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Skew-symmetric endomorphisms in $\mathbb{M}^{1,n}$: a unified canonical form with applications to conformal geometry

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Abstract

We show the existence of families of orthonormal, future directed bases which allow to cast every skew-symmetric endomorphism of $\mathbb{M}^{1,n}$ ($\text{SkewEnd}(\mathbb{M}^{1,n})$) in a single canonical form depending on a minimal number of parameters. This canonical form is shared by every pair of elements in $\text{SkewEnd}(\mathbb{M}^{1,n})$ differing by an orthochronous Lorentz transformation, i.e. it defines the orbits of the orthochronous Lorentz group $O^+(1, n)$ under the adjoint action on its algebra. Using this form, we obtain the quotient topology of $\text{SkewEnd}(\mathbb{M}^{1,n})/O^+(1, n)$. From known relations between $\text{SkewEnd}(\mathbb{M}^{1,n+1})$ and the conformal Killing vector fields (CKVFs) of the sphere \mathbb{S}^n , a canonical form for CKVFs follows immediately. This form is used to find adapted coordinates to an arbitrary CKVF that covers all cases at the same time. We do the calculation for even n and obtain the case of odd n as a consequence. Finally, we employ the adapted coordinates to obtain a wide class of TT-tensors for $n = 3$, which provide Cauchy data at conformally flat null infinity \mathcal{I} . Specifically, this class of data is characterized for generating $\Lambda > 0$ -vacuum spacetimes with two-symmetries, one of which axial, admitting a conformally flat \mathcal{I} . The class of data is infinite dimensional, depending on two arbitrary functions of one variable as well as a number of constants. Moreover, it contains the data for the Kerr–de Sitter spacetime, which we explicitly identify within.

Keywords: canonical form, conformal Killing vector fields, TT-tensors, skew-symmetric endomorphisms

(Some figures may appear in colour only in the online journal)

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1. Introduction

Having a Lie group G acting on a space X representing a set of physical quantities is always a desirable feature in a physical problem, as Lie groups represent symmetries (either global or gauge) and their presence is often translated into a simplification of the formal aspects of the problem. Roughly speaking, in this situation the ‘relevant’ part for the physics effectively happens in the quotient space X/G . For the study of these quotient spaces, one may be interested in obtaining a unified form to give a representative for every orbit in X/G , i.e. a *canonical form* (also known as normal form). A particularly relevant case is when X is a Lie algebra \mathfrak{g} and G its Lie group acting by the adjoint action, in which case the orbits are also called conjugacy classes of G (see e.g. [5]). In the first part of this paper, we study the conjugacy classes of the pseudo-orthogonal group $O(1, n)$ (or Lorentz group), for which we will obtain a canonical form. Our main interest on this, addressed in the second part of the paper, lies in its relation with the Cauchy problem of general relativity (GR) (cf [11] and e.g. [7, 14]) and more precisely, its formulation at null infinity \mathcal{I} for the case of positive cosmological constant Λ (cf [12, 13]). In the remainder of this introduction we summarize our ideas and results, and we will also briefly review some results on conjugacy classes of Lie groups related to our case, as well as the Cauchy problem of GR with positive Λ .

A typical example of a canonical form in the context described above, is the well-known Jordan form, which represents the conjugacy classes of $GL(n, \mathbb{K})$ (where \mathbb{K} is usually \mathbb{R} , \mathbb{C} or the quaternions \mathbb{H}). Besides this example, the problem of finding a canonical representative for the conjugacy classes of a Lie group has been addressed numerous times in the literature. The reader may find a list of canonical forms for algebras whose groups leave invariant a non-degenerate bilinear form in [9] (this includes symmetric, skew-symmetric and symplectic algebras over \mathbb{R} , \mathbb{C} and \mathbb{H}) as well as the study of the affine orthogonal group (or Poincaré group) in [8] or [19]. Notice that these works deal, either directly or indirectly, with our case of interest $O(1, n)$, whose algebra $\mathfrak{o}(1, n)$ will be represented in this paper as skew-symmetric endomorphisms of Minkowski spacetime $\mathbb{M}^{1,n}$. When giving a canonical form, it is usual to base it on criteria of irreducibility rather than uniformity (e.g. [8, 9, 19]). This is similar to what is done when the Darboux decomposition is applied to two-forms (i.e. elements of $\mathfrak{o}(1, n)$), for example in [24] or for the low dimensional case $n = 3$ (e.g. [18, 30]). As a consequence, all canonical forms found for the case of $\mathfrak{o}(1, n)$ require two different types of matrices to represent all orbits, one and only one fitting a given element. Our first aim in this paper is to give a unique matrix form which represents each element $F \in \mathfrak{o}(1, n)$, depending on a minimal number of parameters that allows one to easily determine its orbit under the adjoint action of $O(1, n)$. This is obviously achieved by losing explicit irreducibility in the form. However, this canonical form will be proven to be fruitful by giving several applications. This same issue has also been addressed in [25] for the case of low dimensions, i.e. $O(1, 2)$ and $O(1, 3)$, where in addition, several applications are worked out. The present work constitutes a natural generalization of the results in [25] to arbitrary dimensions.

The Lorentz group is well-known to be of particular interest in physics, as for example, it is the group of isotropies of the special theory of relativity and the Lorentz–Maxwell electrodynamics (e.g. [23, 28, 30]). Its study in arbitrary dimensions has received renewed interest with theories of high energy physics such as conformal field theories [29] or string theories [19]. Related to the former and for our purposes here, a fact of special relevance is that the orthochronous component $O^+(1, n+1) \subset O(1, n+1)$ is homomorphic to the group of conformal transformations of the n -sphere, $\text{Conf}(\mathbb{S}^n)$. The conformal structure of \mathcal{I} happens to be fundamental for the Cauchy problem at null infinity of GR for spacetimes with positive Λ . Such a set consists of a manifold Σ endowed with a (Riemannian) conformal structure $[\gamma]$,

representing the geometry of null infinity $\mathcal{I} := (\Sigma, [\gamma])$, together with the conformal class $[D]$ of a transverse (i.e. zero divergence), traceless, symmetric tensor D (TT-tensor) of \mathcal{I} . If the spacetime generated by the data is to have a Killing vector field, the TT-tensor must satisfy a conformally covariant equation depending on a conformal Killing vector field (CKVF) of \mathcal{I} , the so-called Killing initial data (KID) equation [26].

The class of data in which $[\gamma]$ contains a constant curvature metric (or alternatively the locally conformally flat case) includes the family of Kerr–de Sitter black holes and its study could be a possible route towards a characterization result for this family of spacetimes. Even in this particular (conformally flat) case, it is difficult to give a complete list of TT-tensors. See for example [3, 32], where the authors (locally) derive all TT tensors, in neighborhoods which are respectively star-shaped and with vanishing second de Rham cohomology, as the image under a certain map from the space of symmetric tensors. In addition [32] extends the solution to more dimensions in the analytic case. The parametrization that both authors give is of course redundant, since the map has a large kernel. Our idea is to derive directly, and with an optimal parametrization, TT tensors corresponding to data of spacetimes with Killing vectors. This is achieved by imposing the KID equations to the data. Using the homomorphism between $O^+(1, n+1)$ and $\text{Conf}(\mathbb{S}^n)$, we induce a canonical form for CKVFs from the canonical form obtained for $\mathfrak{o}(1, n+1)$. Since this form covers all orbits of CKVFs under the adjoint action of $\text{Conf}(\mathbb{S}^n)$, our adapted coordinates fit every CKVF and in addition, since the KID equation is conformally covariant, we can choose a conformal gauge where this CKVF is a Killing vector field, which makes the KID equation trivial. Hence, a remarkable feature from our method is that by solving one simple equation, we are solving many cases at once. This has already been done in the case of \mathbb{S}^2 in [25] and here we extend it to the more interesting and difficult case of (open domains of) \mathbb{S}^3 .

Specifically, we obtain the most general class of TT-tensors on a conformally flat \mathcal{I} such that the $\Lambda > 0$ -vacuum four-dimensional spacetime generated by these data admits two local isometries, one of them axial. It is worth highlighting that this is a broad class (of infinite dimensions as it depends on functions) of TT-tensors and it contains the Kerr–de Sitter Cauchy data at \mathcal{I} . This provides a potentially interesting ‘sandbox’ to try the consistence of possible definitions of (global) mass and angular momentum (see [31] for a review on the state of the art). Recall that symmetries are well-known to be related to conserved quantities, in particular, axial symmetry is related to conservation of angular momentum and time symmetry to conservation of energy. Moreover, for a spacetime to have constant mass, one may require no radiation escaping from or coming within the spacetime, a condition which, following the criterion of [10], is guaranteed by conformal the flatness of \mathcal{I} . Finally, the presence of the Kerr–de Sitter data within the set of TT-tensors contributes to its physical relevance and furnishes the possibility of looking for new characterization results for this family of spacetimes.

As an additional sidenote concerning our results, notice that both the canonical form of CKVFs as well as the adapted coordinates are obtained in arbitrary dimensions, so similar applications may be worked out in arbitrary dimensions which, needless to say, is a considerable harder problem. On the possible extension to more dimensions of this type of TT-tensors, one should mention that the Cauchy problem at \mathcal{I} for positive cosmological constant is known to be well-posed in arbitrary even dimensions [2]. However the KID equations are only known to be a necessary consequence of having symmetries, but sufficiency is an open problem in spacetime dimensions higher than four.

The paper is organized as follows. In order to properly define the canonical form, in section 2 we rederive a classification result for skew-symmetric endomorphisms (cf theorem 2.6), employing only elementary linear algebra methods. The results of this section are known (see e.g. [16, 17, 21]), but the method is original and we believe more direct than other approaches

in the literature. We also include the derivations in order to make the paper self-contained. Section 2 leads to the definition of canonical form in section 3. Section 4 deals with a particular type of skew-symmetric endomorphisms (the so-called *simple*, i.e. of minimal matrix rank), which will be useful for future sections. In section 5 we work out some applications of our canonical form: identifying invariants which characterize the conjugacy classes of the orthochronous Lorentz group (cf theorem 5.1) and obtaining the topological structure of this quotient space (cf section 5.1).

In section 6 we use the homomorphism between $O^+(1, n+1)$ and $\text{Conf}(\mathbb{S}^n)$ and apply the canonical form obtained for skew-symmetric endomorphisms to give a canonical form for CKVFs, together with a decomposed form (cf proposition 6.1) which is analogous to the one given for skew-symmetric endomorphisms. In section 7, we adapt coordinates to CKVFs in canonical form, first in the even dimensional case, from which the odd dimensional case is obtained as a consequence. Finally, in section 8 we employ the adapted coordinates to find the most general class of data at \mathcal{S} corresponding to spacetime dimension four, such that \mathcal{S} is conformally flat and the $(\Lambda > 0)$ -vacuum spacetime they generate admits at least two symmetries, one of which is axial. It is remarkable how easily are these equations solved with all the tools developed before. With this solution at hand, we are able to identify the Kerr–de Sitter family within.

2. Classification of skew-symmetric endomorphisms

In this section we derive a classification result for skew-symmetric endomorphisms of Lorentzian vector spaces. Let V be a d -dimensional vector space endowed with a pseudo-Riemannian metric g . If g is of signature $(-, +, \dots, +)$, then (V, g) is said to be Lorentzian. Scalar product with g is denoted by $\langle \cdot, \cdot \rangle$. An endomorphism $F : V \rightarrow V$ is skew-symmetric when it satisfies

$$\langle x, F(y) \rangle = -\langle F(x), y \rangle \quad \forall x, y \in V.$$

We denote this set by $\text{SkewEnd}(V) \subset \text{End}(V)$. We take, by definition, that eigenvectors of an endomorphism are always non-zero. We use the standard notation for spacelike and timelike vectors as well as for spacelike, timelike and degenerate vector subspaces. In our convention all vectors with vanishing norm are null (in particular, the zero vector is null). We denote $\ker F$ and $\text{Im } F$, respectively, to the kernel and image of $F \in \text{End}(V)$.

Lemma 2.1 (basic facts about skew-symmetric endomorphisms). *Let F be a skew-symmetric endomorphism in a pseudo-Riemannian vector space V . Then*

- (a) $\forall w \in V$, $F(w)$ is perpendicular to w , i.e. $\langle F(w), w \rangle = 0$.
- (b) $\text{Im } F \subset (\ker F)^\perp$ and $\ker F \subset (\text{Im } F)^\perp$.
- (c) If $w \in \ker F \cap \text{Im } F$ then w is null.
- (d) If $w \in V$ is a non-null eigenvector of F , then its eigenvalue is zero.
- (e) If w is an eigenvector of F with zero eigenvalue, then all vectors in $\text{Im } F$ are orthogonal to w , i.e. $\text{Im } F \subset w^\perp$.
- (f) If F restricts to a subspace $U \subset V$ (i.e. $F(U) \subset U$), then it also restricts to U^\perp .

Proof. (a) is immediate from $\langle w, F(w) \rangle = -\langle F(w), w \rangle$. For (b), let $v \in \ker F$ and w be of the form $w = F(u)$ for some $u \in V$, then

$$\langle w, v \rangle = \langle F(u), v \rangle = -\langle u, F(v) \rangle = 0$$

the last equality following because $F(v) = 0$. (c) is a consequence of (b) because w belongs both to $\ker F$ and to its orthogonal, so in particular it must be orthogonal to itself, hence null. (d) is immediate from

$$0 = \langle w, F(w) \rangle = \lambda \langle w, w \rangle$$

so if w is non-null, its eigenvalue λ must be zero. (e) is a corollary of (b) because by hypothesis $w \in \ker F$ so

$$\text{Im } F \subset (\ker F)^\perp \subset w^\perp$$

the last inclusion being a consequence of the general fact $U_1 \subset U_2 \Rightarrow U_2^\perp \subset U_1^\perp$. Finally, (f) is true because for any u in an F -invariant subspace U and $w \in U^\perp$

$$0 = \langle F(u), w \rangle = -\langle u, F(w) \rangle.$$

□

Another well-known property of skew-symmetric endomorphisms that we will use is that $\dim \text{Im } F$ is always even. Equivalently, $\dim \ker F$ has the same parity than $\dim V$. To see this, consider the two-form F assigned to every $F \in \text{SkewEnd}(V)$ by the standard relation

$$F(e, e') = \langle e, F(e') \rangle, \quad \forall e, e' \in V.$$

The matrix representing F is skew in the usual sense, hence the dimension of $\text{Im } F \subset V^*$ (the dual of V) is the rank of that matrix, which is known to be even (see e.g. [15]) and clearly $\dim \text{Im } F = \dim \text{Im } F$.

The strategy that we will follow to classify skew-symmetric endomorphisms of V with g Lorentzian is via F -invariant spacelike planes. Conditions for F -invariance of spacelike planes are stated in the following lemma:

Lemma 2.2. *Let $F \in \text{SkewEnd}(V)$. Then F has an F -invariant spacelike plane Π_s if and only if*

$$F(u) = \mu v, \quad F(v) = -\mu u, \tag{1}$$

for $\Pi_s = \text{span}\{u, v\}$ with $u, v \in V$ spacelike, orthogonal, unit and $\mu \in \mathbb{R}$. Moreover, (1) is satisfied for $\mu \neq 0$ if and only if $\pm i\mu$ are eigenvalues of F with (null) eigenvectors $u \pm iv$, for $u, v \in V$ spacelike, orthogonal with the same square norm.

Proof. If (1) is satisfied for $u, v \in V$ spacelike, orthogonal, unit, then $\Pi_s = \text{span}\{u, v\}$ is obviously F -invariant spacelike. On the other hand, if Π_s is F -invariant, then it must hold that

$$F(u) = a_1 u + a_2 v, \quad F(v) = b_1 u + b_2 v, \quad a_1, a_2, b_1, b_2 \in \mathbb{R},$$

for a pair of orthogonal, unit, spacelike vectors u, v spanning Π_s . Using skew-symmetry and the orthogonality and unitarity of u, v , the constants are readily determined: $a_2 = b_2 = 0$ and $a_2 = -b_1 =: \mu$, which implies (1). This proves the first part of the lemma.

For the second part, it is immediate that if (1) holds with $\mu \neq 0$, then $\pm i\mu$ are eigenvalues of F with respective eigenvectors $u \pm iv$. The orthogonality of u, v follows from $\langle F(u), u \rangle = 0 = \mu \langle v, u \rangle$ and the equality of norm from skew-symmetry $\langle F(u), v \rangle = -\langle u, F(v) \rangle \Rightarrow \mu \langle v, v \rangle = \mu \langle u, u \rangle$. Assume now that F has an eigenvalue $i\mu \neq 0$ with (necessarily null) eigenvector $w = u + iv$, for $u, v \in V$. Since F is real, neither u nor v can be zero. From the nullity property $\langle w, w \rangle = 0$, it follows that $\langle u, u \rangle - \langle v, v \rangle = 0$ and $\langle u, v \rangle = 0$. Hence, u, v are orthogonal with

the same norm, so they are either null and proportional, which can be discarded because it would imply that u (and v) is a real eigenvector with complex eigenvalue; or otherwise u, v are spacelike, thus the lemma follows. \square

There is an analogous result for F -invariant timelike planes:

Lemma 2.3. *Let $F \in \text{SkewEnd}(V)$. Then F has an F -invariant timelike plane Π_t if and only if*

$$F(e) = \mu v, \quad F(v) = \mu e, \tag{2}$$

for $\Pi_t = \text{span}\{e, v\}$ with $e, v \in V$ for e timelike unit orthogonal to v spacelike, unit and $\mu \in \mathbb{R}$. Moreover, (1) is satisfied for $\mu \neq 0$ if and only if $\pm\mu$ are eigenvalues of F with (null) eigenvectors $e \pm v$, for $e, v \in V$ orthogonal, timelike and spacelike respectively with opposite square norm.

Proof. For the first claim, repeat the first part of the proof of lemma 2.2 assuming $u = e$ timelike.

For the second claim, assume (2) is satisfied with $\mu \neq 0$. Then it is immediate that $\langle F(e), e \rangle = 0 = \mu \langle v, e \rangle$, hence e, v are orthogonal and by skew-symmetry $\langle F(e), v \rangle = -\langle e, F(v) \rangle \Rightarrow \mu \langle v, v \rangle = -\mu \langle e, e \rangle$, i.e. must have opposite square norm. Conversely, let $\pm\mu \neq 0$ be a pair of eigenvalues with respective null eigenvectors q_{\pm} , that w.l.o.g. can be chosen future directed. Then $e := q_+ + q_-$ and $v := q_+ - q_-$ are orthogonal, with opposite square norm $\langle e, e \rangle = 2 \langle q_+, q_- \rangle = -\langle v, v \rangle < 0$, and they satisfy (2). \square

The F -invariant spacelike or timelike planes will be often be referred to as ‘eigenplanes’ and μ will be denoted as the ‘eigenvalues’ of Π . Notice that a simple change of order in the vectors switches the sign of the eigenvalue μ . Thus, unless otherwise stated, we will consider the eigenvalues of eigenplanes (both spacelike and timelike) non-negative by default.

The first question we address here is under which conditions such a plane exists (cf proposition 2.1). But before doing so, we need to prove some results first.

Lemma 2.4. *Let V be a Lorentzian vector space and $F \in \text{SkewEnd}(V)$. Then there exist two vectors $x, y \in V$, with $x \neq 0$, such that one of the three following exclusive possibilities hold*

- (a) x is a null eigenvector of F .
- (b) x is a non-null eigenvector (with zero eigenvalue).
- (c) x, y are orthogonal, spacelike and with the same norm, and define an eigenplane of F with non-zero eigenvalue, i.e.

$$F(x) = \mu y, \quad F(y) = -\mu x, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

If, instead, V is Riemannian, only cases (b) and (c) can arise.

Proof. From the Jordan block decomposition theorem we know that there is at least one, possibly complex, eigenvalue $s_1 + is_2$ with eigenvector $x + iy$, that is, $F(x + iy) = (s_1 + is_2)(x + iy)$, or equivalently:

$$F(x) = s_1x - s_2y, \tag{3}$$

$$F(y) = s_2x + s_1y. \tag{4}$$

This system is invariant under the interchange $(x, y) \rightarrow (-y, x)$, so without loss of generality we may assume $x \neq 0$. The respective scalar products of (3) and (4) with x, y yield

$$\left. \begin{aligned} s_1 \langle x, x \rangle - s_2 \langle x, y \rangle &= 0 \\ s_1 \langle y, y \rangle + s_2 \langle x, y \rangle &= 0 \end{aligned} \right\} \Leftrightarrow \begin{pmatrix} \langle x, x \rangle & -\langle x, y \rangle \\ \langle y, y \rangle & \langle x, y \rangle \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5)$$

Observe that if $s_1 + is_2 \neq 0$ the determinant of the matrix must vanish. i.e. $\langle x, y \rangle (\langle x, x \rangle + \langle y, y \rangle) = 0$. Hence, we can distinguish the following possibilities:

(a) $s_1 = s_2 = 0$. Then x is an eigenvector of F with vanishing eigenvalue so we fall into cases (a) or (b).

(b) $s_1 + is_2 \neq 0$. From $\langle x, y \rangle (\langle x, x \rangle + \langle y, y \rangle) = 0$ we distinguish two cases:

1. $\langle x, y \rangle = 0$. If $s_1 \neq 0$ then (5) forces x and y to be both null and, being also orthogonal to each other, there is $a \in \mathbb{R}$ such that $y = ax$ and we fall into case (a). So, we can assume $s_1 = 0$ (and then $s_2 \neq 0$). Let $\mu := -s_2$, thus (c) follows from equations (3), (4) and lemma 2.2.
2. $\langle x, y \rangle \neq 0$. Then $\langle x, x \rangle = -\langle y, y \rangle$ and the matrix problem (5) reduces to

$$s_1 \langle x, x \rangle - s_2 \langle x, y \rangle = 0.$$

In addition, (3) and (4) imply

$$\langle F(x), y \rangle = s_1 \langle x, y \rangle - s_2 \langle y, y \rangle = s_1 \langle x, y \rangle + s_2 \langle x, x \rangle = \langle F(y), x \rangle.$$

But skew-symmetry requires $\langle F(x), y \rangle = -\langle F(y), x \rangle$, so $\langle F(y), x \rangle = 0$ and we conclude

$$s_1 \langle x, y \rangle + s_2 \langle x, x \rangle = 0.$$

Combining with (2) yields

$$\begin{pmatrix} \langle x, x \rangle & -\langle x, y \rangle \\ \langle x, y \rangle & \langle x, x \rangle \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is non-zero which yields a contradiction with $s_1 + is_2 \neq 0$. So this case is empty. □

Remark 2.1. One may wonder why the lemma includes the possibility of having a spacelike eigenplane (case (c)), but not a timelike eigenplane. The reason is that invariant timelike planes, which are indeed possible, fall into case (a) by lemma 2.3, because $e \pm v$ are null eigenvectors.

In the case of Riemannian signature, lemma 2.4 can be reduced to the following single statement:

Corollary 2.4.1. *Let V be Riemannian of dimension d and $F \in \text{SkewEnd}(V)$. If $d = 1$ then $F = 0$ and if $d \geq 2$ then there exist two orthogonal and unit vectors u, v satisfying*

$$F(u) = \mu v, \quad F(v) = -\mu u, \quad \mu \in \mathbb{R} \quad (6)$$

Proof. The case $d = 1$ is trivial, so let us assume $d \geq 2$. By the last statement of lemma 2.4 either there exists an eigenvector x with zero eigenvalue or the pair $\{u, v\}$ claimed in the corollary exists. In the former case, we consider the vector subspace x^\perp . Its dimension is at

least one and F restricts to this space so again either the pair $\{u, v\}$ exists or there is $y \in x^\perp$ satisfying $F(y) = 0$. But then $\{x, y\}$ are orthogonal and non-zero. Normalizing we find a pair $\{u, v\}$ that satisfies (6) with $\mu = 0$. \square

Lemma 2.4 lists a set of cases, one of which must always occur. However, we now show that, if the dimension is sufficiently high, case (a) of that lemma implies one of the other two:

Lemma 2.5. *Let $F \in \text{SkewEnd}(V)$, with V Lorentzian of dimension at least four. If F has a null eigenvector, then it also has either a spacelike eigenvector or a spacelike eigenplane.*

Proof. Let $k \in V$ be a null eigenvector of F . The space $A := k^\perp \subset V$ is a null hyperplane and F restricts to A . On this space we define the standard equivalence relation $y_0 \sim y_1$ iff $y_0 - y_1 = ak, a \in \mathbb{R}$. The quotient A/\sim (which has dimension at least two) inherits a positive definite metric \bar{g} and F also descends to the quotient. More precisely, if we denote the equivalence class of any $y \in A$ by \bar{y} , then for any $\bar{y} \in A/\sim$ and any $y \in \bar{y}$ the expression $\bar{F}(\bar{y}) = \overline{F(y)}$ is well-defined (i.e. independent of the choice of representative y) and hence defines an endomorphism \bar{F} of A/\sim which, moreover, satisfies

$$\langle \bar{F}(\bar{y}_1), \bar{y}_2 \rangle_{\bar{g}} = -\langle \bar{y}_1, \bar{F}(\bar{y}_2) \rangle_{\bar{g}}.$$

In other words \bar{F} is a skew-symmetric endomorphism in the Riemannian vector space A/\sim . By corollary 2.4.1 (here we use that the dimension of A/\sim is at least two) there exists a pair of orthogonal and \bar{g} -unit vectors $\{\bar{e}_1, \bar{e}_2\}$ satisfying

$$\bar{F}(\bar{e}_1) = a\bar{e}_2, \quad \bar{F}(\bar{e}_2) = -a\bar{e}_1, \quad a \in \mathbb{R}.$$

Select representatives $e_1 \in \bar{e}_1$ and $e_2 \in \bar{e}_2$. In terms of F , the condition (2) and the fact that k is eigenvector require the existence of constants σ, a, λ_1 and λ_2 such that

$$F(k) = \sigma k, \quad F(e_1) = ae_2 + \lambda_1 k, \quad F(e_2) = -ae_1 + \lambda_2 k.$$

Whenever $a^2 + \sigma^2 \neq 0$ the vectors

$$u := e_1 - \frac{1}{a^2 + \sigma^2} (a\lambda_2 + \sigma\lambda_1)k, \quad v := e_2 + \frac{1}{a^2 + \sigma^2} (a\lambda_1 - \sigma\lambda_2)k$$

satisfy $F(u) = a$ and $F(v) = -au$. Since u and v are spacelike, unit and orthogonal to each other the claim of the proposition follows (with $\mu = a$). If $\sigma = a = 0$, then either $\lambda_1 = \lambda_2 = 0$ and then $\{e_1, e_2\}$ are directly the vectors $\{u, v\}$ claimed in the proposition (with $\mu = 0$), or at least one of the λ s (say λ_2) is not zero. Then $e := e_1 - \frac{\lambda_1}{\lambda_2}e_2$ is a spacelike eigenvector of F . \square

Now we have all the ingredients to show one of the main results of this section, that will eventually allow us to classify skew-symmetric endomorphisms of Lorentzian vector spaces.

Proposition 2.1. *Let V be a Lorentzian vector space of dimension at least five and $F \in \text{SkewEnd}(V)$. Then, there exists a spacelike eigenplane.*

Proof. We examine each one of the three possibilities described in lemma 2.4. Case (c) yields the result trivially, so we can assume that F has an eigenvector x .

If we are in case (b), the vector x is either spacelike or timelike. If it is timelike we consider the Riemannian space x^\perp where F restricts. We may apply corollary 2.4.1 (note that x^\perp has dimension at least four) and conclude that the vectors $\{u, v\}$ exist. So it remains to consider the case when x is spacelike and F admits no timelike eigenvectors. We restrict to x^\perp which is Lorentzian and of dimension at least four. Applying again lemma 2.4, either there exists a

spacelike eigenplane, or a second eigenvector $y \in x^\perp$, which can only be spacelike or null. If y is spacelike, $\{u := x, v := y\}$ span a spacelike eigenplane with $\mu = 0$. If y is null, we may apply lemma 2.5 to $F|_{x^\perp}$ to conclude that either a spacelike eigenplane exists, or there is a spacelike eigenvector $e \in x^\perp$, so the pair $\{u := e, v := x\}$ satisfies (1) with $\mu = 0$. This concludes the proof of case (b).

In case (a), i.e. when there is a null eigenvector x we can apply lemma 2.5 and conclude that either $\{u, v\}$ exist, or there is a spacelike eigenvector $e \in V$, in which case we are into case (b), already solved. This completes the proof. \square

Proposition 2.1 provides the basic tool to classify systematically skew-symmetric endomorphisms if the dimension d is at least five. The idea is to start looking for a first spacelike eigenplane Π . Then, we restrict to Π^\perp , that is Lorentzian of dimension $d - 2$. If $d - 2 \geq 5$, proposition 2.1 applies again and we can keep on going until we reach a subspace of dimension three if d odd or dimension four if d even. Therefore, for a complete classification it only remains to solve the problem in three and four dimensions. This has already been done in [25], where a canonical form based on the classification of skew-symmetric endomorphisms is introduced. The results from [25] that we shall need are summarized in proposition 2.2 and corollary 2.5.1 and their main consequences in the present context are discussed in remarks 2.2 and 2.3 below, where we also relate the canonical form with the classification of skew-symmetric endomorphisms. For a proof and extended discussion, we refer the reader to [25]. In the remainder, when we explicitly write a matrix of entries F^α_β , where α is the row and β the column, we refer to a linear transformation expressed in a vector basis $\{e_\alpha\}_{\alpha=0}^{d-1}$ acting on the vectors $v = v^\alpha e_\alpha \in V$ by

$$F(v) = F^\alpha_\beta v^\beta e_\alpha.$$

Proposition 2.2. *For every non-zero $F \in \text{SkewEnd}(V)$, with V Lorentzian four-dimensional, there exists an orthonormal basis $B := \{e_0, e_1, e_2, e_3\}$, with e_0 timelike future directed, into which F is*

$$F = \begin{pmatrix} 0 & 0 & -1 + \frac{\sigma}{4} & \frac{\tau}{4} \\ 0 & 0 & -1 - \frac{\sigma}{4} & -\frac{\tau}{4} \\ -1 + \frac{\sigma}{4} & 1 + \frac{\sigma}{4} & 0 & 0 \\ \frac{\tau}{4} & \frac{\tau}{4} & 0 & 0 \end{pmatrix}, \quad \sigma, \tau \in \mathbb{R}, \tag{7}$$

where $\sigma := -\frac{1}{2} \text{Tr} F^2$ and $\tau^2 := -4 \det F$, with $\tau \geq 0$. Moreover, if $\tau = 0$ the vector e_3 can be taken to be any spacelike unit vector lying in the kernel of F .

Corollary 2.5.1. *For every non-zero $F \in \text{SkewEnd}(V)$, with V Lorentzian three-dimensional, there exists an orthonormal basis $B := \{e_0, e_1, e_2\}$, with e_0 timelike future directed, into which F is*

$$F = \begin{pmatrix} 0 & 0 & -1 + \frac{\sigma}{4} \\ 0 & 0 & -1 - \frac{\sigma}{4} \\ -1 + \frac{\sigma}{4} & 1 + \frac{\sigma}{4} & 0 \end{pmatrix}, \quad \sigma := -\frac{1}{2} \text{Tr}(F^2) \in \mathbb{R}. \tag{8}$$

Remark 2.2. A classification result follows because only two exclusive possibilities arise:

- (a) If either σ or τ do not vanish, F has a timelike eigenplane and an orthogonal spacelike eigenplane with respective eigenvalues

$$\mu_t := \sqrt{(-\sigma + \rho)/2} \quad \text{and} \quad \mu_s := \sqrt{(\sigma + \rho)/2} \quad \text{for} \quad \rho := \sqrt{\sigma^2 + \tau^2} \geq 0. \quad (9)$$

The inverse relation between μ_t, μ_s and σ, τ is $\sigma = \mu_s^2 - \mu_t^2$ and $\tau = 2\mu_t\mu_s$.

- (b) Otherwise, $\sigma = \tau = 0$ if and only if $\ker F$ is degenerate two-dimensional. Equivalently, F has null eigenvector orthogonal to a spacelike eigenvector both with vanishing eigenvalue.

One can easily check that when $\tau = 0$, the sign of σ determines the causal character of $\ker F$, namely $\sigma < 0$ if $\ker F$ is spacelike, $\sigma = 0$ if $\ker F$ is degenerate and $\sigma > 0$ if $\ker F$ is timelike. Obviously, $\tau \neq 0$ implies $\ker F = \{0\}$. The characteristic polynomial of F is directly calculated from (7)

$$\mathcal{P}_F(x) = (x^2 - \mu_t^2)(x^2 + \mu_s^2).$$

Remark 2.3. For a classification result in the three dimensional case, one can see by direct calculation that $q := (1 + \sigma/4)e_0 + (1 - \sigma/4)e_1$ generates $\ker F$ and furthermore $\langle q, q \rangle = -\sigma$. Hence, the sign of σ determines the causal character of $\ker F$, namely it is spacelike if $\sigma < 0$, degenerate if $\sigma = 0$ and timelike if $\sigma > 0$. Moreover, when $\sigma \neq 0$, F has an eigenplane with opposite causal character than q and eigenvalue $\sqrt{|\sigma|}$. The characteristic polynomial of F reads

$$\mathcal{P}_F(x) = x(x^2 + \sigma).$$

We have now all the necessary ingredients to give a complete classification of skew-symmetric endomorphisms of Lorentzian vector spaces. In what follows we identify Lorentzian (sub)spaces of d -dimension with the Minkowski space $\mathbb{M}^{1,d-1}$. Also, for any real number $x \in \mathbb{R}$, $[x] \in \mathbb{Z}$ denotes its integer part.

Theorem 2.6 (classification of skew-symmetric endomorphisms in Lorentzian spaces). *Let $F \in \text{SkewEnd}(V)$ with V Lorentzian of dimension $d > 2$. Then V has a set of $[\frac{d-1}{2}] - 1$ mutually orthogonal spacelike eigenplanes $\{\Pi_i\}$, $i = 1, \dots, [\frac{d-1}{2}] - 1$, so that V admits one of the following decompositions into direct sum of F -invariant subspaces:*

- (a) If d even $V = \mathbb{M}^{1,3} \oplus \Pi_{\frac{d-4}{2}} \oplus \dots \oplus \Pi_1$ and either $F|_{\mathbb{M}^{1,3}} = 0$ or otherwise one of the following cases holds:

1. $F|_{\mathbb{M}^{1,3}}$ has a spacelike eigenvector e orthogonal to a null eigenvector with vanishing eigenvalue and then $\mathbb{M}^{1,3} = \mathbb{M}^{1,2} \oplus \text{span}\{e\}$.
2. $F|_{\mathbb{M}^{1,3}}$ has a spacelike eigenplane $\Pi_{\frac{d-2}{2}}$ (as well as a timelike eigenplane $\mathbb{M}^{1,1}$ orthogonal to $\Pi_{\frac{d-2}{2}}$) and then $\mathbb{M}^{1,3} = \mathbb{M}^{1,1} \oplus \Pi_{\frac{d-2}{2}}$.

- (b) If d odd $V = \mathbb{M}^{1,2} \oplus \Pi_{\frac{d-3}{2}} \oplus \dots \oplus \Pi_1$ and either $F|_{\mathbb{M}^{1,2}} = 0$ or otherwise one of the following cases holds:

1. $F|_{\mathbb{M}^{1,2}}$ has a spacelike eigenvector e and then $\mathbb{M}^{1,2} = \mathbb{M}^{1,1} \oplus \text{span}\{e\}$.
2. $F|_{\mathbb{M}^{1,2}}$ timelike eigenvector t and then $\mathbb{M}^{1,2} = \text{span}\{t\} \oplus \Pi_{\frac{d-1}{2}}$.
3. $F|_{\mathbb{M}^{1,2}}$ has a null eigenvector with vanishing eigenvalue.

Proof. The proof is a simple combination of the previous results. First, if $d \geq 5$, we can apply proposition 2.1 to obtain the first spacelike eigenplane Π_1 . Then Π_1^\perp is Lorentzian of dimension $d - 2$. If $d - 2 \geq 5$, we can apply again proposition 2.1 to obtain a second eigenplane Π_2 . Continuing with this process, depending on d , two things can happen:

- (a) If d even, we get $\frac{d-4}{2} (= [\frac{d-1}{2}] - 1)$ spacelike eigenplanes, until we eventually reach a Lorentzian vector subspace of dimension four, $\mathbb{M}^{1,3}$, where proposition 2.1 cannot be applied. In $\mathbb{M}^{1,3}$, either $F|_{\mathbb{M}^{1,3}} = 0$ or otherwise cases (1) and (2) follow from remark 2.2, cases 2 and 1 respectively.
- (b) If d odd, we get $\frac{d-3}{2} (= [\frac{d-1}{2}] - 1)$ spacelike eigenplanes, until we reach a Lorentzian vector subspace of dimension three, $\mathbb{M}^{1,2}$. In $\mathbb{M}^{1,2}$, either $F|_{\mathbb{M}^{1,2}} = 0$ or by remark 2.3 there exists a unique eigenvector σ with vanishing eigenvalue. If σ null, case (3) follows. If it is spacelike $e := \sigma$, F restricts to $e^\perp = \mathbb{M}^{1,1} \subset \mathbb{M}^{1,2}$ and (1) follows. If σ timelike, the same argument applies with $t := \sigma$ and $t^\perp \subset \mathbb{M}^{1,2}$ defines the remaining spacelike plane $\Pi_{\frac{d-1}{2}}$.

□

3. Canonical form for skew-symmetric endomorphisms

Our aim here is to extend the results in proposition 2.2 and corollary 2.5.1 to arbitrary dimensions. To do that, we will employ the classification theorem 2.6 derived in section 2, from which it immediately follows a decomposition of any $F \in \text{SkewEnd}(V)$ into direct sum of skew-symmetric endomorphisms of the subspaces that F restricts to, namely

$$F = F|_{\mathbb{M}^{1,3}} \bigoplus_{i=1}^{[\frac{d-1}{2}]-1} F|_{\Pi_i} \quad \text{if } d \text{ even,} \tag{10}$$

$$F = F|_{\mathbb{M}^{1,2}} \bigoplus_{i=1}^{[\frac{d-1}{2}]-1} F|_{\Pi_i} \quad \text{if } d \text{ odd,} \tag{11}$$

where Π_i are spacelike eigenplanes. In what follows, we will denote

$$p := [(d - 1)/2] - 1.$$

Notice that the blocks $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ may also admit different subdecompositions depending on the case. In order to avoid a case-by-case analysis we leave this part unaltered. It will be convenient for the rest of the paper to give a name to the decompositions (10) and (11):

Definition 3.1. Let $F \in \text{SkewEnd}(V)$ non-zero for V Lorentzian d -dimensional. Then, a decomposition of the form (10) or (11) is called **block form** of F . A basis that realizes a block form is called **block form basis**.

Writing F in block form allows us to work with F as a sum of skew-symmetric endomorphisms of Riemannian two-planes plus one skew-symmetric endomorphism of a three or four dimensional Lorentzian vector space. For the latter we will employ the canonical forms in proposition 2.2 and corollary 2.5.1, and for the former, it is immediate that in every (suitably oriented) orthonormal basis of Π_i

$$F|_{\Pi_i} = \begin{pmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{pmatrix}, \quad 0 \leq \mu_i \in \mathbb{R}. \tag{12}$$

Having defined a canonical form for four, three and two dimensional endomorphisms (i.e. matrices (7), (8) and (12) respectively), the idea is to extend this result to arbitrary dimensions

finding a systematic way to construct a block form (10), (11) such that each of the blocks are in canonical form. This is not immediate, firstly, because the block form does not require the blocks $F|_{\mathbb{M}^{1,3}}$ or $F|_{\mathbb{M}^{1,2}}$ to be non-zero and secondly, because, unlike in the four and three dimensional cases, the parameters σ, τ of the four and three dimensional blocks cannot be invariantly defined as, for example, traces of F^2 or determinant of F . The first of these concerns is easily solved by suitably choosing a block form:

Lemma 3.1. *Let $F \in \text{SkewEnd}(V)$ be non-zero for V Lorentzian of dimension d . Then there exists a block form (10) and (11) such that $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ are non-zero and they either contain no spacelike eigenplanes or they contain one with largest eigenvalue (among all spacelike eigenplanes of F). In addition, the rest of spacelike eigenplanes Π_i are sorted by decreasing value of μ_i^2 , i.e. $\mu_1^2 \geq \mu_2^2 \geq \dots \geq \mu_p^2$.*

Proof. If $\ker F$ is degenerate¹, it must correspond with cases 1 (d even) or 3 (d odd) of theorem 2.6. Hence, in any block form the blocks $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ are non-zero and they do not contain any spacelike eigenplane, as claimed in the lemma. So let us assume that $\ker F$ is non-degenerate or zero, which discards cases 1 and 3 of theorem 2.6. In all possible cases, any block form admits the following splitting in

$$F|_{\mathbb{M}^{1,3}} = F|_{\Pi_t} \oplus F|_{\Pi_s}, \quad F|_{\mathbb{M}^{1,2}} = F|_{\text{span}\{v\}} \oplus F|_{v^\perp}, \tag{13}$$

with Π_s, Π_t spacelike and timelike eigenplanes with (possibly zero) respective eigenvalues μ_s and μ_t , v a timelike or spacelike eigenvector (in $\ker F$) and $v^\perp \subset \mathbb{M}^{1,2}$ an eigenplane with opposite causal character than v . If v is spacelike, then either $F|_{v^\perp}$ is non-zero, in which case $F|_{\mathbb{M}^{1,2}} \neq 0$ and clearly contains no spacelike eigenplanes (which is one of the possibilities in the lemma), or $F|_{v^\perp} = 0$ and then $F|_{\mathbb{M}^{1,2}} = 0$, so we can rearrange the decomposition (13) using some timelike vector $v' \in v^\perp$ instead of v , i.e. $F|_{\mathbb{M}^{1,2}} = F|_{\text{span}\{v'\}} \oplus F|_{v'^\perp}$. Hence, in the case of d odd, we may assume that v is timelike and $v^\perp \subset \mathbb{M}^{1,2}$ is a spacelike eigenplane. Let Π_μ be a spacelike eigenplane of F with largest eigenvalue μ among Π_s (d even) or v^\perp (d odd) and Π_1, \dots, Π_p . Then, switching $F|_{\Pi_s}$ or $F|_{\Pi_{v^\perp}}$ by $F|_{\Pi_\mu}$ we construct

$$\hat{F}|_{\mathbb{M}^{1,3}} := F|_{\Pi_t} \oplus F|_{\Pi_\mu}, \quad \hat{F}|_{\mathbb{M}^{1,2}} := F|_{\text{span}\{v\}} \oplus F|_{\Pi_\mu}.$$

The resulting matrix is still in block form and has non-zero blocks $\hat{F}|_{\mathbb{M}^{1,3}}$, $\hat{F}|_{\mathbb{M}^{1,2}}$ containing a spacelike eigenplane with largest eigenvalue, which is the other possibility in the lemma. The last claim follows by simply rearranging the remaining spacelike eigenplanes Π_i by decreasing order of μ_i^2 . \square

With a skew-symmetric endomorphism F in the block form given in lemma 3.1 we can take each of the blocks to its respective canonical form. Let us denote $F_{\sigma\tau} := F|_{\mathbb{M}^{1,3}}$ (if d even), $F_\sigma := F|_{\mathbb{M}^{1,2}}$ (if d odd) and $F_{\mu_i} := F|_{\Pi_i}$ when written in the canonical forms (7), (8) and (12) respectively. Consequently

$$F = F_{\sigma\tau} \bigoplus_{i=1}^p F_{\mu_i} \quad (d \text{ even}), \quad F = F_\sigma \bigoplus_{i=1}^p F_{\mu_i} \quad (d \text{ odd}), \tag{14}$$

where, notice, each of the blocks is written in an orthonormal basis of the corresponding subspace, which moreover is future directed if the subspace is Lorentzian, i.e. $\mathbb{M}^{1,3}$ or $\mathbb{M}^{1,2}$ (cf

¹By degenerate we always mean degenerate signature and dimension at least one.

proposition 2.2 and corollary 2.5.1). Hence, the form given in (14) corresponds to a future directed, orthonormal basis of $\mathbb{M}^{1,d-1}$.

Our aim now is to give an invariant definition of σ, τ, μ_i . A possible way to do this is through the eigenvalues of F^2 . One may wonder why not to use directly the eigenvalues of F . One reason is that since we are interested in real Lorentzian vector spaces V (although, for practical reasons, we may rely on the complexification $V_{\mathbb{C}}$ for some proofs), it is more consistent to give our canonical form in terms of real quantities, while the eigenvalues of F may be complex. In addition, the canonical form will require to sort them in some way, for which using real numbers is better suited.

The characteristic polynomial of F is known (e.g. [24]) to possess the following parity:

$$\mathcal{P}_F(x) = (-1)^d \mathcal{P}_F(-x). \tag{15}$$

Thus, a simple calculation relates the characteristic polynomials of F and F^2

$$\begin{aligned} \mathcal{P}_{F^2}(x) &= \det(xId_d - F^2) = \det(\sqrt{x}Id_d - F) \det(\sqrt{x}Id_d + F) \\ &= (-1)^d \mathcal{P}_F(\sqrt{x}) \mathcal{P}_F(-\sqrt{x}) = (\mathcal{P}_F(\sqrt{x}))^2, \end{aligned} \tag{16}$$

\sqrt{x} being any of the square roots of x in \mathbb{C} and Id_d the $d \times d$ identity matrix. We can extract some conclusions from (16):

Lemma 3.2. *Let $F \in \text{SkewEnd}(V)$ for V Lorentzian of dimension d . Then the non-zero eigenvalues of F^2 have even multiplicity m_a and the zero eigenvalue has multiplicity m_0 with the parity of d . In addition, F possesses p_a (resp. exactly one) spacelike (resp. timelike) eigenplanes with eigenvalue $\mu \neq 0$ if and only if F^2 has a negative (resp. positive) non-zero eigenvalue $-\mu^2$ (resp. μ^2) with multiplicity $m_a := 2p_a$ (resp. exactly two).*

Proof. It is an immediate consequence of equation (16) that non-zero eigenvalues of F^2 must have even multiplicity m_a . Moreover, since the sum of all multiplicities adds up to the dimension d , the multiplicity of the zero m_0 has the parity of d .

Combining lemma 2.2 and equation (16), F has a spacelike eigenplane Π with non-zero eigenvalue μ if and only if F^2 has a negative double² eigenvalue $-\mu^2$. If $d \leq 4$, there cannot be any other spacelike eigenplanes in Π^\perp , so applying the same argument to $F|_{\Pi^\perp} \in \text{SkewEnd}(\Pi^\perp)$, the multiplicity m_a of $-\mu^2$ must be $m_a = 2$. If $d > 4$ and $m_a \geq 4$, then $-\mu^2$ is an eigenvalue of $(F|_{\Pi^\perp})^2$ with multiplicity $m_a - 2$, thus F has a second spacelike eigenplane with eigenvalue μ in Π^\perp . Repeating this argument, F^2 has a negative eigenvalue $-\mu^2$ with multiplicity m_a if and only if F has $p_a = m_a/2$ spacelike eigenplanes with eigenvalue μ .

Finally, by lemma 2.3 and equation (16), F has a timelike eigenplane Π with non-zero eigenvalue μ if and only if F^2 has a positive double eigenvalue μ^2 . Obviously, the maximum number of timelike eigenplanes that F can have is one. Thus, $F|_{\Pi^\perp}$ cannot have timelike eigenplanes and hence $(F|_{\Pi^\perp})^2$ has no additional positive eigenvalues. Consequently, the multiplicity of μ^2 is exactly two. \square

Taking into account lemma 3.2, we will employ the eigenvalues of $-F^2$ rather than those of F^2 , so we assign positive eigenvalues of F^2 with spacelike eigenplanes and negative eigenvalues to timelike eigenplanes. This amounts to employ the roots of the characteristic polynomial $\mathcal{P}_{F^2}(-x)$.

² We adopt the convention that a root with multiplicity $m \geq 2$ is also double.

We now discuss how to invariantly define the parameters σ, τ, μ_i for d even and σ, μ_i for d odd. The result of the argument is formalized below in definition 3.2. Recall that the characteristic polynomial of a direct sum of two or more endomorphisms is the product of their individual characteristic polynomials, in particular, the characteristic polynomial of $-F^2$ equals to the product of the characteristic polynomials of $-F_{\sigma\tau}^2$ or $-F_{\sigma}^2$ times those of each $-F_{\mu_i}^2$ (cf equation (14)). Let us define:

$$\begin{aligned} \mathcal{Q}_{F^2}(x) &:= (\mathcal{P}_{F^2}(-x))^{1/2} \quad (d \text{ even}), \\ \mathcal{Q}_{F^2}(x) &:= \left(\frac{\mathcal{P}_{F^2}(-x)}{x}\right)^{1/2} \quad (d \text{ odd}), \end{aligned} \tag{17}$$

Starting with d even, from formula (14) it is immediate that μ_i^2 are double roots of $\mathcal{P}_{F^2}(-x^2)$, which by lemma 3.1 satisfy $\mu_1^2 \geq \dots \geq \mu_p^2 \geq 0$. On the other hand, let $\mu_t := \sqrt{(-\sigma + \rho)/2}$ and $i\mu_s := i\sqrt{(\sigma + \rho)/2}$ with $\rho := \sqrt{\sigma^2 + \tau^2} \geq 0$, that by remark 2.2, are roots of $\mathcal{P}_{F_{\sigma\tau}}(x)$, thus roots of $\mathcal{P}_F(x)$. By equation (16), $-\mu_t^2, \mu_s^2$ are double roots of $\mathcal{P}_{F^2}(-x)$. The set $\{-\mu_t^2, \mu_s^2, \mu_1^2, \dots, \mu_p^2\}$ are in total $p + 2 = [(d - 1)/2] + 1 = d/2$ elements, each of which is a double root of $\mathcal{P}_{F^2}(-x)$. In other words, $\{-\mu_t^2, \mu_s^2, \mu_1^2, \dots, \mu_p^2\}$ is the set of all roots of the polynomial³ $\mathcal{Q}_{F^2}(x)$. If $\ker F$ is degenerate, then $\ker F_{\sigma\tau}$ is degenerate and by remark 2.2 it must happen $\mu_t = \mu_s = 0$. Hence $\mu_1^2 \geq \mu_2^2 \geq \dots \geq \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$. Otherwise, also by remark 2.2, $F_{\sigma\tau}$ contains a spacelike eigenplane with eigenvalue μ_s (which by lemma 3.1 is the largest) as well as a timelike eigenplane with eigenvalue μ_t . In this case $\mu_s^2 \geq \mu_1^2 \geq \dots \geq \mu_p^2 \geq 0 \geq -\mu_t^2$.

We next discuss σ, μ_i for d odd. Again, from (14) we have that μ_i^2 are double roots of $\mathcal{P}_{F^2}(-x^2)$, which by lemma 3.1 also satisfy $\mu_1^2 \geq \dots \geq \mu_p^2 \geq 0$. By remark 2.3, $\sqrt{\sigma}$ is a root of $\mathcal{P}_{F_{\sigma}}(x)$, thus a root of $\mathcal{P}_F(x)$, so by formula (16), σ is a double root of $\mathcal{P}_{F^2}(-x)$. Also, $\mathcal{P}_{F^2}(-x)$ has at least one zero root and hence, $\mathcal{P}_{F^2}(-x)/x$ is a polynomial with $d - 1$ roots (counting multiplicity). Then, the set $\{\sigma, \mu_1^2, \dots, \mu_p^2\}$ are all double roots of $\mathcal{P}_{F^2}(-x)/x$, which are $p + 1 = [(d - 1)/2] + 1 = (d - 1)/2 + 1 = d/2$ elements. Therefore \mathcal{Q}_{F^2} as defined in (17) is also a polynomial and $\{\sigma, \mu_1^2, \dots, \mu_p^2\}$ is the set of all its roots. If $\ker F$ is timelike, then $\ker F_{\sigma}$ is timelike, which happens if and only if $\sigma > 0$ (cf remark 2.3) and also F_{σ} has a spacelike eigenplane with eigenvalue $\sqrt{|\sigma|}$, that by lemma 3.1 is the largest eigenvalue among spacelike eigenplanes. Thus $\sigma \geq \mu_1^2 \geq \dots \geq \mu_p^2$. In the case $\ker F$ not timelike, the inequalities become $\mu_1^2 \geq \dots \geq \mu_p^2 \geq 0 \geq \sigma$.

Summarizing, the parameters σ, τ, μ_i correspond to the set of all roots of \mathcal{Q}_{F^2} sorted in a certain order fully determined by the causal character of $\ker F$. This allows us to put forward the following definition:

Definition 3.2. Let $\text{Roots}(\mathcal{Q}_{F^2})$ denote the set of roots of $\mathcal{Q}_{F^2}(x)$ repeated as many times as their multiplicity. Then

- (a) If d odd, $\{\sigma; \mu_1^2, \dots, \mu_p^2\} := \text{Roots}(\mathcal{Q}_{F^2})$ sorted by $\sigma \geq \mu_1^2 \geq \dots \geq \mu_p^2$ if $\ker F$ is timelike and $\mu_1^2 \geq \dots \geq \mu_p^2 \geq 0 \geq \sigma$ otherwise.
- (b) If d even, $\sigma := \mu_s^2 - \mu_t^2, \tau := 2|\mu_t\mu_s|$ with $\{-\mu_t^2, \mu_s^2; \mu_1^2, \dots, \mu_p^2\} := \text{Roots}(\mathcal{Q}_{F^2})$ sorted by $\mu_1^2 \geq \dots \geq \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$ if $\ker F$ is degenerate and $\mu_s^2 \geq \mu_1^2 \geq \dots \geq \mu_p^2 \geq 0 \geq -\mu_t^2$ otherwise.

³ $\mathcal{Q}_{F^2}(x)$ is a polynomial because all the roots of $\mathcal{P}_{F^2}(-x)$ are double.

In addition, we also summarize the results concerning the canonical form in the following theorem:

Theorem 3.3. *Let $F \in \text{SkewEnd}(V)$ non-zero, with V Lorentzian of dimension $d \geq 3$ and $p := [(d - 1)/2] - 1$. Then there exists an orthonormal, future oriented basis such that F is given by (14) where $F_{\sigma\tau} := F|_{\mathbb{M}^{1,3}}$, $F_\sigma := F|_{\mathbb{M}^{1,2}}$, $F_{\mu_i} := F|_{\Pi_i}$ are given by (7), (8), (12) respectively and σ, τ, μ_i are given in definition 3.2. In particular, $F_{\sigma\tau}, F_\sigma$ are non-zero and they either do not contain a spacelike eigenplane or they contain one with maximal eigenvalue (among all spacelike eigenplanes of F) and the eigenvalues μ_i are sorted by $\mu_1^2 \geq \mu_2^2 \geq \dots \geq \mu_p^2$.*

Definition 3.3. For any $F \in \text{SkewEnd}(V)$, for V Lorentzian d -dimensional, the form of F given in theorem 3.3 is called **canonical form** and the basis realizing it is called **canonical basis**.

The first and obvious reason why the canonical form is useful is that it allows one to work with all elements $F \in \text{SkewEnd}(V)$ at once. The fact that we can give a canonical form for every element without splitting into cases is a great strength, since we can perform a general analysis just in terms of the parameters that define the canonical form. Moreover, as we will show in section 5, this form is the same for all the elements in the orbit generated by the adjoint action of the orthochronous Lorentz group $O^+(1, d - 1)$. Thus, the canonical form is specially suited for problems with $O^+(1, d - 1)$ invariance (or covariance) which, as discussed in section 6, is directly related to certain conformally covariant problems in GR.

We finish this section with two corollaries that will be useful later. The first one is trivial from the canonical form (14)

Corollary 3.3.1. *The characteristic polynomial of $F \in \text{SkewEnd}(V)$ is*

$$\begin{aligned} \mathcal{P}_F(x) &= (x^2 - \mu_t^2)(x^2 + \mu_s^2) \prod_{i=1}^p (x^2 + \mu_i^2) \quad (d \text{ even}), \\ \mathcal{P}_F(x) &= x(x^2 + \sigma) \prod_{i=1}^p (x^2 + \mu_i^2) \quad (d \text{ odd}), \end{aligned} \tag{18}$$

where $-2\mu_t^2 := \sigma - \sqrt{\sigma^2 + \tau^2}$, $2\mu_s^2 := \sigma + \sqrt{\sigma^2 + \tau^2}$.

The second gives a formula for the rank of F . We base our proof in the canonical form (14) because it is straightforward. However, we remark that this corollary can also be regarded as a consequence of theorem 2.6.

Corollary 3.3.2. *Let $F \in \text{SkewEnd}(V)$, with V Lorentzian of dimension d and m_0 the multiplicity of the zero eigenvalue. Then, only of the following exclusive cases hold:*

- (a) $\ker F$ is non-degenerate or zero if and only if $\text{rank } F = d - m_0$.
- (b) $\ker F$ is degenerate if and only if $m_0 > 2$ and $\text{rank } F = d - m_0 + 2$.

Proof. Consider F in canonical form (14) and let $k \in \mathbb{N}$ be the number of parameters μ_i that vanish. For d even we have $\dim \ker F = 2k + \dim \ker F_{\sigma\tau}$. On the one hand, $\ker F$ degenerate implies $\ker F_{\sigma\tau}$ degenerate, which by remark 2.2 happens if and only if $\sigma = \tau = 0$ and in addition $\dim \ker F_{\sigma\tau} = 2$. Therefore $\dim \ker F = 2k + 2$ and by (18), $m_0 = 2k + 4 (> 2)$. Thus $\text{rank } F = d - \dim \ker F = d - m_0 + 2$. On the other hand, $\ker F$ non-degenerate if at most one of σ or τ vanish. If $\tau \neq 0$ (so that $\mu_s \neq 0$ and $\mu_t \neq 0$), $\dim \ker F_{\sigma\tau} = 0$ and $m_0 = 2k = \dim \ker F$. Consequently $\text{rank } F = d - m_0$. If $\tau = 0$ (and $\sigma \neq 0$, so that exactly one of μ_s, μ_t

vanish), by remark 2.2 $\dim \ker F_{\sigma\tau} = 2$ and by (18) $m_0 = 2k + 2$. Hence $\dim \ker F = 2k + 2$ and $\text{rank } F = d - m_0$.

For d odd, we have $\dim \ker F = 2k + \dim \ker F_\sigma = 2k + 1$, because $\dim \ker F_\sigma = 1$ (cf remark 2.3). $\ker F$ is degenerate if and only if $\ker F_\sigma$ is degenerate, which by remark 2.3 occurs if and only if $\sigma = 0$. Hence, by equation (18), $m_0 = 2k + 3 (> 2)$ and $\text{rank } F = d - \dim \ker F = d - m_0 + 2$. For the $\ker F$ non-degenerate case, $\sigma \neq 0$ and also by (18) $m_0 = 2k + 1 = \dim \ker F$. Therefore $\text{rank } F = d - m_0$. \square

4. Simple endomorphisms

By *simple skew-symmetric endomorphism* we mean a $G \in \text{SkewEnd}(V)$ satisfying $\text{rank } G = 2$. As usual $e_b \equiv \langle e, \cdot \rangle$ is the one-form obtained by lowering index to a vector $e \in V$. Then, a simple skew-symmetric endomorphism can be always written as

$$G = e \otimes v_b - v \otimes e_b$$

for two linearly independent vectors $e, v \in V$ and its action on any vector $w \in V$ is

$$G(w) = \langle v, w \rangle e - \langle e, w \rangle v.$$

Since the two-form associated to a simple endomorphism is $\mathbf{G} = e_b \wedge v_b$, it follows from elementary algebra that two simple skew-symmetric endomorphisms $G = e \otimes v_b - v \otimes e_b$ and $G' = e' \otimes v'_b - v' \otimes e'_b$ are proportional if and only if $\text{span}\{e, v\} = \text{span}\{e', v'\}$. This freedom in the pair $\{e, v\}$ defining G can be used to choose them orthogonal.

Lemma 4.1. *Let $G \in \text{SkewEnd}(V)$ be simple. Then there exist two non-zero orthogonal vectors $e, v \in V$ such that $G = e \otimes v_b - v \otimes e_b$ with v spacelike.*

Proof. By definition $G = \tilde{e} \otimes \tilde{v}_b - \tilde{v} \otimes \tilde{e}_b$ for two linearly independent vectors $\tilde{e}, \tilde{v} \in V$. If one of them is non-null, we set $\tilde{v} := v$ and decompose $V = \text{span}\{v\} \oplus v^\perp$. Thus $\tilde{e} = av + e$ with $a \in \mathbb{R}$ and $e \in v^\perp$ and G takes the form $G = (av + e) \otimes v_b - v \otimes (av + e)_b = e \otimes v_b - v \otimes e_b$, as claimed. If \tilde{e} and \tilde{v} are both null, consider $V = \text{span}\{\tilde{e}\} \oplus (\tilde{e})^\perp$ (we use \oplus because this direct sum is not by orthogonal spaces) where $(\tilde{e})^\perp$ is a spacelike complement of $\text{span}\{\tilde{e}\}$. Then we can write $\tilde{v} = a\tilde{e} + v'$, with $a \in \mathbb{R}$ and $v' \in (\tilde{e})^\perp$ non-null. Thus $G = \tilde{e} \otimes v'_b - v' \otimes \tilde{e}_b$, with v' non-null and we fall into the previous case. All in all, $G = e \otimes v_b - v \otimes e_b$ with e, v orthogonal. Consequently, either one of the vectors is spacelike or both are null and proportional which would imply $G = 0$, against our hypothesis $\text{rank } G = 2$. \square

The decomposition $G = e \otimes v_b - v \otimes e_b$ is not unique even with the restriction of v being spacelike unit and orthogonal to e . One can easily show that the remaining freedom is given by the transformation $e' = ae - b \langle e, e \rangle v$, $v' = be + av$ with $a, b \in \mathbb{R}$ restricted to $a^2 + b^2 \langle e, e \rangle = 1$. Nevertheless, the square norm $\langle e', e' \rangle$ is invariant under this change, so the following definition makes sense:

Definition 4.1. Let $G \in \text{SkewEnd}(V)$ be simple, with $G = e \otimes v_b - v \otimes e_b$, $e, v \in V$ orthogonal with v spacelike unit. Then G is said to be *spacelike*, *timelike* or *null* if the vector e is *spacelike*, *timelike* or *null* respectively. In the non-null case, G is called spacelike (resp. timelike) unit whenever $\langle e, e \rangle = +1$ (resp. $\langle e, e \rangle = -1$).

By lemma 4.1, it is immediate that definition 4.1 comprises any possible simple endomorphism (up to a multiplicative factor).

We next obtain the necessary and sufficient conditions for a simple endomorphism G to commute with a given $F \in \text{SkewEnd}(V)$. We first make the simple observation that the composition of a one-form e_b and a skew-symmetric endomorphism F satisfies (simply apply for sides to any $w \in V$)

$$e_b \circ F = -F(e)_b.$$

An immediate consequence is that for any pair of vectors $e, v \in V$ and $F \in \text{SkewEnd}(V)$ it holds

$$F \circ (e \otimes v_b) = F(e) \otimes v_b, \quad (e \otimes v_b) \circ F = -e \otimes F(v)_b \tag{19}$$

The following commutation result will be used later.

Lemma 4.2. *Let $F, G \in \text{SkewEnd}(V)$ with $G = e \otimes v_b - v \otimes e_b$ simple and $e, v \in V$ as in definition 4.1. Then $[F, G] = 0$ if and only if there exist $\mu \in \mathbb{R}$ such that:*

$$F(e) = \langle e, e \rangle \mu v, \quad F(v) = -\mu e. \tag{20}$$

Proof. The commutator is

$$\begin{aligned} [F, G] &= F \circ G - G \circ F = F \circ (e \otimes v_b - v \otimes e_b) - (e \otimes v_b - v \otimes e_b) \circ F \\ &= F(e) \otimes v_b - F(v) \otimes e_b + e \otimes F(v)_b - v \otimes F(e)_b, \end{aligned} \tag{21}$$

where we have used (19). The ‘if’ part is obtained by direct calculation inserting (20) in (21). To prove the ‘only if’ part, the condition $[F, G] = 0$ requires the two endomorphisms $F(e) \otimes v_b - v \otimes F(e)_b$ and $F(v) \otimes e_b - e \otimes F(v)_b$ to be equal. One such endomorphism is either identically zero or simple. This implies that $\text{span}\{F(e), v\}$ and $\text{span}\{e, F(v)\}$ are either both one dimensional or both two-dimensional and equal. In the first case, $F(v) = -\mu e$ and $F(e) = \alpha v$ for $\mu, \alpha \in \mathbb{R}$, which are determined by skew-symmetry to satisfy $\alpha = \mu \langle e, e \rangle$, so the lemma follows. The second case is empty, for it is necessary that $v = ae + bF(v)$ with $a, b \in \mathbb{R}$, which implies $\langle v, v \rangle = \langle ae + bF(v), v \rangle = b \langle F(v), v \rangle = 0$, against the hypothesis of v being spacelike. \square

Corollary 4.2.1. *Let $G, G' \in \text{SkewEnd}(V)$ be simple, spacelike and linearly independent. Let $\{e, v\}, \{e', v'\}$ be orthogonal spacelike vectors such that $G = e \otimes v_b - v \otimes e_b$ and $G' = e' \otimes v'_b - v' \otimes e'_b$. Then $[G, G'] = 0$ if and only if $\{e, v, e', v'\}$ are mutually orthogonal.*

Proof. By the previous lemma $[G, G'] = 0$ if and only if there exist $\mu \in \mathbb{R}$ such that

$$G(e') = \langle e', v \rangle e - \langle e', e \rangle v = \mu v', \quad G(v') = \langle v', v \rangle e - \langle v', e \rangle v = -\mu e'. \tag{22}$$

If $\mu \neq 0$, then $\text{span}\{e, v\} = \text{span}\{e', v'\}$ and G and G' are proportional, against hypothesis. Thus, $\mu = 0$ and by (22) the set $\{e, v, e', v'\}$ is mutually orthogonal. \square

5. $O^+(1, d-1)$ -classes

In this section we use the canonical form of section 3 to characterize skew-symmetric endomorphisms of V under the adjoint action of the orthochronous Lorentz group $O^+(1, d-1)$. Recall that this is the subgroup of $O(1, d-1)$ preserving time orientation. The corresponding classes of skew-symmetric endomorphisms are also known as the adjoint orbits or conjugacy

classes and we denote them by $[F]_{O^+}$ for a given element $F \in \text{SkewEnd}(V)$. The characterization of these orbits by a set of independent invariants is known and it can be found in [24] in terms of two-forms and other references such as [5]. What we do here is, first, to give an alternative way to characterize the orbits $[F]_{O^+}$ by a convenient set of invariants and second, to show that the canonical form is the same for every element in a given orbit. This makes the canonical form specially useful as a tool for problems with $O(1, d - 1)$ invariance.

We formulate this section in terms of the orthochronous component $O^+(1, d - 1)$ because of its relation with conformal transformations of the sphere \mathbb{S}^{d-2} (see section 6), but note that the orbits of the full group $O(1, d - 1)$ are exactly the same as those of $O^+(1, d - 1)$. Recall that the time-reversing component $O^-(1, d - 1)$ is one-to-one with $O^+(1, d - 1)$. We can map elements $\Lambda^- \in O^-(1, d - 1)$ to elements in $\Lambda^+ \in O^+(1, d - 1)$ by e.g. $\Lambda^+ := \Lambda^- \Lambda_0$, where $\Lambda_0 = -Id_d$. Then

$$\Lambda^+ F (\Lambda^+)^{-1} = \Lambda^- \Lambda_0 F \Lambda_0 (\Lambda^-)^{-1} = \Lambda^- F (\Lambda^-)^{-1},$$

which clearly implies that the orbits generated by the full group $O(1, d - 1)$ coincide with the orbits generated by the subgroup $O^+(1, d - 1)$.

A consequence of equation (15) is that the characteristic polynomial of $F \in \text{SkewEnd}(V)$ must have the form

$$\mathcal{P}_F(x) = x^d + \sum_{b=1}^q c_b x^{d-2b}, \tag{23}$$

where we have introduced $q := \lfloor \frac{d}{2} \rfloor$. The coefficients c_b can be obtained using the Faddeev–LeVerrier algorithm, summarized by the following matrix determinant [15]:

$$c_b = \frac{1}{(2b)!} \begin{vmatrix} \text{Tr } F & 2b - 1 & 0 & \dots & 0 \\ \text{Tr } F^2 & \text{Tr } F & 2b - 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Tr } F^{2b-1} & \text{Tr } F^{2b-2} & \dots & \dots & 1 \\ \text{Tr } F^{2b} & \text{Tr } F^{2b-1} & \dots & \dots & \text{Tr } F \end{vmatrix}.$$

Since the traces of odd powers vanish by skew-symmetry, the coefficients c_b depend on the entries of F only through the traces of the squared powers of F :

$$I_b = \frac{1}{2} \text{Tr}(F^{2b}), \quad b = 1, \dots, q.$$

Recall that the adjoint representation Ad of a matrix Lie group G is a linear representation of G on its Lie algebra automorphisms $\text{Aut}(\mathfrak{g})$ given by

$$\begin{aligned} \text{Ad} : G &\rightarrow \text{Aut}(\mathfrak{g}) \\ g &\mapsto \text{Ad}(g) := \text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g} \quad X \mapsto gXg^{-1}. \end{aligned}$$

The traces I_b are obviously invariant under the adjoint action of $O^+(1, d - 1)$ and so are the coefficients c_b . Another invariant that plays an important role in the classification of conjugacy classes is the rank of F . Since this is always even, we denote it by

$$\text{rank } F = 2r,$$

and clearly $r \leq q$. From now we say **rank parameter** to refer to r . In the following proposition we show that this set of invariants actually identifies the canonical form.

Proposition 5.1. *Let $F, \tilde{F} \in \text{SkewEnd}(V)$, for V Lorentzian of dimension d . Then the invariants $\{c_b, r\}$ and $\{\tilde{c}_b, \tilde{r}\}$ of F and \tilde{F} respectively are equal if and only if their canonical forms given by theorem 3.3 are the same.*

Proof. The ‘if’ part (\Leftarrow) is trivial, because the invariants c_b, r are independent on the basis, so they can be calculated in a canonical basis. Hence, same canonical form implies same invariants. For the ‘only if’ part (\Rightarrow), we notice that if the coefficients c_b and \tilde{c}_b of \mathcal{P}_F and $\mathcal{P}_{\tilde{F}}$ are equal, so are their characteristic polynomials, the multiplicities of their zero eigenvalue and the polynomials \mathcal{Q}_{F^2} and $\mathcal{Q}_{\tilde{F}^2}$ (equation (17)). Since $\text{rank } F = \text{rank } \tilde{F}$, corollary 3.3.2 implies that $\ker F$ and $\ker \tilde{F}$ must have the same causal character. The canonical form only depends on the roots \mathcal{Q}_{F^2} and the causal character of $\ker F$ through definition 3.2. Thus, F and \tilde{F} must have the same canonical form. \square

We now characterize the classes $[F]_{O^+}$ in terms of the same invariants given in proposition 5.1. As mentioned above, this result is known [24], but we give here an alternative and very simple proof based on our canonical form:

Theorem 5.1. [24] *Let $F, \tilde{F} \in \text{SkewEnd}(V)$, for V Lorentzian of dimension d . Then their invariants $\{c_b, r\}$ and $\{\tilde{c}_b, \tilde{r}\}$ are the same if and only if F and \tilde{F} are $O^+(1, d - 1)$ -related.*

Proof. The if (\Leftarrow) part is immediate, since it is trivial from their definitions that the quantities $\{c_b, r\}$ are Lorentz invariant. To prove the ‘only if’ (\Rightarrow), by proposition 5.1, F and \tilde{F} have the same canonical form in canonical bases B and \tilde{B} respectively. By definition (cf theorem 3.3), these bases are unit, future oriented and orthonormal. Thus, the transformation taking B to \tilde{B} transforms F into \tilde{F} and both must be $O^+(1, d - 1)$ -related. \square

Theorem 5.1 establishes the necessary and sufficient conditions for two endomorphisms to be $O^+(1, d - 1)$ -related. Combining this result with proposition 5.1, we find that the canonical form (hence the parameters σ, μ_i^2 or σ, τ, μ_i^2) totally define the equivalence class of skew-symmetric endomorphisms up to $O^+(1, d - 1)$ transformations. Moreover, we emphasize that this form is the same for every equivalence class, unlike other canonical (or normal) forms based on the classification of $\text{SkewEnd}(V)$, such as the one in [9], where they seek irreducibility of the blocks, so they must give two different forms to cover every case.

Next, we discuss some facts about the coefficients of the characteristic polynomial, also stated in [24], where the proof is only indicated, and which can now be easily proven using the canonical form.

Lemma 5.2. *Let $F \in \text{SkewEnd}(V)$ be non-zero and let $2r = \text{rank } F$. Then $c_r > 0$, $c_r = 0$, $c_r < 0$ if and only if $\ker F$ is timelike, null or spacelike (or zero) respectively. Moreover, if $r < q$, $c_q = c_{q-1} = \dots = c_{r+1} = 0$.*

Proof. Taking into account that the parities of d and m_0 are equal (lemma 3.2), $q - \lfloor \frac{m_0}{2} \rfloor = \lfloor \frac{d}{2} \rfloor - \lfloor \frac{m_0}{2} \rfloor = \frac{d-m_0}{2}$, so equation (23) can be rewritten

$$\mathcal{P}_F(x) = x^{m_0} \left(x^{d-m_0} + \sum_{b=1}^{q-\lfloor \frac{m_0}{2} \rfloor} c_b x^{d-m_0-2b} \right) = x^{m_0} \left(x^{d-m_0} + \sum_{b=1}^{\frac{d-m_0}{2}} c_b x^{d-m_0-2b} \right), \quad (24)$$

where we have explicitly substituted all zero coefficients by extracting the common factor x^{m_0} , thus the remaining coefficients $c_b \neq 0$ for $b = 1, \dots, (d - m_0)/2$. By corollary 3.3.2, $\ker F$ is degenerate if and only if $2r = d - m_0 + 2$ and $m_0 > 2$, so the sum in (24) runs up to $(d - m_0)/2 = r - 1$, which means $c_r = c_{r+1} = \dots = c_q = 0$, as stated in the lemma. Also by corollary 3.3.2, $\ker F$ non-degenerate if and only if $2r = d - m_0$. In this case, the sum in (24) runs up to $(d - m_0)/2 = r$, hence $c_r \neq 0$ and if $r < q$, the next coefficients vanish $c_{r+1} = c_{r+2} = \dots = c_q = 0$. In addition c_r is the independent term in the polynomial in parentheses. Let $\mu_1, \dots, \mu_\lambda$ be all the non-zero parameters among the $\{\mu_i\}$ of the canonical form of F given in (14). By equation (18), c_r can be written for d odd:

$$c_r = \sigma \mu_1^2 \dots \mu_\lambda^2.$$

Then, the sign of σ determines the sign of c_r and, by remark 2.3, also the causal character of $\ker F_\sigma$, hence, the causal character of $\ker F$ in accordance with the statement of the lemma. For d even, also from (18) we have

$$c_r = -\frac{\tau^2}{4} \mu_1^2 \dots \mu_\lambda^2 < 0 \quad (\tau \neq 0), \quad c_r = \sigma \mu_1^2 \dots \mu_\lambda^2 \quad (\tau = 0),$$

where the expression for $\tau = 0$ follows because in this case either μ_t or μ_s (or both) vanish, hence either $c_r = \mu_s^2 \mu_1^2 \dots \mu_\lambda^2$ or $c_r = -\mu_t^2 \mu_1^2 \dots \mu_\lambda^2$ and σ equals μ_s^2 in the first situation and $-\mu_t^2$ in the second. By remark 2.2, when $\tau \neq 0$ we have $\ker F_{\sigma\tau} = \{0\}$ and hence $\ker F$ is always spacelike or zero and when $\tau = 0$, the causal character of $\ker F_{\sigma\tau}$ (and that of $\ker F$) is determined by the sign of σ in accordance with the statement of the lemma. \square

Remark 5.1. A converse version of lemma 5.2 also holds, in the sense that the number ν of last vanishing coefficients restricts the allowed rank parameters r . Let ν be defined by $\nu = 0$ if $c_q \neq 0$ and, otherwise, by the largest natural number satisfying $c_q = c_{q-1} = \dots = c_{q-\nu+1} = 0$. By equation (24) it follows $\nu = [m_0/2]$, and since the dimension d and m_0 have the same parity (cf lemma 3.2), $d - m_0 = 2[d/2] - 2[m_0/2] = 2(q - \nu)$ which in particular shows that ν determines m_0 uniquely. If $m_0 > 2$, by corollary 3.3.2 the rank parameter admits two possibilities $r = \{q - \nu, q - \nu + 1\}$, each of which determined by the causal character of $\ker F$. If $m_0 \leq 2$, also by corollary 3.3.2 the $\ker F$ degenerate case cannot occur and $r = (q - \nu)$ is uniquely determined. In particular, if $d = 4$, r is always determined by c_1, c_2 , because $r = 2$ happens if and only if $\nu = 0$ and otherwise $r = 1$ (unless F is identically zero, in which case $r = 0$).

5.1. Structure of $\text{SkewEnd}(V)/O^+(1, d - 1)$

By theorem 5.1, the q -tuple (c_1, \dots, c_q) corresponding to the coefficients of the characteristic polynomial of a skew-symmetric endomorphism, does not suffice to determine a point in the quotient space $\text{SkewEnd}(V)/O^+(1, d - 1)$, since generically two ranks are possible (dimensions three and four are an exception). As dicussed in remark 5.1, for a number ν of last vanishing coefficients c_b , the allowed rank parameters are $r \in \{q - \nu, q - \nu + 1\}$, and $r = q - \nu + 1$ is only possible provided $m_0 > 2$ (in particular, when $c_q \neq 0$ then necessarily $r = q$). One says that there is a *degeneracy* for the value of the rank at certain points in the space of coefficients c_b . In the submanifold $\{c_q = \dots = c_{q-\nu+1} = 0, c_{q-\nu} \neq 0\}$, the possible rank parameters are $r \in \{q - \nu, q - \nu + 1\}$. When a boundary point where the number of last vanishing coefficients increases by exactly one is approached, the rank parameter may remain equal to $q - \nu$ or jump to $q - \nu - 1$ (note that while the coefficients c_i are continuous functions of F , the rank is only lower semicontinuous, e.g. [22]). As we shall see in this section, this

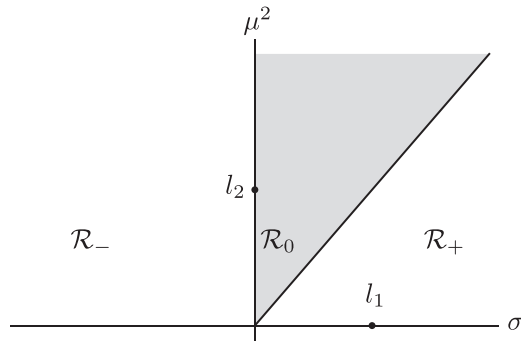


Figure 1. Representation of $\text{SkewEnd}(V)/O^+(1, d - 1)$ in the region $\mathcal{A} \subset \mathbb{R}^2$. The shadowed region is not included.

behaviour gives rise to special limit points which make the space of parameters defining the canonical form (i.e. the space of conjugacy classes) a non-Hausdorff topological space, when endowed with the natural quotient topology. Let us start by locating these limit points using the canonical form. Degeneracies can only occur in dimensions $d = 5$ or larger because in dimension three the rank is two for any non-trivial F and in dimension four the rank is uniquely determined by the invariants (cf remark 5.1). We thus consider first the case $d = 5$ and then extend to all values $d \geq 5$. In $d = 5$ the space of parameters \mathcal{A} defining the $[F]_{O^+}$ classes is (see figure 1)

$$\mathcal{A} := \{(\sigma, \mu^2) \in \mathbb{R} \times \mathbb{R}^+ \mid \sigma \geq \mu^2 \text{ if } \sigma > 0\}.$$

Consider a $[F]_{O^+}$ in the region $\mathcal{R}_+ := \{\sigma \geq \mu^2 > 0\}$ and let F be a representative of $[F]_{O^+}$ in a canonical basis $B = \{e_\alpha\}_{\alpha=0,\dots,4}$, that is

$$F = \begin{pmatrix} 0 & 0 & -1 + \frac{\sigma}{4} \\ 0 & 0 & -1 - \frac{\sigma}{4} \\ -1 + \frac{\sigma}{4} & 1 + \frac{\sigma}{4} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}. \tag{25}$$

Let us define the functions $C_\pm(x) := \frac{1}{x} \pm \frac{x}{4}$. Then, the following change of basis to $B' = \{e'_\alpha\}$ is well defined in \mathcal{R}_+ :

$$\begin{aligned} e'_0 &= C_+(\mu)(C_+(\sqrt{\sigma})e_0 + C_-(\sqrt{\sigma})e_1) - C_-(\mu)e_4, & e'_2 &= -e_3, & e'_4 &= C_-(\sqrt{\sigma})e_0 + C_+(\sqrt{\sigma})e_1 \\ e'_1 &= -C_-(\mu)(C_+(\sqrt{\sigma})e_0 + C_-(\sqrt{\sigma})e_1) + C_+(\mu)e_4, & e'_3 &= -e_2. \end{aligned} \tag{26}$$

By direct calculation, F is written in basis B' as

$$F = \begin{pmatrix} 0 & 0 & -1 + \frac{\mu^2}{4} \\ 0 & 0 & -1 - \frac{\mu^2}{4} \\ -1 + \frac{\mu^2}{4} & 1 + \frac{\mu^2}{4} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\sqrt{\sigma} \\ \sqrt{\sigma} & 0 \end{pmatrix}. \tag{27}$$

The basis B' is non-canonical because $\mu^2 < \sigma$. However, if we vary the parameters so that $\mu \rightarrow 0$ (keeping σ unchanged), the matrix (27) becomes canonical (i.e. of the form (14)) in the limit and the class $[\lim_{\mu \rightarrow 0} F]_{O^+}$ is given by $l_1 = (0, \sigma)$. On the other hand, F in canonical form (25) also admits a limit $\mu \rightarrow 0$, which is also canonical and whose representative $[\lim_{\mu \rightarrow 0} F]_{O^+}$ is given by $l_2 = (\sigma, 0)$. Both limits are defined by the same sequence of points, because the transformation (26) is invertible in \mathcal{R}_+ . However this sequence has two different limit points. As a consequence, the space of canonical matrices, and therefore the quotient space $\text{SkewEnd}(V)/O^+(1, d-1)$, inherits a non-Hausdorff topology.

Something similar happens in the region $\mathcal{R}_- := \{\sigma < 0, \mu > 0\}$. Let F be a representative in canonical form of a point $[F]_{O^+}$ in this region. Then, F has a timelike eigenplane Π_t with eigenvalue $\sqrt{|\sigma|}$ (cf remark 2.3), a spacelike eigenvector e as well as spacelike eigenplane Π_s with eigenvalue μ . Thus $V = \Pi_t \oplus \text{span}\{e\} \oplus \Pi_s$ and there exist a (non-canonical) basis B' adapted to this decomposition, into which F takes the form

$$F = \begin{pmatrix} 0 & \sqrt{|\sigma|} & 0 \\ \sqrt{|\sigma|} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}. \tag{28}$$

Keeping μ unchanged, expression (28) has a limit $\sigma \rightarrow 0$, which has a spacelike eigenplane Π_s of eigenvalue μ and it is identically zero on Π^\perp . Hence, $\ker F$ is timelike and using definition 3.2, the canonical form of this limit $\lim_{\sigma \rightarrow 0} F$ is given by $\sigma' = \mu^2$ and $\mu' = 0$. Thus $[\lim_{\sigma \rightarrow 0} F]_{O^+}$ is represented by the point $l_2 = (\mu^2, 0)$. On the other hand, in a canonical basis (25), F also admits a limit $\sigma \rightarrow 0$, whose class $[\lim_{\sigma \rightarrow 0} F]_{O^+}$ is obviously represented by the point $l_1 = (0, \mu^2)$.

The same reasoning can be carried out to arbitrary odd dimension. First, define the regions

$$\mathcal{R}_+^{(d,0)} := \{\sigma \geq \mu_1^2 \geq \dots \geq \mu_p^2 > 0\} \quad \text{and} \quad \mathcal{R}_-^{(d,0)} := \{\sigma < 0, \mu_1^2 \geq \dots \geq \mu_p^2 > 0\}$$

and also the limit regions

$$\mathcal{R}_0^{(d,0)} := \{\sigma = 0, \mu_1^2 \geq \dots \geq \mu_p^2 > 0\} \quad \text{and} \quad \mathcal{R}_+^{(d,1)} := \{\sigma \geq \mu_1^2 \geq \dots \geq \mu_{p-1}^2 > \mu_p^2 = 0\}.$$

Consider representatives F_+ and F_- (in canonical form) of points $(\sigma^+, (\mu_1^+)^2, \dots, (\mu_p^+)^2)$ and $(\sigma^-, (\mu_1^-)^2, \dots, (\mu_p^-)^2)$ in the regions $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_-^{(d,0)}$ respectively. Then F_+ has a spacelike eigenplane Π_s^+ with eigenvalue μ_p^+ as well as timelike eigenvector e^+ and spacelike eigenplane Π_t^+ with eigenvalue $\sqrt{\sigma^+}$. Restricting to the subspace $W^+ = \text{span}\{e^+\} \oplus \Pi_t^+ \oplus \Pi_s^+$ we can repeat the procedure followed for the five dimensional case and conclude that $[\lim_{\mu_p^+ \rightarrow 0} F_+]$ has simultaneously limits at the points $(\sigma^+, (\mu_1^+)^2, \dots, (\mu_{p-1}^+)^2, 0) \in \mathcal{R}_+^{(d,1)}$ and $(0, (\mu_1^+)^2, \dots, (\mu_p^+)^2) \in \mathcal{R}_0^{(d,0)}$. Analogously F_- has a spacelike eigenplane Π_s^- with eigenvalue μ_p^- as well as spacelike eigenvector e^- and timelike eigenplane $\Pi_t'^-$ with eigenvalue $\sqrt{|\sigma^-|}$. Restricting to the subspace $W^- = \Pi_s^- \oplus \text{span}\{e^-\} \oplus \Pi_t'^-$, the above arguments for the five dimensional case show that $[\lim_{\sigma^- \rightarrow 0} F_-]$ has simultaneous limits on the points $((\mu_p^-)^2, (\mu_1^-)^2, \dots, (\mu_{p-1}^-)^2, 0) \in \mathcal{R}_+^{(d,1)}$ and $(0, (\mu_1^-)^2, \dots, (\mu_p^-)^2) \in \mathcal{R}_0^{(d,0)}$. Thus the regions $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_-^{(d,0)}$ limit simultaneously with $\mathcal{R}_+^{(d,1)}$ and $\mathcal{R}_0^{(d,0)}$ as μ_p and σ tend to zero respectively. Indeed, the same ideas can be applied again to $\mathcal{R}_+^{(d,1)}$ and $\mathcal{R}_-^{(d,1)} := \{\sigma < 0, \mu_1^2 \geq \dots \geq \mu_{p-1}^2 > \mu_p^2 = 0\}$, so that they also limit simultaneously, as μ_{p-1} and σ go to zero respectively, with $\mathcal{R}_0^{(d,1)} := \{\sigma = 0, \mu_1^2 \geq \dots \geq \mu_{p-1}^2 > \mu_p^2 = 0\}$ and

$\mathcal{R}_+^{(d,2)} := \{\sigma > 0, \mu_1^2 \geq \dots \geq \mu_{p-1}^2 > \mu_p^2 = 0\}$. In general, the regions $\mathcal{R}_\pm^{(d,i)}$ analogously defined, i.e. where i gives the number of vanishing parameters $\mu_p = \dots = \mu_{p-i} = 0$ and the subindex \pm gives the sign of σ , have simultaneous limits in $\mathcal{R}_0^{(d,i)}$ and $\mathcal{R}_+^{(d,i+1)}$, where the subindex 0 stands for vanishing σ .

For the even dimensional case (with $d \geq 6$), notice that the canonical form (14) with $\tau = 0$ is equivalent to the odd dimensional case direct sum with a one dimensional zero endomorphism (of a Riemannian line). Hence, the previous reasoning for odd dimensions also applies for even dimensions and $\tau = 0$. For example, consider in $d = 6$ dimensions the regions $\mathcal{R}_+ = \{\tau = 0, \sigma \geq \mu^2 > 0\}$ and $\mathcal{R}_- = \{\tau = 0, \sigma < 0, \mu^2 > 0\}$. Then they both assume limit in $\mathcal{R}_0 = \{\tau = 0, \sigma = 0, \mu^2 > 0\}$ and $\mathcal{R}_+^{(d,1)} = \{\tau = 0, \sigma > \mu^2 = 0\}$. Notice that if we keep $\tau \neq 0$ no degenerate limits of this kind occur. This can be justified as follows. Let μ_t, μ_s be defined as in (9). Then, it can be readily checked that $\det F = -\mu_t^2 \mu_s^2 \mu^2$, so if $\sigma, \tau, \mu \neq 0$, then $\text{rank } F = 6$. If we keep $\tau \neq 0$ (thus both μ_s, μ_t are different from zero), $\mu \neq 0$ and make $\sigma \rightarrow 0$, the limit must have always $\text{rank } F = 6$. Hence, it is not possible that a limit $\sigma \rightarrow 0$ ends at two points with different rank. Similarly, keeping $\tau \neq 0$, the limit $\mu \rightarrow 0$ always has $\text{rank } F = 4$ and therefore, $\mu \rightarrow 0$ limits cannot be degenerate either. The generalization to arbitrary even dimensions with $\tau = 0$ is also straightforward from the odd dimensional case discussed above, which we now summarize in the following remark:

Remark 5.2. In the case of d odd, consider the subset of \mathbb{R}^q given by

$$\mathcal{A}^{(\text{odd})} := \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathbb{R} \times (\mathbb{R}^+)^p \mid \mu_1^2 \geq \dots \geq \mu_{p-1}^2 \text{ and if } \sigma > 0, \sigma \geq \mu_1^2 \geq \dots \geq \mu_p^2\}.$$

Define also the subsets of $\mathcal{A}^{(\text{odd})}$ given by

$$\begin{aligned} \mathcal{R}_+^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{odd})} \mid \sigma \geq \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}, \\ \mathcal{R}_-^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{odd})} \mid \sigma < 0, \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}, \\ \mathcal{R}_0^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{odd})} \mid \sigma = 0, \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}. \end{aligned}$$

Then in the quotient topology of $\text{SkewEnd}(V)/O^+(1, d-1)$ the sequences of $\mathcal{R}_\pm^{(p-i)}$ with limit at $\mathcal{R}_0^{(d,i)}$ also have limit at $\mathcal{R}_+^{(d,i+1)}$.

In the case of d even, first define μ_s as in (9) and let $\mathcal{A}^{(\text{even})}$ be the subspace of \mathbb{R}^q given by:

$$\mathcal{A}^{(\text{even})} := \{(\sigma, \tau, \mu_1^2, \dots, \mu_p^2) \in \mathbb{R} \times (\mathbb{R}^+)^p \mid \mu_1^2 \geq \dots \geq \mu_p^2 \text{ and if } \tau \neq 0 \text{ or } \sigma > 0, \mu_s^2 \geq \mu_1^2 \geq \dots \geq \mu_p^2\}.$$

Define also the following subsets of $\mathcal{A}^{(\text{even})}$

$$\begin{aligned} \mathcal{R}_+^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{even})} \mid \tau = 0, \sigma \geq \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}, \\ \mathcal{R}_-^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{even})} \mid \tau = 0, \sigma < 0, \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}, \\ \mathcal{R}_0^{(d,i)} &:= \{(\sigma, \mu_1^2, \dots, \mu_p^2) \in \mathcal{A}^{(\text{even})} \mid \tau = 0, \sigma = 0, \mu_1^2 \geq \dots \geq \mu_{p-i}^2 > \mu_{p-i+1}^2 = \dots = \mu_p^2 = 0\}. \end{aligned}$$

Then in the quotient topology of $\text{SkewEnd}(V)/O^+(1, d-1)$ the sequences of $\mathcal{R}_\pm^{(d,i)}$ with limit at $\mathcal{R}_0^{(d,i)}$ also have limit at $\mathcal{R}_+^{(d,i+1)}$.

6. Conformal vector fields

One interesting application of our previous results is based on the relation between skew-symmetric endomorphisms and the set of CKVFs of the n -sphere, $\text{CKill}(\mathbb{S}^n)$. These vector

fields are the generators of the conformal transformations of the n -sphere $\text{Conf}(\mathbb{S}^n)$, i.e. the group of transformations ψ_Λ that scale the spherical metric γ , $\psi_\Lambda^*(\gamma) = \Omega^2\gamma$, where Ω is a smooth positive function of \mathbb{S}^n . A standard technique to describe these transformations consists in viewing \mathbb{S}^n as the (real) projectivization of the null future cone in $\mathbb{M}^{1,n+1}$, in such a way that $\text{Conf}(\mathbb{S}^n)$ is induced from the isometries of $\mathbb{M}^{1,n+1}$. This is discussed in detail for the four dimensional case in [27] and in arbitrary dimensions in [24] and in [29] (the latter considers arbitrary signature and the projectivization of the null ‘cone’ in $\mathbb{M}^{p+1,q+1}$, giving $\mathbb{S}^p \times \mathbb{S}^q$). This procedure establishes a group homomorphism $\psi : O(1, n + 1) \rightarrow \text{Conf}(\mathbb{S}^n)$, $\Lambda \mapsto \psi_\Lambda$, which is one-to-one when restricted to the orthochronous component $O^+(1, n + 1) \subset O(1, n + 1)$. Also, see [20] for an alternative way to establish this isomorphism for the $n = 3$ case by means of conformal Killing–Yano tensors.

The Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, g_E)$ and \mathbb{S}^n are well-known to be conformally related via the stereographic projection $St_N : \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{E}^n$, where N denotes the point w.r.t. which the projection is taken. Observe that the stereographic projection depends not only on the point N but also on the (signed) distance between N and the plane onto which the projection is performed. We do not reflect this dependence in the notation for simplicity.

Hence, the composition of a transformation $\psi_\Lambda \in \text{Conf}(\mathbb{S}^n)$ with the stereographic projection yields $St_N \circ \psi_\Lambda \circ St_N^{-1} =: \phi_\Lambda \in \text{Conf}(\mathbb{E}^n)$, which is a conformal transformation of \mathbb{E}^n . Strictly speaking, these transformations are not diffeomorphisms of \mathbb{E}^n , as they require to remove the two points $p_1, p_2 \in \mathbb{E}^n$ satisfying $\psi_\Lambda \circ St_N^{-1}(p_1) = N$ and $\psi_\Lambda^{-1} \circ St_N^{-1}(p_2) = N$, which are the ‘preimage’ and the ‘image’ of infinity under ϕ_Λ respectively. Nevertheless, since $\psi : O(1, n + 1) \rightarrow \text{Conf}(\mathbb{S}^n)$ is a group homomorphism, so is $\phi : O(1, n + 1) \rightarrow \text{Conf}(\mathbb{E}^n)$, $\Lambda \mapsto \phi_\Lambda$ as well as the map which assigns $\psi_\Lambda \mapsto \phi_\Lambda$. In that sense $\text{Conf}(\mathbb{S}^n)$ and $\text{Conf}(\mathbb{E}^n)$ are the same. These group homomorphisms, induce Lie algebra homomorphisms between $\text{SkewEnd}(\mathbb{M}^{1,n+1})$, $\text{CKill}(\mathbb{S}^n)$ and $\text{CKill}(\mathbb{E}^n)$ (the vector fields generating $\text{Conf}(\mathbb{E}^n)$). The precise form of these maps depends, firstly, on the representative used to describe the projective cone (i.e. \mathbb{S}^n) and secondly on the point N as well as on the signed distance from this point to the plane. In [24], the morphism

$$\begin{aligned} \xi := \phi_* : \text{SkewEnd}(\mathbb{M}^{1,n+1}) &\longmapsto \text{CKill}(\mathbb{E}^n), \\ F &\longmapsto \xi(F) =: \xi_F, \end{aligned} \tag{29}$$

is constructed⁴ related to each other using the representative with $\{x^0 = 1\} \cap \{x_\alpha x^\alpha = 0\}$ for the projective cone, where $\{x^\alpha\}$ ($\alpha, \beta = 0, \dots, n + 1$) are Minkowskian coordinates of $\mathbb{M}^{1,n+1}$, N is the point with coordinates $\{x^0 = -x^1 = 1, x^{A+1} = 0\}$ ($A, B = 1, \dots, n$) and the image plane for the stereographic projection is $\{x^0 = x^1 = 1\}$. The result is a representation of $\text{CKill}(\mathbb{E}^n)$ where the vector vector fields are expressed in Cartesian coordinates $\{y^A\}$ induced from the Minkowskian coordinates by means of $\{x^0 = x^1 = 1, x^{A+1} = y^A\}$.

Theorem 6.1. [24] *Let $\mathbb{M}^{1,n+1}$ be endowed with Minkowskian coordinates $\{x^\alpha\}$ and consider any element $F \in \text{SkewEnd}(\mathbb{M}^{1,n+1})$ written in the basis $\{\partial_{x^\alpha}\}$ in the form*

$$F = \begin{pmatrix} 0 & -\nu & -a^t + b^t/2 \\ -\nu & 0 & -a^t - b^t/2 \\ -a + b/2 & a + b/2 & -\omega \end{pmatrix}, \tag{30}$$

⁴The method in [24] is based on the unit spacelike hyperboloid in Minkowski instead of on the null cone. However, the two methods are easily seen to be equivalent to the one we describe.

where $a, b \in \mathbb{R}^n$ are column vectors, t stands for the transpose and ω is a skew-symmetric $n \times n$ matrix ($\omega = -\omega^t$). Then, in the Cartesian coordinates $\{y^A\}$ of \mathbb{E}^n defined by the embedding $i : \mathbb{E}^n \hookrightarrow \mathbb{M}^{1,n+1}$, $i(\mathbb{E}^n) = \{x^0 = x^1 = 1, x^{A+1} = y^A\}$, the CKVFs of \mathbb{E}^n are

$$\xi_F = \left(b^A + \nu y^A + (a_B y^B) y^A - \frac{1}{2} (y_B y^B) a^A - \omega^A_{\ B} y^B \right) \partial_{y^A}. \tag{31}$$

Moreover, $\xi_{Ad_\Lambda(F)} = \phi_{\Lambda*}(\xi_F)$ for every $\Lambda \in O^+(1, n + 1)$ and ξ is a Lie algebra antihomomorphism, i.e. $[\xi_F, \xi_G] = -\xi_{[F,G]}$.

Remark 6.1. For later use, we write explicitly the parameters of the vector field $\nu, a^A, b^A, \omega^A_{\ B}$ in terms of the entries F^α_β of the endomorphism F :

$$\begin{aligned} \nu &= -F^0_{\ 1}, & a_A &= -\frac{1}{2} (F^0_{\ A+1} + F^1_{\ A+1}), \\ b_A &= \frac{1}{2} (F^0_{\ A+1} - F^1_{\ A+1}), & \omega^A_{\ B} &= -F^{A+1}_{\ B+1}. \end{aligned} \tag{32}$$

where capital Latin indices are lowered with the Kronecker δ_{AB} . Unless otherwise stated, ξ without subindex refers to the map ξ given in (29) while ξ_F refers to the CKVF which is image under ξ of the skew-symmetric endomorphism F .

The freedom of choosing a representative for \mathbb{S}^n (as well as the point N and the projection stereographic plane) can be also seen in a more ‘passive’ picture. Consider two different sets of Minkowskian coordinates $\{x^\alpha\}$ and $\{x'^\alpha\}$ related by a $O^+(1, n + 1)$ transformation Λ , $x'^\alpha = \Lambda^\alpha_\beta x^\beta$. Using theorem 6.1, we obtain two different embeddings $i, i' : \mathbb{E}^n \hookrightarrow \mathbb{M}^{1,n+1}$ associated to $\{x^\alpha\}$ and $\{x'^\alpha\}$ respectively, for which $i(\mathbb{E}^n) = \{x^0 = x^1 = 1, x^{A+1} = y^A\}$ and $i'(\mathbb{E}^n) = \{x'^0 = x'^1 = 1, x'^{A+1} = y'^A\}$, as well as two associated maps ξ, ξ' . Let $F \in \text{SkewEnd}(\mathbb{M}^{1,n+1})$, defined by (30) with parameters $\{\nu, a^A, b^A, \omega^A_{\ B}\}$ and $\{\nu', a'^A, b'^A, \omega'^A_{\ B}\}$ in the bases $\{\partial_{x^\alpha}\}$ and $\{\partial_{x'^\alpha}\}$ respectively. Then, F can be associated to two vector fields

$$\begin{aligned} \xi_F &= \left(b^A + \nu y^A + (a_B y^B) y^A - \frac{1}{2} (y_B y^B) a^A - \omega^A_{\ B} y^B \right) \partial_{y^A}, \\ \xi'_F &= \left(b'^A + \nu' y'^A + (a'_B y'^B) y'^A - \frac{1}{2} (y'_B y'^B) a'^A - \omega'^A_{\ B} y'^B \right) \partial_{y'^A}, \end{aligned}$$

which are equal in the following sense. If we transform the representative $\mathbb{S}^n = \{x^0 = 1\} \cap \{x'_\alpha x'^\alpha = 0\}$ with Λ , we obtain a new representative of the projective cone which in coordinates x^α is precisely $\mathbb{S}^n = \{x^0 = 1\} \cap \{x_\alpha x^\alpha = 0\}$. Abusing the notation, the map $\chi_\Lambda := St_N \circ \Lambda \circ St_{N'}^{-1}$ is such that $\chi_{\Lambda*}(\xi'_F) = \xi_F$. Then, considering $i(\mathbb{E}^n)$ and $i'(\mathbb{E}^n)$ as representations of the same space in two different global charts (y^A, \mathbb{R}^n) and (y'^A, \mathbb{R}^n) , χ_Λ can be seen as a change of coordinates $y^A = (\chi_\Lambda(y'))^A$, with the property that the Euclidean metric in coordinates $\{y'^A\}$ transforms as

$$g_E = \delta_{AB} dy'^A dy'^B = \Omega^2(y) \delta_{AB} dy^A dy^B$$

for a smooth positive function Ω . In other words, changing to different Minkowskian coordinates in $\mathbb{M}^{1,n+1}$ induces a change of coordinates in \mathbb{E}^n in such a way that the form (31) of the map ξ is preserved. Notice that a similar result holds if we change the point w.r.t. which we take the stereographic projection, because any two $N, N' \in \mathbb{S}^n$ must be related by a $SO(n) \subset O^+(1, n + 1)$ transformation.

Therefore, for the rest of this section, we will often adapt our choice of Minkowskian coordinates $\{x^\alpha\}$ of $\mathbb{M}^{1,n+1}$ to simplify the problem at hand. With this choice, it comes a corresponding set of Cartesian coordinates $\{y^A\}$ of \mathbb{E}^n such that ξ_F is given by equation (31) and the Euclidean metric is $g_E = \Omega(y)^2 \delta_{AB} dy^A dy^B$. Which coordinates are adequate obviously depends on the problem. For example, from the block form (10) and (11) of skew-symmetric endomorphisms, consider each of the blocks $F|_{\mathbb{M}^{1,3}}$ $F|_{\mathbb{M}^{1,2}}$ as endomorphisms of $\mathbb{M}^{1,n+1}$, extended as the zero map in $(\mathbb{M}^{1,3})^\perp$ and $(\mathbb{M}^{1,2})^\perp$ respectively, and similarly for each $F|_{\Pi_i}$. If we denote by $\xi_{F|_{\mathbb{M}^{1,3}}}$, $\xi_{F|_{\mathbb{M}^{1,2}}}$ and $\xi_{F|_{\Pi_i}}$ the corresponding images by ξ , one readily gets following decomposition:

$$\xi_F = \xi_{F|_{\mathbb{M}^{1,3}}} + \sum_{i=1}^p \xi_{F|_{\Pi_i}} \quad (n \text{ even}), \quad \xi_F = \xi_{F|_{\mathbb{M}^{1,2}}} + \sum_{i=1}^p \xi_{F|_{\Pi_i}} \quad (n \text{ odd}), \tag{33}$$

where in terms of n, p is given by

$$p = \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \tag{34}$$

(because the dimension of the Minkowski space where F is defined is $d = n + 2$, cf theorem 6.1). The explicit form of each of the terms in (33) is direct from (32). Namely, the terms $\xi_{F|_{\mathbb{M}^{1,3}}}$ and $\xi_{F|_{\mathbb{M}^{1,2}}}$ are given by (31) with vanishing parameters a^A, b^A, ω^A_B for $A, B \geq 3$ and $A, B \geq 2$ respectively, and each $\xi_{F|_{\Pi_i}}$ is proportional to a vector field of the form

$$\eta := y^{A_0} \partial_{y^{B_0}} - y^{B_0} \partial_{y^{A_0}} \tag{35}$$

with $A_0, B_0 \in \{1, \dots, n\}$ such that $A_0 \neq B_0$. More specifically, $\xi_{F|_{\Pi_i}} = \mu_i \eta_i$, where η_i is given by equation (35) with $B_0 = A_0 + 1$ and $A_0 = 2i$ if n even while $A_0 = 2i + 1$ if n odd. Vector fields of the form (35) will play an important role in the following analysis. They have the form of axial Killing vector fields, although in general they are CKVFs because of the conformal factor in $g_E = \Omega(y)^2 \delta_{AB} dy^A dy^B$. From the previous discussion, it follows that there exists a conformal transformation $\chi_\Lambda \in \text{Conf}(\mathbb{E}^n)$ such that $g'_E := \chi_\Lambda^*(g_E) = \delta_{AB} dy^A dy^B$. Then by the properties of the Lie derivative it is immediate

$$0 = \mathcal{L}_\eta \chi_\Lambda^*(g_E) = \mathcal{L}_{\chi_{\Lambda*} \eta} g_E.$$

In other words, η is an axial Killing vector of g'_E and $\chi_{\Lambda*} \eta$ is an axial Killing vector of g_E . Thus, we define:

Definition 6.1. A CKVF of an Euclidean metric g_E, η , is said to be a **conformally axial Killing vector field (CAKVF)** if and only if the exist a $\chi_\Lambda \in \text{Conf}(\mathbb{E}^n)$ such that $\chi_{\Lambda*}(\eta)$ is an axial Killing vector field of g_E . Equivalently, η is a CAKVF if and only if it is an axial Killing vector field of $\chi_\Lambda^*(g_E)$.

Remark 6.2. Using theorem 6.1, it is immediate to verify that a CKVF is a CAKVF if and only if it is the image under ξ of a simple unit spacelike endomorphism G .

Notice that the terms in (33) form a commutative subset of $\text{CKill}(\mathbb{E}^n)$. This is an immediate consequence of the fact that ξ is a Lie algebra antihomomorphism (cf theorem 6.1) and the

blocks $F|_{\mathbb{M}^{1,2}}$ (resp. $F|_{\mathbb{M}^{1,3}}$) and $F|_{\Pi_i}$ are pairwise commuting. In addition, a straightforward calculation shows that they form an orthogonal set

$$g_E(\tilde{\xi}, \eta_i) = 0, \quad g_E(\eta_i, \eta_j) = 0 \quad (i \neq j)$$

where $\tilde{\xi} := \xi_F|_{\mathbb{M}^{1,3}}$ for n even and $\tilde{\xi} := \xi_F|_{\mathbb{M}^{1,2}}$ for n odd. In fact, as we show next, orthogonality of two CKVFs implies commutativity provided one of them is a CAKVF. If both are CAKVF, then orthogonality turns out to be equivalent to commutativity.

Lemma 6.2. *Let η, η' be non-proportional CAKVFs and ξ_F a CKVF. Then $[\eta, \eta'] = 0$ if and only if there exist Cartesian coordinates such that $\eta = y^{n-2}\partial_{y^{n-3}} - y^{n-3}\partial_{y^{n-2}}$ and $\eta' = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. Equivalently $[\eta, \eta'] = 0$ if and only if $g_E(\eta, \eta') = 0$. In addition, $[\xi_F, \eta] = 0$ if $g_E(\xi_F, \eta) = 0$.*

Proof. Let $G, G' \in \text{SkewEnd}(\mathbb{M}^{1,n+1})$ be such that $\xi(G) = \eta, \xi(G') = \eta'$. Since G and G' are simple, spacelike and unit (cf remark 6.2), we can write $G = e \otimes v_b - v \otimes e_b$ and $G' = e' \otimes v'_b - v' \otimes e'_b$ for spacelike, unit vectors $\{e, e', v, v'\}$, such that $0 = \langle e, v \rangle = \langle e', v' \rangle$. By corollary 4.2.1, it follows that $[G, G'] = 0$ if and only if $\{e, e', v, v'\}$ are mutually orthogonal. Let us take Cartesian coordinates of $\mathbb{M}^{1,n+1}$ such that $e = \partial_{x^{n-2}}, v = \partial_{x^{n-1}}, e' = \partial_{x^n}, v' = \partial_{x^{n+1}}$. Then, in the associated coordinates $\{y^A\}$ of \mathbb{E}^n it follows $\eta = y^{n-2}\partial_{y^{n-3}} - y^{n-3}\partial_{y^{n-2}}$ and $\eta' = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. This proves the first part of the lemma. From this result, it is trivial that $[\eta, \eta'] = 0$ implies $g_E(\eta, \eta') = 0$.

To prove that $g_E(\eta, \xi_F) = 0$ implies $[\eta, \xi_F] = 0$ (which in particular establishes the converse $g_E(\eta, \eta') = 0 \implies [\eta, \eta'] = 0$ for CAKVFs), let us take coordinates $\{y^A\}$ such that $\eta = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. Then, writing ξ_F as a general CKVF (31), we obtain by direct calculation:

$$g_E(\eta, \xi_F) = \Omega^2 \left(y^n b^{n-1} - y^{n-1} b^n - \frac{y_B y^B}{2} (a^n y^{n-1} - a^{n-1} y^n) + \omega^{n-1}{}_B y^B y^n - \omega^n{}_B y^B y^{n-1} \right) = 0.$$

Therefore $a^n, a^{n-1}, b^n, b^{n-1}, \omega^n{}_B, \omega^{n-1}{}_B$ must vanish. This implies that the associated endomorphisms G and F to η and ξ_F adopt a block structure from which it easily follows that $[G, F] = 0$ and hence $[\eta, \xi_F] = 0$. \square

Definition 6.2. Let $\xi_F \in \text{CKill}(\mathbb{E}^n)$. Then a **decomposed form** of ξ_F is $\xi_F = \tilde{\xi} + \sum_{i=1}^p \mu_i \eta_i$ for an orthogonal subset $\{\tilde{\xi}, \eta_i\}$, where η_i are CAKVFs, $\mu_i \in \mathbb{R}$ for $i = 1, \dots, p$. A set of Cartesian coordinates $\{y^A\}$ such that $\eta_i = y^{A_i} \partial_{y^{A_i+1}} - y^{A_i+1} \partial_{y^{A_i}}$, for $A_i = 2i$ for n odd and $A_i = 2i + 1$ for n even, is called a set of **decomposed** coordinates.

Remark 6.3. Observe that the $\tilde{\xi}$ is a CKVF. By lemma 6.2 and its proof, the parameters $\{\nu, a, b, \omega\}$ defining $\tilde{\xi}$ in a set of decomposed coordinates must all vanish except possibly $\{\nu, a^1, a^2, b^1, b^2, \omega^1{}_2 = -\omega^2{}_1\}$ when n is even or $\{\nu, a^1, b^1\}$ when n is odd. This means that there is a skew-symmetric endomorphism \tilde{F} which restricts to $\mathbb{M}^{1,3} \subset \mathbb{M}^{1,n+1}$ (n even) or $\mathbb{M}^{1,2} \subset \mathbb{M}^{1,n+1}$ (n odd) and vanishes identically on their respective orthogonal complements such that $\tilde{\xi} = \xi_{\tilde{F}}$. We will exploit this fact in an essential way below.

With the definition of decomposed form of CKVFs, we can reformulate theorem 2.6 in terms of CKVFs.

Proposition 6.1. *Let $\xi_F \in \text{CKill}(\mathbb{E}^n)$. Then there exist an orthogonal set $\{\eta_i\}_{i=1}^p$ of CAKVFs such that $[\xi_F, \eta_i] = 0$. For every such a set $\{\eta_j\}_{j=1}^p$ and $i \in \{1, \dots, p\}$ there exist $\mu_i \in \mathbb{R}$ such that $g_E(\eta_i, \eta_i) \mu_i = g_E(\xi_F, \eta_i)$. In addition, with the definition $\tilde{\xi} := \xi_F - \sum \mu_i \eta_i$ the expression $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$ provides a decomposed form of ξ_F .*

Proof. The existence of p commuting CAKVF's is a direct consequence of decompositions (10) and (11) of the associated skew-symmetric endomorphism F , for n even and odd respectively. Indeed, for each such decomposition of F , it follows a set of p CAKVF's commuting with ξ_F . Let us denote $\{\eta_i\}$ any such set. Each η_i is associated to a simple, spacelike unit endomorphism G_i that commutes with F . By lemma 4.2, G_i defines a spacelike eigenplane Π_i of F . The orthogonality of any two such eigenplanes $\Pi_i, \Pi_j, i \neq j$ is a consequence of corollary 4.2.1 because $[G_i, G_j] = 0$. In other words, given a set of p CAKVF's commuting with ξ_F , we have a block form of F , thus, defining $\tilde{\xi} := \xi_F - \sum \mu_i \eta_i$, it is immediate that $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$ is a decomposed form with $g_E(\eta_i, \eta_i) \mu_i = g_E(\xi_F, \eta_i)$. \square

The next step now is to give a definition of canonical form for CKVF's, which we induce from the canonical form of the associated skew-symmetric endomorphism.

Definition 6.3. A CKVF ξ_F is in **canonical form** if it is the image of a skew-symmetric endomorphism F in canonical form, i.e. $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$ such that $\tilde{\xi}$ is given, in a Cartesian set of coordinates $\{y^A\}$ denoted **canonical coordinates**, by the parameters $a^1 = 1, b^1 = \sigma/2, a^2 = 0, b^2 = \tau/2$ if n even and $a^1 = 1, b^1 = \sigma/2$ if n odd (the non-specified parameters all vanish) and η_i are CAKVF's $\eta_i = y^{A_i} \partial_{y^{A_i+1}} - y^{A_i+1} \partial_{y^{A_i}}$, for $A_i = 2i$ for n odd and $A_i = 2i + 1$ for n even, and where σ, τ, μ_i are given by definition 3.2.

Given a CKVF ξ_F , the existence of a canonical form and canonical coordinates are guaranteed by theorem 3.3. By theorem 6.1, the conformal class $[\xi_F]_{\text{Conf}}$ of a CKVF ξ_F is equivalent to the equivalence class $[F]_{O^+}$ of F under the adjoint action of $O^+(1, n + 1)$, and this is determined by the canonical form of F (cf theorem 5.1). This argument together with the results of section 5 yield the following statement.

Theorem 6.3. Let $\xi_F \in \text{CKill}(\mathbb{S}^n)$ be in canonical form. Then its conformal class $[\xi_F]_{\text{Conf}}$ is determined by (σ, τ, μ_i^2) if n even and (σ, μ_i^2) if n odd. Moreover, the structure of $\text{CKill}(\mathbb{S}^n)/\text{Conf}(\mathbb{S}^n)$ corresponds with that of remark 5.2.

Given a canonical form $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$ the set of vectors $\{\tilde{\xi}, \eta_i\}$ are pairwise commuting and linearly independent. As we will next prove, in the case of odd dimension this set is a maximal (linearly independent) pairwise commuting set of CKVF's commuting with ξ (i.e. it is not contained in a larger set of linearly independent vectors commuting with ξ). In the case of even dimension it is not maximal. By remark 6.3, $\tilde{\xi} = \tilde{\xi}(\nu, a^1, a^2, b^1, b^2, \omega)$, where the right-hand side denotes a CKVF of the form (35) whose parameters vanish, except possibly $\{\nu, a^1, a^2, b^1, b^2, \omega := \omega^1_2\}$. As mentioned in the remark, the corresponding skew-symmetric endomorphism \tilde{F} satisfying $\xi_{\tilde{F}} = \tilde{\xi}$ can be understood as an element $\tilde{F} \in \text{SkewEnd}(\mathbb{M}^{1,3})$, with $\mathbb{M}^{1,3} = \text{span}\{e_0, e_1, e_2, e_3\}$, that is identically zero in $(\mathbb{M}^{1,3})^\perp$. Fix the orientation in $\mathbb{M}^{1,3}$ so that the basis $\{e_0, e_1, e_2, e_3\}$ is positively oriented. The Hodge star maps two-forms into two-forms. This defines a natural map

$$\begin{aligned} \star : \text{SkewEnd}(\mathbb{M}^{1,3}) &\longrightarrow \text{SkewEnd}(\mathbb{M}^{1,3}), \\ \tilde{F} &\longmapsto \tilde{F}^\star. \end{aligned}$$

From standard properties of two-forms, (see also [25]) it follows that \tilde{F}^\star commutes with \tilde{F} . We may extend \tilde{F}^\star to an endomorphism on $\mathbb{M}^{1,n+1}$ that vanishes identically on $(\mathbb{M}^{1,3})^\perp$, just as \tilde{F} . It is clear that the commutation property is preserved by this extension. The image of \tilde{F}^\star under ξ is the vector field

$$\tilde{\xi}^\star := \left(\tilde{\xi}(\nu, a^1, a^2, b^1, b^2, \omega) \right)^\star = \tilde{\xi}(-\omega, a^2, -a^1, -b^2, b^1, \nu),$$

which by construction commutes with $\tilde{\xi}$. In the case that $\tilde{\xi}$ is the first element in a decomposed form $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$, it is immediately true that $\tilde{\xi}^*$ also commutes with all of the CAKVF's η_i . Hence, $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ is a pairwise commuting set, all of them commuting with ξ . This set can be proven to be maximal:

Proposition 6.2. *Let $\xi_F = \tilde{\xi} + \sum \mu_i \eta_i$ be a CKVF in canonical form. If n is odd, $\{\tilde{\xi}, \eta_i\}$ is a maximal linearly independent pairwise commuting set of elements that commute with ξ_F . If n is even, $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ is a maximal linearly independent pairwise commuting set of elements that commute with ξ_F .*

Proof. Suppose that there is an additional CKVF ξ' commuting with each element in $\{\tilde{\xi}, \eta_i\}$ if n odd or $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ if n even (in either case ξ' clearly commutes with ξ_F also). Since it commutes with each η_i , by proposition 6.1, it admits a decomposed form $\xi' = \tilde{\xi}' + \sum_{i=1}^p \mu'_i \eta_i$, where $\tilde{\xi}'$ is a CKVF orthogonal to each η_i and which must verify $[\tilde{\xi}', \tilde{\xi}] = 0$. Equivalently, their associated endomorphisms satisfy $\tilde{F}' \in \mathcal{C}(\tilde{F})$, where $\mathcal{C}(\tilde{F})$ denotes the centralizer of F , i.e. the set of all skew-symmetric endomorphisms that commute with F . From the results in [25], $\mathcal{C}(\tilde{F}|_{\mathbb{M}^{1,2}}) = \text{span}\{\tilde{F}|_{\mathbb{M}^{1,2}}\}$ when n is odd and $\mathcal{C}(\tilde{F}|_{\mathbb{M}^{1,3}}) = \text{span}\{\tilde{F}|_{\mathbb{M}^{1,3}}, \tilde{F}^*|_{\mathbb{M}^{1,3}}\}$ when n is even. Here, \tilde{F}^* is the skew-symmetric endomorphism associated with $\tilde{\xi}^*$ and we restrict to $\mathbb{M}^{1,3}$ because the action of the endomorphisms is identically zero in $(\mathbb{M}^{1,3})^\perp$. Thus $\tilde{\xi}' = a\tilde{\xi}$, $a \in \mathbb{R}$, if n odd and $\xi' = b\tilde{\xi} + c\tilde{\xi}^*$, $b, c \in \mathbb{R}$ if n even. \square

7. Adapted coordinates

In the previous section we obtained a canonical form for each CKVF of the Euclidean space based on the canonical form of skew-symmetric endomorphisms in section 3. As an application, we consider in this section the problem of adapting coordinates in \mathbb{E}^n to a given CKVF ξ_F . The use of the canonical form will allow us to solve the problem for every possible ξ_F essentially in one go. Actually it will suffice to consider the case of even dimension n and assume that at least one of the parameters σ, τ in the canonical form of ξ_F is non-zero. The case where both σ and τ vanish will be obtained as a limit (and we will check that this limit does solve the required equations). The case of odd dimension n will be obtained from the even dimensional one by exploiting the property that \mathbb{E}^{2m+1} can be viewed as a hyperplane of \mathbb{E}^{2m+2} in such a way that the given CKVF ξ_F in \mathbb{E}^{2m+1} extends conveniently to \mathbb{E}^{2m+2} . Restricting the adapted coordinates already obtained in the even dimensional case to the appropriate hyperplane we will be able to infer the odd dimensional case. As we will justify the process of adapting coordinates is different for $n = 2$ and $n \geq 4$ even. The case $n = 2$ has been treated in detail in [25], so it will suffice to consider even $n \geq 4$ here.

7.1. Calculation of the adapted coordinates

We start by integrating the PDEs which yield adapted coordinates to an arbitrary CKVF in the case of even n . Consider \mathbb{E}^n endowed with a CKVF ξ_F . First of all, we adapt the Cartesian coordinates of \mathbb{E}^n so that ξ_F takes its canonical form and we fix the metric of \mathbb{E}^n to take the explicitly flat form in these coordinates. We further assume (for the moment) that n is even. For notational reasons it is convenient to rename the canonical coordinates⁵ as

⁵ The fact that we tag the coordinates $\{z_1, z_2, x_i, y_i\}$ with lower indices has no particular meaning. It is simply to avoid a notational clash of upper indices and powers that will appear later.

$z_1 := y^1, z_2 := y^2$ and $x_i := y^{2i+1}, y_i := y^{2i+2}$ for $i = 1, \dots, p$, where in the even case $p = n/2 - 1$ (see (34)). By proposition 6.1, ξ_F can be decomposed as a sum of CKVFs $\tilde{\xi}$ and η_i and, additionally one can construct canonically yet another CKVF $\tilde{\xi}^*$. This collection of CKVFs defines a maximal commutative set. Moreover, $\{\eta_i\}$ are all mutually orthogonal and perpendicular to $\tilde{\xi}$ and $\tilde{\xi}^*$. It is therefore most natural to try and find coordinates adapted simultaneously to the whole family $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$. This will lead a (collection of) coordinate systems where the components of ξ_F are simply constants. From here one can immediately find coordinates that rectify ξ_F , if necessary. It is important to emphasize that selecting the whole set $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ to adapt coordinates provides enough restrictions so that the coordinate change(s) can be fully determined. Imposing the (much weaker) condition that the system of coordinates rectifies only ξ_F is just a too poor condition to solve the problem. This is an interesting example where the structure of the canonical decomposition of ξ_F (or of F) is exploited in full.

By theorem 6.1, the explicit form of $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ in the canonical coordinates is

$$\begin{aligned} \tilde{\xi} &= \left(\frac{\sigma}{2} + \frac{1}{2} \left(z_1^2 - z_2^2 - \sum_{i=1}^p (x_i^2 + y_i^2) \right) \right) \partial_{z_1} \\ &\quad + \left(\frac{\tau}{2} + z_1 z_2 \right) \partial_{z_2} + z_1 \sum_{i=1}^p (x_i \partial_{x_i} + y_i \partial_{y_i}) \\ \tilde{\xi}^* &= - \left(\frac{\tau}{2} + z_1 z_2 \right) \partial_{z_1} + \left(\frac{\sigma}{2} - \frac{1}{2} \left(z_2^2 - z_1^2 - \sum_{i=1}^p (x_i^2 + y_i^2) \right) \right) \partial_{z_2} \\ &\quad - z_2 \sum_{i=1}^p (x_i \partial_{x_i} + y_i \partial_{y_i}) \\ \eta_i &= x_i \partial_{y_i} - y_i \partial_{x_i}. \end{aligned} \tag{36}$$

We are seeking coordinates $\{t_1, t_2, \phi_i, v_i\}$ adapted to these vector fields, i.e. such that $\partial_{t_1} = \tilde{\xi}$, $\partial_{t_2} = \tilde{\xi}^*$, $\partial_{\phi_i} = \eta_i$. It is clear that if $\{t_1, t_2, \phi_i, v_i\}$ is an adapted coordinate system, so it is $\{t_1 - t_{0,1}(v), t_2 - t_{0,2}(v), \phi_i - \phi_{0,i}(v), v_i\}$ for arbitrary functions $t_{0,1}(v)$, $t_{0,2}(v)$ and $\phi_{0,i}(v)$, where $v = (v_1, \dots, v_p)$. This will be used to simplify the process of integration. This freedom, may be restored at the end if so desired. Hence

$$\frac{\partial z_1}{\partial t_1} = \frac{\sigma}{2} + \frac{1}{2} \left(z_1^2 - z_2^2 - \sum_{i=1}^p (x_i^2 + y_i^2) \right), \quad \frac{\partial z_2}{\partial t_1} = \frac{\tau}{2} + z_1 z_2, \quad \frac{\partial x_i}{\partial t_1} = z_1 x_i, \quad \frac{\partial y_i}{\partial t_1} = z_1 y_i, \tag{37}$$

$$\frac{\partial z_2}{\partial t_2} = \frac{\sigma}{2} - \frac{1}{2} \left(z_2^2 - z_1^2 - \sum_{i=1}^p (x_i^2 + y_i^2) \right), \quad \frac{\partial z_1}{\partial t_2} = -\frac{\tau}{2} - z_1 z_2, \quad \frac{\partial x_i}{\partial t_2} = -z_2 x_i, \quad \frac{\partial y_i}{\partial t_2} = -z_2 y_i, \tag{38}$$

$$\frac{\partial z_1}{\partial \phi_i} = 0 \quad \frac{\partial z_2}{\partial \phi_i} = 0 \quad \frac{\partial x_i}{\partial \phi_i} = -y_i \quad \frac{\partial y_i}{\partial \phi_i} = x_i. \tag{39}$$

The additional p coordinates v_i , will appear through functions of integration. It is clear that the structure of the equations is different for $n = 2$, where there are no $\{x_i, y_i\}$, which implies that the process of integration follows a different route. The case $n = 2$ has been treated in full detail in [25], where the complex structure of \mathbb{S}^2 can be exploited to simplify the problem. Here we address the problem for $n \geq 4$ which we assume from now on.

We may start by integrating (39). The first pair gives $z_1 = z_1(t_1, t_2, v)$, $z_2 = z_2(t_1, t_2, v)$, so that the second pair becomes a harmonic oscillator in x_i, y_i , whose solution is

$$\begin{aligned} x_i &= \rho_i(t_1, t_2, v) \cos(\phi_i - \phi_{0,i}(t_1, t_2, v)), \\ y_i &= \rho_i(t_1, t_2, v) \sin(\phi_i - \phi_{0,i}(t_1, t_2, v)), \end{aligned} \tag{40}$$

where ρ_i and $\phi_{0,i}$ are arbitrary functions (depending only on the variables indicated) and ρ_i is not identically zero.

Inserting (40) in any of the two right-most equations of (37) and (38) and equating terms multiplying $\sin(\phi_i + \phi_{0,i})$ and $\cos(\phi_i + \phi_{0,i})$ yields:

$$z_1 = \frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t_1}, \quad z_2 = -\frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t_2}, \quad \frac{\partial \phi_{0,i}}{\partial t_1} = 0, \quad \frac{\partial \phi_{0,i}}{\partial t_2} = 0.$$

Thus, $\phi_{0,i}$ is a function only of v , which may be absorbed on the coordinate ϕ_i as discussed above. The two first equations imply

$$\frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t_1} = \frac{1}{\rho_j} \frac{\partial \rho_j}{\partial t_1}, \quad \frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t_2} = \frac{1}{\rho_j} \frac{\partial \rho_j}{\partial t_2} \iff \rho_i = \hat{\alpha}_i(v) \hat{\rho}(t_1, t_2, v),$$

for arbitrary (non-zero) functions $\hat{\alpha}_i$ and $\hat{\rho}$. Defining $\rho^2 := \sum_{i=1}^p \rho_i^2 = \left(\sum_{i=1}^p \hat{\alpha}_i^2 \right) \hat{\rho}^2$ we can write

$$\rho_i = \hat{\alpha}_i \hat{\rho} = \frac{\hat{\alpha}_i \epsilon}{\sqrt{\sum_{j=1}^p \hat{\alpha}_j^2}} \rho = \alpha_i \rho,$$

where $\alpha_i := \hat{\alpha}_i \epsilon / \sqrt{\sum_{j=1}^p \hat{\alpha}_j^2}$, with $\epsilon^2 = 1$, form a set of arbitrary (non-zero) functions of v such that $\sum_{i=1}^p \alpha_i^2 = 1$. The function ρ satisfies

$$z_1 = \frac{1}{\rho} \frac{\partial \rho}{\partial t_1}, \quad z_2 = -\frac{1}{\rho} \frac{\partial \rho}{\partial t_2}. \tag{41}$$

Inserting (41) in the two left-most equations in (37) and (38), with the change of variable $U = \rho^{-1}$, we obtain after some algebra the following covariant system of PDEs (indices $a, b = 1, 2$ refer to $\{t_1, t_2\}$)

$$\begin{aligned} \nabla_a \nabla_b U &= U A_{ab} + \frac{1}{2U} (1 + \nabla_c U \nabla^c U) g_{ab} \quad \text{with} \\ A &= \frac{1}{2} (-\sigma dt_1^2 + \sigma dt_2^2 + 2\tau dt_1 dt_2), \quad g = dt_1^2 + dt_2^2, \end{aligned} \tag{42}$$

and where ∇ is the Levi-Civita covariant derivative of g .

Lemma 7.1. *Up to shifts $t_1 \rightarrow t_1 - t_{0,1}(v)$ and $t_2 \rightarrow t_2 - t_{0,2}(v)$, the general solution of (42) with either σ or τ non-zero is given by*

$$U = \frac{\epsilon}{\mu_t^2 + \mu_s^2} (\beta \cosh(t_+) - \alpha \cos(t_-)) \quad \text{with } \beta = \sqrt{\alpha^2 + \mu_t^2 + \mu_s^2} \tag{43}$$

where α is a function of integration (depending on v), $\epsilon^2 = 1$ and $t_+ := \mu_t t_1 + \mu_s t_2$, $t_- := \mu_t t_2 - \mu_s t_1$, with μ_s, μ_t given by (9). The solution (43) admits a limit $\sigma = \tau = 0$ (i.e. $\mu_t = \mu_s = 0$) provided $\alpha > 0$, which is

$$\lim_{\mu_s \mu_t \rightarrow 0} U = \epsilon \frac{\alpha}{2} (t_1^2 + t_2^2) + \frac{\epsilon}{2\alpha}. \tag{44}$$

Up to shifts $t_1 \rightarrow t_1 - t_{0,1}(v)$ and $t_2 \rightarrow t_2 - t_{0,2}(v)$, this function is the general solution of (42) for $\sigma = \tau = 0$.

Proof. The coordinates t_+, t_- defined in the lemma diagonalize A and g simultaneously and yield

$$A = \frac{1}{2} (dt_+^2 - dt_-^2), \quad g = \frac{1}{\mu_s^2 + \mu_t^2} (dt_+^2 + dt_-^2).$$

From this and equation (42) it follows that $\partial^2 U / \partial t_+ \partial t_- = 0$ or, equivalently, $U(t_+, t_-) = U_+(t_+) + U_-(t_-)$. Subtracting the $\{t_+, t_+\}$ and $\{t_-, t_-\}$ components of (42) one obtains

$$\frac{d^2 U_+}{dt_+^2} - \frac{d^2 U_-}{dt_-^2} = U = U_+ + U_- \implies \frac{d^2 U_+}{dt_+^2} - U_+ = \frac{d^2 U_-}{dt_-^2} + U_- = \hat{a}$$

for an arbitrary separation function $\hat{a}(v)$. The general solution is clearly

$$U_+ = -\hat{a} + a \cosh(t_+) + b \sinh(t_-) \quad U_- = \hat{a} + c \cos(t_- - \delta), \tag{45}$$

where a, b, c, δ are also functions of v . Since \hat{a} drops out in $U = U_+ + U_-$ we may set $\hat{a} = 0$ w.l.o.g. Inserting (45) in (any of) the diagonal terms of (42) and one simply gets

$$a^2 - b^2 = \frac{1}{\mu_s^2 + \mu_t^2} + c^2.$$

Hence $|a| > |b|$ and we may use the freedom of translating t_+ by a function of v to write $U_+ = a \cosh(t_+)$ (i.e. $b = 0$). A similar translation in t_- sets $\delta = 0$. Rescaling the functions a, c as $a = (\mu_s^2 + \mu_t^2)^{-1} \beta$ and $c = -(\mu_s^2 + \mu_t^2)^{-1} \alpha$ we get

$$U = U_+ + U_- = \frac{\beta}{\mu_s^2 + \mu_t^2} \cosh(t_+) - \frac{\alpha}{\mu_s^2 + \mu_t^2} \cos(t_-), \quad \beta^2 = \mu_s^2 + \mu_t^2 + \alpha^2. \tag{46}$$

It is obvious that $\text{sign}(U) = \text{sign}(\beta)$. Thus taking β as the positive root $\beta = \sqrt{\alpha^2 + \mu_s^2 + \mu_t^2}$ and adding a multiplicative sign ϵ in (46), we obtain (43). To evaluate the convergence as both σ, τ tend to zero, or equivalently $\mu_s, \mu_t \rightarrow 0$, consider the series expansion

$$\begin{aligned} \beta \cosh(t_+) &= \left(|\alpha| + \frac{\mu_s^2 + \mu_t^2}{2|\alpha|} + o_{\mu_t, \mu_s}^{(4)} \right) \left(1 + \frac{(\mu_s t_2 + \mu_t t_1)^2}{2} + o_{\mu_t, \mu_s}^{(4)} \right), \\ \alpha \cos(t_-) &= \alpha - \alpha \frac{(\mu_t t_2 - \mu_s t_1)^2}{2} + o_{\mu_t, \mu_s}^{(4)}, \end{aligned}$$

where $o_{\mu_t, \mu_s}^{(4)}$ denotes a sum of homogeneous polynomials in μ_t, μ_s starting at order four, whose coefficients may depend on t_1, t_2 and α . Then, the expansion of U is

$$U = \frac{\epsilon}{\mu_s^2 + \mu_t^2} \left((|\alpha| - \alpha)(1 + \mu_s \mu_t t_1 t_2) + \frac{|\alpha| \mu_s^2 + \alpha \mu_t^2}{2} t_2^2 + \frac{|\alpha| \mu_t^2 + \alpha \mu_s^2}{2} t_1^2 + \frac{\mu_s^2 + \mu_t^2}{2|\alpha|} + o_{\mu_t, \mu_s}^{(4)} \right).$$

It is clear that $\lim_{\mu_s, \mu_t \rightarrow 0} o_{\mu_t, \mu_s}^{(4)} / (\mu_s^2 + \mu_t^2) = 0$ and the rest of the equation converges if and only if $\alpha > 0$ in which case the limit is (44). An easy calculation shows that this limit is (up to shifts in t_1, t_2) is the general solution of (42) when $\sigma, \tau = 0$. \square

Having the general solution (43) of (42) we can give the expression of the adapted coordinates

$$z_1 = -\frac{1}{U} \frac{\partial U}{\partial t_1} = \left| \frac{1}{U} \right| \frac{\alpha \mu_s \sin(t_-) - \beta \mu_t \sinh(t_+)}{\mu_s^2 + \mu_t^2}, \tag{47}$$

$$z_2 = \frac{1}{U} \frac{\partial U}{\partial t_2} = \left| \frac{1}{U} \right| \frac{\alpha \mu_t \sin(t_-) + \beta \mu_s \sinh(t_+)}{\mu_s^2 + \mu_t^2}, \tag{48}$$

$$x_i = \frac{\alpha_i}{U} \cos(\phi_i), \quad y_i = \frac{\alpha_i}{U} \sin(\phi_i), \tag{49}$$

where no sign of α is in principle assumed⁶, except for the case $\mu_s = \mu_t = 0$, where U must be understood as the limit (with $\alpha > 0$) (44) and $z_1 = -U^{-1} \partial U / \partial t_1, z_2 = U^{-1} \partial U / \partial t_2$. This coincides with the limit of the rhs expressions (47), (48), which is

$$z_1 = \frac{-2\alpha^2 t_1}{1 + \alpha^2(t_1^2 + t_2^2)}, \quad z_2 = \frac{2\alpha^2 t_2}{1 + \alpha^2(t_1^2 + t_2^2)}. \tag{50}$$

From equations (47)–(49) it is obvious that the sign ϵ is not relevant in the definition of the adapted coordinates. This is because the two branches $\epsilon = 1$ and $\epsilon = -1$ correspond to $U > 0$ and $U < 0$ respectively, which in terms of the adapted coordinates, is equivalent to a rotation of π in the ϕ_i angles. Hence, w.l.o.g. we consider $\epsilon = 1$, i.e. $U > 0$. Also notice that the dependence on the variables v_i appears through the functions α_i and α , with $\sum_{i=1}^p \alpha_i^2 = 1$. The set $\{\alpha_i, \alpha\}$ define p independent arbitrary functions of the variables v_i , so it is natural to use as coordinates $\{\alpha_i, \alpha\}$ themselves, provided they are restricted to satisfy $\sum_{i=1}^p \alpha_i^2 = 1$.

7.2. Region covered by the adapted coordinates

We now calculate the region of \mathbb{E}^n covered by the adapted coordinates. It is clear that in no case this region can include neither the zeros of the vector fields $\tilde{\xi}$ and $\tilde{\xi}^*$ and η_i nor the points where these $p + 2$ vectors are linearly dependent. We therefore start by locating those points. Denoting the loci of the zeros of $\tilde{\xi}$ and $\tilde{\xi}^*$ and η_i by $\mathcal{Z}(\tilde{\xi}), \mathcal{Z}(\tilde{\xi}^*)$ and $\mathcal{Z}(\eta_i)$ respectively, a simple calculation gives

$$\begin{aligned} \mathcal{Z}(\tilde{\xi}) &= \left(\left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\} \cap \{z_1 = \pm \mu_t, z_2 = \mp \mu_s\} \right) \cup \\ &\left(\{z_1 = 0\} \cap \left\{ z_2^2 + \sum_{j=1}^p (x_j^2 + y_j^2) = \mu_s^2 - \mu_t^2 \right\} \text{ if } \mu_s \mu_t = 0 \right), \tag{51} \\ \mathcal{Z}(\tilde{\xi}^*) &= \left(\left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\} \cap \{z_1 = \pm \mu_t, z_2 = \mp \mu_s\} \right) \cup \end{aligned}$$

⁶ The domain of definition of α will be later restricted under the condition that the adapted coordinates define a one to one map.

$$\left(\{z_2 = 0\} \cap \left\{ z_1^2 + \sum_{j=1}^p (x_j^2 + y_j^2) = \mu_t^2 - \mu_s^2 \right\} \text{ if } \mu_s \mu_t = 0 \right),$$

$$\mathcal{Z}(\eta_i) = \{x_i = y_i = 0\}. \tag{52}$$

These expressions are valid for every value of μ_s, μ_t and imply that in the case $\mu_s = \mu_t = 0$, $\mathcal{Z}(\tilde{\xi}) = \mathcal{Z}(\tilde{\xi}^*) = \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\} \cap \{z_1 = z_2 = 0\}$, which is contained in each $\mathcal{Z}(\eta_i) = \{x_i = y_i = 0\}$.

On the other hand, since $\{\tilde{\xi}, \eta_i\}$ is an orthogonal set of CKVFs (cf lemma 6.2), they are pointwise linearly independent at all points where they do not vanish. Similarly, $\{\tilde{\xi}^*, \eta_i\}$ is also an orthogonal set, so linear independence is guaranteed away from the zero set. Away from this set, the set of vectors $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ is linearly dependent only at points where $\tilde{\xi}$ and $\tilde{\xi}^*$ are proportional to each other with a non-zero proportionality factor, $\tilde{\xi} = a\tilde{\xi}^*$, $a \neq 0$. One easily checks that, away from $\mathcal{Z}(\tilde{\xi})$ and $\mathcal{Z}(\tilde{\xi}^*)$, the set of point where $\tilde{\xi} - a\tilde{\xi}^*$ vanishes is empty except when $\mu_s \neq 0, \mu_t \neq 0$ and $a = \frac{\mu_t}{\mu_s}$. It turns out to be useful to determine the set of points where $\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^* = 0$ when at least one of $\{\mu_s, \mu_t\}$ is non-zero. We call this set $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$, and a straightforward analysis gives

$$\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*) = \begin{cases} \left\{ \mu_s z_1 = -\mu_t z_2 \right\} \cap \left\{ (\mu_s^2 + \mu_t^2) z_2^2 + \mu_s^2 \sum_{i=1}^p (x_i^2 + y_i^2) = (\mu_s^2 + \mu_t^2) \mu_s^2 \right\} & \text{if } \mu_s \neq 0 \\ \left\{ \mu_s z_1 = -\mu_t z_2 \right\} \cap \left\{ (\mu_s^2 + \mu_t^2) z_1^2 + \mu_t^2 \sum_{i=1}^p (x_i^2 + y_i^2) = (\mu_s^2 + \mu_t^2) \mu_t^2 \right\} & \text{if } \mu_t \neq 0 \end{cases} \tag{53}$$

Obviously, the two expressions are equivalent when both μ_s and μ_t are non-zero. The interest of this set is that it happens to always contain $\mathcal{Z}(\tilde{\xi})$ and $\mathcal{Z}(\tilde{\xi}^*)$. This, together with the fact that when $\mu_s = \mu_t = 0$ these sets are contained in the axes $\mathcal{Z}(\eta_i)$ will allow us to ignore them altogether.

Lemma 7.2. *Assume that at least one of $\{\mu_s, \mu_t\}$ is non-zero. Then $\mathcal{Z}(\tilde{\xi}), \mathcal{Z}(\tilde{\xi}^*) \subset \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$.*

Proof. Consider first $\mu_s, \mu_t \neq 0$. Then at $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*) \cap \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\}$ we have that $z_1 = \pm \mu_t$ and $z_2 = \mp \mu_s$, which establishes $\mathcal{Z}(\tilde{\xi}), \mathcal{Z}(\tilde{\xi}^*) \subset \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$ in this case. When $\mu_t = 0, \mu_s \neq 0$, by definition of the respective sets we have $\mathcal{Z}(\tilde{\xi}) = \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$. Moreover, directly from (52) one finds

$$\mathcal{Z}(\tilde{\xi}^*) = \bigcap_{j=1}^p \{x_j = y_j = 0\} \cap \{z_1 = 0, z_2 = \pm \mu_s\},$$

which (cf the first expression in (53)) is clearly contained in $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$. An analogous argument applies in the case $\mu_t \neq 0, \mu_s = 0$. \square

Let us define the following auxiliary coordinates

$$\widehat{z}_+ := \frac{\mu_s z_1 + \mu_t z_2}{\sqrt{\sum_{i=1}^p (x_i^2 + y_i^2)}}, \quad \widehat{z}_- := \frac{\mu_s z_2 - \mu_t z_1}{\sqrt{\sum_{i=1}^p (x_i^2 + y_i^2)}}, \quad \widehat{x}_i := x_i, \quad \widehat{y}_i := y_i.$$

Except for the case $\mu_s = \mu_t = 0$ (which will be analysed later) the coordinates $\{\hat{z}_+, \hat{z}_-, \hat{x}_i, \hat{y}_i\}$ obviously cover $\mathbb{R}^n \setminus \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\}$. In terms of the adapted coordinates, they read

$$\hat{z}_+ = \alpha \sin(t_-), \quad \hat{z}_- = \beta \sinh(t_+) \quad \hat{x}_i = \frac{\alpha_i}{U} \cos(\phi_i), \quad \hat{y}_i = \frac{\alpha_i}{U} \sin(\phi_i). \quad (54)$$

Let us analyse the points where (54) fails to be a change of coordinates and hence restrict the domain of definition of $\{\alpha, t_-, t_+, \alpha_i, \phi_i\}$. The first thing to notice is that a change of sign in the coordinate α_i is equivalent to a rotation of angle π in the coordinate ϕ_i . Moreover, at points where $\alpha_i = 0$, i.e. the axis of η_i , the coordinate ϕ_i is completely degenerate, which obviously excludes $\bigcup_{j=1}^p \{x_j = y_j = 0\}$ from the region covered by the adapted coordinates. To avoid duplications, we must restrict $\alpha_i \in (0, 1)$ and $\phi \in [-\pi, \pi)$ or alternatively $\alpha_i \in (-1, 1) \setminus \{0\}$ and $\phi_i \in [0, \pi)$. We choose the former for definiteness.

The hypersurface $\{\alpha = \text{const}, t_- = \text{const}, t_+ = \text{const}\}$ is an $n - 3$ dimensional sphere of radius U^{-1} , namely $\{\hat{z}_- = \text{const}, \hat{z}_+ = \text{const}\} \cap \{\sum_{i=1}^p (x_i^2 + y_i^2) = U^{-2} = \text{const}\}$. This gives a straightforward splitting of $\mathbb{R}^n \setminus \{0_{n-2}\}$, with $0_{n-2} := \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\}$, into $\mathbb{R}^2 \times (\mathbb{R}^{n-2} \setminus \{0_{n-2}\})$, where $\mathbb{R}^{n-2} \setminus \{0_{n-2}\}$ is foliated by $n - 3$ dimensional spheres. The set $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$ respects this foliation, so it descends to $\mathbb{R}^2 \times \mathbb{R}^+$ (the last factor is the radius of the $n - 3$ sphere). To avoid extra notation we also use $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$ to denote this quotient set. We next show that the adapted coordinates actually cover the largest possible domain, namely $\mathbb{R}^n \setminus \left\{ \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*) \cup \bigcup_{j=1}^p \{x_j = y_j = 0\} \right\}$. From the previous discussion, this is a consequence of the following result.

Lemma 7.3. *Assume that at least one of $\{\mu_s, \mu_t\}$ is not zero. Then the transformation*

$$\begin{aligned} (\hat{z}_+, \hat{z}_-, U) : \mathbb{R} \times [-\pi, \pi) \times \mathbb{R}^+ &\rightarrow (\mathbb{R}^2 \times \mathbb{R}^+) \setminus \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*) \\ (t_+, t_-, \alpha) &\mapsto (\hat{z}_+, \hat{z}_-, U). \end{aligned} \quad (55)$$

is a diffeomorphism.

Proof. The determinant of the Jacobian of (55) reads

$$\left| \frac{\partial(\hat{z}_+, \hat{z}_-, U)}{\partial(t_+, t_-, \alpha)} \right| = \alpha U.$$

Since U is strictly positive (cf (43) and recall that we chose $\epsilon = 1$ w.l.o.g.), the conflictive points are $\alpha = 0$. To calculate the locus $\{\alpha = 0\}$ we obtain the inverse transformation of α in terms of U, \hat{z}_+, \hat{z}_- by solving (43) and the first two in (54). The result is, after a straightforward computation,

$$\alpha = \pm \left(\hat{z}_+^2 + \frac{1}{4U^2(\mu_s^2 + \mu_t^2)^2} (\hat{z}_+^2 + \hat{z}_-^2 - U^2(\mu_s^2 + \mu_t^2)^2 + (\mu_s^2 + \mu_t^2))^2 \right)^{1/2}. \quad (56)$$

It follows that $\alpha = 0$ is equivalent to $\hat{z}_+ = 0$ and $\hat{z}_-^2 + \mu_s^2 + \mu_t^2 = U^2(\mu_s^2 + \mu_t^2)^2$. When translated into the original coordinates $\{z_1, z_2, x_i, y_i\}$ this set is precisely $\mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$. Also, from

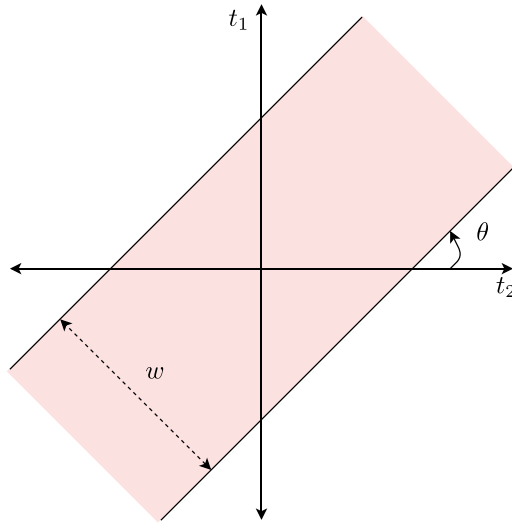


Figure 2. Band $B(\mu_s, \mu_t)$ where the coordinates t_1, t_2 are defined. The tilt is given by $\theta = \arctan\left(\frac{\mu_s}{\mu_t}\right)$ and the width w is $2\pi/\mu_t$ if $\mu_t \neq 0$, $2\pi/\mu_s$ if $\mu_t = 0, \mu_s \neq 0$ and $w \rightarrow \infty$ if $\mu_s = \mu_t = 0$.

(56) it is obvious that α is multivalued, which also implies that t_- is multivalued after substituting α as a function of \hat{z}_+, \hat{z}_-, U in the first equation in (54).⁷ We solve this issue by restricting α to be strictly positive and let t_- take values in $[-\pi, \pi)$. \square

We have shown that the adapted coordinates cover all \mathbb{R}^n except $\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*)$. The domain of definition of the coordinates t_1, t_2 depends on μ_t and μ_s , because $-\pi \leq t_- = \mu_t t_2 - \mu_s t_1 < \pi$. This defines a band $B(\mu_s, \mu_t) := \{-\pi \leq t_- = \mu_t t_2 - \mu_s t_1 < \pi\}$, whose width and tilt is determined by σ, τ through μ_s, μ_t (see figure 2). Nevertheless, the coordinate change is well defined for all values of t_1 and t_2 and involves only periodic functions of t_- . Thus, we can extend the domain of definition of t_1, t_2 to all of \mathbb{R}^2 . This defines a covering of the original space $\mathbb{R}^n \setminus (\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*))$ which unwraps completely the orbits of $\tilde{\xi}$ and $\tilde{\xi}^*$. It is not the universal covering because it does not unwrap the orbits of the axial vectors. This result is a generalization to higher dimensions of the covering discussed in detail in [25].

The limit case $\mu_s = \mu_t = 0$ (that is $\sigma = \tau = 0$) corresponds with a band of infinite width, i.e. $B(\mu_s, \mu_t) = \mathbb{R}^2$. In this case, the adapted coordinates also cover the largest possible set $\mathbb{R}^n \setminus (\bigcup_{j=1}^p \mathcal{Z}(\eta_j))$. Recall that in this case the only points where $\{\tilde{\xi}, \tilde{\xi}^*, \eta_i\}$ is not a linearly independent set is the union of $\mathcal{Z}(\tilde{\xi}), \mathcal{Z}(\tilde{\xi}^*)$, and $\mathcal{Z}(\eta_i)$ and we have already seen that in this case $\mathcal{Z}(\tilde{\xi}) = \mathcal{Z}(\tilde{\xi}^*) \subset \mathcal{Z}(\eta_i)$, for $i = 1, \dots, p$. This limit case is the same result that we would have obtained, had we performed a direct analysis using U as given by (44).

⁷ This was already evident by observing that a change of sign in α is cancelled by a rotation of π in t_- .

7.3. Conformally flat metrics in adapted coordinates

Once we have determined the adapted coordinates and the region they cover, we may proceed to calculate the expression of the Euclidean metric

$$g_E = dz_1^2 + dz_2^2 + \sum_{i=1}^p (dx_i^2 + dy_i^2). \tag{57}$$

in adapted coordinates. We start with the term $\sum_{i=1}^p (dx_i^2 + dy_i^2)$, which is straightforward

$$\begin{aligned} \sum_{i=1}^p (dx_i^2 + dy_i^2) &= \frac{dU^2}{U^4} + \frac{1}{U^2} \sum_{i=1}^p (d\alpha_i^2 + \alpha_i^2 d\phi_i^2) \Big|_{\sum_{i=1}^p \alpha_i^2 = 1} \\ &\quad - \frac{2 dU}{U^2} \left(\sum_{i=1}^p \alpha_i d\alpha_i \right) = \frac{dU}{U^4} + \frac{1}{U^2} \gamma_{\mathbb{S}^{n-3}}, \end{aligned} \tag{58}$$

where in the last equality we used $\sum_{i=1}^p \alpha_i d\alpha_i = 0$, which follows from $\sum_{i=1}^p \alpha_i^2 = 1$ and we have defined

$$\gamma_{\mathbb{S}^{n-3}} := \sum_{i=1}^p (d\alpha_i^2 + \alpha_i^2 d\phi_i^2) \Big|_{\sum_{i=1}^p \alpha_i^2 = 1}. \tag{59}$$

The notation is justified because the right-hand side corresponds to the standard unit metric on \mathbb{S}^{n-3} . This follows because $\sum_{i=1}^p (d\alpha_i^2 + \alpha_i^2 d\phi_i^2)$ is obviously flat and the restriction $\sum_{i=1}^p \alpha_i^2 = 1$ defines a unit sphere. We emphasize, however that for the rest of the paper the notation $\gamma_{\mathbb{S}^{n-3}}$ refers to the quadratic form above, not to the spherical metric in any other coordinate system. Observe also that dU in (58) should be understood as a short name for the explicit differential of U in terms of $dt_1, dt_2, d\alpha$. Using (57) and (58), we have

$$g_{t_1 t_1} = \left(\frac{\partial z_1}{\partial t_1} \right)^2 + \left(\frac{\partial z_2}{\partial t_1} \right)^2 + \frac{1}{U^4} \left(\frac{\partial U}{\partial t_1} \right)^2,$$

which after an explicit calculation reduces to

$$g_{t_1 t_1} = \frac{\alpha^2 + \mu_t^2}{U^2}.$$

Notice that $g_{t_1 t_1} = g_E(\tilde{\xi}, \tilde{\xi})$, $g_{t_2 t_2} = g_E(\tilde{\xi}^*, \tilde{\xi}^*)$ and $g_{t_1 t_2} = g_E(\tilde{\xi}, \tilde{\xi}^*)$. From the expressions in Cartesian coordinates it is straightforward to show

$$g_E(\tilde{\xi}, \tilde{\xi}) = g_E(\tilde{\xi}^*, \tilde{\xi}^*) - \sigma \sum_{i=1}^p (x_i^2 + y_i^2) = g_E(\tilde{\xi}^*, \tilde{\xi}^*) - \frac{\sigma}{U^2},$$

$$g_E(\tilde{\xi}, \tilde{\xi}^*) = \frac{\tau}{2} \sum_{i=1}^p (x_i^2 + y_i^2) = \frac{\tau}{2U^2}$$

where we have used $U^{-2} = \sum_{i=1}^p (x_i^2 + y_i^2)$ (see (49)). Thus

$$g_{t_2 t_2} = g_{t_1 t_1} + \frac{\sigma}{U^2} = \frac{\alpha^2 + \mu_s^2}{U^2}, \quad g_{t_1 t_2} = \frac{\tau}{2U^2} = \frac{\mu_s \mu_t}{U^2}.$$

The remaining terms are rather long to calculate. With the aid of a computer algebra system one gets

$$\begin{aligned}
 g_{\alpha\alpha} &= \left(\frac{\partial z_1}{\partial \alpha}\right)^2 + \left(\frac{\partial z_2}{\partial \alpha}\right)^2 + \frac{1}{U^4} \left(\frac{\partial U}{\partial \alpha}\right)^2 = \frac{1}{\beta^2 U^2} \\
 g_{\alpha t_1} &= \frac{\partial z_1}{\partial \alpha} \frac{\partial z_1}{\partial t_1} + \frac{\partial z_2}{\partial \alpha} \frac{\partial z_2}{\partial t_1} + \frac{1}{U^4} \frac{\partial U}{\partial \alpha} \frac{\partial U}{\partial t_1} = 0, \\
 g_{\alpha t_2} &= \frac{\partial z_1}{\partial \alpha} \frac{\partial z_1}{\partial t_2} + \frac{\partial z_2}{\partial \alpha} \frac{\partial z_2}{\partial t_2} + \frac{1}{U^4} \frac{\partial U}{\partial \alpha} \frac{\partial U}{\partial t_2} = 0.
 \end{aligned}$$

Notice that no terms in $d\alpha_i, d\phi_i$ appear but those in $\gamma_{\mathbb{S}^{n-3}}$, since neither U nor z_1, z_2 depend on α_i, ϕ_i . Putting all these results together we obtain the following expression:

Lemma 7.4. *In adapted coordinates $\{t_1, t_2, \alpha, \alpha_i, \phi_i\}$, the Euclidean metric g_E takes the form*

$$g_E = \frac{1}{U^2} \left((\alpha^2 + \mu_t^2) dt_1^2 + (\alpha^2 + \mu_s^2) dt_2^2 + 2\mu_s \mu_t dt_1 dt_2 + \frac{d\alpha^2}{\alpha^2 + \mu_s^2 + \mu_t^2} + \gamma_{\mathbb{S}^{n-3}} \right). \quad (60)$$

We would like to stress the simplicity of this result. Except in the global conformal factor, the metric does not depend in t_1 and t_2 (so, both $\tilde{\xi}$ and $\tilde{\xi}^*$ are Killing vectors of $U^2 g_E$). The dependence in the coordinate α and the conformal class constants $\{\mu_s, \mu_t\}$ is also extremely simple. Even more, the fact that all dependence in $\{\alpha_i, \phi_i\}$ arises only in $\gamma_{\mathbb{S}^{n-3}}$ allows us to use any other coordinate system on the unit \mathbb{S}^{n-3} . Any such coordinate system is still adapted to $\tilde{\xi}$ and $\tilde{\xi}^*$ but (in general) no longer to $\{\eta_i\}$. This enlargement to partially adapted coordinates is an interesting consequence of the foliation of \mathbb{R}^n by $(n - 3)$ -spheres described above.

7.4. *Odd dimensional case and adapted coordinates theorem*

We now work out the odd n case. As already discussed, we will base the analysis on the even dimensional case by restricting to a suitable a hyperplane. The underlying reason why this is possible is given in the following lemma.

Lemma 7.5. *Fix $n \geq 3$ odd. Let ξ_F be a CKVF of \mathbb{E}^n in canonical form and let $\{z_1, x_i, y_i\}$ be canonical coordinates. Consider the embedding $\mathbb{E}^n \hookrightarrow \mathbb{E}^{n+1}$ where \mathbb{E}^n is identified with the hyperplane $\{z_2 = 0\}$, for a Cartesian coordinate z_2 of \mathbb{E}^{n+1} . Then ξ_F extends to a CKVF of \mathbb{E}^{n+1} with the same value of σ, μ_i and $\tau = 0$.*

Proof. By remark 6.3 and theorem 6.1, the expression of ξ_F in the canonical coordinates $\{z_1, x_i, y_i\}$ is

$$\begin{aligned}
 \xi_F &= \left(\frac{\sigma}{2} + \frac{1}{2} \left(z_1^2 - \sum_{i=1}^p (x_i^2 + y_i^2) \right) \right) \partial_{z_1} + z_1 \sum_{i=1}^p (x_i \partial_{x_i} + y_i \partial_{y_i}) \\
 &\quad + \sum_{i=1}^p \mu_i (x_i \partial_{y_i} - y_i \partial_{x_i}) := \tilde{\xi} + \sum_{i=1}^p \mu_i \eta_i.
 \end{aligned}$$

Define ξ'_F on \mathbb{E}^{n+1} in Cartesian coordinates $\{z_1, z_2, x_i, y_i\}$ by $\xi'_F = \tilde{\xi} + \sum_{i=1}^p \mu_i (x_i \partial_{y_i} - y_i \partial_{x_i})$

where $\tilde{\xi}$ is given by (36) with $\tau = 0$. It is clear that this vector is a CKVF of \mathbb{E}^{n+1} written in canonical form, that it is tangent to the hyperplane $z_2 = 0$ and that it agrees with ξ_F on this submanifold. \square

Consequently, introducing adapted coordinates for the extended CKVF and restricting to $\{z_2 = 0\}$ will provide adapted coordinates for ξ_F . The restriction will obviously reduce the domain of definition of the adapted coordinates $(t_1, t_2, \alpha, \alpha_i, \phi_i)$ to a hypersurface. It is straightforward from equation (48) and the second equation in (50) that for the three cases $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$, the hyperplane $\{z_2 = 0\}$ corresponds to $\{t_2 = 0\}$. It follows that the remaining coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ are adapted to $\tilde{\xi}$ and all η_i . Their domain of definition is $t_1 \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$, $\alpha_i \in (0, 1)$, $\phi_i \in [-\pi, \pi)$ and the coordinate change is given by (47) (or the first in (50)) together with (49) after setting $\tau = 0$ and $t_2 = 0$. Depending on the sign of σ one gets for z_1

$$z_1 = \begin{cases} \frac{-1}{|U^+|} \frac{\alpha \sin(\sqrt{\sigma}t_1)}{\sqrt{\sigma}}, & \sigma > 0 \\ \frac{-1}{|U^-|} \frac{\sqrt{\alpha^2 + |\sigma|} \sinh(\sqrt{|\sigma|}t_1)}{\sqrt{|\sigma|}}, & \sigma < 0, \\ \frac{-1}{|U^0|} \alpha t_1, & \sigma = 0 \end{cases} \tag{61}$$

where

$$U^+ := \frac{1}{\sigma} \left(\sqrt{\alpha^2 + \sigma} - \alpha \cos(\sqrt{\sigma}t_1) \right),$$

$$U^- := \frac{1}{-\sigma} \left(\sqrt{\alpha^2 - \sigma} \cosh(\sqrt{-\sigma}t_1) - \alpha \right), \quad U^0 := \frac{1}{2} \left(\alpha t_1^2 + \frac{1}{\alpha} \right),$$

and for all three cases

$$x_i = \frac{\alpha_i}{U^\epsilon} \cos(\phi_i), \quad y_i = \frac{\alpha_i}{U^\epsilon} \sin(\phi_i), \tag{62}$$

where we write U^ϵ for the function U^+ , U^- or U^0 according with sign of σ .

The range of variation of $\{t_1, \alpha, \alpha_i, \phi_i\}$ was inferred before from the corresponding range of variation of $\{t_1, t_2, \alpha, \alpha_i, \phi_i\}$ in \mathbb{E}^{n+1} . It may happen, however, that when we restrict to the hyperplane $\{z_2 = 0\}$, the range gets enlarged and additional points get covered by the adapted coordinate system. The underlying reason is that, in effect, we are no longer adapting coordinates to $\tilde{\xi}^*$, so the points on $z_2 = 0$ where this vector is linearly dependent to $\tilde{\xi}'$ (or zero) are no longer problematic. When $\tau = 0$, one has

$$(\mu_s = \sqrt{\sigma}, \mu_t = 0) \quad \text{if } \sigma \geq 0, \quad (\mu_s = 0, \mu_t = \sqrt{|\sigma|}) \quad \text{if } \sigma \leq 0.$$

We may ignore the case $\sigma = 0$ because then $\mathcal{Z}(\tilde{\xi}') = \mathcal{Z}(\tilde{\xi}^*)$. It follows from (51) and (53) that

$$\mathcal{Z}(\tilde{\xi}')|_{z_2=0} = \begin{cases} \{z_1 = 0\} \cap \left\{ \sum_{i=1}^p (x_i^2 + y_i^2) = \sigma \right\} & \text{if } \sigma > 0 \\ \bigcap_{j=1}^p \{x_j = y_j = 0\} \cap \{z_1 = \pm \sqrt{|\sigma|}\} & \text{if } \sigma < 0 \end{cases}$$

$$\mathcal{Z}(\mu_s \tilde{\xi}' - \mu_t \tilde{\xi}^*)|_{z_2=0} = \begin{cases} \{z_1 = 0\} \cap \left\{ \sum_{i=1}^p (x_i^2 + y_i^2) = \sigma \right\} & \text{if } \sigma > 0 \\ \left\{ z_1^2 + \sum_{i=1}^p (x_i^2 + y_i^2) = |\sigma| \right\} & \text{if } \sigma < 0. \end{cases}$$

When $\sigma > 0$, the two sets are the same and no extension of the coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ is possible. However, when $\sigma < 0$, the set $\mathcal{Z}(\mu_s \tilde{\xi}' - \mu_t \tilde{\xi}'^*)|_{z_2=0}$ is strictly larger than $\mathcal{Z}(\tilde{\xi}')|_{z_2=0}$. From expressions (61) and (62) one checks that $\mathcal{Z}(\mu_s \tilde{\xi}' - \mu_t \tilde{\xi}'^*)|_{z_2=0} \setminus \mathcal{Z}(\tilde{\xi}')|_{z_2=0}$ corresponds exactly to the value $\alpha = 0$ and that $\mathcal{Z}(\tilde{\xi}) = \mathcal{Z}(\tilde{\xi}')|_{z_2=0}$ is at the limit $t_1 \rightarrow \pm\infty$. Thus, *a priori* there is the possibility that the adapted coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ can be extended regularly to $\alpha = 0$ when $\sigma < 0$. It follows directly from (61) that this is indeed the case (observe that, to the contrary, the limit $\alpha \rightarrow 0$ in (61) is singular when $\sigma \geq 0$, in agreement with the previous discussion). Thus, the range of definition of α is $[0, \infty)$ when $\sigma < 0$. The conclusion is that, irrespectively of the value of σ , the adapted coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ cover the largest possible domain of \mathbb{E}^n , namely all points where $\tilde{\xi}$ is non-zero away from the axes of $\{\eta_i\}$.

To obtain the Euclidean metric in \mathbb{E}^n for n odd in adapted coordinates we simply restrict (60) (with $n \rightarrow n + 1$) to the hypersurface $t_2 = 0$, and get

$$g_E^\epsilon = \frac{1}{(U^\epsilon)^2} \left(\left(\alpha^2 + \frac{(1-\epsilon)|\sigma|}{2} \right) dt_1^2 + \frac{d\alpha^2}{\alpha^2 + |\sigma|} + \gamma_{\mathbb{S}^{n-2}} \right), \tag{63}$$

where $\epsilon = -1, 0, 1$ respectively if $\sigma < 0, \sigma = 0, \sigma > 0$.

Remark 7.1. The three odd dimensional cases can be unified into one. The function U^0 coincides with the limits of U^+ and U^- when $\sigma \rightarrow 0$. However, the analytical continuation of U^+ to negative values of σ does not directly yield U^- . To solve this we introduce the function

$$W_1(y) = \frac{1}{\sigma} \left(\sqrt{y^2 + \sigma} - y \cos(\sqrt{\sigma} t_1) \right),$$

which is analytic in σ and takes real values for real σ . We observe that $U^+(\alpha = y) = W_1(y)$ for $\sigma > 0$, $U^0(\alpha = y) = W_1(y)$ ($\sigma = 0$) and $U^-(\alpha = +\sqrt{y^2 + \sigma}) = W_1(y)$ ($\sigma < 0$). This suggests introducing the coordinate change $\alpha = y$ for $\sigma \geq 0$ and $\alpha = +\sqrt{y^2 + \sigma}$ for $\sigma < 0$. From the domain of α , it follows that y takes values in $y > 0$ when $\sigma \geq 0$ and $y \geq \sqrt{-\sigma}$ when $\sigma < 0$. In terms of y , the three metrics metric g^ϵ take the unified form

$$g_E^\epsilon = \frac{1}{W_1(y)^2} \left(y^2 dt_1^2 + \frac{dy^2}{y^2 + \sigma} + \gamma_{\mathbb{S}^{n-2}} \right).$$

The function W_1 is the analytic continuation of U^+ to negative values of σ . We could have started with U^- and continued analytically to positive values of σ . Instead of repeating the argument, we simply introduce a new variable z defined by $y = \sqrt{z^2 - \sigma}$ with range of variation $z > \sqrt{\sigma}$ for $\sigma \geq 0$ and $z \geq 0$ for $\sigma < 0$. The metric takes the (also unified and even more symmetric) form

$$g_E^\epsilon = \frac{1}{W_2(z)^2} \left((z^2 - \sigma) dt_1^2 + \frac{dz^2}{z^2 - \sigma} + \gamma_{\mathbb{S}^{n-2}} \right), \quad W_2(z) := \frac{1}{\sigma} \left(z - \sqrt{z^2 - \sigma} \cos(\sqrt{\sigma} t_1) \right).$$

The function $W_2(z)$ is again analytic in σ , takes real values on the real line, and now it extends U^- . More specifically, $U^-(\alpha = z) = W_2(z)$ ($\sigma < 0$), $U^0(\alpha = z) = W_2(z)$ ($\sigma = 0$) and $U^+(\alpha = \sqrt{z^2 - \sigma}) = W_2(z)$ ($\sigma > 0$).

Remark 7.1 allows us to work with all the odd dimensional cases at once, which will be useful for section 8. However, this unified form does not arise naturally when the odd dimensional case is viewed as a consequence of the $n + 1$ even dimensional case. So, leaving aside this remark for section 8, we summarize the results of this section in the following theorem.

Theorem 7.6. *Given a CKVF ξ_F of \mathbb{E}^n , with $n \geq 4$ even, in canonical form $\xi_F = \tilde{\xi} + \sum_{i=1}^p \mu_i \eta_i$, the coordinates $t_1, t_2, \phi_i, \alpha, \alpha_i$, for $i = 1, \dots, p$ and $\sum_{i=1}^p \alpha_i^2 = 1$, defined by*

$$z_1 = -\frac{1}{U} \frac{\partial U}{\partial t_1}, \quad z_2 = \frac{1}{U} \frac{\partial U}{\partial t_2} \quad x_i = \frac{\alpha_i}{U} \cos(\phi_i), \quad y_i = \frac{\alpha_i}{U} \sin(\phi_i)$$

with

$$U = \frac{\sqrt{\alpha^2 + \mu_t^2 + \mu_s^2} \cosh(\mu_t t_1 + \mu_s t_2) - \alpha \cos(\mu_t t_2 - \mu_s t_1)}{\mu_t^2 + \mu_s^2},$$

which admits a limit $\lim_{\mu_s, \mu_t \rightarrow 0} U = \frac{\alpha}{2}(t_1^2 + t_2^2) + \frac{1}{2\alpha}$, furnish adapted coordinates to $\tilde{\xi} = \partial_{t_1}$, $\tilde{\xi}^* = \partial_{t_2}$, $\eta_i = \partial_{\phi_i}$, which cover the maximal possible domain, namely $\mathbb{E}^n \setminus \left(\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s \tilde{\xi} - \mu_t \tilde{\xi}^*) \right)$ for $t_1, t_2 \in B(\mu_s, \mu_t)$, $\phi_i \in [-\pi, \pi)$, $\alpha_i \in (0, 1)$ and $\alpha \in \mathbb{R}^+$. Moreover, the metric g_E , which is flat in canonical Cartesian coordinates, is given by

$$g_E = \frac{1}{U^2} \left((\alpha^2 + \mu_t^2) dt_1^2 + (\alpha^2 + \mu_s^2) dt_2^2 + 2\mu_s \mu_t dt_1 dt_2 + \frac{d\alpha^2}{\alpha^2 + \mu_s^2 + \mu_t^2} + \sum_{i=1}^p (d\alpha_i^2 + \alpha_i^2 d\phi_i^2) \Big|_{\sum_{i=1}^p \alpha_i^2 = 1} \right). \tag{64}$$

If $n \geq 3$ is odd and ξ_F is in canonical form, $\xi_F = \tilde{\xi} + \sum_{i=1}^p \mu_i \eta_i$, the coordinates $\{t_1, \phi_i, \alpha, \alpha_i\}$ adapted to $\tilde{\xi} = \partial_{t_1}$, $\eta_i = \partial_{\phi_i}$ are given by the case of $n + 1$ (even) dimensions, for $\tau = 0$ restricted to $t_2 = 0$ (which defines the embedding $\mathbb{E}^n = \{z_2 = 0\} \subset \mathbb{E}^{n+1}$) and cover again the maximal possible domain, given by $\mathbb{E}^n \setminus \left(\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\tilde{\xi}) \right)$ for $t_1 \in \mathbb{R}$, $\phi_i \in [-\pi, \pi)$, $\alpha_i \in (0, 1)$ and $\alpha \in \mathbb{R}^+$ when $\sigma \geq 0$ and $\alpha \in \mathbb{R}^+ \cup \{0\}$ when $\sigma < 0$. Moreover, the metric g_E , which is flat in canonical Cartesian coordinates, is given by the pull-back of (64) at $t_2 = 0$ after setting $\tau = 0$. Explicitly g_E is, depending on the sign of σ , given by (63) with $\gamma_{\mathbb{S}^{n-2}}$ as in (59).

8. TT-tensors

The adapted coordinates derived in section 7 provide a useful tool to solve geometric equations involving CKVFs. In this section we give an example of this in the context of Λ -vacuum spacetimes admitting a smooth null conformal infinity in the $n = 3$ case. Our aim is to give a simple yet interesting application of the formalism developed in the previous sections. We stress that these methods can also be used in the higher dimensional case (with a considerable amount of extra work).

Recall that for such spacetimes the data at \mathcal{S} is a conformal class $[g]$ of Riemannian metrics and a conformal class of transverse and divergence-free tensors. More specifically, for a representative metric g in the conformal class, there is associated a symmetric tensor D^{AB}

satisfying $g_{AB}D^{AB} = 0$ (divergence-free) and $\nabla_A D^{AB} = 0$ (transverse). For any other metric $\tilde{g} = \Omega^2 g$ in the conformal class, the associated tensor is $\Omega^{-(n+2)} D^{AB}$, which is again a TT tensor with respect to \tilde{g} . In dimension $n = 3$, it has been shown in [26] that the spacetime generated by the Cauchy data at \mathcal{S} admits a Killing vector if and only if the metric g admits a CKV ξ (which is the restriction of the Killing vector to \mathcal{S}) and D satisfies the so-called KID equation. This equation admits a natural generalization to arbitrary dimension which is

$$\mathcal{L}_\xi D^{AB} + \frac{n+2}{n} (\text{div}_g \xi) D^{AB} = 0, \tag{65}$$

where $\text{div}_g \xi$ is the divergence of ξ . Equation (65) reduces to the KID equation of Paetz in dimension $n = 3$ and it is conformally covariant, i.e. if $\{g_{AB}, D^{AB}, \xi^A\}$ is a solution, then so it is $\{\Omega^2 g_{AB}, \Omega^{-(n+2)} D^{AB}, \xi^A\}$. We emphasize however, that in higher dimension ($n \geq 4$) it is not known whether a spacetime admitting a smooth \mathcal{S} such that the corresponding data at null infinity solves the KID equation for some CKV ξ , must necessarily admit a Killing vector.

A CKVF satisfying (65) will be called KID vector for short. An important property of KID vectors is that they form a Lie subalgebra of CKVFs, i.e. if ξ, ξ' are KIDs for a given TT tensor D , then $[\xi, \xi']$ is also a KID for D . The problem of obtaining all TT-tensors with generality for a given conformal structure is hard, even in the conformally flat case (see e.g. [3, 32]). In this section we exploit the results above to obtain the general solution of the KID equations in dimension $n = 3$ for spacetimes which possess two commuting symmetries, one of which is axial. This case is specially relevant since $n = 3$ corresponds to the physical case of four spacetime dimensions and the class necessarily contains the Kerr–de Sitter family of spacetimes, which is a particularly interesting explicit family of spacetimes. Our strategy is to take an arbitrary CKVF ξ , derive its canonical form $\xi_F = \tilde{\xi} + \mu\eta$, adapt coordinates to $\tilde{\xi}$ and η and impose the KID equations to $\tilde{\xi}$ and η .⁸

The problem simplifies notably in the conformal gauge where $g := (U^\epsilon)^2 g_E^\epsilon$ because both $\tilde{\xi}$ and η become Killing vector fields. From remark 7.1, we may treat all cases $\sigma < 0, \sigma = 0, \sigma > 0$ at the same time by using the form of the metric

$$g = \frac{dz^2}{z^2 - \sigma} + (z^2 - \sigma) dt^2 + d\phi^2, \quad \tilde{\xi} = \partial_t, \quad \eta = \partial_\phi. \tag{66}$$

We remark that even though we solve the problem by fixing the coordinates and conformal gauge, we shall write the final result in fully covariant form (cf theorem 8.2 below). Also notice that, assuming that we have coordinates adapted to two orthogonal CKVFs $\partial_t, \partial_\phi$, and knowing that these vectors are orthogonal, the vanishing of the Cotton tensor reduces to an ODE in z (in the conformal gauge where $\partial_t, \partial_\phi$ are KVFs and $g_{\phi\phi=1} = 1$) which yields a metric of the form of (66).

In the conformal gauge of g , the condition that a TT-tensor D satisfies KID equations for both $\tilde{\xi}$ and η (which is equivalent to imposing that ξ and η are KID vectors) is trivial in the adapted coordinates obtained in the previous section:

$$\mathcal{L}_{\tilde{\xi}} D^{AB} = \partial_t D^{AB} = 0, \quad \mathcal{L}_\eta D^{AB} = \partial_\phi D^{AB} = 0.$$

⁸ In higher dimensions one could impose the KID equations, for $\tilde{\xi}$ and each η_i still yielding a tractable problem. One can also enlarge the class by suppressing some of the KIDs. Obviously, the less KID equations one imposes the more difficult the problem becomes.

Thus, D^{AB} are only functions of z . The transversality condition is also quite simple in adapted coordinates:

$$\frac{dD^{zz}}{dz} - z \left(\frac{D^{zz}}{z^2 - \sigma} + (z^2 - \sigma)D'' \right) = 0, \tag{67}$$

$$\frac{dD^{zt}}{dz} + \frac{2z}{z^2 - \sigma}D^{zt} = 0 \tag{68}$$

$$\frac{dD^{z\phi}}{dz} = 0, \tag{69}$$

while the traceless condition imposes

$$g_{AB}D^{AB} = \frac{D^{zz}}{z^2 - \sigma} + (z^2 - \sigma)D'' + D^{\phi\phi} = 0. \tag{70}$$

There are no equations for $D^{t\phi}$ so $D^{t\phi} = h(z)$ with $h(z)$ an arbitrary function. The general solution of equations (68) and (69) is obtained at once and reads

$$D^{zt} = \frac{K_1}{z^2 - \sigma}, \quad D^{z\phi} = K_2, \quad K_1, K_2 \in \mathbb{R}.$$

For equations (67) and (70), we let $D^{zz} =: f(z)$ be an arbitrary function and obtain the remaining components

$$D^{\phi\phi} = -\frac{1}{z} \frac{df}{dz}, \quad D'' = \frac{1}{z(z^2 - \sigma)} \frac{df}{dz} - \frac{f}{(z^2 - \sigma)^2}.$$

Summarizing

Lemma 8.1. *In the three-dimensional conformally flat class $[g]$, let ξ_F be a CKVF. Decompose ξ in canonical form $\xi = \tilde{\xi} + \mu\eta$ and fix the conformal gauge so that g given by (66). Then the most general symmetric TT-tensor D satisfying the KID equations for ξ and η simultaneously is, in adapted coordinates $\{z, t, \phi\}$, a combination (with constants) of the following tensors*

$$D_f := f \partial_z \otimes \partial_z + \left(\frac{1}{z(z^2 - \sigma)} \frac{df}{dz} - \frac{f}{(z^2 - \sigma)^2} \right) \partial_t \otimes \partial_t - \frac{1}{z} \frac{df}{dz} \partial_\phi \otimes \partial_\phi,$$

$$D_h := h(\partial_t \otimes \partial_\phi + \partial_\phi \otimes \partial_t), \quad D_{\tilde{\xi}} := \frac{1}{z^2 - \sigma}(\partial_z \otimes \partial_t + \partial_t \otimes \partial_z),$$

$$D_\eta = \partial_z \otimes \partial_\phi + \partial_\phi \otimes \partial_z,$$

where f and h are arbitrary functions of z .

Having obtained the general solution in a particular gauge, our next aim is to give a (diffeomorphism and conformal) covariant form of the generators in lemma 8.1. From [24], we know that, for any CKV ξ of any n -dimensional metric g (not necessarily conformally flat) the following tensors are TT w.r.t. to g and satisfy the KID equation with respect to ξ .

$$\mathcal{D}(\xi) = \frac{1}{|\xi|_g^{n+2}} \left(\xi \otimes \xi - \frac{|\xi|_g^2}{n} g^\sharp \right),$$

where $|\cdot|_g$ denotes the norm w.r.t. g and g^\sharp the contravariant form of g . Thus, we can rewrite D_f as

$$D_f = \left(-2(z^2 - \sigma)^{1/2} f + \frac{(z^2 - \sigma)^{3/2}}{z} \frac{df}{dz} \right) \mathcal{D}(\tilde{\xi}) - \left(\frac{f}{z^2 - \sigma} + \frac{1}{z} \frac{df}{dz} \right) \mathcal{D}(\eta).$$

We now restore the conformal gauge freedom by considering the metric $\hat{g} = \Omega^2 g$ and $\hat{D}_f = D_f/\Omega^5$, for any (positive) conformal factor Ω . Since the tensors $\mathcal{D}(\tilde{\xi}), \mathcal{D}(\eta)$ are already conformal and diffeomorphism covariant, we must impose their multiplicative factors in $\{D_f$ to be conformal and diffeomorphism invariant. With the gauge freedom restored, the norms of the CKVFs now are

$$|\tilde{\xi}|_{\hat{g}} = \Omega \sqrt{z^2 - \sigma}, \quad |\eta|_{\hat{g}} = \Omega.$$

Then, considering $f =: \sqrt{X} \hat{f}(X)$ as function of the conformal invariant quantity $X = |\tilde{\xi}|_{\hat{g}}/|\eta|_{\hat{g}} = \sqrt{z^2 - \sigma}$, one can directly cast \hat{D}_f in the following form:

$$\hat{D}_f = X^4 \frac{d}{dX} \left(\frac{\hat{f}(X)}{X^{3/2}} \right) \mathcal{D}(\tilde{\xi}) - \frac{1}{X^2} \frac{d}{dX} \left(X^{3/2} \hat{f}(X) \right) \mathcal{D}(\eta),$$

which is a conformal and diffeomorphism covariant expression. Notice that the expression is symmetric under the interchange $\tilde{\xi} \leftrightarrow \eta$ because the coefficient of $\mathcal{D}(\eta)$ expressed in the variable $Y = X^{-1}$ is identical in form to the coefficient of $\mathcal{D}(\tilde{\xi})$.

For the tensor $\hat{D}_h := D_h/\Omega^5$, redefining $h =: \hat{h} |\tilde{\xi}|^{-5/2}$, it is immediate to write

$$\hat{D}_h = \hat{D}_{\hat{h}} := \frac{\hat{h}}{|\eta|_{\hat{g}}^{5/2} |\tilde{\xi}|_{\hat{g}}^{5/2}} (\tilde{\xi} \otimes \eta + \eta \otimes \tilde{\xi}), \tag{71}$$

which is obviously conformal and diffeomorphism covariant if and only if \hat{h} is conformal invariant, e.g. considering $\hat{h} \equiv \hat{h}(X)$. We remark that the form (71) already appeared (with different powers due to the different dimension) in the classification [25] of TT tensors in dimension two satisfying the KID equation.

For the remaining tensors $\hat{\mathcal{D}}_\xi := D_\xi/\Omega^5$ and $\hat{D}_\eta := D_\eta/\Omega^5$, we define a conformal class of vector fields χ , which in the original gauge coincides with $\chi := \partial_z$. This vector is divergence-free $\nabla_A \chi^A = 0$, and this equation is conformally invariant provided the conformal weight of χ is -3 (i.e. for $\hat{g} = \Omega^2 g$, the corresponding vector is $\hat{\chi} = \Omega^{-3} \chi$). We therefore impose this conformal behaviour of χ .⁹ The direction of χ is fixed by orthogonality to $\tilde{\xi}$ and η . The combination of norms that has this conformal weight and recovers the appropriate expression in the gauge of lemma 8.1 is $|\chi|_{\hat{g}} := |\tilde{\xi}|_{\hat{g}}^{-1} |\eta|_{\hat{g}}^{-2}$ (note that the orthogonality and norm conditions fix

⁹This choice may appear somewhat ad hoc at this point. However, the condition of vanishing divergence appears naturally when studying (for more general metrics) under which conditions a tensor $\xi \otimes W + W \otimes \xi$ is a TT tensor satisfying the KID equation for ξ . We leave this general analysis for a future work.

χ uniquely up to an irrelevant sign in any gauge). Thus, we may write

$$D_{\tilde{\xi}} = \frac{1}{|\tilde{\xi}|_{\tilde{g}}^2}(\chi \otimes \tilde{\xi} + \tilde{\xi} \otimes \chi), \quad D_{\eta} = \frac{1}{|\eta|_{\tilde{g}}^2}(\chi \otimes \eta + \eta \otimes \chi),$$

which are conformally covariant expressions. Therefore, we get to the final result:

Theorem 8.2. *Let ξ be a CKVF of the class of three dimensional conformally flat metrics and let $\xi = \tilde{\xi} + \mu\eta$ a canonical form. For each conformal gauge, let us define a vector field χ with norm $|\chi|_{\tilde{g}} := |\tilde{\xi}|_{\tilde{g}}^{-1}|\eta|_{\tilde{g}}^{-2}$, orthogonal to $\tilde{\xi}$ and η . Then, any TT-tensor satisfying the KID equation (65) for $\tilde{\xi}$ and η is a combination (with constants) of the following tensors:*

$$\begin{aligned} \widehat{D}_{\tilde{f}} &= X^4 \frac{d}{dX} \left(\frac{\widehat{f}(X)}{X^{3/2}} \right) \mathcal{D}(\tilde{\xi}) - \frac{1}{X^2} \frac{d}{dX} \left(X^{3/2} \widehat{f}(X) \right) \mathcal{D}(\eta), \\ \widehat{D}_{\widehat{h}} &= \frac{\widehat{h}}{|\eta|_{\tilde{g}}^{5/2} |\tilde{\xi}|_{\tilde{g}}^{5/2}} (\tilde{\xi} \otimes \eta + \eta \otimes \tilde{\xi}), \\ D_{\tilde{\xi}} &= \frac{1}{|\tilde{\xi}|_{\tilde{g}}^2} (\chi \otimes \tilde{\xi} + \tilde{\xi} \otimes \chi), \\ D_{\eta} &= \frac{1}{|\eta|_{\tilde{g}}^2} (\chi \otimes \eta + \eta \otimes \chi), \end{aligned}$$

for arbitrary functions \widehat{f} and \widehat{h} of $X = |\tilde{\xi}|_{\tilde{g}}/|\eta|_{\tilde{g}}$.

Remark 8.1. The vector field χ defined in this theorem is divergence-free. This property would have been difficult to guess (and even to prove) in the original Cartesian coordinate system.

Remark 8.2. A corollary of this theorem is that the general solution of the Λ -vacuum Einstein field equation in four dimensions with a smooth conformally flat null infinity and admitting an axial symmetric and a second commuting Killing vector can be parametrized by two functions of one variable and two constants. Recall that in the $\Lambda = 0$ case, the general asymptotically flat stationary and axially symmetric solution of the Einstein field equations can be parametrized (in a neighbourhood of spacelike infinity, by two numerable sets of mass and angular multipole moments (satisfying appropriate convergence properties), see [1, 4, 6] for details. There is an intriguing parallelism between the two situations, at least at the level of crude counting of degrees of freedom. This suggests that maybe in the $\Lambda > 0$ case it is possible to define a set of multipole-type moments that characterizes the data at null infinity (and hence the spacetime), at least in the case of a conformally flat null infinity. For example, the contraction of an arbitrary TT tensor D with of any CKVF gives a conserved current. In particular $D^{\alpha}_{\beta} \tilde{\xi}^{\beta}$ and $D^{\alpha}_{\beta} \eta^{\beta}$ integrated over the surface $\mathcal{S}_t = \{t = \text{const.}\}$ give finite conserved charges (under suitable assumptions on \widehat{f} and \widehat{h}), which one could attempt relate to energy and/or angular momenta. This is an interesting problem, but beyond the scope of the present paper.

Remark 8.3. It is natural to ask whether theorem 8.2 is general for TT-tensors admitting two commuting KIDs, $\tilde{\xi}, \eta$, without the condition of η being conformally axial. In appendix C of [24] one can explicitly find, for an arbitrary CKVF ξ , the set $\mathcal{C}(\xi)$ of elements that commute with ξ . Then, from a case by case analysis, one concludes that except in one special situation, for any linearly independent pair ξ, ξ' , with $\xi' \in \mathcal{C}(\xi)$ it is the case that there is a CAKVF

$\eta \in \mathcal{C}(\xi)$ such that $\text{span}\{\xi, \eta\} = \text{span}\{\xi, \xi'\}$. Thus, all these cases are covered by theorem 8.2. The exceptional case is when ξ, ξ' are conformal to translations. It is immediate to solve the TT and KID equations for such a case directly in Cartesian coordinates.

The solution given in theorem 8.2 provides a large class of initial data, which we know must contain the so-called Kerr–de Sitter-like *class* with conformally flat \mathcal{S} (see [24] for precise definition and properties of this class), which in turn contains the Kerr–de Sitter *family* of spacetimes. It is interesting to identify this class within the general solution given in theorem 8.2. The characterizing property of the Kerr–de Sitter-like class in the conformally flat case is $D = D(\xi)$ for some CKVF ξ , where moreover, only the conformal class of ξ matters to determine the family associated to the data. Decomposing canonically $\xi = \tilde{\xi} + \mu\eta$, a straightforward computation yields

$$D(\xi) = \frac{X^5}{(X^2 + \mu^2)^{5/2}} \mathcal{D}(\tilde{\xi}) + \frac{\mu^2}{(X^2 + \mu^2)^{5/2}} \mathcal{D}(\eta) + \frac{\mu X^{5/2}}{(X^2 + \mu^2)^{5/2}} \hat{D}_{\hat{h}=1},$$

which comparing with theorem 8.2 yields the following corollary: It is also of interest to identify the the Kerr–de Sitter family. To that aim we combine the results in [24] to those in the present paper to show that this family corresponds to $\sigma < 0$. The classification of conformal classes of ξ in [24] is done in terms of the invariants $\hat{c} = -c_1$ and $\hat{k} = -c_2$ together with the rank parameter r , where c_1 and c_2 are the coefficients of the characteristic polynomial of the skew-symmetric endomorphism F associated to ξ . In terms of these objects, it is shown in [24] that the Kerr–de Sitter family corresponds to either $\mathcal{S}_1 = \{\hat{k} > 0, \hat{c} \in \mathbb{R} \text{ and } r = 2\}$, or $\mathcal{S}_2 = \{\hat{k} = 0, \hat{c} > 0 \text{ and } r = 1\}$, the latter defining the Schwarzschild–de Sitter family. It is immediate to verify that, since (cf corollary 3.3.1) $\hat{k} = -\sigma\mu^2 < 0$ and $\hat{c} = -\sigma - \mu^2$, then $\mathcal{S}_1 = \{\sigma < 0, \mu \neq 0\}$ and $\mathcal{S}_2 = \{\sigma < 0, \mu = 0\}$ (the condition $\mu \neq 0$ implies $r = 2$ and $\mu = 0$ implies $r = 1$). Thus, in terms of the classification developed in this paper, the Kerr–de Sitter family corresponds to $\sigma < 0$. It is interesting that in the present scheme we no longer need to specify the rank parameter to identify the Kerr–de Sitter family (unlike in [24]) and that the whole family is represented by an open domain. We emphasise that the dependence in σ in the solutions given in theorem 8.2 and corollary 8.2.1 is implicit through the norm of $\tilde{\xi}$.

Corollary 8.2.1. *The Kerr–de Sitter-like class with conformally flat \mathcal{S} is determined by the TT-tensor $D_{KdS} = \hat{D}_f + \hat{D}_h$ with*

$$\hat{f} = -\frac{1}{3} \frac{X^{3/2}}{(X^2 + \mu^2)^{3/2}}, \quad \hat{h} = \mu \frac{X^{5/2}}{(X^2 + \mu^2)^{5/2}}.$$

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Data availability statement

No new data were created or analysed in this study.

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