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CAMPUS OF INTERNATIONAL EXCELLENCE

On the structure of plane continuous curves

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A León,
por todo lo que hiciste, sin tú saberlo,
durante catorce años, seis meses y dos días.

Contents

Chapter 1

Introduction

> Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.
>
> *Felix Klein*

Intuition has always had a key place in the process of mathematical reasoning and, for that matter, in all of science; it is, in fact, key to any process of research. However, following a cartesian way of thinking, intuition can be polluted by the fact of experience.

One of the most famous results in classical mathematics which seems obvious to our intuition is the so-called Jordan Curve Theorem (JCT), which states that any homeomorphic transformation of the circle into the plane separates the plane into two connected regions, one bounded and one unbounded. Despite seeming obvious, most mathematicians, even professional ones, have never read a proof of it; this text began as an excuse to provide a readable proof of the Jordan Curve Theorem that satisfied both the reader and the writer of the document, simple in the sense of using non-elevated concepts of topology, trying always to use the right tool for the right task.

Still, the sake of motivating the need for a proof of the JCT has turned this text into a path that takes you from the notions of topology, complex analysis and measure theory that are obtained via a degree in mathematics, to some mathematical "monsters", pathological cases of curves, such as nowhere-differentiable curves and space-filling curves, and from there to a proof of the JCT.

More precisely, the content is structured as follows:

In chapter 2 we provide general notation, nomenclature and definitions that are followed throughout the text. As it plays a role in more than section, we state and prove Baire's Category Theorem.

In chapter 3 three different nowhere differentiable curves are presented: the Weierstrass curve, the Koch curve and the Lynch curve. Proper definitions of each one of those curves are given, as well as different properties that will help us prove that they are continuous and nowhere differentiable.

In chapter 4 two space-filling and injective curves are studied in detail: the Osgood curve and Knopp's Osgood curve. We provide the locus, a parametrization and some results about the conditions they need to fulfill in order to be space-filling.

In chapter 5 we present an overview on the relation between the family of space-filling curves and the family of nowhere differentiable curves. As main results, it is shown that the family of space-filling curves is a subset of the family of nowhere-differentiable curves, that it is dense in the set of continuous curves $C(I, \mathbb{C})$ and that the family of nowhere-differentiable curves is of the second category in $C(I, \mathbb{C})$.

Finally, chapter 6 contains, as main result, a proof of the JCT which relies on Brouwer's fixed point theorem for the disk, which we prove using the notion of degree of a continuous map over the circle. We provide all the details and show that it is an invariant under homotopy.

Where possible, the arguments provided have been supported with images that are intended to help the reader to follow the train of thought. Most of those pictures represent the locus of the different particular curves that we talk about, up to a reasonable degree of precision, and that is because the curves that we deal with are of fractal nature. In order to print them, the system of iterated functions of each one of those curves has been implemented in Mathematica code. In the appendix the codes for all such curves are presented. However, the reader will realize that not every picture in the document has its code in the mentioned appendix. This is because some images have been produced, totally or partially, using Geogebra.

A special effort has been put into stating all results in $\mathbb{C}$ and not in $\mathbb{R}^2$, as we consider it the most natural way of thinking of the plane.

As a last comment, we find it convenient to recall that this text, although it presents the structure of a printable book, contains many hyperlinks that connect different parts of the document that make it ideal to be read in electronic format.

Chapter 2

Notation and previous results

As usual, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of real numbers will be denoted by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. We understand $\mathbb{C}$ as the normed space $(\mathbb{C}, |\cdot|)$, where $|\cdot|$ is the modulus of a complex number and $\mathbb{R}$ the normed space $(\mathbb{R}, |\cdot|)$ with $|\cdot|$ the modulus of a real number. The interval $[0,1] \subset \mathbb{R}$ will be denoted by I. We will denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} \ / \ |z| \leq 1\}$. In that sense, $\mathbb{S}^1$ will denote the unit circle, $\mathbb{S}^1 := \{z \in \mathbb{C} \ / \ |z| = 1\}$.

The cardinal of a set A will be denoted by $\#A$. If A is a subset of a set X, we will denote the complement of A by A^c or $X - A$ indistinctly. If A, B are two topological spaces, we will denote by $C(A, B)$ the set of all continuous functions with domain in A and image in B. In general, we will work with the set $C([a, b], \mathbb{C})$, which is known to be a Banach space under the norm

$$\|f\|_\infty = \max_{t \in I} |f(t)|$$

which corresponds to uniform convergence. Every function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ is given by two functions $\gamma_1, \gamma_2 : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t), \ \gamma_1 = \mathrm{Re}(\gamma), \ \gamma_2 = \mathrm{Im}(\gamma).$$

The derivative of a function $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ at $t \in \mathbb{R}$ is the limit, if it exists,

$$\gamma'(t) = \lim_{s \to t} \frac{\gamma(s) - \gamma(t)}{s - t} = \lim_{\delta \to 0} \frac{\gamma(t + \delta) - \gamma(t)}{\delta} \in \mathbb{C}$$

and the theory of real valued functions of multiple variables asserts that $\gamma'(t)$ exists if and only if $\gamma_1'(t), \gamma_2'(t)$ exist, with $\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t)$. We will refer to the elements of $C([a, b], \mathbb{C})$ as curves, as well as to their images $f([a, b])$ for $f \in C([a, b], \mathbb{C})$ and the elements of $C(\mathbb{R}, \mathbb{R})$ as functions, maps or applications. Unless anything else is specified, $[a, b] = I$. Hence, we will say that a curve γ is nowhere differentiable if there is no t in its domain such that $\gamma'(t)$ exists.

Given two curves $\alpha : [a,b] \longrightarrow \mathbb{C}$, $\beta : [c,d] \longrightarrow \mathbb{C}$, we will say that they are concatenable if $\alpha(b) = \beta(c)$. Without any loss of generality, if $\alpha, \beta \in C(I, \mathbb{C})$, the curve

$$(\alpha \uplus \beta)(t) \stackrel{\text{def}}{=} \begin{cases} \alpha(2t) & t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is the concatenation of α and β.

In the literature it is common to see two different definitions of a plane Jordan curve:

Definition 2.0.1. *A (plane) Jordan curve is a homeomorphic image of $\mathbb{S}^1$ in $\mathbb{C}$.*

Definition 2.0.2. *A Jordan curve is the image of a continuous curve $f : [0,1] \longrightarrow \mathbb{C}$ such that $f_{|[0,1)}$ is injective and $f(0) = f(1)$.*

Proposition 2.0.1. *Definition 2.0.1 and Definition 2.0.2 are equivalent.*

Proof. We make use of the map $\phi : [0,1] \longrightarrow \mathbb{S}^1$ defined by $\phi(t) = e^{2\pi i t}$ which is clearly continuous and *surjective*, with $\phi(0) = \phi(1) = 1_{\mathbb{C}}$, and $\phi_{|[0,1)} : [0,1) \longrightarrow \mathbb{S}^1$ is continuous and *bijective*. The only problem is that the inverse map $\phi^{-1} : \mathbb{S}^1 \to [0,1)$, given by the winding number $z = e^{2\pi i t} \to t$, is continuous except at $z = 1_{\mathbb{C}}$. Indeed $\phi^{-1}(1_{\mathbb{C}}) = 0$ but if $z_n = \phi(t_n)$ with $t_n \to 1^-$ then $z_n \to 1_{\mathbb{C}}$ but $\phi^{-1}(z_n) \to 1 \neq \phi^{-1}(1_{\mathbb{C}})$.

We also recall the basic result in Topology, that a continuous bijective map from a compact space (like $\mathbb{S}^1$) to a Hausdorff space (here, the image curve K as a subspace of $\mathbb{C}$) is automatically a homeomorphism, that is, its inverse is continuous.

- Definition 2.0.1 implies Definition 2.0.2: let K be the homeomorphic image of $\mathbb{S}^1$ via an injective and continuous map $h : \mathbb{S}^1 \longrightarrow \mathbb{C}$. By the remarks above, it is clear that the composition $f = h \circ \phi : [0,1] \longrightarrow K$ is continuous and surjective, with $f(0) = f(1) = h(1_{\mathbb{C}})$ and $f_{|[0,1)} : [0,1) \longrightarrow K$ is continuous and bijective.
- Definition 2.0.2 implies Definition 2.0.1: let $f : [0,1] \longrightarrow \mathbb{C}$ be continuous with image K such that $f_{|[0,1)}$ is injective and $f(0) = f(1)$. The composition $h = f \circ \phi^{-1} : \mathbb{S}^1 \longrightarrow [0,1) \longrightarrow K$ is bijective and continuous except possibly at $1_{\mathbb{C}}$ where $1_{\mathbb{C}}$ represents $1 \in \mathbb{C}$. It remains to check it is also continuous at $1_{\mathbb{C}}$. Suppose otherwise. Define $p = h(1_{\mathbb{C}})$, which is the common value $f(0) = f(1)$. Then there is a sequence $z_n \in \mathbb{S}^1$ with $z_n \to 1_{\mathbb{C}}$ but $h(z_n)$ does not converge to p. Passing to a subsequence, we may assume that there is an open neighborhood $V \ni p$ such that $h(z_n) \in V^c$ for all n. In particular $z_n \neq 1_{\mathbb{C}}$ for all n, so $t_n = \phi^{-1}(z_n) \in (0,1)$. Since $[0,1]$ is compact, passing to another subsequence, we may assume that $t_n \to t \in [0,1]$. But then $1_{\mathbb{C}} = \lim z_n = \lim \phi(t_n) = \phi(t)$ by continuity of ϕ on $[0,1]$. Hence $t = 0$ or $t = 1$. In either case, by continuity of f on $[0,1]$ and the equality $f(0) = f(1) = p$, we derive the contradiction $h(z_n) = f(t_n) \to p$.

□

If $[a,b] \subset \mathbb{R}$, we say that an *arc* (or Jordan arc) is a homeomorphic image of $[a,b]$.

The Lebesgue measure in $\mathbb{R}^2$ will be denoted by μ. We will work with the Lebesgue measure in $\mathbb{C}$ by identifying the complex plane with $\mathbb{R}^2$ via the morphism Id defined by $a+bi \mapsto (a,b)$, and, therefore, $\Omega \subset \mathbb{C}$ is measurable if $\mathrm{Id}(\Omega)$ is measurable and its measure is that of $\mathrm{Id}(\Omega)$.

Throughout the text, we make use of the expression of a real number written in a certain base. If b is a number between 0 and 1, its representation in base n will be denoted by

$$b \equiv (0.b_1 b_2 \ldots)_n = \sum_{j \geq 0} \frac{b_j}{n^j}, \ b_j \in \{0, 1, \ldots, n-1\}.$$

Also, upper bars are used to denote periods, as in $(0.b_1\bar{b}_2)_n = (0.b_1b_2b_2b_2b_2\ldots)_n$.

Let M be a subset of a metric space X. Then, we say that:

1. M is *nowhere dense* in X if its closure has no interior.
2. M is *of the first category* in X if M is the union of countably many sets each of which is nowhere dense in X.
3. M is *of the second category* in X if M is not of the first category in X.

The letter Γ will denote the Cantor set and we will use the term *Cantor-type set* referring to a set that is nowhere dense, compact and perfect (it is closed and has no isolated points).

A result that happens to be useful through various parts of the text is Baire's Cathegory Theorem:

Theorem 2.0.2 (Baire's Category Theorem). *Let $X \neq \emptyset$ be a complete metric space. Then X is of the second category in itself.*

Proof. The proof given here is from [9].

By contradiction, assume $X \neq \emptyset$ is of the first category in itself, i.e., $X = \bigcup_{k\geq 1} M_k$ where M_k is nowhere dense in X. By assumption, M_1 is nowhere dense in X and, therefore, $\bar{M}_1$ does not contain any nonempty open set. However, X is open, meaning that $X \neq \bar{M}_1$. This gives us that $\bar{M}_1^c = X - \bar{M}_1$ is open and not empty. Then, let $p_1 \in \bar{M}_1^c$ and B_1 be the ball centered at p_1 with radius $\epsilon_1 \in (0, \frac{1}{2})$ such that $B_1 \subset \bar{M}_1^c$.

Now again, by assumption, M_2 is nowhere dense in X, so that $\bar{M}_2$ does not contain a nonempty open set. Hence, it does not contain the open ball $B(p_1, \frac{1}{2}\epsilon_1)$. It is then implied that $\bar{M}_2^c \cap B(p_1, \frac{1}{2}\epsilon_1)$ is open and not empty, so we may choose

$$B_2 = B(p_2, \epsilon_2) \subset \bar{M}_2^c \cap B(p_1, \frac{1}{2}\epsilon_1), \ \epsilon_2 < \frac{1}{2}\epsilon_1.$$

We obtain, by induction, a sequence of balls $B_k = B(p_k, \epsilon_k), \ \epsilon_k < \frac{1}{2^k}$ satisfying $B_k \cap M_k = \emptyset$ and $B_{k+1} \subset B(p_k, \frac{1}{2}\epsilon_k) \subset B_k, \ k \geq 1$. Since $\epsilon_k < 2^{-k}$, that means that the sequence of centers $\{p_k\}_{k\geq 1}$ is a Cauchy sequence and therefore converges to a certain $p \in X$, since X is complete. Also, for every $n > m$ we have $B_n \subset B(p_m, \frac{1}{2}\epsilon_m)$ and that means

$$d(p_m, p) \leq d(p_m, p_n) + d(p_n, p) < \frac{1}{2}\epsilon_m + d(p_n, p) \longrightarrow \frac{1}{2}\epsilon_m$$

as $n \to \infty$. Therefore, $p \in B_m$, $\forall m$. Since $B_m \subset \bar{M}_m^c$, we obtain that $p \notin M_m$ for every m, but this is absurd since $p \in X = \bigcup M_m$, and so we conclude. $\square$

Chapter 3

Continuous Nowhere Differentiable Curves

When a student is first introduced to differential calculus, it is usually done by following Newton's mechanical approach, understanding the slope of a continuous curve at a given point as the velocity a particle would have at such a point if its position was given by such a curve. Then, it is shown that continuity does not imply differentiability, and that is done by showing examples like the function $f(x) = |x|$, that has no well-defined slope at $x = 0$. It is at this precise moment in a mathematician's education that intuition is vitiated: it sticks on our mind that a continuous function must have a finite or perhaps at most a countable number of points where the slope is not well-defined, because, otherwise, *how would I draw it without lifting my pen?*

This false intuition of the well-behaved nature of continuous curves was also common among mathematicians from the 18th and 19th centuries. In fact, Ampère tried to prove this fact, basing his arguments in this well-behaved nature. It was not until Weierstrass that the mathematical world was provided with the first ever known globally continuous function that had no well-defined tangent at any point. This function was

$$f(x) = \sum_{n\geq 1} a^n \cos(b^n \pi x),\ \ x \in \mathbb{R}$$

with $a \in (0, 1)$, $b > 1$ an odd integer and $ab > 1 + \frac{3\pi}{2}$.

In this chapter, we construct three different curves that are nowhere differentiable, each of them with a different nature: the Weierstrass curve (of an analytic nature), the von Koch curve (of a geometric nature) and the Lynch curve (of a topological nature).

3.1 The Weierstrass Function

As has been said before, until 1875 it was considered that *every continuous function was differentiable except at a "few" isolated points.* Back then, this was known as *Ampère's*

Theorem. However, in 1875, Weierstrass published for the first time in [5] an example of a continuous function that was nowhere differentiable. We shall now define it in its classical form and provide a proof for both continuity and nowhere differentiability.

Theorem 3.1.1 (Weierstrass's Function)**.** *The function*

$$W(x) := \sum_{k\geq 0} a^k \cos(b^k \pi x),$$

is continuous and nowhere differentiable in $\mathbb{R}$*, provided that* $0 < a < 1,\ ab > 1 + \frac{3\pi}{2},\ b > 1$ *such that* b *is an odd integer. We refer to* $W(x)$ *as "the Weierstrass function" or "Weierstrass's function".*

Proof. First, let's prove continuity.

Having $a \in (0,1)$ means that $\sum_{k\geq 0} a^k = \frac{1}{1-a} < \infty$. It is also obvious that

$$\sup_{x\in\mathbb{R}} |a^n \cos(b^n \pi x)| \leq a^n.$$

Now, Weierstrass' M-test assures us that the sequence of partial sums converges uniformly to $W(x)$, i.e.,

$$\sum_{k=0}^{n} a^k \cos(b^k \pi x) \longrightarrow W(x),$$

whence W is continuous in $\mathbb{R}$.

Now, let's prove it is nowhere differentiable. Let $x_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ and let $\alpha_m \in \mathbb{Z}$ such that $b^m x_0 - \alpha_m \in (\frac{-1}{2}, \frac{1}{2}]$. Define

- $x_{m+1} := b^m x_0 - \alpha_m$
- $y_m := \frac{\alpha_m - 1}{b^m}$
- $z_m = \frac{\alpha_m + 1}{b^m}$

which gives us

$$y_m - x_0 = -\frac{1 + x_{m+1}}{b^m} < 0 < \frac{1 - x_{m+1}}{b^m} = z_m - x_0 \Rightarrow y_m < x_0 < z_m.$$

As m increases, both $y_m,\ z_m \longrightarrow x_0$. Considere the left-hand difference quotient

$$\begin{aligned}\frac{W(y_m) - W(x_0)}{y_m - x_0} &= \sum_{n=0}^{\infty} \left(a^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{y_m - x_0} \right) \\ &= \sum_{n=0}^{m-1} \left((ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} \right) \\ &\quad + \sum_{n=0}^{\infty} \left(a^{m+n} \frac{\cos(b^{m+n} \pi y_m) - \cos(b^{m+n} \pi x_0)}{y_m - x_0} \right) = S_1 + S_2.\end{aligned}$$

We deal with S_1 and S_2 separately. Let's start with S_1. Noting that $\frac{|\sin(x)|}{|x|} \leq 1$ and using the well know trigonometric identity

$$\sin(x)\sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2},$$

S_1 can be bounded as follows:

$$\begin{aligned} |S_1| &= \left| \sum_{n=0}^{m-1} (ab)^n (-\pi) \sin\left(\frac{b^n \pi \left(y_m + x_0\right)}{2}\right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{b^n \pi \frac{y_m - x_0}{2}} \right| \\ &\leq \sum_{n=0}^{m-1} \pi (ab)^n = \frac{\pi\left((ab)^m - 1\right)}{ab-1} \leq \frac{\pi (ab)^m}{ab-1}. \end{aligned}$$

Now, for S_2, we proceed as follows: since $b > 1$ is an odd integer and $\alpha_m \in \mathbb{Z}$,

$$\begin{aligned} \cos\left(b^{m+n} \pi y_m\right) &= \cos\left(b^{m+n} \pi \frac{\alpha_m - 1}{b^m}\right) = \cos\left(b^n \pi \left(\alpha_m - 1\right)\right) \\ &= \left[(-1)^{b^n}\right]^{\alpha_m - 1} = -(-1)^{\alpha_m} \end{aligned}$$

and

$$\begin{aligned} \cos\left(b^{m+n} \pi x_0\right) &= \cos\left(b^{m+n} \pi \frac{\alpha_m + x_{m+1}}{b^m}\right) \\ &= \cos\left(b^n \pi \alpha_m\right) \cos\left(b^n \pi x_{m+1}\right) - \sin\left(b^n \pi \alpha_m\right) \sin\left(b^n \pi x_{m+1}\right) \\ &= \left[(-1)^{b^n}\right]^{\alpha_m} \cos\left(b^n \pi x_{m+1}\right) - 0 = (-1)^{\alpha_m} \cos\left(b^n \pi x_{m+1}\right) \end{aligned}$$

which can be used to write S_2 as

$$\begin{aligned} S_2 &= \sum_{n=0}^{\infty} a^{m+n} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m} \cos\left(b^n \pi x_{m+1}\right)}{-\frac{1 + x_{m+1}}{b^m}} \\ &= (ab)^m (-1)^{\alpha_m} \sum_{n=0}^{\infty} a^n \frac{1 + \cos\left(b^n \pi x_{m+1}\right)}{1 + x_{m+1}} \end{aligned}$$

Oberserve that every term appearing in the series just obtained is non-negative and $x_{m+1} \in (\frac{-1}{2}, \frac{1}{2}]$, meaning that

$$\sum_{n \geq 0} a^n \frac{1 + \cos\left(b^n \pi x_{m+1}\right)}{1 + x_{m+1}} \geq \frac{1 + \cos\left(\pi x_{m+1}\right)}{1 + x_{m+1}} \geq \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \tag{3.1}$$

allows us to find a lower bound.

Both the upper and the lower bound that we have found helps us assure the existence of an $\epsilon_1 \in [-1, 1]$ and $\eta_1 > 1$ satisfying

$$\frac{W(y_m) - W(x_0)}{y_m - x_0} = (-1)^{\alpha_m}(ab)^m \eta_1 \left(\frac{2}{3} + \epsilon_1 \frac{\pi}{ab-1}\right). \tag{3.2}$$

Regarding the right-hand quotient

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = S_1' + S_2'$$

an analogous argument is followed and we can conclude that there exists an $\epsilon_2 \in [-1, 1]$ and $\eta_2 > 1$ satisfying

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = -(-1)^{\alpha_m}(ab)^m \eta_2 \left(\frac{2}{3} + \epsilon_2 \frac{\pi}{ab-1}\right) \tag{3.3}$$

Since we had assumed $ab > 1 + \frac{3\pi}{2}$, 3.2 and 3.3 have opposite signs. Also, $(ab)^m \longrightarrow \infty$ as $m \to \infty$ implies that W has no derivative at x_0. We conclude that W is nowhere-differentiable on $\mathbb{R}$.

□

Weierstrass' function was impossible to visualize until the appearance of computer graphics. In Figure 3.1, an image generated by computer can be seen.

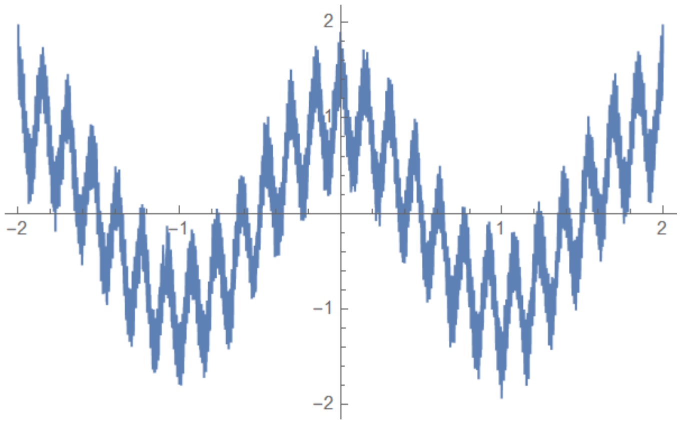


Figure 3.1: Graphic of the Weierstrass function (up to seven partial sums) over the interval $[-2, 2]$. Values of $a = \frac{1}{2},\ b = 13$.

In order to create a curve that preserves the behaviour of the Weierstrass function, it will suffice us to do it the next way:

First, since $W(x) = W(x+2),\ x \in \mathbb{R}$, pick any interval $[x_0, x_0 + 2]$. Now, the curve

$$\begin{aligned} W_{\mathbb{C}} :& [x_0, x_0 + 2] \longrightarrow \mathbb{C} \\ & t \longmapsto t + iW(t) \end{aligned}$$

is a complex curve that has no derivative anywhere. Also, we can create a Jordan curve just by considering $g_{\frac{\pi}{2}}$ the $\frac{\pi}{2}$ anticlockwise rotation around the origin and the curves $W_{x_i} : [x_i, x_i + 2] \longrightarrow \mathbb{C}$ defined by $t \longmapsto A_{x_i}(t-2) + \tau_{x_i}$, where $x_i = x_0 + i,\ i = 0, 2, 4, 6$ and

- $W_{x_0} = W_{\mathbb{C}\,|[x_0,x_1]}$.
- $A_{x_i} = g_{\frac{\pi}{2}} \circ W_{x_{i-1}}$.
- $\tau_{x_i} = W_{x_{i-1}}(x_i) - A_{x_i}(x_{i-1})$.

The concatenation of this for curves gives us a jordan curve as shown in Figure 3.2, where $[x_0, x_1] = [0, 2]$ and the values of a, b are chosen as in Figure 3.1.

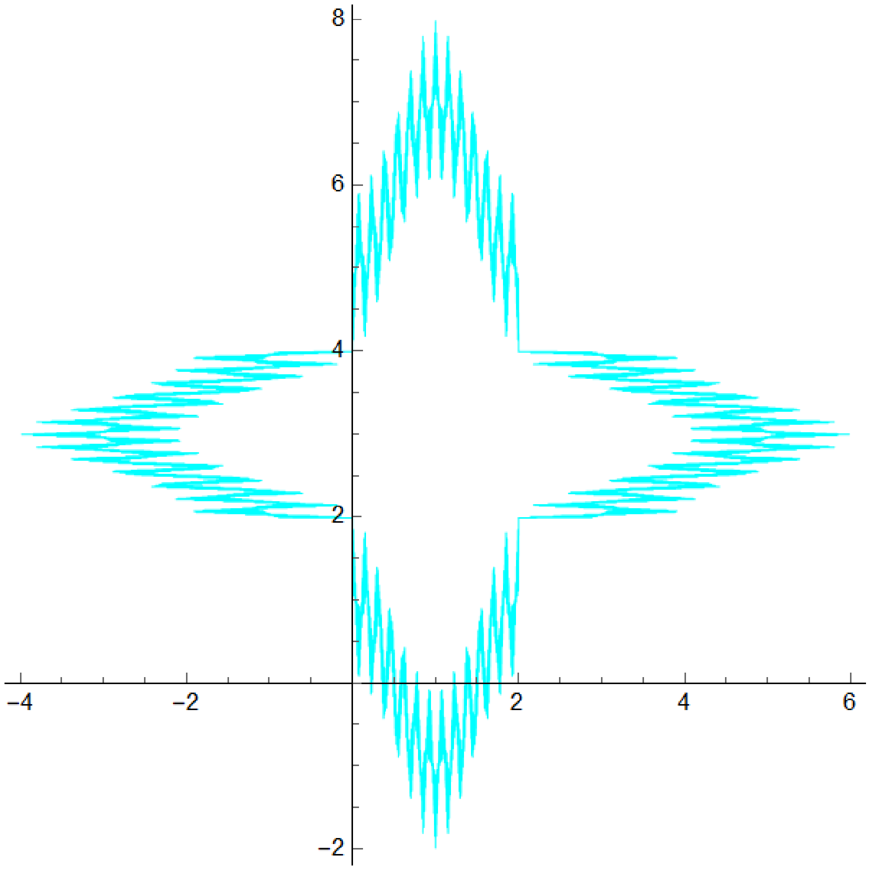


Figure 3.2: The Weierstrass closed curve.

3.2 The von Koch curve

In 1904, Koch published in [8] the curve we are about to study. His motivation to think of that curve can be read in [6]: he found that, despite having served to correct the misconception of curves having tangent everywhere but for some particular points, the analytic nature of the Weierstrass function hid the geometrical nature of the curve, not allowing you to see why it has no tangent.

The von Koch curve is created as follows: start with the line segment $[0, 1] \subset \mathbb{C}$ and divide it in thirds. Create an equilateral triangle with base the middle third so that it "points outwards" and remove its base. Repeat indefinitely for every segment and the Koch curve is obtained at the limit. This provides a well-defined locus for the curve; we aim to parametrize and prove that, despite being continuous, it is nowhere differentiable.

First of all, let K be the Koch curve and K_n be the n-th construction of K, where K_0 is just the line segment $[0, 1]$. In order to provide a parametrization of K, we will create a parametrization of K_n, namely κ_n, so that it is continuous and injective. The limit of the sequence $\{\kappa_n\}_{n\geq 0}$ will be our parametrization.

3.2.1 Parametrizing K_n.

Observe that K_1 has 4 different parts: a horizontal line segment, one side of the equilateral triangle, the other side of the equilateral triangle and another horizontal line segment, all having a common point two by two. Since K_2 is created by copying K_1 into the sides of K_1, it follows that K_2 is formed by 4^2 line segments. In general, its obvious that K_n is formed by 4^n.

In order to parametrize K_n, we give a total order on its sides. For K_1 we number them as in Figure 3.3.

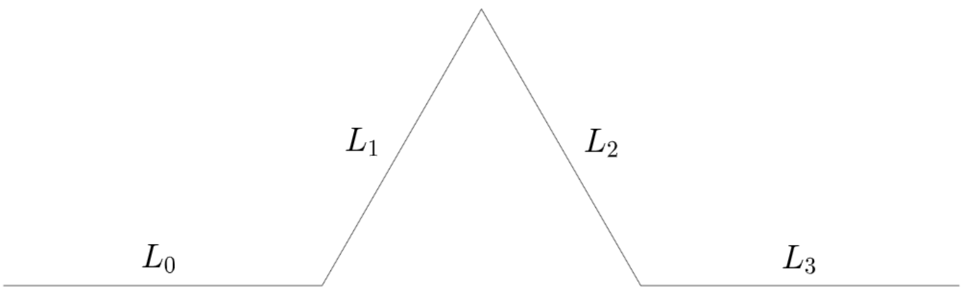


Figure 3.3: Numbering of K_1.

Now, the elements on K_2:

The segments produced by applying the procedure on L_0 are as follows: L_{00} is the line segment on K_2 that has the vertex 0; L_{01} the one that shares a common point with it, L_{02} the one that shares a common point with L_{01} that is not L_{00} and L_{03} the one that shares a common point with L_{02} that is not L_{01}. The same goes on for the segments produced by applying the procedure to L_1, L_2 and L_3.

In general, the segments produced at $(n + 1)$-th iteration are numbered as follows: if the line segments have been produced by applying the procedure on $L_{b_1 b_2 \ldots b_n}$, then $L_{b_1 b_2 \ldots b_n 0}$ is the one that shares a common point with the previous segment $L_{b_1 b_2 \ldots b_{n-1}}$ or that has vertex 0 if $b_i = 0,\ i = 1, \ldots, n$; $L_{b_1 b_2 \ldots b_n 1}$ is the one connected to $L_{b_1 b_2 \ldots b_n 0}$ that lies on the same previous line segment, etc. The numbering for K_2 is shown in Figure 3.4 to exemplify this better.

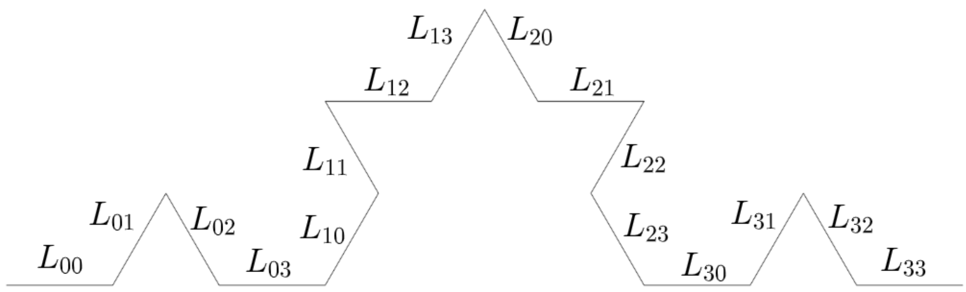


Figure 3.4: Numbering of K_2.

Now, since K_n has 4^n different parts, our parametrization of K_n can be done in the next way. Considering $t \in I$ in its representation in base 4, $t = (0.t_1 t_2 \ldots)_4$, the function

$$\kappa_n : I \longrightarrow K_n, \; n \geq 0 \tag{3.4}$$

that maps $[(0.b_1 \ldots b_n)_4, (0.b_1 \ldots (b_n+1))_4]$ linearly into $L_{b_1 \ldots b_n}$ parametrizes K_n. Actually, κ_n is continuous and injective by construction, which means that it is a homeomorphism onto K_n.

3.2.2 The sequence κ_n converges uniformly.

In order to show that $\kappa := \lim_{n\to\infty} \kappa_n$ exists and is continuous, we will show that $\{\kappa_n\}_{n\geq 1}$ is a Cauchy sequence.

Observe that

$$\|\kappa_1 - \kappa_0\|_\infty = \frac{\sqrt{3}}{6}$$

. Since every step scales the previous step by a factor of $\frac{1}{3}$, we have that

$$\|\kappa_{n+1} - \kappa_n\|_\infty = \frac{\sqrt{6}}{3^n 6}.$$

Thus, for fixed n and arbitrary j we have

$$\|\kappa_{n+j} - \kappa_n\|_\infty \leq \sum_{i=0}^{j-1} \|\kappa_{n+i+1} - \kappa_{n+i}\|_\infty = \frac{\sqrt{3}}{6} \sum_{i=0}^{j-1} \frac{1}{3^{n+i}} < \frac{\sqrt{3}}{3^n 6} \sum_{i=0}^{j-1} \frac{1}{3^i} = \frac{\sqrt{3}}{3^n 4} \tag{3.5}$$

which vanishes as n increases. Therefore, $\{\kappa_n\}_{n\geq 0}$ is a Cauchy sequence and, therefore, there exists a function κ such that it is continuous and satisfies $\lim_{n\to\infty} \kappa_n = \kappa$. We must now show that $\kappa(I) = K$.

3.2.3 Proving $\kappa(I) = K$.

We have stated that $\kappa_n \longrightarrow k$ uniformly, which implies that $\kappa(I) \subset K$. We must now prove that $K \subset \kappa(I)$. For that matter, let us give the next definition:

Definition 3.2.1. *We will say that $p \in K_n$ is a nodal point of K_n if p is vertex 0 or 1 or is common to two consecutive intervals $L_{a_1 \dots a_n}$ and $L_{b_1 \dots b_n}$. We will denote the set of nodal points of K_n by $\mathcal{N}(K_n)$.*

It is trivial to observe that the set of nodal points of K_n is

$$\mathcal{N}(K_n) = \{p \in K_n / p = \kappa_n(\frac{m}{4^n}),\ m = 0, \dots, 4^n\}$$

in view of the definition of κ_n. Also, $\mathcal{N}(K_n) \subsetneq \mathcal{N}(K_{n+1})$.

Since K is the limit of the approximating polygons K_n, the set of nodal points of K can be defined as

$$\mathcal{N}(K) := \bigcup_{n \geq 0} \mathcal{N}(K_n).$$

Now, following the notation of [20], denote

$$\pi_n := \{t \in I \ / \ t = \frac{m}{4^n},\ m = 0, \dots, 4^n\}.$$

It is clear that $\kappa_n(\pi_n) = \mathcal{N}(K_n)$. Now, it will come in handy to show the next result:

Theorem 3.2.1. *The set*

$$\pi := \bigcup_{n \geq 0} \pi_n$$

is dense in $[0, 1]$.

Proof. Let $t_1, t_2 \in \pi_n$ such that $t_1 = \frac{m}{4^n}, t_2 = \frac{m+1}{4^n},\ m = 0, \dots, 4^n - 1$. It is obvious that $|t_1 - t_2| = \frac{1}{4^n}$. Let now $a, b \in [0, 1]$ with $a < b$ and $n \geq 0$ satisfying $|a - b| > \frac{1}{4^n}$. Then, there is an $r \in \pi_n (\subset \pi)$ such that $a < r < b$. □

Since κ_n is a homeomorphism between $[0, 1]$ and K_n,

Theorem 3.2.2. *The set of nodal points of K is countable.*

Proof. Having that $\mathcal{N}(K_n)$ is a countable set and $\mathcal{N}(K)$ is the countable union of countable sets, implies that $\mathcal{N}(K)$ is itself a countable set. □

Theorem 3.2.3. *K has the cardinality of the continuum.*

Proof. First, note that $\kappa(I) \subset K$ which means that $\#K \geq \mathfrak{c} = \#K$. Since $K \subset \mathbb{D}$, and since $\mathbb{D}$ has the cardinal of the continuum, we conclude that $\#K = \mathfrak{c}$. □

Observe that the relevance of Theorem 3.2.3 lies in the fact that, together with Theorem 3.2.2, helps us conclude that not every point in K is a nodal point. We still need the next proposition to fully be capable of showing that $\kappa(I) = K$.

Theorem 3.2.4. *Not every point in K belongs to one of the approximating polygons K_n.*

Proof. Considere the construction of K_1 and observe that it is the middle third that changes. The same goes for every side in K_n: it is the middle third that will change in the creation of the next one. This means that the points of a side $L_{b_1 \dots b_n}$ of K_n that belong to K is homeomorphic to the Cantor set, which is nowhere dense. Therefore, the set of points of K that belong to K_n is a countable union of those nowhere dense sets. It follows from the Baire Category theorem that there are points that do not belong to any approximating polygon K_n. □

Finally, we can now show that $\kappa(I) = K$.

Theorem 3.2.5. $\kappa(I) = K$

Proof. Let then $p \in K$ be any point. We have two possibilities:

- The point p is a never changing point of K. If so, that means that p appeared for the first time at iteration $n_0 \geq 0$. Hence, there exists $t_{n_0} \in I$ such that $\kappa_{n_0}(t_{n_0}) = p$ and, therefore, it means that $\kappa_n(t_{n_0}) = p,\ n \geq n_0$. We conclude that $\kappa(t_{n_0}) = p$ by continuity.

- The point p does not belong to the never changing points of K. Since triangles are always created "to the outside", this means that the point p does not belong to K_n for any $n \geq 0$. In such case, without any loss of generality, say that p lies in the triangle T_n whose two of its sides are $L_{b_1 \dots b_n}$ and $L_{b_1 \dots (b_n+1)}$. It is clear that there exists a sequence $\{T_n\}_{n \geq 0}$ of triangles satisfying that $p \in T_n$, $\forall n \geq 0$ and $T_n \supset T_{n+1}$.

  Because the length of $L_{b_1 \dots b_n}$ is $\frac{1}{3^n}$ and the length of the major side of T_n is $\frac{\sqrt{3}}{3^n}$, that means that T_n shrinks into a point $\Rightarrow T_n \longrightarrow p$. Now, for every point $p_n \in L_{b_1 \dots b_n}$ there exists a $t_n \ /\ \kappa_n(t_n) = p_n$. This means that $p_n \longrightarrow p$. Thanks to compactness, there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ such that $t_{n_k} \longrightarrow t$ satisfying

  $$\kappa_{n_k}(t_{n_k}) \longrightarrow p.$$

  Also, since $\kappa_{n_k}(t_{n_k}) \longrightarrow \kappa(t)$, we conclude that $p = \kappa(t)$.

□

We can summarize what we have done as:

Proposition 3.2.6. *There exists a continuous surjective function $\kappa \in C(I, \mathbb{C})$ whose image is the Koch curve K.*

3.2.4 The function κ is injective.

Observe that Proposition Theorem 3.2.6 together with Theorem 3.2.1 shows that $\mathcal{N}(K)$ is dense in K. Also, they show that π is countable (although it could have been proved straight). For this reason, we will call the elements of π *rational points.*

In order to prove the injectivity of κ, observe that K_1 is contained in the closed triangle of vertices $\{0, \frac{1}{2} + \frac{\sqrt{3}}{6}i, 1\}$. Denote such triangle by $T_{1,1}$. Now, K_2 is contained in the union of the closed triangles $T_{2,1}, T_{2,2}, T_{2,3}, T_{2,4}$ as shown in Figure 3.5.

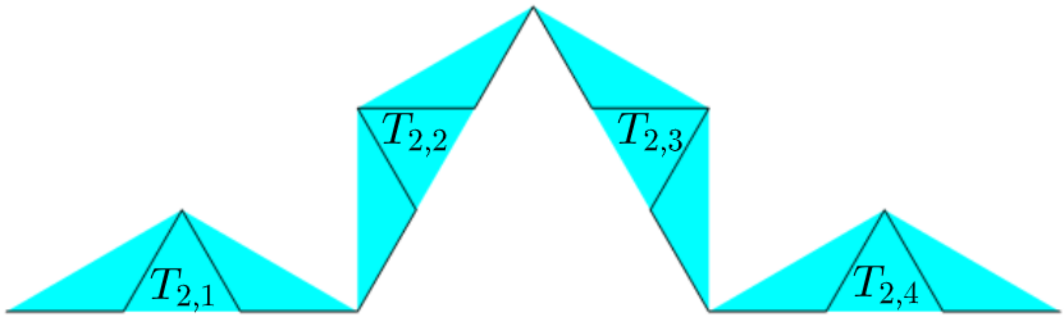


Figure 3.5: Triangles $T_{2,1}, T_{2,2}, T_{2,3}, T_{2,4}$ and K_2.

Now, let $r \in \pi$ and let $n_r = \inf\{n \in \mathbb{N} \ / \ r \in \pi_n\}$. Then, $\kappa_n(r) = \kappa_{n_r}(r)$ for ever $n \geq n_r$, whence $\kappa(r) = \kappa_{n_r}(r)$ for every $r \in \pi$. If $r_1 \neq r_2$ are rational points and $n \geq 0$ is such that $r_1, r_2 \in \pi_n$, since κ_n is injective,

$$\kappa(r_1) = \kappa_n(r_1) \neq \kappa_n(r_2) = \kappa(r_2)$$

which gives us that $\kappa_{|\pi}$ is injective. Note that if r_i, r_{i+1} are two neighbouring points of π_n, for every $t \in [r_i, r_{i+1}]$, $\kappa_n(t) \in T_{n,i}$. Moreover, it is also true that $\kappa_m(t) \in T_{n,i}$ for $n \geq n$. This means that once the image of t gets into some $T_{n,i}$ it does not leave it anymore. Therefore, $\kappa(t) \in T_{n,i}$, since these sets are closed. In fact, $\kappa(t)$ is the unique point in the intersection of all $T_{n,i}$ that contain the image of t under κ.

Let $0 < t_1 < t_2 < 1$. Given that π is dense in $[0, 1]$, there exists an $n \geq 0$ such that there are $r_1, r_2 \in \pi_n$ satisfying $t_1 < r_1 < r_2 < t_2$. Let

$$r_i := \min\{r \in \pi_n \ / \ t_1 < r\}$$

$$r_j := \max\{r \in \pi_n \ / \ t_2 > r_2\}.$$

Then, $\kappa_n(t_1) \in T_{n,i-1}$ and $\kappa_n(t_2) \in T_{n,j}$ and therefore $\kappa(t_1) \in T_{n,i-1}$, $\kappa(t_2) \in T_{n,j}$. Since $D_{n,i-1}$ and $D_{n,j}$ are disjoint, $\kappa(t_1) \neq \kappa(t_2)$ and we conclude that κ in injective.

Summing up, we can claim that we have found a function $\kappa \in C(I, \mathbb{C})$ that is a homeomorphism onto its image K.

3.2.5 The Koch curve has no well-defined tangent lines

The argument will be as follows: we will pick points of K and show that there are, at least, two tangent lines to it.

Theorem 3.2.7. *No point of K has a well-defined tangent line.*

Proof. We will divide the proof in two parts.

Nodal points do not have a well-defined tangent line.

Let $p \in \mathcal{N}(K)$ be a nodal point. It is clear that there exists

$$n_0 = \min\{n \in \mathbb{N} \ / \ p \in \mathcal{N}(K_n)\}$$

and let $p \in L_{b_1 \dots b_{n_0}}$. Denote by p'_{n_0} the other nodal point of $L_{b_1 \dots b_{n_0}}$. At the next iteration, $L_{b_1 \dots b_{n_0}}$ generates for new segments, with one nodal point not lying in $L_{b_1 \dots b_{n_0}}$; denote such point by p''_{n_0}. Now, $p \in L_{b_1 \dots b_{n_0} b_{n_1}}$ and denote by p'_{n_1} the other nodal point lying in $L_{b_1 \dots b_{n_0} b_{n_1}}$. This way, we have generated two sequences $\{p'_n\}_{n \geq n_0}, \{p''_n\}_{n \geq n_0}$ such that

$$p'_n, \ p''_n \longrightarrow p.$$

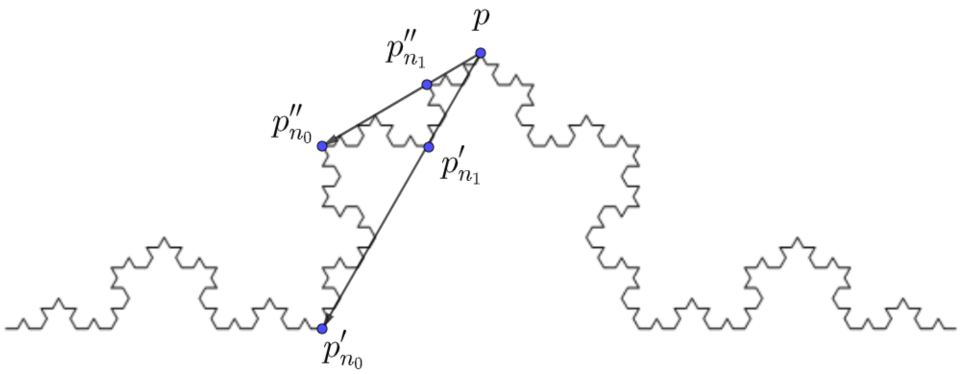


Figure 3.6: Representation of $p'_{n_0}, p'_{n_1}, p''_{n_0}, p''_{n_1}$ for a particular p and n_0.

Let $t'_n, t''_n \in I$ be such that

$$\kappa(t'_n) = p'_n, \ \kappa(t''_n) = p''_n$$

and $t \in I$ satisfying $\kappa(t) = p$. For every $n \geq n_0$, we can define vectors

$$v'_n := \frac{p'_n - p}{t'_n - t}$$
$$v''_n := \frac{p''_n - p}{t''_n - t}$$

such that v'_n and v'_{n+1} are colinear and v''_n and v''_{n+1} are colinear too. That means that they define the same unit vector

$$v' := \frac{v'_n}{|v'_n|}, \ v'' := \frac{v''_n}{|v''_n|},$$

whence

$$\frac{v_n'}{|v_n'|} \longrightarrow v', \ \frac{v_n''}{|v_n''|} \longrightarrow v''$$

If K had a well-defined tangent line at p, the vectors v' and v'' would be collinear, but, by construction, $\measuredangle(v', v'') \equiv \frac{\pi}{6}$ $(mod\ \pi)$, which concludes this part of the proof.

The points of $K - \mathcal{N}(K)$ do not have a well-defined tangent line.

Let $p \in K - \mathcal{N}(K)$ and let $t_1, t_2 \in \pi_n$, (with $t_1 = \frac{m_1}{4^n}$, $t_2 = \frac{m_1+1}{4^n}$, $m = 0, \ldots, 4^n - 1$) such that $t_1 < \kappa^{-1}(p) < t_2$ for some $n \geq 0$. Since $p \notin \mathcal{N}(K) \Longrightarrow a_n := |p - \kappa(t_1)| \neq |p - \kappa(t_2)| =: y_n'$. Let's say $a_n < y_n'$. At the next step, $[\kappa(t_1), \kappa(t_2)]$ produces four new segments, one of those being $[\kappa(t_1), b_n]$ and y_n'' the nodal point produced not lying on $[\kappa(t_1), \kappa(t_2)]$. Hence, p lies in the interior of the triangle with vertices $\{\kappa(t_1), b_n, y_n''\}$, which implies that the vectors

$$y_n' - p, y_n'' - p$$

form an angle between $\frac{\pi}{6}$ and π.

Now, let t_n', t_n'' and t be such that

$$\kappa(t_n') = y_n', \ \kappa(t_n'') = y_n'', \ \kappa(t) = p.$$

It is obvious that $y_n', \ y_n'' \longrightarrow p$ which means that if K had a tangent line at p, the limits

$$\lim_{n\to\infty} \frac{y_n' - p}{t_n' - t}, \ \lim_{n\to\infty} \frac{y_n'' - p}{t_n'' - t}$$

would have to exist and coincide. But, if they exist, they have to form an angle between $\frac{\pi}{6}$ and π, contradicting the hypothesis. Therefore, there is no well-defined tangent line at $p \in K - \mathcal{N}(K)$ either. □

We can conclude this section by saying that:

Theorem 3.2.8. *The function $\kappa \in C(I, \mathbb{C})$, which is a homeomorphism onto its image K, is continuous and nowhere-differentiable.*

3.2.6 The Koch snowflake.

A Jordan curve that is nowhere differentiable can be then generated by the concatenation of three von Koch curves. The result is known in literature as the "Koch snowflake".

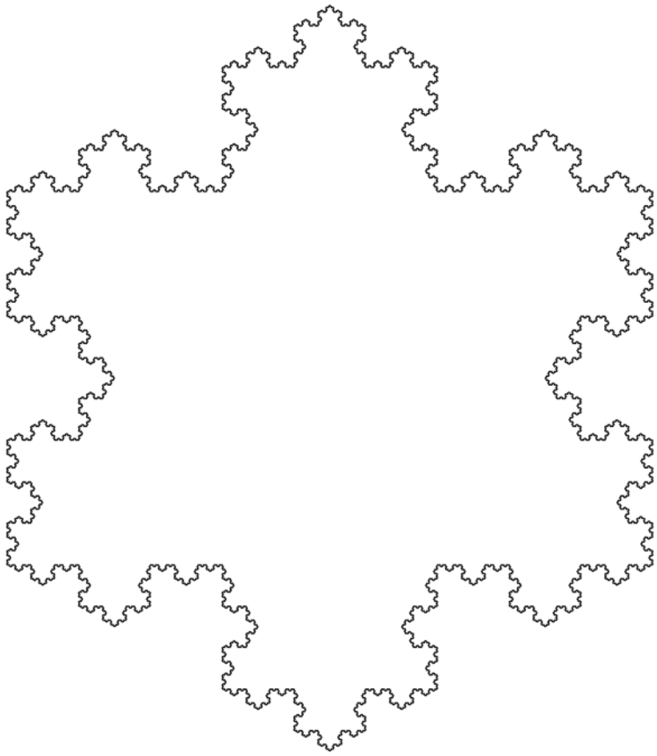


Figure 3.7: The Koch snowflake.

3.3 The Lynch Function

Through this text, and through most texts involving continuous nowhere differentiable functions, usually two notions are key: the uniform limit of a sequence of functions (when dealing with then in an analytical manner) and the Baire Category theorem (when a more topological approach is made). These, despite being standard knowledge for a mathematician, are still pretty sophisticated concepts that can give the impression that continuous nowhere differentiable functions are *rare monsters*. In that sense, Lynch provided in [10] an example of a continuous nowhere differentiable function needing only basic concepts of topology. It is for that reason that we show it here. Also, it will be helpful in section 5.2.

Let $T : \mathbb{C} \longrightarrow \mathbb{R}$ be such that $T(z) = \mathrm{Re}(z)$. For any $x \in \mathbb{R}$ and $\Omega \subset \mathbb{C}$, let $\Omega[x] := \{y \in \mathbb{R} \ / \ x + iy \in \Omega\}$. We will define a sequence $\{C_n\}$ with the next properties:

1. C_n is compact for $n \in \mathbb{N}$.
2. $C_{n+1} \subset C_n \subset \mathbb{C}$.
3. $T(C_n) = [0, 1], \ \forall n \in \mathbb{N}$.
4. $\mathrm{diam}(C_n) < \frac{1}{n}$, for every $x \in I$ and $n \in \mathbb{N}$.
5. For every $x_1 \in I, \ \exists x_2 \in I$ with $0 < |x - y| < 1\frac{1}{n}$ such that $y_1 \in C_n[x_1], \ y_2 \in C_n[x_2]$

implies that

$$\frac{|y_1 - y_2|}{|x_1 - x_2|} > n.$$

The sets C_n will be chosen as the closure of band neighbourhoods of the graph of polygonal arcs defined on I. Property 3 holds trivially and property 4 can be obtained by choosing the thickness of the bands appropriately to compensate for the steepness of each segment. However, before actually constructing them, we will prove that property 5 holds for closed band neighbourhoods of straight line segments.

Proposition 3.3.1. *Property 5 holds for closed band neighbourhoods of straight line segments.*

Proof. Let $n \in \mathbb{N}$ and m is such that $m \neq n$. Now, let $f(x) = mx + b$ and $\delta > 0,\ x_1 \in I$. If $m > n$, choose $x_2 = x_1 + \delta$ and take a band neighbourhood $N_\epsilon(f)$ of the graph of f. For $y_1 \in \overline{N_\epsilon(f)[x]}$ and $y_2 \in \overline{N_\epsilon(f)[x_2]}$ we have that $\frac{|y_1 - y_2|}{|x_1 - x_2|}$ is the absolute value of the slope of the line between points $x_1 + y_1 i,\ x_2 + y_2 i$. The minimum of this slope is obtained when

$$y_1 = mx_1 + b + \epsilon,\ y_2 = mx_2 + b - \epsilon.$$

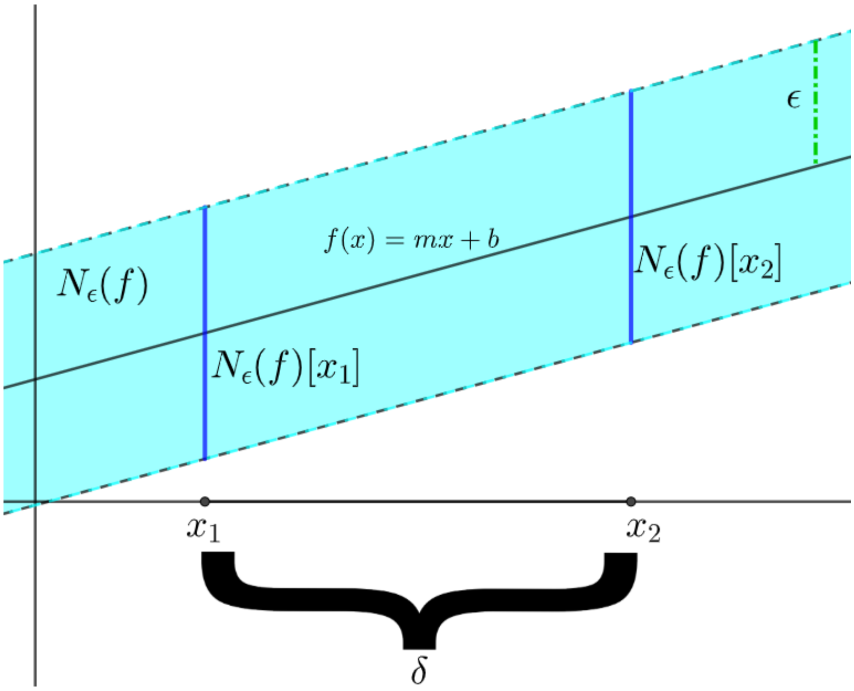


Figure 3.8: Line segment f for Lynch's function with band neighbourhood $N_\epsilon(f)$.

We can then choose $\epsilon > 0$ sufficiently small so that

$$\frac{|y_1 - y_2|}{|x_1 - x_2|} = |m - \frac{2\epsilon}{\delta}| > n.$$

If $m < n$ we would only have to repeat the argument, *mutatis mutandis*. Therefore, the proof is finished. □

Now, say C_n has been constructed. We construct C_{n+1} in the next way. Let P be a polygonal arc lying in the interior of C_n where each segment P_n has a slope whose absolute value exceeds n. For each $i \in \{0, \dots, k\}$ let δ_i satisfying

$$0 < \delta < \min\{\frac{|T(P_i)|}{2}, \frac{1}{n}\}.$$

Proposition Theorem 3.3.1 assures us that there is an ϵ_i-neighbourhood for each P_i. Observe that, since $\delta_i < \frac{|T(P_i)|}{2}$, we can always pick an $y \in T(P_i)$ for every $x \in T(P_i)$.

Define $\epsilon := \min\{\epsilon_i,\ i = 1, \dots, k\}$. Then, $\overline{N_\epsilon(P)}$ is a closed neighbourhood of P satisfying property 5. Observe that you can always pick a sufficiently small $\epsilon > 0$ such that $\overline{N_\epsilon(P)} \subset C_n$ and properties 3, 4, and 5 are satisfied. Define $C_{n+1} = \overline{N_\epsilon(P)}$.

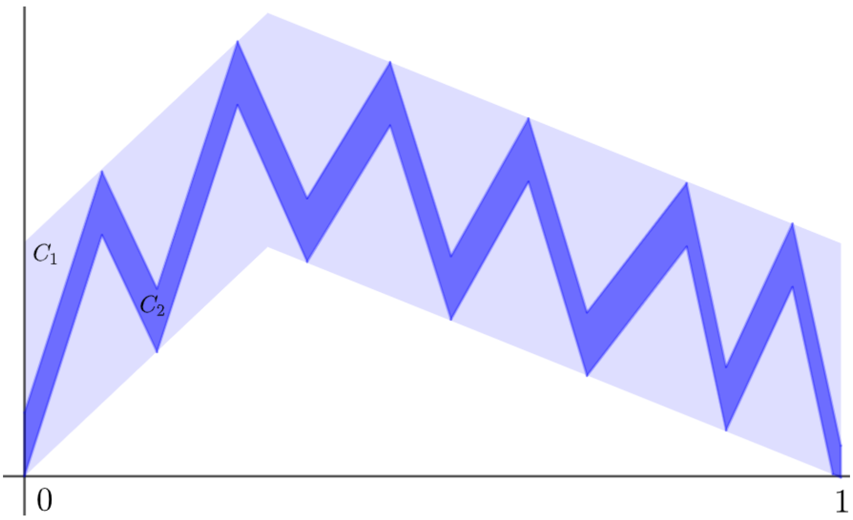


Figure 3.9: Scheme of C_1 and C_2 as above described.

We have created a sequence of bands with greater slope and narrower for each element in the sequence. We can now define the Lynch function.

Let $C = \bigcap C_n$. It is obvious that $\text{diam}(C[x]) = 0\ \forall x \in I$ so that C is the graph of a function $L : I \longrightarrow \mathbb{R}$. Since C is compact, L is necessarily continuous.

Theorem 3.3.2. *The function $L : I \longrightarrow \mathbb{R}$ as above described is nowhere differentiable on I.*

Proof. Let $x_1 \in I,\ \delta > 0$ and let $n \in N$ such that $\delta > \frac{1}{n}$. Property 5 implies the existence of an $x_2 \in I$ with $0 < |x_1 - x_2| < \frac{1}{n}$ such that $y_1 \in C_n[x_1],\ y_2 \in C_n[x_2]$ implies that

$$|\frac{y_1 - y_2}{x_1 - x_2}| > n.$$

Since $L(x_1) \in C_n[x_1]$ and $L(x_2) \in C_n[x_2]$, ther is no bound for the difference quotient

$$|\frac{L(x_1) - L(x_2)}{x_1 - x_2}|$$

as $\delta \Longrightarrow 0$. We conclude that L is not differentiable at x_1.

□

Of course, the Lynch function does not present a curve *per se*. The Lynch curve is, therefore, defined in the following way: if $L : I \longrightarrow \mathbb{R}$ is the Lynch function, then

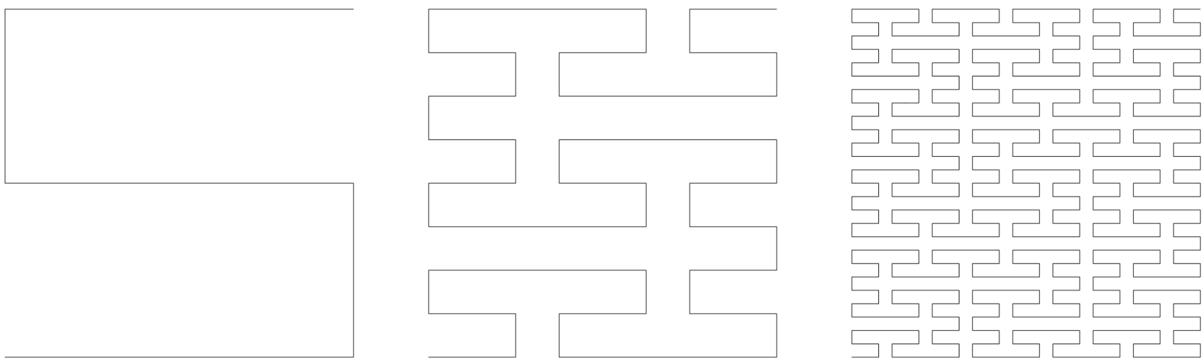
$$\begin{aligned} L_{\mathbb{C}} : & I \longrightarrow \mathbb{C} \\ & t \longmapsto t + L(t)i \end{aligned}$$

is the Lynch (complex) curve. Since L is nowhere differentiable, so is $L_{\mathbb{C}}$. We have, then, showed our last example.

Chapter 4

Space-Filling Curves

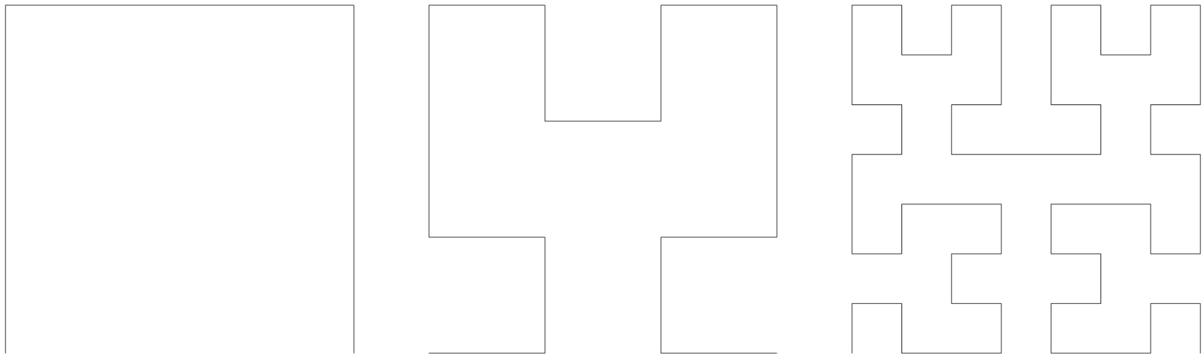
Another one of the greatest *mathematical monsters* is that of space-filling curves. It was the year 1878 when Cantor proved that any two given smooth manifolds of finite dimensions (without needing them to have equal dimension) had the same cardinality [4], which, *a priori*, means that you could find a bijective map between a line segment and a square. Just a year later, Netto proved that in order to map the interval $[0,1]$ into the square $[0,1]^2$ bijectively, such a mapping had to be discontinuous [14]. Peano constructed the very first continuous curve that passes through every point of a two-dimensional region with positive Lebesgue measure [16]. Such curve is constructed iteratively and is continuous and surjective from I to $Q = \{z \in \mathbb{C} \ / \ 0 \leq \mathrm{Re}(z), \mathrm{Im}(z) \leq 1\}$ and nowhere differentiable. For a proof of this, see [18], chapter 3. Although Peano never provided drawings of the construction of his curve, the first three iterations are shown below.



(a) First iteration. (b) Second iteration. (c) Third iteration.

Figure 4.1: Three first iterations of the Peano curve.

However, this is not the only curve of its kind: Hilbert provided in a paper in 1891 the start to the visual satisfaction of surface-filling curves. The curve he provided there is, as Peano's, constructed iteratively, continuous and surjective from I to Q and nowhere differentiable. Below, the three first iterations of it.



(a) First iteration. (b) Second iteration. (c) Third iteration.

Figure 4.2: Three first iterations of the Hilbert curve.

In [3] it is shown that if $p, q \in \Omega$ (where $\Omega \subset \mathbb{C}$ is a homeomorphic image of $\mathbb{D}$), then there does always exists a continuous surjective map $\phi : [a, b] \subset \mathbb{R} \longrightarrow \Omega$ satisfying $\phi(a) = p,\ \phi(b) = q$. Hilbert's and Peano's are particular cases of this phenomena. Despite being this the most famous cases of the so called *space-filling curves* (term that we will define later) they will not conform the corpus of this chapter. It is in the next step of this field of research that this chapter works on: continuous injective maps from $[0, 1]$ into $\mathbb{C}$ with positive Lebesgue measure.

We are going to give construction, properties and proofs of two of the curves that satisfy this condition: Osgood's curve (section 4.1) and Knopp's curve (section 4.2). For that matter, we will need the next definition:

Definition 4.0.1. *Let $\gamma\colon I \longrightarrow \mathbb{C}$ be continuous. We will say that γ is a* ***space-fillig curve*** *if $\mu(\gamma(I)) > 0$.*

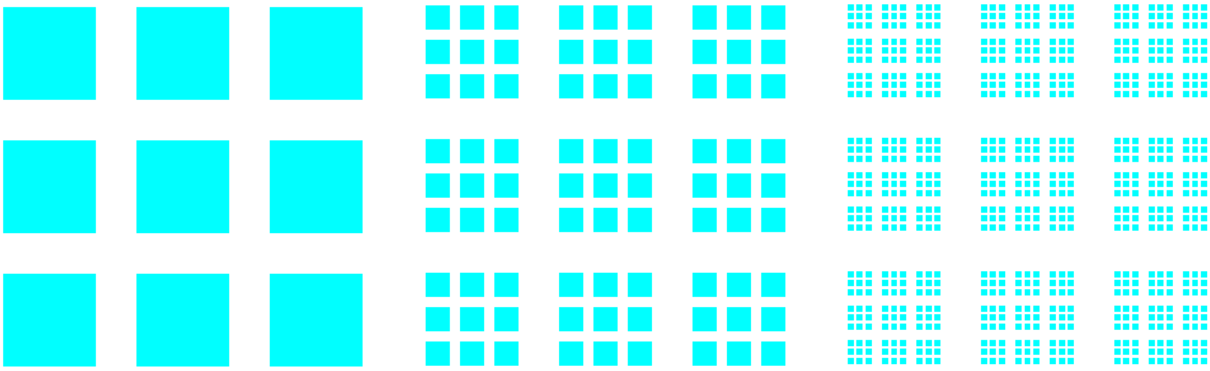
4.1 Osgood's Curve

In 1903, Osgood published the very first construction of a Jordan arc with positive Lebesgue measure, although he used the word *area*. He does actually construct a whole family of such curves, and as is said in [18], such curves are strongly inspired by the geometric generation of the Peano curve [16]. We provide an up-to-date and rigorous approach to the construction of his curve, since it does not usually appear in the literature except for mentioning its existence and, in fact, it is only treated in some detail in [18] and Osgood's original paper [15] and some references that only treat it in a more informal way.

First, a geometric construction of the locus of the curve will be given, following an iterative procedure, which will immediately lead us to find that such locus has positive outer Lebesgue measure. Second, we will construct a Cantor-type set and provide a total order on the points of the locus so that the identification with the elements of the Cantor-type set and its complementary set is natural, allowing us to interpret the locus as the trace of our parameterized curve and finally, after such a parametrization is done, continuity and injectivity will follow.

4.1.1 Construction of the locus for Osgood's Curve

Let $P \subset \mathbb{C}$ be a square of side length 1 with vertices $\{0, 1, 1+i, i\}$. Our will is to construct a curve satisfying that has Lebesgue measure $\lambda \in (0,1)$; such thing will be accomplished by subtracting some rectangular areas (smaller in each iteration), so that the sum of the rectangular areas adds up to $1 - \lambda$ at the limit. Within every iteration, the removed rectangular shapes leave, by construction, new squares, all identical to each other. In the image below, the first iteration is shown. This very same process is repeated in every one of the sub-squares obtained at the previous iteration. In figures Figure 4.3a, Figure 4.3b, Figure 4.3c, the blue shaded squares are what is left after the first, second and third iteration respectively for a given width of the rectangles.



(a) Firs iteration. (b) Second iteration. (c) Third iteration.

Figure 4.3: Three first iterations of the Osgood mesh.

Definition 4.1.1. *We will say that the width of a removed rectangle at the* $n+1$*-th iteration is* w_n*, for* $n \geq 0$.

Definition 4.1.2. *We will say that a square created at the* n*-th iteration has side length* l_n*, for* $n \geq 1$.

Our goal now is to obtain the side length of a working square in a particular iteration and the distance among two subsquares of the same partition. Since in every iteration we create two vertical rectangular shapes and two horizontal rectangular shapes in every subsquare created at the previous iteration, 9^n more sub-squares are created at iteration n. In each one of those squares, the removed area at iteration n, denoted by S_n, follows the next rule:

$$S_n = 4w_n^2 + 12w_n l_{n+1} \tag{4.1}$$

which comes from the fact that you are subtracting the 12 green bars and the 4 black squares as shown in Figure 4.4. Also, notice that the side of the previous square is three times the side of a subsquare plus two times width of a rectangle created at the last iteration, i.e.:

$$l_n = 3l_{n+1} + 2w_n, \;\; n \geq 0. \tag{4.2}$$

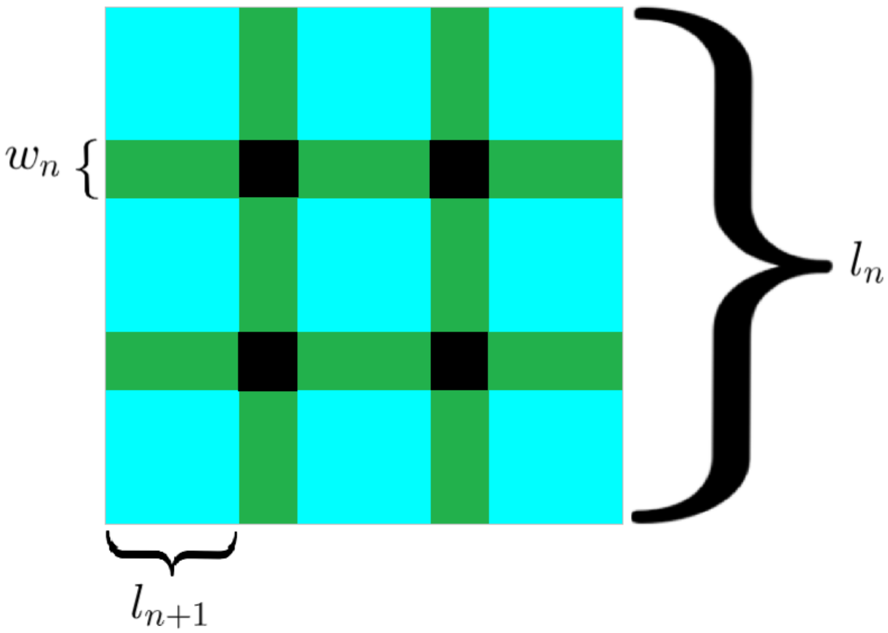


Figure 4.4: Scheme representing w_n, l_n and l_{n+1} at the $n+1$ iteration in a given subsquare of the last iteration.

Therefore, we want to find sequences $\{l_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0} \subset \mathbb{R}$ satisfying the following relationship:

$$1 - \lambda = \sum_{n\geq 0} 9^n S_n, \ \ \lambda \in (0,1) \tag{4.3}$$

In [18], the author claims that

> "This may be accomplished as follows: At the first step, we choose the width of the bars to be $(1-\sqrt{\lambda})/4$. At the next step, $(1-\sqrt{\lambda})/24$, and at the next step, $(1-\sqrt{\lambda})/144$, etc..."

However, he does not provide a proof of this nor a motivation to choose such values. Instead of proving this, we will prove that, for every $c > 3$, there exist two different values for w_0 so that relationship (4.3) is satisfied, provided that $\frac{w_{n+1}}{w_n} = \frac{1}{c}$, which does not appear anywhere in the literature.

Theorem 4.1.1. *Let $c > 3$ and $\lambda \in (0,1)$, and let $\{w_n\}_{n\geq 0}$ be a sequence satisfying $w_n = \frac{w_0}{c^n}$. Then, there exist two different values of w_0 such that the series $\sum_{n\geq 0} 9^n S_n$ converges to $1-\lambda$. Those two values are the solutions to the next quadratic equation:*

$$cw_0^2 + (3-c)w_0 + \frac{(1-\lambda)(c-3)^2}{4c} = 0 \tag{4.4}$$

Proof. We must realize that $l_0 = 1$, since we start with only one square and since $l_n = 3l_{n+1} + 2w_n$, $n \geq 0$, proceeding by recursion, we obtain:

$$\begin{aligned} l_{n+1} &= \frac{1}{3}l_n - \frac{2}{3}w_n \\ &= \frac{1}{3}(\frac{1}{3}l_{n-1} - \frac{2}{3}w_{n-1}) - \frac{2}{3}w_n = \frac{1}{3^2}l_{n-1} - \frac{2}{3}(w_n + \frac{1}{3}w_{n-1}) \\ &= \frac{1}{3^{n+1}}l_0 - \frac{2}{3}\sum_{i=0}^{n}\frac{1}{3^i}w_{n-i} = \frac{1}{3^{n+1}} - \frac{2}{3}w_0\sum_{i=0}^{n}\frac{1}{3^i}\frac{1}{c^{n-i}} \\ &= \frac{1}{3^{n+1}} - \frac{2}{3c^n}w_0\sum_{i=0}^{n}\left(\frac{c}{3}\right)^i \\ &= \frac{1}{3^{n+1}} + \frac{2w_0\left(3^{n+1} - c^{n+1}\right)}{3^{n+1}c^n(c-3)} \end{aligned}$$

which vanishes as n tends to infinity. This leads us the expression

$$\begin{aligned} S_n = 4w_n^2 + 12l_{n+1}w_n &= \frac{4w_0^2}{c^{2n}} + \frac{12w_0}{c^n}\left(\frac{1}{3^{n+1}} + \frac{2w_0(3^{n+1} - c^{n+1})}{3^{n+1}c^n(c-3)}\right) \\ = \frac{4w_0^2}{c^{2n}} + \frac{4w_0}{3^n c^n} + \frac{8w_0^2(3^{n+1} - c^{n+1})}{3^n c^{2n}(c-3)}. \end{aligned}$$

This tells us that our series is not defined for $c = 3$. Now,

$$\begin{aligned} 9^n S^n &= 9^n\left(\frac{4w_0^2}{c^{2n}} + \frac{4w_0}{3^n c^n} + \frac{8w_0^2(3^{n+1} - c^{n+1})}{3^n c^{2n}(c-3)}\right) \\ &= \frac{4w_0^2 3^{2n}}{c^{2n}} + \frac{4w_0 3^n}{c^n} + \frac{8w_0^2 3^n(3^{n+1} - c^{n+1})}{c^{2n}(c-3)} \end{aligned}$$

Which tells us that the series $A = \sum_{n\geq 0} 9^n S^n$ will be convergent if and only if series $A_1 = \sum_{n\geq 0}\frac{3^{2n}}{c^{2n}}$, $A_2 = \sum_{n\geq 0}\frac{3^n}{c^n}$ and $A_3 = \sum_{n\geq 0}\frac{3^n(3^{n+1}-c^{n+1})}{c^{2n}}$ are convergent, since $A = 4w_0^2 A_1 + 4w_0 A_2 + \frac{8w_0^2}{c-3}A_3$. By d'Alembert's criteria, we can assure convergence for A_1, A_2 and A_3 for $c > 3$ and divergence for $c < 3$. Since we were excluding $c = 3$ so that S_n is defined, we can claim that A is convergent if and only if $c > 3$.

Now, finding the values of A_1, A_2 and A_3 for any $c > 3$ is easy: those values are:

1. $A_1 = \dfrac{c^2}{c^2 - 9}$
2. $A_2 = \dfrac{c}{c - 3}$
3. $A_3 = \dfrac{c^3}{9 - c^2}$

Therefore, after some manipulations, we obtain:

$$\sum_{n\geq 0} 9^n S_n = \frac{4w_0^2c^2}{c^2-9} + \frac{4w_0c}{c-3} + \frac{8w_0c^3}{(c-3)(9-c^2)} = \frac{-4cw_0(3 + c(w_0 - 1))}{(c-3)^2} \tag{4.5}$$

and as we wish (4.3) to be satisfied, we impose

$$1 - \lambda = \frac{-4cw_0(3 + c(w_0 - 1))}{(c-3)^2} \tag{4.6}$$

which leads to equation $cw_0^2 + (3-c)w_0 + \dfrac{(1-\lambda)(c-3)^2}{4c} = 0$ and as its discriminant is equal to $(c-3)^2(1-\lambda)$ (which is always positive), it does always have two real solutions, which are $w_0 = \dfrac{(c-3)(1 \pm \sqrt{\lambda})}{2c}$. □

In order to construct a curve as the one we want, we need to connect the squares in a particular manner. In Figure 4.5 the joins between squares of the first partition are represented. In general, at the n-th iteration, joins are created as follows:

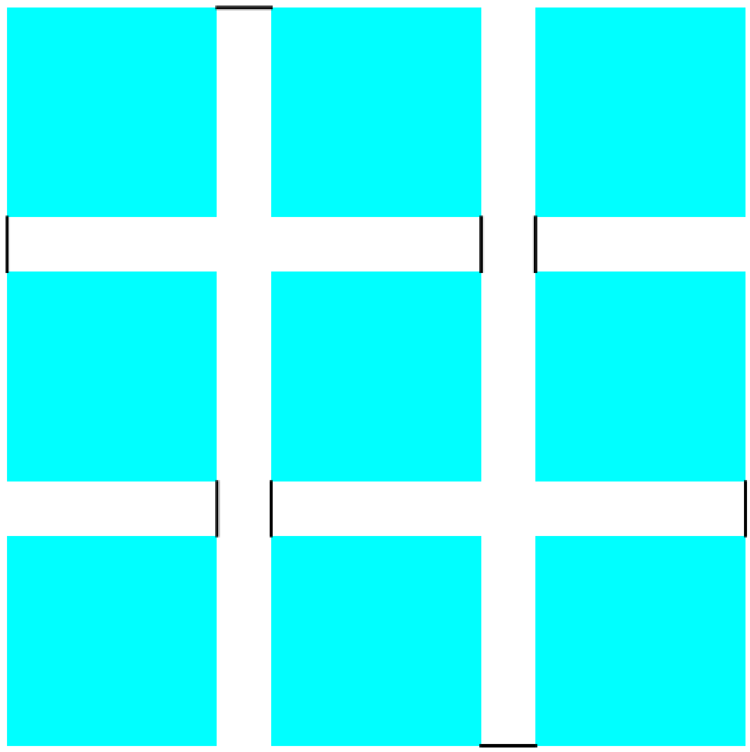


Figure 4.5: The black segments represent the joins between squares at the first iteration.

Let's say that we want to join the subsquares within a square created at a given iteration. Then, either two of the vertices of that square serve as mooring points for the two joins such square has (namely a and b) or it has one join and one of its vertex is 0 or $1+i$. Either way, the procedure is the same. We want first to join squares that are in the same colmun, second the column that is in the middle and finally the column that is left, considering that those columns must be joined. For that matter, start at a (or at vertex 0 or $1+i$ if applicable), move through two consecutive sides of that subsquare so that you arrive at the vertex of the subsquare that is in the same diagonal as your starting point. Join this point with the nearest vertex of the subsquare that is in its same column. Now, move through two consecutive sides of that subsquare so that you arrive to the vertex of the subsquare that is in the same diagonal as your starting point. Join such point with the nearest vertex of the subsquare that is in its same column. As you have now gone through the three rows of such column, we need to join it to the middle column. Join that vertex to the nearest vertex of the subsquare from the middle column. By repeating the process, we arrive at b. Observe that, once created, joins do not change.

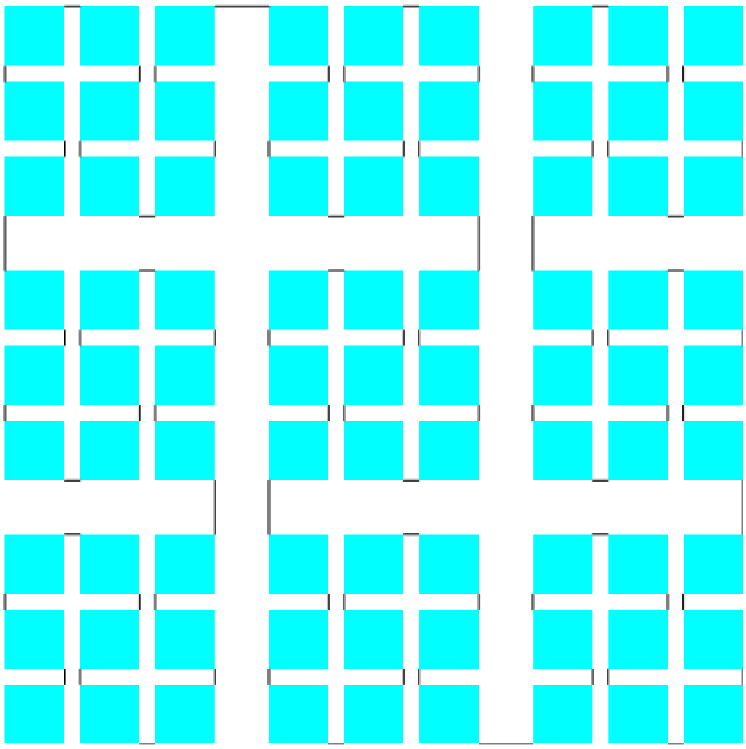


Figure 4.6: Joins of the first and second iteration.

4.1.2 Calculating the Lebesgue measure for Osgood's Curve.

Denote by $\mathcal{C}_n(\lambda)$ the union of all the squares and joins left after the n-th iteration. This set has Lebesgue measure

$$1 - \sum_{k=0}^{n} 9^k S_k$$

and therefore the set $\mathcal{C} = \bigcap_{n\geq 1} \mathcal{C}_n(\lambda)$ has Lebesgue measure

$$\lim_{n\to\infty} 1 - \sum_{k=0}^{n} 9^k S_k = 1 - \lim_{n\to\infty} \sum_{k=0}^{n} 9^k S_k = 1 - (1-\lambda) = \lambda\,.$$

We want the set $\mathcal{C}$ to be the trace of our curve, which I shall now construct.

Observe that in the first iteration of the construction of $\mathcal{C}$ we have 17 different parts: 9 squares and 8 joins. Within every iteration, 9 more subsquares and 8 more joins are created inside every square created at the previous iterations and the previous joins remain unchanged. This suggests that we should create a domain of definition that captures this behavior. Such thing shall be done as follows:

4.1.3 Parametrization of the locus and its continuity.

First, we provide an order on the elements of $\mathcal{C}$. For $\mathcal{C}_1(\lambda)$ we obtain 17 different parts (9 squares and 8 joins) and we number them as in the picture below.

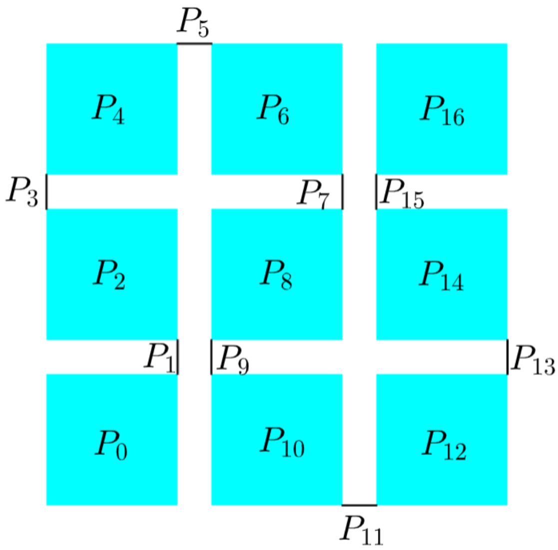


Figure 4.7: Ordering of the elements of the first partition.

At the second iteration, square $P_{(2b_1)}$ is divided into 17 new different parts (9 squares and 8 joins), which we enumerate as $P_{(2b_1)b_j}$, $j = 0, \ldots, 16$, in the following way: $P_{(2b_1)0}$ is the subsquare that is joined to the (previous) square $P_{2(b_1-1)}$ if $b_1 \geq 1$ or the one that has the vertex 0 of E in case that $b_1 = 0$. $P_{(2b_1)2}$ is the one connected to $P_{(2b_1)0}$, $P_{(2b_1)4}$ is the one connected to $P_{(2b_1)2}$ that is not $P_{(2b_1)0}$... and $P_{(2b_1)16}$ the one that connects to the next set of subsquares by P_{2b_1+1}. Similarly, $P_{(2b_1)(2b_j+1)}$ is the join connecting squares $P_{(2b_1)(2b_j)}$ and $P_{(2b_1)(2b_j+2)}$. The same process is repeated ad infinitum. This provides an order in $\mathcal{C}$ in base 17. In fact, squares only have even figures and joins have n even figures, one odd figure and infinite zeros, as joins, once created, do not change any more.

We can now define Γ_8 as

$$\Gamma_8 = \{x \in [0,1] \ / \ x = (0.(2b_1)(2b_2)\ldots)_{17}, \ b_i = 0, \ldots, 8\}$$

so that we want to map ${\Gamma_8}^c$ into the joins and Γ_8 into the rest of the points of $\mathcal{C}$. Let's start by doing that last thing and then extend it to ${\Gamma_8}^c$.

Since

$$P_{2b_1} \supset P_{(2b_1)(2b_2)} \supset P_{(2b_1)(2b_2)(2b_3)} \cdots \tag{4.7}$$

is a nested sequence of closed squares that shrinks into points,

$$P_{(2b_1)} \bigcap P_{(2b_1)(2b_2)} \bigcap P_{(2b_1)(2b_2)(2b_3)} \cdots \tag{4.8}$$

defines a point which, by construction, lies on $\mathcal{C}$ (and not in any join). Then, we define

$$f\,(\ (0.(2b_1)(2b_2)(2b_3)\ldots)_{17})\ ) := P_{(2b_1)} \bigcap P_{(2b_1)(2b_2)} \bigcap P_{(2b_1)(2b_2)(2b_3)} \cdots \tag{4.9}$$

which is a function on Γ_8 and we extend it to I. Such thing is done by linear interpolation in the next sense: if (a_n, b_n) is an interval removed from I at a certain iteration of the construction of Γ_8, we define

$$F(t) = \frac{f(a_n)(b_n - t) + f(b_n)(t - a_n)}{b_n - a_n},\ t \in (a_n, b_n).$$

Hence, the extension is the map $\gamma\colon\ I \longrightarrow \mathcal{C}$ defined by

$$\gamma(t) := \begin{cases} f(t) & if \quad t \in \Gamma_8 \\ \\ F(t) & if \quad t \in {\Gamma_8}^c \end{cases} \tag{4.10}$$

is both continuous and injective. In order to proof this, we need a couple of previous lemma.

Lemma 4.1.2. *Consider* ${\Gamma_8}^1$ *defined as*

$${\Gamma_8}^1 := I - \bigcup_{k=1}^{8} \left(\frac{2k-1}{17}, \frac{2k}{17}\right)$$

and with that, define

$${\Gamma_8}^j := {\Gamma_8}^{j-1} - \bigcup_{k=1}^{\frac{17^j - 1}{2}} \left(\frac{2k-1}{17^j}, \frac{2k}{17^j}\right),\ j \geq 1.$$

Then, the set Γ_8 *is the intersection of all* ${\Gamma_8}^j$*, i.e.,*

$$\Gamma_8 = \bigcap_{j\geq 1} {\Gamma_8}^j\,.$$

Proof. First, consider $t \in \Gamma_8$, that is, $t = (0.(2b_1)(2b_2)\ldots)_{17},\ b_i = 0, \ldots, 8$. Let $t_j = (0.(2b_1)(2b_2)\ldots(2b_j))_{17}$ be the truncation of t after j digits. It is clear that $\lim\limits_{j\to\infty} t_j = t$. Actually,

$$t_j \leq t \leq t_j + \frac{1}{17^j},\ j \geq 1.$$

Observe that the numbers in I that have j decimal (non-zero) digits and whose digits are even are exactly the left endpoints of the intervals that conform ${\Gamma_8}^j$, meaning that $\left[t_j, t_j + \frac{1}{17^j}\right] \subset {\Gamma_8}^j \Rightarrow t \in {\Gamma_8}^j,\ \forall j \geq 1$ and, therefore, $t \in \bigcap_{j\geq 1} {\Gamma_8}^j$.

Conversely, suppose $t \in \bigcap_{j\geq 1} {\Gamma_8}^j$, which, obviously, means that $t \in {\Gamma_8}^j,\ j \geq 1$. Now, again, observe that the elements of ${\Gamma_8}^j$ are the numbers in I such that their j-th truncation uses only even figures in base 17 (see Figure 4.8). Thus, every truncation of t uses only even figures in base 17 and as t is the limit of the sequence $\{t_j\}_{j\geq 1}$ (with t_j the truncation of t after j decimal digits), it implies that t has only even figures in base 17. Hence, $t \in \Gamma_8$ and this concludes the proof. □

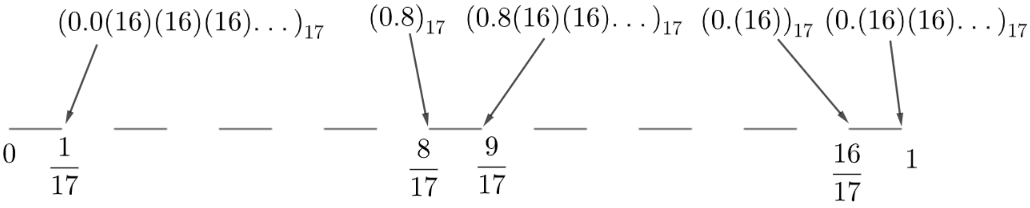


Figure 4.8: The segments represent each of the intervals contained in ${\Gamma_8}^1$. Below the segments, the value the boundary points of some intervals; above, their expression in base 17 (finite expression if the numerator of its corresponding fraction is even, infinite expression otherwise).

Lemma 4.1.3. *The set Γ_8 is a Cantor-type set.*

Proof. Let $t \in \Gamma_8 \Rightarrow t \in {\Gamma_8}^j,\ j \geq 1$. Let now $j_0 \geq 1$. If $t \in {\Gamma_8}^{j_0}$, that means that it belongs to one and only one of its contained subintervals. Denote such interval by J_0 and define $t_{j_0} \in J_0$ in the next way: if t is not the left extreme of J_0, then t_{j_0} is the left extreme of J_0; otherwise, t_{j_0} is the right extreme of J_0. As $|J_0| = \dfrac{1}{17^{j_0}} \Rightarrow |t - t_{j_0}| \leq \dfrac{1}{17^{j_0}}$. Now again, as t is in Γ_8 and it belongs to $J_0 \subset {\Gamma_8}^{j_0}$, it will belong to ${\Gamma_8}^{j_0+1}$ and, in particular, to one of the subintervals in which J_0 is divided, namely J_1, and we can then define t_{j_1} analogously to how it was done with t_{j_0} . Proceeding by iteration, we end up constructing a sequence $t_{j_i},\ i \geq 0$ so that it converges to t (since by construction $|t - t_{j_i}| \leq \frac{1}{17^{j_i}} \to 0$ as $i \to \infty$). Therefore, t is an accumulation point, give the fact that the extremes of the intervals always remain in Γ_8. Since t was chosen arbitrarily, every point in Γ_8 is an accumulation point.

Now, Γ_8 is closed because we have shown in Lemma Theorem 4.1.2 that it is a countable intersection of closed sets; the fact that we have just shown that it has no isolated points makes it a perfect set. Also, being closed, together with the fact that it is a subset of I makes it bounded, and by Heine-Borel's theorem it is compact.

Now, observe that Γ_8 contains no subinterval of I: since Γ_8 can be constructed be subtraction of open subintervals of length $\frac{1}{17^n}$, that means that any fixed but arbitrary subinterval $[a, b] \subset I$ contains, at least, one of those open subintervals of length $\frac{1}{17^m} < |b - a|$ or is, at least, partially contained into one of those, and we can conclude that Γ_8 contains no subintervals.

Finally, the closure of Γ_8 is itself since it is closed and, as it contains no intervals, it has empty interior, whence nowhere dense.

□

We are now capable of proving that our function γ is continuous and injective.

Theorem 4.1.4. *The function γ defined in* 4.10 *is continuous and injective.*

Proof. Let's first prove injectivity: any point on $\mathcal{C}$ lies on a join or it does not. If so, it is clear by construction that there is one and only one preimage in one of the elements of Γ_8^c. If not, it must lie on P_{b1} and on $P_{b_1b_2}$ and on $P_{b_1b_2b_3}$ and so on and so forth. The point satisfies that it cannot lie in two squares that belong to the same partition. Hence, the point can be defined as a sequence of nested closed squares (which is unique) that collapses into a point, which, in turn, correspond to a unique sequence of closed intervals that shrink to a point and defines a unique element in Γ_8, as had been stated in the definition of f.

Now, let's prove continuity. This will be done in various steps: first, we provide the approximating polygons, then, the functions that have those approximating polygons as image. With that, we show that those functions converge uniformly, therefore its limit is continuous and we show that, in fact, such limit is γ.

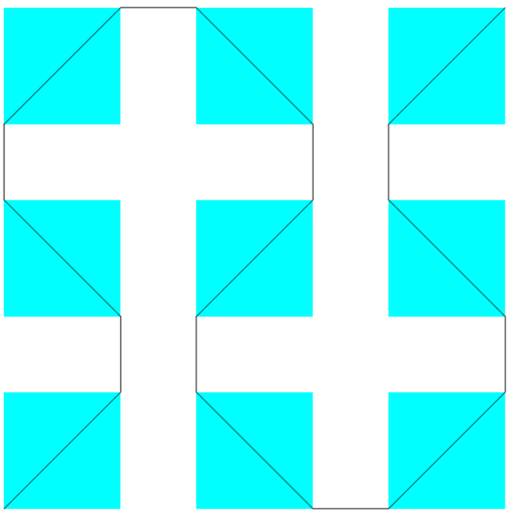


Figure 4.9: Q_1 and $\mathcal{C}_1(\lambda)$.

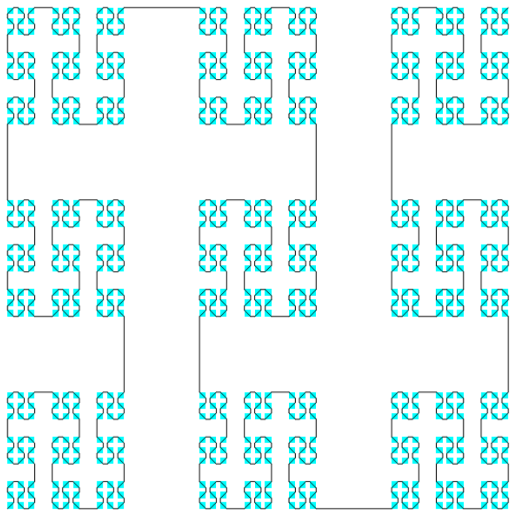


Figure 4.10: Q_3 and $\mathcal{C}_3(\lambda)$.

Constructing the polygons.

Considere $\mathcal{C}_n(\lambda)$ as before. Every square in $\mathcal{C}_n(\lambda)$ has two joins but $P_{0\ldots0}$ and $P_{(16)\ldots(16)}$ that have only one. For squares $P_{0\ldots0}$ and $P_{(16)\ldots(16)}$ draw the diagonal that vertex 0 and the vertex that has the join have in common and the diagonal that vertex $1+i$ and the vertex that has the join have in common respectively. For any other square, draw the diagonal that connects the vertices that have joins. The resulting shape of combining the diagonals and the joins is the approximating polygon of order n, namely, Q_n. See Figure 4.9 and Figure 4.10 for an image of Q_1 and Q_3 in context with $\mathcal{C}_1(\lambda)$ and $\mathcal{C}_3(\lambda)$. As long as Q_n is defined after the construction of $\mathcal{C}_n(\lambda)$, the sequence $\{Q_n\}_{n\in\mathbb{N}}$ converges to the locus of the Osgood curve. See Figure 4.9 and Figure 4.10 for a scheme of this.

Parametrizing the polygons.

In order to parametrize Q_n, consider ${\Gamma_8}^n$ and $({\Gamma_8}^n)^c$ and let γ_n be such that:

- it maps each interval contained in ${\Gamma_8}^n$ to the diagonal segments of Q_n as follows: $[0, \frac{1}{17^n}]$ is mapped into the diagonal of $P_{0\ldots0}$, $[\frac{2}{17^n}, \frac{3}{17^n}]$ is mapped into the diagonal of $P_{0\ldots02}$, etc.
- it maps each interval contained in $({\Gamma_8}^n)^c$ to the vertical and horizontal segments of Q_n (the joins of $\mathcal{C}_n(\lambda)$) as follows: $\left(\frac{1}{17^n}, \frac{2}{17^n}\right)$ is mapped into $P_{0\ldots01}$, $\left(\frac{3}{17^n}, \frac{4}{17^n}\right)$ is mapped into $P_{0\ldots03}$,etc.

Since it is picewise linear, $\gamma_n \in C(I, E)$. We have then parametrized it continuously.

The limit of the sequence $\{\gamma_n\}_{n\geq1}$.

We will now prove that the sequence $\{\gamma_n\}_{ngeq1} \subset C(I, E)$ is a Cauchy sequence. Observe that γ_n and γ_{n+1} coincide at the joins of $\mathcal{C}_n(\lambda)$. Thus, they can only differ over the squares of $\mathcal{C}_n(\lambda)$, which means that $\|\gamma_{n+1} - \gamma_n\|_\infty < \sqrt{2}l_n$. Hence, for a fixed $n \in \mathbb{N}$ and an arbitrary $j \in \mathbb{N}$,

$$\begin{aligned}\|\gamma_{n+j} - \gamma_n\|_\infty &= \|(\gamma_{n+j} - \gamma_{n+j-1}) + (\gamma_{n+j-1} - \gamma_{n+j-2}) + \cdots + (\gamma_{n+1} - \gamma_n)\|_\infty \\ &< \sqrt{2} \sum_{i=n}^{n+j-1} l_i \xrightarrow{n\to\infty} 0\end{aligned}$$

which follows from the fact that the series $\sum_{n\geq0} l_n$ is convergent. This proofs that $\{\gamma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore, it has a limit and such limit is a continuous function from I to E, namely $\lim_{n\to\infty} \gamma_n = \tilde{\gamma} \in C(I, E)$.

The limit is our function γ.

So far, the reasoning we have followed has lead us to the fact that the image of I via $\tilde{\gamma}$ is contained in the Osgood curve. Let's show that it is indeed the Osgood curve and, with that, that $\gamma = \tilde{\gamma}$.

For that matter, consider $p \in \mathcal{C}$. That means that there is a $t \in I$ such that $\gamma(t) = p$.

Let first p lie in a join, which means that $t \in {\Gamma_8}^c$. As joins, once created, do not change any more, that means that $p = \gamma_n(t)$ is constant for every $n \geq 1$ and it is therefore in the image of γ_n for every $n \geq 1$ and in the image of γ.

Now, if p does not lie in a join, that means that $t \in \Gamma_8$ such that $p = \gamma(t)$. We have two possible scenarios:

- The point p lies in the diagonal of a given square of its defining sequence of squares as in 4.8 (and, therefore, in the diagonal of every following subsquare), namely $P_{b_1 \ldots b_n}$. Then, there exists a sequence $\{t_k\}_{k=n}^{\infty}$ such that $\gamma_k(t_k) = p$ and it is, as before, both in the image of γ_n and γ.

- The point p does not lie in any diagonal. In such case, observe that if $p = \bigcap_{n \geq 1} P_{b_1 \ldots b_n}$ as in 4.8 and $\{p_n\}_{n \geq 1}$ is a sequence satisfying $p_n \in P_{b_1 \ldots b_n}$, then $p = \lim_{n \to \infty} p_n$. This is actually true, in particular, for p_n lying in the corresponding diagonal of $P_{b_1 \ldots b_n}$. Then, for every $n \geq 1$ there is a $t_n \in {\Gamma_8}^n$ such that $\gamma_n(t_n) = p_n \Rightarrow \lim_{n \to \infty} \gamma_n(t_n) = p = \gamma(t)$. Since I is compact, there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \longrightarrow t$ which implies that $\gamma_{n_k}(t_{n_k}) \longrightarrow \gamma(t) = p$. This concludes that $\tilde{\gamma}$ and γ are the same function.

□

It shall be remarked that Jordan arcs with positive n-dimensional Lebesgue measure are typically referred to as *Osgood curves.*

4.2 Knopp's Osgood Curve

Despite being the first proposal of a Jordan arc with positive two-dimensional Lebesgue measure, Osgood's construction was criticized because of the need for "joins". Sierpiński was the first one to provide an example of an Osgood curve that had no need for joins, although it had kind of a difficult construction. Knopp criticized both Osgood's and Sierpiński's construction in [7]. Let's construct Knopp's Osgood curve. As we did with Osgood's curve, we will first provide the locus for it and, latter, it will be parametrized.

Let $\mathcal{T} = \triangle ABC$ be a triangle. Remove from it a triangle -with one vertex at B and the opposite side to such vertex lying in AC, namely A_1C_1, that has area $r_1\mu(\mathcal{T})$, $r_1 \in (0, 1)$ so that you are left with two triangles $\mathcal{T}_0 = \triangle ABC_1, \mathcal{T}_1 = \triangle A_1BC$, as in Figure 4.11, satisfying

$$\mu(\mathcal{T}) = \mu(\mathcal{T}_0) + \mu(\mathcal{T}_1) + r_1\mu(\mathcal{T}) \Rightarrow \mu(\mathcal{T}_0) + \mu(\mathcal{T}_1) = (1 - r_1)\mu(\mathcal{T}).$$

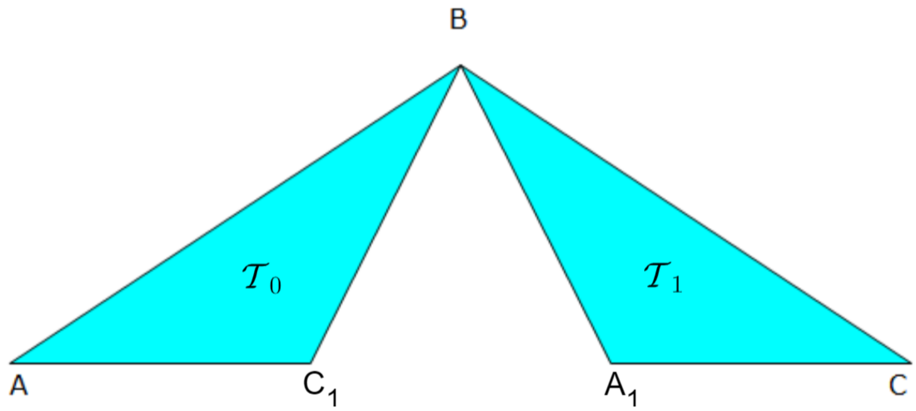


Figure 4.11

Now, remove from $\mathcal{T}_0$ a triangle with area $r_2\mu(\mathcal{T}_0)$ with a vertex in C_1 and its opposite side lying in AB, namely $B_{2,1}A_2$. From $\mathcal{T}_1$ remove a triangle with area $r_2\mu(\mathcal{T}_1)$ with a vertex in A_1 and its opposite side lying in BC, namely $B_{2,2}C_2$. We are left with triangles $\mathcal{T}_{00} = \triangle AB_{2,1}C_1, \mathcal{T}_{01} = \triangle C_1A_2B, \mathcal{T}_{10} = \triangle BC_2A_1, \mathcal{T}_{11} = \triangle A_1B_{2,2}C$ (as in Figure 4.12, which satisfies that

$$\mu(\mathcal{T}) = r_1\mu(\mathcal{T}) + r_2(\mu(\mathcal{T}_0) + \mu(\mathcal{T}_1)) + \sum_{i,j=0}^{1} \mu(\mathcal{T}_{ij})$$

implying that

$$\sum_{i,j=0}^{1} \mu(\mathcal{T}_{ij}) = \mu(\mathcal{T})(1 - r_1)(1 - r_2).$$

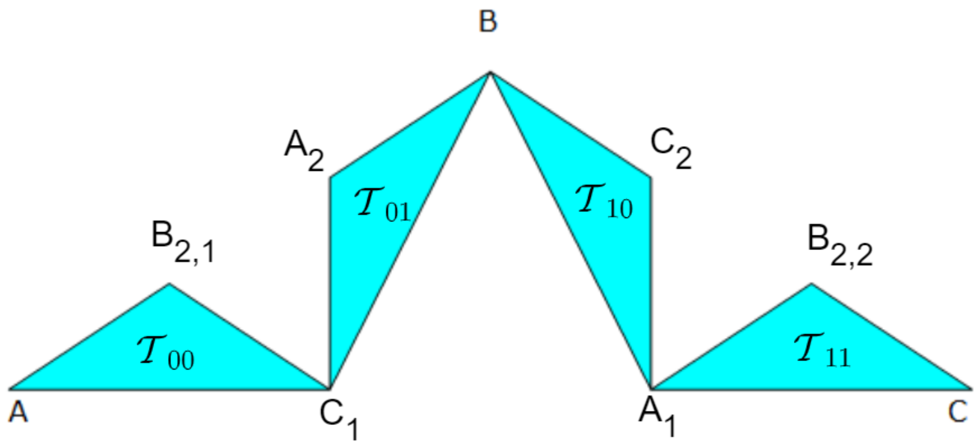


Figure 4.12

In general, following this procedure, we obtain that

$$\sum_{b_1,\ldots,b_n=0}^{1} \mu(\mathcal{T}_{b_1\ldots b_n}) = \mu(\mathcal{T}) \prod_{i=1}^{n} (1 - r_i)$$

which means that, in the limit, the set

$$\mathcal{C} = \left(\mathcal{T}_0 \bigcup \mathcal{T}_1\right) \bigcap \left(\mathcal{T}_{00} \bigcup \mathcal{T}_{01} \bigcup \mathcal{T}_{10} \bigcup \mathcal{T}_{11}\right) \bigcap \cdots$$

has two-dimensional Lebesgue measure

$$\mu(\mathcal{C}) = \mu(\mathcal{T}) \prod_{j \geq 1} (1 - r_j)$$

which converges to a number greater than 0 if and only if $\sum_{j \geq 1} r_j < \infty$ (see [1] for a proof).

If, for example, $\mathcal{T}$ is choosen to be a right isosceles triangle with a base of length 2 (which yields $\mu(\mathcal{T}) = 1$) and $r_j = \frac{r^2}{j^2}$, $r \in (0, 1)$, it is obtained that

$$\mu(\mathcal{C}) = \prod_{j \geq 1} \left(1 - \frac{r^2}{j^2}\right)$$

and, from Weierstrass factorization theorem (see [1]),

$$\mu(\mathcal{C}) = \prod_{j \geq 1} \left(1 - \frac{r^2}{j^2}\right) = \frac{\sin(\pi r)}{\pi r}.$$

It is easily checked that the function

$$\begin{aligned} f : (0, 1) &\longrightarrow (0, 1) \\ x &\longmapsto \frac{\sin(\pi x)}{\pi x} \end{aligned}$$

is both continuous and bijective, hence, equation $\frac{\sin(\pi r)}{\pi r} = \lambda$ has exactly one solution for $r \in (0, 1)$ for any given $\lambda \in (0, 1)$. This means that it suffices to choose r appropriately in order for $\mathcal{C}$ to have two-dimensional Lebesgue measure $\lambda \in (0, 1)$ for any $\lambda \in (0, 1)$ that we wish.

We have discussed the construction and properties of the locus of our desired curve. We now aim to parametrize it: observe that the subindex at triangle $\mathcal{T}_{b_1 \ldots b_n}$ expresses an order in binary, which tells us that we can identify every element of $\mathcal{C}$ with every point in I by expressing such point in its infinite binary form. We then define the function

$$\begin{aligned} f : [0, 1] &\longrightarrow \mathcal{C} \\ t = (0.b_1 b_2 \ldots)_2 &\longmapsto \bigcap_{n \geq 1} \mathcal{T}_{b_1 \ldots b_n} \end{aligned}$$

which is well defined since $\mathcal{T}_{b_1 \ldots b_n} \supset \mathcal{T}_{b_1 \ldots b_n b_{n+1}}$, implying that $\mathcal{T}_{b_1} \supset \mathcal{T}_{b_1 b_2} \supset \mathcal{T}_{b_1 b_2 b_3} \ldots$ is a nested sequence of closed triangles that collapse into points, and, therefore, the intersection $\bigcap_{n \geq 1} \mathcal{T}_{b_1 \ldots b_n}$ defines a unique point in $\mathcal{C}$. Let's prove its continuity and injectivity.

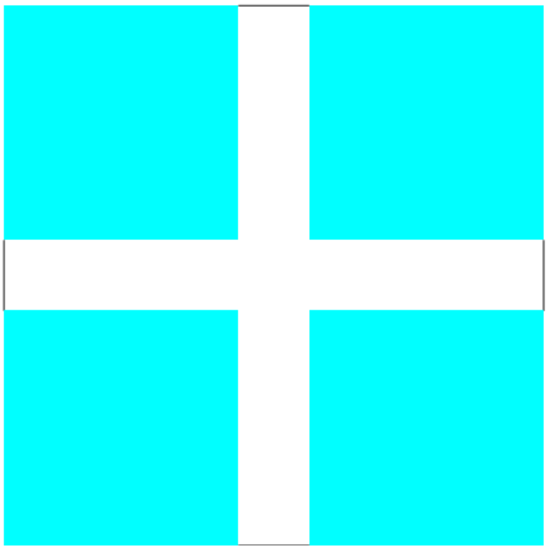
- *continuity*: Let $t_1, t_2 \in I$ so that $|t_1 - t_2| < \frac{1}{2^n}$. In such case, $f(t_1)$ and $f(t_2)$ lie either at the same triangle or in two adjacent triangles. As the length of the sides of the triangles vanishes at the limit, we obtain continuity.
- *injectivity* Let $t_1 = (0.a_1a_2a_3\ldots)_2, t_2 = (0.b_1b_2b_3\ldots)_2 \in I$ be such that $f(t_1) = \bigcap_{n\geq 1} \mathcal{T}_{c_1\ldots c_n} = f(t_2)$. By definition of f, we have that $a_i = c_i = b_i,\ i \geq 1$ and, therefore $t_1 = t_2$, which concludes that our function is injective.

4.3 Closed Osgood Curves

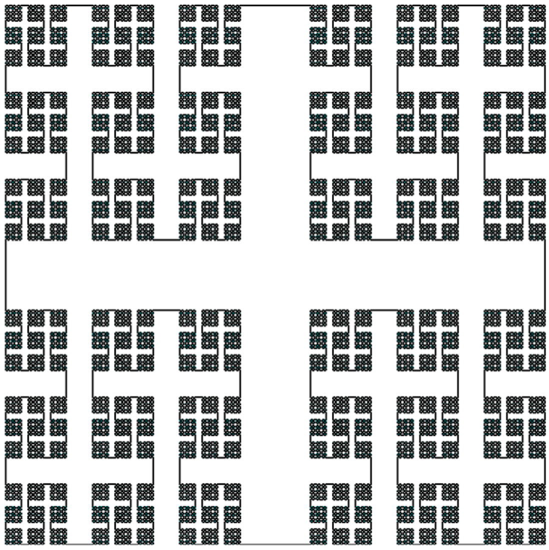
As it is the reason to be of this chapter to provide more pathological cases of Jordan curves that motivate the need of a proof for the JCT, we will, as was done in the previous chapter, construct a Jordan curve out of the Jordan arcs that we have constructed insgood this chapter.

4.3.1 Osgood's closed curve

In order to construct a Jordan curve that derives from Osgood's curve, it suffices us to concatenate for of them. For example, consider the Osgood curve created at section 4.1 and consider the square of side length $L > 2$, $\bar{E} = \{z \in \mathbb{C}\ /\ 0 \leq |\text{Re}(z)|, |\text{Im}(z)| \leq L\}$. In such case, we can place four squares of side length 1 inside it, as shown in Figure 4.13a, so that they share no common points.



(a) Square of side length $L > 2$ with the four squares of side 1.



(b) An approximation of the resulting curve.

Figure 4.13

The sides of $\bar{E}$ that do not belong to any of the four squares will act as joins. Now, the top left square is so that we can create there the curve in section 4.1. The top right square is the reflection of the top left square via the straight line that divides $\bar{E}$ in half vertically. The bottom right square is the reflection of the bottom right square via the straight line that divides $\bar{E}$ in half horizontally and the bottom left square is the reflection of the top left square via the straight line that divides $\bar{E}$ in half horizontally, as shown in Figure 4.13b.

It will, of course, have positive outer 2-dimensional Lebesgue measure: if the top left square has outer Lebesgue measure $\lambda \in (0, 1)$, then this closed version has outer measure $4\,\lambda$. In order to parametrize it, divide the interval $[0, 8]$ in eight intervals of length 1. The interval $[0, 1]$ will correspond to the top left square; $(1, 2)$ to the join between the top left and top right squares; $[2, 3]$ to the top right square... and the construction of the curve within each one of the different squares is like in section 4.1, *mutatis mutandis.* That way, we have created and Osgood curve that is also a closed curve: if the main big square has side length $L = 2 + x,\ x > 0$, the image of $t = 0$ and $t = 8$ are both the bottom left vertex of the top left square, i.e., $(1 + x)i$.

4.3.2 Knopp's closed curve

Knopp's triangle provides us with a Jordan curve too if we place three copies of the same triangle so that their longest sides form a triangle as in Figure 4.14.

Parametrizing this curve is done, essentially, in the same way it was done in the previous one: we will go over it clockwise, starting at the point that the top right triangle and the left triangle share. Now, divide the interval $[0, 3]$ in three intervals of side length 1. Then, $[0, 1)$ will correspond to the right triangle, $[1, 2)$ to the bottom triangle and $[2, 3]$ to the triangle that is left.

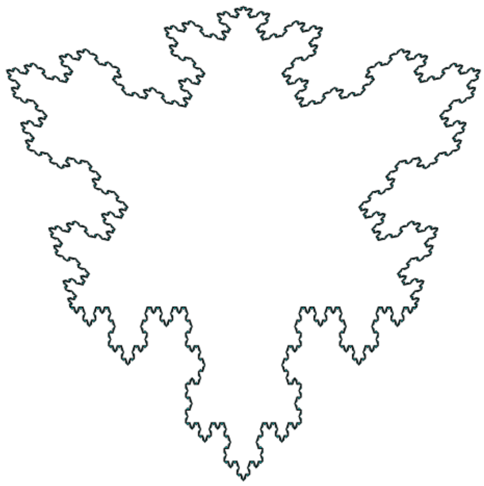


Figure 4.14

Chapter 5

Space-filling and nowhere differentiable curves.

So far, we have shown that there are uncountably many pathological curves that break our intuition of what a curve is. However, this *non-numerability* does not really show "how large" the family of nowhere differentiable curves nor the family of space-filling curves is. Throughout this chapter, we will deal with the topological size of these two families.

For this purpose, let us give the next definitions.

Definition 5.0.1. *We will denote the set of space-filling curves by $\mathcal{SF}$ and the set of nowhere differentiable curves by $\mathcal{ND}$.*

5.1 The family of space-filling curves.

The space-filling curves that have been shown in chapter 4 share a common property with those shown in chapter 3: they are of fractal nature. Also, by the graphic representation of both of them, one can ask about the differentiability of space-filling curves. It can be shown by a direct proof that neither Osgood's Curve nor Knopp's Osgood Curve are differentiable; as a matter of fact, Osgood's Curve is not differentiable on Γ_8 but everywhere differentiable on ${\Gamma_8}^c$ and Knopp's Curve is not differentiable. However, Morayne gave in [13] the next theorem.

Theorem 5.1.1 (Morayne's Theorem)**.** *Let $F = (f_1, f_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$ be such that f_1 is Lebesgue-measurable and either $f_1'(t)$ or $f_2'(t)$ exists for every $t \in \mathbb{R}$. If $F(\mathbb{R})$ is Lebesgue-measurable, then $\mu(F(\mathbb{R})) = 0$.*

Proof. See [13]. □

Theorem 5.1.1 provides a fast way of proving a beautiful and remarkable result for us, which can be found in [18], p. 79.

Theorem 5.1.2. *No space-filling curve is differentiable.*

Proof. By *absurdum reductio*, let $F \in SF$ and suppose without loss of generality that F is differentiable on $(0,1)$ and let

$$d : \mathbb{R} \longrightarrow (0,1)$$

be a diffeomorphism. Then $\mu(F((0,1)) > 0$ and $F \circ d : \mathbb{R} \longrightarrow \mathbb{R}^2$ satisfies Theorem 5.1.1. Therefore,

$$\mu(F \circ d) = \mu(F((0,1)) = \mu(F(I)) = 0$$

which contradicts our hypothesis. We conclude that there is no differentiable space-filling curve □

Theorem 5.1.2 gives us, precisely, what was said at the beginning of the section: the Osgood curve is not differentiable at Γ_8 and differentiable at ${\Gamma_8}^c$; Knopp's Osgood curve is not differentiable.

In [3], the authors did a magnificent work proving some algebraic and topological properties of space-filling curves, with lots of emphasis put on the size in both topological and algebraic senses. In this section we treat some of the topological properties that they provide.

Proposition 5.1.3. *The set $\mathcal{SF} \subset C(I, \mathbb{C})$ is non-closed.*

Proof. Let $f \in \mathcal{SF}$ and let $f_n := \frac{1}{n} f$. Clearly, $f_n \in SF, n \geq 0$ but $\{f_n\}_{n \geq 0}$ converges to $0 \notin \mathcal{SF}$, □

Proposition 5.1.4. *$\mathcal{SF}$ is dense in $C(I, \mathbb{C})$.*

Proof. Let $f \in C(I, \mathbb{C})$ be uniformly continuos and let $B(f, \epsilon)$ be a ball. The uniform continuity of f on I implies that there is $\delta > 0$ satisfying

$$\|f(t_1) - f(t_2)\|_\infty < \frac{\epsilon}{2} \text{ if } |t_1 - t_2| < \delta.$$

Let $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ be such that $|t_j - t_{j-1}| < \delta,\ j = 1, \ldots, N$. Hence,

$$\|f(t_j) - f(t_{j-1})\|_\infty < \frac{\epsilon}{2}.$$

Let $R = \{z \in \mathbb{C}\ /Re(z) \in [a,b],\ Im(z) \in [c,d]\}$ be a closed non-degenerate rectangle such that

$$\max\{b-a, d-c\} < \frac{\epsilon}{2}$$

and $\{f(t_0), f(t_1)\} \subset R$. Let ϕ be a continuous and surjective map from $[t_0, t_1]$ to R such that $\phi(t_1) = f(t_1)$, which we know that exists from what was said in the third paragraph of chapter 4. Let g be a map satisfying that $g_{|[t_0,t_1]} = \phi$ and that it is affine-linear in each segment $[t_j, t_{j+1}]$ and such that $g(t_j) = f(t_j)$. On the one hand, we have that $g \in \mathcal{SF}$ as $\mu(g(I)) = \mu(\phi(I)) = (b-a)(d-c) > 0$ and, on the other hand, to prove that $g \in B(f, \epsilon)$ can be done as follows:

- If $t \in [t_0, t_1]$ we reason in the next fashion: having $|t_0 - t_1| < \delta$ implies that $\|f(t_0) - f(t)\|_\infty < \frac{\epsilon}{2}$ for any $t \in [t_0, t_1]$. For $t \in [t_0, t_1]$ we have that $g(t) \in R$ which implies that $\|g(t) - f(t_0)\|_\infty < \frac{\epsilon}{2}$ and, therefore,

$$\|g(t) - f(t)\|_\infty \leq \|g(t) - f(t_0)\|_\infty + \|f(t_0) - f(t)\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- If $t \in [t_j, t_{j+1}]$ with $j = 1, \ldots, n-1$, we have that

$$\|g(t) - f(t)\|_\infty \leq \|g(t) - g(t_j)\|_\infty + \|g(t_j) - f(t)\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Then,we have that $g \in \mathcal{SF} \cap B(f, \epsilon)$, whence $\mathcal{SF}$ is dense. □

Observe that this result shows that $\mathcal{ND}$ is dense in $C(I, \mathbb{C})$.

5.2 The family of continuous nowhere differentiable curves.

It was the year 1929 when Steinhaus asked about the category of $\mathcal{ND}[a, b]$ in the space of all continuous functions in $[a, b]$. Both Banach and Mazurkiewicz provided a proof that $\mathcal{ND}[a, b]$ is of second category in [2] and [12] respectively. We refer to that as Banach-Manzurkiewicz Theorem. To provide a proof of it, the following lemma is needed.

Lemma 5.2.1. *Let $\mathcal{P}([a, b])$ denote the set of piecewise linear continuous curves defined over $[a, b]$. Hence, $\mathcal{P}([a, b])$ is dense in $C([a, b], \mathbb{C})$.*

Proof. Without any loss of generality, let $[a, b] = I$. Let $g \in C(I, \mathbb{C})$ and considere the partition of I

$$P_n := \{0 = t_0 < t_1 < \cdots < t_N = 1\}.$$

Considere h_n the piecewise linear curve

$$h_n(t) := g(t_j)\frac{t_{j+1} - t}{t_{j+1} - t_j} + g(t_{j+1})\frac{t - t_j}{t_{j+1} - t_j},\ t \in [t_j, t_{j+1}].$$

We have that $h_n \in P([a, b])$ for each P_n. Now, let $\epsilon > 0$. The fact that g is cotinuous in I means that

$$\forall \epsilon \geq 0 \exists \delta > 0 \ / \ |x - y| < \delta \Rightarrow \|S_n(x) - S_n(y)\|_\infty < \frac{\epsilon}{4}.$$

Also, we can choose $n \geq 0$ so that $|t_{j+1} - t_j| < \delta,\ j = 0, \ldots, n-1$ for any given $\delta > 0$.

Therefore, for $x \in [t_j, t_{j+1}]$ we have

$$\begin{aligned}
|g(t) - h_n(t)| &= \left| g(t) - \frac{1}{t_{j+1} - t_j} \left[t_{j+1} g(t_j) - t_j g(t_{j+1}) + t\left(g(t_{j+1}) - g(t_j)\right) \right] \right. \\
&= \left| g(t) - \frac{t_{j+1} g(t_j) - t_j g(t_{j+1})}{t_{j+1} - t_j} - t \frac{g(t_{j+1}) - g(t_j)}{t_{j+1} - t_j} \right| \\
&= \left| g(t) - g(t_{j+1}) - \frac{t_{j+1} - t}{t_{j+1} - t_j} \left(g(t_j) - g(t_{j+1}) \right) \right| \\
&\leq |g(t) - g(t_{j+1})| + \left| \frac{t_{j+1} - t}{t_{j+1} - t_j} \right| |g(t_{j+1}) - g(t_j)| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Finally,

$$\|g - h_n\|_\infty \leq \max_{j=0,\ldots,n-1} \left\{ \sup_{t \in [t_j, t_{j+1}]} |g(t) - h_n(t)| \right\} \leq \frac{\epsilon}{2} < \epsilon$$

and we conclude the proof. □

Theorem 5.2.2 (Banach-Mazurkiewicz)**.** *$\mathcal{ND}[a,b]$ is of the second category in $C([a,b],\mathbb{C})$.*

Proof. Assume $[a,b] = I$. Denote by

$$E_n := \{ f \in C(I,\mathbb{C}) \ / \ \exists x \in [0, 1 - \frac{1}{n}] \ / \ \forall h \in (0, 1-x) \Rightarrow |f(x+h) - f(x)| \leq nh \}.$$

Let's proof that the sets E_n are closed for all $n \in \mathbb{N}$.

Let $f \in \overline{E_n}$ and $\{f_k\}_{k \geq 0} \subset E_n$ such that $f_k \to f$ uniformly on I. Since $f_k \in E_n$, there exists $t_k \in \left[0, 1 - \frac{1}{n}\right]$ for every $k \in \mathbb{N}$ by the definition of E_n. The sequence $\{t_k\}$ is clearly bounded, therefore there exists a subsequence $\{t_{k_l}\}$ such that

$$t_{k_l} \longrightarrow x$$

for some $t \in \left[0, 1 - \frac{1}{n}\right]$. Let $\{f_{k_l}\}$ be the corresponding subsequence of $\{f_k\}$. By the construction, $|f_{k_l}(t_{k_l} + h) - f_{k_l}(t_{k_l})| \leq nh$ for every $0 < h < 1 - t_{k_l}$. Since $t_{k_l} \to t$ and $0 < h < 1 - t$ we can always choose some $l_0 \in \mathbb{N}$ sufficiently large so that $0 < h < 1 - t_{k_l}$ for $l > l_0$. Hence,

$$\begin{aligned}
|f(t+h) - f(t)| \leq & |f(t+h) - f(t_{k_l} + h)| + |f(t_{k_l} + h) - f_{k_l}(t_{k_l} + h)| \\
& + |f_{k_l}(t_{k_l} + h) - f_{k_l}(t_{k_l})| + |f_{k_l}(t_{k_l}) - f(t_{k_l})| \\
& + |f(t_{k_l}) - f(t)| \\
\leq & |f(t+h) - f(t_{k_l} + h)| + \|f - f_{k_l}\|_\infty + nh + \|f_{k_l} - f\|_\infty \\
& + |f(t_{k_l}) - f(t)|.
\end{aligned}$$

Letting $l \to \infty$, the continuity of f at both x and $x+h$, together with the convergence of f_{k_k} gives the inequality $\|f(x+h) - f(x)\|_\infty \leq nh,\ \forall 0 < h < 1-x$ implying that $f \in E_n$. Therefore, E_n is closed.

Also, E_n is nowhere dense; we have shown in Theorem 5.2.1 that $P(I)$ is dense in $C(I)$, so to prove that E_n is nowhere dense, it will suffice us to prove that

$$\forall g \in P(I) \text{ and } \epsilon > 0\ \exists\ h \in C(I,\mathbb{C}) - E_n\ /\ \|g - h\|_\infty < \epsilon.$$

Recall that the graph of g is made out of linear segments; if L_i is one of those linear segments, denote by m_i its slope. Now, define

$$M := \max\{m_i\}.$$

Now, pick $m \in \mathbb{N}$ satisfying $m\epsilon > n + M$ and let

$$\phi(t) := \inf_{k\in\mathbb{Z}} |t-k|,\ \ \Phi(t) := t + \phi(t)i$$

be the "saw-tooth" function and curve respectively. We define now the curve

$$h(t) := g(t) + \epsilon\Phi(mt)$$

which clearly satisfies $h \in C(I,\mathbb{C})$. Then, for all $t \in [0,1)$, the right-hand side derivative of h, $h'^{+}(t)$, satisfies that

$$|h'^{+}(t)| = |g'^{+}(t) + \epsilon m \Phi'^{+}(mt)| > n$$

since $m\epsilon > n + M$, whence $h \in C(I) - E_n$. We have too that

$$\|g - h\|_\infty = \sup_{t\in I} |g(t) - (g(t) + \epsilon\Phi(mt))| = \epsilon \sup_{t\in I} |\Phi(mt)| = \frac{\epsilon}{2} < \epsilon$$

which shows that E_n is nowhere dense in $C(I,\mathbb{C})$.

Due to the fact that E_n is nowhere dense, the set $E := \bigcup_{n\geq 1} E_n$ needs to be of the first category in $C(I,\mathbb{C})$. This is the set of all elements in $C(I,\mathbb{C})$ with bounded right hand difference quotients at some point $t \in [0,1)$. Similarly, it is proven that the set of all functions that have bounded left hand difference quotients at some $t \in (0,1]$ is of the first category. Indeed, the union of this two sets includes all functions in $C(I,\mathbb{C})$ that have a finite one-sided derivative somewhere in I. Having that $C(I,\mathbb{C})$ is complete, and by Baire's theorem is of the second category, we conclude that $\mathcal{ND}(I)$ is of the second category.

□

Chapter 6

The Jordan Curve Theorem

In this chapter a quick overview of a rather simple proof of the Jordan Curve Theorem is provided. As Tverberg said, *"Although the JCT is one of the best known topological theorems, there are many, even among professional mathematicians, who have never read a proof of it."*[19] and it is in that context that a proof is intended to be provided in this chapter. As many of the topics covered all the way throughout this text, the statement of this theorem was popularized among mathematicians during the nineteenth century and, ever since, new proofs with different approaches have been published. Although proofs based on algebraic topology are the most popular as an illustration of the power of the theory, here we follow the ideas of Maehara in [11] (filling the gaps that have been spotted), which is based on a basic knowledge of topology.

We recall a couple facts that will be useful.

1. If F is a closed set of $\mathbb{C}$, then any component of $\mathbb{C}-F$ is opened and arcwise connected. We call this components the complementary domains of F.
2. If K is compact in $\mathbb{C}$, then $\mathbb{C}-K$ has exactly one unbounded component and shall refer to it as the exterior component of K.

The cornerstone of the proof we provide to the JCT is Brouwer's Fixed Point Theorem. However, in order to prove it, we need the degree of a continuous map over the circle, which we are define and give its main properties.

6.1 Degree of a continuous map over the circle.

Denote by $\mathbb{E}$ the function

$$\mathbb{E}:\mathbb{R}\longrightarrow\mathbb{S}^1,\ \mathbb{E}(t)=e^{2\pi it}.$$

Recall that $\mathbb{E}(t+N)=\mathbb{E}(t)$, for any given $N\in\mathbb{Z}$, and that, actually, $\mathbb{E}(t_1)=\mathbb{E}(t_2)\Leftrightarrow t_1-t_2\in\mathbb{Z}$. In that sense, we have next the definition and results.

Definition 6.1.1 (Lift)**.** *Let X be a topological space and $f:X\longrightarrow\mathbb{S}^1$ a continuous map. We will say that $\tilde{f}:X\longrightarrow\mathbb{R}$ is a lift of f if it is continuous and $\mathbb{E}\circ\tilde{f}=f$.*

Lemma 6.1.1 (Existence of a lift of a continuous map). *Every continuous map*

$$g : I \longrightarrow \mathbb{S}^1$$

admits a lift.

Proof. Having $\mathbb{E} \circ \tilde{g} = g, \ t \in I$, is equivalent to having

$$2\pi i \tilde{g}(t) = \{\log(g(t))\} \Leftrightarrow \tilde{g}(t) = \frac{1}{2\pi i}\{\log(g(t))\}.$$

Now, $g(t)$ is a continuous map that avoids the origin, and therefore, by construction of a continuous determination of a logarithm (equivalent to constructing a continuous argument) as studied in many courses in complex analysis, there exists a continuous function $l : I \longrightarrow \mathbb{C}$ such that $g(t) = e^{l(t)}$. Now, $\tilde{g}(t) = \frac{1}{2\pi i} l(t)$ is a lift of g:

$$e^{2\pi i \tilde{g}(t)} = e^{\frac{2\pi i}{2\pi i} l(t)} = e^{l(t)} = g(t), \ t \in I.$$

□

Proposition 6.1.2 (Properties of a lift). *Let $g : I \longrightarrow \mathbb{S}^1$ be a continuous map. The next properties hold:*

1. *If $\tilde{g} : I \longrightarrow \mathbb{R}$ is a lift of g, so is $\tilde{g} + N, \ N \in \mathbb{Z}$.*
2. *Any two lifts of g differ on an integer constant.*

Proof. 1. Let $\tilde{g}$ be a lift of g and define

$$\hat{g}(t) := \tilde{g}(t) + N, \ N \in \mathbb{Z}\,.$$

Now, $\mathbb{E}(\hat{g}(t)) = \mathbb{E}(\tilde{g}(t) + N) = \mathbb{E}(tildeg(t)) = g(t)$ and we have this part shown.

2. Assume $\tilde{g}_1, \tilde{g}_2$ are lifts of g. That means that $e^{2\pi i \tilde{g}_1(t)} = g(t) = e^{2\pi i \tilde{g}_2(t)}$. Hence,

$$1 = e^{2\pi i (\tilde{g}_2(t) - \tilde{g}_1(t))}, \ t \in I$$

which means that $\Delta(t) := \tilde{g}_2(t) - \tilde{g}_1(t) \in \mathbb{Z}$ for every $t \in I$. Actually, $\Delta : I \longrightarrow \mathbb{Z}$ is continuous, therefore its image is connected whence Δ is constant:

$$\tilde{g}_2(t) - \tilde{g}_1(t) = \Delta(t) = \Delta \in \mathbb{Z}\,.$$

□

Corollary 6.1.2.1. *If two lifts of a continuous map $g : I \longrightarrow \mathbb{S}^1$ coincide at $0 \in I$, then they coincide everywhere in I.*

Proof. Let $\tilde{g}_1, \ \tilde{g}_2$ be two lifts of g. By the previous proposition, we know that $\tilde{g}_2 = \tilde{g}_1 + N, \ N \in Z$. Assume that $\tilde{g}_1(0) = \tilde{g}_2(0)$. Then, we have

$$\tilde{g}_2(0) = \tilde{g}_1(0) + N = \tilde{g}_1(0),$$

therefore $N = 0$ and we conclude. □

We have then the next lemmas as an immediate corollary of this results:

Lemma 6.1.3 (Path-Lifting Lemma). *Let $g : I \longrightarrow \mathbb{S}^1$ be a continuous map and let $x \in \mathbb{R}$ be such that $\mathbb{E}(x) = g(0)$. Then, there is a unique lift $\tilde{g} : I \longrightarrow \mathbb{R}$ of g satisfying*

1. $\mathbb{E}(\tilde{g}(t)) = g(t),\ t \in I$.
2. $\tilde{g}(0) = x$.

Lemma 6.1.4 (Homotopy-Lifting Lemma). *Let $F : I \times I \longrightarrow \mathbb{S}^1$ be a continuous map and let $x \in \mathbb{R}$ such that $\mathbb{E}(x) = F(0,0)$. Then, there is a unique continuous map $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ such that*

1. $\mathbb{E}(\tilde{F}(t,s)) = F(t,s),\ t, s \in I$.
2. $\tilde{F}(0,0) = x$.

We are now ready to construct the degree of a circle map. For that matter, let $\phi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a continuous map. The composite map

$$g = \phi \circ \mathbb{E} : I \longrightarrow \mathbb{S}^1$$

is obviously a continuous map and it satisfies $g(0) = g(1)$ because $\mathbb{E}(0) = \mathbb{E}(1) = 1$. Then, Lemma 6.1.3 says that g has a lift $\tilde{g} : I \longrightarrow \mathbb{R}$, i.e., $\mathbb{E} \circ \tilde{g} = g$. All of this gives us that

$$\mathbb{E}(\tilde{g}(0)) = \mathbb{E}(\tilde{g}(1)) \Rightarrow \tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z}\,.$$

We can now define the degree.

Definition 6.1.2 (Degree of a circle map). *Following the notation used above, if $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous map, we define its degree as*

$$deg(f) = \tilde{g}(1) - \tilde{g}(0).$$

Proposition 6.1.5. *Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ me a continuous map. Then, $deg(f)$ does not depend on the choice of a particular lift of $g = f \circ \mathbb{E}$, i.e., $deg(f)$ is well-defined.*

Proof. Let $\tilde{g}$ be a lift of $g = f \circ \mathbb{E}$. Then, $deg(f) = \tilde{g}(1) - \tilde{g}(0)$. Now, let $\hat{g}$ be other lift of g. We have shown previously that

$$\hat{g} = \tilde{g} + N,\ N \in \mathbb{Z}\,.$$

Therefore,

$$\hat{g}(1) - \hat{g}(0) = (\tilde{g}(1) + N) - (\tilde{g}(0) + N) = \tilde{g}(1) - \tilde{g}(0) = deg(f)$$

and the proof is finished. □

Remark. *It is important to observe that $deg(\varphi) = ind(\varphi \circ \mathbb{E} : I \longrightarrow \mathbb{S}^1, 0)$, that is, the degree of a circle map φ is the index of 0 with respect to $\varphi \circ \mathbb{E} : I \longrightarrow \mathbb{S}^1$.*

Definition 6.1.3 (Homotopy). *Let X, Y be two topological spaces and $f, g : X \longrightarrow Y$ continuous maps. A homotopy between f and g is a continuous map $H : X \times I \longrightarrow Y$ satisfying $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, $\forall x \in X$. If such H exists, we say that f and g are homotopic.*

It is easy to show that being homotopic is an equivalence relation on $C(X, Y)$.

Theorem 6.1.6 (Invariance of the degree under homotopies). *Let $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be two continuous maps. If f, g are homotopic, then $def(f) = deg(g)$.*

Proof. Let $H : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1$ be a homotopy from f to g. That means

$$H(z, 0) = f(z),\ H(z, 1) = g(z),\ z \in \mathbb{S}^1 .$$

Let $F : I \times I \longrightarrow \mathbb{S}^1$ be such that

$$F = H \circ (\mathbb{E} \times Id)_{|I \times I}.$$

Lemma Theorem 6.1.4 says that F lifts to a map $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ satisfying $F = \mathbb{E} \circ \tilde{F}$. It is natural to see that the map $t \longmapsto \tilde{F}(t, 0)$ is a lift of f and $t \longmapsto \tilde{F}(t, 1)$ is a lift of g. Then, we have:

$$deg(f) = \tilde{F}(1, 0) - \tilde{F}(0, 0)$$
$$deg(g) = \tilde{F}(1, 1) - \tilde{F}(0, 1)$$

Consider the map $\Delta : I \longrightarrow \mathbb{Z}$ defined by $\Delta(t) = \tilde{F}(1, t) - \tilde{F}(0, t)$. Such map is naturally continuous and having that I is connected and $\mathbb{Z}$ is discrete implies that Δ is a constant function. Whence,

$$deg(f) = \Delta(0) = \Delta(1) = deg(g)$$

and we conclude.

□

Proposition 6.1.7 (Degree of a constant map over the circle). *Let $z^* \in \mathbb{S}^1$ and $f(z) = z^*$ for any $z \in \mathbb{S}^1$. Then, $deg(f) = 0$.*

Proof. Let $z^* \in \mathbb{S}^1$ be fixed but arbitrary and let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be such that $f(z) = z^*$ for any $z \in \mathbb{S}^1$. If $z^* \in \mathbb{S}^1$ then there is a value $t^* \in I$ satisfying $\mathbb{E}(t^*) = z^*$. This means that the map $\tilde{g}(t) = t^*$ is a lift of $g = f \circ \mathbb{E}$: on the one hand, we have $g(t) = f(\mathbb{E}(t)) = f(e^{2\pi i t}) = z^*$ and, on the other hand, that $\mathbb{E}(\tilde{g}(t)) = \mathbb{E}(t^*) = z^*$, giving us that $\tilde{g}$ is a lift of g.

The fact that $\tilde{g}$ is a lift of g means that $deg(f) = \tilde{g}(1) - \tilde{g}(0) = t^* - t^* = 0$ and we conclude. □

6.2 Brouwer's Fixed Point Theorem and the JCT

Theorem 6.2.1 (Brouwer's Fixed Point Theorem)**.** *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a continuous map. Thus, there exists at least one $z \in \mathbb{D}$ satisfying $f(z) = z$.*

Proof. We will follow by *reductio ad absurdum.* Suppose f has no fixed point, that is, $f(z) \neq z,\ \forall z \in \mathbb{D}$. Now, the map $\Delta(z) = f(z) - z \neq 0$ for all $z \in \mathbb{D}$. We have then that, for every $z \in \mathbb{S}^1$,

$$|\Delta(z) + z| = |f(z)| \leq 1 = |z| < |\Delta(z)| + |z|$$

and this is equivalent to having $|\frac{\Delta(z)}{z} + 1| < |\frac{\Delta(z)}{z}| + 1,\ z \in \mathbb{S}^1$. Now, the map

$$q(z) = \frac{\Delta(z)}{z},\ z \in \mathbb{S}^1$$

maps $\mathbb{S}^1$ into $\mathbb{C}^* - (0, \infty)$ because, otherwise, that would mean that $\mathrm{Im}(\frac{f(z)}{z}) = 0$ and since $\frac{f(z)}{z} \in \mathbb{S}^1$ that implies that $\frac{f(z)}{z} = \pm 1$; $f(z) = -z$ gives no trouble but $f(z) = z$ is a contradiction with our hypothesis. However, $\mathbb{C}^* - (0, \infty)$ is simply connected, and therefore $deg(q) = ind(q(\mathbb{S}^1), 0) = 0$. But, $H(t, z) = f(tz) - tz \in \mathbb{C}^*$ is a homotopy betweeen $H(1, z) = \Delta(z)$ and $H(0, z) = f(0) \in \mathbb{C}^*$. Therefore, $deg(\Delta) = deg(f(0)) = 0$. Since $deg(q) = deg(\Delta) - 1 = 0 \Rightarrow deg(\Delta) = 1$, which is a contradiction. Whence, there must be $z \in \mathbb{S}^1\ /\ f(z) = z$.

□

Definition 6.2.1 (Retraction)**.** *Let X be a metric space, $A \subset X$ a subspace and*

$$r \colon X \longrightarrow A$$

*a continuous map. We say that r is a **retraction** if $r_{|A} = Id_A$.*

The next theorem will be proven using Brouwer's Fixed Point Theorem, but it is seen in literature that they are actually equivalent.

Theorem 6.2.2 (No-Retraction Theorem)**.** *There exists no retraction of $\mathbb{D}$ into $\mathbb{S}^1$.*

Proof. Let $r \colon \mathbb{D} \longrightarrow \mathbb{S}^1$ be a continuous map satisfying $r(z) = z,\ \forall z \in \mathbb{S}^1$. Now, let s be a map from $\mathbb{S}^1$ into itself such that $s(z) = -z$. It is then clear that $s \circ r$ is continuos and has no fixed points, which is a contradiction with the previous theorem. □

Definition 6.2.2. *Let $E = \{z \in \mathbb{C} /\ |\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq 1\}$. We say that a **path** γ in E is a continuous function $\gamma \colon [-1, 1] \longrightarrow E$.*

Notice that if K is a convex and non-empty compact set of $\mathbb{C}$ with non-empty interior, then it is homeomorphic to $\mathbb{D}$. Let me show you this in a very intuitive way. It is obvious that it suffices to consider that $0 \in K$ and that $K \subset \mathbb{D}$: otherwise, we only need a translation and/or a homothety to be in the considered case (and such transformations are bijective and linear in a finite dimension space, hence homeomorphisms). The argument is as follows: as K is compact it contains its own boundary and every point in ∂K is conected to 0 via a line segment and every point in $K - \{0\}$ belongs to one and only one of this line segments. In that sense, if $q \in K - \{0\}$, denote $p_q \in \partial K$ such that $q \in [0, p_q]$ and then the function $K \longrightarrow \mathbb{D}$ that maps $K \ni q \mapsto p = \dfrac{q}{|p_q|}$ is well-defined and it is a homeomorphism between K and $\mathbb{D}$, as it maps any point in $K - \{0\}$ to one and only one in $\mathbb{D} - \{0\}$ and it maps $0 \in K$ to itself.

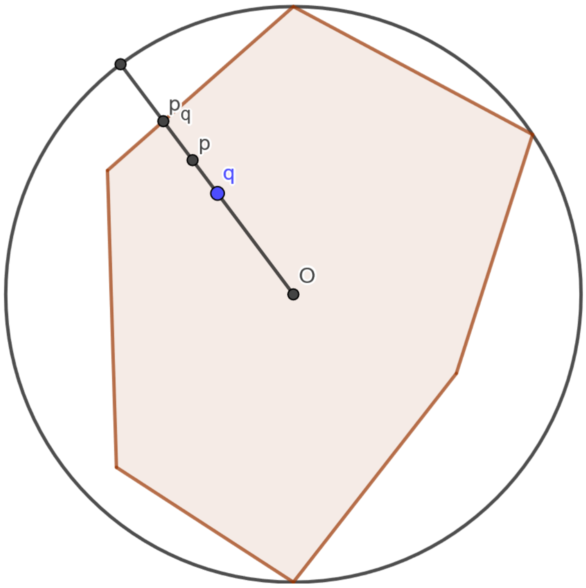


Figure 6.1: An inlustration of the map defined previously.

This result shows us that Brouwer's theorem is valid for every $K \subset \mathbb{C}$ considered as above. Let us announce it properly:

Corollary 6.2.2.1. *Brouwer's fixed point theorem holds for every non-empty compact and convex set of $\mathbb{C}$ with non-empty interior.*

Proof. Let $K \subset \mathbb{C}$ be a non-empty compact and convex set of $\mathbb{C}$ with non-empty interior. We have shown before that it must be homeomorphic to $\mathbb{D}$. Let $\varphi : \mathbb{D} \longrightarrow K$ be an homeomorphism and $g : K \longrightarrow K$ be a continuous map. Then, the composite

$$h = \varphi^{-1} \circ g \circ \varphi : \mathbb{D} \longrightarrow \mathbb{D}$$

is continuous and by Brouwer's fixed point theorem has a fixed point, i.e.,

$$\exists z \in \mathbb{D} \ / \ h(z) = z.$$

As φ is bijective, that means that there exists a unique point $p \in K$ such that $p = \varphi(z)$. Now, let $q = g(p)$. It is clear that $\varphi^{-1}(q) = z$ because $\varphi^{-1}(q) = \varphi(g(p)) = \varphi(g(\varphi(p))) = z$. Because φ is bijective and p, q have the same image via φ^{-1}, we have that $p = q = g(p)$ and, therefore p is a fixed point of $g : K \longrightarrow K$, and so we conclude. □

Remark. *What has been shown in the previous corollary is actually stronger than what was claimed. Observe that convexity has not been used in the proof and that what we have* *actually required of the set K is to be homeomorphic to $\mathbb{D}$. Hence, Brouwer's fixed point theorem is valid for every homeomorphic transformation of $\mathbb{D}$, but to our case of study it is enough to consider non-empty compact and convex sets with non-empty interior.*

Then, the next Lemma can be proven with ease.

Lemma 6.2.3. *Considere two paths γ_1, γ_2 such that*

$$\gamma_1(-1) = -1 + b_{-1}i, \ \gamma_1(1) = 1 + b_1 i, \ b_1, b_{-1} \in [-1, 1]$$

$$\gamma_2(-1) = a_{-1} - i, \ \gamma_2(1) = a_1 + i, \ a_1, a_{-1} \in [-1, 1]$$

Then, there exists $(s, t) \in [-1, 1]^2$ such that $\gamma_1(s) = \gamma_2(t)$.

Proof. Let's say that $\gamma_1(s) = x_1(s) + y_1(s)i, \ \gamma_2(s) = x_2(s) + y_2(s)i$ such that

$$x_1(-1) = -1, \ x_1(1) = 1, \ y_2(-1) = -1, \ y_2(1) = 1$$

and assume that $\gamma_1(s) \neq \gamma_2(t)$ for every $(s, t) \in [-1, 1]^2$.

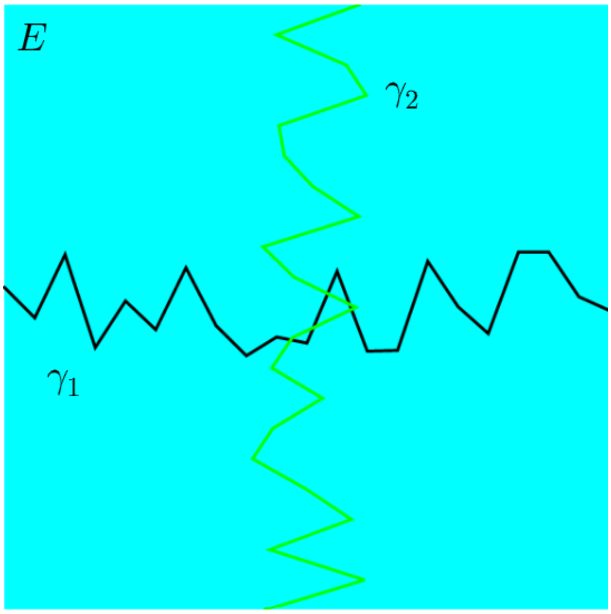


Figure 6.2: A particular case of the described phenomena in Lemma Theorem 6.2.3

It is then clear that

$$N(s + ti) := \max\left(|x_1(s) - x_2(t)|, |y_1(s) - y_2(t)|\right) > 0, \ \forall s + ti \in E$$

and, therefor, the function $f : E \longrightarrow \Gamma \subset E$ (where $\Gamma = \partial E$) given by

$$f(s + ti) = \frac{1}{N(s + ti)} \left((x_2(t) - x_1(s)) + (y_1(s) - y_2(t)i)\right.$$

can be shown to have no fixed points. Let's see it:

If $f(s+ti)$ does have a fixed point, that means that there exists a point $s+ti \in E$ satisfying

$$s+ti = \frac{1}{N(s+ti)}\left((x_2(t)-x_1(s))+(y_1(s)-y_2(t)i\right)$$

which is equivalent to having

$$s = \frac{1}{N(s+ti)}\left(x_2(t)-x_1(s)\right)$$

$$t = \frac{1}{N(s+ti)}\left(y_1(s)-y_2(t)\right).$$

For that matter, assume $N(s+ti) = |x_1(s)-x_2(t)|$. Hence, we have two different cases: $s=1$ or $s=-1$.

1. $s = 1 \Rightarrow x_2(t) = 1+|x_2(t)-1|$. Since $-1 \leq x_2(t) \leq 1$, the only possibility por $x_2(t)$ would be $x_2(t) = 1$ but then $N(s+ti) = 0$, which is a contradiction with our hypothesis.
2. $s = -1 \Rightarrow x_2(t) = -1-|x_2(t)+1|$. Since $-1 \leq x_2(t) \leq 1$, the only possibility por $x_2(t)$ would be $x_2(t) = -1$ but then $N(s+ti) = 0$, which is a contradiction with our hypothesis.

Following a perfectly analogous argument the case $N(s+ti) = |y_1(s)-y_2(t)|$ is proven.

We have found a continuous map that has no fixed points, which contradicts Brouwer's theorem, therefore we have reached a conrtadiction $\Rightarrow \exists (s,t) \in [-1,1]^2 \ / \ \gamma_1(s) = \gamma_2(t)$. □

Notice that the technique followed in this proof is very similar to the one we followed at the No-Retraction Theorem: we have built a function that sends every point of the set to its boundary and have shown that it has no fixed points.

The next notion is need for a propper formulation of the Jordan Curve Theorem.

Definition 6.2.3. *A closed set $F \subset \mathbb{C}$ is said to* ***separate*** *(or that it* ***separates****) $\mathbb{C}$ if $\mathbb{C}-F$ has at least two components.*

Lemma 6.2.4. *No arc α separates the plane*

Proof. Let's say that there exists an arc α that does separate the plane. Then, $\mathbb{C}-\alpha$ has exactly one unbounded component and at least one bounded component, namely U and W respectively. This fact implies that ∂U, $\partial W \subset \alpha$. Considere $x_0 \in W$ and $D = \bar{B}(x_0, R)$ such that $D^{\text{o}} \supset W \cup \alpha$. Applying the Tietze Extension Theorem (see Introducción a la Topología, 5.3.8, Outerelo y Margalef), we can extened Id_α continuosly to a retraction $r\colon D \longrightarrow \alpha$. Now, define $q\colon D \to D$ such that

$$q(x) = \begin{cases} x & if \quad x \in W^c \cap D \\ \\ r(x) & if \qquad x \in \bar{W} \end{cases}$$

which is continuous. Then, considering $p(x) = \dfrac{q(x)-x_0}{|q(x)-x_0|}R,\ x \in D - x_0$, the map $p \circ q \colon D \to \partial D$ is a retraction of D into its frontiere, which is a contradiction with the No-Retraction Theorem.

□

We are finally ready to enounce and proof the main resault of this chapter.

Theorem 6.2.5 (Jordan Curve Theorem)**.** *Let J be any Jordan curve. Then, $\mathbb{C}-J$ has exactly two components: one conected and unbounded (which we shall call exterior) and one conected and bounded (which we shall call interior) and both have J as its boundary.*

Proof. Since J is compact in $\mathbb{C}$, assertion 2 (from the ones made at the begginig of this chapter) tells us that J has only one unbounded complement. Hence, it suffices us to prove that J has one and only one bounded complement and that J is the boundary of them both.

Let's first see that J separates the plane. Being J compact implies the existence of two points $a, b \in J$ such that $|a-b| = \max_{x,y\in J}(|x-y|)$. Without any loss of generality, we may assume that $a=-i,\ b=i \in \mathbb{C}$.

Let $\mathcal{E} = \{z \in \mathbb{C} \,/ |\mathrm{Re}(z)| \leq 2,\ |\mathrm{Im}(z)| \leq 1\}$ be a rectangle and let Γ be its frontier. We have that $a, b \in \Gamma \cap J$. Let $E=-2, W=2 \in \mathcal{E}$ and $[E,W]$ the line segment that joins E, W. Since J joins the horizontal sides of $\mathcal{E}$ and $[E,W]$ joins the vertical sides of $\mathcal{E}$, Lemma 6.2.3 tells us that $[E,W] \cap J \neq \emptyset$. Now, let $l \in [E,W] \cap J$ satisfying

$$|E-l| = \min_{x\in[E,W]\cap J}(|E-x|)\,.$$

As a, b define two different paths in J, we denote J_E the one path that contains l and J_W to the other path. Let now $m \in [E,W] \cap J_E$ such that

$$|E-m| = \max_{x\in[E,W]\cap J_E}(|E-x|)\,.$$

The segment $[m,W]$ meets J_W at least once, otherwise, $[E,l] + \alpha(l,m) + [m,W]$ would not meet J_W, contradicting Lemma 6.2.3. Let $p \in J_W \cap [m,W]$ such that

$$|p-W| = \max_{x\in J_W\cap[m,W]}(|W-x|)$$

and $q \in J_W \cap [m,W]$ satisfying

$$|q-W| = \min_{x\in J_W\cap[m,W]}(|W-x|)$$

Finally, let $z = \frac{m+p}{2}$ and U the component of $\mathbb{C}-J$ that contains z. We are going to prove that U is bounded.

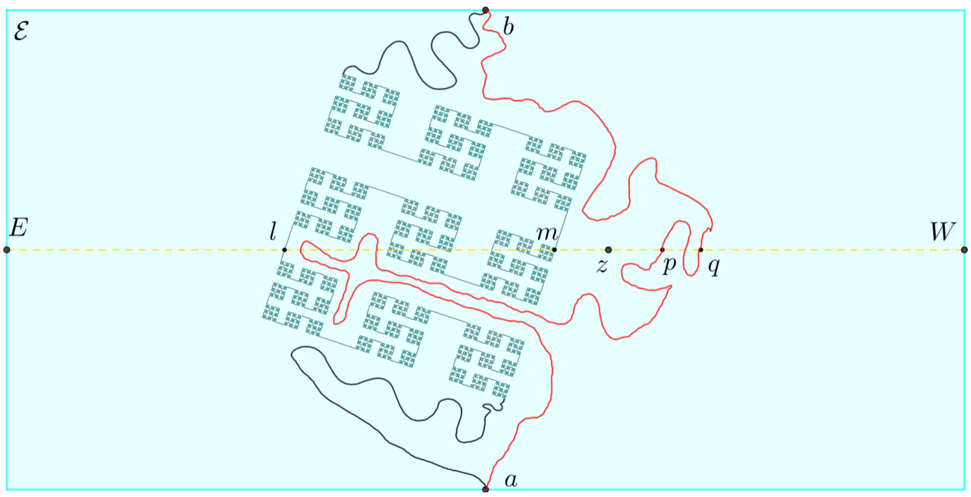


Figure 6.3: The black line is J_E and the red line is J_W for a particular Jordan curve. The dotted line in gold color represents $[E, W]$.

Let's suppose U is not bounded. Then, as U is arc-conected, there exists a path γ that joins z to a point ρ in $\mathbb{C} - \mathcal{E}$. Because γ is continuous, γ has at least one point in common with Γ and at least one of this points, namely w, satisfies that $\alpha(z, w) \subset \mathcal{E}$, where $\alpha(z, w)$ is the subarc on γ that joins z and w.

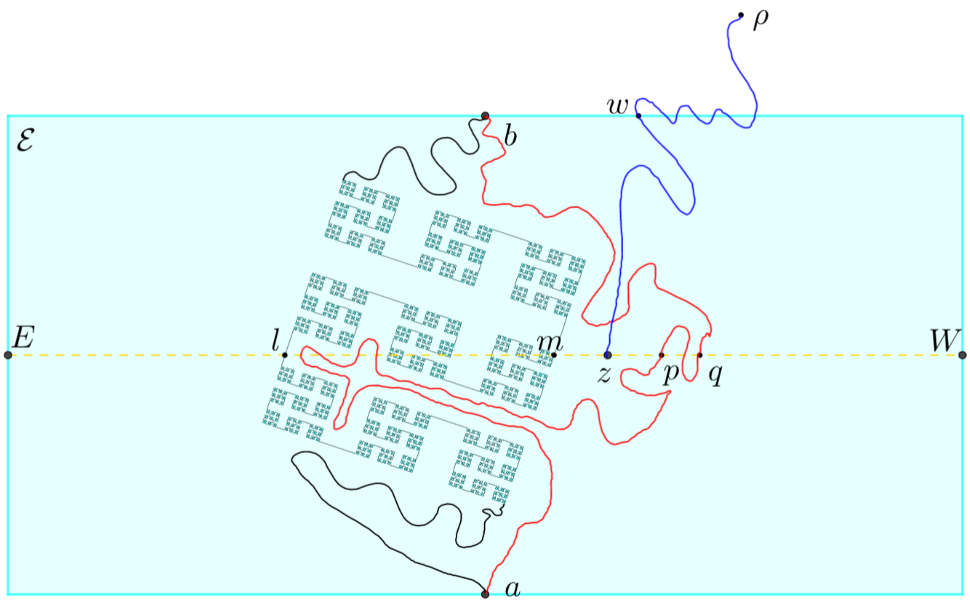


Figure 6.4: Representation of ρ, γ as stated, w and $\alpha(z, w)$.

Assume $\text{Im}(\rho) > 1$ and observe that, as $\gamma(-1) = z, \gamma(1) = \rho$, the set of common points between γ and Γ is the set $\{\gamma(t) \ / \ \text{Im}(\gamma(t)) = 1\}$. Since $t \in [-1, 1]$, the set $T = \{t \in [-1, 1] \ / \ \text{Im}(\gamma(t)) = 1\}$ has both upper and lower bound. Let then t_0 be the lower bound of T. So, we choose $w = \gamma(t_0)$.

Observe that $w \neq b$ because $w \in U \subset \mathbb{C} - J$ and $b \in J$. If $\mathrm{Re}(w) > 0$, we can find a path between w and W so that nor a or b belongs to the path: it suffices to considere the path defined by w and W over Γ that does not contain a, namely $\alpha(w, W)$. Considere now the path $[E, l] + \alpha(l, m) + [m, z] + \alpha(z, w) + \alpha(w, W)$, meaning the concatenation of all those paths. Such path does not meet J_S, which contradicts Lemma 6.2.3. Therefore, U is bounded. Similarly it is proven for $\mathrm{Re}(w) < 0$ and the two analogous cases if $\mathrm{Im}(\rho) < -1$. We have then proved that J separates the plane. See Figure 6.4 for a detailed scheme of the previous argument.

We are now going to prove that J is the boundary of any of its complements. Let U be a complement of J and $x \in U$. Obviously, $\partial U \subset J$. If $\partial U \subset J$, there exists and arc α which contains entirely ∂U as it is, at most, J. Since J separates the plane, there is a point $y \in \mathbb{C} - J$ such that x and y are separated by ∂U, which contradicts Lemma 6.2.4. We then have $\partial U = J$. It is only left to prove that there are no more bounded components.

Let $V \neq U$ be another bounded component of $\mathbb{C} - J$. It would have to satisfy $V \subset \mathcal{E}$. Let β be the path $[N, l] + \alpha(l, m) + [m, p] + \alpha(p, q) + [q, S]$, being $\alpha(p, q)$ the subarc of J_W going from p to q. That means $\beta \cap V = \emptyset$, because that lie inside a bounded component of $\mathbb{C} - J$ are the points of $[m, p]$, which is a subset of U and we had stated that $V \neq U$. Since $a, b \notin \beta$ there are circular neighbourhoods V_a, V_b of a and b respectively such that $V_a, V_b \cap \beta = \emptyset$. As $\partial V = J$, there are $a_1 \in V \cap V_a$, $b_1 \in V \cap V_b$ and let $\alpha(a_1, b_1)$ be a path in V from a_1 to b_1. Then, the path $[a, a_1] + \alpha(a_1, b_1) + [b, b_1]$ does not meet β, which contradicts Lemma 6.2.3. $\square$

Mathematica source code

Appendix A

Mathematica source code for figures in chapter 3

A.1 Figure 3.1

```
a = 0.5; b = 13.;
W[x_] := Sum[a^k*Cos[b^k*x*Pi], {k, 0, 7}]

Plot[W[x], {x, -2, 2}]
```

A.2 Figure 3.2

```
a = 0.5; b = 13.;

W[x_] := Sum[a^k*Cos[b^k*x*Pi], {k, 0, 7}]

W1 = Table[{x, W[x]}, {x, 0., 2., 0.001}];

W2 = RotationTransform[1.5*Pi][W1];
W2 = TranslationTransform[W1[[1]] - Last[W2]][W2];
W3 = RotationTransform[1.5*Pi][W2];
W3 = TranslationTransform[W2[[1]] - Last[W3]][W3];
W4 = RotationTransform[1.5*Pi][W3];
W4 = TranslationTransform[W3[[1]] - Last[W4]][W4];


Show[Graphics[{Cyan, Line[W1], Line[W2], Line[W3], Line[W4]}],
 Axes -> True, AxesStyle -> Black]
```

A.3 Figure 3.3, Figure 3.4 and Figure 3.6 respectively

The numbering of the sides in Figure 3.3, Figure 3.4 and the points and lines in Figure 3.6 were introduced later with Geogebra.

```
(* First step of the generation of the Koch curve*)
Graphics[KochCurve[1]]

(*Second step of the generation of the Koch curve*)
Graphics[KochCurve[2]]

(*Fourth step of the generation of the Koch curve*)
Graphics[KochCurve[4]]
```

A.4 Figure 3.5

```
a = Triangle[{{0.‘, 0.‘}, {0.1666666666666666‘,
     0.09622504486493763‘}, {0.3333333333333333‘, 0.‘}}];

b = Triangle[{{0.3333333333333333‘, 0.‘}, {0.3333333333333333‘,
    0.1924500897298753‘}, {0.5‘, 0.2886751345948129‘}}];
c = ReflectionTransform[{-1, 0}][b];
c = TranslationTransform[3 {1/3 + (0.5‘-1./3.), 0}][c];
d = ReflectionTransform[{-1, 0}][a];
d = TranslationTransform[3 {1/3 + (0.5‘-1./3.), 0}][d];

Graphics[{{Cyan, {a, b, c, d}}, KochCurve[2]}]
```

A.5 Figure 3.7

```
A = KochCurve[6];
B = ReflectionTransform[{Cos[Pi/6], Sin[Pi/6]}][
   RotationTransform[5 Pi/3][A]];
R = ReflectionTransform[{Cos[Pi/6], -Sin[Pi/6]}, {1, 0}][
   RotationTransform[Pi/3, {1, 0}][A]];
Graphics[{A, B, R}]
```

Appendix B

Mathematica source code for figures in chapter 4

B.1 Figure 4.1

```
(*First iteration*)
Graphics[PeanoCurve[1]]

(*Second iteration*)
Graphics[PeanoCurve[2]]

(*Third iteration*)
Graphics[PeanoCurve[3]]
```

B.2 Figure 4.2

```
(*First iteration*)
Graphics[HilbertCurve[1]]

(*Second iteration*)
Graphics[HilbertCurve[2]]

(*Third iteration*)
Graphics[HilbertCurve[3]]
```

B.3 Figure 4.3

```
(* We define a function that generates the mesh of order n given the order

OSGOOD[n_] :=
 Block[{square, p0, p2, p4, p6, p8, p10, p12, p14, p16, W, L},
  W = 0.11306936062370848`;
  L = 0.2579537595841943`;
  square = {{{0., 0.}, {0., 1.}, {1., 1.}, {1., 0.}}};
  Do[
   p0 = ScalingTransform[{L, L}][square];
   p2 = TranslationTransform[{0, L + W}][p0];
   p4 = TranslationTransform[{0, 2 L + 2 W}][p0];
   p6 = TranslationTransform[(L + W) {1., 1.}][p2];
   p8 = TranslationTransform[(L + W) {1., 1.}][p0];
   p10 = TranslationTransform[(L + W) {1., -1.}][p2];
   p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
   p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
   p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];

   square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}], n];

  Graphics[{Cyan, Polygon[square]}, Frame -> False]
  ]

  (* First iteration *)
  OSGOOD[1]

  (* Second iteration *)
  OSGOOD[2]

  (* Third iteration *)
  OSGOOD[3]
```

B.4 Figure 4.5 and Figure 4.6

We define a function that requires the iteration that you wish to print (n), how much area you want it to cover (λ), the proportion between the width of the bars of two consecutive iterations (c) and whether you want the positive or negative root that Theorem 4.1.1 provides ($sign$).

```
OsgoodCurve[n_, \[Lambda]_, c_,
  sign_](*n\[GreaterEqual]1 is the iteration, \[Lambda] the area \
covered at infinity (0<\[Lambda]<1), c the ratio between W_i and \
W_{i+1} (which needs to be c>3) and sign=+-1 in order to choose the \
positive or the negative root*):=
 Block[{square, p0, p2, p4, p6, p8, p10, p12, p14, p16, , j0, j2, j4,
   j6, j8, j10, j12, j14, j16, join1, join2, join3, join4, join5,
   join6, join7, join8, joins, listjoins, Listjoins, W, L},
  (*Initializing constants*)
  L = 1./3. - 2*W/3.;
  W = (c - 3.) (1. + sign*Sqrt[\[Lambda]])/(2 c);
  (*initializing square and joins, p for squares, j for joins*)
  square = {{{0., 0.}, {0., 1.}, {1., 1.}, {1., 0.}}};
  p0 = ScalingTransform[{L, L}][square];
  p2 = TranslationTransform[{0., L + W}][p0];
  p4 = TranslationTransform[{0., 2 L + 2 W}][p0];
  p6 = TranslationTransform[(L + W) {1., 1.}][p2];
  p8 = TranslationTransform[(L + W) {1., 1.}][p0];
  p10 = TranslationTransform[(L + W) {1., -1.}][p2];
  p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
  p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
  p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
  square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}];
  join6 = {{2 L + W, 0.}, {2 L + 2 W, 0}};
  join7 = {{1., L}, {1., L + W}};
  join8 = {{2 L + 2 W, 2 L + W}, {2 L + 2 W, 2 L + 2 W}};
  join2 = {{0., 2 L + W}, {0., 2 L + 2 W}};
  join1 = {{L, L}, {L, L + W}};
  join3 = {{L, 1.}, {L + W, 1.}};
  join4 = {{2 L + W, 2 L + 2 W}, {2 L + W, 2 L + W}};
  join5 = {{L + W, L}, {L + W, L + W}};
  joins = {join1, join2, join3, join4, join5, join6, join7, join8};
  listjoins = {joins};
  Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];

  If[n == 1,
   Graphics[{{Cyan, Polygon[square]}, {Thick, Black,
      Listjoins}}], (*the first iteration of the curve is done \
outside the loop*)
    Do[
     j0 = ScalingTransform[{L, L}][joins];
     j2 = ReflectionTransform[{1., 0.}][j0];
```

```
    j2 = TranslationTransform[{L, L + W}][j2];
    j4 = TranslationTransform[{0, 2 L + 2 W}][j0];
    j6 = TranslationTransform[(L + W) {1., 1.}][j2];
    j8 = TranslationTransform[(L + W) {1., 1.}][j0];
    j10 = TranslationTransform[(L + W) {1., -1.}][j2];
    j12 = TranslationTransform[2 (L + W) {1., 0.}][j0];
    j14 = TranslationTransform[2 (L + W) {1., 0.}][j2];
    j16 = TranslationTransform[2 (L + W) {1., 0.}][j4];
    joins = Catenate[{j0, j2, j4, j6, j8, j10, j12, j14, j16}];
    AppendTo[listjoins, joins];
    p0 = ScalingTransform[{L, L}][square];
    p2 = TranslationTransform[{0., L + W}][p0];
    p4 = TranslationTransform[{0, 2 L + 2 W}][p0];
    p6 = TranslationTransform[(L + W) {1., 1.}][p2];
    p8 = TranslationTransform[(L + W) {1., 1.}][p0];
    p10 = TranslationTransform[(L + W) {1., -1.}][p2];
    p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
    p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
    p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
    square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}],
    n - 1];
   Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];
   Graphics[{{Cyan, Polygon[square]}, {Thin, Black, Listjoins}}]
   ]
  ]


(* Joins and squares of the first iteration *)
  OsgoodCurve[1, 0.5, 6, -1]

(* Joins and squares of the second iteration *)
  OsgoodCurve[2, 0.5, 6, -1]
```

B.5 Figure 4.8

```
\[CapitalGamma]8[n_] := Block[{r, r1, r2, r3, r4, r5, r6, r7, r8},
  r = Table[{{2.*k, 0.}, {2. k + 1., 0.}}, {k, 0., 8.}];
  If[n == 1, Graphics[Line[r]],
   Do[
    r0 = ScalingTransform[{1./17., 0.}][r];
    r1 = TranslationTransform[{2., 0}][r0];
    r2 = TranslationTransform[{2., 0}][r1];
    r3 = TranslationTransform[{2., 0}][r2];
    r4 = TranslationTransform[{2., 0}][r3];
    r5 = TranslationTransform[{2., 0}][r4];
    r6 = TranslationTransform[{2., 0}][r5];
    r7 = TranslationTransform[{2., 0}][r6];
```

```
    r8 = TranslationTransform[{2., 0}][r7];
    r = Catenate[{r1, r2, r3, r4, r5, r6, r7, r8}], n]];
  Graphics[Line[r]]
  ]
```

B.6 Figure 4.9 and Figure 4.10

```
OsgoodPolygon[n_, \[Lambda]_, c_,
  sign_](*n\[GreaterEqual]1 is the iteration, \[Lambda] the area \
covered at infinity (0<\[Lambda]<1), c the ratio between W_i and \
W_{i+1} (which needs to be c>3) and sign=+-1 in order to choose the \
positive or the negative root*):=
 Block[{square, d0, d2, d4, d6, d8, d10, d12, d14, d16, p0, p2, p4,
   p6, p8, p10, p12, p14, p16, , j0, j2, j4, j6, j8, j10, j12, j14,
   j16, join1, join2, join3, join4, join5, join6, join7, join8, diag1,
    diag2, diag3, diag4, diag5, diag6, diag7, diag8, diag9, diags,
   listdiags, Listdiags, joins, listjoins, Listjoins, W, L},
  (*Initializing constants*)
  L = 1./3. - 2*W/3.;
  W = (c - 3.) (1. + sign*Sqrt[\[Lambda]])/(2 c);
  (*initializing square and joins, p for squares, j for joins*)
  square = {{{0., 0.}, {0., 1.}, {1., 1.}, {1., 0.}}};
  p0 = ScalingTransform[{L, L}][square];
  p2 = TranslationTransform[{0., L + W}][p0];
  p4 = TranslationTransform[{0., 2 L + 2 W}][p0];
  p6 = TranslationTransform[(L + W) {1., 1.}][p2];
  p8 = TranslationTransform[(L + W) {1., 1.}][p0];
  p10 = TranslationTransform[(L + W) {1., -1.}][p2];
  p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
  p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
  p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
  square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}];
  join6 = {{2 L + W, 0.}, {2 L + 2 W, 0}};
  join7 = {{1., L}, {1., L + W}};
  join8 = {{2 L + 2 W, 2 L + W}, {2 L + 2 W, 2 L + 2 W}};
  join2 = {{0., 2 L + W}, {0., 2 L + 2 W}};
  join1 = {{L, L}, {L, L + W}};
  join3 = {{L, 1.}, {L + W, 1.}};
  join4 = {{2 L + W, 2 L + 2 W}, {2 L + W, 2 L + W}};
  join5 = {{L + W, L}, {L + W, L + W}};


  joins = {join1, join2, join3, join4, join5, join6, join7, join8};
  diag1 = {{0., 0}, {L, L}};
  diag2 = {{L, L + W}, {0., 2 L + W}};
  diag3 = {{0., 2 L + 2 W}, {L, 1.}};
  diag4 = {{L + W, 1}, {2 L + W, 2 (L + W)}};
  diag5 = {{2 L + W, 2 L + W}, {L + W, L + W}};
```

```
  diag6 = {{L + W, L}, {2 L + W, 0}};
  diag7 = {{2 L + 2 W, 0}, {1, L}};
  diag8 = {{1, L + W}, {2 L + 2 W, 2 L + W}};
  diag9 = {{2 L + 2 W, 2 L + 2 W}, {1, 1}};
  diags = {diag1, diag2, diag3, diag4, diag5, diag6, diag7, diag8,
    diag9};
  listjoins = {joins};
  Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];
  listdiags = {diags};
  Listdiags = Table[Line[listdiags[[i]]], {i, Length[listdiags]}];

  If[n == 1,
   Graphics[{{Cyan, Polygon[square]}, {Thin, Black, Listjoins}, {Thin,
       Black, Listdiags}}], (*the first iteration of the curve is \
done outside the loop*)
   Do[
    j0 = ScalingTransform[{L, L}][joins];
    j2 = ReflectionTransform[{1., 0.}][j0];
    j2 = TranslationTransform[{L, L + W}][j2];
    j4 = TranslationTransform[{0, 2 L + 2 W}][j0];
    j6 = TranslationTransform[(L + W) {1., 1.}][j2];
    j8 = TranslationTransform[(L + W) {1., 1.}][j0];
    j10 = TranslationTransform[(L + W) {1., -1.}][j2];
    j12 = TranslationTransform[2 (L + W) {1., 0.}][j0];
    j14 = TranslationTransform[2 (L + W) {1., 0.}][j2];
    j16 = TranslationTransform[2 (L + W) {1., 0.}][j4];
    joins = Catenate[{j0, j2, j4, j6, j8, j10, j12, j14, j16}];
    AppendTo[listjoins, joins];
    p0 = ScalingTransform[{L, L}][square];
    p2 = TranslationTransform[{0., L + W}][p0];
    p4 = TranslationTransform[{0, 2 L + 2 W}][p0];
    p6 = TranslationTransform[(L + W) {1., 1.}][p2];
    p8 = TranslationTransform[(L + W) {1., 1.}][p0];
    p10 = TranslationTransform[(L + W) {1., -1.}][p2];
    p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
    p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
    p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
    d0 = ScalingTransform[{L, L}][diags];
    d2 = ReflectionTransform[{1., 0.}][d0];
    d2 = TranslationTransform[{L, L + W}][d2];
    d4 = TranslationTransform[{0, 2 L + 2 W}][d0];
    d6 = TranslationTransform[(L + W) {1., 1.}][d2];
    d8 = TranslationTransform[(L + W) {1., 1.}][d0];
    d10 = TranslationTransform[(L + W) {1., -1.}][d2];
    d12 = TranslationTransform[2 (L + W) {1., 0.}][d0];
    d14 = TranslationTransform[2 (L + W) {1., 0.}][d2];
    d16 = TranslationTransform[2 (L + W) {1., 0.}][d4];
    diags = Catenate[{d0, d2, d4, d6, d8, d10, d12, d14, d16}];
    square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}],
```

```
    n - 1];
   Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];
   Graphics[{{Cyan, Polygon[square]}, {Thin, Black, Listjoins}, {Thin,
       Black, Line[diags]}}]
   ]
  ]

(* First iteration *)
OsgoodPolygon[1, 0.3, 9., -1]

(* Second iteration *)
OsgoodPolygon[3, 0.3, 9., -1]
```

B.7 Figure 4.11 and Figure 4.12

Modified from [17]. Ordering added with Geogebra.

```
dist[{x0_, y0_}, {x1_, y1_}] := Sqrt[(x0 - x1)^2 + (y0 - y1)^2]

triangleSplit[\[ScriptCapitalT][p1_, p2_, p3_, level_], r_, rconst_] :=
  Module[{c = dist[p1, p2], a = dist[p2, p3],
   b = dist[p3, p1], \[Alpha], \[Gamma], \[Theta], originalArea,
   targetArea, b1, b2, p1a, p3a},
  \[Alpha] = ArcCos[(b^2 + c^2 - a^2)/(2 b c)];
  \[Gamma] = ArcCos[(a^2 + b^2 - c^2)/(2 a b)];
   \[Theta] = ArcTan[(p3[[1]] - p1[[1]]), (p3[[2]] - p1[[2]])];
  originalArea = Sqrt[(a^2 + b^2 + c^2)^2 - 2 (a^4 + b^4 + c^4)]/4;
  (* if we're removing r_j of the area,
  each resulting triangle wants half the remaining area *)
  targetArea = (originalArea/2)*If[rconst, (1 - r), (1 - (r/level)^2)];
  (* we know area, one side, one angle, so get other side *)
  b1 = (2 targetArea)/(c Sin[\[Alpha]]);
  b2 = (2 targetArea)/(a Sin[\[Gamma]]);
  (* finally, get the coordinates of the third vertex *)
  p1a = p1 + b1*{Cos[\[Theta]], Sin[\[Theta]]};
  p3a = p3 - b2*{Cos[\[Theta]], Sin[\[Theta]]};
  (* return the two resulting triangle objects *)
  {\[ScriptCapitalT][p1, p1a, p2,
    If[rconst, level + 1, 1]], \[ScriptCapitalT][p2, p3a, p3,
    If[rconst, level + 1, 1]]}]

triGraphic[p1_, p2_, p3_, _] := Polygon[{p1, p2, p3}]

r = 0.333333333333;
rconst = "r";
```

```
(* First iteration *)
lvl=1;

Graphics[
 {EdgeForm[Thin], Cyan,
  Nest[Flatten[
      Map[Function[x,
        triangleSplit[x, r,
         rconst == "r"]], #]] &, {\[ScriptCapitalT][{-1, 0}, {0,
       Tan[30 Degree]}, {1, 0}, 1]}, lvl] /. \[ScriptCapitalT] ->
    triGraphic}]

 (* Second iteration *)
lvl=2;

Graphics[
 {EdgeForm[Thin], Cyan,
  Nest[Flatten[
      Map[Function[x,
        triangleSplit[x, r,
         rconst == "r"]], #]] &, {\[ScriptCapitalT][{-1, 0}, {0,
       Tan[30 Degree]}, {1, 0}, 1]}, lvl] /. \[ScriptCapitalT] ->
    triGraphic}]
```

B.8 Figure 4.13a

```
L = 1./3. - 2*W/3.;
W = (c - 3.) (1. + sign*Sqrt[\[Lambda]])/(2 c);

J1 = {{1., 0.}, {1. + separation, 0}};
J2 = {{1., 2. + separation}, {1. + separation, 2. + separation}};
J3 = {{0., 1.}, {0., 1. + separation}};
J4 = {{2. + separation, 1.}, {2. + separation, 1. + separation}};
primordialjoins = {J1, J2, J3, J4};
listprimordialjoins = {primordialjoins};

c = 6; sign = -1; \[Lambda] = 0.3; separation = 0.3;

square = {{{0., 0.}, {0., 1.}, {1., 1.}, {1., 0.}}};
p0 = square;
p2 = TranslationTransform[{0., 1 + separation}][p0];
p4 = TranslationTransform[{1 + separation, 0.}][p0];
p6 = TranslationTransform[(1 + separation) {1., 1.}][p0];
square = Catenate[{p0, p2, p4, p6}]; Graphics[{Line[
   primordialjoins], {Cyan, Polygon[square]}}]
```

B.9 Figure 4.13b

```
ClosedOsgoodCurve[separation_, n_, \[Lambda]_, c_, sign_] :=

 (*n\[GreaterEqual]1 is the iteration, \[Lambda] the area covered at \
infinity (0<\[Lambda]<1), c the ratio between W_i and W_{i+1} (which \
needs to be c>3) and sign=+-1 in order to choose the positive or the \
negative root*)


 Block[{square, square1, square2, square3, p0, p2, p4, p6, p8, p10,
   p12, p14, p16, , j0, j2, j4, j6, j8, j10, j12, j14, j16, join1,
   join2, join3, join4, join5, join6, join7, join8, joins, listjoins,
   Listjoins, p, J1, J2, J3, J4, listprimordialjoins, primordialjoins,
    PrimordialJoins, joins1, joins2, joins3, listjoins1, listjoins2,
   listjoins3, Listjoins1, Listjoins2, Listjoins3, W, L},
  (*Initializing constants*)
  L = 1./3. - 2*W/3.;
  W = (c - 3.) (1. + sign*Sqrt[\[Lambda]])/(2 c);

  J1 = {{1., 0.}, {1. + separation, 0}};
  J2 = {{1., 2. + separation}, {1. + separation, 2. + separation}};
  J3 = {{0., 1.}, {0., 1. + separation}};
  J4 = {{2. + separation, 1.}, {2. + separation, 1. + separation}};
  primordialjoins = {J1, J2, J3, J4};
  listprimordialjoins = {primordialjoins};

  PrimordialJoins =
   Table[Line[listprimordialjoins[[i]]], {i,
     Length[listprimordialjoins]}];
  (*initializing square and joins, p for squares, j for joins*)
  join1 = {{0., L}, {0., L + W}};
  join2 = {{L, 0.}, {L + W, 0.}};
  join3 = {{2 L + W, L}, {2 L + W, L + W}};
  join4 = {{2 L + 2 W, L}, {2 L + 2 W, L + W}};
  join5 = {{1., 2 L + W}, {1., 2 (L + W)}};
  join6 = {{L, 2 L + W}, {L, 2 L + 2 W}};
  join7 = {{L + W, 2 L + W}, {L + W, 2 L + 2 W}};
  join8 = {{2 L + W, 1.}, {2 (L + W), 1.}};
  joins = {join1, join2, join3, join4, join5, join6, join7, join8};
  listjoins = {joins};
  Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];

  square = {{{0., 0.}, {0., 1.}, {1., 1.}, {1., 0.}}};
  p0 = ScalingTransform[{L, L}][square];
  p2 = TranslationTransform[{0., L + W}][p0];
  p4 = TranslationTransform[{0., 2 L + 2 W}][p0];
  p6 = TranslationTransform[(L + W) {1., 1.}][p2];
  p8 = TranslationTransform[(L + W) {1., 1.}][p0];
```

```
  p10 = TranslationTransform[(L + W) {1., -1.}][p2];
  p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
  p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
  p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
  square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}];
  square1 = TranslationTransform[(1. + separation) {1., 1.}][square];
  square2 = ReflectionTransform[{1., 0.}][square];
  square2 = TranslationTransform[{1., 1. + separation}][square2];
  square3 = TranslationTransform[(1. + separation) {1, -1}][square2];

  joins1 = TranslationTransform[(1. + separation) {1., 1.}][joins];
  joins2 = ReflectionTransform[{1., 0.}][joins];
  joins2 = TranslationTransform[{1., 1. + separation}][joins2];
  joins3 = TranslationTransform[(1. + separation) {1, -1}][joins2];
  listjoins1 = {joins1};
  listjoins2 = {joins2};
  listjoins3 = {joins3};

  Listjoins1 = Table[Line[listjoins1[[i]]], {i, Length[listjoins1]}];
  Listjoins2 = Table[Line[listjoins2[[i]]], {i, Length[listjoins1]}];
  Listjoins3 = Table[Line[listjoins3[[i]]], {i, Length[listjoins1]}];

  If[n == 1,
   Graphics[{{Cyan, Polygon[square]}, {Cyan, Polygon[square1]}, {Cyan,
       Polygon[square2]}, {Cyan, Polygon[square3]}, {Thick, Black,
      Listjoins}, {Thick, Black, Listjoins1}, {Thick, Black,
      Listjoins2}, {Thick, Black, Listjoins3}, {Thick, Black,
      PrimordialJoins}}], (*the first iteration of the curve is done \
outside the loop*)
   Do[
    j0 = ScalingTransform[{L, L}][joins];
    j2 = ReflectionTransform[{1., 0.}][j0];
    j2 = TranslationTransform[{L, L + W}][j2];
    j4 = TranslationTransform[{0, 2 L + 2 W}][j0];
    j6 = TranslationTransform[(L + W) {1., 1.}][j2];
    j8 = TranslationTransform[(L + W) {1., 1.}][j0];
    j10 = TranslationTransform[(L + W) {1., -1.}][j2];
    j12 = TranslationTransform[2 (L + W) {1., 0.}][j0];
    j14 = TranslationTransform[2 (L + W) {1., 0.}][j2];
    j16 = TranslationTransform[2 (L + W) {1., 0.}][j4];
    joins = Catenate[{j0, j2, j4, j6, j8, j10, j12, j14, j16}];
    joins1 = TranslationTransform[(1. + separation) {1., 1.}][joins];
    joins2 = ReflectionTransform[{1., 0.}][joins];
    joins2 = TranslationTransform[{1., 1. + separation}][joins2];
    joins3 = TranslationTransform[(1. + separation) {1, -1}][joins2];
    AppendTo[listjoins, joins];
    AppendTo[listjoins1, joins1];
    AppendTo[listjoins2, joins2];
    AppendTo[listjoins3, joins3];
```

```
    p0 = ScalingTransform[{L, L}][square];
    p2 = TranslationTransform[{0., L + W}][p0];
    p4 = TranslationTransform[{0, 2 L + 2 W}][p0];
    p6 = TranslationTransform[(L + W) {1., 1.}][p2];
    p8 = TranslationTransform[(L + W) {1., 1.}][p0];
    p10 = TranslationTransform[(L + W) {1., -1.}][p2];
    p12 = TranslationTransform[2 (L + W) {1., 0.}][p0];
    p14 = TranslationTransform[2 (L + W) {1., 0.}][p2];
    p16 = TranslationTransform[2 (L + W) {1., 0.}][p4];
    square = Catenate[{p0, p2, p4, p6, p8, p10, p12, p14, p16}],
    n - 1];
   (*In the loop I have created the bottom left square.
   Now I create the three other squares by transforming that one*)
   square1 = TranslationTransform[(1. + separation) {1., 1.}][square];
   square2 = ReflectionTransform[{1., 0.}][square];
   square2 = TranslationTransform[{1., 1. + separation}][square2];
   square3 = TranslationTransform[(1. + separation) {1, -1}][square2];


   Listjoins = Table[Line[listjoins[[i]]], {i, Length[listjoins]}];
   Listjoins1 = Table[Line[listjoins1[[i]]], {i, Length[listjoins1]}];
   Listjoins2 = Table[Line[listjoins2[[i]]], {i, Length[listjoins1]}];
   Listjoins3 = Table[Line[listjoins3[[i]]], {i, Length[listjoins1]}];
   Graphics[{{Cyan, Polygon[square]}, {Cyan, Polygon[square1]}, {Cyan,
       Polygon[square2]}, {Cyan, Polygon[square3]}, {Thick, Black,
      Listjoins}, {Thick, Black, Listjoins1}, {Thick, Black,
      Listjoins2}, {Thick, Black, Listjoins3}, {Thick, Black,
      PrimordialJoins}}]
   ]
  ]


ClosedOsgoodCurve[0.3, 4, 0.3, 6, -1]
```

B.10 Figure 4.14

Modified from [17].

```
dist[{x0_, y0_}, {x1_, y1_}] := Sqrt[(x0 - x1)^2 + (y0 - y1)^2]

triangleSplit[\[ScriptCapitalT][p1_, p2_, p3_, level_], r_, rconst_] :=
  Module[{c = dist[p1, p2], a = dist[p2, p3],
   b = dist[p3, p1], \[Alpha], \[Gamma], \[Theta], originalArea,
   targetArea, b1, b2, p1a, p3a},
  \[Alpha] = ArcCos[(b^2 + c^2 - a^2)/(2 b c)];
  \[Gamma] = ArcCos[(a^2 + b^2 - c^2)/(2 a b)];
   \[Theta] = ArcTan[(p3[[1]] - p1[[1]]), (p3[[2]] - p1[[2]])];
  originalArea = Sqrt[(a^2 + b^2 + c^2)^2 - 2 (a^4 + b^4 + c^4)]/4;
```

```
  (* if we're removing r_j of the area,
  each resulting triangle wants half the remaining area *)
  targetArea = (originalArea/2)*If[rconst, (1 - r), (1 - (r/level)^2)];
  (* we know area, one side, one angle, so get other side *)
  b1 = (2 targetArea)/(c Sin[\[Alpha]]);
  b2 = (2 targetArea)/(a Sin[\[Gamma]]);
  (* finally, get the coordinates of the third vertex *)
  p1a = p1 + b1*{Cos[\[Theta]], Sin[\[Theta]]};
  p3a = p3 - b2*{Cos[\[Theta]], Sin[\[Theta]]};
  (* return the two resulting triangle objects *)
  {\[ScriptCapitalT][p1, p1a, p2,
    If[rconst, level + 1, 1]], \[ScriptCapitalT][p2, p3a, p3,
    If[rconst, level + 1, 1]]}]

triGraphic[p1_, p2_, p3_, _] := Polygon[{p1, p2, p3}]

r = 1./3.;
lvl = 6;
rconst = "r";

Graphics[{
  {EdgeForm[Thin], Cyan,
   Nest[Flatten[
       Map[Function[x,
         triangleSplit[x, r,
          rconst == "r"]], #]] &, {\[ScriptCapitalT][{-1,
        0}, {-1 + Sqrt[2]*Cos[105 Degree],
        Sqrt[2]*Sin[105 Degree]}, {0, Tan[Pi/3]}, 1]},
     lvl] /. \[ScriptCapitalT] -> triGraphic},
  {EdgeForm[Thin], Cyan,
   Nest[Flatten[
       Map[Function[x,
         triangleSplit[x, r,
          rconst == "r"]], #]] &, {\[ScriptCapitalT][{0,
        Tan[Pi/3]}, {1 + Sqrt[2]*Cos[75 Degree],
        Sqrt[2]*Sin[75 Degree]}, {1, 0}, 1]},
     lvl] /. \[ScriptCapitalT] -> triGraphic},
  {EdgeForm[Thin], Cyan,
   Nest[Flatten[
       Map[Function[x,
         triangleSplit[x, r,
          rconst == "r"]], #]] &, {\[ScriptCapitalT][{-1,
        0}, {0, -1}, {1, 0}, 1]}, lvl] /. \[ScriptCapitalT] ->
     triGraphic}},
 ImageSize -> {400, 400}, PlotRange -> {{-2, 2}, {-2, 2}}]
```

Appendix C

Mathematica source code for figures in chapter 6

C.1 Figure 6.2

```
square = {Cyan, Rectangle[]};
g1 = Table[{t, RandomReal[{0.4, 0.6}]}, {t, 0, 1, 0.05}];
g2 = Table[{RandomReal[{0.4, 0.6}], t}, {t, 0, 1, 0.05}];
Graphics[{square, {Thick, Line[g1]}, {Green, Thick, Line[g2]}}]
```

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