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Skew-symmetric endomorphisms in $\mathbb{M}^{1,3}$: a unified canonical form with applications to conformal geometry

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Abstract

We derive a canonical form for skew-symmetric endomorphisms F in Lorentzian vector spaces of dimension three and four which covers all non-trivial cases at once. We analyze its invariance group, as well as the connection of this canonical form with duality rotations of two-forms. After reviewing the relation between these endomorphisms and the algebra of conformal Killing vectors of $\mathbb{S}^2$, CKill $\left(\mathbb{S}^2\right)$, we are able to also give a canonical form for an arbitrary element $\xi \in$ CKill $\left(\mathbb{S}^2\right)$ along with its invariance group. The construction allows us to obtain explicitly the change of basis that transforms any given F into its canonical form. For any non-trivial ξ we construct, via its canonical form, adapted coordinates that allow us to study its properties in depth. Two applications are worked out: we determine explicitly for which metrics, among a natural class of spaces of constant curvature, a given ξ is a Killing vector and solve all local traceless and transverse tensors that satisfy the Killing initial data equation for ξ. In addition to their own interest, the present results will be a basic ingredient for a subsequent generalization to arbitrary dimensions.

Keywords: skew-symmetric, canonical, conformal Killing, conformal, TT tensor, KID equations at scri

(Some figures may appear in colour only in the online journal)

1. Introduction

Finding a canonical form for the elements of a certain set is often an interesting problem to solve, since it is a powerful tool for both computations and mathematical analysis. By canonical

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form (sometimes also called normal form) of the elements x of a set X one usually understands a specific form, depending on a number of parameters, that every element x can be carried to. The value of such parameters is obviously determined by x. The most common examples are canonical forms of matrices, such as the echelon form or the Jordan form. However, the same concept arises in other sets, such as smooth fields on a manifold or even systems of differential equations (e.g. canonical coordinates for Hamiltonian systems). A canonical form must be somehow useful either to simplify the calculations or to make explicit some information we may want to exploit. Taking an element to its canonical form requires showing the existence of (and ideally also finding explicitly) a transformation, namely, a change of basis, coordinates, etc that brings the element into its canonical form, and which need not to be unique.

When dealing with Lie algebras $\mathfrak{g}$, one may attempt to find a canonical form for every element $F \in \mathfrak{g}$ that captures all the information of its orbit under the (e.g. adjoint) action of the Lie group G. For example, this is the case of the already mentioned Jordan canonical form, regarded as the matrix form (up to permutations of the blocks) that encodes all the information of the $\mathfrak{gl}(n, \mathbb{C})$ orbits under the adjoint action of the group $GL(n, \mathbb{C})$. Identifying these orbits, and the more general problem of the orbits generated by an algebraic group action on a set, is an active field of research in different fields of mathematics and it is already well-understood for the case of classical Lie groups. We refer the reader to [4] and references therein for an extensive review of this problem and other references such as [1, 3, 6, 11, 15].

From the point of view of physics, it is of particular importance the study of the pseudo-orthogonal group $O(1, n+1)$ because of its role in the theory of relativity and other physical theories. First, it is the group of isotropies in the special theory of relativity and in the Lorentz–Maxwell electrodynamics. For the latter, the elements of the Lie algebra $\mathfrak{o}(1, n+1)$, represented here as skew-symmetric endomorphisms of Minkowski $\mathbb{M}^{1,n+1}$ (or equivalently the two-forms of the same space), also represent the electromagnetic field (e.g. [16]). Besides, and this is of great importance in our approach, in general relativity the pseudo-orthogonal group is related to the group of conformal transformations of certain spaces [22, 24]. Also, techniques in conformal geometry allow to recast the Einstein field equations (in fact, an equivalent set thereof) as a Cauchy or characteristic problem in a hypersurface $\mathscr{I}$ ([8, 9] and references therein) representing 'infinity' in a physically precise sense. We are specially interested in the case of positive cosmological constant, where this Cauchy problem is always well-posed and $\mathscr{I}$ happens to be Riemannian. The initial data consist of a metric γ in $\mathscr{I}$ and a symmetric 'TT' tensor D of $\mathscr{I}$, i.e. traceless and transverse (zero divergence). If the solution spacetime is to have a Killing vector, then the so called Killing initial data (KID) equations must be satisfied [21], and this involves a conformal Killing vector (CKV) of γ. Moreover, only the conformal class of the data matters and of particular importance is the case of γ conformal to the standard metric of the sphere, in particular because of its relation with black hole spacetimes such as Kerr-de Sitter [17]. We will expand on this later in this introduction.

In the physics literature a 'canonical' form for the $\mathfrak{o}(1, n+1)$ elements is often employed mostly in four dimensions [26] but also in arbitrary dimensions [14, 17]. This form requires identifying the causal character of the eigenvectors of a given element $F \in \mathfrak{o}(1, n+1)$ and gives rise to two different types of canonical forms, one and only one admitted by each given F. Something similar is done in more generality in [6] where, from a powerful classification result, a list of canonical forms for a wide sample of Lie algebras is given, but the pseudo-orthogonal case still requires two different forms. All these forms contain sufficient information to identify the orbit generated by the adjoint action of the group acting on the given element. However, it is surprising that, to the best of the authors' knowledge, there are no previous attempts to find a unified canonical form to which any single element of the algebra $\mathfrak{o}(1, n+1)$ can be carried to. In the present paper, we address and solve the problem of finding a unified canonical form for skew-symmetric endomorphisms in three ($n = 1$) and four ($n = 2$) dimensions.

As mentioned above, one aspect of the relevance of pseudo-orthogonal groups (or any signature) lies in their relation with the conformal group of a related space. For $O(1, n+1)$ this is the conformal group of the sphere $\mathbb{S}^n$, that we denote $\mathrm{Conf}(\mathbb{S}^n)$. More specifically, the orthochronous subgroup (i.e. the one preserving time orientation) $O^+(1, n+1)$ is isomorphic to $\mathrm{Conf}(\mathbb{S}^n)$ [17, 22], and so it is the lie algebra $\mathfrak{o}(1, n+1)$ to the CKV fields $\mathrm{CKill}(\mathbb{S}^n)$. Thus, finding a canonical form for the elements of $\mathfrak{o}(1, n+1)$, in turn implies a canonical form for the elements of $\mathrm{CKill}(\mathbb{S}^n)$. Amongst other applications, it is particularly useful to employ the canonical form to find adapted coordinates to an arbitrary $\xi \in \mathrm{CKill}(\mathbb{S}^n)$. In these coordinates, the KID equations are straightforward to solve with generality, which is a first step in order to obtain all TT tensors that generate spacetimes with at least one symmetry. This is a possible route to obtain a new characterization result for Kerr-de Sitter, specially relevant for the physical $n = 3$ case. Indeed, given any CKV ξ on a pseudo-riemannian manifold of any dimension there exists [17] a special TT tensor that can be canonically built out of ξ. This TT tensor is moreover a KID with respect to ξ. Kerr-de Sitter (in dimension 3) can be characterized [17] by the properties of having a conformally flat scri with a TT tensor built canonically from a CKV lying in an appropriate conformal class. The idea we want to explore in the future is to find sufficient conditions that force a TT tensor satisfying the KID equations to adopt the special form we just mentioned. Here we study in detail the $n = 2$ case, where in addition we prove that there always exist an element $\xi^\perp \in \mathrm{CKill}(\mathbb{S}^n)$, which is everywhere orthogonal to ξ, with the same norm and such that $[\xi, \xi^\perp] = 0$ (cf lemma 3 below), so it is convenient to adapt coordinates simultaneously to ξ, $\xi^\perp$. With these coordinates at hand, we obtain all TT tensors satisfying the KID equation in a very simple and elegant form (cf section 9). It turns out that a basis of such tensors can be written down explicitly in terms of ξ and $\xi^\perp$ and that one of its elements is precisely the TT tensor canonically built from ξ that arises in the characterization of Kerr-de Sitter. Thus identifying sufficient conditions under which the TT tensor adopts the canonical form becomes explicit in dimension $n = 2$. This gives support to our programme of characterizing Kerr-de Sitter in the physical dimension $n = 3$ by means of a detailed classification of TT tensors satisfying the KID equation.

This will obviously require generalizing the results obtained here to higher dimensions. These results will be presented in a subsequent work [18]. The problem in higher dimensions is considerably harder and relies to a large extent on the results obtained here. Presenting the low dimensional case in a separate paper is convenient for several reasons. First, the most relevant physical dimension for a spacetime is four, so studying this case in detail is particularly important and intrinsically interesting. Second, as already mentioned, the results presented here turn out to be an essential building block for the generalization to arbitrary dimensions. As it will be shown in [18], the canonical form for any dimension will follow by combining the results in this paper and well-known classification theorems of pseudo-orthogonal algebras. In addition, dealing with low dimensions allows us to analyze some of the questions in more depth and get additional insights into the problem. This perspective also provides clues about the possible solutions to the problem in more dimensions. Finally, although simpler than in higher dimensions, even the low dimensional case is far from trivial, so it makes sense the present this case in a separate work.

This paper is intended to be self-contained and only requires elementary knowledge of algebra and differential geometry. Our intention is to make our results accessible for readers with different backgrounds. The paper is organized as follows. Sections 2–4 are devoted to the obtention and analysis of a canonical form for any given (non-zero) element $F \in \mathfrak{o}(1, 3)$. In section 2 we obtain our canonical form in four dimensions, i.e. for $\mathfrak{o}(1, 3)$ and show its universal validity for every non-trivial F. The change of basis that yields to the canonical form is not unique. This implies the existence of an invariance group, that we derive in section 3. In section 4 we analyze the generators of the invariance group and obtain a decomposition of the element F in terms of these. We also make a connection between this decomposition and the standard duality rotations for two-forms. In all these sections, the three-dimensional case is obtained and discussed as a corollary of the four-dimensional one.

The following sections 5–8 are devoted to the study of so-called global CKVs (GCKV) defined on Euclidean space $\mathbb{E}^2$, and which are directly related to CKV on the sphere $\mathbb{S}^2$. Section 5 defines such vectors and section 6 describes a known relation between them and the Lie algebra $\mathfrak{o}(1,3)$. In section 7 we apply all the results for the $\mathfrak{o}(1,3)$ algebra to the CKVs of the sphere, namely, the obtention of a canonical form and its invariance group. As a useful consequence of the two viewpoints, we are able (corollary 4) to obtain in a fully explicit form the change of basis that transforms any given F into its canonical form. Finally, section 8 gives a set of coordinates adapted to an arbitrary ξ and its orthogonal $\xi^{\perp}$. The results concerning the canonical form of GCKV and the adapted coordinates are summarized in theorem 1. Our last section 9 gives two interesting applications for the previous results. First, given a GCKV ξ, theorem 2 gives a list of all metrics, conformal to the metric of a two-sphere, for which ξ is a Killing vector. Second, theorem 3 gives an elegant solution of the TT tensors satisfying the KID equations in $\mathbb{S}^2$.

2. Canonical form of skew-symmetric endomorphisms in $\mathbb{M}^{1,3}$

In this section we consider Lorentzian four-vector spaces (V, g), i.e a four dimensional vector space V endowed with a pseudo-Riemannian metric g of signature $\{-,+,+,+\}$. The inner product with g is denoted by $\langle\cdot,\cdot\rangle$. We will often identify Lorentzian vector spaces of dimension n with Minkowski $\mathbb{M}^{1,n-1}$. Null vectors are vectors with vanishing norm (in particular, the zero vector is null in our conventions). An endomorphism $F : V \to V$ is skew-symmetric when it satisfies

$$\langle e, F(e')\rangle = -\langle F(e), e'\rangle, \quad \forall\ e, e' \in V. \tag{1}$$

This subset of $\mathrm{End}\,(V)$ is denoted by $\mathrm{SkewEnd}\,(V)$. We take, by definition, that eigenvectors of an endomorphism are always non-zero. $\ker F$ and $\mathrm{Im}\,F$ denote, respectively, the kernel and image of $F \in \mathrm{End}\,(V)$.

We now briefly discuss a few basic properties of skew-symmetric endomorphisms that we will be referring to. First, it is immediate from (1) that every vector $e \in V$ is perpendicular to its image, i.e. $\langle F(e), e\rangle = 0$. Second, consider a, possibly complex, eigenvalue $\lambda \in \mathbb{C}$ and its eigenvector $w \in V_{\mathbb{C}}$ (the complexification of V). By the previous property, w must be null if $\lambda \neq 0$, because $\langle F(e), e\rangle = \lambda\,\langle e, e\rangle = 0$. Eigenvectors with zero eigenvalue may be both null and non-null. Since F is real, the complex conjugate $\lambda^\star \in \mathbb{C}$ is an eigenvalue with eigenvector $w^\star \in V_{\mathbb{C}}$, so

$$\langle F(w), w^\star\rangle = \lambda\,\langle w, w^\star\rangle = -\lambda^\star\,\langle w, w^\star\rangle\,.$$

Thus, either λ is purely imaginary (including zero) or, if not, $w, w^\star$ are a pair of null vectors orthogonal to each other. Suppose the latter and denote $w = u + \mathrm{i}v$ for $u, v \in V$. Then the nullity condition implies $\langle u, v\rangle = 0$ and $\langle u, u\rangle = \langle v, v\rangle$ and orthogonality to $w^\star$ implies $\langle u, u\rangle = -\langle v, v\rangle$. Hence u, v are null and proportional, i.e. $u = av$ for some $a \in \mathbb{R}$, in consequence $w = (a + \mathrm{i})v$. Therefore, $v \in V$ is a real null eigenvector and its corresponding eigenvalue λ must be real. Summarizing, F has only real or purely imaginary eigenvalues and their corresponding eigenvectors must be null for non-zero eigenvalues.

It will be useful to work with two-dimensional subspaces which are invariant under the action of F, which we will call 'eigenplanes'. Let span $\{e, e'\} = \Pi$ be a spacelike eigenplane for a pair of spacelike, orthogonal, unit vectors e, e'. Then by F-invariance

$$F(e) = a_1 e + a_2 e', \qquad F(e') = b_1 e + b_2 e', \quad a_1, a_2, b_1, b_2 \in \mathbb{R},$$

by skew-symmetry $a_1 = \langle F(e), e\rangle = 0, b_2 = \langle F(e'), e'\rangle = 0$ and $a_2 = \langle F(e), e'\rangle = -\langle e, F(e')\rangle = -b_1 =: \mu$. Hence

$$F(e) = \mu e', \qquad F(e') = -\mu e, \quad \mu \in \mathbb{R}, \tag{2}$$

which is equivalent to the following eigenequations

$$F(e + \mathrm{i}e') = -\mathrm{i}\mu(e + \mathrm{i}e'), \qquad F(e - \mathrm{i}e') = \mathrm{i}\mu(e - \mathrm{i}e'). \tag{3}$$

In a similar way, for a pair of orthogonal vectors e_0, e_1 spanning a timelike eigenplane, with e_0 unit timelike and e_1 unit spacelike, one can immediately verify

$$F(e_0) = \mu e_1, \qquad F(e_1) = \mu e_0, \quad \mu \in \mathbb{R} \tag{4}$$

and

$$F(e_0 + e_1) = \mu(e_0 + e_1), \qquad F(e_0 - e_1) = -\mu(e_0 - e_1). \tag{5}$$

If F admits an invariant subspace U of any dimension, F also leaves the orthogonal space $U^\perp$ invariant. This follows immediately from

$$0 = \langle F(u), v\rangle = -\langle u, F(v)\rangle \quad \forall\, u \in U,\ \forall\, v \in U^\perp.$$

In particular, in four dimensions the existence of a timelike eigenplane is equivalent to the existence of an (orthogonal) spacelike eigenplane.

Another well-known property of skew-symmetric endomorphisms is that $\dim \operatorname{Im} F$ is always even. Equivalently, in four dimensions $\dim \ker F$ is also even (in arbitrary dimension V, $\dim \ker F$ has the same parity as $\dim V$). To see this, consider the two-form $\boldsymbol{F}$ associated to $F \in \mathrm{SkewEnd}\,(V)$ by the standard relation

$$\boldsymbol{F}(e, e') = \langle e, F(e')\rangle, \quad \forall\, e, e' \in V. \tag{6}$$

The matrix representing $\boldsymbol{F}$ is skew in the usual sense. The dimension of $\operatorname{Im}\boldsymbol{F} \subset V^\star$ (the dual of V) is the rank of this matrix, which is known to be even (see e.g. [10]), and clearly $\dim \operatorname{Im} \boldsymbol{F} = \dim \operatorname{Im} F$.

The first step towards our canonical form for F is the following classification result, which relies on the properties described above.

Lemma 1 (Classification of SkewEnd $(\mathbb{M}^{1,3})$). *Let $F \in \mathrm{SkewEnd}\,(V)$ in a Lorentzian vector space (V, g) of dimension four. If $F \neq 0$ then one of the following exclusive possibilities hold:*

(*a*) *F has a spacelike eigenvector orthogonal to a null eigenvector, both with vanishing eigenvalue.*

(*b*) *F has a spacelike eigenplane (as well as a timelike orthogonal eigenplane).*

Proof Since F is not identically zero, $\dim \ker F$ only can be either 2 or 0. Consider first $\dim \ker F = 0$ and let us prove that (b) must happen. We show this by proving that equations (3) and (5) must be satisfied. Since $\ker\ F = \{0\}$, F can only have non-zero eigenvalues, and we already know that they are either real or purely imaginary. The existence of a purely imaginary one leads to equation (3), which in turn implies (5). Suppose now that all eigenvalues are real non-zero. If there exist two different real eigenvalues $\mu,\ \mu'$ their respective eigenvectors w, w' (which recall are null) must satisfy

$$\langle F(w), w'\rangle = \mu \langle w, w'\rangle = -\mu' \langle w, w'\rangle .$$

The product $\langle w, w'\rangle$ cannot be zero, as otherwise w, w' would be proportional and the eigenvalues μ and μ' would be the same. Thus, $\mu = -\mu'$, and hence (5), and also (3), hold. The remaining case is when all eigenvalues are equal, i.e. the characteristic polynomial is $p_F = (F - I\mu)^4$. By the Cayley–Hamilton theorem $\langle p_f(u), v\rangle = 0, \quad \forall\ u, v \in V$. In particular, $\langle p_f(u), v\rangle = \langle p_f(v), u\rangle , \quad \forall\ u, v \in V$. By skew-symmetry the even powers on each side cancel out and we are left with

$$\begin{aligned} -4\ \mu \left\langle F^3(u), v\right\rangle - 4\ \mu^3 \langle F(u), v\rangle &= -4\ \mu \left\langle F^3(v), u\right\rangle - 4\ \mu^3 \langle F(v), u\rangle \\ &= 4\ \mu \left\langle F^3(u), v\right\rangle + 4\ \mu^3 \langle F(u), v\rangle , \quad \forall\ u, v \in V. \end{aligned}$$

Since we are in the case $\mu \in \mathbb{R} \setminus \{0\}$ we conclude that $F(F^2 +\ \mu^2) = 0$, and since F is invertible ($\ker\ F = \{0\}$) also $F^2 +\ \mu^2 = 0$. But this means that F admits a complex eigenvalue, which is a contradiction, and we have exhausted all possible cases with $\dim \ker F = 0$.

Now let $\dim \ker F = 2$. According to the causal character of $\ker F$, either $\ker F$ is null, and we are in case (a) of the lemma or $\ker F$ is non-degenerate, and we are in case (b). The fact that cases (a) and (b) are mutually exclusive is obvious. □

The classification in lemma 1 contains two possible cases. It is common to use this result to find simple forms for each case, for example, in case (a) by including in the basis two orthogonal vectors $k, e \in\ \ker F$; or in case (b), by combining bases in the orthogonal and timelike eigenplanes, so that F is explicitly a direct sum of two two-dimensional endomorphisms. In the following proposition we find a canonical form which includes cases (a) and (b) simultaneously, and which depends on two parameters only.

Proposition 1 *For every non-zero* $F \in \mathrm{SkewEnd}(V)$*, with* (V, g) *a four-dimensional Lorentzian vector space with a choice of time orientation, there exists an orthonormal unit basis* $B := \{e_0, e_1, e_2, e_3\}$*, with* e_0 *timelike future directed such that*

$$\begin{pmatrix} F(e_0) \\ F(e_1) \\ F(e_2) \\ F(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 + \frac{\sigma}{4} & \frac{\tau}{4} \\ 0 & 0 & 1 + \frac{\sigma}{4} & \frac{\tau}{4} \\ -1 + \frac{\sigma}{4} & -1 - \frac{\sigma}{4} & 0 & 0 \\ \frac{\tau}{4} & -\frac{\tau}{4} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \sigma, \tau \in \mathbb{R}, \tag{7}$$

where $\sigma := -\frac{1}{2}\ \mathrm{Tr}\ F^2$ *and* $\tau^2 := -4 \det F$*, with* $\tau \geqslant 0$*. Moreover, if* $\tau = 0$ *the vector* e_3 *can be taken to be any spacelike unit vector lying in the kernel of* F*.*

Proof By lemma 1 there exist two possible cases. We start proving the proposition assuming that we are in case (a). Let $\mathrm{span}\,\{k, e\} = \ker\ F$, with $k, e \in V$ a pair of orthogonal null and spacelike unit vectors respectively. We can complete them to a semi-null basis $B = \{k, l, w, e\}$, i.e. such that $\langle k, l\rangle = -2$, $\langle w, w\rangle = \langle e, e\rangle = 1$ and the rest of scalar products all zero. Using these orthogonality relations and skew-symmetry of F we can calculate:

$$F(k) = 0, \qquad F(l) = aw, \qquad F(w) = \frac{a}{2}k, \qquad F(e) = 0,$$

for a constant $a \in \mathbb{R}\setminus\{0\}$. Redefine a new basis $\{l', k', w', e'\}$, with $k' := \frac{\epsilon a}{2}k, l' := \frac{2\epsilon}{a}l$, $w' := -\epsilon w, e' := e$, where $\epsilon^2 = 1$ is chosen so that k', l' are future directed. Then

$$F(k') = 0, \qquad F(l') = w', \qquad F(w') = k', \qquad F(e') = 0,$$

which in the orthonormal basis $B = \{e_0, e_1, e_2, e_3\}$ given by $k' = e_0 + e_1, l' = e_0 - e_1$, $w' = e_2, e' = e_3$ is

$$F(e_0) = -e_2, \qquad F(e_1) = e_2, \qquad F(e_2) = -e_0 - e_1, \qquad F(e_3) = 0.$$

This corresponds to expression (7) with $\sigma = \tau = 0$.

It remains to prove the proposition for case (b). In this case, there exist timelike and spacelike eigenplanes, $\Pi_t = \text{span}\{e'_0, e'_1\}$ and $\Pi_s = \text{span}\{e'_2, e'_3\}$ respectively, i.e. fulfilling equations (2) and (4) for respective eigenvalues μ_0 and μ_1, such that at most one of them vanishes. We can take the bases of Π_t, Π_s so that that $B' := \{e'_0, e'_1, e'_2, e'_3\}$ is an orthonormal basis of V, with e'_0 past directed and the eigenvalues μ_0 and μ_1 are positive or (at most one) zero. Then, the following change of basis is well-defined:

$$e_0 = \frac{-1}{\sqrt{\mu_0^2 + \mu_1^2}}\left[\left(1 + \frac{\mu_0^2+\mu_1^2}{4}\right)e'_0 + \left(1 - \frac{\mu_0^2+\mu_1^2}{4}\right)e'_2\right], \; e_2 = \frac{1}{\sqrt{\mu_0^2+\mu_1^2}}\left(\mu_0 e'_1 + \mu_1 e'_3\right),$$

$$e_1 = \frac{1}{\sqrt{\mu_0^2 + \mu_1^2}}\left[\left(1 - \frac{\mu_0^2+\mu_1^2}{4}\right)e'_0 + \left(1 + \frac{\mu_0^2+\mu_1^2}{4}\right)e'_2\right], \; e_3 = \frac{1}{\sqrt{\mu_0^2+\mu_1^2}}\left(-\mu_1 e'_1 + \mu_0 e'_3\right).$$

One checks by explicit computation that $B := \{e_0, e_1, e_2, e_3\}$ is an orthonormal basis, with e_0 timelike and future directed (because $\langle e_0, e'_0\rangle > 0$). It is also a matter of direct calculation to see that

$$F(e_0) = \left(-1 + \frac{\sigma}{4}\right)e_2 + \frac{\tau}{4}e_3, \qquad F(e_1) = \left(1 + \frac{\sigma}{4}\right)e_2 + \frac{\tau}{4}e_3,$$

$$F(e_2) = \left(-1 + \frac{\sigma}{4}\right)e_0 - \left(1 + \frac{\sigma}{4}\right)e_1, \qquad F(e_3) = \frac{\tau}{4}\left(e_0 - e_1\right),$$

where the parameters $\sigma, \tau \in \mathbb{R}$ are $\sigma = \mu_1^2 - \mu_0^2$ and $\tau = 2\mu_0\mu_1 \geqslant 0$. This corresponds to (7) with at most one of the parameters σ, τ vanishing.

To show the last statement, a simple computation shows that (when $\tau = 0$) the kernel of F is given by

$$\ker\ F = \left\{a\left(1 + \frac{\sigma}{4}\right)e_0 + a\left(1 - \frac{\sigma}{4}\right)e_1 + be_3, \quad a, b \in \mathbb{R}\right\}.$$

The subset of spacelike unit vectors in $\ker F$ is given by $1 + a^2\sigma > 0$ and $b = \epsilon\sqrt{1 + a^2\sigma}$, $\epsilon = \pm 1$. We introduce the four vectors

$$e'_0 = \left(\frac{b+\epsilon}{2} + \left(1 + \frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}\right)e_0 + \left(1 - \frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}e_1 + a\left(1 + \frac{\sigma}{4}\right)e_3,$$

$$
\begin{aligned}
e_1' &= -\left(1-\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}e_0 + \left(\frac{b+\epsilon}{2} - \left(1+\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}\right)e_1 + a\left(-1+\frac{\sigma}{4}\right)e_3,\\
e_2' &= \epsilon e_2,\\
e_3' &= a\left(1+\frac{\sigma}{4}\right)e_0 + a\left(1-\frac{\sigma}{4}\right)e_1 + be_3,
\end{aligned}
$$

and observe that they are well-defined for all values of σ, including zero. A straightforward computation shows that this is an orthonormal basis, and that (7) holds with $\tau = 0$. The last statement of the proposition follows. $\square$

Obtaining a canonical form in the three-dimensional case is much easier, the main reason being that any two-form in three-dimensions is simple, i.e. $\boldsymbol{F} \wedge \boldsymbol{F} = 0$ or, in other words, that $\boldsymbol{F}$ is of rank one as a differential form. So, the reader may wonder why it has not been treated before. The reason is that we can obtain the three dimensional case as a direct corollary of the four-dimensional one. The construction is as follows. Let $F \in \mathrm{SkewEnd}\,(V)$ with V Lorentzian three-dimensional. From F we may define an auxiliary skew-symmetric endomorphism $\hat{F}$ defined on $V \oplus \mathbb{E}_1$ endowed with the product metric ($\mathbb{E}_1$ is the one-dimensional Euclidean space). It is obvious that this space is a Lorentzian four-dimensional vector space. We denote by E_3 a unit vector in $\mathbb{E}_1$ and define $\hat{F}$ simply by $\hat{F}(u + aE_3) = F(u) + 0$, for all $u \in V$ and $a \in \mathbb{R}$ (we will identify $u \in V$ with $u + 0 \in V \oplus \mathbb{E}_1$ from now on). It is immediate to check that $\hat{F}$ is skew-symmetric. Moreover, it has $\tau = 0$, by construction. Then, the following corollary is immediate:

Corollary 1 *For every non-zero $F \in \mathrm{SkewEnd}\,(V)$, with (V, g) a Lorentzian three-dimensional vector space with a choice of time orientation, there exists an orthonormal unit basis $B := \{e_0, e_1, e_2\}$, with e_0 timelike future directed such that*

$$
\begin{pmatrix} F(e_0)\\ F(e_1)\\ F(e_2)\end{pmatrix} = \begin{pmatrix} 0 & 0 & -1+\frac{\sigma}{4}\\ 0 & 0 & -1-\frac{\sigma}{4}\\ -1+\frac{\sigma}{4} & 1+\frac{\sigma}{4} & 0\end{pmatrix}\begin{pmatrix} e_0\\ e_1\\ e_2\end{pmatrix}, \quad \sigma := -\frac{1}{2}\,\mathrm{Tr}\left(F^2\right) \in \mathbb{R}. \tag{8}
$$

Proof By the last statement of proposition 1, the canonical basis $B = \{e_0, e_1, e_2, e_3\}$ of $\hat{F}$ can be taken with $e_3 = E_3$, which means that $\{e_0, e_1, e_2\}$ is a basis of V. $\square$

Remark 1 (Classification from the canonical form). For the canonical forms (8) and (7) we can derive a classification result for skew-symmetric endomorphisms and recover lemma 1 in terms of σ, τ. For $F \in \mathrm{SkewEnd}\left(\mathbb{M}^{1,2}\right)$ non-zero it is straightforward that $q := (1 + \sigma/4)$ $e_0 + (1 - \sigma/4)e_1$ generates $\ker F$ and furthermore $\langle q, q\rangle = -\sigma$. Hence, the sign of σ determines the causal character of the kernel, namely spacelike for $\sigma < 0$, null for $\sigma = 0$ and timelike for $\sigma > 0$. In the four-dimensional case, if $\tau \neq 0$, then $\ker\ F = \{0\}$ and we must be in case (b) of lemma 1. If $\tau = 0$, then $e_3 \in\ \ker F$ (spacelike) and the sign of σ determines the causal character of any non-zero vector $q \in \mathrm{span}\,\{e_0, e_1, e_2\} \cap \ker\ F$ just like in the previous case. That is, $\tau = 0$ and $\sigma = 0$ corresponds with case (a) of lemma 1 and otherwise we are in case (b).

At this point, it is convenient to comment on the relation between our results and previous canonical forms of skew-symmetric endomorphisms. It is standard in the literature to work with two-forms of $\mathbb{M}^{1,3}$, also called bivectors, instead of skew-symmetric endomorphisms.

The usual classification of two-forms in $\mathbb{M}^{1,3}$ (which can be found in e.g. [13, 26]) reduces to two cases with their respective canonical forms, namely

$$\boldsymbol{F} = a\boldsymbol{e} \wedge \boldsymbol{w} + b\boldsymbol{u} \wedge \boldsymbol{v}, \qquad \boldsymbol{F} = \boldsymbol{k} \wedge \boldsymbol{v}, \quad a, b \in \mathbb{R} \tag{9}$$

where $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}$ are spacelike, unit and orthogonal to each other, $\boldsymbol{e}$ is unit and orthogonal to all of them and $\boldsymbol{k}$ is null and orthogonal to $\boldsymbol{v}$. Our main improvement is that we no longer need to distinguish two cases and we are able to cover every case with one single canonical form. The first of the canonical forms in (9) obviously corresponds to a skew-symmetric endomorphism which admits a timelike eigenplane with eigenvalue a and a spacelike eigenplane with eigenvalue b. These endomorphisms correspond to a canonical form (7) in which at least one of the parameters σ, τ is not zero (cf remark 1). From (9) it follows easily that a, b are directly related to the eigenvalues of F, specifically it holds $|a| = \mu_0$ and $|b| = \mu_1$. The second canonical form in (9) corresponds with a skew-symmetric endomorphism that has a null eigenvector orthogonal to a spacelike eigenvector, both with zero eigenvalue, which in our canonical form is $\sigma = \tau = 0$ (cf remark 1). We also remark that our result is valid only for real skew-symmetric endomorphisms, because it relies on lemma 1. For the complex case see [12] where, however, the classification is also done in a case by case basis.

The three dimensional case is always simple (i.e. of rank one) and thus can be written as product of two one-forms, whose causal character will determine the classification. In this paper we have treated this case as a corollary of the four-dimensional one. This approach will be useful in our extension of the classification results to the higher dimensional case [18].

3. Group of invariance of the canonical form

In this section F is a non-zero skew-symmetric endomorphism in a four-dimensional vector space, and $B = \{e_0, e_1, e_2, e_3\}$ is a canonical basis, i.e. one where e_0 is future directed and (7) holds. It is useful to introduce the semi-null basis $\{\ell, k, e_2, e_3\}$ defined by $\ell = e_0 + e_1$, $k = e_0 - e_1$. In this basis the endomorphism F takes the form

$$F(\ell) = \frac{\sigma}{2}e_2 + \frac{\tau}{2}e_3, \qquad F(k) = -2e_2, \qquad F(e_2) = -\ell + \frac{\sigma}{4}k, \qquad F(e_3) = \frac{\tau}{4}k. \tag{10}$$

We are interested in finding the most general orthochronous Lorentz transformation which transforms B into a basis $B' = \{e'_0, e'_1, e'_2, e'_3\}$ in which F takes the same form. In terms of the corresponding semi-null basis $\{\ell', k', e'_2, e'_3\}$ we must impose (10) with primed vectors. We start with the following lemma:

Lemma 2 *Let F be skew-symmetric and $\{\ell, k, e_2, e_3\}$ be a semi-null basis that satisfies*

$$F(k) = -2e_2, \qquad F(e_2) = -\ell + \frac{\sigma}{4}k \tag{11}$$

and

$$\langle F(\ell), F(\ell)\rangle = \frac{\sigma^2 + \tau^2}{4}. \tag{12}$$

Then either the semi-null basis $\{\ell, k, e_2, e_3\}$ or $\{\ell, k, e_2, -e_3\}$ fulfills (10), and both do whenever $\tau = 0$.

Proof Skew-symmetry imposes $F(\ell)$ and $F(e_3)$ to satisfy

$$F(\ell) = \frac{\sigma}{2}e_2 + \frac{q}{2}e_3, \qquad F(e_3) = \frac{q}{4}k', \quad q \in \mathbb{R}.$$

Condition (12) imposes $q^2 = \tau^2$. Thus $q = \pm\tau$. Since reflecting e_3 replaces q by $-q$, either the basis $\{\ell, k, e_2, e_3\}$ or the basis $\{\ell, k, e_2, -e_3\}$ satisfies (10) with $\tau \geqslant 0$ (and both do in case $\tau = 0$). $\square$

Thus, to understand the group of invariance of (10) it suffices to impose (11) and (12) for $\{\ell', k', e'_2\}$. Let us decompose k' in the original basis as

$$k' = Ak + B\ell + c_2 e_2 + c_3 e_3. \tag{13}$$

Observe that $A, B \geqslant 0$ as a consequence of k' being future directed. Let us introduce two vectors e'_2 and ℓ' so that (11) are satisfied, namely

$$e'_2 := -\frac{1}{2}F(k') = \left(A - \frac{B\sigma}{4}\right) e_2 - \frac{B\tau}{4} e_3 + \frac{c_2}{2}\ell - \frac{1}{8}\left(\sigma c_2 + \tau c_3\right) k, \tag{14}$$

$$\ell' := \frac{\sigma}{4}k' - F(e'_2) = \frac{B\left(\sigma^2 + \tau^2\right)}{16} k + A\ell - \frac{1}{4}\left(\sigma c_2 + \tau c_3\right) e_2 + \frac{1}{4}\left(\sigma c_3 - \tau c_2\right) e_3. \tag{15}$$

The conditions of k' being null, future directed and e'_2 spacelike and unit are easily found to be equivalent to

$$-4AB + \|c\|^2 = 0, \quad A, B \geqslant 0, \tag{16}$$

$$A^2 + \frac{\sigma^2 + \tau^2}{16} B^2 + \frac{\sigma}{8}\left(c_2^2 - c_3^2\right) + \frac{\tau}{4} c_2 c_3 = 1, \tag{17}$$

where we have set $\|c\|^2 = c_2^2 + c_3^2$. Under (16) and (17) one easily checks that the conditions $\langle e'_2, k'\rangle = 0$, $\langle e'_2, \ell'\rangle = 0$, $\langle \ell', \ell'\rangle = 0$ and $\langle \ell', k'\rangle = -2$ are all identically satisfied. Thus, $\{\ell', k', e'_2\}$ defines a timelike hyperplane and we can introduce e'_3 as one of its two unit normals. By construction, the semi-null basis $\{\ell', k', e'_2, e'_3\}$ satisfies (11). By lemma 2, this basis or the one defined with the reversed e'_3 will be a canonical basis of F if and only if (12) is satisfied. By skew-symmetry, this condition is equivalent to

$$\langle \ell', F^2(\ell')\rangle + \frac{\sigma^2 + \tau^2}{4} = 0. \tag{18}$$

Directly from (10) we compute

$$F^2(\ell) = -\frac{\sigma}{2}\ell + \frac{\sigma^2 + \tau^2}{8} k, \quad F^2(k) = 2\ell - \frac{\sigma}{2}k,$$
$$F^2(e_2) = -\sigma e_2 - \frac{\tau}{2} e_3, \qquad F^2(e_3) = -\frac{\tau}{2} e_2,$$

from where it follows

$$F^2(\ell') = \frac{1}{2}\left(\frac{(\sigma^2 + \tau^2)B}{4} - \sigma A\right)\ell + \frac{\sigma^2 + \tau^2}{8}\left(A - \frac{1}{4}\sigma B\right) k$$
$$+ \frac{\left(2\sigma^2 + \tau^2\right) c_2 + \sigma\tau c_3}{8} e_2 + \frac{\tau(\sigma c_2 + \tau c_3)}{8} e_3.$$

One easily checks that (18) is identically satisfied when (16) and (17) hold. Thus, it only remains to solve this algebraic system. To that aim, it is convenient to introduce $Q \geqslant 0$ and an angle $\theta \in [0, \frac{\pi}{2}]$ defined by

$$\sigma = Q\cos(2\theta), \qquad \tau = Q\sin(2\theta). \tag{19}$$

When $\sigma^2 + \tau^2 > 0$, $\{Q, \theta\}$ are uniquely defined. When $\sigma = \tau = 0$, then $Q = 0$ and θ can take any value. Define also λ_2, λ_3 by

$$c_2 = 2\lambda_2 \cos\theta - 2\lambda_3 \sin\theta, \qquad c_3 = 2\lambda_2 \sin\theta + 2\lambda_3 \cos\theta.$$

In terms of the new variables, equations (16) and (17) become (with obvious meaning for $\|\lambda\|^2$)

$$AB - \|\lambda\|^2 = 0, \qquad 16A^2 + Q^2B^2 + 8Q\left(\lambda_2^2 - \lambda_3^2\right) - 16 = 0 \quad A, B \geqslant 0.$$

When $Q = 0$, the solution is clearly $A = 1, B = \|\lambda\|^2$, with unrestricted λ_2, λ_3. When $Q > 0$, we may multiply the first equation by Q and find the equivalent problem

$$(4A + QB)^2 = 16(1 + Q\lambda_3^2), \qquad (4A - QB)^2 = 16(1 - Q\lambda_2^2), \quad A, B \geqslant 0.$$

This system is solvable if and only if

$$|\lambda_2| \leqslant \frac{1}{\sqrt{Q}} \tag{20}$$

and the solution is given by

$$A = \frac{1}{2}\left(\sqrt{1 + Q\lambda_3^2} + \epsilon\sqrt{1 - Q\lambda_2^2}\right), \qquad B = \frac{2}{Q}\left(\sqrt{1 + Q\lambda_3^2} - \epsilon\sqrt{1 - Q\lambda_2^2}\right), \tag{21}$$

where $\epsilon = \pm 1$. Observe that the branches $\epsilon = 1$ and $\epsilon = -1$ are connected to each other across the set $|\lambda_2| = 1/\sqrt{Q}$. Note also that the case $Q = 0$ is included as a limit $Q \to 0$ in the branch $\epsilon = 1$ (and then the bound (20) becomes vacuous, in accordance with the unrestricted values of $\{\lambda_2, \lambda_3\}$ when $Q = 0$). We can now write down explicitly the vectors ℓ', k', e_2' defined in (13)–(15). It is useful to introduce the two spacelike, orthogonal and unit vectors

$$u_2 = \cos\theta e_2 + \sin\theta e_3, \qquad u_3 = -\sin\theta e_2 + \cos\theta e_3$$

which simplify the expression to

$$\begin{aligned}
\ell' &= \frac{Q^2}{16}Bk + A\ell + \frac{Q}{2}\left(-\lambda_2 u_2 + \lambda_3 u_3\right), \\
k' &= Ak + B\ell + 2\lambda_2 u_2 + 2\lambda_3 u_3, \\
e_2' &= \left(\lambda_2 \cos\theta - \lambda_3 \sin\theta\right)\ell - \frac{Q}{4}\left(\lambda_2 \cos\theta + \lambda_3 \sin\theta\right)k \\
&\quad + \epsilon \cos\theta\sqrt{1 - Q\lambda_2^2}\, u_2 - \sin\theta\sqrt{1 + Q\lambda_3^2}\, u_3,
\end{aligned}$$

where A, B must be understood as given by (21) (including the limiting case $Q = 0$). The fourth vector e_3' is unit and orthogonal to all of them. The following pair of vectors satisfy these properties (and of course there are no others).

$$\begin{aligned}
e_3' = \widehat{\epsilon}\Bigg(&\left(\lambda_3 \cos\theta + \lambda_2 \sin\theta\right)\ell + \frac{Q}{4}\left(\lambda_3 \cos\theta - \lambda_2 \sin\theta\right)k \\
&+ \epsilon \sin\theta\sqrt{1 - Q\lambda_2^2}\, u_2 + \cos\theta\sqrt{1 + Q\lambda_3^2}\, u_3\Bigg)
\end{aligned} \tag{22}$$

where $\widehat{\epsilon} = \pm 1$. It is also straightforward to check that $F(e'_3) = \widehat{\epsilon}(\tau/4)k'$. Thus, if $\tau \neq 0$, we must choose $\widehat{\epsilon} = 1$ while in the case $\tau = 0$ both signs are possible (in accordance with lemma 2). Summarizing, the most general orthochronous Lorentz transformation that transforms a canonical semi-null basis of F into another one is given by

$$\begin{pmatrix} \ell' \\ k' \\ e'_2 \\ \widehat{\epsilon}e'_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\sqrt{1+Q\lambda_3^2}+\epsilon\sqrt{1-Q\lambda_2^2}\right) & \frac{Q}{8}\left(\sqrt{1+Q\lambda_3^2}-\epsilon\sqrt{1-Q\lambda_2^2}\right) & -Q\lambda_2/2 & Q\lambda_3/2 \\ \frac{2}{Q}\left(\sqrt{1+Q\lambda_3^2}-\epsilon\sqrt{1-Q\lambda_2^2}\right) & \frac{1}{2}\left(\sqrt{1+Q\lambda_3^2}+\epsilon\sqrt{1-Q\lambda_2^2}\right) & 2\lambda_2 & 2\lambda_3 \\ \lambda_2\cos\theta - \lambda_3\sin\theta & -Q(\lambda_2\cos\theta + \lambda_3\sin\theta)/4 & \epsilon\cos\theta\sqrt{1-Q\lambda_2^2} & -\sin\theta\sqrt{1+Q\lambda_3^2} \\ \lambda_3\cos\theta + \lambda_2\sin\theta & Q(\lambda_3\cos\theta - \lambda_2\sin\theta)/4 & \epsilon\sin\theta\sqrt{1-Q\lambda_2^2} & \cos\theta\sqrt{1+Q\lambda_3^2} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix} =: \mathcal{T}_F(\lambda_2, \lambda_3, \epsilon) \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix}$$

where $\widehat{\epsilon} = 1$, unless $\tau = 0$ in which case $\widehat{\epsilon} = \pm 1$. Concerning the global structure of the group, recall that λ_3 takes any value in the real line, while $|\lambda_2| \leqslant 1/\sqrt{Q}$. We have already mentioned that as long as $Q \neq 0$, the two branches $\epsilon = \pm 1$ are connected to each other through the values $|\lambda_2| = 1/\sqrt{Q}$. The topology of the group is therefore $\mathbb{R} \times \mathbb{S}^1$ (hence connected) when $Q \neq 0$ and $\tau \neq 0$. When $Q \neq 0$, $\tau = 0$ the group has two connected components (one corresponding to each value of $\widehat{\epsilon}$) each one with the topology of $\mathbb{R} \times \mathbb{S}^1$. Finally, when $Q = 0$, the group has two connected components (again one for each value of $\widehat{\epsilon}$) and the topology of each component is $\mathbb{R}^2$. By construction all elements of the group (in all cases) are orthochronous Lorentz transformations. Moreover, it is immediate to check that the determinant of $\mathcal{T}_F(\lambda_2, \lambda_3, \epsilon)$ is one for all values of $\lambda_2, \lambda_3, \epsilon$. Thus, all elements with $\widehat{\epsilon} = 1$ preserve orientation, while the elements with $\widehat{\epsilon} = -1$ reverse orientation.

3.1. Invariance group in the three-dimensional case

We have found before that for any non-zero skew-symmetric endomorphism F in $\mathbb{M}^{1,2}$ there exists an orthonormal, future directed basis $B_3 = \{e_0, e_1, e_2\}$ where F takes the canonical form (8). As in the previous case it is natural to ask what is the group of invariance of F, i.e. the most general orthochronous Lorentz transformation which transforms B into a basis where F takes the same form. From F, recall the auxiliary skew-symmetric endomorphism $\hat{F}$ defined on $\mathbb{M}^{1,2} \oplus \mathbb{E}_1$ that was introduced before corollary 1, that is, the endomorphism that acts as $\hat{F}(u + ae_3) = F(u) + 0$, for all $u \in \mathbb{M}^{1,2}$ and $a \in \mathbb{R}$ where $\mathbb{E}_1 = \mathrm{span}\,\{E_3\}$, with E_3 unit. Moreover, the basis $B := \{e_0, e_1, e_2, e_3 = E_3\}$ is canonical for $\hat{F}$ in the sense of (7) and in addition $\tau = 0$. It is clear that there exists a bijection between the set of orthonormal, future directed bases $B'_3 = \{e'_0, e'_1, e'_2\}$ where F takes its canonical form and the set of future directed orthonormal bases B' in $\mathbb{M}^{1,2} \oplus \mathbb{E}_1$ where $\hat{F}$ takes its canonical form and the last element of B' is E_3. Thus, in order to determine the group of invariance of F it suffices to study the subgroup of invariance of $\hat{F}$ which preserves the vector e_3. Since $\tau = 0$ we must impose

$$B = Q\sin(2\theta) = 2Q\cos\theta\ \sin\theta = 0$$

and three separate cases arise: (case 1) when $Q \neq 0, \theta = 0$, (case 2) when $Q = 0$ and (case 3) when $Q \neq 0, \theta = \pi/2$. Equivalently, cases 1, 2 and 3 correspond respectively to $\sigma > 0, \sigma = 0$ and $\sigma < 0$. Recall also that when $Q = 0$ we may choose any value of $\theta \in [0, \pi/2]$ w.l.o.g. We choose $\theta = 0$ in this case. Recall also that the case $Q = 0$ is recovered as a limit $Q \to 0$ after setting $\epsilon = 1$.

We only need to impose the condition $e'_3 = e_3$ in each case. Directly from (22) one finds (we also use that $Q = |\sigma|$)

$$e'_3 = \widehat{\epsilon}\left(\lambda_3\ell + \frac{|\sigma|}{4}\lambda_3 k + \sqrt{1+|\sigma|\lambda_3^2}\, e_3\right), \qquad \text{Case 1}$$

$$e'_3 = \widehat{\epsilon}(\lambda_3\ell + e_3), \qquad \text{Case 2}$$

$$e'_3 = \widehat{\epsilon}\left(\lambda_2\ell - \frac{|\sigma|}{4}\lambda_2 k + \epsilon\sqrt{1-|\sigma|\lambda_2^2}\right) e_3, \qquad \text{Case 3}$$

Thus, cases 1 and 2 require $\widehat{\epsilon} = 1, \lambda_3 = 0$ and in case 3 we must set $\widehat{\epsilon} = \epsilon, \lambda_2 = 0$. Inserting these values in the group of invariance of $\hat{F}$ one finds the most general orthochronous Lorentz transformation that preserves the form of F. We express the result in the canonically associated semi-null bases $\ell = e_0 + e_1, k = e_0 - e_1, e_2 = e_2$. Renaming λ_2, λ_3 as λ, the three cases can be written in the following form

$$\sigma \geqslant 0: \begin{pmatrix} \ell' \\ k' \\ e'_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(1+\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & \frac{|\sigma|}{8}\left(1-\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & -\frac{|\sigma|\lambda}{2} \\ \frac{2}{|\sigma|}\left(1-\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & \frac{1}{2}\left(1+\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & 2\lambda \\ \lambda & -\frac{|\sigma|\lambda}{4} & \epsilon\sqrt{1-|\sigma|\lambda^2} \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \end{pmatrix}$$

$$\sigma < 0: \begin{pmatrix} \ell' \\ k' \\ e'_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\epsilon+\sqrt{1+|\sigma|\lambda^2}\right) & \frac{|\sigma|}{8}\left(\sqrt{1+|\sigma|\lambda^2}-\epsilon\right) & -\frac{|\sigma|\lambda}{2} \\ \frac{2}{|\sigma|}\left(\sqrt{1+|\sigma|\lambda^2}-\epsilon\right) & \frac{1}{2}\left(\sqrt{1+|\sigma|\lambda^2}+\epsilon\right) & -2\lambda \\ -\lambda & -\frac{|\sigma|\lambda}{4} & \sqrt{1+|\sigma|\lambda^2} \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \end{pmatrix}$$

with the understanding that the case $\sigma = 0$ is obtained from the first expression by setting $\epsilon = 1$ and then performing the limit $\sigma \to 0$.

When $\sigma > 0$, the parameter λ is restricted to $|\lambda| \leqslant 1/|\sigma|$ and the two branches $\epsilon = 1$ and $\epsilon = -1$ are connected through $|\lambda| = |\sigma|$. The group is connected and has topology $\mathbb{S}^1$. As an immediate consequence all the elements in the group are not only orthochronous Lorentz transformations (by construction) but also orientation preserving, as they are all connected to the identity. This can also be checked by computing the determinant of its matrix representation, which is one irrespectively of the value of λ and ϵ. When $\sigma = 0$ the parameter λ takes values in the real line and the group has $\mathbb{R}$-topology. Again all its elements are orientation preserving. In fact, in this case the group is simply the set of null rotations preserving ℓ. Finally, in the case $\sigma < 0$, λ also takes values in the real line and the group has two connected components (corresponding to the two values of ϵ). Each component has topology $\mathbb{R}$. The determinant of the matrix representation is now ϵ, so the Lorentz transformations with $\epsilon = 1$ preserve orientation (and define the connected component to the identity) while $\epsilon = -1$ reverse orientation.

4. Generators of the invariance group

Returning to the four dimensional case, the identity element $\boldsymbol{e}$ of the group of invariance corresponds to $\lambda_2 = \lambda_3 = 0$ and $\epsilon = \widehat{\epsilon} = 1$. We may compute the Lie algebra that generates it by taking derivatives of the group transformation with respect to λ_2 and λ_3 respectively and evaluating at $\boldsymbol{e}$. This defines two skew-symmetric endomorphisms

$$h_2 := \left.\frac{\partial \mathcal{T}_F(\lambda_2,\lambda_3,\epsilon)}{\partial \lambda_2}\right|_{\boldsymbol{e}}, \qquad h_3 := \left.\frac{\partial \mathcal{T}_F(\lambda_2,\lambda_3,\epsilon)}{\partial \lambda_3}\right|_{\boldsymbol{e}}.$$

It is immediate to obtain their explicit expression

$$\begin{pmatrix} h_2(\ell) \\ h_2(k) \\ h_2(e_2) \\ h_2(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{Q}{2}\cos\theta & -\frac{Q}{2}\sin\theta \\ 0 & 0 & 2\cos\theta & 2\sin\theta \\ \cos\theta & -\frac{Q}{4}\cos\theta & 0 & 0 \\ \sin\theta & -\frac{Q}{4}\sin\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix},$$

$$\begin{pmatrix} h_3(\ell) \\ h_3(k) \\ h_3(e_2) \\ h_3(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{Q}{2}\sin\theta & \frac{Q}{2}\cos\theta \\ 0 & 0 & -2\sin\theta & 2\cos\theta \\ -\sin\theta & -\frac{Q}{4}\sin\theta & 0 & 0 \\ \cos\theta & \frac{Q}{4}\cos\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix}.$$

Note that any skew-symmetric endomorphism G that commutes with F generates a one-parameter subgroup of Lorentz transformations that leaves the form of F invariant. It follows that this uniparametric group is necessarily a subgroup of the full invariance group of F. Hence G must belong to the Lie algebra generated by h_2 and h_3. Conversely, h_2, h_3 (and any linear combination thereof) defines a skew-symmetric endomorphism that commutes with F. In other words, $\mathcal{C}_F := \mathrm{span}\{h_2, h_3\}$ defines the Lie subalgebra of $so(1,3)$ formed by the elements that commute with F. This Lie subalgebra is called the *centralizer* of F (e.g. [15]) and, as we have just shown, it is two-dimensional for any non-zero F. An easy computation shows that $[h_2, h_3] = 0$, so the centralizer of F is an Abelian Lie algebra. With these properties, it is not difficult to obtain the exponentiated form of the group elements. Define the two C^1 functions $t_\epsilon(s), t_3(s)$ (prime denotes derivative with respect to s)

$$t'_\epsilon = \epsilon\sqrt{1 - Qt_\epsilon^2}, \qquad t_\epsilon(s=0) = 0,$$
$$t'_3 = \sqrt{1 + Qt_3^2}, \qquad t_3(s=0) = 0,$$

and set

$$\mathcal{T}_\epsilon(s) := \begin{pmatrix} \frac{1}{2}\left(1 + t'_\epsilon\right) & \frac{Q}{8}\left(1 - t'_\epsilon\right) & -\frac{Q}{2}\cos\theta\, t_\epsilon & -\frac{Q}{2}\sin\theta\, t_\epsilon \\ \frac{2}{Q}\left(1 - t'_\epsilon\right) & \frac{1}{2}\left(1 + t'_\epsilon\right) & 2\cos\theta\, t_\epsilon & 2\sin\theta\, t_\epsilon \\ \cos\theta\, t_\epsilon & -\frac{Q}{4}\cos\theta\, t_\epsilon & \cos^2\theta\, t'_\epsilon + \sin^2\theta & \sin\theta\cos\theta\left(t'_\epsilon - 1\right) \\ \sin\theta\, t_\epsilon & -\frac{Q}{4}\sin\theta\, t_\epsilon & \sin\theta\cos\theta\left(t'_\epsilon - 1\right) & \sin^2\theta\, t'_\epsilon + \cos^2\theta \end{pmatrix}$$

$$\mathcal{T}_3(s) := \begin{pmatrix} \frac{1}{2}\left(1+t_3'\right) & \frac{Q}{8}\left(t_3'-1\right) & -\frac{Q}{2}\sin\theta\, t_3 & \frac{Q}{2}\cos\theta\, t_3 \\ \frac{2}{Q}\left(t_3'-1\right) & \frac{1}{2}\left(1+t_3'\right) & -2\sin\theta\, t_3 & 2\cos\theta\, t_3 \\ -\sin\theta\, t_3 & -\frac{Q}{4}\sin\theta\, t_3 & \cos^2\theta + t_3'\,\sin^2\theta & \sin\theta\cos\theta\left(1-t_3'\right) \\ \cos\theta\, t_3 & \frac{Q}{4}\cos\theta\, t_3 & \sin\theta\cos\theta\left(1-t_3'\right) & \sin^2\theta + \cos^2\theta\, t_3' \end{pmatrix}$$

(in the right-hand sides t_ϵ, t_ϵ' etc are to be understood evaluated at s). By direct computation one checks that (Id stands for the 4×4 identity matrix)

$$\frac{\mathrm{d}\mathcal{T}_\epsilon}{\mathrm{d}s} = h_2\mathcal{T}_\epsilon, \qquad \mathcal{T}_{\epsilon=1}(s=0) = \mathrm{Id},$$
$$\frac{\mathrm{d}\mathcal{T}_3}{\mathrm{d}s} = h_3\mathcal{T}_3, \qquad \mathcal{T}_3(s=0) = \mathrm{Id},$$
$$\mathcal{T}_F(\lambda_2,\lambda_3,\epsilon)\big|_{\lambda_2=t_\epsilon(s_1),\lambda_3=t_3(s_2)} = \mathcal{T}_\epsilon(s_1)\mathcal{T}_3(s_2) = \mathcal{T}_3(s_2)\mathcal{T}_\epsilon(s_1).$$

This shows in particular that $\mathcal{T}_{\epsilon=1}(s) = \exp(sh_2)$ and $\mathcal{T}_3(s) = \exp(sh_3)$. Observe also that (in agreement with a previous discussion), when $Q\neq 0$ the branch $\mathcal{T}_{\epsilon=-1}$ is connected to the branch $\mathcal{T}_{\epsilon=1}$ because in this case

$$t_{\epsilon=1}(s) = \frac{\sin(\sqrt{Q}s)}{\sqrt{Q}}, \qquad s\in\left[-\frac{\pi}{2\sqrt{Q}},\frac{\pi}{2\sqrt{Q}}\right],$$
$$t_{\epsilon=-1}(s) = -\frac{\sin(\sqrt{Q}s)}{\sqrt{Q}}, \quad s\in\left[-\frac{\pi}{2\sqrt{Q}},\frac{\pi}{2\sqrt{Q}}\right],$$

so that $s=\pm\pi/(2\sqrt{Q})$ in the first branch is smoothly connected to $s=\mp\pi/(2\sqrt{Q})$ in the second branch.

From the matrix representation of h_2 and h_3 it is obvious (the last two columns are linearly dependent) that $\det(h_2)=\det(h_3)=0$ so both h_2, h_3 are simple, i.e. of matrix rank two. Moreover,

$$-\mathrm{tr}\,(h_2\circ h_2) = \mathrm{tr}\,(h_3\circ h_3) = 2Q \tag{23}$$

and $\mathrm{tr}\,(h_2\circ h_3)=0$. Given that F commutes with itself, i.e. $F\in\mathcal{C}_F$, it must be a linear combination of h_2 and h_3. Indeed, it is immediate to check that

$$F = -\cos\theta\, h_2 + \sin\theta\, h_3. \tag{24}$$

This expression suggests that the connection between F and the basis $\{h_2, h_3\}$ is via a duality rotation. To show that this is indeed the case, we define the one-forms $\{\boldsymbol{\ell},\boldsymbol{k},\boldsymbol{e_2},\boldsymbol{e_3}\}$ metrically associated to the semi-null basis $\{\ell,k,e_2,e_3\}$. Also, for any skew-symmetric endomorphism F, we associate the two-form $\boldsymbol{F}$ by the standard relation (6). It is straightforward to find the explicit forms of $\boldsymbol{h_2}$ and $\boldsymbol{h_3}$ to be[1]

$$\boldsymbol{h_2} = \left(\boldsymbol{\ell} - \frac{Q}{4}\boldsymbol{k}\right)\wedge\left(\cos\theta\,\boldsymbol{e_2} + \sin\theta\,\boldsymbol{e_3}\right),$$

[1] Our convention for the exterior product is $\boldsymbol{u}\wedge\boldsymbol{v} := \boldsymbol{u}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{u}$.

$$\boldsymbol{h_3} = \left(\boldsymbol{\ell} + \frac{Q}{4}\boldsymbol{k}\right) \wedge (-\sin\theta\, \boldsymbol{e_2} + \cos\theta\, \boldsymbol{e_3})\,. \tag{25}$$

Duality rotations of a two-form are defined in terms of the Hodge-dual operator, which in turn depends in a choice of orientation in the vector space. To keep the comparison fully general, we let $\kappa = +1$ ($\kappa = -1$) when the orientation in $\mathbb{M}^{1,3}$ is such that the basis $\{\ell, k, e_2, e_3\}$ is positively (negatively) oriented. Equivalently, if $\boldsymbol{\eta}$ is the volume form that defines the orientation, κ is given by

$$\boldsymbol{\eta}(\ell, k, e_2, e_3) = 2\kappa. \tag{26}$$

Let $\boldsymbol{G}^\star$ denote the Hodge dual[2] associated to $\boldsymbol{G}$. It is then immediate to check that

$$\boldsymbol{h_2^\star} = \kappa \boldsymbol{h_3}.$$

Defining $\boldsymbol{f} := -\boldsymbol{h_2}$ and $\mu := -\kappa\theta$, we may rewrite (24) as

$$\boldsymbol{F} = \cos\mu \boldsymbol{f} + \sin\mu \boldsymbol{f}^\star, \tag{27}$$

which indeed shows that $\boldsymbol{F}$ is obtained from the simple form $\boldsymbol{f}$ by a duality rotation of angle μ. Notice that $f_{\alpha\beta}f^{\alpha\beta} = 2Q \geqslant 0$ (by (23)). For later use, we observe that the most general linear combination $\boldsymbol{f} = a_0\boldsymbol{h_2} + b_0\boldsymbol{h_3}$ that defines a simple two-form such that $f_{\alpha\beta}f^{\alpha\beta} \geqslant 0$ and (27) holds for some value of μ is:

$$\begin{aligned} &Q = 0: \ \boldsymbol{f} = -\cos(\theta + \kappa\mu)\boldsymbol{h_2} + \sin(\theta + \kappa\mu)\boldsymbol{h_3}, \quad \mu \in \mathbb{R} \\ &Q > 0: \ \boldsymbol{f} = -\cos(n\pi)\boldsymbol{h_2}, \quad \mu = -\kappa\theta + n\pi, \quad n \in \mathbb{N}. \end{aligned} \tag{28}$$

This can be proved easily from the explicit expressions of $\boldsymbol{h_2}, \boldsymbol{h_3}$ and the fact that they are linearly independent simple two-forms.

One may wonder whether this connection with duality rotations could have been used as the starting point to obtain in an easy and natural way the canonical form of F. We will argue that this alternative approach, although possible, it is far from obvious and cannot be regarded as natural.

We fix a skew-symmetric endomorphism F in a four-dimensional vector space with a Lorentzian metric, and let $\boldsymbol{F}$ be the metrically associated two-form. Define as before $\sigma := -\frac{1}{2}\operatorname{trace}(F^2)$ and $\tau^2 = -4\det(F)$, $\tau > 0$ where the determinant is taken for any matrix representation of F in an orthonormal basis. The invariants σ and τ are directly related to the two algebraic invariants of $\boldsymbol{F}$ as

$$\sigma = \frac{1}{2}F_{\alpha\beta}\boldsymbol{F}^{\alpha\beta}, \qquad \tau = \frac{1}{2}\operatorname{abs}\left(F_{\alpha\beta}\boldsymbol{F}^{\star\alpha\beta}\right). \tag{29}$$

The first one follows trivially from the definition of σ. The second is a well-known algebraic identity that can be found e.g. in [16]. Given F, a duality rotation of angle $-\mu$ defines the two-form $\overset{\mu}{\boldsymbol{F}}$ as [19, 23],

$$\overset{\mu}{\boldsymbol{F}} := \cos\mu\, \boldsymbol{F} - \sin\mu\, \boldsymbol{F}^*. \tag{30}$$

[2] In abstract index notation $\boldsymbol{G}^\star_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\boldsymbol{G}^{\mu\nu}$.

A simple computation shows that $\overset{\mu}{\boldsymbol{F}}$ is simple (i.e. $\overset{\mu}{F}_{\alpha\beta}\overset{\mu}{F}{}^{\star\alpha\beta} = 0$) and satisfies $\overset{\mu}{F}_{\alpha\beta}\overset{\mu}{F}{}^{\star\alpha\beta} \geqslant 0$ if and only if (see [19])

$$\begin{aligned}
&\sigma \ \sin(2\mu) + \widehat{\kappa}\tau \ \cos(2\mu) = 0, \\
&\sigma \ \cos(2\mu) - \widehat{\kappa}\tau \ \sin(2\mu) \geqslant 0,
\end{aligned} \tag{31}$$

where $\widehat{\kappa}$ is the sign defined by $\frac{1}{2}F_{\alpha\beta}F^{\star\alpha\beta} = \widehat{\kappa}\tau$ (when $\tau = 0$, $\widehat{\kappa}$ can take any value $\widehat{\kappa} = \pm 1$). Inserting (19) we find that whenever $Q = 0$ all values of μ solve (31) (which reflects the fact that $\boldsymbol{F}$ is null, and so are all its duality rotated two-forms). When $Q \neq 0$, the solutions of (31) are $\mu = -\widehat{\kappa}\theta + n\pi$, $n \in \mathbb{N}$. Thus, we recover the expression in (28) provided we can ensure that $\widehat{\kappa} = \kappa$. Note that the sign of $F_{\alpha\beta}F^{\star\alpha\beta}$ only depends on F and the choice of orientation. It is a matter of direct checking that F as given in (10) with the choice of orientation where (26) holds satisfies $F_{\alpha\beta}F^{\star\alpha\beta} = 2\kappa\tau$, so that indeed $\widehat{\kappa} = \kappa$ follows (unless $\tau = 0$, of course, in which case $\widehat{\kappa} = \pm 1$).

We can now show how the canonical basis can be constructed from F using a duality rotation approach. Fixed an orientation on the vector space (i.e. a choice of volume form $\boldsymbol{\eta}$, and its associated Hodge dual) define σ and τ as in (29). Let $\widehat{\kappa} \in \{-1, 1\}$ be such that $2\widehat{\kappa} = F_{\alpha\beta}F^{\star\alpha\beta}$ (if $\tau = 0$, we allow any sign for $\widehat{\kappa}$). Introduce θ so that (19) holds with $\theta \in [0, \pi/2]$ (if $\sigma = \tau = 0$ then θ can take any value in this interval). Define then $\mu = -\widehat{\kappa}\theta$ and construct $\overset{\mu}{\boldsymbol{F}}$ by (30). We let $\boldsymbol{h_2} := -\overset{\mu}{\boldsymbol{F}}$. Since this two-form is simple, there exist two linearly independent vectors a, b such that $\boldsymbol{h_2} = \boldsymbol{a} \wedge \boldsymbol{b}$. These vectors are obviously not unique, but certainly at least one of them must be spacelike. It can also be taken unit. We let $E_2 := b$ have this property. Exploiting the freedom $a \to a + sE_2$, $s \in \mathbb{R}$ we may take a perpendicular to E_2. By construction $(h_2)_{\alpha\beta}(h_2)^{\alpha\beta} \geqslant 0$ (recall (31)) which is equivalent to $\langle a, a\rangle \geqslant 0$, i.e. a is spacelike or null. Let $Q \geqslant 0$ be defined by $Q = \langle a, a\rangle$. It is clear that there exists a timelike plane Π containing a and orthogonal to E_2 (this plane is obviously non-unique). Fixed Π, it is easy to show that there exists a future directed a null basis $\{\ell, k\}$ on Π satisfying $\langle \ell, k\rangle = -2$ and such that $\boldsymbol{a} = \boldsymbol{\ell} - (1/4)Q\boldsymbol{k}$. Finally, consider the timelike hyperplane defined by span$\{\ell, k, E_2\}$ and select the unique unit normal E_3 to this hyperplane satisfying the orientation requirement (cf (26))

$$\eta(\ell, k, E_2, E_3) = 2\widehat{\kappa}.$$

So far, from a non-zero F we have constructed a (collection of) semi-null basis $\{\ell, k, E_2, E_3\}$ in quite a natural way. Observe that when $\sigma = \tau = 0$, the angle θ is arbitrary, so the semi-null basis has extra additional freedom in this case. What appears to be hard to guess from this construction is that instead of $\{E_2, E_3\}$ we should introduce $\{e_2, e_3\}$ by means of the θ-dependent rotation (cf (25))

$$E_2 = \cos\theta e_2 + \sin\theta e_3, \qquad E_3 = -\sin\theta e_2 + \cos\theta e_3. \tag{32}$$

It is by using this transformation that the form of F in the basis $\{\ell, k, e_2, e_3\}$ takes a form that depends only on the invariants σ, τ. It is remarkable that the θ-freedom inherent to the case $\sigma = \tau = 0$ (i.e. when F is null) drops out after performing the rotation (32), and we get a canonical form that covers all cases and depends only on σ and τ, irrespectively of which values these invariants may take.

5. Global conformal Killing vectors on the plane

In the following sections we connect our previous results with the Lie algebra of CKV fields of the sphere and the group of motions they generate, i.e. the Möbius group. In our analysis, it is useful to employ the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Although we will rederive some of the results we need here, we refer the reader to [20, 25] for more details about the Möbius transformations on the Riemann sphere. Some of the contents may also be found in other more general references such as [22, 24]. Regarding Lie groups and Lie algebras, most of the results we will employ can be found in introductory level textbooks such as [13], but other references [11, 15] are also appropriate.

Consider the euclidean plane $\mathbb{E}^2 = (\mathbb{R}^2, g_E)$ and select Cartesian coordinates $\{x, y\}$. It is well-known that the set of CKV on $\mathbb{E}^2$ is given by

$$\xi = U(x, y)\partial_x + V(x, y)\partial_y,$$

where U, V satisfy the Cauchy–Riemann conditions $\partial_x U = \partial_y V$, $\partial_y U = -\partial_x V$. These vector fields satisfy

$$\mathcal{L}_\xi g_E = \left(\partial_x U + \partial_y V\right) g_E.$$

Consider the one-point compactification of $\mathbb{E}^2$ into the Riemann sphere $\mathbb{S}^2$. It is also standard that the set of CKV that extend smoothly to $\mathbb{S}^2$ is given by the subset of CKV for which U and V are polynomials of degree at most two. We name them *global conformal Killing vectors* (GCKV). Thus, the set of GCKV is parametrized by six real constants $\{b_x, b_y, \nu, \omega, a_x, a_y\}$ and take the form

$$\begin{aligned}\xi &= \left(b_x + \nu x - \omega y + \frac{1}{2}a_x\left(x^2 - y^2\right) + a_y xy\right)\partial_x \\ &\quad + \left(b_y + \nu y + \omega x + \frac{1}{2}a_y\left(y^2 - x^2\right) + a_x xy\right)\partial_x \\ &= \xi(a_x, b_x, \nu, \omega, b_x, b_y)\end{aligned} \tag{33}$$

It is clear that the use of complex coordinates is advantageous in this context. For reasons that will be clear later, it is convenient for us to introduce the complex coordinate $z = \frac{1}{2}(x - \mathrm{i}y)$. In terms of z, the set of CKV is given by $\xi = f\partial_z + \overline{f}\partial_{\overline{z}}$ (recall that bar denotes complex conjugation) where f is a holomorphic function of z, while U, V are defined by $2f = U - \mathrm{i}V$. The set of GCKV is parametrized by three complex constants $\{\mu_0,\ \mu_1,\ \mu_2\}$ as

$$\xi = \left(\mu_0 + \mu_1 z + \frac{1}{2}\mu_2 z^2\right)\partial_z + \left(\overline{\mu_0} + \overline{\mu_1 z} + \frac{1}{2}\overline{\mu_2 z}^2\right)\partial_{\overline{z}}. \tag{34}$$

The relationship between the two sets of parameters is immediately checked to be (we emphasize that this specific form depends on our choice of complex coordinate z)

$$\mu_0 = \frac{1}{2}\left(b_x - \mathrm{i}b_y\right), \qquad \mu_1 = \nu - \mathrm{i}\omega, \qquad \mu_2 = 2\left(a_x + \mathrm{i}a_y\right). \tag{35}$$

We denote the GCKV with parameters $\mu := (\mu_0,\ \mu_1,\ \mu_2)$ as $\xi_{\{\mu\}}$. We shall need the following lemma concerning orthogonal and commuting GCKV. The result should be known but we did not find an appropriate reference.

Lemma 3 *Let $\xi_{\{\mu\}}, \xi_{\{\sigma\}}$ be GCKV fields on $\mathbb{E}^2$ with corresponding parameters $\mu = \{\mu_0, \mu_1, \ \mu_2\}, \sigma = \{\sigma_0, \sigma_1, \sigma_2\}$. Assume that $\xi_{\{\mu\}}$ is not the zero vector field. Then*

(*a*) *$\xi_{\{\sigma\}}$ is everywhere perpendicular to $\xi_{\{\mu\}}$ if and only if $\sigma = \mathrm{i}r\mu$ with $r \in \mathbb{R}$.*
(*b*) *$\xi_{\{\sigma\}}$ commutes with $\xi_{\{\mu\}}$ if and only if $\sigma = c\mu$ with $c \in \mathbb{C}$.*

Moreover, $\xi_{c\mu}$ has Euclidean norm

$$g_E(\xi_{\{c\mu\}}, \xi_{\{c\mu\}})|_p = |c|^2 g_E(\xi_{\{\mu\}}, \xi_{\{\mu\}})|_p, \quad \forall \ p \in \mathbb{E}^2.$$

Proof Let $f_\mu = \mu_0 + \mu_1 z + \frac{1}{2}\mu_2 z^2$ so that $\xi_{\{\mu\}} = f_\mu \partial_z + \overline{f_\mu}\partial_{\overline{z}}$ and define f_σ correspondingly. The euclidean metric is $g_E = 4 \ \mathrm{d}z \ \mathrm{d}\overline{z}$, so

$$g_E(\xi_{\{\mu\}}, \xi_{\{\mu\}})|_p = 2 \left(f_\mu \overline{f_\sigma} + \overline{f_\mu} f_\sigma \right) |_{z(p)}. \tag{36}$$

The condition of orthogonality is equivalent to $f_\mu \overline{f_\sigma} + \overline{f_\mu} f_\sigma = 0$. This is a polynomial in $\{z, \overline{z}\}$, so its vanishing is equivalent to the vanishing of all its coefficients. Expanding, we find

$$\mu_0\overline{\sigma_0} + \overline{\mu_0}\sigma_0 = 0, \qquad \mu_1\overline{\sigma_1} + \overline{\mu_1}\sigma_1 = 0, \qquad \mu_2\overline{\sigma_2} + \overline{\mu_2}\sigma_2 = 0, \tag{37}$$

$$\mu_1\overline{\sigma_0} + \overline{\mu_0}\sigma_1 = 0, \qquad \mu_2\overline{\sigma_0} + \overline{\mu_0}\sigma_2 = 0, \qquad \mu_2\overline{\sigma_1} + \overline{\mu_1}\sigma_2 = 0. \tag{38}$$

Equations (37) are equivalent to the existence of three real numbers $\{q_1, q_2, q_3\}$ such that $\mu_a\overline{\sigma_a} = \mathrm{i}q_a$, $a = 0, 1, 2$. Multiplying the equations in (38) respectively by $\mu_0\overline{\mu_1}$, $\mu_0\overline{\mu_2}$ and $\mu_1\overline{\mu_2}$ one finds

$$\begin{aligned} &q_0|\mu_1|^2 - q_1|\mu_0|^2 = 0, \qquad q_0|\mu_2|^2 - q_2|\mu_0|^2 = 0, \\ &q_1|\mu_2|^2 - q_2|\mu_1|^2 = 0 \quad \Longleftrightarrow \quad (q_0, q_1, q_2) \times \left(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2\right) = (0, 0, 0), \end{aligned}$$

where $\times$ stands for the standard cross product. Since $(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2) \neq (0, 0, 0)$ (from our assumption that $\xi_{\{\mu\}}$ is not identically zero) there exists a real number r such that $(q_0, q_1, q_2) = -r(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2)$. Thus $\mu_a\overline{\sigma_a} = -\mathrm{i}r|\mu_a|^2$. Fix $a \in \{0, 1, 2\}$. If $\mu_a \neq 0$, it follows that $\overline{\sigma_a} = -\mathrm{i}r\overline{\mu_a}$. If, instead, $\mu_a = 0$ then it follows from (38) (since at least of the μ's is not zero) that $\sigma_a = 0$. In either case we have $\sigma_a = \mathrm{i}r\mu_a$. This proves point (a) in the lemma.

For point (b) we compute the Lie bracket and find

$$\left[\xi_{\{\mu\}}, \xi_{\{\sigma\}}\right] = \left(f_\mu \frac{\mathrm{d}f_\sigma}{\mathrm{d}z} - f_\sigma \frac{\mathrm{d}f_\mu}{\mathrm{d}z} \right) \partial_z + \left(\overline{f_\mu} \frac{\mathrm{d}\overline{f_\sigma}}{\mathrm{d}\overline{z}} - \overline{f_\sigma} \frac{\mathrm{d}\overline{f_\mu}}{\mathrm{d}\overline{z}} \right) \partial_{\overline{z}}.$$

The two vectors commute iff

$$\begin{aligned} f_\mu \frac{\mathrm{d}f_\sigma}{\mathrm{d}z} - f_\sigma \frac{\mathrm{d}f_\mu}{\mathrm{d}z} &= \mu_0\sigma_1 - \mu_1\sigma_0 + (\mu_0\sigma_2 - \mu_2\sigma_0) z + \frac{1}{2}(\mu_1\sigma_2 - \mu_2\sigma_2) z^2 = 0 \\ &\Longleftrightarrow \quad (\sigma_0, \sigma_1, \sigma_2) \propto (\mu_0, \mu_1, \mu_2), \end{aligned}$$

and point (b) is proved. The last claim of the lemma follows from (36) and the linearity $f_{c\mu} = cf_\mu$. □

An immediate corollary of this result is that the set of GCKV that commute with a given GCKV $\xi_{\{\mu\}}$ is two-dimensional and generated by $\xi_{\{\mu\}}$ and $\xi^\perp_{\{\mu\}} := \xi_{\{-\mathrm{i}\mu\}}$.

Recall that a Möbius transformation is a diffeomorphism of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ of the form

$$\begin{aligned} \chi^{\mathbb{A}} : \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} \\ z \mapsto \chi^{\mathbb{A}}(z) &= \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \mathbb{A} := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \alpha\delta - \beta\gamma = 1. \end{aligned} \tag{39}$$

The set of Möbius transformations forms a group under composition, which we denote by Moeb, and the map $\chi : SL(2,\mathbb{C}) \to \text{Moeb}$ defined by $\chi(\mathbb{A}) = \chi^{\mathbb{A}}$ is a group morphism. The kernel of this morphism is $K := \{\mathbb{I}_2, -\mathbb{I}_2\}$ and in fact χ descends to an isomorphism between $PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/K$ and Moeb. In geometric terms, the Möbius group corresponds to the set of orientation-preserving conformal diffeomorphisms of the standard sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$ (recall that a diffeomorphism $\chi := \mathbb{S}^2 \to \mathbb{S}^2$ is conformal if $\chi^\star(g_{\mathbb{S}^2}) = \Omega^2 g_{\mathbb{S}^2}$ for some $\Omega \in C^\infty(\mathbb{S}^2, \mathbb{R}^+)$). The Möbius group thus transforms CKV of $\mathbb{S}^2$ into themselves, and, hence it also transforms global GCKV of $\mathbb{E}^2$ into themselves. In other words, given a GCKV $\xi_{\{\mu\}}$, the vector field $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}})$ is also a GCKV[3]. Let $\mu' := (\mu'_0,\ \mu'_1,\ \mu'_2)$ be the set of parameters of $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}}) =: \xi_{\{\mu'\}}$. A straightforward computation shows that

$$\begin{pmatrix} \mu'_0 \\ \mu'_1 \\ \mu'_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha^2 & -\alpha\beta & \frac{1}{2}\beta^2 \\ -2\alpha\gamma & \alpha\delta + \beta\gamma & -\beta\delta \\ 2\gamma^2 & -2\gamma\delta & \delta^2 \end{pmatrix}}_{:= \mathbb{Q}_{\mathbb{A}}} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{pmatrix}. \tag{40}$$

The determinant of this matrix is one, so $\mathbb{Q}_{\mathbb{A}} \in SL(3,\mathbb{C})$. As a consequence of $\chi^{\mathbb{A}_1} \circ \chi^{\mathbb{A}_2} = \chi^{\mathbb{A}_1 \cdot \mathbb{A}_2}$ (where $\cdot$ denotes product of matrices), it follows that the map $\mathbb{Q} : SL(2,\mathbb{C}) \to SL(3,\mathbb{C})$ defined by $\mathbb{Q}(\mathbb{A}) = \mathbb{Q}_{\mathbb{A}}$ is a morphism of groups, i.e. $\mathbb{Q}_{\mathbb{A}_1} \cdot \mathbb{Q}_{\mathbb{A}_2} = \mathbb{Q}_{\mathbb{A}_1 \cdot \mathbb{A}_2}$. This property can also be confirmed by explicit computation. In particular $\mathbb{Q}$ defines a representation of the group $SL(2,\mathbb{C})$ on $\mathbb{C}^3$. It is easy to show that this representation is actually isomorphic to the adjoint representation. Recall that for matrix Lie group G (i.e. a Lie subgroup of $GL(n,\mathbb{C})$), the adjoint representation Ad takes the explicit form (e.g. [13])

$$\begin{aligned} \text{Ad} \colon G &\to \text{Aut}(\mathfrak{g}) \\ g \mapsto \text{Ad}(g) := \text{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto gXg^{-1} \end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of G and $\text{Aut}(\mathfrak{g})$ is the set of automorphisms of $\mathfrak{g}$. The isomorphism between $\mathbb{Q}$ and Ad is as follows. Let us choose the basis of $sl(2,\mathbb{C})$ given by

$$\mathfrak{w}^0 := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{w}^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathfrak{w}^2 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and define the vector space isomorphism $h : \mathbb{C}^3 \to sl(2,\mathbb{C})$ defined by $h(\mu_0, \mu_1, \mu_2) = \mu_a \mathfrak{w}^a$ $(a, b, \ldots = 0, 1, 2)$. One then checks easily by explicit computation that $h^{-1} \circ \text{Ad}_g \circ h = \mathbb{Q}(g)$, for all $g \in SL(2,\mathbb{C})$.

Recall that the Killing form of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear map on $\mathfrak{g}$ defined by $B(\mathfrak{a}_1, \mathfrak{a}_2) := \text{Tr}\,(\text{ad}(\mathfrak{a}_1) \circ \text{ad}(\mathfrak{a}_2))$ where $\text{ad}(\mathfrak{a})$, $\mathfrak{a} \in \mathfrak{g}$ is the adjoint endomorphism $\text{ad}(\mathfrak{a}) : \mathfrak{g} \to \mathfrak{g}$

[3] Note that $\chi^{\mathbb{A}}$ has singularities as a map from $\mathbb{E}^2$ into $\mathbb{E}^2$, but $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}})$ extends smoothly to all $\mathbb{E}^2$, and in fact to the whole Riemann sphere. Again this is standard and well-understood, so we will abuse the notation and write $\chi^A_\star$ as if the map χ^A were well-defined everywhere on $\mathbb{E}^2$

defined by $\mathrm{ad}(\mathfrak{a})(\mathfrak{b}) := [\mathfrak{a}, \mathfrak{b}]$. The Lie algebra $sl(2, \mathbb{C})$ is semi-simple, so its Killing form is non-degenerate (e.g. [15]). The explicit form in the basis $\{\mathfrak{w}_0, \mathfrak{w}_1, \mathfrak{w}_2\}$ is given by

$$B(\mu_a \mathfrak{w}^a, \sigma_a \mathfrak{w}^a) = 8\,(\mu_1\sigma_1 - \mu_0\sigma_2 - \mu_2\sigma_0)\,.$$

A fundamental property of the Killing form is that it is invariant under automorphisms (see e.g. [27]), so in particular under the adjoint representation $B(\mathrm{Ad}_g(\mathfrak{a}), \mathrm{Ad}_g(\mathfrak{b})) = B(\mathfrak{a}, \mathfrak{b})$ for all $g \in G$. Given $\{\mu\}$ we define two real quantities $\sigma_{\{\mu\}}$, $\tau_{\{\mu\}}$ by

$$\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}} := 2\mu_0\mu_2 - \mu_1^2.$$

As a consequence of the discussion above, the quantities $\sigma_{\{\mu\}}$, $\tau_{\{\mu\}}$ associated to a GCKV $\xi_{\{\mu\}}$ are invariant under Möbius transformations. We have now all necessary ingredients to determine the set of Möbius transformations that transform a GCKV into its canonical form. Before doing so, however, we summarize known results on the relationship between GCKV and skew-symmetric endomorphism in the Minkowski spacetime.

6. GCKV and skew-symmetric endomorphisms

It is well-known that conformal diffeomorphisms on the standard sphere of dimension $n \geqslant 2$, $\mathbb{S}^n$, are in one-to-one correspondence with orthochronous Lorentz transformations in the Minkowski spacetime $\mathbb{M}^{1,n+1}$. The underlying reason (see e.g. [22] or [24]) is that such Lorentz transformations leave invariant the future null cone, and the set of null semi-lines in the cone admits a differentiable structure and a metric that makes it isometric to $\mathbb{S}^n$. The action of the orthochronous Lorentz group on the set of future directed null semi-lines gives rise to a conformal transformation, defining a map that turns out to be one-to-one. This property translates, at the infinitesimal level, to the existence of a one-to-one map between CKV of $\mathbb{S}^n$ and the set of skew-symmetric endomorphisms in $\mathbb{M}^{1,n+1}$. The explicit form of these two maps depends on the choice of isometry between the set of null-semilines and $\mathbb{S}^n$. This freedom amounts, essentially to fixing a future directed orthonormal Lorentz frame $\{e_\alpha\}$ with associated Minkowskian coordinates $\{T, X^i\}$ in $\mathbb{M}^{1,n+1}$ and selecting a unit spacelike direction $u = u^i e_i$ with respect to which one performs a stereographic projection of the sphere $\{T = 1, \sum_{i=1}^{n+1}(X^i)^2 = 1\}$ minus the point $p_u := \{X^i = u^i\}$ onto an n-dimensional spacelike plane Π_u that lies in the hyperplane $\{T = 1\}$, is orthogonal to u and does not contain the point p_u (such a plane is uniquely defined by the signed euclidean distance from Π_u and p_u in the Euclidean plane $\{T = 1\}$). The final choice is a set of Cartesian coordinates in Π_u.

The construction above can also be done using the hyperboloid of timelike unit future vectors $\mathcal{H} \subset \mathbb{M}^{1,n+1}$, whose isometries are the orthochronous Lorentz transformations. The boundary $\partial\mathcal{H}$ of the conformal compactification of the hyperboloid (which represents 'infinity' of $\mathcal{H}$) is a standard sphere, where the action of the Lorentz group can be extended and it turns out to generate conformal transformations. Details of this construction can be found e.g. in appendix A of [17]. As in the other representation, the details of the map depend on how the sphere at infinity is introduced. The way how the explicit construction was carried out in [17] corresponds, in the description above, to choosing the vector $u = -e_1$, the plane $\Pi_u = \{T = 1, X^1 = 1\}$ and Cartesian coordinates in Π_u given by $\{X^2, \ldots, X^{n+1}\}$. With these choices, and restricting to dimension $n = 2$, the explicit map between the set of skew-symmetric endomorphisms $\mathrm{SkewEnd}\left(\mathbb{M}^{1,3}\right)$ and the set of global conformal Killings vectors on $\mathbb{R}^2$ (denoted by $\mathrm{GCKV}(\mathbb{R}^2)$) is

$$
\Psi := \mathrm{SkewEnd}\left(\mathbb{M}^{1,3}\right) \to \mathrm{GCKV}(\mathbb{R}^2)
$$

$$
F = \begin{pmatrix} 0 & -\nu & -a_x + \dfrac{b_x}{2} & -a_y + \dfrac{b_y}{2} \\ -\nu & 0 & -a_x - \dfrac{b_x}{2} & -a_y - \dfrac{b_y}{2} \\ -a_x + \dfrac{b_x}{2} & a_x + \dfrac{b_x}{2} & 0 & -\omega \\ -a_y + \dfrac{b_y}{2} & a_y + \dfrac{b_y}{2} & \omega & 0 \end{pmatrix} \mapsto \xi_F := \xi(b_x, b_y, \nu, \omega, a_x, a_y), \tag{41}
$$

where $F \in \mathrm{Skew}(\mathbb{M}^{1,3})$ is expressed in the orthonormal basis $\{e_\alpha\}$ (specifically $F(e_\nu) = F^\mu{}_\nu e_\mu$ with $F^\mu{}_\nu$ being the row μ, column ν of the matrix above), $\xi(b_x, b_y, \nu, \omega, a_x, a_y)$ is given by (33) and the coordinates of the plane Π_u are renamed as $\{x := X^2, y := X^3\}$.

Given an (active) orthochronous Lorentz transformation $\Lambda(e_\mu) = \Lambda^\nu{}_\mu e_\nu$, we may consider the skew-symmetric endomorphism $F_\Lambda := \Lambda \circ F \circ \Lambda^{-1}$. The construction above guarantees that

$$
\xi_{F_\Lambda} = \Xi^\Lambda_\star(\xi_F),
$$

where Ξ^Λ is the conformal diffeomorphism associated to the Lorentz transformation Λ. Let us restrict from now on to proper (i.e. orthochronous and orientation preserving) Lorentz transformations. Thus, Ξ^Λ is an orientation preserving conformal diffeomorphism, and having fixed the coordinate system $\{x, y\} \in \mathbb{R}^2$, as well as $z = \frac{1}{2}(x - \mathrm{i}y)$, Ξ^Λ is a Möbius transformation. Thus there exists a pair $\pm\mathbb{A} \in SL(2,\mathbb{C})$ such that $\chi^{\pm\mathbb{A}(\Lambda)} = \Xi^\Lambda$. We are interested in determining the explicit form of $\mathbb{A}(\Lambda)$ (actually of its inverse map $\Lambda(\mathbb{A})$). Having also fixed a future directed orthonormal basis $\{e_\alpha\}$, we may represent a proper Lorentz transformation as an element of $SO^\uparrow(1,3)$ (the connected component of the identity of $SO(1,3)$). The aim is, thus, to determine the map $\mathcal{O} : SL(2,\mathbb{C}) \to SO^\uparrow(1,3)$ satisfying $\Xi^{\mathcal{O}(\mathbb{A})} = \chi^{\mathbb{A}}$. Of course, this maps depends on the choices we have made concerning the unit spacelike direction u and plane Π_u to perform the stereographic projection.

As discussed at length in many references, (see e.g. [22], pp 8–24), when the vector u is chosen to be e_z, the plane is selected to be $\{T = 1, X^3 = 0\}$ and the complex coordinate z' in this plane is taken as $z' = X^1 + \mathrm{i}X^2$, the corresponding map $\mathcal{O}'$ is (we parametrize $\mathbb{A}$ is in (39))

$$
\mathcal{O}'(\mathbb{A}) = \frac{1}{2} \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\beta} + \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma}) & \mathrm{i}(\alpha\overline{\beta} - \beta\overline{\alpha} + \gamma\overline{\delta} - \delta\overline{\gamma}) & \alpha\overline{\alpha} - \beta\overline{\beta} + \gamma\overline{\gamma} - \delta\overline{\delta} \\ \alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} + \delta\overline{\beta} & \alpha\overline{\delta} + \beta\overline{\gamma} + \gamma\overline{\beta} + \delta\overline{\alpha} & \mathrm{i}(\alpha\overline{\delta} - \beta\overline{\gamma} + \gamma\overline{\beta} - \delta\overline{\alpha}) & \alpha\overline{\gamma} - \beta\overline{\delta} + \gamma\overline{\alpha} - \delta\overline{\beta} \\ \mathrm{i}(-\alpha\overline{\gamma} - \beta\overline{\delta} + \gamma\overline{\alpha} + \delta\overline{\beta}) & \mathrm{i}(-\alpha\overline{\delta} - \beta\overline{\gamma} + \gamma\overline{\beta} + \delta\overline{\alpha}) & \alpha\overline{\delta} - \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha} & \mathrm{i}(-\alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} - \delta\overline{\beta}) \\ \alpha\overline{\alpha} + \beta\overline{\beta} - \gamma\overline{\gamma} - \delta\overline{\delta} & \alpha\overline{\beta} + \beta\overline{\alpha} - \gamma\overline{\delta} - \delta\overline{\gamma} & \mathrm{i}(\alpha\overline{\beta} - \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) & \alpha\overline{\alpha} - \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} \end{pmatrix}.
$$

We may take advantage of this fact to determine our $\mathcal{O}(\mathbb{A})$. To do that we simply need to relate the action of the Möbius group in the plane $\Pi_u := \{X^1 = 1\}$ (in the coordinate z) with the corresponding action on the plane $\Pi'_u := \{X^3 = 0\}$ in the coordinate z'. At this point we can explain the reason why we have chosen $z = \frac{1}{2}(x - \mathrm{i}y)$. The reason for the factor 2 comes from the fact that the plane Π_u lies at distance 2 from the point of stereographic projection, while the plane Π'_u lies at distance 1 of its corresponding stereographic point. The sign is introduced because the basis $\{-e_1, e_2, e_3\}$ (with respect to which the point u and the coordinates $\{x, y\}$ are defined) has opposite orientation than the basis $\{e_3, e_1, e_2\}$ with respect to which the point u' and the coordinates $\{X^1, X^2\}$ are built. By introducing a minus sign in z we make sure that the transformation ψ of $\mathbb{S}^2$ defined by $\{z(p) = z'(\psi(p))\}$ is orientation preserving (where $z(p)$

and $z'(p)$ stand for the two respective stereographic projections of $\mathbb{S}^2$ onto $\mathbb{C}^2 \cup \{\infty\}$). Now, a straightforward computation shows that an orientation preserving conformal diffeomorphism $\chi : \mathbb{S}^2 \to \mathbb{S}^2$ which in the plane Π_u takes the form

$$z(\chi(p)) = \frac{\alpha z(p) + \beta}{\gamma z(p) + \delta}, \quad \alpha\delta - \beta\gamma = 1,\ p \in \mathbb{S}^2,$$

has the following form in the Π'_u plane

$$z'(\chi(p)) = \frac{\alpha' z'(p) + \beta'}{\gamma' z'(p) + \delta'},$$

where

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} U, \quad U := \frac{1}{2} \begin{pmatrix} 1 - \mathrm{i} & -1 + \mathrm{i} \\ 1 + \mathrm{i} & 1 + \mathrm{i} \end{pmatrix}.$$

Since the map $\mathcal{O}'$ is a morphisms of groups, it follows that the Lorentz transformation $\mathcal{O}(\mathbb{A})$ is given by

$$\mathcal{O}(\mathbb{A}) = \mathcal{O}'(\mathbb{A}') = \mathcal{O}'(U)^{-1}\mathcal{O}'(\mathbb{A})\mathcal{O}'(U)$$

The $SO^{\uparrow}(1,3)$ Lorentz matrix $\mathcal{O}'(U)$ is the rotation

$$\mathcal{O}'(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and we conclude that the Lorentz transformation $\mathcal{O}(\mathbb{A})$ takes the explicit form

$$\mathcal{O}(\mathbb{A}) = \frac{1}{2} \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & -\alpha\overline{\alpha} + \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\beta} + \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma} & \mathrm{i}(-\alpha\overline{\beta} + \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) \\ -\alpha\overline{\alpha} - \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\alpha} - \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} & -\alpha\overline{\beta} - \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma} & \mathrm{i}(\alpha\overline{\beta} - \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) \\ \alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} + \delta\overline{\beta} & -\alpha\overline{\gamma} + \beta\overline{\delta} - \gamma\overline{\alpha} + \delta\overline{\beta} & \alpha\overline{\delta} + \beta\overline{\gamma} + \gamma\overline{\beta} + \delta\overline{\alpha} & \mathrm{i}(-\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha}) \\ \mathrm{i}(\alpha\overline{\gamma} + \beta\overline{\delta} - \gamma\overline{\alpha} - \delta\overline{\beta}) & \mathrm{i}(-\alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} - \delta\overline{\beta}) & \mathrm{i}(\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} - \delta\overline{\alpha}) & \alpha\overline{\delta} - \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha} \end{pmatrix} \tag{42}$$

(to avoid ambiguities, recall that the Lorentz transformation defined by this matrix is $\Lambda(e_\mu) = \Lambda^\nu{}_\mu e_\nu$ with $\Lambda^\nu{}_\mu$ the row ν and column μ).

7. Canonical form of the GCKV

We start with a definition motivated by the canonical form of skew-symmetric endomorphisms discussed in section 2.

Definition 1 Let $\mathbb{E}^2$ be Euclidean space and $\{x, y\}$ a Cartesian coordinate system. A GCKV ξ is called **canonical** with respect to $\{x, y\}$ if it has the form

$$\xi = (\mu_0 + z^2)\partial_z + \left(\overline{\mu_0} + \overline{z}^2\right)\partial_{\overline{z}}, \quad z := \frac{1}{2}(x - \mathrm{i}y),\ \mu_0 \in \mathbb{C}.$$

Equivalently, a GCKV is canonical with respect to $\{x, y\}$ whenever its corresponding form (34) has $\mu_1 = 0$ and $\mu_2 = 2$. We next characterize the class of Möbius transformations $\chi^{\mathbb{A}}$ which send a given GCKV into its canonical form.

Proposition 2 *Let $\{x, y\}$ be a Cartesian coordinate system in $\mathbb{E}^2$. Let ξ be a non-trivial GCKV and define the complex constants $\{\mu_0,\ \mu_1,\ \mu_2\}$ such that $\xi = \xi_{\{\mu\}}$ when expressed in the complex coordinate $z = (x - \mathrm{i}y)/2$ and its complex conjugate. Then $\chi^{\mathbb{A}} \in Moeb$ has the property that $\chi^{\mathbb{A}}_\star(\xi)$ is written in canonical form with respect to $\{x, y\}$ if and only if*

$$\mathbb{A} = \begin{pmatrix} \frac{1}{2}(\delta\mu_2 - \gamma\mu_1) & \frac{1}{2}\delta\mu_1 - \gamma\mu_0 \\ \gamma & \delta \end{pmatrix}, \qquad \frac{1}{2}\delta^2\mu_2 - \gamma\delta\mu_1 + \gamma^2\mu_0 = 1. \tag{43}$$

Moreover, for any such $\mathbb{A}$, it holds

$$\chi^{\mathbb{A}}_\star(\xi) = \left(\frac{1}{4}\left(\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}}\right) + z^2\right)\partial_z + \left(\frac{1}{4}\left(\sigma_{\{\mu\}} + \mathrm{i}\tau_{\{\mu\}}\right) + \bar{z}^2\right)\partial_{\bar{z}}.$$

Proof From (40) and the fact that the canonical form has $\mu'_1 = 0$ and $\mu'_2 = 2$, we need to find the most general $\alpha, \beta, \gamma, \delta$ subject to $\alpha\delta - \beta\gamma = 1$ such that

$$-2\alpha\gamma\mu_0 + (\alpha\delta + \beta\gamma)\,\mu_1 - \beta\delta\mu_2 = 0, \tag{44}$$

$$2\gamma^2\mu_0 - 2\gamma\delta\mu_1 + \delta^2\mu_2 = 2. \tag{45}$$

The first can be written, using the determinant condition $\alpha\delta - \beta\gamma = 1$, as $-2\alpha\gamma\mu_0 + (1 + 2\beta\gamma)\mu_1 - \beta\delta\mu_2 = 0$. Multiplying by δ yields

$$\begin{aligned} 0 &= -2\alpha\delta\gamma\mu_0 + \delta\mu_1 + \beta\left(2\gamma\delta\mu_1 - \delta^2\mu_2\right) = -2\alpha\delta\gamma\mu_0 + \delta\mu_1 + \beta\left(2\gamma^2\mu_0 - 2\right) \\ &= -2\gamma\mu_0 + \delta\mu_1 - 2\beta \quad \Longrightarrow \quad \beta = \frac{1}{2}\delta\mu_1 - \gamma\mu_0, \end{aligned} \tag{46}$$

where in the second equality we used (45) and in the third one we inserted the determinant condition. To determine α we compute

$$\begin{aligned} \alpha\delta = 1 + \beta\gamma = 1 + \frac{1}{2}\gamma\delta\mu_1 - \gamma^2\mu_0 = \frac{1}{2}\delta\left(\delta\mu_2 - \gamma\mu_1\right) \\ \Longrightarrow \quad \delta\left(\alpha + \frac{1}{2}\gamma\mu_1 - \frac{1}{2}\delta\mu_2\right) = 0, \end{aligned}$$

where in the third equality we used (45) to replace $\gamma^2\mu_0$. If $\delta \neq 0$ we conclude that $\alpha = (1/2)(\gamma\mu_1 - \delta\mu_2)$, and the form of $\mathbb{A}$ is necessarily as given in (43). If, on the other hand, $\delta = 0$, then the determinant condition forces $\gamma \neq 0$. Thus, equation (44) gives $-2\alpha\mu'_0 + \beta\mu_1 = 0$, which after using (46) implies $\alpha = -(1/2)\gamma\mu_1$, so (43) also follows. This proves the 'only if' part of the statement. For the 'if' part one simple checks that β and α obtained above indeed satisfy (44) and (45), as soon as γ, δ satisfy the determinant condition given in (43).

The second part of the proposition is immediate form the fact that $2\mu_0\mu_2 - \mu_1^2$ is invariant under (40). Thus, $\chi^{\mathbb{A}}_\star(\xi)$ has μ'_0 satisfying

$$4\mu'_0 = 2\mu'_0\mu'_2 - \mu'^2_1 = 2\mu_0\mu_2 - \mu_1^2 = \sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}}. \tag{47}$$

□

Corollary 2 *The subgroup of $SL(2, \mathbb{C})$ that leaves invariant a GCKV field in canonical form with parameter μ_0 is given by*

$$\mathbb{A}_{\mu_0} = \left\{\begin{pmatrix} \delta & -\gamma\mu_0 \\ \gamma & \delta \end{pmatrix}, \qquad \delta^2 + \mu_0\gamma^2 = 1\right\}.$$

Proof Insert $\mu_1 = 0$ and $\mu_2 = 2$ into (43). $\square$

Corollary 3 *Given any GCKV ξ as in proposition 2, the set of elements $\mathbb{A} \in SL(2,\mathbb{C})$ such that $\chi_\star^{\mathbb{A}}(\xi)$ takes the canonical form is*

$$\mathbb{A}_{\frac{1}{4}(\sigma_{\{\mu\}}-\mathrm{i}\tau_{\{\mu\}})} \cdot \mathbb{A}_0$$

where $\mathbb{A}_0$ is any element of $SL(2,\mathbb{C})$ satisfying (43).

Proof Fix $\mathbb{A}_0$ satisfying (43). Any other element $\mathbb{A}_1$ will satisfy (43) if and only if $\mathbb{A}_1\mathbb{A}_0^{-1}$ leaves invariant the column vector $(\mu'_0, 0, 2)$, $4\mu'_0 := \sigma_{\{\mu\}} - \tau_{\{\mu\}}$, i.e. if and only if $\mathbb{A}_1 \cdot \mathbb{A}_0 \in \mathbb{A}_{\mu'_0}$.

$\square$

Corollary 4 *Let F be a non-zero skew-symmetric endomorphism in $\mathbb{M}^{1,3}$ and let the matrix (F) be defined by $F(e_\mu) = F^\nu{}_\mu e_\nu$ where $\{e_\mu\}$ is an orthonormal basis. Define $\{b_x, b_y, \nu, \omega, a_x, a_y\}$ so that (F) reads as in (41). Define $\mu_0,\ \mu_1,\ \mu_2$ by means of (35) and let $\Lambda := \mathcal{O}(\mathbb{A})$, where $\mathbb{A}$ is any of the matrices defined in proposition 2. Then, in the basis $e'_\nu := \Lambda^\mu{}_\nu e_\mu$, the endomorphism F takes the canonical form (7) with $\sigma - \mathrm{i}\tau = 2\mu_0\mu_2 - \mu_1^2$.*

In proposition 1 we showed the existence of the canonical form of $F \in \mathrm{SkewEnd}\left(\mathbb{M}^{1,3}\right)$, and this motivated the definition 1 of canonical form of GCKVFs. However, it is only in corollary 4 that we have been able to (easily) find the explicit change of basis that takes F to its canonical form. This is possible because we are dealing with low dimensions and the GCKVFs take a very simple expression in complex coordinates of the Riemann sphere, but this is a much more difficult problem in higher dimensions.

We can however easily derive the three-dimensional case as a simple consequence. For that we consider, as usual, the extension $\widehat{F} \in \mathrm{SkewEnd}\left(\mathbb{M}^{1,3}\right)$ of $F \in \mathrm{SkewEnd}\left(\mathbb{M}^{1,2}\right)$ described before corollary 1. In the basis $\{e_0, e_1, e_2, e_3 := E_3\}$, $\widehat{F}$ has $a_y = b_y = \omega = 0$, so the quantities $\mu_0,\ \mu_1,\ \mu_2$ defined in (35) are real. In order to apply corollary 4 to find the change of orthonormal basis $\{e_0, e_1, e_2\}$ that brings F into its canonical form we simply need to impose that $e'_3 = e_3$, which amounts to $\Lambda^0{}_3 = \Lambda^1{}_3 = \Lambda^2{}_3 = 0$ and $\Lambda^0{}_3 = 1$. It is easy to show (recall that α, β are expressed in terms of γ, δ in the matrix $\mathbb{A}$ of corollary 4) that the general solution to the first three equations is $\gamma\bar{\delta} = \bar{\gamma}\delta$. The condition $\Lambda^0{}_3 = 1$ is then

$$\frac{1}{2}\delta\bar{\delta}\mu_2 - \gamma\bar{\delta}\mu_1 + \gamma\bar{\gamma}\mu_0 = 1.$$

Multiplying by δ and using the determinant condition in (43) implies $\delta = \bar{\delta}$, while multiplying by γ gives $\gamma = \bar{\gamma}$, and then $\Lambda^0{}_3 = 1$ is just identical to the determinant condition so no more consequences can be extracted. Thus all parameters $\alpha, \beta, \gamma, \delta$ are real. Summarizing:

Corollary 5 *Let F be a non-zero skew-symmetric endomorphism of $\mathbb{M}^{1,2}$ and the matrix (F) be defined by $F(e_i) = F^j{}_i e_j$ where $\{e_i\}_{i=0,1,2}$ is an orthonormal basis. Define $\mu_0 := (F^1{}_3 - F^2{}_3)/2, \mu_1 := -F^1{}_2, \mu_2 := -(F^1{}_3 + F^2{}_3)$. For any pair of real numbers γ, δ satisfying $\delta^2\mu_2 - 2\gamma\delta\mu_1 + 2\gamma^2\mu_0 = 2$, let $\alpha := (\delta\mu_2 - \gamma\mu_1)/2$ and $\beta := \delta\mu_1/2 - \gamma\mu_0$. Then, in the basis $e'_i := \Lambda^j{}_i e_j$, with*

$$\Lambda := \begin{pmatrix} \frac{1}{2}\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right) & \frac{1}{2}\left(-\alpha^2+\beta^2-\gamma^2+\delta^2\right) & \alpha\beta+\gamma\delta \\ \frac{1}{2}\left(-\alpha^2-\beta^2+\gamma^2+\delta^2\right) & \frac{1}{2}\left(\alpha^2-\beta^2-\gamma^2+\delta^2\right) & -\alpha\beta+\gamma\delta \\ \alpha\gamma+\beta\delta & -\alpha\gamma+\beta\delta & \alpha\delta+\beta\gamma \end{pmatrix},$$

the endomorphism F takes the canonical form (8) *with* $\sigma = 2\mu_0\mu_2 - \mu_1^2$.

8. Adapted coordinates to a GKCV

So far we have explored the action of the Möbius group on a GCKV and have found that for any such vector, there exists a set of transformations that brings it into a canonical form. The perspective so far has been active. We now change the point of view and exploit the previous results to find coordinate systems in (appropriate subsets of) $\mathbb{E}^2$ that rectify a given (and fixed) GKCV ξ.

Consider $\mathbb{E}^2$ and fix a non-trivial GCKV field ξ. Let us select a Cartesian coordinate system $\{x, y\}$ and define, as before $z = (1/2)(x - \mathrm{i}y)$ and $\overline{z} = (1/2)(x + \mathrm{i}y)$. When expressed in the $\{z, \overline{z}\}$ coordinate system ξ will be $\xi = \xi_{\{\mu\}}$ for some triple of complex numbers $\{\mu\} = \{\mu_0,\ \mu_1,\ \mu_2\}$. We now view the Möbius transformation as a change of coordinates. Specifically, given $\alpha, \beta, \gamma, \delta$ complex constants satisfying $\alpha\delta - \beta\gamma = 1$, the quantity

$$\omega = \frac{\alpha z + \beta}{\gamma z + \delta}$$

and its complex conjugate $\overline{\omega}$ define a coordinate system on $\mathbb{R}^2 \setminus \{\gamma z + \delta = 0\}$. The inverse of this coordinate transformation is, obviously,

$$z = \frac{\delta\omega - \beta}{-\gamma\omega + \alpha}. \tag{48}$$

It is well-known that transformations of a manifold can be dually seen as coordinate changes in suitable restricted coordinate patches. We will refer to (48) as a Möbius coordinate change. With this point of view, we may express ξ in the coordinate system $\{\omega, \overline{\omega}\}$ and the duality above implies that ξ takes the form

$$\xi = \left(\mu_0' + \mu_1'\omega + \frac{1}{2}\mu_2'\omega^2\right)\partial_\omega + \left(\overline{\mu_0'} + \overline{\mu_1'}\,\overline{\omega} + \frac{1}{2}\overline{\mu_2'}\,\overline{\omega}^2\right)\partial_{\overline{\omega}},$$

with $\{\mu_0',\ \mu_1',\ \mu_2'\}$ given by (40) (this can also be checked by direct computation).

We may now take $\{\alpha, \beta, \gamma, \delta\}$ so that corresponding matrix $\mathbb{A}$ satisfies (43). It follows that ξ takes the canonical form

$$\xi := \left(\frac{1}{4}\left(\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}}\right) + \omega^2\right)\partial_\omega + \left(\frac{1}{4}\left(\sigma_{\{\mu\}} + \mathrm{i}\tau_{\{\mu\}}\right) + \overline{\omega}^2\right)\partial_{\overline{\omega}}.$$

By lemma 3, the vector $\xi^\perp$ defined by $\xi^\perp := \xi_{\{\mathrm{i}\mu\}}$ is a GCKV orthogonal to ξ everywhere, with the same pointwise norm as ξ and satisfying $[\xi, \xi^\perp] = 0$. In particular ξ and $\xi^\perp$ are linearly independent except at points where both vanish identically. As a consequence, it makes sense to tackle the problem of finding coordinates that rectify ξ by trying to determine a coordinate system $\{v_1, v_2\}$ (on a suitable subset of $\mathbb{R}^2$) such that

$$\xi = \partial_{v_1}, \qquad \xi^\perp = \partial_{v_2}.$$

Assume that we have already transformed into the coordinates $\{\omega, \overline{\omega}\}$ where ξ (and also $\xi^\perp$) take their canonical forms

$$\xi = \left(\frac{1}{4}Q\mathrm{e}^{-2\mathrm{i}\theta} + \omega^2\right)\partial_\omega + \mathrm{c.c}, \qquad \xi^\perp = \left(\frac{\mathrm{i}}{4}Q\mathrm{e}^{-2\mathrm{i}\theta} + \mathrm{i}\omega^2\right)\partial_\omega + \mathrm{c.c} \tag{49}$$

where we have defined the real constants $Q \geqslant 0$ and $\theta \in [0, \pi)$ by

$$\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}} = Q\ \mathrm{e}^{-2\mathrm{i}\theta} \tag{50}$$

and where c.c. stands for complex conjugate of the previous term. We are seeking a coordinate system $\{\zeta, \overline{\zeta}\}$ defined by

$$\zeta := \frac{1}{2}(v_1 + \mathrm{i}v_2)$$

such that

$$\xi - \mathrm{i}\xi^{\perp} = \partial_{\zeta}$$

(this is because $\partial_{\zeta} = \partial_{v_1} - \mathrm{i}\partial_{v_2}$). Since $\xi - \mathrm{i}\xi^{\perp} = 2\left(\frac{1}{4}Q\ \mathrm{e}^{-2\mathrm{i}\theta} + \omega^2\right)\partial_{\omega}$ the coordinate change musty satisfy the ODE

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\omega} = \frac{1}{2\omega^2 + \frac{Q}{2}\ \mathrm{e}^{-2\mathrm{i}\theta}}.$$

This equation can be integrated immediately. The result is

$$\zeta(\omega) = \zeta_0 + \frac{-\mathrm{i}\ \mathrm{e}^{\mathrm{i}\theta}}{2\sqrt{Q}}\ \ln\left(\frac{\omega - \mathrm{i}\frac{\sqrt{Q}}{2}\ \mathrm{e}^{-\mathrm{i}\theta}}{\omega + \mathrm{i}\frac{\sqrt{Q}}{2}\ \mathrm{e}^{-\mathrm{i}\theta}}\right) \quad \Longleftrightarrow$$

$$\omega(\zeta;\zeta_0) = \frac{\mathrm{i}\sqrt{Q}\ \mathrm{e}^{-\mathrm{i}\theta}}{2}\frac{1 + \mathrm{e}^{2\mathrm{i}\sqrt{Q}\mathrm{e}^{-\mathrm{i}\theta}(\zeta-\zeta_0)}}{1 - \mathrm{e}^{2\mathrm{i}\sqrt{Q}\mathrm{e}^{-\mathrm{i}\theta}(\zeta-\zeta_0)}}, \tag{51}$$

where ζ_0 is an arbitrary complex constant. These expressions include the case $Q = 0$ as a limit. Explicitly

$$\zeta - \zeta_0 = -\frac{1}{2\omega} \quad \Longleftrightarrow \quad \omega = -\frac{1}{2(\zeta - \zeta_0)}. \tag{52}$$

Since the logarithm is a multivalued complex function, one needs to be careful concerning the domain and range of this coordinate change. In the $\{\omega, \overline{\omega}\}$ plane, the vector field ξ vanishes at the two points (cf (49)) $\omega = \pm\mathrm{i}\frac{\sqrt{Q}}{2}\ \mathrm{e}^{-\mathrm{i}\theta}$ (which degenerate to the point at the origin when $Q = 0$). It is clear that neither of these points will be covered by the $\{\zeta, \overline{\zeta}\}$ coordinate system. The case $Q = 0$ is very simple because, from (52), it is clear that the $\{\zeta, \overline{\zeta}\}$ coordinate system covers the whole $\{\omega, \overline{\omega}\}$ plane except the origin. Since the point at infinity in the ω-plane is sent to the point ζ_0 in the ζ-plane we conclude that the $\{\zeta, \overline{\zeta}\}$ coordinate covers the whole Riemann sphere except the single point where ξ vanishes.

When $Q \neq 0$, the situation is more interesting. The reason in the multivaluedness of the logarithm. This suggests that the coordinate change may in fact define a larger manifold that covers the original one. In order to discuss this, let is introduce the auxiliary function

$$\mathfrak{z} := \frac{\omega - \mathrm{i}\frac{\sqrt{Q}}{2}\ \mathrm{e}^{-\mathrm{i}\theta}}{\omega + \mathrm{i}\frac{\sqrt{Q}}{2}\ \mathrm{e}^{-\mathrm{i}\theta}}.$$

This is a Möbius transformation, so it maps diffeomorphically $\mathbb{C} \cup \{\infty\}$ onto itself. The two zeroes of ξ are mapped respectively to the origin and infinity in the $\mathfrak{z}$ variable. Since (51) can be written as $\zeta - \zeta_0 = -\mathrm{i}\ \mathrm{e}^{\mathrm{i}\theta}\ \ln(\mathfrak{z})/(2\sqrt{Q})$ and $\ln(\mathfrak{z}) = \ln|\mathfrak{z}| + \mathrm{i}(\theta(\mathfrak{z}) + 2\pi m), \quad m \in \mathbb{N}$, a single value of $\mathfrak{z}$ may be mapped to an infinite number of points depending on the branch on the branch of logarithm one takes. One may decide to restrict the $\{\zeta, \overline{\zeta}\}$-domain to be the band $B := \{\zeta \in \mathbb{C} : \mathrm{Im}(2\mathrm{i}\sqrt{Q}\ \mathrm{e}^{-\mathrm{i}\theta}(\zeta - \zeta_0)) \in (0, 2\pi)\}$ and then the coordinate change $\zeta(\mathfrak{z})$ defines a diffeomorphism between $\mathbb{C} \setminus \{\mathfrak{z} = (r, 0), \quad r \geqslant 0\}$ into B. Let $\partial_1 B$ be the connected component of ∂B defined by $\mathrm{Im}(2\mathrm{i}\sqrt{Q}\ \mathrm{e}^{-\mathrm{i}\theta}(\zeta - \zeta_0)) = 0$ and $\partial_2 B$ the other component $\partial_2 B := \{\mathrm{Im}(2\mathrm{i}\sqrt{Q}\ \mathrm{e}^{-\mathrm{i}\theta}(\zeta - \zeta_0)) = 2\pi\}$, then the semi-line $\{\mathfrak{z} = r\}$, with r real and positive and $\theta(\mathfrak{z}) \in \{0, 2\pi\}$, is mapped to the respective points $\zeta_1(r) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \ln(r)/(2\sqrt{Q}) \in \partial_1 B$ and $\zeta_2(r) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \ln(r)/(2\sqrt{Q}) + \pi\ \mathrm{e}^{\mathrm{i}\theta}/\sqrt{Q} \in \partial_2 B$. This shows that these two boundaries are to identified by means of the translation defined by the shift

$$\zeta_t := \pi\ \mathrm{e}^{\mathrm{i}\theta}/\sqrt{Q}. \tag{53}$$

The topology of the resulting manifold is $\mathbb{R} \times \mathbb{S}^1$. This is in agreement with the fact that ξ vanishes at precisely two points of the Riemann sphere, and the complement of two points on a sphere is indeed a cylinder. The alternative is to let ζ take values in all $\mathbb{C}$ and consider the inverse map

$$\mathfrak{z}(\zeta) := \mathrm{e}^{2\mathrm{i}\sqrt{Q}\mathrm{e}^{-\mathrm{i}\theta}(\zeta - \zeta_0)}.$$

It is clear that this defines an infinite covering of the $\mathfrak{z}$-punctured complex plane $\mathbb{C} \setminus \{0\}$. As described above, the fundamental domain of this covering is the (open) band B limited by the lines (see figure 1, where we have set $\zeta_0 = 0$ for definiteness)

$$\zeta_1(s) = \zeta_0 + \frac{-\mathrm{i}\ \mathrm{e}^{\mathrm{i}\theta} s}{2\sqrt{Q}}, \qquad s \in \mathbb{R},$$

$$\zeta_2(s) = \zeta_0 + \frac{-\mathrm{i}\ \mathrm{e}^{\mathrm{i}\theta} s}{2\sqrt{Q}} + \zeta_t, \quad s \in \mathbb{R}.$$

The ζ-complex plane therefore corresponds to the complete unwrapping of the cylinder, i.e. to its universal covering. In the $\{\zeta, \overline{\zeta}\}$ coordinate system we have

$$\xi = \frac{1}{2}\left(\partial_\zeta + \partial_{\overline{\zeta}}\right), \qquad \xi^\perp = \frac{\mathrm{i}}{2}\left(\partial_\zeta - \partial_{\overline{\zeta}}\right),$$

so ξ points along the real axis and $\xi^\perp$ into the imaginary axis. The angle of the boundaries $\partial_1 B$ (and $\partial_2 B$) with the real axis is $\frac{\pi}{2} + \theta$. For generic values of θ it follows that the integral lines of ξ descend to the quotient $\overline{B}$ (with the boundaries identified as above) as open lines that asymptote to the two points at infinity along the band (as in figure 2). Observe that these two asymptotic values correspond to $\mathfrak{z} = 0$ or $\mathfrak{z} = \infty$, which correspond to the two zeros of ξ. Thus, the integral lines of ξ start asymptotically at one of its zeros and approaches asymptotically the other zero. Along the way, the integral lines circle each zero an infinite number of times (because the projection to the lines parallel to the real axis descend to the quotient in such a way that they intersect the boundaries of B an infinite number of times). The only exception to this behaviour is when $\theta = \frac{\pi}{2}$ or when $\theta = 0$ (recall that by construction $\theta \in [0, \pi)$). In the former case, the integral lines of ξ, never leave the fundamental domain. This means that the curves asymptote to the two zeros of ξ and they never encircle them along the way. The case $\theta = 0$ corresponds to the situation when the projection of the integral lines of ξ define closed curves on $\overline{B}$ with the boundaries identified. This is the situation when the integral curves of ξ in the original $\{\omega, \overline{\omega}\}$ plane are topological circles (which degenerate to points at the zeroes of ξ).

It is interesting to see how the limit $Q = 0$ is recovered in this setting. The translation vector that identifies points in the boundary $\partial_1 B$ with points in the boundary $\partial_2 B$ diverges as $Q \to 0$.

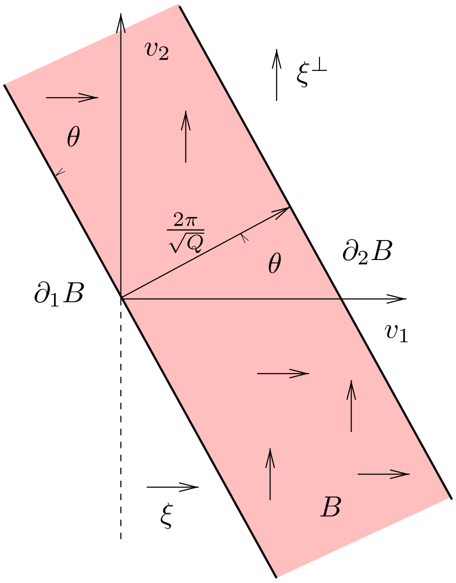

Figure 1. Domain of the complex coordinate $\zeta = \frac{1}{2}(v_1 + \mathrm{i}v_2)$ adapted to $\xi = \partial_{v_1}$ and $\xi^\perp = \partial_{v_2}$. The parameters Q and θ determine the width and tilt of the band respectively. The factor two in the distance between the boundaries (compare (53)) arises because $\zeta = \frac{1}{2}(v_1 + \mathrm{i}v_2)$.

Thus, the band B becomes larger and larger until it covers the whole ζ-plane in the limit. On other words, the ζ-coordinate is no longer a covering of the original ω-coordinate. In the limit, ξ vanishes at only one point in the ω-plane (the origin) which is sent to infinity in the ζ-coordinates. It is by the process of the band B becoming wider and wider that the limits at infinity along the band, which correspond to two points for any non-zero value of Q, merge into a single point when $Q = 0$. The process also explains in which sense the parameter θ, which measures the inclination of the band B becomes irrelevant in the limit $Q = 0$, in agreement with the fact that (50) lets θ take any value when $\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}}$ (and hence also Q) vanishes.

In all the expressions above we have maintained the additive integration constant ζ_0, instead of setting it to zero as the simplest choice. The reason is that ζ_0 can be directly connected with the freedom one has in performing the coordinate change (48) that brings ξ into its canonical form. To understand this we simply note that, from (51) one can check that the following identity holds

$$\omega(\zeta;\zeta_0) = \frac{\cos\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right)\omega(\zeta;0) - \frac{\sqrt{Q}}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\,\sin\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right)}{\frac{2}{\sqrt{Q}}\,\mathrm{e}^{\mathrm{i}\theta}\,\sin\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right)\omega(\zeta;0) + \cos\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right)}.$$

Thus, the relation between $\omega(\zeta;0)$ and $\omega(\zeta;\zeta_0)$ is a Möbius transformation defined by the matrix

$$\begin{pmatrix} \cos\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right) & -\dfrac{\sqrt{Q}}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\sin\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right) \\ \dfrac{2}{\sqrt{Q}}\,\mathrm{e}^{\mathrm{i}\theta}\sin\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right) & \cos\left(\sqrt{Q}\,\mathrm{e}^{-\mathrm{i}\theta}\zeta_0\right) \end{pmatrix}.$$

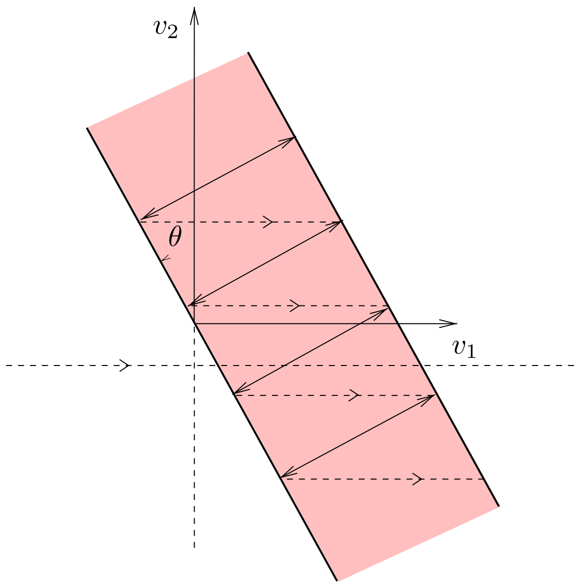

Figure 2. Integral lines of ξ (dashed line). The points joint by arrows are identified by the translation defined by (53).

It is immediate to check that, letting ζ_0 take any value, one runs along the full subgroup $\mathbb{A}_{\frac{1}{4}Q\mathrm{e}^{-2\mathrm{i}\theta}}$ defined in corollary 2. Thus, by corollary 3, the freedom in performing the coordinate change (48) that transforms ξ into its canonical form can be absorbed into the additive constant ζ_0, and vice-versa. Having understood this, we will set $\zeta_0 = 0$ from now on.

So far we have considered ξ without referring to any specific metric. We now endow $\mathbb{R}^2$ coordinated by $\{x, y\}$ (or $\{z, \overline{z}\}$) with the following class of metrics. Let $u := \{u_0, u_1, u_2, u_3\} \in \mathbb{R}^4$, $u \neq 0$, and define

$$
\begin{aligned}
g_u &:= \frac{1}{\Omega_u^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) = \frac{1}{\Omega_u^2} 4\mathrm{d}z d\overline{z},\\
\Omega_u &:= u_0 + u_1 + u_2 x + u_3 y + \frac{1}{4}(u_0 - u_1)(x^2 + y^2)\\
&= u_0(1 + z\overline{z}) + u_1(1 - z\overline{z}) + u_2(z + \overline{z}) + u_3 \mathrm{i}(z - \overline{z}).
\end{aligned}
\tag{54}
$$

The Gauss curvature of g_u is $\kappa_u := u_0^2 - u_1^2 - u_2^2 - u_3^2$. Since $g_{-u} = g_u$, there is a sign freedom in u that we must keep in mind. When $\kappa_u \geqslant 0$, then it must be that $u_0 \neq 0$ and the sign freedom may be fixed by the requirement $u_0 > 0$. However, this is no longer possible when $\kappa_u < 0$.

Observe that $g_{\{u_0 = \frac{1}{2}, u_1 = \frac{1}{2}, u_2 = 0, u_3 = 0\}} = g_E := 4\mathrm{d}z d\overline{z}$. Under a Möbius coordinate change (48), the metric g_u takes the form

$$
\begin{aligned}
g_u &= \frac{1}{\Omega_{u'}^2} 4\mathrm{d}\omega\mathrm{d}\overline{\omega},\\
\Omega_{u'} &= u_0'(1 + \omega\overline{\omega}) + u_1'(1 - \omega\overline{\omega}) + u_2'(\omega + \overline{\omega}) + u_3'\mathrm{i}(\omega - \overline{\omega}),
\end{aligned}
$$

where the constants $u' := \{u_0', u_1', u_2', u_3'\}$ are obtained from $u = \{u_0, u_1, u_2, u_3\}$ by the transformation

$$\epsilon \begin{pmatrix} u'_0 \\ u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} \alpha\overline{\alpha}+\beta\overline{\beta}+\gamma\overline{\gamma}+\delta\overline{\delta} & \alpha\overline{\alpha}-\beta\overline{\beta}+\gamma\overline{\gamma}-\delta\overline{\delta} & -\alpha\overline{\beta}-\beta\overline{\alpha}-\gamma\overline{\delta}-\delta\overline{\gamma} & \mathrm{i}(\alpha\overline{\beta}-\beta\overline{\alpha}+\gamma\overline{\delta}-\delta\overline{\gamma}) \\ \alpha\overline{\alpha}+\beta\overline{\beta}-\gamma\overline{\gamma}-\delta\overline{\delta} & \alpha\overline{\alpha}-\beta\overline{\beta}-\gamma\overline{\gamma}+\delta\overline{\delta} & -\alpha\overline{\beta}-\beta\overline{\alpha}+\gamma\overline{\delta}+\delta\overline{\gamma} & \mathrm{i}(\alpha\overline{\beta}-\beta\overline{\alpha}-\gamma\overline{\delta}+\delta\overline{\gamma}) \\ -(\alpha\overline{\gamma}+\beta\overline{\delta}+\gamma\overline{\alpha}++\delta\overline{\beta}) & -\alpha\overline{\gamma}+\beta\overline{\delta}-\gamma\overline{\alpha}+\delta\overline{\beta} & \alpha\overline{\delta}+\beta\overline{\gamma}+\gamma\overline{\beta}+\delta\overline{\alpha} & \mathrm{i}(-\alpha\overline{\delta}+\beta\overline{\gamma}-\gamma\overline{\beta}+\delta\overline{\alpha}) \\ \mathrm{i}(-\alpha\overline{\gamma}-\beta\overline{\delta}+\gamma\overline{\alpha}+\delta\overline{\beta}) & \mathrm{i}(-\alpha\overline{\gamma}+\beta\overline{\delta}+\gamma\overline{\alpha}-\delta\overline{\beta}) & \mathrm{i}(\alpha\overline{\delta}+\beta\overline{\gamma}-\gamma\overline{\beta}-\delta\overline{\alpha}) & \alpha\overline{\delta}-\beta\overline{\gamma}-\gamma\overline{\beta}+\delta\overline{\alpha} \end{pmatrix}}_{\Lambda_{(\alpha,\beta,\gamma,\delta)}} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where $\epsilon := \pm 1$. This sign reflects the impossibility (in general) of choosing between u and $-u$. One can check that $\Lambda_{(\alpha,\beta,\gamma,\delta)} = \mathcal{O}(\mathbb{A}^{-1})^{\mathrm{T}}$ (42) where $\mathbb{A}$ is as in (39) and $^{\mathrm{T}}$ denotes transpose. It follows that $\Lambda(\alpha,\beta,\gamma,\delta)$ defines a morphism of groups between $SL(2,\mathbb{C})$ and $SO^{\uparrow}(1,3)$ and that u transforms as the components of a covector in the Minkowski spacetime. Also observe that when u is timelike or null (i.e. $\kappa_u \geqslant 0$), the choice $u_0, u'_0 > 0$ selects $\epsilon = 1$.

In order to express the metric in the coordinates $\{v_2, v_2\}$ we need to compute the functions $\omega\overline{\omega}$, $\omega + \overline{\omega}$ and $\mathrm{i}(\omega - \overline{\omega})$ in terms of these variables. For notational simplicity we introduce the auxiliary quantities

$$h_1 := v_1 \cos\theta + v_2 \sin\theta, \qquad h_2 := v_2 \cos\theta - v_1 \sin\theta. \tag{55}$$

From (51) with $\zeta_0 = 0$, a straightforward computation that uses basic trigonometry yields

$$\omega\overline{\omega} = \frac{Q\left(\cosh\left(\sqrt{Q}h_2\right) + \cos\left(\sqrt{Q}h_1\right)\right)}{4\left(\cosh\left(\sqrt{Q}h_2\right) - \cos\left(\sqrt{Q}h_1\right)\right)},$$

$$\omega + \overline{\omega} = \frac{\sqrt{Q}\ \sin\theta \sinh\left(\sqrt{Q}h_2\right) - \sqrt{Q}\ \cos\theta \sin\left(\sqrt{Q}h_1\right)}{\cosh\left(\sqrt{Q}h_2\right) - \cos\left(\sqrt{Q}h_1\right)},$$

$$\mathrm{i}\,(\omega - \overline{\omega}) = -\frac{\sqrt{Q}\ \cos\theta \sinh\left(\sqrt{Q}h_2\right) + \sqrt{Q}\ \sin\theta \sin\left(\sqrt{Q}h_1\right)}{\cosh\left(\sqrt{Q}h_2\right) - \cos\left(\sqrt{Q}h_1\right)}.$$

Since $\mathrm{d}\omega = \frac{\mathrm{d}\omega}{\mathrm{d}\zeta}\mathrm{d}\zeta = 2(\omega^2 + \frac{Q}{4}\ \mathrm{e}^{-2\mathrm{i}\theta})\mathrm{d}\zeta$, determining the line-element $\mathrm{d}\omega\ \mathrm{d}\overline{\omega}$ requires expressing $|\omega^2 + Q/4\mathrm{e}^{-2\mathrm{i}\theta}|^2$ in terms of $\{v_1, v_2\}$. The result is obtained by a direct computation,

$$4\left(\omega^2 + \frac{Q}{4}\ \mathrm{e}^{-2\mathrm{i}\theta}\right)\left(\overline{\omega}^2 + \frac{Q}{4}\ \mathrm{e}^{2\mathrm{i}\theta}\ \right) = \frac{Q^2}{\left(\cosh(\sqrt{Q}h_2) - \cos(\sqrt{Q}h_1)\right)^2}.$$

Let us introduce the functions

$$f_+(v_1, v_2) := \frac{1}{4}\left(\cosh(\sqrt{Q}h_2) + \cos(\sqrt{Q}h_1)\right),$$

$$f_-(v_1, v_2) := \frac{1}{Q}\left(\cosh(\sqrt{Q}h_2) - \cos(\sqrt{Q}h_1)\right),$$

$$f_2(v_1, v_2) := \frac{1}{\sqrt{Q}}\left(\sin\theta\ \sinh(\sqrt{Q}h_2) - \cos\theta\ \sin(\sqrt{Q}h_1)\right),$$

$$f_3(v_1, v_2) := \frac{-1}{\sqrt{Q}}\left(\cos\theta\ \sinh(\sqrt{Q}h_2) + \sin\theta\ \sin(\sqrt{Q}h_1)\right), \tag{56}$$

so that we may express

$$\omega\overline{\omega} = \frac{f_+}{f_-}, \qquad \omega + \overline{\omega} = \frac{f_2}{f_-}, \qquad \mathrm{i}\,(\omega - \overline{\omega}) = \frac{f_3}{f_-}.$$

All these function admit smooth limits at $Q \to 0$, with corresponding expressions

$$f_+(v_1, v_2) = \frac{1}{2},$$
$$f_2(v_1, v_2) = -v_1,$$
$$f_3(v_2, v_2) = -v_2,$$
$$f_-(v_1, v_2) = \frac{1}{2}\left(v_1^2 + v_2^2\right).$$

For $Q \neq 0$, the functions $\{f_+, f_-, f_2, f_3\}$ are all periodic in the variable h_1 with periodicity $2\pi/\sqrt{Q}$. This corresponds to the fact that the ζ-plane is a covering of the ω-plane, with the identification defined by the translation ζ_t.

Thus, in the adapted coordinates $\{v_1, v_2\}$ where $\xi = \partial_{v_1}$ and $\xi^\perp = \partial_{v_2}$, the metric $g_0 := 4\, \mathrm{d}\omega\, \mathrm{d}\overline{\omega}$ takes the form

$$g_0 = \frac{4}{f_-^2} \mathrm{d}\zeta\, \mathrm{d}\overline{\zeta} = \frac{Q^2}{\left(\cosh\left(\sqrt{Q} h_2\right) - \cos\left(\sqrt{Q} h_1\right)\right)^2}\left(\mathrm{d}v_1^2 + \mathrm{d}v_2^2\right).$$

Hence, the metric g_u becomes

$$\begin{aligned} g_u &= \frac{1}{\left((u_0' - u_1') f_+ + (u_0' + u_1') f_- + u_2' f_2 + u_3' f_3\right)^2}\left(\mathrm{d}v_1^2 + \mathrm{d}v_2^2\right) \\ &:= \frac{1}{\widehat{\Omega}^2(v_1, v_2)}\left(\mathrm{d}v_1^2 + \mathrm{d}v_2^2\right). \end{aligned} \tag{57}$$

We may now summarize the results obtained so far concerning GCKV.

Theorem 1 *Let $\mathbb{E}_2$ be the euclidean plane and $\{x, y\}$ be Cartesian coordinates. Let ξ be a GCKV in this space and define the complex constants $\{\mu_0, \mu_1, \mu_2\}$ by means of the expression of ξ given by (34) in the complex coordinates $z = \frac{1}{2}(x - \mathrm{i}y)$, $\overline{z} = \frac{1}{2}(x - \mathrm{i}y)$. Define*

$$\alpha = \frac{1}{2}\left(\delta\mu_2 - \gamma\mu_1\right), \qquad \beta = \frac{1}{2}\delta\mu_1 - \gamma\mu_0,$$

where γ and δ are any pair of complex constants satisfying

$$\frac{1}{2}\delta^2\mu_2 - \gamma\delta\mu_1 + \gamma^2\mu_0 = 1.$$

Then ξ takes its canonical form (cf proposition 2)

$$\xi = \left(\mu_0' + \omega^2\right)\partial_\omega + \left(\overline{\mu_0'} + \overline{\omega}^2\right)\partial_{\overline{\omega}}, \qquad 4\mu_0' := 2\mu_0\mu_2 - \mu_1^2,$$

in the coordinate system $\{\omega, \overline{\omega}\}$ defined by $\omega = (\alpha z + \beta)/(\gamma z + \delta)$. Any other coordinate system $\{\omega', \overline{\omega}'\}$ where ξ is in canonical form is related to $\{\omega, \overline{\omega}\}$ by (cf corollary 2)

$$\omega' = \frac{\delta'\omega - \gamma'\mu_0'}{\gamma'\omega + \delta'}, \qquad \delta'^2 + \mu_0'\gamma'^2 = 1.$$

In addition, the real coordinates $\{v_1, v_2\}$ defined by $\zeta := v_1 + \mathrm{i}v_2$ together with (51) and $4\mu_0' := \sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}} = Q\mathrm{e}^{-2\mathrm{i}\theta}$ are adapted to ξ and $\xi^\perp := \xi_{\{\mathrm{i}\mu\}}$ (cf lemma 3), namely $\xi = \partial_{v_1}$ and $\xi^\perp = \partial_{v_2}$. Moreover, the class of metrics (54) is written in adapted coordinates as (57).

We mentioned above that the freedom in the coordinate change that brings ξ into its canonical form can be translated into the freedom of a constant shift in the coordinates $\{v_1, v_2\}$. Given $\{\tilde{v}_1, \tilde{v}_2\}$ let $\tilde{h}_1$ and $\tilde{h}_2$ by defined exactly by the same expression as (55) but with $\{v_1, v_2\}$ replaced by $\{\tilde{v}_1, \tilde{v}_2\}$. Similarly, we introduce four functions $\left\{\tilde{f}_+(\tilde{v}_1, \tilde{v}_2), \tilde{f}_-(\tilde{v}_1, \tilde{v}_2), \tilde{f}_2(\tilde{v}_1, \tilde{v}_2), \tilde{f}_3(\tilde{v}_1, \tilde{v}_2)\right\}$ by the same definition as (56), with $\{h_1, h_2\}$ replaced by $\{\tilde{h}_1, \tilde{h}_2\}$. Let us now consider the coordinate change

$$\begin{cases} v_1 = \tilde{v}_1 - \cos\theta\,\ell_1 + \sin\theta\ \ell_2 \\ v_2 = \tilde{v}_2 - \sin\theta\,\ell_1 - \cos\theta\,\ell_2 \end{cases} \tag{58}$$

where ℓ_1 and ℓ_2 are constants. Then $h_1 = \tilde{h}_1 - \ell_1$ and $h_2 = \widetilde{h}_2 - \ell_2$ and we may relate the functions $\{f\}$ written in terms of $\{\tilde{v}_1, \tilde{v}_2\}$ with the functions $\{\tilde{f}\}$. The result is

$$\begin{pmatrix} 2f_+ \\ 2f_- \\ f_2 \\ f_3 \end{pmatrix}_{\tilde{v}_1,\tilde{v}_2} = \begin{pmatrix} \frac{1}{2}(\mathrm{Coh}+\mathrm{Co}) & \frac{Q}{8}(\mathrm{Coh}-\mathrm{Co}) & -\frac{\sqrt{Q}}{2}\mathrm{Si} & \frac{\sqrt{Q}}{2}\mathrm{Sih} \\ \frac{2}{Q}(\mathrm{Coh}-\mathrm{Co}) & \frac{1}{2}(\mathrm{Coh}+\mathrm{Co}) & \frac{2}{\sqrt{Q}}\mathrm{Si} & \frac{2}{\sqrt{Q}}\mathrm{Sih} \\ \frac{1}{\sqrt{Q}}(\cos\theta\ \mathrm{Si}-\sin\theta\ \mathrm{Sih}) & -\frac{\sqrt{Q}}{4}(\cos\theta\ \mathrm{Si}+\sin\theta\ \mathrm{Sih}) & \cos\theta\ \mathrm{Co} & -\sin\theta\ \mathrm{Coh} \\ \frac{1}{\sqrt{Q}}(\cos\theta\ \mathrm{Sih}+\sin\theta\ \mathrm{Si}) & \frac{\sqrt{Q}}{4}(\cos\theta\ \mathrm{Sih}-\sin\theta\ \mathrm{Si}) & \sin\theta\ \mathrm{Co} & \cos\theta\ \mathrm{Coh} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2\tilde{f}_+ \\ 2\tilde{f}_- \\ \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} := W(\ell_1, \ell_2) \begin{pmatrix} 2\tilde{f}_+ \\ 2\tilde{f}_- \\ \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \tag{59}$$

where for notational simplicity we have introduced $\mathrm{Co} = \cos(\sqrt{Q}\ell_1), \mathrm{Coh} = \cosh(\sqrt{Q}\ell_2)$, $\mathrm{Si} = \sin(\sqrt{Q}\ell_1), \mathrm{Sih} = \sinh(\sqrt{Q}\ell_2)$. If we compare $W(\ell_1, \ell_2)$ and $\mathcal{T}(\lambda_2, \lambda_3, \epsilon)$ we see that the matrices are identical after setting

$$\lambda_2 = \frac{1}{\sqrt{Q}}\ \sin(\sqrt{Q}\ell_1), \qquad \lambda_3 = \frac{1}{\sqrt{Q}}\ \sinh(\sqrt{Q}\ell_2), \qquad \epsilon\sqrt{1 - Q\lambda_2^2} = \cos(\sqrt{Q}\ell_1). \tag{60}$$

Of course this does not happen by chance. We have seen before that the shift in ζ corresponds to the subgroup of Möbius transformation that leaves the canonical form of ξ invariant. By the relationship between GCKV and skew-symmetric endomorphism in $\mathbb{M}^{1,3}$ described in section 6, this Möbius subgroup corresponds to the set of orthochronous Lorentz transformations that leave the skew-symmetric endomorphism invariant, and this is precisely the group $\{\mathcal{T}(\lambda_2, \lambda_3, \epsilon)\}$. With the choice we have made of the shift constants (58), the relationship between the parameters $\{\ell_1, \ell_2\}$ and $\{\lambda_2, \lambda_3\}$ take the remarkably simple form given by (60). Note that the map $(\ell_1, \ell_2) \to (\lambda_2, \lambda_3, \epsilon)$ is again a covering. If we let ℓ_2 be periodic with periodicity $\frac{2\pi}{\sqrt{Q}}$, the map is a bijection. Observe that, to make the comparison work, we have inserted a factor 2 in front of $f_\pm$ in the column vector (59). The reason is easy to understand. The constants $\{u'_0, u'_1, u'_2, u'_3\}$ in the conformal factor $\widehat{\Omega}$ in the metric g_u define a Lorentz covector of length $-u'^2_0 + u'^2_1 + u'^2_2 + u'^2_3 = -(u'_0 + u'_1)(u'_0 - u'_1) + u'^2_2 + u'^2_3$. This means that, viewed as vectors in a Lorentz space, the basis $\{f_+, f_-, f_2, f_3\}$ is semi-null, but with scalar product $\langle f_+, f_- \rangle = \frac{1}{2}$ However, the transformation law $\mathcal{T}(\lambda_2, \lambda_3, \epsilon)$ was written in a semi-null basis $\{\ell, k, e_2, e_3\}$ with normalization $\langle \ell, k \rangle = -2$, which is precisely the normalization of the basis $\{2f_+, 2f_-, f_2, f_3\}$.

Having obtained the transformation law for $\{f_+, f_-, f_2, f_3\}$ it follows immediately that under the coordinate transformation (58), the metric g_u becomes

$$g_u = \frac{1}{\left((\tilde{u}_0 - \tilde{u}_1)\tilde{f}_+ + (\tilde{u}_0 + \tilde{u}_1)\tilde{f}_- + \tilde{u}_2\tilde{f}_1 + \tilde{u}_3\tilde{f}_2\right)^2}\left(\mathrm{d}\tilde{v}_1^2 + \mathrm{d}\tilde{v}_2^2\right)$$

where the constants $\{\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ are given by

$$\begin{pmatrix} \frac{1}{2}(\tilde{u}_0 - \tilde{u}_1) \\ \frac{1}{2}(\tilde{u}_0 + \tilde{u}_1) \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} = \epsilon (W(\ell_1, \ell_2))^{\mathrm{T}} \begin{pmatrix} \frac{1}{2}\left(u'_0 - u'_1\right) \\ \frac{1}{2}\left(u'_0 + u'_1\right) \\ u'_2 \\ u'_3 \end{pmatrix}$$

(the reason for the sign ϵ is the same as discussed before).

9. Applications

9.1. Killing vectors of g_u

Our aim is to determine under which conditions ξ is a Killing vector of the metric g_u. We will address the question by analyzing the situation in the adapted coordinates. Since $\xi = \partial_{v_1}$, ξ will be a Killing vector of g_u if and only if the function $\widehat{\Omega}$ satisfies $\partial_{v_1}\widehat{\Omega} = 0$. It is straightforward to check that

$$\begin{aligned}
\partial_{v_1} f_+ &= \frac{Q}{4}\left(\cos(2\theta) f_2 + \sin(2\theta) f_3\right), \\
\partial_{v_1} f_- &= -f_2, \\
\partial_{v_1} f_2 &= -2f_+ + \frac{Q}{2}\cos(2\theta) f_-, \\
\partial_{v_1} f_3 &= \frac{Q}{2}\sin(2\theta) f_-,
\end{aligned}$$

which imply

$$\begin{aligned}
\partial_{v_1}\widehat{\Omega} = {} & -2u'_2 f_+ + \frac{Q}{2}\left(\cos(2\theta)u'_2 + \sin(2\theta)u'_3\right) f_- \\
& + \left(\frac{Q}{2}\cos(2\theta)u_- - 2u'_+\right) f_2 + \frac{Q}{2}\sin(2\theta)u'_- f_3,
\end{aligned}$$

where we have set $u'_\pm := \frac{1}{2}(u'_0 \pm u'_1)$. The functions $\{f_+, f_-, f_2, f_3\}$ are linearly independent, so this derivative will vanish if and only if each coefficient vanishes. If $Q\sin(2\theta) \neq 0$, it is immediate that the only solution is $u'_+ = u'_- = u'_2 = u'_3 = 0$, which is not possible for a metric g_u. Thus, a necessary condition for ξ to be a Killing vector of (any) g_u is that the invariant (see (50)) $\sigma_{\{\mu\}} - \mathrm{i}\tau_{\{\mu\}}$ be real (i.e. $\tau_{\{\mu\}} = 0$). When $Q \neq 0$, the condition $\sin(2\theta) = 0$ is $\theta \in \{0, \frac{\pi}{2}\}$ (recall that $\theta \in [0, \pi)$ by construction). To cover all cases at once we set $\cos\theta = \hat{\epsilon}$ and $\sin\theta = 1 - \hat{\epsilon}$, with $\hat{\epsilon}^2 = \hat{\epsilon}$. Then $\cos(2\theta) = 2\hat{\epsilon} - 1$ (this choice is also valid when $Q = 0$ because θ can be fixed to any value). Then

$$\partial_{v_1}\widehat{\Omega}=0 \quad\Longleftrightarrow\quad (u'_-,u'_+,u'_2,u'_3)=s_1\underbrace{\left(1,\frac{Q}{4}(2\hat{\epsilon}-1),0,0\right)}_{w_1}+s_2\underbrace{(0,0,0,1)}_{w_2},\qquad s_1,s_2\in\mathbb{R}.$$

The Lorentzian norm of this vector is $-4u'_+u'_-+u'^2_2+u'^2_3=-(2\hat{\epsilon}-1)Qs_1^2+s_2^2$. Under the constant shift given by ℓ_1,ℓ_2, the two-dimensional vector space spanned by w_1 and w_2 remains invariant, and the vector $s_1w_1+s_2w_2$ transforms to $\tilde{s}_1w_2+\tilde{s}_2w_3$ with

$$\begin{pmatrix}\tilde{s}_1\\ \tilde{s}_2\end{pmatrix}=\epsilon\begin{pmatrix}\hat{\epsilon}\ \cosh(\sqrt{Q}\ell_2)+\cos(\sqrt{Q}\ell_1)(1-\hat{\epsilon}) & \frac{1}{\sqrt{Q}}\left(\sinh(\sqrt{Q}\ell_2)\hat{\epsilon}+\sin(\sqrt{Q}\ell_1)(1-\hat{\epsilon})\right)\\ \sqrt{Q}\left(\sinh(\sqrt{Q}\ell_2)\hat{\epsilon}-\sin(\sqrt{Q}\ell_1)(1-\hat{\epsilon})\right) & \hat{\epsilon}\ \cosh(\sqrt{Q}\ell_2)+\cos(\sqrt{Q}\ell_1)(1-\hat{\epsilon})\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}.$$

This transformation leaves the norm $-(2\hat{\epsilon}-1)Qs_1^2+s_2^2$ invariant (as it must) and defines a group which is one-dimensional when $Q\neq 0$ and two-dimensional when $Q=0$. Thus, when transforming the vector u into the original coordinate system $\{z,\bar{z}\}$ we may ignore the action of the invariance group that leaves the canonical form of ξ invariant provided we let u take all non-zero values in the vector space span $\{w_1,w_2\}$. We may summarize the result in the following theorem.

Theorem 2 *Given a non-identically zero GCKV ξ in two-dimensional Euclidean space and let $\{\mu\}:=\{\mu_0,\ \mu_1,\ \mu_2\}$ be the set of parameters such that $\xi=\xi_{\{\mu\}}$ in the coordinate system $\{z,\bar{z}\}$. Let $U\subset\mathbb{R}^4\setminus\{0\}$ be defined by the property that for all $u\in U$, ξ is a Killing vector of the metric g_u (defined in (54)). Then*

- *If $2\mu_0\mu_2-\mu_1^2\notin\mathbb{R}$ then $U=\emptyset$.*
- *If $2\mu_0\mu_2-\mu_1^2\in\mathbb{R}$, let δ,γ be any pair of complex numbers satisfying*

$$\frac{1}{2}\delta^2\mu_2-\gamma\delta\mu_1+\gamma^2\mu_0=1$$

and set $\alpha=\frac{1}{2}(\delta\mu_2-\gamma\mu_1)$ and $\beta=\frac{1}{2}\delta\mu_1-\gamma\mu_0$. Then $u\in U$ if and only if

$$\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=\mathcal{O}(\mathbb{A})^{\mathrm{T}}\begin{pmatrix}s_1\left(\frac{1}{4}(2\mu_0\mu_2-\mu_1^2)+1\right)\\ s_1\left(\frac{1}{4}(2\mu_0\mu_2-\mu_1^2)-1\right)\\ 0\\ s_2\end{pmatrix},$$

where $(s_1,s_2)\in\mathbb{R}^2\setminus\{0\}$, $\mathbb{A}$ is the matrix (39) and $\mathcal{O}(\mathbb{A})$ was defined in (42).

Moreover, such g_u has constant curvature κ_u given by

$$\kappa_u=s_1^2(2\mu_0\mu_2-\mu_1^2)-s_2^2.$$

Proof We only need to check that $w_1=(1,\frac{1}{4}(2\mu_0\mu_2-\mu_1^2),0,0)$, This is an immediate consequence of the definitions (50) and (47), which in the case $\cos\theta=\hat{\epsilon}$ and $\sin\theta=1-\hat{\epsilon}$ imply

$$Q\,(2\hat{\epsilon}-1)=2\mu_0\mu_2-\mu_1^2.$$

□

One may wonder why this problem has no been addressed in the original coordinate system $\{z, \overline{z}\}$. The Lie derivative of a metric $g_\Psi := 4\Psi^{-2}\mathrm{d}z\ \mathrm{d}\overline{z}$ along $\xi_{\{\mu\}}$ (given by (34)) is

$$\pounds_{\xi_{\{\mu\}}} g_\Psi = \left(-2\xi_{\{\mu\}}(\Psi) + \Psi\,(\mu_1 + \overline{\mu_1} + \mu_2 z + \overline{\mu_2 z})\right) g_\Psi.$$

Thus $\xi_{\{\mu\}}$ is a Killing vector of g_u if and only if

$$-2\xi_{\{\mu\}}(\Omega_u) + \Omega_u\,(\mu_1 + \overline{\mu_1} + \mu_2 z + \overline{\mu_2 z}) = 0.$$

The computation gives a polynomial in $\{z, \overline{z}\}$ of degree two. Equating each coefficient to zero, one finds that the conditions that need to be satisfied can be written in the form

$$\begin{pmatrix} 0 & -\nu & -a_x + \dfrac{b_x}{2} & -a_y + \dfrac{b_y}{2} \\ -\nu & 0 & -a_x - \dfrac{b_x}{2} & -a_y - \dfrac{b_y}{2} \\ -a_x + \dfrac{b_x}{2} & a_x + \dfrac{b_x}{2} & 0 & -\omega \\ -a_y + \dfrac{b_y}{2} & a_y + \dfrac{b_y}{2} & \omega & 0 \end{pmatrix} \begin{pmatrix} -u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{61}$$

where we have expressed $\{\mu\}$ in terms of its real and imaginary parts by means of (35). Recalling the relationship between GCKV ξ and skew-symmetric endomorphisms F_ξ we conclude that $\xi_{\{\mu\}}$ is a Killing vector of g_u if and only if the non-zero Lorentz vector $(-u_0, u_1, u_2, u_3)$ lies in the kernel of F_ξ (observe that this vector is obtained from the covector u by raising indices with the Minkowski metric). Being skew-symmetric and not identically zero, F_ξ can only have rank two or four, so in order to admit a non-trivial kernel, the rank must be two. This corresponds to the condition $\tau_{\{\mu\}} = 0 \Leftrightarrow \mathrm{Im}(2\mu_0\mu_2 - \mu_1^2) = 0$. So, the kernel is two-dimensional, which recovers the statement in theorem 2 that the set $U \cup \{0\}$ is a two-dimensional vector space. Thus, the problem becomes geometrically very neat in the original coordinate system. However, in theorem 2 we have been able to determine explicitly the vector subspace $U \cup \{0\}$ (equivalently the kernel of F_ξ, after index raising) in a way that covers all cases at once. It is not so clear how to achieve the same by a direct attempt of solving (61) in such a way that the solution covers all possible values of $\{b_x, b_y, \nu, \omega, a_x, a_y\}$ under the restriction $b_x a_y - b_y a_x + \nu\omega = 0$ (namely $\mathrm{Im}(2\mu_0\mu_2 - \mu_1^2) = 0$).

The issue addressed in theorem 2 is to determine for which metrics g_u a given GCKV is Killing. A complementary problem is to fix g_u and determine all GCKV which are Killings of g_u. This problem may be approached in the language of skew-symmetric endomorphisms. A skew-symmetric endomorphism F in $\mathbb{M}^{1,3}$ of rank two is necessarily of the form $F = q_1 \otimes \boldsymbol{q_2} - q_2 \otimes \boldsymbol{q_1}$ where q_1 and q_2 are linearly independent Lorentz vectors and boldface denote the metrically related one-form. A vector u lies in the kernel of F if and only if it is orthogonal to q_1 and q_2. Thus, the set of Killing vectors of g_u is obtained from all skew-symmetric endomorphisms

$$F_{u^\perp} := \{F = q_1 \otimes \boldsymbol{q_2} - q_2 \otimes \boldsymbol{q_1}; \quad \mathrm{span}\{q_1, q_2\} = u^\perp\}.$$

where $u^\perp$ stands for the set of vectors in the kernel of the covector (u_0, u_1, u_2, u_3). We do not attempt to find an explicitly parametrization of all Killing vectors of g_u that covers at once all possible choices of u (this problem does not appear to be simple either in terms of endomorphisms, or by using canonical forms of ξ).

9.2. Transverse and traceless and Lie constant tensors on $\mathbb{E}^2$

Transverse and traceless (TT) symmetric two-covariant tensors, namely, tensors $D_{\alpha\beta} = D_{\beta\alpha}$ satisfying (indices are raised with a metric g and ∇ is the corresponding Levi-Civita connection)

$$\nabla_\alpha D^{\alpha\beta} = 0 \quad \text{(transverse)}, \qquad D^\alpha{}_\alpha = 0 \quad \text{(traceless)}$$

play a prominent role in General Relativity, in several circumstances. For example, they are fundamental for the construction of initial data in spacelike slices with prescribed regularity at spacelike infinity [5] or black hole initial data [2]. Another example is the free data at null infinity for Λ-vacuum spacetimes with positive cosmological constant (see the original work [7] or more modern reviews [8, 9]). In this setup, an interesting subclass that arises when the spacetime admits Killing vectors is the subclass of TT tensors which satisfy the so-called KID equation [21]. In dimension n, this equation is

$$\pounds_\xi D_{\alpha\beta} + \frac{n-2}{n}(\mathrm{div}_g\ \xi) D_{\alpha\beta} = 0,$$

where ξ is a CKV of g and $\pounds_\xi$, $\mathrm{div}_g \xi$ stand respectively for the Lie derivative along ξ and the divergence of ξ with respect to g. In dimension $n = 2$ the general solution of (local) TT tensors satisfying the KID equation can be explicitly solved. Although this dimension is not particularly interesting from a physical point of view, there are several motivations for presenting the result. Firstly, dimensional reduction is a useful tool in many geometric problems, so it is not unlikely that the case of dimension two may find applications in higher dimensions. Also, the $n = 2$ case may serve as a toy model to address the (much more difficult) problem in higher dimensions. In addition, the solution we find turns out to admit an interesting generalization in arbitrary dimension. And lastly, it is remarkable, that the problem is so simple in dimension $n = 2$ that its general solution can be explicitly given.

A key property of the TT conditions and of the KID equations is their conformal covariance. If $D_{\alpha\beta}$ is a TT tensor with respect to g then $\Psi^{2-n} D_{\alpha\beta}$ is a TT tensor with respect to $\Psi^2 g$. Also, if D satisfies the KID equation for g, then $\Psi^{2-n} D$ also satisfies the KID equation for $\Psi^2 g$. In dimension $n = 2$ one actually has conformal invariance. Since all two-dimensional metrics are locally conformal to the flat metric, and we are interested in solving the (more general) local problem, we may assume that $g = 4\mathrm{d}z\mathrm{d}\bar{z}$. As already mentioned, a vector field ξ is conformal of this metric if and only if $\xi = f(z)\partial_z + \overline{f}(\bar{z})\partial_{\bar{z}}$. We expand $D = D_{zz}\mathrm{d}z^2 + D_{\bar{z}\bar{z}}\mathrm{d}\bar{z}^2 + 2D_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$. The condition of being traceless is $D_{z\bar{z}} = 0$ and D real requires $D_{\bar{z}\bar{z}} = \overline{D_{zz}}$, With these restrictions, the transverse equations take the following explicit and simple form

$$\partial_z D_{\bar{z}\bar{z}} = 0, \qquad \partial_{\bar{z}} D_{zz} = 0,$$

so D_{zz} is a holomorphic function of z. Imposing TT as well as the reality condition, the KID equations read

$$f\frac{\mathrm{d}D_{zz}}{\mathrm{d}z} + 2D_{zz}\frac{\mathrm{d}f}{\mathrm{d}z} = 0,$$

which integrates to $D_{zz} = \frac{q}{f^2}$, $\quad q \in \mathbb{C}$. Writing $q = q_1 + \mathrm{i}q_2$, with real q_1, q_2, we conclude that the most general (real) TT tensor that satisfies the KID equation is a linear combination of (we add the factor 4 for convenience)

$$D_1 := \frac{1}{4}\left(\frac{1}{f^2}\mathrm{d}z^2 + \frac{1}{\overline{f}^2}\mathrm{d}\bar{z}^2\right), \qquad D_2 = \frac{\mathrm{i}}{4}\left(\frac{1}{\overline{f}^2}\mathrm{d}\bar{z}^2 - \frac{1}{f^2}\mathrm{d}z^2\right).$$

These expressions are valid in the coordinate system $\{z, \overline{z}\}$. We are interested in covariant expressions that are valid in any coordinate system, and are explicitly invariant under conformal transformations. To achieve this, we introduce the vector field

$$\xi^{\perp} := \mathrm{i}\left(f\partial_z - \overline{f}\partial_{\overline{z}}\right). \tag{62}$$

This is everywhere orthogonal to ξ and has the same norm at every point. If the zeros of ξ do not separate the manifold, these two properties define $\xi^{\perp}$ in terms of ξ uniquely except for a global sign. If the zeroes of ξ separate the manifold, $\xi^{\perp}$ is still uniquely defined (up to a sign) if one adds the condition that $\xi^{\perp}$ is a CKV of g (which (62) clearly is). Thus, we may speak of $\xi^{\perp}$ unambiguously (up to global sign), once ξ has been fixed. Next we note that, in the $\{z, \overline{z}\}$ coordinate system and with respect to the metric $g_E := 4\mathrm{d}z\mathrm{d}\overline{z}$ we have

$$\begin{aligned} \boldsymbol{\xi} &= 2f\,\mathrm{d}\overline{z} + 2\overline{f}\,\mathrm{d}\overline{z}, \qquad |\xi|^2_{g_E} := g_E(\xi,\xi) = 4f\overline{f}, \\ \boldsymbol{\xi}^{\perp} &= 2\mathrm{i}f\,\mathrm{d}\overline{z} - 2\mathrm{i}\overline{f}\,\mathrm{d}z, \qquad |\xi^{\perp}|^2_{g_E} = 4f\overline{f} \end{aligned}$$

and then we may write

$$\begin{aligned} D_1 &= \frac{1}{|\xi|^4_{g_E}}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi} - \frac{1}{2}|\xi|^2_{g_E} g_E\right), \\ D_2 &= \frac{1}{2|\xi|^4_{g_E}}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi}^{\perp} + \boldsymbol{\xi}^{\perp}\otimes\boldsymbol{\xi}\right). \end{aligned}$$

These expressions are obviously coordinate independent and also conformally invariant. Thus, D_1 and D_2 take this form also for the original metric g. Notice that at the fixed points of ξ, i.e. those points where ξ vanishes, the general solution $D = c_1 D_1 + c_2 D_2$ for $c_1, c_2 \in \mathbb{R}$ diverges unless $c_1 = c_2 = 0$. This follows from the fact that the square norm of D is

$$D_{\alpha\beta}D^{\alpha\beta} = \frac{1}{2|\xi|^4_{g_E}}(c_1^2 + c_2^2),$$

which is regular at the fixed points of ξ only if $c_1 = c_2 = 0$. Summarizing, we have proved the following theorem.

Theorem 3 *Let (M, g) be a two-dimensional Riemannian manifold and ξ a CKV of g. Let D be a (real) TT symmetric, two-covariant tensor that satisfies the KID equation with respect to ξ. Then D is a linear combination (with constants) of*

$$\begin{aligned} D_{\xi} &:= \frac{1}{|\xi|^4_g}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi} - \frac{1}{2}|\xi|^2_g g\right), \\ D_{\xi,\xi^{\perp}} &:= \frac{1}{2|\xi|^2_g|\xi^{\perp}|^2_g}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi}^{\perp} + \boldsymbol{\xi}^{\perp}\otimes\boldsymbol{\xi}\right), \end{aligned}$$

where $\xi^{\perp}$ is defined as described above and $\boldsymbol{\xi} := g(\xi, \cdot)$, $\boldsymbol{\xi}^{\perp} := g(\xi^{\perp}, \cdot)$. Moreover, the only solution regular at any of the fixed points of ξ is the zero tensor.

The tensor D_{ξ} is the TT tensor (specialized to dimension $n = 2$) built canonically out of the CKV ξ mentioned in the introduction and which plays an important role in the characterization result of Ker-de Sitter in dimension $n = 3$ obtained in [17]. As far as we know, the tensor $D_{\xi,\xi^{\perp}}$ has not appeared in the literature yet. As for D_{ξ}, it admits an extension to higher dimensions by simply replacing ξ by any CKV η that commutes with ξ. The details of this extension will be presented in our subsequent work in higher dimensions [18].

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References

[1] Abellanas L and Alonso L M 1975 A general setting for Casimir invariants *J. Math. Phys.* **16** 1580–4
[2] Beig R 1997 *TT-tensors and Conformally Flat Structures on 3-manifolds* ed P T Chrusciel vol 41 (Warsaw: Banach Center Publications) pp 109–18
[3] Burgoyne N and Cushman R 1977 Conjugacy classes in linear groups *J. Algebra* **44** 339–62
[4] Crooks P 2019 Complex adjoint orbits in Lie theory and geometry *Expo. Math.* **37** 104–44
[5] Dain S and Friedrich H 2001 Asymptotically flat initial data with prescribed regularity at infinity *Commun. Math. Phys.* **222** 569–609
[6] Djokovi D Z *et al* 1983 Normal forms of elements of classical real and complex Lie and Jordan algebras *J. Math. Phys.* **24** 1363–74
[7] Friedrich H 1986 Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant *J. Geom. Phys.* **3** 101–17
[8] Friedrich H 2002 Conformal Einstein evolution *The Conformal Structure of Space-Time: Geometry, Analysis, Numerics* ed J Frauendiener and H Friedrich (Berlin: Springer) pp 1–50
[9] Friedrich H 2014 Geometric asymptotics and beyond (arXiv:1411.3854 [gr-Qc])
[10] Gantmacher F R 1960 The theory of matrices *The Theory of Matrices* vol 1 (New York: Chelsea)
[11] Goodman R and Wallach N R 2009 Symmetry, representations, and invariants *Graduate Texts in Mathematics* (New York: Springer)
[12] Hall G S 1986 The classification of complex bivectors in space-time *Acta Phys. Pol.* **17** 139–49
[13] Hall B C 2003 *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* (*Graduate Texts in Mathematics*) (Berlin: Springer)
[14] Ida D 2020 Complete integrability of cohomogeneity-one strings in $\mathbb{R}^{n,1}$ and canonical form of Killing vector algebra *J. Math. Phys.* **61** 082501
[15] Knapp A W 2002 Lie groups beyond an introduction *Progress in Mathematics* (Boston: Birkhäuser)
[16] Lichnerowicz A 1955 *Théories Relativistes de la Gravitation et de l'électromagnétisme: Relativité Générale et Théories Unitaires* (*Collection d'ouvrages de Mathématiques à l'usage des Physiciens*) (Paris: Masson)
[17] Mars M, Paetz T-T and Senovilla J M M 2017 Classification of Kerr-de Sitter-like spacetimes with conformally flat $\mathscr{I}$ *Class. Quantum Grav.* **34** 095010
[18] Mars M and Peón-Nieto C 2020 in preparation
[19] Misner C W and Wheeler J A 1957 Classical physics as geometry *Ann. Phys., NY* **2** 525–603
[20] Needham T 1997 *Visual Complex Analysis* (Oxford: Clarendon)
[21] Paetz T-T 2016 Killing initial data on spacelike conformal boundaries *J. Geom. Phys.* **106** 51–69
[22] Penrose R and Rindler W 1984 *Spinors and Space-Time: Two-Spinor Calculus and Relativistic Fields Vol 1* (*Cambridge Monographs on Mathematical Physics*) (Cambridge: Cambridge University Press)
[23] Rainich G Y 1924 Electrodynamics in the general relativity theory *Proc. Natl Acad. Sci.* **10** 124–7

[24] Schottenloher M 2008 *A Mathematical Introduction to Conformal Field Theory* (*Lecture Notes in Physics*) (Berlin: Springer)
[25] Schwerdtfeger H 1979 *Geometry of Complex Numbers* (New York: Dover)
[26] Synge J L 1956 *Relativity: The Special Theory* (*Series in Physics*) (Amsterdam: North-Holland)
[27] Čap A and Slovák J 2009 *Parabolic Geometries: Background and General Theory* (*Mathematical Surveys and Monographs*) vol 154 (Providence, RI: American Mathematical Society)