18}$$

though, as is generally the case with Frechet spaces, other families of semi-norms the reader may be familiar with such as

$$p_{k,\alpha} := \sup_{x \in \mathbb{R}^d} |x|^k |D^\alpha u(x)|; \quad \text{or} \quad p'_{k,\alpha} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^k |D^\alpha u(x)|$$

induce equivalent topologies. We note that with this metric $\mathcal{S}(\mathbb{R}^d \to \mathbb{C})$ is a Fréchet space (that is a complete Hausdorff topological vector space). For a quick proof based on the fundamental theorem of calculus see for example [5] page 237.

With this we move on to discuss space of tempered distributions which we define as the dual of $\mathcal{S}(\mathbb{R}^d \to \mathbb{C})$ and write as $\mathcal{S}'(\mathbb{R}^d \to \mathbb{C})$. One may verify that we have the inclusion

$$L^p(\mathbb{R}^d \to \mathbb{C}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d \to \mathbb{C}); \quad f \mapsto T_f \tag{2.19}$$

where given $u \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C})$ we define

$$T_f(u) := \langle u, f \rangle := \int_{\mathbb{R}^d} u\overline{f}.$$

Given two Schwartz functions u, v a simple application of Fubini gives

$$T_{\mathcal{F}v}(u) = \langle u, \mathcal{F}v \rangle = \langle \mathcal{F}u, v \rangle = (\mathcal{F}^t T_v)(u)$$

and integration by parts gives

$$T_{D^\alpha v}(u) = \langle u, D^\alpha v \rangle = (-1)^{|\alpha|} \langle D^\alpha u, v \rangle = T_v((-1)^{|\alpha|} D^\alpha u) \quad \forall \alpha \in \mathbb{N}^d.$$

This gives us a way of extending the Fourier transform and differentiation to the space of tempered distributions. Given $w \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$ we define the *(distributional) Fourier transform* of w by

$$\mathcal{F}w := \mathcal{F}^t w$$

where we have used the notation T^t for the transpose of a linear function T (remember that the Fourier transform is an endomorphism of the Schwartz space) and the *(weak) α'th* derivative of w by

$$D^\alpha w := w \circ ((-1)^{|\alpha|} D^\alpha).$$

Due to our previous discussion we have that, with this definition, given $u \in \mathcal{S}(\mathbb{R}^d)$

$$\mathcal{F}T_u = T_{\mathcal{F}u}; \quad D^\alpha T_u = T_{D^\alpha u} \tag{2.20}$$

as one would desire.

Two other operations that are permitted on $\mathcal{S}(\mathbb{R}^d)$ are multiplication by functions of polynomial growth and the application of the inverse Fourier transform which we shall, as for L^2 functions, denote by $\mathcal{F}^*$. Both definitions once again being given by duality.

Before ending our discussion of (scalar) tempered distributions we comment on some generalizations. We first note that the previous discussion works equivalently for vector valued distributions, i.e. elements of the dual to $\mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$, which we shall denote by

$$\mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m)$$

where the only change is that the inclusion (2.19) is now given by

$$L^p(\mathbb{R}^d \to \mathbb{C}^m) \hookrightarrow \mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m); \quad f \mapsto T_f \tag{2.21}$$

with T_f defined by

$$T_f(u) := \int_{\mathbb{R}^d} u \cdot \bar{f} \qquad \forall u \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m).$$

In both cases we have that, by duality, due to the formulas derived in (2.10) given a tempered distribution w and $\alpha \in \mathbb{N}^d$

$$D^\alpha \mathcal{F} w = (-2\pi i)^{|\alpha|} \mathcal{F} x^\alpha w; \quad \mathcal{F} D^\alpha w = (2\pi i)^{|\alpha|} x^\alpha \mathcal{F} w \tag{2.22}$$

and of course, since Plancherel's theorem holds for all Schwartz functions, we have that

$$\mathcal{F}^* \mathcal{F} w = \mathcal{F} \mathcal{F}^* w \tag{2.23}$$

The reader will observe that we have not as of yet placed a topology on the space of tempered distributions. And though we shall have no need of it in this work we mention here that the topology one usually considers is the weak*-topology with which

$$w_n \xrightarrow{*} w \iff \langle u, w_n \rangle \to \langle u, w \rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m).$$

This topology makes all the above operations (that is $\mathcal{F}, \mathcal{F}^*, D^\alpha$ and multiplication by function of polynomial growth) continuous as a quick verification will show.

Finally in addition to "changing the image" of our distributions we may also "change the domain" by considering, for example, periodic tempered distributions. Where now, as we saw in the section on the Fourier transform, $C^\infty(\mathbb{T}^d \to \mathbb{C}^m)$ takes the place of the Schwartz space and where we place on $C^\infty(\mathbb{T}^d \to \mathbb{C}^m)$ the topology defined by the countable family of semi-norms:

$$q_k(u) := \sum_{|\alpha| \le k} \sup_{x \in \mathbb{T}^d} |D^\alpha u|$$

and denote its dual by $\mathcal{S}'(\mathbb{T}^d \to \mathbb{C}^m)$. By defining as is natural the Fourier series of a periodic distribution w by the sequence (which can be shown to be of polynomial growth)

$$\hat{w}(k) := \langle e^{-2\pi i k x}, w \rangle \quad k \in \mathbb{Z}^d \tag{2.24}$$

and its α-th distributional derivative by

$$D^\alpha w := w \circ (-1)^{|\alpha|} D^\alpha$$

we derive formulas analogous to the ones seen in the section on the Fourier transform for "periodic" distributions as well. Namely:

$$w = \sum_{k \in \mathbb{Z}^d} \hat{w}(k) e^{2\pi i k x}; \quad \widehat{D^\alpha w}(k) = (2\pi i k)^\alpha \hat{w}(k) \tag{2.25}$$

Essentially, due to the natural inclusion in (2.19) and the analogous inclusion in the periodic case, the notion of Fourier transform and differentiation of tempered distributions allows us to manipulate rough functions (periodic or non-periodic) as if they had Fourier transforms and were smooth. As we shall see this will prove of great use when obtaining "distributional solutions" to some PDE's.

Definition 2.20. *Consider a mapping*

$$P : C^\infty(\mathbb{R}^d \to \mathbb{C}^m) \to C^\infty(\mathbb{R}^d \to \mathbb{C}^n)$$

that extends to

$$P : S \to \mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^n)$$

where S is some subset of $\mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m)$ containing $C^\infty(\mathbb{R}^d \to \mathbb{C}^m)$. Then we say that a distributional solution *to P is any tempered distribution $w \in S$ verifying*

$$P(w) = 0$$

where the above equality is a distributional sense, i.e. $\langle \phi, P(w) \rangle = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}^m)$.

In the above definition P typically defines a linear or non-linear differential equation. Note that the above definition may be extended without any difficulty to the case of (periodic) distributional solutions in the case where $P : C^\infty(\mathbb{T}^d \to \mathbb{C}^m) \to C^\infty(\mathbb{T}^d \to \mathbb{R}^n)$.

This said we now move on to discuss Sobolev spaces:

Definition 2.21. *Given $k \in \mathbb{N}^+$ we define the* Sobolev space $H^k(\mathbb{R}^d \to \mathbb{C}^m)$ *as:*

$$H^k(\mathbb{R}^d \to \mathbb{C}^m) := \{f \in L^2(\mathbb{R}^d \to \mathbb{C}^m) : D^\alpha f \in L^2(\mathbb{R}^d \to \mathbb{C}^m) \hookrightarrow \mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m) \quad \forall |\alpha| \leq k\}$$

where, due to (2.21), we consider $L^2(\mathbb{R}^d \to \mathbb{C}^m)$ as a subspace of $\mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m)$.

We may interpret the Sobolev space $H^k(\mathbb{R}^d \to \mathbb{C}^m)$ as the space of k times differentiable functions in $L^2(\mathbb{R}^d \to \mathbb{C}^m)$ and we give it the norm:

$$\|f\|_{H^k(\mathbb{R}^d \to \mathbb{C}^m)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} \qquad f \in H^k(\mathbb{R}^d \to \mathbb{C}^m).$$

Now, due to the fact that, as we saw in (2.23), the Fourier transform is an automorphism of $\mathcal{S}'(\mathbb{R}^d \to \mathbb{C}^m)$, by using property (2.22) we deduce that

$$D^\alpha f \in L^2(\mathbb{R}^d \to \mathbb{C}^m) \iff \mathcal{F}(D^\alpha f) = (2\pi)^{|\alpha|} |\xi^\alpha| \hat{f}(\xi) \in L^2(\mathbb{R}^d \to \mathbb{C}^m)$$

from which we deduce that

$$f \in H^k(\mathbb{R}^d \to \mathbb{C}^m) \iff \sum_{|\alpha| \leq k} (2\pi i)^{|\alpha|} |\xi^\alpha| \hat{f}(\xi) \sim_k \langle \xi \rangle^k \hat{f} \in L^2(\mathbb{R}^d \to \mathbb{C}^m) \tag{2.26}$$

In fact, since the Fourier transform is a unitary transformation on $L^2(\mathbb{R}^d \to \mathbb{C}^m)$, the same reasoning gives

$$\|f\|_{H^k(\mathbb{R}^d \to \mathbb{C}^m)} \sim_k \left\| \langle \xi \rangle^k \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^d \to \mathbb{C}^m)} \tag{2.27}$$

From (2.26) and (2.27) we deduce that if we define for a given real number s the s-th order Sobolev space as

$$H^s(\mathbb{R}^d \to \mathbb{C}^m) := \{f \in L^2(\mathbb{R}^d \to \mathbb{C}^m) : \langle \xi \rangle^s \hat{f}(\xi) \in L^2(\mathbb{R}^d \to \mathbb{C}^m)\}$$

and give it the norm

$$\|f\|_{H^k(\mathbb{R}^d\to\mathbb{C}^m)} := \left\|\langle\xi\rangle^s \hat{f}(\xi)\right\|_{L^2(\mathbb{R}^d\to\mathbb{C}^m)}$$

then our new definition is equivalent to the the previous one in (2.21) when s is a positive integer. We have thus found how to generalize the concept of Sobolev space to all real orders and obtained a useful way of characterizing them and giving a neat expression for their norm. Nonetheless, it will always be useful to retain the first definition based on derivatives, as it carries with it the motivation behind the definition of Sobolev spaces.

As was the case with tempered distributions we are able to extend the concept of Sobolev space to periodic domains by defining given an integer k the Sobolev space $H^k(\mathbb{T}^d \to \mathbb{C}^m)$ as the space of square integrable $\mathbb{Z}^d$ periodic functions with distributional derivatives themselves square integrable. Explicitly we define:

$$\begin{aligned} &H^k(\mathbb{T}^d \to \mathbb{C}^m) := \\ &\qquad \{f \in L^2(\mathbb{T}^d \to \mathbb{C}^m) : D^\alpha f \in L^2(\mathbb{T}^d \to \mathbb{C}^m) \hookrightarrow \mathcal{S}'(\mathbb{T}^d \to \mathbb{C}^m) \quad \forall\, |\alpha| \le k\} \end{aligned} \tag{2.28}$$

Using the same method as before, this time by Proposition 2.19 and equation (2.25), we deduce that

$$D^\alpha f \in H^k(\mathbb{T}^d \to \mathbb{C}^m) \iff \widehat{D^\alpha f}(k) = |k^\alpha| \hat{f}(k) \in l^2(\mathbb{Z}^d \to \mathbb{C}^m)$$

which leads us as in the previous case to defining for $s \in \mathbb{R}$ the more general Sobolev space

$$H^s(\mathbb{T}^d \to \mathbb{C}^m) := \{f \in L^2(\mathbb{R}^d \to \mathbb{C}^m) : \langle k\rangle^s \hat{f}(k) \in l^2(\mathbb{Z}^d \to \mathbb{C}^m)\}$$

and to giving it the norm

$$\|f\|_{H^s(\mathbb{T}^d\to\mathbb{C}^m)} := \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2\right)^{\frac{1}{2}} \tag{2.29}$$

where of course the definitions in (2.28) and (2.29) coincide for $s \in \mathbb{N}$. Note that, by the previous discussion, we have that both in the euclidean and periodic case

$$f \in H^s(\mathbb{R}^d \to \mathbb{C}^m) \implies D^\alpha f \in H^{s-|\alpha|}(\mathbb{R}^d \to \mathbb{C}^m) \qquad \forall |\alpha| \le s \tag{2.30}$$

$$f \in H^s(\mathbb{T}^d \to \mathbb{C}^m) \implies D^\alpha f \in H^{s-|\alpha|}(\mathbb{T}^d \to \mathbb{C}^m) \qquad \forall |\alpha| \le s \tag{2.31}$$

In the future we shall have use of the following lemma which will allow us to justify the existence of certain integrals and also allow us to interchange derivatives.

Lemma 2.22. *Given $f \in H^s(\mathbb{T}^d \to \mathbb{C}^m)$ with $s > d/2$. Then the Fourier series of f is absolutely convergent and $f \in C(\mathbb{T}^d \to \mathbb{C}^m)$ with the bound*

$$\|f\|_{L^\infty(\mathbb{T}^d\to\mathbb{C}^m)} \lesssim_{d,s} \|f\|_{H^s(\mathbb{T}^d\to\mathbb{C}^m)}$$

Proof. The proof is an application of the Cauchy-Schwartz inequality and (2.7). We have that

$$\sum_{k\in\mathbb{Z}^d} \left|\hat{f}(k)\right| = \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-s}\langle k\rangle^{s}\left|\hat{f}(k)\right| \le \left(\sum_{k\in\mathbb{Z}^d} \frac{\langle k\rangle^{-2s}}{2}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^d} \frac{\langle k\rangle^{2s}}{2}\left|\hat{f}(k)\right|^2\right)^{\frac{1}{2}}$$

$$\lesssim_{d,s} \|f\|_{H^s(\mathbb{T}^d\to\mathbb{C}^m)} < \infty.$$

In consequence, the sum

$$\sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{2\pi i k\cdot x} \tag{2.32}$$

converges absolutely. Since by Plancherel's Theorem the above sum also converges in $L^2(\mathbb{T}^d \to \mathbb{C}^m)$ to f we deduce that (2.32) converges almost everywhere to f (for example by taking a subsequence of the above sum that converges almost everywhere to f). Therefore

$$\|f\|_{L^\infty(\mathbb{T}^d\to\mathbb{C}^m)} = \left\|\sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{2\pi i k\cdot x}\right\|_{L^\infty(\mathbb{T}^d\to\mathbb{C}^m)}$$

which is

$$\leq \sum_{k\in\mathbb{Z}^d} \left\|\hat{f}(k)e^{2\pi i k\cdot x}\right\|_{L^\infty(\mathbb{T}^d\to\mathbb{C}^m)} = \sum_{k\in\mathbb{Z}^d} \left|\hat{f}(k)\right| \lesssim_{d,s} \|f\|_{H^s(\mathbb{T}^d\to\mathbb{C}^m)}.$$

The continuity of f follows from the point-wise equality

$$f(x) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{2\pi i k\cdot x} = \int_{\mathbb{Z}^d} \hat{f}(k)e^{2\pi i k\cdot x}dk \tag{2.33}$$

together with Proposition 2.3 applied to $\mathbb{Z}^d$ with the counting measure dk. □

As a corollary of this we have the following two results

Proposition 2.23 (**Sobolev embedding**). *Let $f \in H^s(\mathbb{T}^d \to \mathbb{C}^m)$ where $s > \frac{d}{2} + k$. Then $f \in C^k(\mathbb{T}^d \to \mathbb{C}^m)$.*

Proof. By (2.31), we may apply the previous proposition to deduce that $D^\alpha f$ is continuous for all $|\alpha| \leq k$. Therefore it suffices to show that for $|\alpha| \leq k$ the distributional derivatives $D^\alpha f$ are also the classical derivatives of f which we denote by f_α.

By the hypothesis placed on f we have that the series

$$\sum_{k\in\mathbb{Z}^d} (2\pi i k)^\alpha \hat{f}(k)e^{2\pi i k\cdot x}$$

is absolutely convergent (by Lemma 2.22), and hence, we may commute the derivatives of f with the the sum in its Fourier series to deduce the point-wise equality

$$f_\alpha(x) = \sum_{k\in\mathbb{Z}^d} (2\pi i k)^\alpha \hat{f}(k)e^{2\pi i k\cdot x}.$$

Now, note that by using (2.25) for the Fourier coefficients of distributions we also have that the equality

$$D^\alpha f(x) = \sum_{k\in\mathbb{Z}^d} (2\pi i k)^\alpha \hat{f}(k)e^{2\pi i k\cdot x}$$

holds in $L^2(\mathbb{T}^d \to \mathbb{C}^m)$. From these last two equalities we deduce that $f_\alpha = D^\alpha f$ almost everywhere which concludes our proof. □

As before the previous results also have an euclidean analogue whose proof is identical on replacing all of the above sums over $\mathbb{Z}^d$ with integrals over $\mathbb{R}^d$. In fact the proofs are if anything simpler as we no longer have to deal with the convergence of infinite sums. We now conclude our preliminaries by proving one final lemma which we shall use on occasion.

Lemma 2.24. *Let* $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$ *be a Schwartz function. Then*

$$\partial_t \int_{\mathbb{R}^d} f(t)dx = \int_{\mathbb{R}^d} \partial_t f(t)dx$$

Proof. Let $s > \frac{d}{2}$. By the definition of the Schwartz Space we have that

$$|\langle x\rangle^s \partial_t f(t,x)| \leq \|\langle x\rangle^s \partial_t f(t,x)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}^m)} < \infty \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

which implies that

$$|\partial_t f(t,x)| \leq \langle x\rangle^{-s} \|\langle x\rangle^s \partial_t f(t,x)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}^m)} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

Since $\langle x\rangle^{-s}$ is integrable this implies that f verifies properties $a)$ and $b)$ of Proposition 2.4 which proves our lemma. □

Chapter 3. The heat equation

Here we begin our study of the (generalized multicomponent) *heat equation*. Which is given by the equation:

$$\begin{aligned} \partial_t u &= \nu \Delta u + F \\ u(0,x) &= u_0(x) \end{aligned} \tag{3.1}$$

where ν is a positive constant, the solution u is to be defined on the space-time region $[0,T] \times \mathbb{R}^d$ with values in $\mathbb{R}^m$ and where $u_0 : \mathbb{R}^d \to \mathbb{R}^m$ and $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ are both known functions. Depending on whether $F = 0$ or $F \neq 0$ we will call the equation resulting from (3.1) the homogeneous or inhomogeneous heat equation respectively.

We begin our study of the heat equation by observing that, as noted by Tychonoff in [19], the solutions to (3.1) are in general not unique for any dimension $d \in \mathbb{N}$ even if $u_0 = 0$ and u is required to be smooth. In general we must set (in addition to some degree of regularity) a hypothesis of decay at infinity of u to obtain uniqueness of solutions to (3.1). Our next theorem proves exactly this for the case of smooth solutions:

Proposition 3.1. *If u_1, u_2 are solutions to (3.1) which are C^1 in time (i.e, $\partial_t^1 u$ is in $C^0([0,T] \times \mathbb{R}^d \to \mathbb{R}^m)$) obeying the bounds:*

$$\|u_i\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^d \to \mathbb{R}^m)} := \sup_{t\in[0,T]} \|u_i(t)\|_{L^2(\mathbb{R}^d \to \mathbb{R}^m)} < \infty \qquad i = 1,2 \tag{3.2}$$

then $u_1 = u_2$ (recall that $u_i(t)$ denotes the function $u_i(t,\cdot)$).

Proof. Let us set the difference of the two solutions as $u := u_1 - u_2$. Then we observe the three following facts: by the linearity of Δ, u solves the homogeneous heat equation with initial condition $u_0 = 0$, u is C^1 in time and u obeys the energy bound of (3.2). Therefore our goal becomes proving that any function verifying these three properties is identically zero. We observe that by working component by component we may suppose that $m = 1$. Let us set the "local energy" as:

$$E(t,R) := \int_{\mathbb{R}^d} u(t,x)^2 \phi(x/R)\, dx,$$

where $\phi \in C_c^\infty(\mathbb{R}^d)$ is a smooth non-negative bump function compactly supported in say: $B(0,2)$ such that $\phi\Big|_{B(0,1)} = 1$. The reason for introducing such a bump function is to be able to exchange a derivative with the integral sign and to be able to integrate by parts in subsequent expressions, as we shall soon see.

Since (by the decay condition) for each $t \in \mathbb{R}$ $u(t) \in L^2(\mathbb{R}^d)$, we may apply the dominated convergence theorem to obtain:

$$\lim_{R\to\infty} E(t,R) = \|u(t)\|^2_{L^2(\mathbb{R}^d)} \qquad \forall t \in \mathbb{R} \tag{3.3}$$

Thus to prove our Proposition it is sufficient to see that the above limit is zero. We have that for each $R \in \mathbb{R}^d$:

$$\begin{aligned} \partial_t(u(t,x)^2\phi(x/R)) &= 2u(t,x)\partial_t u(t,x)\phi(x/R) \\ &\leq 2\|\phi\|_\infty \|u\partial_t u\|_{L^\infty([0,T]\times B(0,2R))} \chi_{B(0,2R)}(x) =: g_R(x) \end{aligned} \tag{3.4}$$

Due to the time regularity of u_i we have that, for each R, $g_R(x)$ is well defined and $g_R \in L^2(\mathbb{R}^d)$. Hence we can differentiate under the integral sign to obtain:

$$\partial_t E(t,R) = \int_{\mathbb{R}^d} 2u(t,x)\partial_t u(t,x)\phi(x/R)dx = \int_{\mathbb{R}^d} 2u(t,x)\nu\Delta u(t,x)\phi(x/R)dx \tag{3.5}$$

By now integrating by parts various times (which we may do as the derivatives of ϕ are all of course compactly supported) we obtain:

$$\partial_t E(t,R) \leq -\nu \int_{\mathbb{R}^d} 2u(t,x)\partial_i u(t,x)\partial_i(\phi(x/R))dx = -\nu \int_{\mathbb{R}^d} \partial_i(u(t,x)^2)\partial_i(\phi(x/R))dx$$

$$= \nu \int_{\mathbb{R}^d} u(t,x)^2\Delta(\phi(x/R))dx = \nu R^{-2} \int_{\mathbb{R}^d} u(t,x)^2(\Delta\phi)(x/R)dx.$$

$\Delta\phi$ is of compact support and continuous, hence it is bounded and we thus obtain the bound:

$$|\partial_t E(t,R)| \lesssim_{\phi,\nu} R^{-2}\|u\|^2_{L^\infty_t L^2_x([0,T]\times\mathbb{R}^d)}.$$

We now use that, as can be seen by (3.5), $E(0,R) = 0$ and $E(t,R)$ is C^1 in time, to conclude from the mean value theorem that:

$$|E(t,R)| \lesssim_{\phi,\nu,T} R^{-2}\|u\|^2_{L^\infty_t L^2_x([0,T]\times\mathbb{R}^d)}$$

and thus the limit in (3.3) vanishes, concluding our proof. □

While we used in the preceding proof that the interval $[0,T]$ is compact we note that if we impose the stronger decay condition

$$\|u_i\|_{L^\infty_t L^2_x([0,\infty)\times\mathbb{R}^d\to\mathbb{R}^m)} < \infty$$

we obtain via the previous theorem uniqueness of solutions to (3.1) on each compact interval $[0,T]$ and hence, uniqueness of solutions on $[0,\infty)\times\mathbb{R}^d$.

This said we now turn to the existence of solutions to the heat equation. We reason as follows: suppose we are given a very well behaved solution u to the heat equation (3.1), say for example $u \in \mathcal{S}([0,T]\times\mathbb{R}^d \to \mathbb{R}^m)$. Then we have that, taking the spatial Fourier transform

$$\hat{u}(t,\xi) := \int_{\mathbb{R}^d} u(t,x)e^{-2\pi i\xi\cdot x}dx$$

(this is by definition just $\widehat{u(t)}(\xi)$) and differentiating with respect to time

$$\partial_t\hat{u}(t,\xi) \overset{(i)}{=} \int_{\mathbb{R}^d} \partial_t u(t,x)e^{-2\pi i\xi\cdot x}dx = \int_{\mathbb{R}^d}(\nu\Delta u(t,x) + F(t,x))e^{-2\pi i\xi\cdot x}dx$$

$$\overset{(ii)}{=} -4\pi^2\nu|\xi|^2\hat{u}(t,\xi) + \hat{F}(t,\xi)$$

where the derivation under the integral sign in (i) is justified by lemma 2.24, as u is a Schwartz function, and in (ii) we used 2.13 (note that all Schwartz functions are integrable). In addition to this we have that the initial condition $u(0,x) = u_0(x)$ gives an initial condition for $\hat{u}$:

$$\hat{u}(0,\xi) = \int_{\mathbb{R}^d} u_0(x)e^{-2\pi i\xi\cdot x}dx = \hat{u}_0(\xi).$$

We have thus obtained a non homogeneous ODE (dependent on the parameter ξ) for $\hat{u}$. For each $\xi \in \mathbb{R}^d$ this has the unique solution given by Lemma 2.2 (Duhamel's formula)

$$\hat{u}(t,\xi) = e^{-4\pi^2\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-4\pi^2\nu|\xi|^2(t-t')}\hat{F}(t',\xi)dt' \tag{3.6}$$

Applying the Fourier inversion formula (and Fubini's theorem) we finally obtain that:

$$u(t,x) = \int_{\mathbb{R}^d} e^{-4\pi^2\nu|\xi|^2 t}\hat{u}_0(\xi)e^{2\pi i\xi\cdot x}d\xi + \int_0^t \left(\int_{\mathbb{R}^d} e^{-4\pi^2\nu|\xi|^2(t-t')}\hat{F}(t',\xi)e^{2\pi i\xi\cdot x}d\xi\right)dt' \tag{3.7}$$

We shall write this in a slightly more succinct fashion as:

$$u(t,x) = e^{\nu t\Delta}u_0(x) + \int_0^t e^{\nu(t-t')\Delta}F(t',x)dt' \tag{3.8}$$

where, given $t \in [0,T]$, $e^{\nu t\Delta}$ is the operator defined on functions $f \in S(\mathbb{R}^d \to \mathbb{R}^m)$ by:

$$e^{\nu t\Delta}f(x) := \int_{\mathbb{R}^d} e^{-4\pi^2\nu|\xi|^2 t}\hat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi \quad \forall x \in \mathbb{R}^d$$

and where for each t'

$$e^{\nu(t-t')\Delta}F(t',x) := (e^{\nu(t-t')\Delta}F(t'))(x).$$

We have thus obtained via (3.8) (or equivalently (3.7)) a guess at what our solutions to the heat equation should look like. It is simple to see that if u_0 and F are Schwartz then, for u defined as in (3.7) , u is smooth in time and space and one may integrate under the integral sign and apply Leibniz's Integration Rule to obtain that u verifies the heat equation (3.1) as desired. In addition to this, by using (3.6), and the fact that the Fourier transform is a unitary tranformation on $L^2(\mathbb{R}^d \to \mathbb{R}^m)$ it is clear that u verifies the estimate

$$\|u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^d\to\mathbb{R}^m)} \le \|u_0\|_{L^2(\mathbb{R}^d\to\mathbb{R}^m)} + T\|F\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^d\to\mathbb{R}^m)} < \infty \quad \forall T \in \mathbb{R}^+$$

which, due to Proposition 3.1, implies that, for all $T \in \mathbb{R}^+$, u is the unique solution to (3.1) in $L_t^\infty L_x^2([0,T] \times \mathbb{R}^d \to \mathbb{R}^m)$.

We now proceed to extend these results to less regular functions. Given a Schwartz function f we have that given $s \ge 0$

$$\left\|e^{\nu t\Delta}f\right\|_{H^s(\mathbb{R}^d\to\mathbb{R}^m)} = \left\|\langle\xi\rangle^s \widehat{e^{\nu t\Delta}f}(\xi)\right\|_{L^2(\mathbb{R}^d\to\mathbb{R}^m)} = \left\|\langle\xi\rangle^s e^{-4\pi^2\nu|\xi|^2 t}\hat{f}(\xi)\right\|_{L^2(\mathbb{R}^d\to\mathbb{R}^m)} \le \|f\|_{H^s(\mathbb{R}^d\to\mathbb{R}^m)}$$

and therefore $e^{\nu t\Delta}$ may be extended by density to a non-expansive operator[1]

$$e^{\nu t\Delta} : H^s(\mathbb{R}^d \to \mathbb{R}^m) \to H^s(\mathbb{R}^d \to \mathbb{R}^m).$$

[1]Given two metric spaces $(X, \|\|_X), (Y, \|\|_Y)$ a *non-expansive operator* is a function $f : X \to Y$ such that for some constant $C \in \mathbb{R} : \|f(x)\|_Y \le C\|x\|_X \quad \forall x \in X$.

The Periodic heat equation

Our discussion of the heat equation goes through more more or less identically in the periodic case. That is, when we assume that the functions u_0, F are periodic (of period for example $\mathbb{Z}^d$). In terms of regularity we begin our discussion by supposing that F and u_0 are smooth and we will search for a smooth periodic solution u. Since we will now be dealing with the periodic Fourier transform, the analogous formula to the one we derived in (3.7) is :

$$u(t,x) = \sum_{k\in\mathbb{Z}^d} e^{-4\pi^2\nu|k|^2 t}\hat{u}_0(k)e^{2\pi i k\cdot x} + \int_0^t \left(\sum_{k\in\mathbb{Z}^d} e^{-4\pi^2\nu|k|^2(t-t')}\hat{F}(t',k)e^{2\pi i k\cdot x}\right)dt' \tag{3.9}$$

which by abuse of notation we will once again write as

$$u(t,x) = e^{\nu t\Delta}u_0(x) + \int_0^t e^{\nu(t-t')\Delta}F(t',x)dt' \tag{3.10}$$

where for each $t \in [0,T]$ the operator $e^{\nu t\Delta}$ is defined on smooth periodic functions by

$$\widehat{e^{\nu t\Delta}f}(k) := e^{-4\pi^2\nu|k|^2 t}\hat{f}(k) \quad \forall f \in C^\infty(\mathbb{T}^d \to \mathbb{R}^m) \tag{3.11}$$

Using the expression in (3.11) we may, just as in the non-periodic case, extend $e^{\nu t\Delta}f$ to a non expansive operator by defining

$$\widehat{e^{\nu t\Delta}f}(k) := e^{-4\pi^2\nu|k|^2 t}\hat{f}(k) \quad \forall f \in H^s(\mathbb{T}^d \to \mathbb{R}^m) \tag{3.12}$$

We shall find use for this later when we prove the existence of ("mild") solutions to the Navier-Stokes equations. For now though, we deal with the case of the smooth periodic functions u_0 and F. We have that, for each $x \in \mathbb{T}^d$, the series of absolute values of the derivatives with respect to t of the terms in the sums in (3.9) verify

$$\sum_{k\in\mathbb{Z}^d} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 t}|\hat{u}_0(k)| \lesssim \sum_{k\in\mathbb{Z}^d}|k|^2|\hat{u}_0(k)| \tag{3.13}$$

$$\sum_{k\in\mathbb{Z}^d} 4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2(t-t')}\left|\hat{F}(t',k)\right| \lesssim \sum_{k\in\mathbb{Z}^d}|k|^2\left|\hat{F}(t',k)\right| \tag{3.14}$$

and, since $\hat{u}_0$ and $\hat{F}(t')$ are of rapid decrease for each $t' \in [0,T]$, these two series converge and hence, by Proposition 2.4 applied to the counting measure on $\mathbb{Z}^d$, we may commute the derivative with respect to t with the sums in (3.9). We also observe that taking $s > \frac{d}{2}$ and applying the Cauchy-Schwartz inequality together with Lemma 2.7 gives

$$\sum_{k\in\mathbb{Z}^d}|k|^2\left|\hat{F}(t',k)\right| \lesssim_{d,s} \sum_{k\in\mathbb{Z}^d}\frac{|k|^4\langle k\rangle^{2s}}{2}\left|\hat{F}(t',k)\right|^2 \le \left\|\Delta^{s+2}F(t')\right\|^2_{L^2(\mathbb{T}^d\to\mathbb{R}^d)} \tag{3.15}$$

where the function of t' in (3.15) is integrable on $[0,T]$ by the smoothness of F. Hence we may apply Proposition 2.5 (Leibnitz's integral rule) to obtain that

$$\begin{aligned}\partial_t u(t,x) &= \sum_{k\in\mathbb{Z}^d} -4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2 t}\hat{u}_0(k)e^{2\pi i k\cdot x}\\ &+ \int_0^t\left(\sum_{k\in\mathbb{Z}^d} -4\pi^2\nu|k|^2 e^{-4\pi^2\nu|k|^2(t-t')}\hat{F}(t',k)e^{2\pi i k\cdot x}\right)dt' + \sum_{k\in\mathbb{Z}^d}\hat{F}(t,k)e^{2\pi i k\cdot x}\end{aligned} \tag{3.16}$$

The reasoning just carried out is identical to the one necessary to justify the interchange of Δ with the sums and integrals in (3.9) (as the sums in (3.13) and (3.14) don't change when we consider Δ instead of the differential with t). Hence we obtain from (3.16) and Plancherel's theorem that

$$\partial_t u(t,x) = \Delta u(t,x) + F(t,x)$$

as desired. Furthermore an identical reasoning shows us that we may commute the derivatives ∂_t^m and D^α for arbitrary $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}^d$. Therefore the solution to the heat equation in (3.9) is smooth. We end the our study of the heat equation with the periodic analogue for Proposition 3.1 which, in particular, proves that the solution in (3.9) is unique. The proof is if anything simpler as we now work on the compact set $[0,T] \times \mathbb{T}^d$.

Proposition 3.2. *Let u_1 u_2 be two solutions to the periodic heat equation that are C^1 in time or C^2 in space and such that $u_1(0) = u_2(0)$. Then $u_1 = u_2$.*

Proof. Once again it is sufficient to prove our proposition in the case where $m = 1$. As before we consider the difference $u := u_1 - u_2$ and it's energy, which we now define as

$$E(t) := \int_{\mathbb{T}^d} u(t,x)^2 dx = \|u(t)\|^2_{L^2(\mathbb{T}^d \to \mathbb{R}^d)}$$

we observe that u satisfies the heat equation with "forcing term" $F = 0$ and therefore

$$\partial_t u(t,x)^2 = 2u(t,x)\partial_t u(t,x) = 2u(t,x)\Delta u(t,x)$$

which is continuous by hypothesis and hence, bounded by some constant M (which will of course be integrable on $\mathbb{T}^d$). Therefore, by Proposition 2.4 (differentiation under the integral sign), we may commute the derivative with respect to t with the integral that defines $E(t)$. Obtaining on integrating by parts that

$$\partial_t E(t) = 2\int_{\mathbb{T}^d} u(t,x)\Delta u(t,x) = -2\int_{\mathbb{T}^d} \partial_i u(t,x)\partial_i u(t,x) \leq 0 \tag{3.17}$$

and in consequence, since $E(0) = 0$, we deduce that $E = 0$ which of course implies that $u = 0$, concluding our proof. □

Finally we end this section by noting that, due to the effect of the negative exponential $e^{-4\pi|k|^2 t}$ in the Fourier transform of u, it is not difficult to show the following smoothing effect for $t > 0$, (see [12] page 240 for instance).

Proposition 3.3 (Instantaneous smoothing for the heat equation). *Let u_0 be a function in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$. Then $e^{\nu t\Delta}u_0$ is smooth on $(0,T] \times \mathbb{T}^d$ and verifies*

$$\partial_t e^{\nu t\Delta}u_0(x) = \nu\Delta e^{\nu t\Delta}u_0(x) \quad \forall (t,x) \in (0,T] \times \mathbb{T}^d.$$

An analogous statement also holds in the non-periodic case where one only need change the torus $\mathbb{T}^d$ to $\mathbb{R}^d$. As we shall see later on, this instantaneous smoothing effect will be reproduced by solutions to the Navier-Stokes equations.

Chapter 4. The Leray Equations

Here we study the system given by the equations:

$$\begin{aligned} v =& F - \nabla p \\ \nabla\cdot v =& 0 \end{aligned} \tag{4.1}$$

where $v : \mathbb{R}^d \to \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}$ are to be determined and $F : \mathbb{R}^d \to \mathbb{R}^d$ is given.

We will call the system of equations given by (4.1) the *Leray equations* and we will begin by studying the existence of solutions to these equations in the periodic setting. As with the heat equation, we will commence by imposing a high degree of regularity on our functions, supposing F, v, p to be smooth (and periodic) so as to be able to appropriately manipulate their Fourier series

$$F(x) = \sum_{k\in\mathbb{Z}^d} \hat{F}(k)e^{2\pi ik\cdot x}; \quad v(x) = \sum_{k\in\mathbb{Z}^d} \hat{v}(k)e^{2\pi ik\cdot x}; \quad p(x) = \sum_{k\in\mathbb{Z}^d} \hat{p}(k)e^{2\pi ik\cdot x}.$$

As we saw in the preliminaries (Proposition 2.18), the smoothness of said functions implies that we may differentiate them term by term to obtain the new and equivalent "diagonalised" system where only Fourier coefficients of the same frequency interact:

$$\hat{v}(k) = \hat{F}(k) - 2\pi ik\hat{p}(k); \quad 2\pi ik\cdot\hat{v}(k) = 0 \tag{4.2}$$

where k ranges through $\mathbb{Z}^d$. We can use the second set of equations in (4.2) to "eliminate" v from the first equation by taking the inner product with k. Thus obtaining:

$$\hat{F}(k)\cdot k = 2\pi ik\hat{p}(k)\cdot k = 2\pi i|k|^2\hat{p}(k).$$

from which we deduce that, for $k \neq 0$:

$$\hat{p}(k) = \frac{\hat{F}(k)\cdot k}{2\pi i|k|^2},$$

and where we recover v from (4.2) as:

$$\hat{v}(k) = \hat{F}(k) - \frac{\hat{F}(k)\cdot k}{|k|^2}k \qquad k \neq 0.$$

and where necessarily (again by (4.2)), we have that $\hat{v}(0) = \hat{F}(0)$. Finally $\hat{p}(0)$ may be an arbitrary real number C and hence we obtain that the smooth solutions to (4.1) must be given by:

$$v(x) = \hat{F}(0) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left(\hat{F}(k) - \frac{\hat{F}(k)\cdot k}{|k|^2}k\right) e^{2\pi ik\cdot x}; \quad p(x) = C + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{\hat{F}(k)\cdot k}{2\pi i|k|^2}e^{2\pi ik\cdot x} \tag{4.3}$$

We will write this succinctly as:

$$v = \mathbb{P}(F); \quad p = C + \Delta^{-1}\nabla\cdot F \tag{4.4}$$

where we define:

Definition 4.1. *Given $f : \mathbb{T}^d \to \mathbb{R}$, a smooth periodic function of mean 0 (i.e. such that $\hat{f}(0) = 0$), the* inverse laplacian *of f is*

$$\widehat{\Delta^{-1} f}(k) := -\frac{\hat{f}(k)}{4\pi^2 |k|^2} \qquad k \neq 0; \quad \widehat{\Delta^{-1} f}(0) := 0 \tag{4.5}$$

Definition 4.2. *Given a smooth periodic function $F : \mathbb{T}^d \to \mathbb{R}^d$, the* Leray projection *of F is*

$$\mathbb{P}(F) := F - \nabla \Delta^{-1} \nabla \cdot F$$

or equivalently in terms of Fourier coefficients

$$\widehat{\mathbb{P}(F)}(k) := \hat{F}(k) - \frac{\hat{F}(k) \cdot k}{|k|^2} k \qquad k \neq 0; \quad \widehat{\mathbb{P}(F)}(0) := \hat{F}(0) \tag{4.6}$$

Notice that since f is smooth $\widehat{\Delta^{-1} f}(k)$ is in $l^2(\mathbb{Z}^d \to \mathbb{R})$ (in fact it is in $s(\mathbb{Z}^d \to \mathbb{R})$) and hence, by Plancherel's theorem, (4.5) implicitly defines $\Delta^{-1} f$. The analogous statement also being true for $\mathbb{P}(F)$.

By a direct calculation of Fourier coefficients by means of (2.13) it is simple to see that the functions given by (4.3) solve the Leray equations (4.1). We now extend these results to Sobolev functions. By the expression in (4.6) we have that, given a smooth function F

$$\begin{aligned} \|\mathbb{P}(F)\|^2_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} &\leq \|F\|^2_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{2s} \left| \frac{\hat{F}(k) \cdot k}{|k|^2} k \right|^2 \\ &\leq \|F\|^2_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{2s} \left| \hat{F}(k) \right|^2 \lesssim \|F\|^2_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} \end{aligned} \tag{4.7}$$

and:

$$\begin{aligned} \left\| \Delta^{-1} \nabla \cdot F \right\|^2_{H^{s+1}(\mathbb{T}^d \to \mathbb{R}^d)} &= \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{2(s+1)} \left| \frac{\hat{F}(k) \cdot k}{2\pi |k|^2} \right|^2 \leq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{2s} \frac{1 + |k|^2}{4\pi^2 |k|^2} \left| \hat{F}(k) \right|^2 \\ &\leq \frac{2}{4\pi^2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{2s} \left| \hat{F}(k) \right|^2 \leq \|F\|^2_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} \end{aligned} \tag{4.8}$$

Thus we may also extend $\mathbb{P}$ and $\Delta^{-1} \nabla \cdot$ to the Sobolev spaces $H^s(\mathbb{T}^d \to \mathbb{R}^d)$ for all $s \in \mathbb{R}$ as non-expansive operators with image respectively in $H^s(\mathbb{T}^d \to \mathbb{R}^d)$ and $H^{s+1}(\mathbb{T}^d \to \mathbb{R}^d)$. We now state some further properties of the Leray Projection which will serve us later on among other things to eliminate the pressure term from the Navier-Stokes equations.

Lemma 4.3. *The following statements hold:*
a) Let $p \in H^1(\mathbb{R}^d \to \mathbb{R})$ then $\mathbb{P}(\nabla p) = 0$.
b) Let $v \in H^1(\mathbb{R}^d \to \mathbb{R}^d)$ be divergence free, then $\mathbb{P}(v) = v$ and $\nabla \cdot \mathbb{P}(v) = 0$.

Proof. Both properties may be proved by a direct calculation of Fourier coefficients by means of the formula for derivatives of distributions in (2.25). In the case of a) we have that

$$\widehat{\mathbb{P}(\nabla p)}(k) = \widehat{\nabla p}(k) - \frac{\widehat{\nabla p}(k) \cdot k}{|k|^2} k = \hat{p}(k) k - \hat{p}(k) k = 0, \quad k \neq 0$$

$$\widehat{\mathbb{P}(\nabla p)}(0) = \widehat{\nabla p}(0) = \hat{p}(0) 0 = 0,$$

and for b) we have that

$$\begin{aligned}
\widehat{\mathbb{P}(v)}(k) &= \hat{v}(k) - \frac{\hat{v}(k)\cdot k}{|k|^2}k = \hat{v}(k) - \frac{\widehat{\nabla\cdot v}(k)}{|k|^2}k = \hat{v}(k), \quad k \neq 0 \\
\widehat{\mathbb{P}(v)}(0) &= \hat{v}(0), \\
\widehat{\nabla\cdot\mathbb{P}(v)}(k) &= \hat{v}(k)\cdot k - \frac{\hat{v}(k)\cdot k}{|k|^2}k\cdot k = \hat{v}(k)\cdot k - \hat{v}(k)\cdot k = 0, \quad k \neq 0 \\
\widehat{\nabla\cdot\mathbb{P}(v)}(0) &= \widehat{\mathbb{P}(v)}\cdot 0 = 0
\end{aligned}$$

which allows us to conclude by Plancherel's theorem. □

All the previous arguments may be extended for the non-periodic setting if we assume that $d > 1$ so that $|x|^{-1}$ be locally integrable. Given F in the Schwartz space $\mathcal{S}(\mathbb{R}^d \to \mathbb{R}^d)$ we define

$$\widehat{\Delta^{-1}\nabla\cdot F}(\xi) := \frac{-i\xi\cdot\hat{F}(\xi)}{2\pi|\xi|^2} \tag{4.9}$$

The function defined on the right side of (4.9) is in $L^1(\mathbb{R}^d \to \mathbb{R})$ as

$$\left\| \frac{-i\xi\cdot\hat{F}(\xi)}{2\pi|\xi|^2} \right\|_{L^1(\mathbb{R}^d\to\mathbb{R})} \leq \|F\|_{L^1(\mathbb{R}^d\to\mathbb{R}^d)} \int_{B(0,1)} \frac{d\xi}{|\xi|} + \int_{B(0,1)^c} \left|\hat{F}(\xi)\right| d\xi < \infty$$

where we used that $d > 1$ and the fact that $\hat{F}$ is Schwartz and hence of rapid decrease. In consequence the function on the right hand side of (4.9) defines a tempered distribution and by Fourier inversion for tempered distributions (4.9) implicitly defines the tempered distribution $\Delta^{-1}\nabla\cdot F$. Defining as before

$$\mathbb{P}(F) := F - \nabla\Delta^{-1}\nabla\cdot F$$

or equivalently

$$\widehat{\mathbb{P}(F)}(\xi) := \hat{F}(\xi) - \frac{\xi\cdot\hat{F}(\xi}{|\xi|^2}\xi \tag{4.10}$$

we have by the formulas derived in (2.25) that the tempered distributions

$$v = \mathbb{P}(F); \quad p = \Delta^{-1}\nabla\cdot F$$

solve (4.1) in the distributional sense, i.e.

$$\langle u, v\rangle = \langle u, F - \nabla p\rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}^d).$$

Note that since, by hypothesis, $\hat{F}(\xi)$ is rapidly decreasing then by (4.10) and the euclidean analogue of Proposition 2.23 we have that $\mathbb{P}(F)$ and $\nabla\Delta^{-1}\nabla\cdot F$ are in fact smooth and hence (v, p) are not only a distributional solution but also a classical solution to (4.1). Additionally, the equation (4.10) for the Fourier coefficients of $\mathbb{P}(F)$ show that, as in the periodic case, $\mathbb{P}$ may be extended to a non-expansive operator on the Sobolev space $H^s(\mathbb{R}^d \to \mathbb{R}^d)$.

Chapter 5. The Navier-Stokes Equations

In the physical derivation of the preliminaries we deduced the Navier-Stokes equations for an incompressible fluid. For simplicity, we will normalize the fluid density ρ to be one in these equations. That is, we will take the *Navier-Stokes equations* (N.S. for short) to be:

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ \nabla \cdot u &= 0 \end{aligned} \tag{5.1}$$

where $\nu \in \mathbb{R}^+$ is a strictly positive real constant called the kinematic viscosity and :

$$u : I \times \mathbb{R}^d \to \mathbb{R}^d; \quad p : I \times \mathbb{R}^d \to \mathbb{R}$$

with $I = [0, T] \subset \mathbb{R}$ are both unknown. We shall call u the velocity and p the pressure.

5.1 Uniqueness of solutions to the Navier-Stokes Equations

We begin our study of the Navier-Stokes equations by discussing the uniqueness of solutions to said equations. In the case where $d = 1$ we have that $\nabla \cdot u = \partial_x u = \nabla u$ and $\Delta u = \partial_{xx} u$ and hence the Navier-Stokes equations are reduced to

$$\partial_t u = -\partial_x p; \qquad \partial_x u = 0 \tag{5.2}$$

The second equation of (5.2) implies that u is constant in space and, therefore, its solutions are given by taking u to be any differentiable function of time constant in space and

$$p(t, x) = -\partial_t u(t) x + f(t) \tag{5.3}$$

where f is any differentiable function that is constant in the space variable. Hence, for $d = 1$ we have very little uniqueness to the Navier-Stokes equations. On the other hand, if we impose that the pressure p be bounded in space, then, from (5.3) we deduce that $\partial_t u = 0$ and therefore, u is constant and p is just a function of time.

We now go on to study the case when $d > 1$ and where u is supposed to be space periodic of period $\mathbb{Z}^d$. In terms of regularity we will require that u be C^1 in time and C^2 in space (which will of course mean by (5.1) that p will be C^1 in space and continuous in time). We abbreviate this in this section by saying u and p are smooth. This said, from the first equation in N.S., we easily deduce that ∇p is also space periodic, from where we deduce that there exists a smooth family of functions only of time $\{a_k\}_{k \in \mathbb{Z}^d}$ such that

$$p(t, x + k) - p(t, x) = a_k(t).$$

By now applying induction on $|k| := k_1 + \ldots + k_d$ we deduce that there exists a continuous function of time $a(t)$ such that

$$p(t, x + k) - p(t, x) = k \cdot a(t) \tag{5.4}$$

expression from which we deduce that $p_1(t,x) := p(t,x) - x{\cdot}a(t)$ is $\mathbb{Z}^d$ periodic as:

$$p_1(t,x+k) - p_1(t,x) = k{\cdot}a(t) - ((x+k){\cdot}a(t) - x{\cdot}a(t)) = 0 \quad \forall k \in \mathbb{Z}^d.$$

By subtracting off the mean of p_1; $\quad r(t) := \int_{\mathbb{T}^d} p_1(t,x)dx$ and defining

$$p_0(t,x) = p_1(t,x) - r(t) = p(t,x) - x{\cdot}a(t) - r(t) \tag{5.5}$$

p_0 remains ($\mathbb{Z}^d$) periodic and we may write:

$$p(t,x) = p_0(t,x) + x{\cdot}a(t) + r(t) \tag{5.6}$$

where p_0 is a smooth periodic function of mean 0. Observe that this way of writing p is unique as if we have:

$$p(t,x) = p_0'(t,x) + x{\cdot}a'(t) + r'(t) \tag{5.7}$$

with p_0' periodic and of mean 0 then we deduce from (5.6) and (5.7) that

$$x{\cdot}(a'(t) - a(t)) = p_0(t,x) - p_0'(t,x) + r(t) - r'(t) \tag{5.8}$$

which implies that $x{\cdot}(a'(t) - a(t))$ is periodic and in consequence that $a'(t) - a(t) = 0$. Now taking the mean on both sides of (5.8) we obtain

$$0 = \int_{\mathbb{T}^d} r(t) - r'(t) = r(t) - r'(t).$$

We have thus obtained $a(t) = a'(t); r(t) = r'(t)$ and therefore the functions in (5.6) are unique as desired. From the uniqueness of (5.6) we obtain that

$$p \text{ is periodic and of mean } 0 \iff a(t) = r(t) = 0 \tag{5.9}$$

In this case we will say that the pressure p is *normalised*.

We now prove the following lemma that will show non-uniqueness of (u) periodic solutions to N.S. by using the expression derived for p_0 in (5.5) to create other solutions to N.S. To do this we will apply something resembling a Galilean transformation of the spatial coordinate to cancel out the affine term $-x{\cdot}a(t) - r(t)$.

Lemma 5.1. *Let p_0 be as in (5.5) where a is continuous and (u,p) is a smooth u-periodic solution to N.S. Then by setting*

$$v(t) := \int_0^t a(s)ds; \quad X(t) := \int_0^t v(s)ds$$

and $u_2(t,x) := u(t,x - X(t)) + v(t)$; $\quad p_2(t,x) := p_0(t,x - X(t))$ we have that u_2, p_2 is a smooth periodic solution to N.S. with $u_2(0,x) = u(0,x)$.

Proof. We have that by the chain rule and the fundamental theorem of calculus

$$\begin{aligned}\partial_t u_2(t,x) &= \partial_t(t, x - X(t)) \cdot (\partial_t u(t,x - X(t)), \nabla u(t,x - X(t))) + a(t)\\ &= \partial_t u(t,x - X(t)) - v(t){\cdot}\nabla u(t,x - X(t)) + a(t)\end{aligned}$$

$$(u_2{\cdot}\nabla u_2)(t,x) = (u{\cdot}\nabla u)(t,x - X(t)) + v(t){\cdot}\nabla u(t,x - X(t))$$

$$\nu\Delta u_2(t,x) = \nu\Delta u(t,x - X(t))$$

$$-\nabla p_2(t,x) = -\nabla p(t,x - X(t)) + a(t).$$

Since (u,p) is a solution to N.S. we therefore have that, by using our previous calculations, so is (u_2,p_2) as:

$$\begin{aligned}\partial_t u_2(t,x) + (u_2\cdot\nabla u_2)(t,x) &= \partial_t u(t,x-X(t)) + a(t) + (u\cdot\nabla u)(t,x-X(t))\\ &= \nu\Delta u(t,x-X(t)) - \nabla\cdot p(t,x-X(t)) + a(t) = \nu\Delta u_2(t,x) - \nabla p_2(t,x)\end{aligned}$$

as desired as, by definition of u_2, we have that $u_2(0,x) = u(0,x)$. □

As an immediate consequence of our lemma we have the following proposition

Proposition 5.2. *The u-periodic Navier-Stokes equations do not have an unique solution.*

By varying $a(t)$ and $r(t)$ in (5.5) (and by applying a suitable transformation) we were able to obtain many solutions to the Navier-Stokes equations (once we have one). We now consider the case in which the pressure is normalised, and hence, $a(t)$ and $r(t)$ are fixed as zero. As we shall see in the following theorem, in this case, the uniqueness of solutions to the periodic N.S. is guaranteed.

Theorem 5.3. *There exists at most one smooth solution (u,p) to the Navier-Stokes equations subject to the following constraints:*
a) u is periodic with initial data $u(0,x) = u_0(x)$.
b) p is periodic of mean zero.

Proof. As in the heat equation, to prove uniqueness we will use the energy method. Proving that two such solutions (u_1,p_1), (u_2,p_2) must be the same. Let us set $w := u_1 - u_2$. Since (u_1,p_1) and (u_2,p_2) are both solutions to N.S. we have that:

$$\partial_t w = -u_1\cdot\nabla u_1 + u_2\cdot\nabla u_2 + \nu\Delta w - \nabla(p_1 - p_2)$$

and since

$$u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2 = u_1\cdot\nabla w + w\cdot\nabla u_2$$

we thus have that w is a smooth solution to

$$\partial_t w = -u_1\cdot\nabla w - w\cdot\nabla u_2 + \nu\Delta w - \nabla(p_1 - p_2); \quad \nabla\cdot w = 0 \tag{5.10}$$

We now set the energy to

$$E(t) := \int_{\mathbb{T}^d} |w(t,x)|^2 dx$$

and observe that, as w is C^1 in time, the derivative with respect to t of the integrand is bounded on $\mathbb{T}^d$. Hence, we may differentiate under the integral sign, obtaining:

$$\partial_t E(t) = \int_{\mathbb{T}^d} \partial_t |w(t,x)|^2 dx = 2\int_{\mathbb{T}^d} \partial_t(w_i(t,x)w_i(t,x))dx = 2\int_{\mathbb{T}^d} \partial_t w(t,x)\cdot w(t,x)dx.$$

Now using (5.10) gives

$$\partial_t E(t) = \lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) \tag{5.11}$$

where (omitting t and x to simplify the notation in the remaining expressions), λ_i are defined by

$$\lambda_1 := -2\int_{\mathbb{T}^d}(u_1\cdot\nabla w)\cdot w; \quad \lambda_2 := -2\int_{\mathbb{T}^d}(w\cdot\nabla u_2)\cdot w$$

$$\lambda_3 := 2\int_{\mathbb{T}^d}\nu\Delta w\cdot w; \quad \lambda_4 := -2\int_{\mathbb{T}^d}\nabla(p_1-p_2)\cdot w.$$

To prove our theorem we wish to see that $E(t)$ vanishes. To accomplish this we will prove that the sum in (5.11) is smaller or equal to $CE(t)$ for some constant C . If we achieve this it follows by Lemma 2.1 (Gronwall's inequality) that $E(t) \leq e^{Ct}E(0) = 0$ and since, by definition, the energy $E(t)$ is positive this will imply that $E(t) = 0$ and hence $u_1 = u_2$. By (5.1) this implies that $\nabla p_1 = \nabla p_2$ and therefore, since p_1 and p_2 have the same mean, $p_1 = p_2$ concluding our proof. We now set our plan in action. Observe that

$$2(u_1\cdot\nabla w)\cdot w = 2(u_{1,j}\partial_j w)_i w_i = 2(u_{1,j}\partial_j w_i)w_i = u_{1,j}(2(\partial_j w_i)w_i) = u_{1,j}(\partial_j(w_i w_i))$$

and thus, by integration by parts, as u_1 and w are $\mathbb{Z}^d$ periodic and u_1 is divergence free

$$\lambda_1 = -\int_{\mathbb{T}^d}\partial_j(u_{1,j}w_i w_i) + \int_{\mathbb{T}^d}\partial_j(u_{1,j})|w|^2 = 0 + \int_{\mathbb{T}^d}(\nabla\cdot u_1)|w|^2 = 0 \tag{5.12}$$

Using the periodicity of p_1 and p_2 and the fact that $w = u_1 - u_2$ is divergence free we may again integrate by parts to obtain that

$$\lambda_4 = -2\int_{\mathbb{T}^d}(\partial_i(p_1-p_2))w_i = 2\int_{\mathbb{T}^d}(p_1-p_2)\partial_i w_i = 2\int(p_1-p_2)\nabla\cdot w = 0 \tag{5.13}$$

Integrating by parts for λ_3 in turn gives

$$\lambda_3(t) = 2\nu\int_{\mathbb{T}^d}\partial_i\partial_i w_j w_j = -2\nu\int_{\mathbb{T}^d}\partial_i w_j\partial_i w_j \leq 0 \tag{5.14}$$

and finally since,

$$\nabla u_2 : [0,T]\times\mathbb{R}^d \to \mathcal{M}_{d,d}(\mathbb{R})$$

is a continuous (as u_2 is C^2 and in particular C^1 in space) function to the space $\mathcal{M}_{d,d}(\mathbb{R})$ of $d\times d$ matrices with real coefficients, its norm as a bilinear function defined by

$$\nabla u_2(t,x)(y_1,y_2) := y_{2,j}y_{1,i}\partial_i u_{2,j}(t,x) \quad y_1,y_2\in\mathbb{R}^d$$

is bounded by some constant C, i.e.

$$|\nabla u_2(t,x)(y_1,y_2)| \leq C|y_1||y_2|, \quad \forall y_1,y_2\in\mathbb{R}^d \quad \forall(t,x)\in[0,T]\times\mathbb{T}^d.$$

Which gives

$$\lambda_2 = -2\int_{\mathbb{T}^d}\nabla u_2(w,w) \leq -2C\int_{\mathbb{T}^d}|w|^2 = -2CE \tag{5.15}$$

We now have all the necessary ingredients for our proof as (5.12-5.15) now give

$$\partial_t E \leq \lambda_2 \leq -2CE$$

from which we conclude due to Gronwall's inequality, as in our initial discussion. □

5.2 An equivalent pressure free integral equation

We now study the existence of solutions to N.S. We will suppose once more that u is periodic and that p is normalised and we will search for smooth solutions u. Where this time by smooth we mean smooth in the classical C^∞ sense. We begin our study by showing that, by using the Leray Projection and with some help from the heat equation, we may eliminate the pressure term from N.S., leaving the pressure as a function of the velocity. It is for this reason that in our previous theorem (and in all subsequent ones) as initial data we will give an initial velocity u_0 without specifying an initial pressure.

We first rewrite the term $u\cdot\nabla u$ (also known as the transport term), we have that, since u is divergence free,

$$u\cdot\nabla u = u_i\partial_i u = \partial_i(u_i u) - (\partial_i u_i)u = \nabla\cdot(u\otimes u) - (\nabla\cdot u)u = \nabla\cdot(u\otimes u)$$

where $(u\otimes u)$ is the rank two contravariant tensor defined by: $(u\otimes u)_{i,j} := u_i u_j$ and $\nabla\cdot(u\otimes u)$ is the divergence of $(u\otimes u)$ defined by $(\nabla\cdot(u\otimes u))_{i,j} := \partial_j(u\otimes u)_{j,i}$.

Now, due to our smoothness hypothesis we have that all the terms in N.S. are smooth, and hence, we may apply the Leray Projection to them. By applying Lemma 4.3 we deduce that u solves

$$\partial_t u + \mathbb{P}(\nabla\cdot(u\otimes u)) = \nu\Delta u; \quad u(0) = u_0 \tag{5.16}$$

where $\mathbb{P}(\nabla\cdot(u\otimes u)(t,x) := (\mathbb{P}(\nabla\cdot(u\otimes u)(t))(x)$ The equations (5.16) (where u_0 is divergence free) are the equations we wish to reduce our study to. To do so we must show that from a smooth periodic solution to (5.16) we may construct a smooth and periodic solution (u,p) to N.S where p is normalized. We now show exactly this. Let u be a smooth periodic solution to (5.16) then, on taking divergence in (5.16), we obtain by Lemma 4.3,

$$\partial_t(\nabla\cdot u) = \nu\Delta(\nabla\cdot u); \quad (\nabla\cdot u)(0,x) = 0 \tag{5.17}$$

and in consequence $\nabla\cdot u$ is solution to the heat equation (5.17). In Proposition 3.2 we proved the uniqueness of smooth solutions to the periodic heat equation from which we deduce thay, since the identically null function is trivially a solution to (5.17)

$$\nabla\cdot u = 0.$$

That is, u verifies the divergence free condition in N.S. We now construct the pressure p that will make (u,p) a solution to N.S. Expanding (5.16) we obtain that

$$\partial_t u + \nabla\cdot(u\otimes u) = \nu\Delta u + \nabla(\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u))$$

from which we deduce that if we set

$$p = -\Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u) \tag{5.18}$$

then (u,p) solve the Navier Stokes equations.

Proposition 5.4. *Let $p : [0,T]\times\mathbb{T}^d$ be defined as in (5.18) where u is a smooth periodic solution to (5.16). Then (u,p) is a smooth periodic solution to the Navier-Stokes equation with p normalized. That is, solving the Navier-Stokes equations for smooth (u,p) with p normalized is equivalent to obtaining a smooth solution to the pressure free equation*

$$\partial_t u + \mathbb{P}(\nabla\cdot(u\otimes u)) = \nu\Delta u; \quad u(0) = u_0$$

Proof. Due to our previous discussion it only remains to show that p is smooth and normalized. We begin by proving that p is smooth. Since $\nabla\cdot(u\otimes u)$ is smooth we have that so is $\nabla\cdot\nabla\cdot(u\otimes u)$. In consequence it is sufficient to show that if $f \in C^\infty([0,T]\times\mathbb{T}^d \to \mathbb{R})$ is of mean zero then $\Delta^{-1}f$ is also smooth of mean zero. Consider $m \in \mathbb{N}$. Since f is smooth on the compact set $[0,T]\times\mathbb{T}^d$ its derivative $\partial_t^m f$ takes a maximum M (which is integrable of course) and hence we deduce from Proposition 2.4 that we may integrate under the integral sign to obtain

$$\partial_t^m \hat{f}(t,k) = \int_{\mathbb{T}^d} \partial_t^m f(t,k) e^{2\pi i k\cdot x} dx = \widehat{\partial_t^m f}(t,k) \quad \forall (t,k) \in [0,T]\times\mathbb{Z}^d.$$

Therefore we also have that

$$\partial_t^m \widehat{\Delta^{-1} f}(t,k) = -\frac{\widehat{(\partial_t^m f)}(t,k)}{4\pi^2|k|^2} = \widehat{\Delta^{-1}\partial_t^m f}(t,k), \quad k \neq 0$$

which, by Plancherel's Theorem (Theorem 2.19) shows that $\Delta^{-1}p$ is smooth in time with

$$\partial_t^m \Delta^{-1} f(t,x) = -\sum_{k\in\mathbb{Z}^d/\{0\}} \frac{\widehat{\partial_t^m f}(t,k)}{4\pi^2|k|^2} e^{2\pi i k\cdot x}.$$

The smoothness in space is quickly deduced as, by the smoothness of f, the sequence

$$\frac{\hat{f}(t,k)}{4\pi^2|k|^2} \quad k \neq 0$$

is rapidly decreasing for each t and hence, by the theory already seen in the Fourier series preliminaries in equation (2.14), $(\Delta^{-1}f)(t)$ and in consequence $\Delta^{-1}f$ is smooth in space. With which we conclude the smoothness of p. Finally p is normalized practically by defintion of Δ^{-1} as, given f as before,

$$\int_{\mathbb{T}^d} \Delta^{-1} f(t,x)dx = -\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d/\{0\}} \frac{\hat{f}(t,k)}{4\pi^2 k^2} e^{2\pi i k\cdot x} dx = -\sum_{k\in\mathbb{Z}^d/\{0\}} \frac{\hat{f}(t,k)}{4\pi^2 k^2} \int_{\mathbb{T}^d} e^{2\pi i k\cdot x} dx = 0$$

where the exchange of the sum and the integral is justified due to the absolute convergence of the series being integrated for each $t \in [0,T]$. With which we conclude as desired that solving N.S. for smooth periodic velocity and normalized pressure is equivalent to solving the pressure free expression (5.16). □

It is interesting to note that we have just proved that p is not actually an additional unknown function, as its value may be determined solely from u via equation (5.18). Now, we have that equation (5.16) is in the form of a (generalized multicomponent) non-homogeneous periodic heat equation

$$\partial_t u = \nu\Delta u + F; \quad u(0) = u_0$$

where in our case $F = -\mathbb{P}(\nabla\cdot(u\otimes u))$. As we deduced in Chapter 3 when we studied the heat equation, the unique smooth solution to this equation is given by

$$u(t,x) = e^{\nu t\Delta}u_0(x) + \int_0^t e^{\nu(t-t')\Delta} F(t',x)dt'$$

where we had defined the operator $e^{\nu t\Delta}$ on the space of Sobolev functions by

$$\widehat{e^{\nu t\Delta} f}(k) := e^{-4\pi^2\nu|k|^2 t}\hat{f}(k) \quad \forall f \in H^s(\mathbb{T}^d \to \mathbb{R}^m).$$

From this we may derive the following:

Proposition 5.5. *Solving the Navier-Stokes equations for smooth (u, p) with p normalized is equivalent to obtaining a smooth solution to the pressure free integral equation*

$$u(t,x) = e^{\nu t\Delta}u_0(x) - \int_0^t e^{\nu(t-t')\Delta}\mathbb{P}(\nabla\cdot(u\otimes u))(t',x)dt' \tag{5.19}$$

Proof. Due to our preceding argument it only remains to show that if u is a smooth function verifying the integral equation (5.19) then u also solves (5.16). To do so note that since u is smooth so is $\mathbb{P}(\nabla\cdot(u\otimes u))$ and hence we may, in the same way as justified in our argument in Chapter 3 on the heat equation, commute the derivative with respect to t with the sums defined by the Fourier series of the operator $e^{\nu t\Delta}$ and use Leibnitz's rule to deduce that u will be a smooth solution to (5.16) with initial velocity u_0. □

Having reduced our problem thus far, from this moment on our study will center on the integral equation in (5.19). Our strategy will be to, in subsequent sections, study this equation when u_0 is not necessarily smooth, finding solutions to said equation and showing that these solutions are smooth if the initial data u_0 is smooth as well.

5.3 Mild solutions and their maximal Cauchy development

To prove the existence of smooth periodic solutions with normalised pressure to N.S. we will take inspiration in the proof of the Picard uniqueness theorem for ODE'S. Concretely, we will begin by supposing that u_0 has only a finite amount of regularity and then apply a fixed point method in a suitable Banach space consisting of finitely regular functions. In this way we will obtain a (non-smooth) solution u which we shall then prove to be smooth in the case that u_0 be smooth. The Banach space we shall be working with will be

$$X^s_T := C^0_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)\cap L^2_t H^{s+1}_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) \tag{5.20}$$

where $T > 0$ and where

$$C^0_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) := C^0([0,T]\to H^s(\mathbb{T}^d\to\mathbb{R}^d))$$

is the space of continuous functions from [0,T] to the Sobolev space $H^s(\mathbb{T}^d\to\mathbb{R}^d)$ and

$$L^2_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) := \{u : [0,T]\to H^s(\mathbb{T}^d\to\mathbb{R}^d) : \int_0^T \|u(t)\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} < \infty\}$$

is the space of "square integrable functions" from $[0,T]$ to $H^s(\mathbb{T}^d\to\mathbb{R}^d)$, where the integral considered is the Bochner integral. We endow X^s_T with the norm

$$\|u\|_{X^s_T} := \|u\|_{C^0_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{1/2}\|\nabla u\|_{L^2_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}$$

The first order of business is of course proving that the space in (5.20) is in fact a Banach space. The only non-trivial matter being the completeness of said space. We prove this now

Proposition 5.6. *X^s_T is a complete and hence a Banach space.*

Proof. We begin by noting that $C_t^0 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ is a Banach space as $[0,T]$ is compact and $H^s(\mathbb{T}^d \to \mathbb{R}^d)$ is complete. Likewise, due to Proposition 2.9, $L_t^2 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ is also complete. In consequence, given a Cauchy sequence $\{u_n\}_{n=1}^\infty \subset X_T^s$, by definition of the norm on X_T^s, $\{u_n\}_{n=1}^\infty$ and $\{\nabla u_n\}_{n=1}^\infty$ are Cauchy sequences in each of these spaces respectively and hence, converge to functions u, f respectively. It only remains to see that $f = \nabla u$ almost everywhere as then we will have that

$$\|u_n - u\|_{X_T^s} = \|u_n - u\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{1/2}\|\nabla u_n - f\|_{L_t^2 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \xrightarrow{n\to\infty} 0.$$

As stated in Proposition 2.9 we may extract a subsequence $\{\nabla u_{n_k}\}_{k=1}^\infty$ that converges for almost all t to f. Therefore, given a Schwartz function $\phi \in \mathcal{S}(\mathbb{T}^d \to \mathbb{R}^d)$, we have that, using our distributional notation of the preliminaries,

$$\langle \phi, f_j(t)\rangle = \lim_{k\to\infty} \langle \phi, \partial_j u_{n_k}(t)\rangle = \lim_{k\to\infty} -\langle \partial_j \phi, u_{n_k}\rangle = -\langle \partial_j \phi, u(t)\rangle \quad \text{for almost all } t \in [0,T].$$

That is, for almost all t, $\nabla u = f$ with which we conclude our proof.

□

This proved, we now formally introduce the concept of mild solutions to the Navier-Stokes equations.

Definition 5.7. *Let $T > 0$, $s \geq 0$ and $u_0 \in H^s(\mathbb{T}^d \to \mathbb{R}^d)$ be divergence free. An H^s* mild solution *on $[0,T]$ to the Navier-Stokes equations with initial data u_0 is a function*

$$u \in X_T^s$$

such that for each $t \in [0,T]$ u is a distributional solution to (5.19). Additionally if u is an H^s mild solution on $[0,T]$ with initial data u_0 for all $0 < T < T^*$ we shall say that u is an H^s mild solution on $[0,T^*)$ with initial data u_0.

If the value of s and the initial velocity u_0 are understood to be fixed we shall sometimes call such solutions mild solutions on $[0,T]$, $[0,T^*)$ respectively. If, furthermore, the value of T (respectively T_*) is also fixed, we shall call said solutions mild solutions for short.

Note that given $T < T'$ we have that $X_T^s \cap X_{T'}^s = \emptyset$ and hence there is no intersection between mild solutions on $[0,T]$ and mild solutions on $[0,T']$. Nonetheless given $u \in X_{T'}^s$ it is clear that the restricted function $u\big|_{[0,T]\times\mathbb{T}^d} \in X_T^s$ and hence we will, by abuse of notation, also consider u as an element of X_T^s and say that u is a mild solution on $[0,T]$ as well as on $[0.T']$. This said, we now wish to prove the existence of H^s mild solutions to the Navier-Stokes equations. To do so we will employ the following results:

Lemma 5.8 (Product estimate). *Let u, v be functions in $H^s(\mathbb{T}^d \to \mathbb{R}) \cap L^\infty(\mathbb{T}^d \to \mathbb{R})$. Then $uv \in H^s(\mathbb{T}^d \to \mathbb{R})$ with*

$$\|uv\|_{H^s(\mathbb{T}^d\to\mathbb{R})} \lesssim_{d,s} \|u\|_{H^s(\mathbb{T}^d\to\mathbb{R})}\|v\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})} + \|u\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})}\|v\|_{H^s(\mathbb{T}^d\to\mathbb{R})} \tag{5.21}$$

Proof sketch. It is sufficient to prove this for smooth u and v as then we have that taking u_n, v_n to be smooth sequences converging to u , v in both $H^s(\mathbb{T}^d \to \mathbb{R})$ and $L^\infty(\mathbb{T}^d \to \mathbb{R})$

(the existence of such a sequence is established via convolutions with an approximation of unity whenever one wishes to show that $C^\infty(\mathbb{T}^d \to \mathbb{R})$ is dense in $L^2(\mathbb{T}^d \to \mathbb{R})$

$$\begin{aligned}\|uv\|_{H^s(\mathbb{T}^d\to\mathbb{R})} = \lim_{n\to\infty}\|u_nv_n\|_{H^s(\mathbb{T}^d\to\mathbb{R})} &\lesssim_{d,s} \lim_{n\to\infty}\|u_n\|_{H^s(\mathbb{T}^d\to\mathbb{R})}\|v_n\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})} \\ &\quad + \lim_{n\to\infty}\|u_n\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})}\|v_n\|_{H^s(\mathbb{T}^d\to\mathbb{R})} \\ &= \|u\|_{H^s(\mathbb{T}^d\to\mathbb{R})}\|v\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})} + \|u\|_{L^\infty(\mathbb{T}^d\to\mathbb{R})}\|v\|_{H^s(\mathbb{T}^d\to\mathbb{R})}.\end{aligned}$$

A proof of the inequality for u, v smooth may be proved by utilizing Littlewood-Paley theory, see for example [16], pages 42-43. □

We mention that the euclidean (non-periodic) analogue to the above Lemma also holds. As a quick corollary of the previous result and Lemma 2.22 we have that if u, v are in $H^s(\mathbb{T}^d \to \mathbb{R})$ for $s > \frac{d}{2}$ then

$$\|uv\|_{H^s(\mathbb{T}^d\to\mathbb{R})} \lesssim \|u\|_{H^s(\mathbb{T}^d\to\mathbb{R})}\|v\|_{H^s(\mathbb{T}^d\to\mathbb{R})} \tag{5.22}$$

and therefore $H^s(\mathbb{T}^d \to \mathbb{R})$ is a Banach algebra[1] for $s > \frac{d}{2}$. To prove the existence of mild solutions we shall need the following proposition, which will be our go to inequality in many propositions and theorems to come.

Lemma 5.9 (Main estimate). *Let $T > 0$, $s \geq 0$, $u_0 \in H^s(\mathbb{T}^d \to \mathbb{R}^d)$, $F \in L^1_t H^s_x([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ and $G \in L^2_t H^s_x([0,T] \times \mathbb{T}^d \to \mathbb{R}^{d^2})$ and set*

$$u(t,x) := e^{\nu t\Delta}u_0(x) + \int_0^t e^{\nu(t-t')\Delta}(F + \nabla\cdot G)(t',x)dt' \tag{5.23}$$

we have that $u \in C^0_t H^s_x([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$, $\nabla u \in L^2_t H^s_x([0,T] \times \mathbb{T}^d \to \mathbb{R}^{d^2})$ and u verifies the inequality

$$\begin{aligned}\|u\|_{C^0_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{1/2}\|\nabla u\|_{L^2_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})} &\lesssim_{d,s} \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \\ &\quad + \|F\|_{L^1_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}\|G\|_{L^2_tH^s_x([0,T]\times\mathbb{R}^d\to\mathbb{R}^{d^2})}\end{aligned} \tag{5.24}$$

Proof. We begin by noting that the integral in (5.23) is meant in the sense the Bochner integral of the function $[0,T] \to H^s(\mathbb{T}^d \to \mathbb{R}^d)$; $t \mapsto e^{\nu(t-t')\Delta}(F + \nabla\cdot G)(t')$. We now estimate the terms on the left hand side of (5.24) one by one. To simplify notation we set

$$h(t,x) := \int_0^t e^{\nu(t-t')\Delta}F(t',x)dt'; \quad g(t,x) := \int_0^t e^{\nu(t-t')\Delta}\nabla\cdot G(t',x)dt'.$$

We already now from the heat equation that for each t $e^{\nu t\Delta}$ is a non expansive operator on $H^s(\mathbb{T}^d \to \mathbb{R}^d)$. In particular for each $t \in [0,T]$

$$\left\|e^{\nu t\Delta}u_0\right\|_{H^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \leq \|u_0\|_{H^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \tag{5.25}$$

which of course gives

$$\left\|e^{\nu t\Delta}u_0\right\|_{C^0_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \leq \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}.$$

[1] A Banach algebra is a Banach space $(X, \|\|_X)$ with a product that verifies that for all $f, g \in X$ $\|fg\|_X \leq \|f\|_X\|g\|_X$. See for instance [2].

Once more by the non-expansiveness of $e^{\nu t\Delta}$, and using the fact that the norm of the integral is smaller or equal to the integral of the norm, we obtain that

$$\|h(t)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \le \int_0^t \left\|e^{\nu(t-t')\Delta}F(t')\right\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}dt' \le \int_0^T \|F(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}dt'$$
$$= \|F\|_{L^1_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}$$

and hence also

$$\|h\|_{C^0_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \le \|h\|_{L^1_tH^s_x(\mathbb{T}^d\to\mathbb{R}^d)} \tag{5.26}$$

By using Fubini's theorem (which we may do as $\|G(t')\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}$ is square integrable and hence integrable on the compact interval $[0,t]$) and (2.25) for the Fourier transform of the derivatives of tempered distributions, we obtain that for each $k\in\mathbb{Z}$

$$\begin{aligned}\hat g(t,k) &= \int_{\mathbb{T}^d} g(t,x)e^{-2\pi ik\cdot x}dx = \int_{\mathbb{T}^d}\left(\int_0^t e^{\nu(t-t')\Delta}\nabla\cdot G(t',x)dt'\right)e^{-2\pi ik\cdot x}dx\\ &= \int_0^t\left(\int_{\mathbb{T}^d} e^{\nu(t-t')\Delta}\nabla\cdot G(t',x)e^{-2\pi ik\cdot x}dx\right)dt' = \int_0^t \widehat{e^{\nu(t-t')\Delta}\nabla\cdot G}(t',k)dt'\\ &= \int_0^t e^{-4\pi^2\nu|k|^2(t-t')}2\pi ik\cdot\hat G(t',k)dt'\end{aligned} \tag{5.27}$$

from where we deduce that

$$\begin{aligned}\|g(t)\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} &= \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\left|\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}2\pi ik\cdot\hat G(t',k)dt'\right|^2\\ &\le \sum_{k\in\mathbb{Z}^d}4\pi^2|k|^2\langle k\rangle^{2s}\left(\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}\left|\hat G(t',k)\right|dt'\right)^2\end{aligned} \tag{5.28}$$

Now, by applying Cauchy-Schwartz's inequality we obtain that

$$\|g(t)\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \le \sum_{k\in\mathbb{Z}^d}4\pi^2|k|^2\langle k\rangle^{2s}\int_0^t e^{-8\pi^2\nu|k|^2(t-t')}dt'\int_0^t\left|\hat G(t',k)\right|^2dt'.$$

The first integral in the above is, by a variable change, lower or equal to $(8\pi^2\nu|k|^2)^{-1}$ and in consequence, by an application of the monotone convergence theorem, we have that

$$\|g(t)\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \lesssim \nu^{-1}\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\int_0^t\left|\hat G(t',k)\right|^2dt' = \nu^{-1}\|G\|^2_{L^2_tH^s_x([0,t]\times\mathbb{T}^d\to\mathbb{R}^d)}$$

which gives the type of estimate we are aiming for in

$$\|g\|_{C^0_tH^s_x(\mathbb{T}^d\to\mathbb{R}^d)} \lesssim \nu^{-1/2}\|G\|_{L^2_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}.$$

By applying the monotone convergence theorem and once again the Cauchy-Schwartz inequality we deduce in turn that

$$\begin{aligned}\left\|\nabla e^{\nu t\Delta}u_0\right\|^2_{L^2_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})} &\sim \int_0^T\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}|k|^2e^{-4\pi^2\nu|k|^2t}|\hat u_0(k)|^2dt\\ &\lesssim \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}|k|^2|k|^{-2}\int_0^T|\hat u_0(k)|^2dt = \|u_0\|^2_{L^2_tH^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\end{aligned} \tag{5.29}$$

which gives us the inequality we're looking for in

$$\left\|\nabla e^{\nu t\Delta}u_0\right\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}\lesssim\|u_0\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}.$$

To estimate the term $\|\nabla g(t)\|^2_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}$ we employ Theorem 2.8 (Young's inequality). Let us set in accordance with (5.27)

$$I_k:=\int_0^T\left|\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}\hat{G}(t',k)dt'\right|^2dt=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}K(t,t')\hat{G}(t',k)dt'\right|^2dt$$

where K is defined by

$$K(t,t'):=\begin{cases}e^{-4\pi^2\nu|k|^2(t-t')} & (t,t')\in[0,T]\times[0,t]\\ 0 & (t,t')\notin[0,T]\times[0,t]\end{cases}.$$

We have that

$$\int_{\mathbb{R}}K(t,t')dt=\int_{t'}^T e^{-4\pi^2\nu|k|^2(t-t')}dt\leq\frac{1}{4\pi^2\nu|k|^2}$$
$$\int_{\mathbb{R}}K(t,t')dt'=\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}dt'\leq\frac{1}{4\pi^2\nu|k|^2}$$

and since $\hat{G}(t',k)\in L^2([0,T]\to\mathbb{R}^{d^2})$ so does $\hat{G}(t',k)\chi_{[0,T]}(t')$ and we may apply Young's inequality with $p=q=2$, $r=1$ to obtain that

$$I_k=\left\|\int_{\mathbb{R}}K(t,t')(\hat{G}(t',k)\chi_{[0,T]}(t'))dt'\right\|^2_{L^2(\mathbb{R}\to\mathbb{R}^d)}\lesssim\frac{1}{\nu^2|k|^4}\int_0^T\left|\hat{G}(t',k)\right|^2dt'.$$

This last inequality gives us the desired bound as we have that by (2.25) and by (5.27) to do so we begin by obtaining a bound for the term

$$\|\nabla g\|^2_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}\sim\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}|k|^4I_k\lesssim\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\nu^{-2}\int_0^T\left|\hat{G}(t',k)\right|^2dt' \tag{5.30}$$

which, by commuting the sum with the integral (which is justified by the monotone convergence theorem) and taking square roots on both sides gives

$$\|\nabla g\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}\lesssim\nu^{-1}\|G\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}$$

as desired. We now must bound the term

$$\begin{aligned}\|\nabla h\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}&=\left\|\nabla\int_0^t e^{\nu(t-t')\Delta}F(t',k)dt'\right\|_{L_t^2H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}\\&\sim\left(\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}|k|^2\int_0^T\left|\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}\hat{F}(t',k)dt'\right|^2dt\right)^{\frac{1}{2}}\end{aligned} \tag{5.31}$$

where the last line may be obtained by applying Fubini just as in (5.27). To do so we begin by obtaining a bound for the terms

$$\int_0^T\left|\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}\hat{F}(t',k)dt'\right|^2dt \tag{5.32}$$

To achieve this we will again use the function K and apply Young's inequality, where, we will now set $p = 1$, $q = r = 2$. We have that

$$\|K(t)\|_{L^2(\mathbb{R}\to\mathbb{R})} \lesssim |k|^{-1}; \quad \|K(t')\|_{L^2(\mathbb{R}\to\mathbb{R})} \lesssim |k|^{-1}$$

and hence Young's inequality gives

$$\int_0^T \left|\int_0^t e^{-4\pi^2\nu|k|^2(t-t')}\hat{F}(t',k)dt'\right|^2 dt \lesssim |k|^{-2}\left(\int_0^T \left|\hat{F}(t',k)\right|dt'\right)^2.$$

By substituting this back into (5.31) and now applying Minkowski's integral inequality (2.8) we obtain that

$$\begin{aligned}\|\nabla h\|_{L^2_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})} &\lesssim \left(\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}|k|^2|k|^{-2}\left(\int_0^T\left|\hat{F}(t',k)\right|dt'\right)^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in\mathbb{Z}^d}\left(\int_0^T\langle k\rangle^s\left|\hat{F}(t',k)\right|dt'\right)^2\right)^{\frac{1}{2}} \leq \int_0^T\left(\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2s}\left|\hat{F}(t',k)\right|^2dt'\right)^{\frac{1}{2}} \\ &= \|F\|_{L^1_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \end{aligned} \tag{5.33}$$

as desired. Having obtained all our desired bounds it only remains to justify the continuity in t of the terms $e^{\nu t\Delta}u_0, h(t)$ and $g(t)$. This is done without much difficulty thanks to the the smoothing properties of the negative exponentials $e^{-4\pi^2\nu|k|^2t}$. □

In the smooth case (i.e. u_0, F smooth) we all ready proved that the function u defined in (3.10) solves the inhomogeneous heat equation (3.1) our next proposition generalizes this for the case when u_0, F are now only supposed to be square integrable.

Proposition 5.10 (Distributional solution to the heat equation). *Let u_0 be in $L^2(\mathbb{T}^d\to\mathbb{R}^d), F\in L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$, then the function*

$$u(t,x) = e^{\nu t\Delta}u_0(x) + \int_0^t e^{\nu(t-t')\Delta}F(t',x)dt'$$

is a distributional solution to the inhomogeneous heat equation

$$\partial_t u = \nu\Delta u + F; \quad u(0) = u_0.$$

Proof. As we observed in Proposition 3.3 (the instantaneous smoothing effect for the heat equation), we have that $e^{\nu t\Delta}u_0$ solves the inhomogeneous heat equation for almost all t and hence, it is a distributional solution to the inhomogeneous heat equation. That is, by writing $P := \partial_t - \nu\Delta$ and using our distributional notation introduced in the preliminaries

$$\langle\phi, P(e^{\nu t\Delta}u_0)\rangle := \int_{[0,T]\times\mathbb{T}^d}\phi P(e^{\nu t\Delta}u_0)dtdx = 0 \quad \forall\phi\in C^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d).$$

It therefore only remains to show that

$$\partial_t\int_0^t e^{\nu(t-t')\Delta}F(t',x)dt'; \quad \Delta\int_0^t e^{\nu(t-t')\Delta}F(t',x)dt'$$

exist as tempered distributions and that the distributional equality

$$P\left(\int_0^t e^{\nu(t-t')\Delta}F(t',x)dt'\right) = F$$

holds, as we would then have that $P(u) = F$ as desired. Let us define given $f \in L^2([0,T]\times\mathbb{T}^d \to \mathbb{R}^d)$ the operators S, L by

$$\begin{aligned}(S(f))(t,x) &:= \int_0^t e^{\nu(t-t')\Delta}f(t',x)dt' \\ (L(f))(t,x) &:= \sum_{k\in\mathbb{Z}^d}\left(\int_0^t -4\pi|k|^2 e^{-4\pi^2|k|^2(t-t')}\hat{f}(t',k)dt'\right)e^{2\pi i k\cdot x}\end{aligned}$$

We will first show that the functions on the right hand side are indeed well defined as functions in a suitable space by using the bounds of our main estimate lemma. Note that

$$L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) = L_t^2L_x^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) = L_t^2H_x^0([0,T]\times\mathbb{T}^d\to\mathbb{R}^d).$$

Now, we have that, by using basic inequalities that employ the sup, the inequality in (5.26) (where now $s = 0$) and the Cauchy-Schwartz inequality that

$$\begin{aligned}\|S(f)\|_{L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \le T^{\frac{1}{2}}\|S(f)\|_{C_t^0L_x^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} &\le T^{\frac{1}{2}}\|f\|_{L_t^1L_x^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \\ &\le T\|f\|_{L_t^2L_x^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\end{aligned} \tag{5.34}$$

which justifies that indeed $S(f)$ is in $L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$ and is in particular a distribution. For the second term we have that

$$\begin{aligned}\|L(f)\|_{L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} &= \left(\int_0^T\sum_{k\in\mathbb{Z}^d}\left|\int_0^t -4\pi|k|^2e^{-4\pi^2|k|^2(t-t')}\hat{f}(t',k)dt'\right|^2\right)^{\frac{1}{2}} \\ &\sim \left(\sum_{k\in\mathbb{Z}^d}|k|^4\int_0^T\left|\int_0^t e^{-4\pi^2|k|^2(t-t')}\hat{f}(t',k)dt'\right|^2\right)^{\frac{1}{2}} \lesssim \nu^{-1}\|f\|_{L_t^2L_x^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\end{aligned} \tag{5.35}$$

where in the above we employed the monotone convergence theorem and an identical justification as to the one in (5.30). Now, by our study of the heat equation, we know that if f were smooth then we would have that

$$\Delta S(f) = L(f); \quad \partial_t S(f) = \nu L(f) + f \tag{5.36}$$

that is, $S(f)$ would be a classical, and hence distributional solution, to the inhomogeneous heat equation with initial condition 0 and "forcing term" f. This said, let us now consider a smooth sequence $\{F_n\}_{n=1}^\infty$ converging to F in $L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$. By applying the bounds obtained in (5.34) and (5.35) to $F - F_n$ we deduce that $S(F_n)$ and $L(F_n)$ converge respectively to $S(F)$ and $L(F)$ in $L^2([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$, and hence, by the Cauchy-Schwartz inequality, also as distributions . Since the F_n are smooth this allows us to conclude by (5.36) that for any smooth function ϕ,

$$\begin{aligned}\langle\phi, L(F_n)\rangle = \langle\Delta\phi, S(F_n)\rangle &\xrightarrow{n\to\infty} \langle\Delta\phi, S(F)\rangle \\ \langle\phi, \nu L(F_n) + F_n\rangle = -\langle\partial_t\phi, S(F_n)\rangle &\xrightarrow{n\to\infty} -\langle\partial_t\phi, S(F)\rangle.\end{aligned} \tag{5.37}$$

By again using the distributional convergence of $S(F_n), L(F_n)$ (in addition to that of F_n) we also deduce that

$$\langle \phi, L(F_n)\rangle \xrightarrow{n\to\infty} \langle \phi, L(F)\rangle; \quad \langle \phi, \nu L(F_n) + F_n\rangle \xrightarrow{n\to\infty} \langle \phi, \nu L(F) + F\rangle$$

which combined with (5.37) gives the distributional equality

$$\Delta S(F) = L(F); \quad \partial_t S(F) = \nu L(F) + F$$

with which we conclude as desired that

$$P(u) = P(e^{\nu t\Delta}u_0 + S(F)) = 0 + (\nu L(F) + F) - \nu L(F) = F$$

which concludes our proof as ,by definition of u, $u(0) = u_0$. □

Our next theorem proves the existence of mild solutions to the Navier-Stokes equations. As in the heat equation the presence of the viscosity term $\nu > 0$ provides a dampening effect and will allow us to ensure a greater time existence of mild solutions the greater it is. On the other hand the larger the initial data is, the more localized these solutions will be.

Theorem 5.11 (**Existence of mild solutions**). *Let $s > \frac{d}{2}$ and let $u_0 \in H^s(\mathbb{T}^d \to \mathbb{R}^d)$ be divergence free. Then there exists*

$$T_0 \gtrsim_{d,s} \frac{\nu}{\|u_0\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}}$$

and an H^s mild solution on $[0, T_0]$. Additionally if u_1, u_2 are two H^s mild solutions on $[0, T]$ for some $T \in \mathbb{R}^+$ then $u_1 = u_2$.

Proof. As anticipated we shall use a fixed point method. Given $u \in X^s_T$ we define

$$\Phi(u)(t, x) := e^{\nu t\Delta}u_0(x) - \int_0^t e^{\nu(t-t')}\mathbb{P}(\nabla\cdot(u \otimes u))(t', x)dt' \quad \forall t \in [0, T] \tag{5.38}$$

where, as previously observed, we are using the notation

$$\mathbb{P}(\nabla\cdot(u \otimes u))(t', x) := (\mathbb{P}(\nabla\cdot(u \otimes u)(t'))(x)$$

and where the above integral is (as previously observed in Lemma 5.9) a Bochner integral. We wish to show that Φ defines a contraction

$$\Phi : X^s_T \to X^s_T.$$

To do so we begin by showing that indeed $\Phi(u) \in X^s_T$.

I) $\Phi(u) \in X^T_s$: due to the inequality derived in (5.22) we deduce on taking components that $(u \otimes u)(t) \in H^s(\mathbb{T}^d \to \mathbb{R}^d)$ for each $t \in [0, T]$ with

$$\|(u \otimes u)(t)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^{d^2})} \lesssim_{d,s} \|u(t)\|^2_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \tag{5.39}$$

from where we may deduce that $(u \otimes u) \in C^0_t H^s_x([0, T] \times \mathbb{T}^d \to \mathbb{R}^{d^2})$. To do so fix any time t' and set

$$v_t(\tilde{t}) := u(\tilde{t} + t - t') \quad \forall \tilde{t} \in [t' - t, T + t' - t]$$

by using (5.22) and the fact that $u \in X_T^s$ we have that

$$\lim_{t\to t'} \|(u\otimes u)(t) - (u\otimes u)(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^{d^2})} = \lim_{t\to t'} \|(v_t\otimes v_t)(t') - (u\otimes u)(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^{d^2})}$$
$$= \frac{1}{2}\lim_{t\to t'} \|((v_t - u)\otimes(v_t+u))(t') + ((v_t+u)\otimes(v_t-u))(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^{d^2})}$$
$$\lesssim \lim_{t\to t'} \|(v_t-u)(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}\|(v_t+u)(t')\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} = 0 \quad (5.40)$$

Now, as we saw in (4.7), $\mathbb{P}$ is a non-expansive map on $H^s(\mathbb{T}^d \to \mathbb{R}^{d^2})$ which implies that also

$$\mathbb{P}(u\otimes u) \in C_t^0 H_x^s([0,T]\times\mathbb{T}^d \to \mathbb{R}^{d^2}),$$

where we define $\mathbb{P}((u\otimes u))$ by $\mathbb{P}((u\otimes u))_i := \mathbb{P}(u_i u)$. Observe additionally that

$$C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) \subset L_t^2 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$$

with

$$\|u\|_{L_t^2 H_x^s[0,T]\times\mathbb{R}^d\to\mathbb{R}^d)} \leq \left(T\|u\|^2_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\right)^{1/2} = T^{1/2}\|u\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \quad (5.41)$$

In particular

$$\mathbb{P}((u\otimes u)) \in L_t^2 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2}) \quad (5.42)$$

By a quick verification of Fourier coefficients we have that $\mathbb{P}$ commutes with $\nabla\cdot$ and hence we may apply Lemma 5.9 with $F = 0, G = \mathbb{P}(u\otimes u)(t)$, which is justified by (5.42) and the fact that by hypothesis $u_0 \in H^s(\mathbb{T}^d\to\mathbb{R}^d)$, to deduce that $\Phi(u) \in X_T^s$ and

$$\|\Phi(u)\|_{X_T^s} \lesssim_{d,s} \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}\|\mathbb{P}(u\otimes u)(t)\|_{L_t^2 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})} \quad (5.43)$$

II) Φ (when appropriately restricted) is a contraction: from (5.43), the non-expansiveness of $\mathbb{P}$ and (5.39) we deduce that

$$\|\Phi(u)\|_{X_s^T} \lesssim_{d,s} \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}T^{1/2}\|u\|^2_{C_t^0 H_x^s([0,T]\times\mathbb{R}^d)}$$
$$\leq \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}T^{1/2}\|u\|^2_{X_T^s} \quad (5.44)$$

i.e. there exists some positive constant $\lambda_{d,s}$ depending only on d and s such that

$$\|\Phi(u)\|_{X_T^s} \leq \lambda_{d,s}\left(\|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}T^{1/2}\|u\|^2_{X_T^s}\right) \quad (5.45)$$

If we now set

$$R_0 = 2\lambda_{d,s}\|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}; \quad T_0 \leq (2\lambda_{d,s})^{-4}\nu\|u_0\|^{-2}_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \quad (5.46)$$

we deduce from (5.45) that if $u \in \bar{B}(0,R_0) \subset X_{T_0}^s$ then

$$\|\Phi(u)\|_{X_{T_0}^s} \leq 2\lambda_{d,s}\|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} = R_0$$

and hence that the restriction $\Phi : \bar{B}(0,R_0) \to \bar{B}(0,R_0)$ is an endomorphism. A further estimate gives that, by a similar reasoning as the one used to deduce equation (5.45),

$$\|\Phi(u) - \Phi(v)\|_{X_T^s} \lesssim_{d,s} T^{1/2}\nu^{-1/2}\|(u\otimes u) - (v\otimes v)\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}$$
$$\lesssim_{d,s} T^{1/2}\nu^{-1/2}\|u+v\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\|u-v\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}$$
$$\leq 2T^{1/2}\nu^{-1/2}R_0\|u-v\|_{X_T^s} \quad (5.47)$$

and hence for $T_0 < 16^{-1}\min\{\lambda_{d,s}^{-4}, \lambda_{d,s}^{-2}\}\nu\|u_0\|^{-2}_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}$

$$\Phi : \bar{B}(0, R_0) \to \bar{B}(0, R_0) \subset X^s_{T_0} \tag{5.48}$$

will be a contraction and will therefore have a unique fixed point $u \in \bar{B}(0, R) \subset X^s_{T_0}$. Before moving on to the next point observe that, the constant ν being fixed, in our previous discussion the value of T_0 such that (5.48) is a contraction may be chosen dependent only on R_0, d, s. The value of R_0 in turn being chosen dependent only on $\|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}, d, s$. In the future we will use the notation

$$R_0 = R\left(\|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}, d, s\right); \quad T_0 = T(R_0, d, s)$$

to denote the previously calculated radius and time for which Φ as in (5.48) is a contraction.

III) Uniqueness of mild solutions: since we have only proved that the restriction of Φ is a contraction on $\bar{B}(0, R)$ we have not yet proved the uniqueness of H^s mild solutions. As we have not yet ruled out the existence of another H^s mild solution on $B(0, R')$ for some $R' > R$. Let us consider two H^s mild solutions u_1, u_2 on some interval $[0, T]$. To prove uniqueness we will use our previous observation and carry out an argument similar to the one used in ODE'S to prove Cauchy's existence and uniqueness theorem. Let

$$r := max\{\|u_1\|_{X^s_T}, \|u_2\|_{X^s_T}\}$$

and

$$R_0' := max\{r, R(r, d, s)\}$$

then as we just saw in the previous point, we may choose $T_0 \leq T$ depending only on d, s, R_0' such that (5.48) is a contraction. By hypothesis, u_1 and u_2 are fixed points of the function Φ in (5.48) and therefore, are equal on the time interval $[0, T_0) \subset [0, T)$. We now repeat this argument using the translated functions

$$v_i(t, x) := u_i(t + T_0, x), \quad i = 1, 2$$

which are now in $X^s_{T-T_0}$. We note that the integral equation in (5.19) is translation invariant as, the fact that u_i solve (5.19) implies that

$$\begin{aligned}
e^{\nu t\Delta}v_i(0, x) &- \int_0^t e^{\nu(t-t')\Delta}\mathbb{P}(\nabla\cdot(v_i \otimes v_i))(t', x)dt' \\
&= e^{\nu\Delta t}u_i(T_0, x) - \int_0^t e^{\nu(t-t')\Delta}\mathbb{P}(\nabla\cdot(u_i \otimes u_i))(t' + T_0, x)dt' \\
&= e^{\nu t\Delta}\left(e^{\nu T_0\Delta}u_0(x) - \int_0^{T_0} e^{\nu(T_0-t')\Delta}\mathbb{P}(\nabla\cdot(u_i \otimes u_i))(t', x)dt'\right) \\
&- \int_{T_0}^{t+T_0} e^{\nu(t+T_0-t')\Delta}\mathbb{P}(\nabla\cdot(u_i \otimes u_i))(t', x)dt' \\
&= e^{\nu(t+T_0)\Delta}u_0(x) - \int_0^{t+T_0} e^{\nu(t+T_0-t')\Delta}\mathbb{P}(\nabla\cdot(u_i \otimes u_i))(t', x)dt' \\
&= u_i(t + T_0, x) = v_i(t, x)
\end{aligned}$$

where we used the fact that the Leray projection and the derivatives are taken in the spatial coordinate and hence not affected by the time translation, the variable change $t' \to t' - T_0$ and the equality

$$e^{\nu t\Delta}e^{\nu T_0\Delta}f = e^{\nu(t+T_0)\Delta}f \quad \forall f \in H^s(\mathbb{T}^d \to \mathbb{R}^d)$$

which may be quickly checked by verifying that the Fourier coefficients on both sides are equal. Thus, we have that v_i are fixed points of the "modified" endomorphism

$$\tilde{\Phi} : X^s_{T-T_0} \to X^s_{T-T_0}$$

$$\tilde{\Phi}(u)(t,x) := e^{\nu t\Delta}u(T_0,x) + \int_0^t e^{\nu(t-t')\Delta}\mathbb{P}(\nabla\cdot(u \otimes u))(t',x)dt' \quad \forall t \in [0, T - T_0]$$

Suppose that $T_0 < T - T_0$. Then note that since, for all $u, v \in X^s_{T_0}$

$$\left\|\tilde{\Phi}(u) - \tilde{\Phi}(v)\right\|_{X^s_{T_0}} = \|\Phi(u) - \Phi(v)\|_{X^s_{T_0}}$$

and since

$$\Phi : \bar{B}(0, R_0') \to \bar{B}(0, R_0') \subset X^s_{T_0}$$

is a contraction we will also have that

$$\tilde{\Phi} : \bar{B}(0, R_0') \subset X^s_{T_0} \to X^s_{T_0}$$

will be a contraction (note that we don't actually know that $\tilde{\Phi}$ maps $\bar{B}(0, R_0')$ to itself). Clearly we have that, since $T_0 \leq T$,

$$\|v_i\|_{X^s_{T_0}} \leq \|u_i\|_{X^s_T}$$

and therefore $v_i \in \bar{B}(0, R_0') \subset X^s_{T_0}$, subset on which $\tilde{\Phi}$ is a contraction. Since, due to the translation invariance of (5.19), v_i are fixed points of $\tilde{\Phi}$ which implies that $v_1 = v_2$ on $[0, T_0]$. Where it was used that, though a contractive mapping need does not necessarily have a fixed point, if it has one said fixed point is unique. We deduce by the definition of v_i and by the fact that the u_i already coincided on $[0, T_0)$, that $u_1 = u_2$ on $[0, 2T_0)$. If on the other hand $T_0 \geq T - T_0$ then we have that

$$\left\|\tilde{\Phi}(u) - \tilde{\Phi}(v)\right\|_{X^s_{T-T_0}} = \|\Phi(u) - \Phi(v)\|_{X^s_{T-T_0}} \leq \|\Phi(u) - \Phi(v)\|_{X^s_{T_0}}$$

and hence, we obtain directly by the previous reasoning that $v_1 = v_2$ on $[0, T - T_0]$ and in consequence $u_1 = u_2$. Iterating this process until $T_0 \geq T - nT_0$ we obtain that $u_1 = u_2$ on the whole interval $[0, T]$, concluding our proof. □

In our next proposition we will prove that the mild solution to N.S. that we constructed in our previous theorem depends continuously on the initial data. We will use the notation

$$H^s(\mathbb{T}^d \to \mathbb{R}^d)^{\nabla,0}$$

to denote the subset of functions of $H^s(\mathbb{T}^d \to \mathbb{R}^d)$ that are of divergence zero.

Proposition 5.12. *Let $s > \frac{d}{2}$, $u_0 \in H^s(\mathbb{T}^d \to \mathbb{R}^d)^{\nabla,0}$ and u be the H^s mild solution on $[0,T_0]$ to the Navier-Stokes equations with initial data u_0 constructed in Theorem 5.11. Then, given $T < T_0$, there exists a neighbourhood U of u_0 in $H^s(\mathbb{T}^d \to \mathbb{R}^d)^{\nabla,0}$ and a Lipschitz continuous mapping*

$$F: U \to X^s_T; \quad v_0 \mapsto v$$

that maps v_0 to an H^s mild solution v on $[0,T]$.

Proof. We begin by showing that the mapping F indeed exists. Let us define

$$\hat{T}_0 < 16^{-1} \min\{\lambda_{d,s}^{-4}, \lambda_{d,s}^{-2}\} \nu (\|u_0\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} + \epsilon)^{-2} \tag{5.49}$$

where ϵ is taken small enough that $\hat{T}_0 = T$ (which is possible by the definition of T_0 given in the previous theorem). Now let us set $U := B(u_0, \epsilon) \subset H^s(\mathbb{T}^d \to \mathbb{R}^d)^{\nabla,0}$ and let us consider $v_0 \in U$. We define given $f \in X^s_{\hat{T}_0}$

$$\tilde{\Phi}(f)(t,x) := e^{\nu t \Delta} v_0(x) - \int_0^t e^{\nu(t-t')} \mathbb{P}(\nabla \cdot (f \otimes f))(t',x) dt' \quad \forall t \in [0, T_0] \tag{5.50}$$

As we saw in Theorem 5.11 we have that for

$$\tilde{R}_0 = 2\lambda_{d,s} \|v_0\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}; \quad \tilde{T}_0 < 16^{-1} \min\{\lambda_{d,s}^{-4}, \lambda_{d,s}^{-2}\} \nu \|v_0\|^{-2}_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}, \tag{5.51}$$

where $\lambda_{d,s}$ are as in the previous theorem, we obtain a contraction

$$\tilde{\Phi}: B(0, \tilde{R}_0) \subset X^s_{\tilde{T}_0} \to B(0, \tilde{R}_0) \subset X^s_{\tilde{T}_0}.$$

Since $\hat{T}_0 < \tilde{T}_0$ we also have that

$$\tilde{\Phi}: B(0, \tilde{R}_0) \subset X^s_{\hat{T}_0} \to B(0, \tilde{R}_0) \subset X^s_{\hat{T}_0}. \tag{5.52}$$

is a contraction and thus has a unique fixed point v, which, by definition of $\tilde{\Phi}$ is an H^s mild solution with initial data v_0 to N.S. on $[0, \hat{T}_0]$. We have thus obtained an image v for v_0 under the mapping F. Proving that F exists.

It only remains to see that F is Lipschitz-continuous. Given $w_0 \in U$ let us denote its image by F by $w := F(w_0)$. Now, by the same reasoning used to obtain the inequality in (5.47), we obtain that

$$\|w - v\|_{X^s_{\hat{T}_0}} \leq \lambda_{d,s} \left(\|w_0 - v_0\|_{H^s_x(\mathbb{T}^d \to \mathbb{R}^d)} + \hat{T}_0^{1/2} \nu^{-1/2} \|w + v\|_{X^s_{\hat{T}_0}} \|w - v\|_{X^s_{\hat{T}_0}} \right) \tag{5.53}$$

Now using that, by (5.52), $\|v\|_{X^s_{\hat{T}_0}} \leq 2\lambda_{d,s} \|v_0\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}$ and that, by an analogous reasoning, $\|w\|_{X^s_{\hat{T}_0}} \leq 2\lambda_{d,s} \|w_0\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}$, gives that (5.53) is lower or equal to

$$\lambda_{d,s} \left\{ \|w_0 - v_0\|_{H^s_x(\mathbb{T}^d \to \mathbb{R}^d)} + 2\lambda_{d,s} \hat{T_0}^{1/2} \nu^{-1/2} \left(\|w_0\|_{H^s_x(\mathbb{T}^d \to \mathbb{R}^d)} + \|v_0\|_{H^s_x(\mathbb{T}^d \to \mathbb{R}^d)} \right) \|w - v\|_{X^s_{\hat{T}_0}} \right\} \tag{5.54}$$

Now note that, by (5.49), there exists $0 < \gamma < 1$ (independent of ϵ, u_0 and v_0) such that

$$\hat{T_0}^{1/2} = (1-\gamma) 4^{-1} \lambda_{d,s} \nu^{1/2} \|u_0\|^{-2}_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}.$$

From where we deduce that

$$\begin{aligned} &2\lambda_{d,s}\hat{T_0}^{1/2}\nu^{-1/2}\left(\|w_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}+\|v_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}\right)\\ &\qquad=(1-\gamma)\frac{\|w_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}+\|v_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}}{2\|u_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}}\le(1-\gamma)\frac{2\|u_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}+2\epsilon}{2\|u_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}} \end{aligned} \tag{5.55}$$

which may be made smaller than 1 by taking ϵ as small as necessary. The previous three equations now give that

$$\|w-v\|_{X^s_{\hat{T}_0}}\le\lambda_{d,s}\|w_0-v_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}+K\|w-v\|_{X^s_{\hat{T}_0}}$$

where K is a constant (which may depend on d, s) smaller than 1. From this last expression it is immediate that

$$\|F(w_0)-F(v_0)\|_{X^s_{\hat{T}_0}}=\|w-v\|_{X^s_{\hat{T}_0}}\le\frac{\lambda_{d,s}}{1-K}\|w_0-v_0\|_{H^s_x(\mathbb{T}^d\to\mathbb{R}^d)}.$$

Since $1-K>0$ we deduce the Lipschitz continuity of F, as desired.

□

In the future we will see that H^s mild solutions are unique on the interval where they are defined and hence, the previous proposition will give a stability condition not just for the H^s mild solutions that we previously constructed, but for all of them (as the one that we constructed is the only one on its time interval for a given initial data) on the time interval $[0,T_0]$. Before proving uniqueness of mild solutions we now prove the H^s mild analogue of the maximal Cauchy development theorem for ODE's.

Theorem 5.13. *Given $s>d/2$ and a divergence free function $u_0\in H^s(\mathbb{T}^d\to\mathbb{R}^d)$ there exists $T_*\in\mathbb{R}^+$ and an H^s mild solution u on $[0,T_*)$ such that if $T_*<\infty$ then*

$$\lim_{t\to T_*^-}\|u(t)\|_{H^s(\mathbb{T}^d\to\mathbb{Z}^d)}=\infty.$$

Furthermore T_* and u are unique.

Proof. Let $I\subset\mathbb{R}^+$ be the union of all intervals $[0,T]$ on which mild H^s solutions exist. Clearly I is connected and hence an interval. By the uniqueness of H^s mild solutions, which was proved in the previous theorem, we may "glue together" solutions to obtain an H^s mild solution u on all of I. That is, given $t\in I$ we define

$$u(t,x):=u_t(t,x)$$

where u_t is an H^s mild solution on some interval containing t (which exists by construction of I). The uniqueness of the preceding theorem being what guarantees that u is indeed well defined and independent of the u_t chosen.

Additionally I is open on the right as if $t_0\in I$ then, by considering the time translated problem as in the proof of the previous theorem, we deduce that for

$$T_0=T\left(\|u(t_0)\|_{H^s(\mathbb{R}^d\to\mathbb{R}^d)},d,s\right)$$

we have that $[t_0,t_0+T_0]$ is also in I. We may thus write $I=[0,T_*)$ where T_* is potentially infinite. From here we already see that the only thing preventing us from proving that $T_*=\infty$ by gluing together a sequence of solutions is the fact that the norm $\|u(t_0)\|_{H^s(\mathbb{R}^d\to\mathbb{R}^d)}$ may go to infinity as t_0 becomes larger and larger. This will be the idea we use in the remaining part of our proof.

It remains to see that if T_* is finite then there can be no sequence $\{t_n\}_{n=1}^{\infty} \subset [0, T_*)$ converging to T_* such that $\|u(t_n)\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}$ is bounded. We reason by contradiction. Suppose that such a sequence exists and let

$$M := \sup_{n \in \mathbb{N}} \|u(t_n)\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)}.$$

We now consider the radius and time

$$R_0 := R(M, d, s); \quad T_0 := T(R_0, d, s)$$

and we take some time t_{n_0} with $T_* < T_0 + t_{n_0}$, which is of course possible as $t_n \to T_*$. We now aim to show that we may construct an H^s mild solution on $[0, t_{n_0} + T_0]$. This would contradict the definition of I as, by construction of I, we would necessarily have that $[0, t_{n_0} + T_0] \subset [0, T_*)$ which is absurd as $T_* < t_{n_0} + T_0$.

Now, the existence of such a solution is a direct consequence of what we have seen in Theorem 5.11 as we have that by construction of T_0 and R_0 the function

$$\tilde{\Phi}(u)(t,x) := e^{\nu t \Delta} u(t_{n_0}, x) - \int_0^t e^{\nu(t-t')\Delta} \mathbb{P}(\nabla \cdot (u \otimes u))(t', x) dt' \quad \forall t \in [0, T_0]$$

defines a contraction

$$\tilde{\Phi} : \bar{B}(0, R_0) \to \bar{B}(0, R_0) \subset X^s_{T_0}$$

and therefore there it has a fixed point v. We now "glue" the translation of v to u defining

$$f(t) := \begin{cases} u(t) & t \in [0, t_{n_0}] \\ v(t - t_{n_0}) & t \in [t_{n_0}, t_{n_0} + T_0]. \end{cases}$$

We have that f is well defined as, by construction of v,

$$v(0, x) = \tilde{\Phi}(v)(0, x) = u(t_{n_0}, x) \quad \forall x \in \mathbb{T}^d$$

and hence f is also in $X^s_{t_{n_0}+T_0}$. It remains to see that f is a fixed point of Φ as in (5.38) where now $T = t_{n_0} + T_0$. Let $t \in [t_{n_0}, t_{n_0} + T_0]$ then, by a similar procedure as to when we showed the translation invariance of Φ, we have that

$$\begin{aligned}
\Phi(f)(t,x) &= e^{\nu t \Delta} u_0(x) - \int_0^t e^{\nu(t-t')} \mathbb{P}(\nabla \cdot (f \otimes f))(t', x) dt' \\
&= e^{\nu t \Delta} u_0(x) - \int_0^{t_{n_0}} e^{\nu(t-t')} \mathbb{P}(\nabla \cdot (u \otimes u))(t', x) dt' \\
&\quad - \int_{t_{n_0}}^t e^{\nu(t-t')} \mathbb{P}(\nabla \cdot (v \otimes v))(t' - t_{n_0}, x) dt' \\
&= e^{\nu t \Delta} u(t_{n_0}, x) - \int_0^{t - t_{n_0}} e^{\nu(t - t_{n_0} - t')} \mathbb{P}(\nabla \cdot (v \otimes v))(t', x) dt' \\
&= \tilde{\Phi}(v)(t - t_{n_0}, x) = v(t - t_{n_0}, x) = f(t, x),
\end{aligned}$$

where it was used that u and v are fixed points of Φ and $\tilde{\Phi}$ respectively. By a simple reasoning it is in turn clear that

$$\Phi(f)(t,x) = \Phi(u)(t,x) = u(t,x) \quad \forall t \in [0, t_{n_0}]$$

and hence f is a fixed point of Φ and therefore an H^s mild solution on $[0, t_{n_0} + T_0]$, which is a contradiction as previously discussed, and with which we conclude our proof as the uniqueness of (u, T_*) is trivial. □

5.4 Blow-up time and regularity of solutions

In the preliminaries on ODEs we already discussed how given an ODE its maximal solution is locally defined in time if and only if said solution blows up in finite time. As we have seen in our preceding theorem an analogous statement also holds in the case of mild solutions to the Navier-Stokes equations. We will for this reason use the following terminology

Definition 5.14. *Let $s, d, u_0, u, T*$ be as in the maximal Cauchy development of Theorem 5.13. Then we will say that T_* is the* blow-up time *of the Navier-Stokes equations with initial data u_0.*

Note that in Theorem 5.13 the value of s was fixed and hence it would be conceivable that the maximal time existence T_* for H^s mild solutions would depend on s. A requirement of higher regularity hypothetically implying a smaller time existence for the unique H^s mild solution. But, on the contrary, this is not the case. As we shall see in our next theorem the blow-up time is independent of s (and indeed so are the H^s mild solutions themselves). To prove this we begin by proving an intermediary proposition.

Proposition 5.15. *Let u, u_0, s, d, T_* be as in Theorem 5.13. Then if $T_* < \infty$ we have that*

$$\|u\|_{L^\infty([0,T_*)\times\mathbb{T}^d\to\mathbb{R}^d)} = \infty \tag{5.56}$$

In consequence (whether T_* is finite or not)

$$T_* = \sup\{T \in \mathbb{R} : \|u\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} < \infty\} \tag{5.57}$$

Proof. We begin by proving the part of our proposition corresponding to (5.56). Suppose to the contrary that $T_* < \infty$ and

$$\|u\|_{L^\infty([0,T_*)\times\mathbb{T}^d\to\mathbb{R}^d)} = M < \infty$$

and consider $0 < t_1 < t_2 < T_*$. We define in a similar fashion as to before

$$\|u\|_{X^s_{[t_1,t_2]}} := \|u\|_{C^0_t H^s_x([t_1,t_2]\times\mathbb{T}^d\to\mathbb{R}^d)} + \|\nabla u\|_{L^2_t H^s_x([t_1,t_2]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}$$

which is finite as u is an H^s mild solution on $[0, T_*)$.

We will aim to obtain an inequality similar to the one obtained in equation (5.44) of Theorem 5.11. To do so, as we have often done before, we consider the translation of u by t_1. By the translation invariance of (5.19) we may apply Lemma 5.9 to said translation. This gives

$$\|u\|_{X^s_{[t_1,t_2]}} \lesssim_{d,s} \|u(t_1)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}\|\mathbb{P}(u\otimes u)\|_{L^2_t H^s_x([t_1,t_2]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}.$$

Now, using the non-expansiveness of $\mathbb{P}$ together with a version of the inequality (5.41) (which is deduced in the same way) gives

$$\|u\|_{X^s_{[t_1,t_2]}} \lesssim_{d,s} \|u(t_1)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}(t_2-t_1)^{1/2}\|(u\otimes u)\|_{C^0_t H^s_x([t_1,t_2]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}$$

and finally, applying the product estimate (5.21) gives

$$\|u\|_{X^s_{[t_1,t_2]}} \lesssim_{d,s} \|u(t_1)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}(t_2-t_1)^{1/2}M\|u\|_{C^0_t H^s_x([t_1,t_2]\times\mathbb{T}^d\to\mathbb{R}^{d^2})}.$$

$$\leq \|u(t_1)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}(T_* - t_1)^{1/2} M \|u\|_{X^s_{[t_1,t_2]}} \tag{5.58}$$

By now making $T_* - t_1$ sufficiently small we deduce by subtracting the second part on the right hand side of (5.58) from the left hand side that for all $t_2 \in (t_1, T_*)$

$$\|u\|_{X^s_{[t_1,t_2]}} \lesssim_{d,s} \|u(t_1)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)}$$

and hence

$$\lim_{t_2\to T_*} \|u\|_{X^s_{[t_1,t_2]}} < \infty$$

which contradicts Theorem 5.13 as, from the fact that (by definition of the norm on $X^s_{[t_1,t_2]}$)

$$\|u(t_2)\|_{H^s(\mathbb{R}^d\to\mathbb{R}^d)} \leq \|u(t)\|_{X^s_{[t_1,t_2]}},$$

we derive by taking limits and applying theorem 5.13 that

$$\infty = \lim_{t_2\to T_*} \|u(t_2)\|_{H^s(\mathbb{R}^d\to\mathbb{R}^d)} \leq \lim_{t_2\to T_*} \|u(t)\|_{X^s_{[t_1,t_2]}} < \infty$$

which is impossible. Therefore, necessarily, (5.56) holds. We now use Lemma 2.22 to deduce that

$$\|u\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \leq \|u\|_{C^0_t H^s_x([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \quad \forall T \in [0, T_*) \tag{5.59}$$

and in consequence we obtain the expression

$$T_* = \sup\{T \in \mathbb{R} : \|u\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} < \infty\}$$

as desired, as in the case where T_* is infinite this also holds by (5.59). □

This proved we have the sufficient tools to prove the regularity independence of the blow-up time and the mild solutions to N.S.

Theorem 5.16 (**Maximal Cauchy development is independent of regularity**). *Let $s' > s > \frac{d}{2}$, $u_0 \in H^{s'}(\mathbb{T}^d \to \mathbb{R}^d)$ be divergence free and (u, T^s_*),$(v, T^{s'}_*)$ be the unique $H^s, H^{s'}$ mild solutions and blow-up times to the Navier-Stokes equations with initial data u_0. Then*

$$(u, T^s_*) = (v, T^{s'}_*).$$

Proof. First note that our theorem makes sense as we have that

$$u_0 \in H^{s'}(\mathbb{T}^d \to \mathbb{R}^d) \subset H^s(\mathbb{T}^d \to \mathbb{R}^d).$$

where the inequality

$$\|\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \leq \|\|_{H^{s'}(\mathbb{T}^d\to\mathbb{R}^d)} \tag{5.60}$$

holds. Now, the inequality in (5.60) implies that v is also an H^s mild solution on $[0, T^{s'}_*)$ which implies that by the definition of T^s_*

$$T^{s'}_* \leq T^s_*.$$

Additionally, by the uniqueness part of Theorem 5.11, we deduce that $u = v$ on $[0, T^{s'}_*)$. I.e.

$$u(t) = v(t) \quad \forall t \in [0, T^{s'}_*) \tag{5.61}$$

To conclude our proof it therefore only remains to see that $T_*^s \leq T_*^{s'}$ and hence that $T_*^s = T_*^{s'}$. If $T_*^{s'} = \infty$ this is of course clear. If not we observe that

$$\|u\|_{L^\infty([0,T_*^{s'})\times\mathbb{T}^d\to\mathbb{R}^d)} = \lim_{T\to T_*^{s'}-} \|v\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} = \infty$$

where in the first equality we used (5.61) and in the second we used Theorem 5.13. From here we deduce that

$$\sup\{T \in \mathbb{R} : \|u\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} < \infty\} \leq T_*^{s'}.$$

Since by Proposition 5.15 we have the equality

$$T_*^s = \sup\{T \in \mathbb{R} : \|u\|_{L^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} < \infty\}$$

we deduce that

$$T_*^s \leq T_*^{s'}$$

as desired and with which we conclude our proof. □

We now proceed to show that if u_0 is smooth then our mild solution on $[0, T_*)$ is in fact smooth. In consequence the Clay Millennium Navier-Stokes problem (see [4] for a clear statement of said problem) is equivalent to showing that T_* is infinite.

Theorem 5.17 (Existence of smooth solutions). *If $u_0 : \mathbb{T}^d \to \mathbb{R}^d$ is smooth and divergence free then there exists $T_* > 0$ and a unique smooth solution (u, p) to the periodic Navier-Stokes equations on $[0, T_*)$. Furthermore, if $T_* < \infty$ then $\|u\|_{L^\infty([0,T_*)\times\mathbb{T}^d)\to\mathbb{R}^d} = \infty$.*

Proof. As we have discussed previously obtaining a smooth solution to the Navier-Stokes equations is equivalent to obtaining a smooth solution to (5.19). What we have proved so far via Theorem 5.13 and Theorem 5.16 is that there exists T_* and a unique H^s mild solution u on $[0, T_*)$ for all $s > \frac{d}{2}$. We have also proved that the blow-up criterion $\|u\|_{L^\infty([0,T_*)\times\mathbb{T}^d)} = \infty$ holds if T_* is finite. In consequence it suffices to show that in fact u is a smooth solution to (5.19).

To prove that u is smooth we use the fact that. since u is a distributional solution to (5.19), it is also, by Proposition 5.10, a distributional solution to the equation

$$\partial_t u = \Delta u + \nabla\cdot\mathbb{P}(u \otimes u) \tag{5.62}$$

We deduce from the general property of the Sobolev norms

$$\|\nabla u(t)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \leq \|u(t)\|_{H^{s+1}(\mathbb{T}^d\to\mathbb{R}^d)}$$

and from the linearity of the operators ∇, Δ that given $t_0 \in [0, T_*), s \geq 0$

$$\begin{aligned}
&\|\Delta u(t_0) + \nabla\cdot\mathbb{P}(u \otimes u)(t_0) - \Delta u(t) + \nabla\cdot\mathbb{P}(u \otimes u)(t)\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \\
&\qquad \leq \|\Delta(u(t_0) - u(t))\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \|\nabla\cdot(\mathbb{P}(u \otimes u)(t) - \mathbb{P}(u \otimes u)(t_0))\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} \\
&\qquad\qquad \leq \|u(t) - u(t_0)\|_{H^{s+2}(\mathbb{T}^d\to\mathbb{R}^d)} + \|\mathbb{P}(u \otimes u)(t) - \mathbb{P}(u \otimes u)(t_0)\|_{H^{s+1}(\mathbb{T}^d\to\mathbb{R}^d)}
\end{aligned} \tag{5.63}$$

By hypothesis

$$u \in C_t^0 H_x^{s+2}([0, T] \times \mathbb{T}^d \to \mathbb{R}^d) \quad \forall T < T_*$$

and in consequence

$$\|u(t) - u(t_0)\|_{H^{s+2}(\mathbb{T}^d \to \mathbb{R}^d)} \xrightarrow{t \to t_0} 0.$$

Additionally, as we deduced in the proof point I) of Theorem 5.11, since

$$u \in C_t^0 H_x^{s+1}([0,T] \times \mathbb{T}^d \to \mathbb{R}^d) \quad \forall T < T_*,$$

we also have that

$$\mathbb{P}(u \otimes u) \in C_t^0 H_x^{s+1}([0,T] \times \mathbb{T}^d \to \mathbb{R}^d) \quad \forall T < T_*$$

and hence we also have that the second term in (5.63) verifies that

$$\|\mathbb{P}(u \otimes u)(t) - \mathbb{P}(u \otimes u)(t_0)\|_{H^{s+1}(\mathbb{T}^d \to \mathbb{R}^d)} \xrightarrow{t \to t_0} 0.$$

Therefore, the right-hand side of (5.62) is in $C_t^0 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ for every $s \geq 0$, $T < T_*$. In consequence so is the left hand side from which we deduce that $u \in C_t^1 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ for all $s \geq 0$. Iterating this argument we deduce that

$$u \in C_t^k H_x^s([0,T_*) \times \mathbb{T}^d \to \mathbb{R}^d) \quad \forall s, k \geq 0$$

where the above space is just the space of functions u such that their m-th distributional derivatives with respect to time verify: $\partial_t^m u \in C_t^0 H_x^{s+1}([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)\ \forall m \leq k$. Hence by the Sobolev embedding (Proposition 2.23) we have that u defines a smooth function

$$u : [0,T] \to C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$$

and is therefore smooth in time and space, with which we conclude our proof. □

In the two dimensional case it is well known that the Navier-Stokes equations have a unique smooth solution for smooth initial data. We are now in a position to prove this by using the following lemma:

Lemma 5.18. *Let $u_0 \in C^\infty(\mathbb{T}^2 \to \mathbb{R}^2)$ and let u be the unique smooth solution on $[0,T_*)$ to the Navier-Stokes equations. Then*

$$\|u(t)\|_{L^\infty(\mathbb{T}^2 \to \mathbb{R}^2)} \leq K exp(K e^{Kt}) \quad \forall t \in [0,T_*)$$

where K is a constant that depends only on $\|u_0\|_{L^\infty(\mathbb{T}^2 \to \mathbb{R}^2)}$ and $\|rot\, u_0\|_{L^\infty(\mathbb{T}^2 \to \mathbb{R}^2)}$ where $rot\, u_0 := \partial_1 u_{0,2} + \partial_2 u_{0,1}$ is the curl (rotational) of u_0

Proof sketch. The proof is based on the fact that the *vorticity* defined as $w := rot u$ is a scalar obeying the "maximum principle" $\|w(t)\|_{L^\infty(\mathbb{T}^2 \to \mathbb{R}^2)} \leq \|rot\, u_0\|_{L^\infty(\mathbb{T}^2 \to \mathbb{R}^2)}$ and on a generalized version of Gronwall's inequality. For all the details see [7]. □

Corollary 5.19. *The two dimensional Navier-Stokes equations with smooth initial data have a unique globally defined smooth solution.*

Proof. By Theorem 5.17 we know that there exists a unique smooth solution u on $[0,T_*)$. By now applying Proposition 5.15 in conjunction with the bound in Lemma 5.18 we deduce that $T_* = \infty$, concluding our proof. □

We now return to the general dimensional case. Just as in the heat equation (Proposition 3.3) one may also obtain an instantaneous smoothing effect for the solutions to the Navier-Stokes equations. That is, we will now see that H^s mild solutions to the Navier-Stokes equations are in fact smooth past the initial time $t = 0$. Concretely:

Proposition 5.20. *Let Let u, u_0, s, d, T_* be as in the maximal Cauchy development of Theorem 5.13. Then u is smooth on $(0, T_*)$.*

Proof. Let $0 < \epsilon < T_*$ be arbitrary. Note that by definition of H^s mild solutions we have that $u \in L_t^2 H_x^{s+1}([0, T_*) \times \mathbb{T}^d \to \mathbb{R}^d)$ and in consequence

$$u(t) \in H^{s+1}(\mathbb{T}^d \to \mathbb{R}^d)$$

for almost all $t \in [0, T*)$. Therefore there exists some $0 \leq \epsilon_1 < \epsilon$ such that

$$u(\epsilon_1) \in H^{s+1}(\mathbb{T}^d \to \mathbb{R}^d) \tag{5.64}$$

As we have already seen in the proof of point (III) of Theorem 5.11 the integral equation (5.19) is translation invariant and hence the time translated function

$$u_{\epsilon_1}(t, x) := u(t + \epsilon_1, x)$$

will also be an H^s mild solution on $[0, T_* - \epsilon_1)$, now with initial condition $u(\epsilon_1)$. By (5.64) and Theorem 5.13 we deduce the existence of an H^{s+1} mild solution v that solves the Navier-Stokes with initial data $u(\epsilon_1)$ on some interval $[0, T'_*)$. By Theorem 5.16 we have that in fact $T'_* = T_* - \epsilon_1$ and $u_{\epsilon_1} = v$ and hence we deduce that u_{ϵ_1} is an H^{s+1} mild solution on $[0, T_* - \epsilon_1)$.

Iterating this argument we obtain a sequence $\{\epsilon_n\}_{n=1}^{\infty} \subset [0, \epsilon)$ such that for each n the translated function

$$u_{\epsilon_n}(t, x) := u(t + \epsilon_n, x)$$

is an H^{s+n} mild solution on $[0, T_* - \epsilon) \subset [0, T_* - \epsilon_n)$. From here it is immediate that (due to the $C_t^0 H_x^{s+n}$ part of the definition of H^{s+n} mild solutions)

$$u(\epsilon) \in H^{s+n}(\mathbb{T}^d \to \mathbb{R}^d) \quad \forall n \in \mathbb{N}$$

which by the Sobolev embedding seen in Proposition 2.31 implies that $u(\epsilon)$ is smooth in space. By now using Theorem 5.17 for the existence of smooth solutions (and again the translation invariance of (5.19)) we deduce that u_ϵ is smooth on $[0, T_* - \epsilon)$ and hence u is smooth on (ϵ, T_*) with which we conclude our proof as ϵ was arbitrary. □

Our next lemma is a generalization of Lemma 5.9 for the boundary case $s = \frac{d}{2}$, where one longer has the bound on the L^∞ norm given by Lemma 2.22.

Lemma 5.21. *Let $u_0 \in H^{\frac{d}{2}-1}(\mathbb{T}^d \to \mathbb{R}^d)$ and $F \in L_t^1 H_x^{\frac{d}{2}-1}([0, \infty) \times \mathbb{T}^d \to \mathbb{R}^d)$. Then the function $u : [0, +\infty) \times \mathbb{T}^d \to \mathbb{R}$ defined by*

$$u(t) = e^{\nu t \Delta} u_0 + \int_0^t e^{\nu (t-t')\Delta} F(t')\ dt'$$

verifies the bound

$$\begin{aligned} \|u\|_{C_t^0 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{1/2} \|\nabla u\|_{L_t^2 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^{d^2})} + \nu^{\frac{1}{2}} \|u\|_{L_t^2 L_x^\infty([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \\ \lesssim_d \|u_0\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2} \|F\|_{L_t^1 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d)} \end{aligned} \tag{5.65}$$

Proof. Note that we may use the main estimate of Lemma 5.9 to bound the first two terms on the left hand side of (5.65). Now, by considering

$$\tilde{u}(t,x) := u(\nu^{-1}t, x); \quad \tilde{F}(t,x) := \nu^{-1}F(\nu^{-1}t, x)$$

we may reduce our proof to the case where $\nu = 1$. We begin by obtaining the bound

$$\left\|e^{t\Delta}u_0\right\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \lesssim_d \|u_0\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)}.$$

Consider $t > 0$, we have that

$$e^{t\Delta}u_0(x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{-4\pi^2|k|^2t}\hat{u}_0(k)e^{2\pi i k\cdot x}$$

and since the sequence $e^{-4\pi^2|k|^2t}\hat{u}_0(t,k)$ is of rapid decrease it not only converges in $L^2(\mathbb{R}^d \to \mathbb{R}^d)$ to $e^{t\Delta}u_0$ but also almost everywhere allowing us to deduce that

$$\left\|e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^d\to\mathbb{R}^d)} \leq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{-4\pi^2|k|^2t}|\hat{u}_0(k)| \quad \forall t \in (0,\infty).$$

Now we choose $d-2 < \alpha < d$ and expand

$$e^{-4\pi^2|k|^2t}\hat{u}_0(k) = e^{-2\pi^2|k|^2t}|k|^{-\frac{\alpha}{2}}e^{-2\pi^2|k|^2t}|k|^{\frac{\alpha}{2}}|\hat{u}_0(k)|$$

and apply the Cauchy-Schwartz inequality to deduce that

$$\left\|e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^d\to\mathbb{R}^d)} \leq \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{-4\pi^2|k|^2t}|k|^{-\alpha}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{-4\pi^2|k|^2t}|k|^{\alpha}|\hat{u}_0(k)|^2\right)^{\frac{1}{2}} \forall t > 0.$$

By applying the integral test of convergence of series to the first sum in the above inequality and using generalised spherical coordinates (whose variable change has Jacobian smaller than ρ^{d-1}) we deduce that

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{-4\pi^2|k|^2t}|k|^{-\alpha} \leq \int_{\mathbb{R}^d} e^{-4\pi^2|x|^2t}|x|^{-\alpha} \lesssim_d \int_0^\infty e^{-4\pi^2\rho^2t}\rho^{d-1-\alpha}d\rho$$

which by the variable change $\rho \to (4\pi^2t)^{-\frac{1}{2}}\rho^{\frac{1}{2}}$ is

$$\sim t^{\frac{-d+\alpha}{2}}\int_0^\infty e^{-\rho}\rho^{\frac{d-\alpha}{2}-1} = \Gamma\left(\frac{d-\alpha}{2}\right)t^{-d+\alpha+1} \lesssim_d t^{\frac{-d+\alpha}{2}}.$$

where Γ is the gamma function. Substituting this back in, taking the L^2 norm and applying the monotone convergence theorem gives

$$\left\|e^{t\Delta}u_0\right\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \lesssim_d \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{\alpha}|\hat{u}_0(k)|^2 \int_0^\infty t^{\frac{-d+\alpha}{2}}e^{-4\pi^2|k|^2t}dt\right)^{\frac{1}{2}}$$

which, by once again using a change of variable and the gamma function (note that we had set $\alpha > d-2$), gives

$$\left\|e^{t\Delta}u_0\right\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \lesssim_d \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{d-2}|\hat{u}_0(k)|^2\right)^{\frac{1}{2}} \sim_d \|u_0\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)}$$

as desired. To obtain the remaining bound we apply Minkowski's integral inequality to obtain that for each $x \in \mathbb{T}^d$

$$\begin{aligned}\left(\int_0^\infty \left|\int_0^t e^{(t-t')\Delta}F(t',x)dt'\right|^2 dt\right)^{\frac{1}{2}} &\leq \int_0^\infty \left(\int_0^\infty \left|e^{(t-t')\Delta}F(t',x)\chi_{[0,t]}(t')dt\right|^2 dt'\right)^{\frac{1}{2}} \\ &= \int_0^\infty \left(\int_{-t'}^\infty \left|e^{t\Delta}F(t',x)\chi_{[0,t]}(t')dt\right|^2 dt'\right)^{\frac{1}{2}} \\ &= \int_0^\infty \left(\int_0^\infty \left|e^{t\Delta}F(t',x)\chi_{[0,t]}(t')dt\right|^2 dt'\right)^{\frac{1}{2}} \\ &\leq \int_0^\infty \left\|e^{t\Delta}F(t',x)\right\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} dt' \qquad (5.66)\end{aligned}$$

which by now applying the previous bound for u_0 to $F(t')$ gives that

$$\left(\int_0^\infty \left|\int_0^t e^{(t-t')\Delta}F(t',x)dt'\right|^2 dt\right)^{\frac{1}{2}} \lesssim_d \int_0^\infty \|F(t')\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)} dt' = \|F\|_{L^1_t H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)}$$

with which, as $x \in \mathbb{T}^d$ was any, allows us to conclude

$$\left\|\int_0^t e^{(t-t')\Delta}F(t')\right\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \lesssim_d \|F\|_{L^1_t H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)}$$

as desired. □

As we observed in Theorem 5.11 the smaller the size of the initial data u_0 the larger we can guarantee the time interval of existence of mild solutions to be. Our next theorem proves that, in fact, if u_0 is small enough then these mild solutions are global. Not only this, if u_0 is smooth then we obtain that the mild solution is itself smooth. Which means that the Clay Millennium problem has a positive answer when u_0 is small.

To prove all this we will now change from the Banach space X_T^s to the Banach space X, where

$$X := C_t^0 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d) \cap L_t^2 H_x^{\frac{d}{2}}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d) \cap L_t^2 L_x^\infty([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)$$

and give it, based on our previous bound, the norm

$$\|u\|_X := \|u\|_{C_t^0 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{1/2}\|\nabla u\|_{L_t^2 H_x^{\frac{d}{2}-1}([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^{d^2})} + \nu^{\frac{1}{2}}\|u\|_{L_t^2 L_x^\infty([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)}.$$

We also define the space

$$X_T := C_t^0 H_x^{\frac{d}{2}-1}([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) \cap L_t^2 H_x^{\frac{d}{2}}([0,T]\times\mathbb{T}^d\to\mathbb{R}^d) \cap L_t^2 L_x^\infty([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)$$

with the obvious modifications for its norm $\|\|_{X_T}$.

Theorem 5.22 (Global existence for small data). *Given a divergence free function $u_0 \in H^{\frac{d}{2}-1}(\mathbb{R}^d \to \mathbb{R}^d)$ there exists a constant ϵ_d depending only on d such that if $\|u_0\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)} \leq \nu\epsilon_d$ then there exists an $H^{\frac{d}{2}-1}$ mild solution on $[0,\infty)$ to the Navier-Stokes equations with initial data u_0. Furthermore if u_0 is smooth then so is u.*

Proof. We begin by observing that we may once again simplify to the case $\nu = 1$ by considering $\tilde{u}(t,x) = \nu^{-1}u(\nu^{-1}t, x)$. As in many of our previous theorems our proof will rely on a version of the inequalities derived in Theorem 5.11. By considering Φ as in (5.38) and utilizing the bounds of the previous proposition together with the non-expansiveness of $\mathbb{P}$, the product inequality of (5.21) and the Cauchy-Schwartz inequality we have that given $f \in X$

$$\begin{aligned}
\|\Phi(f)\|_X &\lesssim_d \|u_0\|_{H^{\frac{d}{2}-1}(\mathbb{T}^d\to\mathbb{R}^d)} + \|\mathbb{P}(\nabla\cdot(f\otimes f))\|_{L^1_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \\
&\lesssim_d \epsilon_d + \|\nabla\cdot(f\otimes f)\|_{L^1_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^{d^2})} \\
&\sim \epsilon_d + \|(\nabla\cdot f)f\|_{L^1_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \\
&\lesssim_d \epsilon_d + \|\nabla\cdot f\|_{L^2_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \|f\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \le \epsilon_d + \|f\|_X^2
\end{aligned}$$

a similar reasoning gives that for all $f, g \in X$

$$\begin{aligned}
\|\Phi(f)-\Phi(g)\|_X &\lesssim_d \|\nabla\cdot(f\otimes f) - \nabla\cdot(g\otimes g)\|_{L^1_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^{d^2})} \\
&\sim \|\nabla\cdot((f+g)\otimes(f-g))\|_{L^1_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^{d^2})} \\
&\lesssim_d \|\nabla\cdot(f+g)\|_{L^2_t H^{\frac{d}{2}-1}_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \|f-g\|_{L^2_t L^\infty_x([0,\infty)\times\mathbb{T}^d\to\mathbb{R}^d)} \\
&\le (\|f\|_X + \|g\|_X)\|f-g\|_X
\end{aligned}$$

i.e. there exist constants λ_d, γ_d such that

$$\|\Phi(f)\|_X \le \lambda_d(\epsilon_d + \|f\|_X^2); \quad \|\Phi(f)-\Phi(g)\|_X \le \gamma_d(\|f\|_X + \|g\|_X)\|f-g\|_X \tag{5.67}$$

where we may of course suppose that $1 < \gamma_d < \lambda_d$. We deduce that, if we set for example

$$\epsilon_d = \frac{1}{8\lambda_d^2}; \quad R = \frac{1}{2\lambda_d},$$

then

$$\Phi : \bar{B}(0,R) \subset X \to \bar{B}(0,R) \subset X$$

will be a contraction and hence have a unique fixed point u which will be an $H^{\frac{d}{2}-1}$ mild solution on $[0,\infty)$.

It only remains to prove that if u_0 is smooth so is u. Let v, T_* be the maximal Cauchy development which as we know from Theorem 5.17 is smooth on $[0, T_*)$. The same reasoning as the one carried out to obtain (5.67) gives that for $T < T_*$

$$\|v\|_{X_T} \le \lambda_d(\epsilon_d + \|v\|_{X_T}^2) \tag{5.68}$$

From here we will aim to show that $\|v\|_{X_T} \le R$ for all $T \in [0, T_*)$. Suppose to the contrary that this is false and note that $\|v\|_{X_T}$ is a function only of T, d and (by the dominated convergence theorem) is continuous in T. From the continuity in T and the fact that

$$\|v\|_{X_0} = \|u_0\|_{H^{\frac{d}{2}-1}} \le \epsilon_d$$

we deduce that, as T varies in $[0, T_*)$, $\|v\|_{X_T}$ takes all the values in $[\epsilon_d, R]$ and therefore, we deduce that there exists T' such that $\|v\|_{X_{T'}} = R$. This allows us to reach a contradiction as, since (5.68) holds for all T (and in particular for T'), we have that

$$\frac{1}{2\lambda_d} \leq \lambda_d \left(\frac{1}{8\lambda_d^2} + \frac{1}{4\lambda_d^2} \right) = \frac{1}{8\lambda_d} + \frac{1}{4\lambda_d} = \frac{3}{8\lambda_d}$$

which is impossible. Therefore necessarily $\|v\|_{X_T} \leq R$ for all $T \in [0, T_*)$. It now only remains to show that $T_* = \infty$ as, if this is true, then we will have that $v \in \bar{B}(0, R) \subset X$ will be a fixed point of Φ and hence, by uniqueness of said fixed point, we will have that $v = u$ concluding our proof. To obtain that $T_* = \infty$ we apply in a similar fashion as to before the Cauchy-Schwartz inequality, now in the inequality of Lemma 5.9, to obtain that, for $s > \frac{d}{2}$, we have that

$$\|u\|_{X_T^s} \lesssim_d \|u_0\|_{H^s(\mathbb{T}^d \to \mathbb{R}^d)} + \|u\|_{X_T^s} \|u\|_{X_T}.$$

Since, as we have already proved, $\|u\|_{X_T}$ is bounded by R for all $T \in [0, T_*)$ we deduce that $\|u\|_{X_T^s}$ and in consequence $\|u\|_{L^\infty([0,T_*)\times\mathbb{T}^d \to \mathbb{R}^d)}$ are bounded. This allows us to conclude by Proposition 5.15 that T_* is infinite as desired, concluding our proof. □

5.5 Non-periodic Extension and some Generalizations

We now outline how one may extend the previous results to the non-periodic setting. As in the non-periodic case, by the chain rule, the Navier-Stokes equations are equivalent to

$$\partial_t u + \nabla\cdot(u \otimes u) = \nu \Delta u - \nabla p; \quad \nabla\cdot u = 0 \tag{5.69}$$

where now we have $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\quad p : [0, T] \times \mathbb{R}^d \to \mathbb{R}$.

In order to obtain a good well-posedness theory it is now necessary to impose some growth condition so that we may, as in the periodic case, work with Sobolev spaces. We will say that (u, p) is a *classical solution to the non-periodic Navier-Stokes equations* if u, p are smooth and if for all $s \geq 0$ $u \in C_t^0 H_x^s(\mathbb{R}^d \to \mathbb{R}^d), p \in C_t^0 H_x^s(\mathbb{R}^d \to \mathbb{R})$.

To develop an analogous theory to the periodic case we wish to reduce our study to the equation (5.16) and to normalized pressure which we define as

$$p_0 = -\Delta^{-1} \nabla\cdot\nabla\cdot(u \otimes u) \tag{5.70}$$

where u is the velocity of a classical solution to the Navier-Stokes equations.

By the same reasoning as in the periodic case we have that: if u is smooth and $u \in C_t^0 H_x^s(\mathbb{R}^d \to \mathbb{R}^d)\, \forall s \geq 0$ is a solution to (5.16), then (u, p_0) solves (5.69). Therefore it only remains to see that (u, p_0) is a classical solution. By the non-expansiveness of

$$\Delta^{-1} \nabla\cdot\nabla : H^s(\mathbb{R}^d \to \mathbb{R}^{d^2}) \to H^s(\mathbb{R}^d \to \mathbb{R}^d),$$

and the non-periodic analogue to Lemma 5.8 we have that $p_0 \in C_t^0 H_x^s(\mathbb{R}^d \to \mathbb{R})$ and also that the term

$$\nabla\cdot(u \otimes u) - \nu \Delta u + \nabla p_0 \in C_t^0 H_x^s(\mathbb{R}^d \to \mathbb{R}^d) \quad \forall s \geq 0$$

and hence, by (5.69), $u \in C_t^1 H_x^s(\mathbb{R}^d \to \mathbb{R}^d)$. This implies the analogous statement for p and by recurrence we obtain that p is in $C_t^m H_x^s(\mathbb{R}^d \to \mathbb{R}^d)$ for all m and s and is in particular smooth. Proving that indeed (u, p_0) is a classical solution to (5.69) and hence concluding our reduction. In fact we may prove more. We have that if (u, p) is a classical solution to the non-periodic Navier-Stokes equations then by the mean value property for harmonic functions it is simple to prove that $p = p_0 + C$ for some constant $C \in \mathbb{R}$.

Having completed our desired reduction we now show how one may obtain an analogous theory to that of the periodic case. An identical proof to that of Theorem 5.3 shows that there exists a unique classical solution to the Navier-Stokes equations with periodic pressure (with the caveat that we only need obtain that the velocity is unique as the uniqueness of the velocity follows by definition of the normalised pressure).

Our study of the non-periodic heat equation also allows us to generalize without issue the bound of Lemma 5.9. This allows us to obtain the euclidean analogues to all results we proved from Theorem 5.11 to the global existence for small data of Theorem 5.22. No modifications of the proofs or definitions being required save for replacing all instances of $\mathbb{T}^d$ with $\mathbb{R}^d$ and, in the cases where we require u_0 to be smooth, additionally setting the growth condition $D^\alpha u \in L^2(\mathbb{R}^d \to \mathbb{R}^d)\ \forall \alpha \in \mathbb{Z}^d$ on the derivatives of u.

Finally, we may obtain a generalization of various of our previous results by replacing $\left(X_T^s, \|\|_{X_T^s}\right)$ with the larger Banach space $\left(C_t^0 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d), \|\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)}\right)$. To do so, we note that if u is as in Lemma 5.9 we have that, by this same proposition,

$$\|u\|_{C_t^0 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} \lesssim_{d,s} \|u_0\|_{H^s(\mathbb{T}^d\to\mathbb{R}^d)} + \|F\|_{L_t^1 H_x^s([0,T]\times\mathbb{T}^d\to\mathbb{R}^d)} + \nu^{-1/2}\|G\|_{L_t^2 H_x^s([0,T]\times\mathbb{R}^d\to\mathbb{R}^d)}.$$

This allows us to reason identically to Theorem 5.11 to obtain the key bound given by the first line of equation (5.44), which allows us to reprove Theorem 5.11, where one only need alter the definition of H^s mild solutions, asking them to be in $C_t^0 H_x^s([0,T] \times \mathbb{T}^d \to \mathbb{R}^d)$ instead of X_T^s.

With this modification the proofs of the results from Theorem 5.11 up to the existence of smooth solutions to the Navier-Stokes equations (for a smooth initial condition) of Theorem 5.17 go through identically. Finally, an equivalent discussion may be made in the euclidean case by considering H^s mild solutions in $C_t^0 H_x^s([0,T] \times \mathbb{R}^d \to \mathbb{R}^d)$, allowing us to prove the non-periodic analogues to the results in our previous generalization. Argument which concludes this final section and with it our work.

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