Free data at spacelike $\mathscr{I}$ and characterization of Kerr-de Sitter in all dimensions.

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Abstract

We study the free data in the Fefferman-Graham expansion of asymptotically Einstein metrics with non-zero cosmological constant. We prove that if $\mathscr{I}$ is conformally flat, the rescaled Weyl tensor at $\mathscr{I}$ agrees up to a constant with the free data at $\mathscr{I}$, namely the traceless part of the n-th order coefficient of the expansion. In the non-conformally flat case, the rescaled Weyl tensor is generically divergent at $\mathscr{I}$ but one can still extract the free data in terms of the difference of the Weyl tensors of suitably constructed metrics, in full generality when the spacetime dimension D is even and provided the so-called obstruction tensor at $\mathscr{I}$ is identically zero when D is odd. These results provide a geometric definition of the data, particularly relevant for the asymptotic Cauchy problem of even dimensional Einstein metrics with positive Λ and also for the odd dimensional analytic case irrespectively of the sign of Λ. We establish a Killing initial data equation at spacelike $\mathscr{I}$ in all dimension for analytic data. These results are used to find a geometric characterization of the Kerr-de Sitter metrics in all dimensions in terms of its geometric data at null infinity.

1 Introduction

Globally hyperbolic spacetimes are uniquely determined by their initial configurations. Indeed, the Einstein equations admit a Cauchy problem which is longtime known to be well-posed by the landmarks result of Y. Choquet-Bruhat [9] and Choquet-Bruhat and Geroch [4]. This allows, in particular, to extract interesting properties of the solutions without actually having to deal with the full complexity of the Einstein equations. This Cauchy problem splits the Einstein equations into constraint equations on an initial spacelike hypersurface[1] plus evolution equations, which propagate the fields (and the constraints). This is the classical initial value formulation and a set of initial data is by definition any solution of the constraint equations. Although certainly simpler that the full Einstein equations, they are still pose a difficult problem in geometric analysis (see e.g. [19] and references therein). In addition, the solutions evolving from a set of initial data are local due to the intrinsic hyperbolicity of the evolution equations.

The works of R. Penrose in the 1960s [30], [28], [29] pioneered the use conformal techniques for the analysis of global properties of solutions of the Einstein equations. Eversince, conformal geometry has been widely used in general relativity. For certain conformal transformations $g = \Omega^2 \widetilde{g}$, where Ω is a smooth positive function nowhere vanishing in $\widetilde{M}$, one can extend g to a manifold with boundary $M = \widetilde{M} \cup \partial M$ so that the asymptotic properties of $\widetilde{g}$ become local properties at the submanifold $\mathscr{I} := (\partial M, g\mid_{\partial M})$, known as "conformal infinity".

When written in terms of the conformal metric g, the Einstein equation of $\tilde{g}$ are singular at $\mathscr{I}$. However, in a remarkable achievement H. Friedrich was able (by means of introducing carefully chosen variables) to rewrite the equations in spacetime dimension four as a system of geometric PDE that are regular at $\mathscr{I}$ (see the seminal works [10], [11] and the reviews [13], [14]). This system allows to formulate an asymptotic initial value problem which was also proven by Friedrich [12] to be always well-posed if the cosmological constant is positive, a case to which we will pay special attention in this paper.

[1] It can be also cast as a characteristic initial value problem if the initial hypersurface is null (see [32]).

The Cauchy problem at $\mathscr{I}$ with positive cosmological constant Λ is interesting for several reasons. First, because one can prescribe by hand the asymptotic behaviour of the spacetime. It is also noteworthy that associated to a conformal metric g solving the conformal Friedrich equations, there is a solution to the Einstein equations $\widetilde{g}$ which is "semiglobal" (i.e. the "physical" spacetime $\widetilde{g} = \Omega^{-2} g$ extends infinitely towards the future or past, depending on whether $\mathscr{I}$ is a final or an initial state). In addition, a remarkable simplification occurs in the constraint equations at $\mathscr{I}$ as opposed to the standard constraint equations of the classical initial value problem. The data at $\mathscr{I}$ consist of a conformal Riemannian three-manfiold $(\Sigma, [\gamma])$ which prescribes the (conformal) geometry of $\mathscr{I}$, together with a conformal class of symmetric two-tensors $[D]$ with vanishing trace an divergence, i.e. a conformal class of transverse and traceless (TT) tensors.

The Friedrich conformal field equations are specially taylored to dimension four and does not appear to extend to higher dimensions. The basic problem is that there do not appear to be enough evolution equations that remain regular at $\mathscr{I}$ [13]. Actually, one of the fundamental objects in the conformal Friedrich equations is the rescaled Weyl tensor, which plays a centrar role in this paper. Our analysis shows that in dimension higher than four this object is regular at $\mathscr{I}$ only in few particular cases. Thus, there are reasons to believe that any attempt to find a regular Cauchy problem well-posed at $\mathscr{I}$ based on this object will be unfruitful.

In spite of this, one can actually find in the literature related existence and uniqueness results in higher dimensions [2] based on quite a different approach. This is the framework introduced by Fefferman and Graham, first in the paper [7] and later extended into a monograph [8]. An important part of their work relates to asymptotically Einstein $n+1$-dimensional metrics, namely, conformally extendable metrics which satisfy the Einstein equations (to a certain order) at (n-dimensional) $\mathscr{I}$. This is carried through the study of their asymptotic formal series expansions, usually called Fefferman-Graham (FG) expansion. It is remarkable the qualitative difference which appears between the n odd and even cases. For n even, there is an obstruction to find a smooth metric satisfying the Einstein equations to infinite order at Σ. The expansion introduces logarithmic terms which depend on a certain conformally invariant tensor $\mathcal{O}$ determined by the boundary metric, known as obstruction tensor. No such obstruction occurs for n odd. Interestingly it is the obstruction tensor what allows Anderson [2] to find an asymptotic Cauchy problem for the Einstein equations in the n odd case. For $n+1$ even dimensional metrics $\widetilde{g}$, this tensor provides the differential equation $\mathcal{O} = 0$, which for Lorentzian conformally Einstein metrics, can be cast as a Cauchy problem at $\mathscr{I}$. Anderson proves that solutions of this Cauchy problem exist and are uniquely determined for every pair of symmetric two-tensors $(\gamma, g_{(n)})$, γ positive definite and $g_{(n)}$ traceless and transverse w.r.t. γ. A posteriori, γ determines the geometry of $\mathscr{I}$ and $g_{(n)}$ is n-th order coefficient of the asymptotic expansion of $\widetilde{g}$. This idea is not extendable to the n even case, for no obstruction tensor can be built out of $\widetilde{g}$ when $n+1$ is odd.

Conversely, an existence and uniqueness theorem can be used to characterize spacetimes by means of their Cauchy data. The situation is particularly interesting in the case of the asymptotic Cauchy problem for positive Λ, because of the simplicity of the data, which potentially allows one to achieve classification results for spacetimes whose explicit form need not to be known. For this task, one can use Anderson's theorem when n is odd. However, using the coefficients of the Fefferman-Graham expansion as data has limitations because by construction they are attached to very special coordinate systems where the FG expansion holds. Consequently, computing the data requires first to find these special coordinates, which in general is not an easy task. Thus, it would be a remarkable improvement to determine the asymptotic data in an entirely geometrical way. We note that this is precisely the case when $n = 3$ because γ determines the geometry of $\mathscr{I}$ and $g_{(3)}$ is the electric part of the rescaled Weyl tensor [12]. This the first issue we address in this paper. We prove that the n-th order coefficient of the FG expansion agrees (up to a certain constant) with the electric part of the rescaled Weyl tensor W at $\mathscr{I}$ if γ is conformally flat. We also discuss that this result would actually be an "if and only if" provided one knew that purely magnetic Λ-vacuum spacetimes (i.e. whose electric part of the Weyl tensor vanishes) do not exist besides the case of spacetimes with constant curvature. This non-existence of purely magnetic spacetimes was first conjectured in [26] (see also [3]) and still remains open despite interesting progress in several particular situations. More details will be given in the main text. In the non-conformally flat $\mathscr{I}$ case, W is in general not regular at $\mathscr{I}$ and we show that by removing the divergent terms, the leading order is precisely $g_{(n)}$. How to geometrically characterize this divergent part is left open.

Any characterization result of spacetimes via data (Σ, γ, D) at $\mathscr{I}$ with positive Λ must encode all the information of the corresponding spacetime. In particular, the presence of symmetries must also constraint these data. The case $n = 3$ has been studied in [27] where it is shown that the spacetime admitting a Killing vector is equivalent to the TT tensor D satisfying geometrically neat equation, the so-called Killing initial data (KID) equation. This equation involves a conformal Killing vector field (CKVF) of γ which is a posteriori the Killing vector field at $\mathscr{I}$ in the conformally extended spacetime. Apart from this $n=3$ case, no previous results relating continuous local isometries to initial data at $\mathscr{I}$ were known in more dimensions. In this paper we prove a higher dimensional result, analogous to the $n=3$ one, restricted to the case of analytic metrics with zero obstruction tensor.

The results described above are used in the second part of the paper to characterize the generalized Kerr-de Sitter spacetimes [16]. We find that the data corresponding to Kerr-de Sitter is characterized by the conformal class $[\gamma]$ being locally conformally flat and the tensor $g_{(n)}$ taking the form $g_{(n)} = D_\xi$ where D_ξ is a TT tensor depending on a conformal Killing vector field (CKVF) ξ of γ. More concretely, the CKVF must belong to a specific conformal class which we also explicitly determine. Since by the results in the first part of the paper $g_{(n)}$ admits a clear geometric interpretation in terms of the rescalled Weyl tensor whenever $[\gamma]$ is locally conformally flat, our characterization result of Kerr-de Sitter at $\mathscr{I}$ is fully geometric. The result here is a natural generalization of the already known case for $n=3$, studied in [21] (see [22] for the non-conformally flat $\mathscr{I}$ case, and also [15]).

The contents of the paper are organized as follows. In section 2 we start by setting the basics of the Fefferman-Graham formalism. Some general results of conformal geometry are also proven and two useful formulas for the Weyl tensor are derived, which have several applications in this paper, an example being the calculation of the initial data in the second part of the paper. The main results of the section are Lemma 2.4 which can be used to relate the Weyl tensors at $\mathscr{I}$ of the conformal metric with arbitrary free data and the one with vanishing free data, and Theorem 2.3 which establishes that the n-th order coefficient of the FG expansion coincides (up to a certain constant) with the electric part of the rescaled Weyl tensor in the case when $\mathscr{I}$ is conformally flat and $n>3$ (for $n=3$ this is true in full generality). This theorem finds immediate application in the Cauchy problem of Einstein equations at $\mathscr{I}$ with positive cosmological constant (cf. Corollary 2.3.1). In section 3 we derive a KID equation for analytic data at $\mathscr{I}$ for n odd and even with vanishing obstruction tensor (we indicate that the result should also hold when the obstruction tensor is non-zero, but this requires additional analysis). This equation is necessary and sufficient for the Cauchy development of the data at $\mathscr{I}$ to admit a Killing vector field. Our final section 4 gives an interesting application of the previous results. Namely, we calculate the initial data of the Kerr-de Sitter metrics in all dimensions [16]. As mentioned above, these data are uniquely characterized by the conformal class of a CKVF ξ. In order to give a complete characterization, we identify this conformal class using the results in [23].

Throughout this paper, n refers to the dimension of $\mathscr{I}$ and $n+1$ the dimension of the spacetime. Several results below are valid in arbitrary signature and arbitrary sign of the cosmological costant. We will work in the general setup unless otherwise stated.

2 Initial data and the Weyl tensor

In this section we relate the initial data at spacelike conformally flat $\mathscr{I}$ which appears in Anderson's existence and uniqueness theorem [2] to the electric part of the rescaled Weyl tensor. This theorem relies on the Fefferman-Graham (FG) expansion of Poincaré metrics near $\mathscr{I}$. These are a generalization of the Poincaré metric of the disk model of the hyperbolic space, whose conformal infinity is given by the conformal structure associated to the usual n-sphere. The properties concerning Poincaré metrics that are needed in this paper will be stated next, and we refer to the original publication [7] and to the extended monograph [8] for further details.

Consider an $n+1$-dimensional pseudo-Riemannian manifold (M, g) with boundary ∂M and denote its interior by $\widetilde{M} = \mathrm{Int}(M)$. Then (M, g) is said to be a *conformal extension* of $(\widetilde{M}, \widetilde{g})$ if there exists a smooth function Ω positive on $\widetilde{M}$ such that

$$g = \Omega^2 \widetilde{g} \qquad \text{and} \qquad \partial M = \{\Omega = 0 \cap \mathrm{d}\Omega \neq 0\}.$$

The extended metric g is assumed to be smooth in $\widetilde{M}$ but only to have finite differentiability up to ∂M. When g is smooth also at the boundary, we will call this a *smooth conformal extension.* The submanifold $\mathscr{I} := (\partial M, g|_{\partial M})$, which will be non-degenerate of signature (p, q) in the cases we shall deal with, is called "conformal infinity". Notice that multiplying the conformal factor Ω by any smooth positive function $\widehat{\omega}$ yields a different conformal extension such that $\mathscr{I} = (\partial M, \omega^2 g|_{\partial M})$, where $\omega = \widehat{\omega}|_{\mathscr{I}}$. Hence one usually considers $\mathscr{I}$ as ∂M equipped with the whole conformal class of metrics $[g|_{\partial M}] = \omega^2 g|_{\partial M}$, $\forall \omega \in \mathcal{C}^\infty(\partial M)$, $\omega > 0$. A metric admitting a conformal extension is said to be conformally extendable. Also, throughout this paper by "conformally flat" we mean "locally conformally flat" unless otherwise stated.

The so-called Poincaré metrics associated to a conformal manifold, i.e. a smooth manifold endowed with a conformal structure $(\Sigma, [\gamma])$, are metrics $\widetilde{g}$ admitting a smooth conformal extension $g = \Omega^2 \widetilde{g}$ with prescribed conformal infinity $\mathscr{I} := (\Sigma, [\gamma])$. The extension g is defined in an open neighborhood M of $\Sigma \times \{0\}$ in $\Sigma \times [0, \infty)$ (and $\widetilde{g}$ in $\widetilde{M} = M - \{\Sigma \times \{0\}\}$) satisfying the following conditions. Σ is naturally embedded in M by $i : \Sigma \hookrightarrow \Sigma \times [0, \infty)$, where $i(\Sigma) = \Sigma \times \{0\}$, so we identify Σ and $\Sigma \times \{0\}$ when there is no risk of confusion. In M, Ω is a defining function of $\Sigma = \{\Omega = 0\}$ as before. Moreover, the definition imposes, depending on the parity of n, a certain decay rate of the tensor $Ric(\widetilde{g}) - \lambda n \widetilde{g}$ near $\mathscr{I}$, where $Ric(\widetilde{g})$ is the Ricci tensor of $\widetilde{g}$ and $\lambda := \frac{2}{n(n-1)} \Lambda$ with Λ the cosmological constant, which we assume to be non-zero. Following [7] we say that a symmetric 2-tensor field D is $O^+(\Omega^m)$ if it is $D = O(\Omega^m)$ and $\mathrm{Tr}_\gamma i^\star (\Omega^{-m} D|_{\mathscr{I}}) = 0$. With this notation we can give the formal definitions [7], [8]:

Definition 2.1. *A Poincaré metric for a conformal n-manifold $(\Sigma, [\gamma])$ of signature (p, q), is a metric $\widetilde{g}$ of signature $(p+1, q)$ if $\lambda > 0$ or $(p+1, q)$ if $\lambda < 0$ admitting a smooth conformal extension such that $\mathscr{I} = (\Sigma, [\gamma])$ and*

1. *If $n = 2$ or $n \geq 3$ and odd, $Ric(\widetilde{g}) - \lambda n \widetilde{g}$ vanishes to infinite order at Σ.*
2. *If $n \geq 4$ and even, $Ric(\widetilde{g}) - \lambda n \widetilde{g}$ is $O^+(\Omega^{n-2})$.*

Let $\widetilde{g}$ be a conformally extendable metric $\widetilde{g}$ and g a conformal extension. From the well-known relation between Ricci tensors two conformal metrics (e.g. [35])

$$R_{\alpha\beta} - \widetilde{R}_{\alpha\beta} = -\frac{n-1}{\Omega} \nabla_\alpha \nabla_\beta \Omega - g_{\alpha\beta} \frac{\nabla_\mu \nabla^\mu \Omega}{\Omega} + g_{\alpha\beta} \frac{n}{\Omega^2} \nabla_\mu \Omega \nabla^\mu \Omega. \tag{1}$$

it follows [17] that $\widetilde{g}$ satisfies the Einstein equation $Ric(\widetilde{g}) - \lambda n \widetilde{g} = O(\Omega^{-1})$ if and only if $(\Omega^{-2} \widetilde{g}^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega)|_{\mathscr{I}} = (g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega)\ |_{\mathscr{I}} = -\lambda$. It if this holds, it can be proven [25] that all sectional curvatures of $\widetilde{g}$ take a limit at ∂M equal to λ. In the Riemannian case (which requires $\lambda < 0$) these metrics are called *asymptotically hyperbolic*. In the case with general signature and arbitrary non-zero λ we call them *asymptotically of constant curvature* (ACC). It is obvious from Definition 2.1 that Poincaré metrics are ACC. We define then:

Definition 2.2. *Let $\widetilde{g}$ be an ACC and conformally extendable metric with $\mathscr{I} = (\Sigma, [\gamma])$. Let also be a representative $\gamma \in [\gamma]$. Then $\widetilde{g}$, as well as the conformally extended metric $g = \Omega^2 \widetilde{g}$, are said to be in normal form w.r.t. γ if*

$$\widetilde{g} = \frac{1}{\Omega^2} \left(-\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega\right), \qquad g = \Omega^2 \widetilde{g} = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega, \tag{2}$$

where g_Ω is a family of induced metrics on the leaves $\Sigma_\Omega = \{\Omega = const.\}$ such that $g_\Omega\ |_\Sigma = \gamma$.

Notice that it is always possible to adapt coordinates $\{\Omega, x^i\}$ to the normal form (2), where $\partial_\Omega := \partial / \partial \Omega$ and $\partial_i := \partial / \partial_{x^i}$ are normal and tangent to the Σ_Ω leaves respectively. We denote these as *Gaussian coordinates*[2]. For an ACC metric $\widetilde{g}$ which satisfies the Einstein equations to infinite order at $\mathscr{I}$, the Fefferman-Graham expansion (FG) is a formal expansion in Gaussian coordinates

$$g_\Omega \sim \sum_{r=0}^{(n-1)/2} g_{(2r)} \Omega^{2r} + \sum_{r=n}^{\infty} g_{(r)} \Omega^r, \qquad \text{if } n \text{ is odd}, \tag{3}$$

$$g_\Omega \sim \sum_{r=0}^{\infty} g_{(2r)} \Omega^{2r} + \sum_{r=n}^{\infty} \sum_{s=1}^{m_r} \mathcal{O}_{(r,s)} \Omega^r (\log \Omega)^s, \qquad \text{if } n \text{ is even}, \tag{4}$$

where $m_r \leq r - n + 1$ in an integer for each r, the coefficients $g_{(r)}$ are objects defined at $\mathscr{I}$ and extended to M as independent of Ω and the logarithmic terms arise in the n even case whenever the so-called obstruction tensor $\mathcal{O}_{(n,1)} = \mathcal{O}$ for $[\gamma]$ is non-zero. This tensor is a conformally invariant symmetric and trace-free 2-covariant in even dimensions and its vanishing is a necesssary and sufficent condition for the tensor g_Ω to be smooth up to and including $\mathscr{I}$. The main properties of this expansion follow from a theorem by Fefferman and Graham which characterizes metrics at $\mathscr{I}$.

[2]In the usual definition of Gaussian coordinates, the ∂_Ω vector is unit. The introduction of a constant factor λ does not modify its general properties.

Theorem 2.1 (Fefferman-Graham [8]). *Let (Σ, γ) be a pseudo-Riemannian manifold of signature (p, q) and let h be a symmetric 2-tensor with $\mathrm{Tr}_\gamma h = 0$.*

- *If $n = 2$ and if $\mathrm{div}_\gamma h = -\frac{1}{2}\mathrm{grad}_\gamma \mathrm{Scal}_\gamma$, with Scal_γ the Ricci scalar of γ, there exist an even (i.e. with only non-zero coefficients of even order) Poincaré metric g in normal form w.r.t γ which admits an expansion of the form (4) (with $\mathcal{O}_{(r,s)} = 0$) and the tracefree part of $g_{(2)}$ is $tf(g_{(2)}) = h$. These conditions uniquely determine the coefficients of the expansion.*

- *If $n \geq 3$ is odd and if $\mathrm{div}_\gamma h = 0$, there exist a Poincaré metric g in normal form w.r.t γ, which admits an expansion of the form (3) such that $g_{(n)} = tf(g_{(n)}) = h$. These conditions uniquely determine the coefficients of the expansion.*

- *If $n \geq 4$ is even, there exist a one-form $\mathfrak{b}$ determined by γ is such a way that if $\mathrm{div}_\gamma h = \mathfrak{b}$, then there exists a metric $g = -\mathrm{d}\Omega^2/\lambda + g_\Omega$, such that $Ric(g) - \lambda n g$ vanishes to infinite order and which admits an expansion of the form (4) with $tf(g_{(n)}) = h$. These conditions uniquely determine the coefficients of the expansion and the solution is smooth if and only if the obstruction tensor of γ vanishes.*

Notice that for $n \geq 4$, Theorem 2.1 does not mention Poincaré metrics, because $\widetilde{g}$ fails to be smooth at $\mathscr{I}$. Forcing the metric to satisfy the Einstein equations to infinite order at $\mathscr{I}$ with n even forces the appearance of the logarithmic terms whenever the obstruction tensor is non-zero. In order to distinguish this case from the Poincaré metrics, which are smooth by definition (see above), the metrics of Theorem 2.1 are called Feffeman-Graham-Poincaré (FGP) in this paper. Obviously, whenever n is odd or n is even and $\mathcal{O} = 0$, a FGP metric is Poincaré.

The proof of Theorem 2.1 relies on the fact that for both n odd and even cases, the Einstein equations at $\mathscr{I}$ yield recursive relations which allow to calculate all the coefficients in the expansions in terms of γ and $g_{(n)}$. In turns out that the recursive relations give each coefficient of order $r \neq n$, as a function of previous terms and tangential derivatives up to order $r - 2$. For this reason the expansions (3) and (4) are both even up to order n and the terms $g_{(r)}$ for $r < n$ are uniquely generated by γ. The n-th order coefficient is independent of γ, except for its trace and divergence, which depends on γ

$$\mathrm{Tr}_\gamma g_{(n)} = \mathfrak{a}, \qquad \mathrm{div}_\gamma g_{(n)} = \mathfrak{b},$$

where $\mathfrak{a} = 0$, $\mathfrak{b} = 0$ for n odd and $\mathfrak{a}$ is a scalar and $\mathfrak{b}$ a one-form determined by γ for n even. Also, $\mathcal{O}$ is obtainable from γ and in particular, it vanishes for conformally flat γ, a case which will be central in this paper.

Therefore, by prescribing certain data at Σ, we can determine a unique FGP metric g to infinite order at this hypersurface. Away from Σ there are multiple ways of extending g if no further assumptions are made. Anderson's existence and uniqueness theorem [2] proves that for metrics of Lorentzian signature and n odd, there is a unique way of extending this metric away from Σ so the $\Lambda > 0$ vacuum Einstein field equations hold, namely

$$\widetilde{R}_{\alpha\beta} = n\lambda \widetilde{g}_{\alpha\beta}.$$

In other words, the coefficients $(\gamma, g_{(n)})$, with (Σ, γ) Riemannian, determine a unique Einstein metric for (Σ, γ) in a collar neighborhood of Σ. The proof relies on imposing that the obstruction tensor of g vanishes. This can be done in the n odd case because g is $n + 1$ (even) dimensional. This method is not applicable to the n even case, for no obstruction tensor is associated to g when $n + 1$ is odd. The Cauchy problem has the following equivalence of data is

$$(\Sigma, \gamma, g_{(n)}) \simeq (\Sigma, \omega^2\gamma, \omega^{2-n} g_{(n)})$$

for every smooth positive function ω of Σ. This arises because of the multiple ways in which one can conformally extend an Einstein metric. Thus, strictly speaking, the data are $(\Sigma, [\gamma], [g_{(n)}])$, where $(\Sigma, [\gamma])$ is the manifold Σ endowed with a conformal structure and $[g_{(n)}]$ is a conformal class of TT tensors for $[\gamma]$.

Theorem 2.2 (Anderson [2]). *Let $n \geq 3$ odd. For every choice of asymptotic data $(\Sigma, [\gamma], [g_{(n)}])$, with γ Riemannian, there is a unique Lorentzian metric solving Einstein's equations for $\Lambda > 0$ in a neighborhood of $\mathscr{I}$. This problem is well-posed in suitable Sobolev spaces.*

In this paper, a conformally extended metric (i.e. defined at $\mathscr{I}$) will be denoted by g and the metric from which it is extended by $\widetilde{g}$ (i.e. not defined at $\mathscr{I}$). We use tilde to distiguish geometric objects associated to $\widetilde{g}$ from those associated to g. For instance, $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}$ and ∇ the one of g. The signature of the metric $\widetilde{g}$ and the sign of λ will remain general, namely $(p+1,q)$ if $\lambda > 0$ or $(q,p+1)$ if $\lambda < 0$ unless otherwise specified. However, we note that our main interest is in the positive cosmological constant setting, specially for the applications in the second part of the paper. Therefore, for the sake of simplicity some of our results are given in detail only for this case (namely Lemma 2.8, Proposition 2.1 and Theorem 2.3), while the negative λ will be just indicated.

If g is an ACC metric in normal form w.r.t. a representative $\gamma \in [\gamma]$, it is immediate to verify that the vector field $T := \nabla\Omega$ is geodesic (affinely parametrized). Any conformal extension such that T is geodesic is called a *geodesic conformal extension*. We start by proving some general results about this kind of extensions. In the following, Greek indices α, β running form zero to n are used for spacetime coordinates. For spacelike hypesurfaces (usually $\{\Omega = const.\}$) we use Latin indices $i = 1, \cdots, n$.

Lemma 2.1. *Let $\widetilde{g}$ be an ACC metric. Then, a conformal extension $g = \Omega^2\widetilde{g}$ is geodesic if and only if*

$$\nabla_\alpha\Omega\nabla^\alpha\Omega = -\lambda.$$

Proof. The lemma follows from

$$\nabla^\alpha\Omega\nabla^\beta\Omega\nabla_\alpha\nabla_\beta\Omega = \frac{1}{2}\nabla^\alpha\Omega\nabla_\alpha\left(\nabla_\beta\Omega\nabla^\beta\Omega\right), \tag{5}$$

because if $\nabla_\alpha\Omega\nabla^\alpha\Omega = -\lambda$ the RHS of (5) vanishes and T is geodesic and, converesely, if T is geodesic then (5) is zero along the integral lines of T and, g being ACC, its value on $\Omega = 0$ is $-\lambda$. □

The following result guarantees the existence of geodesic conformal extensions for each choice of boundary metric γ. The proof, which we detail next, is similar to the one for $\Lambda < 0$, which appears in [17] (Lemma 5.2). Before stating the lemma, let us briefly review the non-characteristic condition for a first order PDE Cauchy problem (see [6], Chapter 3). For this it will be convenient to use coordinates $\{x^\alpha\} = \{\Omega, x^i\}$ adapted to the initial hypersurface, that is $\Sigma = \{\Omega = 0\}$. Consider a first order PDE Cauchy problem

$$F(x^\alpha; u, \nabla_\alpha u) = 0, \qquad u\mid_\Sigma = \phi, \tag{6}$$

where u is a scalar function. Two functions $\{\phi, \psi_0\}$ of Σ are a set of *admissible* initial data whenever they satisfy the following *compatibility condition*

$$F(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial\phi}{\partial x^1}, \cdots, \frac{\partial\phi}{\partial x^n}) = 0. \tag{7}$$

Denote $\mathcal{D}_{\nabla_\alpha u}F$ to the derivative of F w.r.t. $\nabla_\alpha u$ and let $V(x^\alpha; u, \nabla_\alpha u)$ be the vector of components $V^\alpha = \mathcal{D}_{\nabla_\alpha u}F$. Also, let T be the normal covector to Σ, i.e. $T_\alpha = \nabla_\alpha\Omega$. Then, for every set of admissible initial data, the Cauchy problem is said to be non-characteristic if

$$T \cdot V(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial\phi}{\partial x^1}, \cdots, \frac{\partial\phi}{\partial x^n}) \neq 0,$$

where $\cdot$ denotes the usual action of a covector on a vector. A non-characteristic Cauchy problem is known to be locally well-posed (e.g. [6]), i.e. that there exists a solution u of (6), satisfying $u|_\Sigma = \phi$, $\partial_0 u|_\Sigma = \psi_0$. After this remark, we can prove the next Lemma.

Lemma 2.2. *Let $\widetilde{g}$ be an ACC metric with conformal infinity $(\Sigma, [\gamma])$. Then, for every representative $\gamma \in [\gamma]$, there exist a geodesic conformal extension $g = \Omega^2\widetilde{g}$ which induces the metric γ at Σ.*

Proof. Consider a conformally extended metric g such that $g = \Omega^2\widetilde{g}$ and $g\mid_{\Omega=0} = \gamma$. Let $\hat{g} \in [\widetilde{g}]$ be such that $\hat{g} = \omega^2 g$ with $\omega > 0$ and $\omega\mid_{\Omega=0} = 1$, so that $\hat{g}$ realizes the same boundary metric γ. Therefore $\hat{g} = \hat{\Omega}^2\widetilde{g}$, with $\hat{\Omega} = \omega\Omega$, so by Lemma 2.1, we have to show that there exists a function ω such that $\hat{\Omega}$ satisfies (5) for the metric $\hat{g}$

$$\hat{g}^{\alpha\beta}\nabla_\alpha\hat{\Omega}\nabla_\beta\hat{\Omega} = \frac{g^{\alpha\beta}}{\omega^2}\nabla_\alpha(\omega\Omega)\nabla_\beta(\omega\Omega) = -\lambda.$$

Expanding the derivatives and defining $u := \log\omega$, this amounts to

$$g^{\alpha\beta}\left(2\nabla_\alpha\Omega\nabla_\beta u + \Omega\nabla_\alpha u\nabla_\beta u\right) = \frac{-\lambda - g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega}{\Omega}. \tag{8}$$

The LHS of (8) is obviously regular at $\Omega = 0$. Also, since g is ACC $g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega \mid_{\Omega=0} = -\lambda$, thus the RHS tends to $-\partial_\Omega\left(g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\right)$ at $\Omega = 0$, which is regular. Hence, we can pose a Cauchy problem at $\{\Omega = 0\}$, for which we must complete $\phi = \log\omega \mid_\Sigma = 0$ to admissible initial data for (8). These data must satisfy (7), thus ψ_0 is fixed to satisfy $2g^{00}\psi_0 = -\partial_\Omega(g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega) \mid_\Sigma$. The vector field V has components $2g^{\alpha\beta}(\nabla_\beta\Omega + \Omega\nabla_\beta u)$ and therefore

$$T \cdot V(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial\phi}{\partial x^1}, \cdots, \frac{\partial\phi}{\partial x^n}) = 2g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega \mid_\Sigma = -2\lambda.$$

Hence the problem is non-characteristic if $\lambda \neq 0$. □

Remark 2.1. *This lemma combined with the use of Gaussian coordinates $\{\Omega, x^i\}$ implies that given an ACC metric $\widetilde{g}$ and any representative γ of its boundary conformal structure, there exists coordinates near ∂M where $\tilde{g}$ is written in normal form w.r.t that representative (see Definition 2.2).*

2.1 Formulas for the Weyl tensor

In this subsection we derive two useful formulas for the Weyl tensor and its electric part (cf. Lemmas 2.3 and 2.4). We start by listing well-known identities relating the geometry of any two metrics g and $\widetilde{g}$ (not necessarily conformal to each other for the moment). Our convention for the Riemann tensor is

$$R^\mu{}_{\alpha\nu\beta}X_\mu = -\nabla_\nu\nabla_\beta X_\alpha + \nabla_\beta\nabla_\nu X_\alpha.$$

The general formula that relates the two Riemann tensors $R^\mu{}_{\alpha\nu\beta}$ and $\widetilde{R}^\mu{}_{\alpha\nu\beta}$ of g and $\widetilde{g}$ respectively is (see e.g. [35]):

$$R^\mu{}_{\alpha\nu\beta} - \widetilde{R}^\mu{}_{\alpha\nu\beta} = 2\nabla_{[\nu}Q^\mu{}_{\beta]\alpha} + 2Q^\sigma{}_{[\nu|\alpha|}Q^\mu{}_{\beta]\sigma} \tag{9}$$

where $Q = \nabla - \widetilde{\nabla}$ is the difference of connections tensor

$$Q^\mu{}_{\alpha\beta} := \frac{1}{2}\widetilde{g}^{\mu\nu}(\nabla_\nu\widetilde{g}_{\alpha\beta} - \nabla_\alpha\widetilde{g}_{\beta\nu} - \nabla_\beta\widetilde{g}_{\alpha\nu}). \tag{10}$$

If they are conformally related $g = \Omega^2\widetilde{g}$, then Q reads

$$Q^\mu{}_{\alpha\beta} = \frac{1}{\Omega}\left(T_\alpha\delta^\mu{}_\beta + T_\beta\delta^\mu{}_\alpha - T^\mu g_{\alpha\beta}\right), \qquad T_\mu := \nabla_\mu\Omega, \quad T^\mu := g^{\mu\nu}T_\nu. \tag{11}$$

Now assume that g defines a geodesic conformal extension. Let us define the following contraction of the Riemann tensor with T

$$(R_T)_{\alpha\beta} := R_{\mu\alpha\nu\beta}T^\mu T^\nu$$

and denote

$$A_{\alpha\beta} := \nabla_\alpha T_\beta, \quad A^2_{\alpha\beta} := \nabla_\alpha T^\mu\nabla_\mu T_\beta.$$

Observe that A is symmetric. Since T is geodesic, we have

$$\begin{aligned}(R_T)_{\alpha\beta} &= T^\nu(-\nabla_\nu\nabla_\beta T_\alpha + \nabla_\beta\nabla_\nu T_\alpha)\\ &= -\nabla_T\nabla_\beta T_\alpha + \nabla_\beta\nabla_T T_\alpha - \nabla_\beta T^\nu\nabla_\nu T_\alpha = -\nabla_T A_{\alpha\beta} - A^2_{\alpha\beta}\end{aligned} \tag{12}$$

Using expressions (9) and (11) we calculate the difference of tensors R_T and $\widetilde{R}_T$ defined analogously for $\widetilde{g}$, specifically

$$(R_T)_{\alpha\beta} := \widetilde{R}_{\mu\alpha\nu\beta}T^\mu T^\nu.$$

First

$$\begin{aligned}2T_\mu T^\nu\nabla_{[\nu}Q^\mu{}_{\beta]\alpha} &= T_\mu T^\nu\nabla_\nu\Big(\frac{1}{\Omega}(T_\alpha\delta^\mu{}_\beta + T_\beta\delta^\mu{}_\alpha - g_{\alpha\beta}T^\mu)\Big) - T_\mu T^\nu\nabla_\beta\Big(\frac{1}{\Omega}(T_\nu\delta^\mu{}_\alpha + T_\alpha\delta^\mu{}_\nu - g_{\alpha\nu}T^\mu)\Big)\\
&= -\frac{1}{\Omega^2}T_\mu T^\nu T_\nu(T_\alpha\delta^\mu{}_\beta + T_\beta\delta^\mu{}_\alpha - g_{\alpha\beta}T^\mu) + \frac{1}{\Omega^2}T_\mu T^\nu T_\beta(T_\nu\delta^\mu{}_\alpha + T_\alpha\delta^\mu{}_\nu - g_{\alpha\nu}T^\mu) - \frac{T_\mu T^\mu}{\Omega}\nabla_\beta T_\alpha\\
&= \frac{\lambda}{\Omega^2}(2T_\alpha T_\beta + \lambda g_{\alpha\beta}) - \frac{\lambda}{\Omega^2}T_\alpha T_\beta + \frac{\lambda}{\Omega}\nabla_\beta T_\alpha = \frac{\lambda}{\Omega}\nabla_\alpha T_\beta + \frac{\lambda}{\Omega^2}\left(T_\alpha T_\beta + \lambda g_{\alpha\beta}\right)\end{aligned}$$

and second

$$2T_\mu T^\nu Q^\sigma{}_{[\nu|\alpha|}Q^\mu{}_{\beta]\sigma} = \frac{T_\mu T^\nu}{\Omega^2}(T_\nu\delta^\sigma{}_\alpha + T_\alpha\delta^\sigma{}_\nu - g_{\alpha\nu}T^\sigma)(T_\beta\delta^\mu{}_\sigma + T_\sigma\delta^\mu{}_\beta - g_{\beta\sigma}T^\mu)$$
$$- \frac{T_\mu T^\nu}{\Omega^2}(T_\beta\delta^\sigma{}_\alpha + T_\alpha\delta^\sigma{}_\beta - g_{\alpha\beta}T^\sigma)(T_\nu\delta^\mu{}_\sigma + T_\sigma\delta^\mu{}_\nu - g_{\nu\sigma}T^\mu)$$
$$= -\frac{\lambda}{\Omega^2}(2T_\alpha T_\beta + \lambda g_{\alpha\beta}) + \frac{\lambda}{\Omega^2}(2T_\alpha T_\beta + \lambda g_{\alpha\beta}) = 0.$$

Recalling $\nabla_\alpha T_\beta = A_{\alpha\beta}$, we have that (9) yields

$$(R_T)_{\alpha\beta} - \Omega^2(\widetilde{R}_T)_{\alpha\beta} = \frac{\lambda}{\Omega}A_{\alpha\beta} + \frac{\lambda}{\Omega^2}\left(T_\alpha T_\beta + \lambda g_{\alpha\beta}\right). \tag{13}$$

Assume now that $\widetilde{g}$ is an ACC metric in normal form w.r.t. to the boundary metric. Thus, from (2), in Gaussian coordinates

$$T^\alpha\partial_\alpha = -\lambda\partial_\Omega.$$

Also, A, which is, up to a constant factor, the second fundamental form of the leaves $\Sigma_\Omega = \{\Omega = const.\}$, is

$$A_{\alpha\beta} = \nabla_\alpha T_\beta = -\Gamma^0_{\alpha\beta} = \frac{g^{00}}{2}\partial_\Omega g_{\alpha\beta} = -\frac{\lambda}{2}\partial_\Omega g_{\alpha\beta} \tag{14}$$

and its covariant derivative w.r.t. T

$$\nabla_T A_{\alpha\beta} = -\lambda\partial_\Omega A_{\alpha\beta} + \lambda\left(\Gamma^\mu_{\alpha 0}A_{\mu\beta} + \Gamma^\mu_{\beta 0}A_{\alpha\mu}\right),$$

with

$$\Gamma^\mu_{\alpha 0} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{0\nu} + \partial_\Omega g_{\alpha\nu} - \partial_\nu g_{0\alpha}\right) = \frac{1}{2}g^{\mu\nu}\partial_\Omega g_{\alpha\nu} = -\frac{1}{\lambda}A^\mu{}_\alpha,$$

so in consequence

$$\nabla_T A = -\lambda\partial_\Omega A - 2A^2. \tag{15}$$

Inserting (12) and (15) in equation (13) yields

$$\Omega^2(\widetilde{R}_T)_{\alpha\beta} = \lambda\partial_\Omega A_{\alpha\beta} + A^2_{\alpha\beta} - \frac{\lambda}{\Omega}A_{\alpha\beta} - \frac{\lambda}{\Omega^2}\left(T_\alpha T_\beta + \lambda g_{\alpha\beta}\right). \tag{16}$$

If furthermore, we assume that $\widetilde{g}$ is Einstein with cosmological constant $\Lambda \neq 0$

$$\widetilde{R}_{\mu\alpha\nu\beta} = \widetilde{C}_{\mu\alpha\nu\beta} + 2\lambda\widetilde{g}_{\mu[\nu}\widetilde{g}_{\beta]\alpha},$$

we can relate $\widetilde{R}_T$ to the following components of the Weyl tensor, which we call *T-electric part*,

$$(C_T)_{\alpha\beta} := C_{\mu\alpha\nu\beta}T^\mu T^\nu = \Omega^2\widetilde{C}_{\mu\alpha\nu\beta}T^\mu T^\nu,$$

by

$$(\widetilde{R}_T)_{\alpha\beta} = \frac{(C_T)_{\alpha\beta}}{\Omega^2} - \lambda\left(\frac{\lambda g_{\alpha\beta} + T_\alpha T_\beta}{\Omega^4}\right). \tag{17}$$

Notice that since $T^\mu T_\mu = -\lambda$, then $C_T = |\lambda|C_{elec}$, where $(C_{elec})_{\alpha\beta} := C^\mu{}_{\alpha\nu\beta}u_\mu u^\nu$ is the electric part of the Weyl tensor, for u unit orthogonal to the leaves Σ_Ω. Combining (16) and (17) gives

$$(C_T)_{\alpha\beta} = \lambda\partial_\Omega A_{\alpha\beta} + A^2_{\alpha\beta} - \frac{\lambda}{\Omega}A_{\alpha\beta}$$

which putting $A_{\alpha\beta}$ in terms of the metric with (14) yields the following result:

Lemma 2.3. *Let $\widetilde{g}$ be a conformally extendable Einstein metric with $\Lambda \neq 0$ and $g = \Omega^2\widetilde{g}$ a geodesic conformal extension. Then, in Gaussian coordinates $\{\Omega, x^i\}$, the T-electric part of the Weyl tensor reads*

$$(C_T)_{ij} = \frac{\lambda^2}{2}\left(\frac{1}{2}\partial_\Omega g_{ik}g^{kl}\partial_\Omega g_{lj} + \frac{1}{\Omega}\partial_\Omega g_{ij} - \partial^2_\Omega g_{ij}\right). \tag{18}$$

where g_Ω is the metric induced by g on the leaves $\{\Omega = const.\}$.

Remark 2.2. *Note that equation* (18) *implies that C_T is always $O(\Omega)$. In particular, in dimension $n=3$ it is always the case that*

$$(\Omega^{-1}C_T)\mid_{\mathscr{I}} = -\frac{3\lambda^2}{2}g_{(3)}$$

which recovers the well-known result by Friedrich [12] that for positive Λ the electric part of the rescaled Weyl tensor corresponds to the free data specifiable at $\mathscr{I}$.

Remark 2.3. *Assume that $\hat{g}$ satisfies the hypothesis of Lemma 2.3 and that admits an expansion of the form*

$$\hat{g} = \sum_{r=0}^{(n-1)/2} g_{(2r)}\Omega^{2r} + \Omega^{n+1}h$$

with n odd and h at least C^2 up to an including $\{\Omega=0\}$. Equation (18) *implies that its T-electric Weyl tensor $\hat{C}_T$ only has even powers of Ω up to and including Ω^{n-1} (higher order terms may be even and odd). As a consequence, the tensor $\Omega^{2-n}\hat{C}_T$ splits as a sum of divergent terms at $\Omega=0$ plus a regular part which vanishes at $\Omega=0$.*

In the remainder of this section we derive a general expression relating the leading order terms of the Weyl tensor of a metric g, related to a given one $\hat{g}$ as in equation (19) below. This will prove to be useful in the conformally flat $\mathscr{I}$ setting. The result (cf. Lemma 2.4) is fully general and thus no requirement such as being FGP or Einstein is imposed to either g nor $\hat{g}$.

Let two $n+1$-dimensional metrics g and $\hat{g}$ be related by the formula

$$g = \hat{g} + \Omega^m q \tag{19}$$

for a natural number $m \geq 2$, where q is a symmetric tensor and all three tensors $g, \hat{g}$ and q are at least $\mathcal{C}^2$ in a neighborhood including $\{\Omega=0\}$. We assume that $\nabla\Omega$ is nowhere zero and has constant causal character at $\Omega=0$ so that we may define a function $F\neq 0$ and a vector $u\neq 0$ by $\nabla\Omega = Fu$, with u unit or null $g(u,u)=\epsilon$, $\epsilon=\pm 1, 0$. First notice that the inverse metrics g^{-1} and $\hat{g}^{-1}$ must be related by a similar formula

$$g^{-1} = \hat{g}^{-1} + \Omega^m l, \tag{20}$$

for a contravariant two-tensor l (also $\mathcal{C}^2$ near $\{\Omega=0\}$, as well as $g^{-1}, \hat{g}^{-1}$), because the presence of any term of order $\Omega^{m'}$, $m'<m$, would imply terms of order $\Omega^{m'}$ in $g^{-1}g$ which could not be cancelled. When using indices, we will omit the $^{-1}$ in the metrics and write upper indices. Also, indices in objects with hats are moved with the metric $\hat{g}$ and its inverse and indices of unhatted tensors are moved with g.

Recall the definition of the Weyl tensor

$$C^\mu{}_{\nu\alpha\beta} = R^\mu{}_{\nu\alpha\beta} + A^\mu{}_{\nu\alpha\beta} \qquad \text{with} \qquad A^\mu{}_{\nu\alpha\beta} := -\frac{2}{n-1}(\delta^\mu{}_{[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R^\mu{}_{|\beta)}) + \frac{2R}{n(n-1)}\delta^\mu{}_{[\alpha}g_{\beta]\nu}. \tag{21}$$

Using the relation of Riemann tensors (9) for g and $\hat{g}$ and (21) we find

$$C^\mu{}_{\nu\alpha\beta} = \hat{C}^\mu{}_{\nu\alpha\beta} + B^\mu{}_{\nu\alpha\beta} + A^\mu{}_{\nu\alpha\beta} - \hat{A}^\mu{}_{\nu\alpha\beta} \qquad \text{with} \qquad B^\mu{}_{\nu\alpha\beta} := 2\nabla_{[\alpha}Q^\mu{}_{\beta]\nu} + Q^\sigma{}_{[\alpha|\nu|}Q^\mu{}_{\beta]\sigma}$$

where Q is the difference of connections tensor (10). We also define $B_{\alpha\beta} = B^\mu{}_{\alpha\mu\beta}$ and $B = g^{\alpha\beta}B_{\alpha\beta}$ so that

$$R_{\alpha\beta} - \hat{R}_{\alpha\beta} = B_{\alpha\beta}, \qquad R^\mu{}_\beta - \hat{R}^\mu{}_\beta = B^\mu{}_\beta + \Omega^m l^{\mu\alpha}\hat{R}_{\alpha\beta}, \qquad R - \hat{R} = B + \Omega^m l^{\mu\beta}\hat{R}_{\mu\beta}.$$

With these definitions we expand $A^\mu{}_{\nu\alpha\beta}$

$$\begin{aligned} A^\mu{}_{\nu\alpha\beta} = &-\frac{2}{n-1}\left(\delta^\mu{}_{[\alpha}\hat{R}_{\beta]\nu} - \hat{g}_{\nu[\alpha}\hat{R}^\mu{}_{\beta]}\right) + \frac{2}{n(n-1)}\hat{R}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} \\ &-\frac{2}{n-1}\left(\delta^\mu{}_{[\alpha}B_{\beta]\nu} - \hat{g}_{\nu[\alpha}B^\mu{}_{\beta]} - \Omega^m\left(\hat{g}_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\mu\sigma} + q_{\nu[\alpha}\hat{R}^\mu{}_{\beta]} + q_{\nu[\alpha}B^\mu{}_{\beta]}\right) - \Omega^{2m}q_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\sigma\mu}\right) \\ &+\frac{2B}{n(n-1)}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + \frac{2\Omega^m}{n(n-1)}\left(l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + (\hat{R}+B)\delta^\mu{}_{[\alpha}q_{\beta]\nu}\right) + \frac{2\Omega^{2m}}{n(n-1)}l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^\mu{}_{[\alpha}q_{\beta]\nu}, \end{aligned}$$

so defining

$$D^\mu{}_{\nu\alpha\beta} := -\frac{2}{n-1}\left(\delta^\mu{}_{[\alpha}B_{\beta]\nu} - \hat{g}_{\nu[\alpha}B^\mu{}_{\beta]} - \Omega^m\left(\hat{g}_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\mu\sigma} + q_{\nu[\alpha}\hat{R}^\mu{}_{\beta]} + q_{\nu[\alpha}B^\mu{}_{\beta]}\right) - \Omega^{2m}q_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\sigma\mu}\right)$$

$$+\frac{2B}{n(n-1)}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + \frac{2\Omega^m}{n(n-1)}\left(l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + (\hat{R}+B)\delta^\mu{}_{[\alpha}q_{\beta]\nu}\right) + \frac{2\Omega^{2m}}{n(n-1)}l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^\mu{}_{[\alpha}q_{\beta]\nu}.$$

gives

$$A^\mu{}_{\nu\alpha\beta} = \hat{A}^\mu{}_{\nu\alpha\beta} + D^\mu{}_{\nu\alpha\beta},$$

from which

$$C^\mu{}_{\nu\alpha\beta} = \hat{C}^\mu{}_{\nu\alpha\beta} + B^\mu{}_{\nu\alpha\beta} + D^\mu{}_{\nu\alpha\beta}. \tag{22}$$

We now analyze the behaviour near $\{\Omega = 0\}$ of the tensors B and D. Using formula (10) (with $\widetilde{g} \to \hat{g} = g - \Omega^m q$) we have

$$\begin{aligned}
\hat{Q}_{\nu\alpha\beta} := \hat{g}_{\mu\nu}Q^\mu{}_{\alpha\beta} &= -F\frac{m}{2}\Omega^{m-1}\left(u_\nu q_{\alpha\beta} - u_\alpha q_{\beta\nu} - u_\beta q_{\alpha\nu}\right) - \frac{\Omega^m}{2}\left(\nabla_\nu q_{\alpha\beta} - \nabla_\alpha q_{\beta\nu} - \nabla_\beta q_{\alpha\nu}\right)\\
&= -F\frac{m}{2}\Omega^{m-1}\left(u_\nu q_{\alpha\beta} - u_\alpha q_{\beta\nu} - u_\beta q_{\alpha\nu}\right) + O(\Omega^m) = O(\Omega^{m-1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\nabla_\mu\hat{Q}_{\nu\alpha\beta} &= -F^2\frac{m(m-1)}{2}\Omega^{m-2}u_\mu\left(u_\nu q_{\alpha\beta} - u_\alpha q_{\beta\nu} - u_\beta q_{\alpha\nu}\right) - \frac{\Omega^m}{2}\nabla_\mu\left(\nabla_\nu q_{\alpha\beta} - \nabla_\alpha q_{\beta\nu} - \nabla_\beta q_{\alpha\nu}\right)\\
&\quad - m\frac{\Omega^{m-1}}{2}\left(\nabla_\mu\big(F\left(u_\nu q_{\alpha\beta} - u_\alpha q_{\beta\nu} - u_\beta q_{\alpha\nu}\right)\big) + Fu_\mu\left(\nabla_\nu q_{\alpha\beta} - \nabla_\alpha q_{\beta\nu} - \nabla_\beta q_{\alpha\nu}\right)\right)\\
&= -F^2\frac{m(m-1)}{2}\Omega^{m-2}u_\mu\left(u_\nu q_{\alpha\beta} - u_\alpha q_{\beta\nu} - u_\beta q_{\alpha\nu}\right) + O(\Omega^{m-1})
\end{aligned}$$

thus

$$\begin{aligned}
\nabla_\mu Q^\nu{}_{\alpha\beta} &= \nabla_\mu(\hat{g}^{\sigma\nu}\hat{Q}_{\sigma\alpha\beta}) = \nabla_\mu\big((g^{\sigma\nu} - \Omega^m l^{\sigma\nu})\hat{Q}_{\sigma\alpha\beta}\big) = g^{\sigma\nu}\nabla_\mu\hat{Q}_{\sigma\alpha\beta} + O(\Omega^{m-1})\\
&= -F^2\frac{m(m-1)}{2}\Omega^{m-2}u_\mu\left(u^\nu q_{\alpha\beta} - u_\alpha q^\nu{}_\beta - u_\beta q^\nu{}_\alpha\right) + O(\Omega^{m-1}).
\end{aligned}$$

Therefore, the leading order terms of B are

$$B^\mu{}_{\nu\alpha\beta} = 2\nabla_{[\alpha}Q^\mu{}_{\beta]\nu} + O(\Omega^{2m-2}) = -m(m-1)F^2\Omega^{m-2}\big(u^\mu u_{[\alpha}q_{\beta]\nu} + q^\mu{}_{[\alpha}u_{\beta]}u_\nu\big) + O(\Omega^{m-1}) = O(\Omega^{m-2}).$$

Next, we calculate the leading order terms of D. Notice that since $\hat{g}$ is $\mathcal{C}^2$ at $\{\Omega = 0\}$, its Ricci tensor is well-defined. Moreover B and all its traces are $O(\Omega^{m-2})$. Thus

$$D^\mu{}_{\nu\alpha\beta} = -\frac{2}{n-1}\big(\delta^\mu{}_{[\alpha}B_{\beta]\nu} - \hat{g}_{\nu[\alpha}B^\mu{}_{\beta]}\big) + \frac{2B}{n(n-1)}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + O(\Omega^m).$$

If u is non-null, i.e. $\epsilon \neq 0$, it is useful to decompose q in terms parallel and orthogonal to u, i.e.

$$q_{\alpha\beta} = Uu_\alpha u_\beta + 2u_{(\alpha}V_{\beta)} + t_{\alpha\beta}, \qquad \text{with} \quad u^\mu V_\mu = 0,\ u^\mu t_{\mu\nu} = 0,$$

which also entails a decomposition of the metrics

$$g_{\alpha\beta} = \epsilon u_\alpha u_\beta + h_{\alpha\beta} \tag{23}$$

which defines $h_{\alpha\beta}$ as the projector orthogonal to u. In terms of these

$$B^\mu{}_{\nu\alpha\beta} = -m(m-1)\Omega^{m-2}F^2\big(u^\mu u_{[\alpha}t_{\beta]\nu} + t^\mu{}_{[\alpha}u_{\beta]}u_\nu\big) + O(\Omega^{m-1}) \tag{24}$$

and

$$\begin{aligned}
B_{\beta\nu} = B^\mu{}_{\beta\mu\nu} &= -\frac{1}{2}m(m-1)\Omega^{m-2}F^2(\epsilon t_{\beta\nu} + tu_\beta u_\nu) + O(\Omega^{m-1})\\
B = B^\mu{}_\mu &= -\frac{1}{2}m(m-1)\Omega^{m-2}F^2(2\epsilon t) + O(\Omega^{m-1})
\end{aligned}$$

where $t = g^{\alpha\beta} t_{\alpha\beta} = h^{\alpha\beta} t_{\alpha\beta}$. In consequence,

$$D^{\mu}{}_{\nu\alpha\beta} = -m(m-1)\Omega^{m-2} F^2 \times \Big(\frac{-1}{n-1} \big(\epsilon \delta^{\mu}{}_{[\alpha} t_{\beta]\nu} + t \delta^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} - \epsilon \hat{g}_{\nu[\alpha} t^{\mu}{}_{\beta]} - t u^{\mu} \hat{g}_{\nu[\alpha} u_{\beta]} \big) \\ + \frac{2\epsilon t}{n(n-1)} \delta^{\mu}{}_{[\alpha} \hat{g}_{\beta]\nu} \Big) + O(\Omega^{m-1}) \tag{25}$$

From (23) one has

$$\hat{g}_{\alpha\beta} = \epsilon u_\alpha u_\beta + h_{\alpha\beta} + O(\Omega^m), \qquad \delta^{\alpha}{}_{\beta} = \epsilon u^{\alpha} u_{\beta} + h^{\alpha}{}_{\beta},$$

so that (25) reads

$$\begin{aligned} D^{\mu}{}_{\nu\alpha\beta} &= -m(m-1)\Omega^{m-2} F^2 \times \Big(\frac{-1}{n-1} \big(u^{\mu} u_{[\alpha} t_{\beta]\nu} + \epsilon h^{\mu}{}_{[\alpha} t_{\beta]\nu} + t h^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} + t^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} \\ &\quad + \epsilon t^{\mu}{}_{[\alpha} h_{\beta]\nu} + t u^{\mu} u_{[\alpha} h_{\beta]\nu} \big) + \frac{2t}{n(n-1)} \big(u^{\mu} u_{[\alpha} h_{\beta]\nu} + h^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} + \epsilon h^{\mu}{}_{[\alpha} h_{\beta]\nu} \big) \Big) + O(\Omega^{m-1}) \\ &= -m(m-1)\Omega^{m-2} F^2 \times \Big(\frac{-1}{n-1} \big(u^{\mu} u_{[\alpha} t_{\beta]\nu} + t^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} \big) - t \frac{n-2}{n(n-1)} \big(u^{\mu} u_{[\alpha} h_{\beta]\nu} + h^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu} \big) \\ &\quad - \frac{\epsilon}{n-1} \big(h^{\mu}{}_{[\alpha} t_{\beta]\nu} - \frac{t}{n} h^{\mu}{}_{[\alpha} h_{\beta]\nu} + t^{\mu}{}_{[\alpha} h_{\beta]\nu} - \frac{t}{n} h^{\mu}{}_{[\alpha} h_{\beta]\nu} \big) \Big) + O(\Omega^{m-1}). \end{aligned} \tag{26}$$

Denote the traceless part of $t_{\alpha\beta}$ by

$$\mathring{t}_{\alpha\beta} = t_{\alpha\beta} - \frac{t}{n} h_{\alpha\beta}.$$

Also, notice that the lower order terms of all expression are $O(\Omega^{m-1}) = o(\Omega^{m-2})$ for $m \geq 2$. Hence, combining (22), (24) and (26) gives the following result

Lemma 2.4. *Let $n \geq 3$ and g, $\hat{g}$ be $(n+1)$-dimensional metrics related by (19), for $m \geq 2$, with $g, \hat{g}, q$ and Ω at least $\mathcal{C}^2$ in a neighborhood of $\{\Omega = 0\}$. Assume that $\nabla\Omega$ is nowhere null at $\Omega = 0$. Then their Weyl tensors satisfy the following equation*

$$C^{\mu}{}_{\nu\alpha\beta} = \hat{C}^{\mu}{}_{\nu\alpha\beta} - K_m(\Omega) \frac{n-2}{n-1} (u^{\mu} u_{[\alpha} \mathring{t}_{\beta]\nu} + \mathring{t}^{\mu}{}_{[\alpha} u_{\beta]} u_{\nu}) + \frac{\epsilon K_m(\Omega)}{n-1} (h^{\mu}{}_{[\alpha} \mathring{t}_{\beta]\nu} + \mathring{t}^{\mu}{}_{[\alpha} h_{\beta]\nu}) + o(\Omega^{m-2}) \tag{27}$$

with

$$K_m(\Omega) = m(m-1)\Omega^{m-2} F^2,$$

and where $\nabla\Omega = Fu$, for $g(u,u) = \epsilon = \pm 1$, $h_{\alpha\beta}$ is the projector orthogonal to u, all indices are raised and lowered with g, $t_{\alpha\beta} = q_{\mu\nu} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta}$ while t and $\mathring{t}_{\alpha\beta}$ are its trace and traceless part respectively.

Remark 2.4. *Lemma 2.4 has an interesting application in the context of data at $\mathscr{I}$. Consider a FGP metric $\widetilde{g}$ and a geodesic conformal extension $g = \Omega^2 \widetilde{g}$ and assume that either n is odd or that the obstruction tensor is identically zero if n is even. The FG expansion of this metric allows one to decompose $g = \hat{g} + \Omega^n q$ where $\hat{g}$ is a metric containing all the terms of the expansion of order strictly lower than n (and possibly also higher order terms, but <u>not</u> the term at order n). The rest of terms are collected in $\Omega^n q$. By construction all these objects are $\mathcal{C}^\infty$ up to and including $\Omega = 0$ (here we use the assumption that the obstruction tensor vanishes in the even case). Hence all the hypothesis of Lemma 2.4 holds with $m = n$. From equation (27), the T-electric part of the Weyl tensors of g and of $\hat{g}$ are related by*

$$(C_T)_{ij} = (\hat{C}_T)_{ij} - \Omega^{n-2} \lambda^2 n(n-2) \mathring{t}_{ij} + o(\Omega^{n-2}), \tag{28}$$

It follows immediately from the FG expansion and the definition of $\mathring{t}$ in Lemma 2.4 that $\mathring{t}_{ij}|_{\Omega=0} = tf(g_{(n)})$ (taking the trace-free part is unnecessary when n is odd because $g_{(n)}$ is always trace-free in that case). The tensor $(\hat{C}_T)_{ij}$ is in general $O(1)$ in Ω, so $\Omega^{2-n}(\hat{C}_T)_{ij}$ will generically contain $[(n-1)/2]$ divergent terms, and the same divergent terms must appear in $\Omega^{2-n}(C_T)_{ij}$ because of (28). Substracting the divergence terms we get

$$\Big(\Omega^{2-n} (C_T)_{ij} - \Omega^{2-n} (\hat{C}_T)_{ij} \Big) |_{\mathscr{I}} = -\lambda^2 n(n-2) tf(g_{(n)}), \tag{29}$$

which provides a general formula for the free data in terms of the electric parts of the Weyl tensors of g and $\hat{g}$ at $\mathscr{I}$. In the case of n odd more can be said because $\hat{g}$ satisfies all the conditions of Remark

2.3. So the regular part of $(\hat{C}_T)_{ij}$ vanishes at $\mathscr{I}$ and (29) establishes that $g_{(n)}$ arises as the value of $(C_T)_{ij}$ at $\mathscr{I}$ once all its divergent terms have been substracted. This last statement is not true in the n even case with zero obstruction tensor, since $\Omega^{2-n}(\hat{C}_T)_{ij}$ may contain regular non-zero terms.

In the next section we will prove that in arbitrary dimension and for conformally flat $\mathscr{I}$, $(\hat{C}_T)_{ij}$ vanishes so the T-electric part of the rescaled Weyl tensor of g actually encodes the free-data $tf(g_n)$.

2.2 Electric part of the Weyl tensor in the FG expansion

The aim of this subsection is to determine the role that the electric part of the rescaled Weyl tensor plays in the FG expansion coefficients, with particular interest in the conformally flat $\mathscr{I}$ case. We will use formula (18) to relate the electric part of the rescaled Weyl tensor to the n-th order coefficient $g_{(n)}$ of the FG expansion. We start with some preliminary results about umbilical submanifolds (also called *totally umbilic*). Recall that a nowhere null submanifold $\Sigma \subset M$ is umbilical if its second fundamental form is

$$K_{ij} = f(x^k)\gamma_{ij}$$

for a smooth function f of Σ and γ the induced metric. This property is well-known to be invariant under conformal scalings of the ambient metric.

The following results are stated imposing the minimal conditions of differentiability required near $\mathscr{I}$. We remark than for the cases of our interest, namely FGP metrics, these conditions are always satisfied.

Lemma 2.5. *Let $n \geq 2$. Every nowhere null umbilical hypersurface (Σ, γ) of a conformally flat $n+1$-manifold $(M, \hat{g})$, where γ is induced by $\hat{g}$, is conformally flat.*

Proof. For $n = 2$ the result is immediate as every 2-surface is locally conformally flat, so let us assume $n \geq 3$. Since umbilical submanifolds remain umbilical w.r.t. to the whole conformal class of the metrics and $\hat{g}$ is conformally flat, then (Σ, γ) is umbilical w.r.t. the flat metric $g_E = \omega^2 \hat{g}$. In this gauge, the Gauss equation and its trace by γ yield

$$R(\gamma)_{ijkl} = -\epsilon(K_{il}K_{jk} - K_{ik}K_{jl}) = -\epsilon(\gamma_{il}\gamma_{jk} - \gamma_{ik}\gamma_{jl})\kappa^2,$$
$$R(\gamma)_{jl} = -\epsilon(K^2_{jl} - KK_{jl}) = -\epsilon(1-n)\kappa^2\gamma_{jl},$$

where $K_{ij} = \kappa\gamma_{ij}$ is the second fundamental form, for $\kappa \in \mathbb{R}$ constant as a consequence of the Codazzi equation and the fact that the ambient metric g_E is flat, and $K^2_{ij} := \gamma^{kl}K_{ik}K_{jl}$, $K := \gamma^{ij}K_{ij}$, $\epsilon = \hat{g}(u, u)$ with u the unit normal to Σ. The Schouten tensor of γ is

$$P(\gamma)_{ij} = \frac{1}{n-2}\left(R(\gamma)_{ij} - \frac{R(\gamma)}{2(n-1)}\gamma_{ij}\right) = \epsilon\frac{\kappa^2}{2}\gamma_{ij}.$$

Thus for $n = 3$ we can calculate the Cotton tensor

$$C(\gamma)_{ijk} = \nabla_k P(\gamma)_{ij} - \nabla_j P(\gamma)_{ik} = 0,$$

and for $n \geq 4$ the Weyl tensor is

$$W(\gamma)_{ijkl} = R(\gamma)_{ijkl} - \gamma_{ik}P(\gamma)_{jl} + \gamma_{jk}P(\gamma)_{il} + \gamma_{il}P(\gamma)_{jk} - \gamma_{jl}P(\gamma)_{ik} = 0.$$

By the standard characterization of locally conformally flat metrics by the vanishing of the Cotton ($n = 3$) or Weyl ($n \geq 4$) tensors, the result follows. □

Lemma 2.6. *Let g and $\hat{g}$ be metrics related by $g = \hat{g} + \Omega^m q$, where Ω is a defining function of $\Sigma = \{\Omega = 0\}$ and $g, \hat{g}$ and q are $\mathcal{C}^1$ in a neighborhood of Σ. Then if $m \geq 2$, Σ is umbilical w.r.t. g if and only if it is umbilical w.r.t. $\hat{g}$.*

Proof. The metrics induced by g and $\hat{g}$ at Σ are the same. Assume that Σ is nowhere null. Thus, the property of being umbilical is preserved if the covariant derivatives ∇u and $\hat{\nabla} u$ w.r.t. the Levi Civita connections of g and $\hat{g}$ respectively of the normal unit (which is the same for g and $\hat{g}$) covector $u \in (T\Sigma)^\perp$ coincide at Σ. The inverse metric g^{-1} is $g^{-1} = \hat{g}^{-1} + \Omega^m l$ for l a contravariant tensor $O(1)$ in Ω (cf. equation (20) and argument below). Then, the Christoffel symbols are

$$\Gamma^\mu_{\alpha\beta} = (\hat{g}^{-1} + \Omega^m l)^{\mu\nu}(\partial_\alpha(\hat{g} + \Omega^m q)_{\beta\nu} + \partial_\beta(\hat{g} + \Omega^m q)_{\alpha\nu} - \partial_\nu(\hat{g} + \Omega^m q)_{\alpha\beta}) = \hat{\Gamma}^\mu_{\alpha\beta} + O(\Omega^{m-1}),$$

from which it follows $\nabla u \mid_\Sigma = \hat{\nabla} u \mid_\Sigma$ if $m \geq 2$. □

Our interest in umbilical submanifolds is because of the (well-known) fact that $\mathscr{I}$ is umbilical for Poincaré or FGP metrics. This results follows immediately from the Einstein equations at $\mathscr{I}$, and will be the base for an interesting decompostion that we will derive later in this section (cf. Proposition 2.1).

Lemma 2.7. *Let $\widetilde{g}$ be a Poincaré or FGP metric for $\mathscr{I} = (\Sigma, [\gamma])$. Then $\mathscr{I}$ is umbilical.*

Proof. For a geodesic conformal extension $g = \Omega^2 \widetilde{g}$, the relation between the Ricci tensors of g and $\widetilde{g}$ is given by (1) with $\nabla_\mu \Omega \nabla^\mu \Omega = -\lambda$ (cf. Lemma 2.1). This expression is not defined at $\Omega = 0$, but it is when multiplied by Ω. Rearranging terms this gives

$$\Omega R_{\alpha\beta} + (n-1)\nabla_\alpha \nabla_\beta \Omega + g_{\alpha\beta} \nabla_\mu \nabla^\mu \Omega = \Omega(\widetilde{R}_{\alpha\beta} - \lambda n \widetilde{g}_{\alpha\beta}), \tag{30}$$

where we have used $g = \Omega^2 \widetilde{g}$ in the RHS. Since $\widetilde{g}$ is a Poincaré or FGP metric, the RHS vanishes at $\mathscr{I}$. This also implies that $g_{\alpha\beta}$ is at least $\mathcal{C}^2$ at $\mathscr{I}$, so $R_{\alpha\beta}$ is defined at $\mathscr{I}$. In addition writing $\nabla_\alpha \Omega = |\lambda|^{1/2} u_\alpha$, where u is the unit normal of the hypesurfaces $\Sigma_\Omega = \{\Omega = const.\}$, then $\nabla_i \nabla_j \Omega \mid_{\mathscr{I}} = |\lambda|^{1/2} K_{ij}$, where K_{ij} is the second fundamental form of $\mathscr{I}$. Thus, equation (30) gives at $\mathscr{I}$

$$(n-1)|\lambda|^{1/2} K_{ij} + f\gamma_{ij} = 0, \qquad \text{with} \qquad f := \nabla_\mu \nabla^\mu \Omega \mid_{\mathscr{I}} .$$

□

For concreteness, in the remainder of this Section, we state and prove our results in the case of positive cosmological constant and Lorentzian signature. However, they also hold with slight modifications for arbitrary signature and non-vanishing cosmological constant (see Remark 2.5 for the specific correspondence).

We start by giving the general form of the FG expansion of the de Sitter spacetime. We refer the reader to [34] for a similar proof in the case of $\lambda < 0$. Also, see a discussion of general case in [5] (in terms of Fefferman-Graham ambient metrics).

Lemma 2.8. *For every Riemmanian conformally flat boundary metric γ of dimension $n \geq 3$ and positive cosmological constant λ, the metric*

$$g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega \qquad g_{ij} = \mathcal{P}^k{}_i \mathcal{P}^l{}_k \gamma_{lj}, \qquad \textit{for} \quad \mathcal{P}^k{}_i := \delta^k{}_i + \frac{\Omega^2}{2} P_{il} \gamma^{lk} \tag{31}$$

with P the Schouten tensor of γ, is locally conformally isometric to de Sitter, i.e. $g = \Omega^2 \widetilde{g}_{dS}$, where $\widetilde{g}_{dS}$ is de Sitter.

Proof. De Sitter spacetime is ACC and its boundary metric γ is (by Lemmas 2.7 and 2.5) necessarily conformally flat. Moreover, given the freedom in scaling any conformal extension by an arbitrary positive function, any conformally flat metric is (locally) a boundary metric for the de Sitter space. In addition, as a consequence of this fact and Lemma 2.2 we have that for any conformally flat metric γ, there exists a local coordinate system of de Sitter near null infinity such that the metric is in normal form with respect to γ (see Remark 2.1). The core of the proof is to verify that this ACC metric in normal form w.r.t any such conformally flat γ takes the explicit form (31).

Therefore, consider a conformally flat boundary metric γ for a geodesic conformal extension of de Sitter g. Since de Sitter metric is also conformally flat, it follows that the T-electric part of the Weyl tensor $C_T = 0$. Using formula (18) we obtain the coefficients of the FG expansion, which give the normal form of g w.r.t. γ. Let us put (18) in matrix notation

$$C_T = \frac{\lambda^2}{2}\left(\frac{1}{2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \frac{1}{\Omega}\dot{g}_\Omega - \ddot{g}_\Omega\right) = 0 \quad \Longrightarrow \quad \ddot{g}_\Omega = \frac{1}{2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \frac{1}{\Omega}\dot{g}_\Omega \tag{32}$$

where a dot denotes derivative w.r.t. Ω. First we calculate

$$\partial_\Omega(\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega) = \ddot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega - \dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \dot{g}_\Omega g_\Omega^{-1}\ddot{g}_\Omega = \frac{2}{\Omega}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega \tag{33}$$

where we have used $\partial_\Omega(g_\Omega^{-1}) = -g_\Omega^{-1}\dot{g}_\Omega g_\Omega^{-1}$ for the first equality and expression of $\ddot{g}_\Omega$ in (32) for the second equality. Then, taking two derivatives in Ω of (32) gives

$$\partial_\Omega^{(4)} g_\Omega = \frac{3}{2\Omega^2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega. \tag{34}$$

Thus, one more derivative in Ω of (34) and combining with (33) gives $\partial_\Omega^{(5)} g_\Omega = 0$ and hence all higher derivates also vanish. Expression (34) evaluated at $\Omega = 0$ gives the expressions for the coefficients (note $\partial_\Omega^{(k)} g_\Omega \mid_{\Omega=0} = k! g_{(k)}$)

$$g_{(4)} = \frac{1}{4!}\frac{3}{2}(2g_{(2)})\gamma^{-1}(2g_{(2)}) = \frac{1}{4} g_{(2)}\gamma^{-1} g_{(2)}.$$

The coefficient $g_{(2)}$ can be directly calculated from the recursive relations for the FG expansion and it always coincides with the Schouten tensor of the boundary metric [1], namely

$$g_{(2)} = \frac{1}{n-2}\left(Ric(\gamma) - \frac{R(\gamma)}{2(n-1)}\gamma\right) = P$$

Having calculated the only non-zero coefficients $g_{(2)}$ and $g_{(4)}$, it is straightforward to verify that the FG expansion of de Sitter take the form (31).

We have shown that for any choice of conformally flat γ, there exists a de Sitter metric $\widetilde{g}_{dS}$ and a choice of conformal factor Ω with associated Gaussian coordinates such that, defining g as in (31), we have $\Omega^{-2} g = \widetilde{g}_{dS}$. Moreover, the metric (31) satisfies all the properties stated in Theorem 2.1 with the choice $h = 0$ (recall that we are assuming $n \geq 3$ and that the obstruction tensor vanishes identically when γ is conformally flat). The Lemma follows as a consequence of the uniqueness part of the FG expansion stated in Theorem 2.1. $\square$

Remark 2.5. *The result generalizes to arbitrary signature and arbitrary sign of λ by changing $g_{(2)} = \mathrm{sign}(\lambda)P$ (see [1]), γ to a conformally flat metric signature (p, q) and g to conformal to a metric of constant curvature (instead of conformal to de Sitter) and signature $(p+1, q)$ if $\lambda > 0$ or $(p, q+1)$ if $\lambda < 0$.*

Remark 2.6. *The proof of Lemma 2.8 shows that the condition $C_T = 0$ suffices to obtain a metric of the form* (31) *with γ in an arbitrary conformal class. The spacetimes satisfying this condition are the so-called "purely magnetic" and they have a long tradition in general relativity (e.g. [3] and references therein). The purely magnetic condition imply restrictive integrability conditions which lead to a conjecture [26] that no Einstein spacetimes exist in the $n = 3$ case, besides the spaces of constant curvature. Although no general proof has been found so far, the conjecture has been established in restricted cases such as Petrov type D, and this not only in dimension four, but in arbitrary dimensions [18]. The explicit form* (31) *that the metric must take whenever $C_T = 0$ gives an avenue to analyze the conjecture in the case of metrics admitting a conformal compactification. This is an interesting problem which, however, falls beyond the scope of the present paper.*

Before proving the main result of this section, namely Theorem 2.3, we state and prove an auxiliary result (Proposition 2.1) which is of independent interest since it provides (when combined with Lemma 2.4) a useful decomposition for calculating leading order terms of the Weyl tensor. This will be useful for the calculation of initial data of spacetimes which admit a smooth conformally flat $\mathscr{I}$ (cf. Corollary 2.3.1).

Proposition 2.1. *Assume $n \geq 3$. Let $\widetilde{g}$ be a FGP metric with λ positive for a Riemannian conformal manifold $\mathscr{I} = (\Sigma, [\gamma])$. Then $\mathscr{I}$ is locally conformally flat if and only if any geodesic conformal extension $g = \Omega^2 \widetilde{g}$, admits the following decomposition*

$$g = \hat{g} + \Omega^n q \tag{35}$$

where $\hat{g}$ is conformally isometric to de Sitter and $\hat{g}$, q and Ω are at least $\mathcal{C}^1$ in a neighborhood of $\{\Omega = 0\}$.

Proof. $\mathscr{I}$ is umbilical w.r.t. g and if g admits the decomposition (35), by Lemma 2.6 $\mathscr{I}$ is also umbilical w.r.t. $\hat{g}$. Since $\hat{g}$ is conformally flat, Lemma 2.5 implies that $\mathscr{I}$ is also conformally flat. This proves the proposition in one direction.

The converse follows by considering the FGP metric in normal form constructed from a representative γ in the conformal structure of $\mathscr{I}$. By assumption, γ is conformally flat. The terms up to order n are uniquely generated by γ (see Theorem 2.1 and discussion below). Thus, by Lemma 2.8

$$g = -\frac{\mathrm{d}\Omega^2}{\lambda} + \mathcal{P}^k{}_i \mathcal{P}^l{}_k \gamma_{lj} + \Omega^n q := \hat{g} + \Omega^n q,$$

where $\hat{g}$ is locally conformally isometric to de Sitter and $\hat{g}$, q and Ω are smooth at $\Omega = 0$ by construction. $\square$

Theorem 2.3. *Assume $n \geq 3$ and let $\widetilde{g}$ be a FGP metric with λ positive for a Riemannian conformal manifold $\mathscr{I} = (\Sigma, [\gamma])$. Then, if $\mathscr{I}$ is conformally flat, the traceless part of the n-th order coefficient of the FG expansion coincides, up to a constant, with the T-electric part of the rescaled Weyl tensor at $\mathscr{I}$ defined by*

$$D := \Omega^{2-n} C_T \mid_{\mathscr{I}} .$$

Proof. By Proposition 2.1, admitting a smooth conformally flat $\mathscr{I}$ amounts to admitting a decomposition of the form (35). Then, by Lemma 2.8, the associated FG expansion has the form

$$g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega = -\frac{\mathrm{d}\Omega^2}{\lambda} + \mathcal{P}^k{}_i \mathcal{P}^l{}_k \gamma_{lj} + \Omega^n g_{(n)} + \cdots = \hat{g} + \Omega^n q,$$

where $q \mid_{\mathscr{I}} = g_{(n)}$ and $\hat{g}$ is conformal to de Sitter. Using the formula (27) of Lemma 2.4 with $m = n$ and putting $T = |\lambda|^{1/2} u$, with u unit normal, one obtains

$$(C_T)_{\alpha\beta} = -\frac{\lambda^2}{2} n(n-2) \mathring{t}_{\alpha\beta} \Omega^{n-2} + o(\Omega^{n-2})$$

and the Theorem follows.

□

Remark 2.7. *It is also interesting to comment on the necessary and sufficient conditions for $g_{(n)}$ and $\Omega^{2-n} C_T \mid_{\mathscr{I}}$ to be the same in the case of Einstein metrics. Just like in the proof of Lemma 2.8, if C_T has a zero of order $m > 3$, we can apply formula* (18) *and find*

$$\partial_\Omega^{(5)} g_\Omega = O(\Omega^{m-3}) \tag{36}$$

and all coefficients of the FG expansion vanish up to order $g_{(m+2)}$. If, like in the conformally flat case, C_T has a zero of order $n-2$, its leading order term determines $g_{(n)}$. If n is odd, we can construct (cf. Theorem 2.2) two solutions of the $\Lambda > 0$ Einstein field equations $\hat{g}$ and g in a neighborhood of $\{\Omega = 0\}$, the first one corresponding to the data $(\mathscr{I}, [\gamma], 0)$ and the second to the data $(\mathscr{I}, [\gamma], [g_{(n)}])$ where $[\gamma]$ is an arbitrary conformal class. By the FG expansion we also have $g = \hat{g} + \Omega^n q$ with $q = g_{(n)} + O(\Omega)$. As a consequence of (36)*, the metric $\hat{g}$ is of the form* (31) *with γ in the given conformal class. Then, from equation* (18) *it follows that $\hat{g}$ is purely magnetic. The converse is also true, namely, if $g = \hat{g} + \Omega^n q$, with $\hat{g}$ a purely magnetic Einstein spacetime and both $\hat{g}$, q and Ω are $\mathcal{C}^2$ near $\{\Omega = 0\}$, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ and $g_{(n)}$ coincide (up to a constant) provided $n > 2$. The proof involves simply taking the T-electric part in* (27).

This proves that, for Einstein metrics with positive Λ, of dimension $n + 1 \geq 4$ and admitting a conformal compactification, $g_{(n)}$ and $C_T \mid_{\mathscr{I}}$ coincide up to a constant if and only if $g = \hat{g} + \Omega^n q$, where $\hat{g}$ is a purely magnetic spacetime Einstein with non-zero cosmological constant. However, as mentioned in Remark 2.6, it is not clear (and not an easy question) whether purely magnetic Einstein spacetimes are isometric to de Sitter or anti-de Sitter.

Note that Theorem 2.3 has been proven for metrics of all dimensions $n \geq 3$ and arbitrary signature. An interesting Corollary arises when applying this to the case of $\Lambda > 0$ Einstein metrics of Lorentzian signature and odd n, because the coefficients of the FG expansion γ and $g_{(n)}$ determine initial data at $\mathscr{I}$ which characterize the spacetime metric [2]. Therefore:

Corollary 2.3.1. *Let $n \geq 3$ be odd. Then for every set of the asymptotic data $(\Sigma, \gamma, g_{(n)})$ of Einstein's vacuum equations with $\Lambda > 0$ and γ conformally flat, $g_{(n)}$ is up to a constant, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ of the corresponding spacetime.*

3 KID for analytic metrics

In this section we prove a result that determines, in the analytic case, the necessary and sufficient conditions for initial data at $\mathscr{I}$ so that the corresponding spacetime metric it generates admits a local isometry. The proof relies in the FG expansion of FGP metrics. It is important to remark that analytic metrics correspond to analytic data [8].

This theorem is a generalization to higher dimensions (but restricted to the analytic case) of a known result [27] in dimension $n = 3$ establishing that a set (Σ, γ, D) of asymptotic data at spacelike $\mathscr{I}$, where

$D = g_{(3)}$ (cf. Remark 2.2) is a TT tensor, generates a spacetime admitting one Killing vector field if and only if $g_{(3)}$ satisfies the following Killing Initial Data (KID) equation for ξ a conformal Killing vector field (CKVF) of γ

$$\mathcal{L}_\xi g_{(3)} + \frac{1}{3}\mathrm{div}_\gamma(\xi) g_{(3)} = 0.$$

A posteriori ξ it is also the restriction to $\mathscr{I}$ of the Killing vector of the spacetime whose existence is guaranteed by the theorem.

Before stating and proving the theorem it is necessary to comment on the convergence of the FG expansion of a FGP metric in the case when the data $(\gamma, g_{(n)})$ at $\mathscr{I}$ are analytic. When n is odd, so that no obstruction tensor nor logarithmic terms arise, the series was shown to converge (in some neighbourhood of $\Omega = 0$) already [7] in full generality, i.e. with no restrictions on the signature of γ nor the sign of λ. In the case of n even, the convergence has been studied under the assumption that γ is positive definite, still keeping an arbitrary sign for λ (the two possible signs are actually dual to each other [1]). In such case the convergence of the FG expansion has been established in [20] irrespectively of whether the obstruction tensor is zero on not (i.e. irrespectively of whether the expansion is a power expansion or includes also logarithmic terms). Thus, for any parity of n and analytic data $(\gamma, g_{(n)})$ with γ positive definite, the formal solution converges and by Theorem 2.2, is the unique formal solution asymptotically solving the Einstein equations to infinite order. Actually, the convergence guarantees that the equations are in fact solved in a sufficiently small neighborhood of $\Omega = 0$, not just to infinite order at $\mathscr{I}$. Analytic data will be called asymptotic data in the analytic class and such data always defines a solution of the Einstein vacuum field equations in a neihgbourhood of $\mathscr{I}$ (irrespectively of the parity of n as long as γ is positive definite). For simplicity (and because this is what we shall need later) we restrict ourselves to γ with vanishing obstruction tensor whenever n is even. At the end of this section, we make a comment concerning the case with non-zero obstruction tensor.

With the above remark, the statement of the Theorem is as follows:

Theorem 3.1. *Let Σ be n dimensional with $n \geq 3$ and let $(\Sigma, \gamma, g_{(n)})$ be asymptotic data in the analytic class, with γ Riemannian and if n even $\mathcal{O} = 0$. Then, the corresponding spacetime admits a Killing vector field if and only if there exist a CKVF ξ of $\mathscr{I}$ satisfying the following Killing initial data (KID) equation*

$$\mathcal{L}_\xi g_{(n)} + \frac{n-2}{n}\mathrm{div}_\gamma(\xi) g_{(n)} = 0. \tag{37}$$

Proof. The proof that (37) is necessary is obtained by direct calculation as follows. Let X be a Killing vector field of $\widetilde{g}$ so that

$$0 = \mathcal{L}_X \widetilde{g} = \mathcal{L}_X(\Omega^{-2} g) = -2\frac{X(\Omega)}{\Omega^3} g + \frac{1}{\Omega^2}\mathcal{L}_X g.$$

It follows that on Int(M), X is a conformal Killing vector of g with a specific right-hand side, namely

$$\mathcal{L}_X g_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 2\frac{\mathrm{div}_g X}{n+1} g_{\alpha\beta}, \qquad X(\Omega) = \frac{\Omega}{n+1}\mathrm{div}_g X. \tag{38}$$

The following argument [12] shows that X must be extendable to $\mathscr{I}$. The terms $\mathcal{L}_X g_{0\beta}$ of (38) imply a linear, homogeneous symmetric hyperbolic system of propagation equations for X. Thus, putting initial data corresponding to X sufficiently close to $\mathscr{I}$ generates a solution whose domain of dependence must reach $\mathscr{I}$ (and possibly beyond if the manifold is extendable across $\mathscr{I}$). Hence X must admit a smooth extension on $\mathscr{I}$, which vanishes near $\mathscr{I}$ only if $X \mid_{\mathscr{I}} = 0$. The rest of equations $\mathcal{L}_X g_{ij}$ are also satisfied at $\mathscr{I}$ by continuity so the extension is a CKVF.

Then, from the second of equations (38), it follows that $X(\Omega) = 0$ when $\Omega = 0$, thus X is tangent to $\mathscr{I}$, so we denote $\xi := X \mid_{\mathscr{I}}$. Putting g in normal form $g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega$ it easily follows that $\Gamma^\alpha_{\alpha j} = \Gamma^i_{ij}$. In consequence, expanding $\mathrm{div}_g X$ and evaluating at $\mathscr{I}$ yields

$$\begin{aligned} \mathrm{div}_g X \mid_{\mathscr{I}} &= \partial_\Omega(X(\Omega)) \mid_{\mathscr{I}} + \partial_j \xi^j + \Gamma^i_{ij} \mid_{\mathscr{I}} \xi^j \\ &= \frac{1}{n+1}\mathrm{div}_g X \mid_{\mathscr{I}} + \mathrm{div}_\gamma \xi \quad \Longrightarrow \quad \mathrm{div}_g X \mid_{\mathscr{I}} = \frac{n+1}{n}\mathrm{div}_\gamma \xi \end{aligned} \tag{39}$$

where we have used the second equation in (38). In addition, the normal form gives the following tangent components of the first equation in (38):

$$\mathcal{L}_X g_\Omega = \frac{2}{n+1}\mathrm{div}_g X g_\Omega.$$

Evaluating this expression at $\mathscr{I}$ and taking into account (39) shows that ξ is a CKVF of γ. Also, using the FG expansion of g_Ω we have the following expansion of $\mathcal{L}_X g_\Omega$:

$$\begin{aligned}\mathcal{L}_X g_\Omega &= X(\Omega)\partial_\Omega g_\Omega + \mathcal{L}_X\gamma + \Omega^2\mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots \\ &= \frac{\Omega}{n+1}(\mathrm{div}_g X)\partial_\Omega g_\Omega + \mathcal{L}_X\gamma + \Omega^2\mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots . \end{aligned} \tag{40}$$

Therefore

$$\mathcal{L}_X\gamma + \Omega^2\mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots = \frac{1}{n+1}(\mathrm{div}_g X)(2g_\Omega - \Omega\partial_\Omega g_\Omega). \tag{41}$$

Equating n-th order terms and evaluating at $\mathscr{I}$ yields (37) after substituting $\mathrm{div}_g X \mid_{\mathscr{I}}$ as in (39).

To prove sufficiency, let us first choose the conformal gauge where ξ is a Killing vector field of $\gamma' = \omega^2\gamma$. Thus, the corresponding KID equation for $g'_{(n)}$ becomes:

$$\mathcal{L}_\xi g'_{(n)} = 0. \tag{42}$$

The remainder of the proof in this gauge, so we drop all the primes. By Lemma 2.2 there exist a geodesic extension which recovers the representative γ at $\mathscr{I}$. In addition, there exists a unique vector field X, extended from ξ at $\mathscr{I}$, which satisfies $[T, X] = 0$. This is obvious in geodesic gaussian coordinates $\{\Omega, x^i\}$, because

$$[T, X]^\alpha = -\lambda\partial_\Omega X^\alpha = 0,$$

with initial conditions $X^\Omega \mid_{\Omega=0} = 0$ and $X^i \mid_{\Omega=0} = \xi^i$ has a unique solution $X^\Omega = 0$ and $X^i = \xi^i$. We now prove that X is a Killing vector field of the physical metric $\widetilde{g}$ provided that (42) holds.

Consider the normal form metric $g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega$. Since $\mathcal{L}_X \mathrm{d}\Omega = \mathrm{d}(X(\Omega)) = 0$, it follows that $\mathcal{L}_X g = \mathcal{L}_X(g_\Omega)$. Using the FG expansion of g_Ω we have

$$\mathcal{L}_X g_\Omega = \mathcal{L}_X\gamma + \Omega^2\mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots .$$

If g is analytic, the value of the coefficients $\mathcal{L}_X g_{(r)}$ determine $\mathcal{L}_X g$ in a neighborhood of $\mathscr{I}$. These are

$$\partial_\Omega^{(r)}\left(\mathcal{L}_X g_\Omega\right)\mid_{\Omega=0} = \mathcal{L}_\xi\left(\partial_\Omega^{(r)} g_\Omega \mid_{\Omega=0}\right) = r!\mathcal{L}_\xi g_{(r)}.$$

We want to show that all these quantities are identically zero, for which we exploit the Feffeman-Graham recursive construction. The fundamental equation that determines recursively the coefficients of the FG expansion takes the form [1]

$$-\Omega\ddot{g}_\Omega + (n-1)\dot{g}_\Omega - 2Hg_\Omega = \Omega L, \qquad L := \frac{2}{\lambda}Ric(g_\Omega) - H\dot{g}_\Omega - (\dot{g}_\Omega)^2 - \frac{2}{\lambda}G^{\parallel}. \tag{43}$$

where $\widetilde{G}_\parallel$ are the tangent components of the tensor $\widetilde{G} = \widetilde{R}ic(\widetilde{g}) - \lambda n\widetilde{g}$ and $\lambda H = \mathrm{Tr}_{g_\Omega} A$. A direct calculation shows that taking the r-th order derivative of equation (43) at $\Omega = 0$ and separating the terms containing highest $(r+1)$ order coefficients from the rest gives an expression of the form:

$$(n-r-1)g_{(r+1)} + \left(\mathrm{Tr}_\gamma g_{(r+1)}\right)\gamma = \mathcal{F}_{(r-1)} \tag{44}$$

where $\mathcal{F}_{(r-1)}$ is a sum of terms containing products of coefficients up to order $r-1$ and tangential derivatives thereof, up to second order. Notice that $\widetilde{G}_\parallel$ vanishes to all orders at $\mathscr{I}$, so it does not contribute to these equations. We now prove by induction that the Lie derivative of all cofficients vanish provided equation (42) is satisfied.

First, the Lie derivative of (43), given that ξ is a Killing of γ, yields

$$(n-r-1)\mathcal{L}_\xi g_{(r+1)} + \left(\mathrm{Tr}_\gamma \mathcal{L}_\xi g_{(r+1)}\right)\gamma = \mathcal{L}_\xi\mathcal{F}_{(r-1)}$$

Assume by hypothesis that the Lie derivative $\mathcal{L}_\xi$ of all the coefficients up to a certain order r is zero (for the moment we do not assume neither $r < n$ nor $r > n$). The Lie derivative $\mathcal{L}_\xi\mathcal{F}^{(r-1)}$ is a sum where each terms is multiplied by either $\mathcal{L}_\xi g_{(s)}$, $\mathcal{L}_\xi\partial_i g_{(s)}$ or $\mathcal{L}_\xi\partial_i\partial_j g_{(s)}$, with $s \leq r-1$. Since ξ commutes with $T = \partial_\Omega$, we can adapt coordinates to both vector fields, namely $\xi = \partial_j$, so that in these coordinates $\mathcal{L}_\xi\partial_i g_{(s)} = \partial_i\mathcal{L}_\xi g_{(s)}$ and $\mathcal{L}_\xi\partial_i\partial_j g_{(s)} = \partial_i\partial_j\mathcal{L}_\xi g_{(s)}$. Thus each term in $\mathcal{L}_\xi\mathcal{F}^{(r-1)}$ contains a Lie derivative $\mathcal{L}_\xi g_{(s)}$ with $s < r-1$, or a tangential derivative thereof up to second order. Thus by the induction hypothesis $\mathcal{L}_\xi\mathcal{F}^{(r-1)} = 0$. Therefore, it follows that $\mathcal{L}_\xi g_{(r+1)} = 0$

The induction hypothesis can be assumed for $r < n-1$ because it is true for the first term $\mathcal{L}_\xi \gamma = 0$ and we have equations for the succesive terms. For $r = n-1$ the fundamental equation does not determine the term $g_{(n)}$ any longer (this is the reason why this terms is free-data in the FG expansion), so the induction hypothesis cannot go further in principle. But since we are imposing the condition $\mathcal{L}_\xi g_{(n)} = 0$, the induction hypothesis can be extended to any value of r. Therefore, all the derivatives $\mathcal{L}_\xi g_{(r+1)}$ vanish so if g is analytic $\mathcal{L}_\xi g = 0$. □

In short, the argument behind the proof of Theorem 3.1 relies on the well-known fact that the recursive relations that determine the coefficients of the FG expansion can be cast in a covariant form, so that ultimately all terms can be expressed in terms of γ, its curvature tensor, $g_{(n)}$ and covariant derivatives thereof. Then the Lie derivative of any coefficient must be zero provided that $\mathcal{L}_\xi \gamma = \mathcal{L}_\xi g_{(n)} = 0$. The case with non-zero obstruction tensor, and hence involving logarithmic terms is likely to admit an analogous proof. However, the recursive equations equivalent to (44) are not so explicit, because taking derivatives of order higher than n yields an expression which mixes up coefficients of the regular part $g_{(r)}$ and logarithmic terms $\mathcal{O}_{(r,s)}$ of the expansion. These expressions are notably more involved (see e.g. [31]). If one showed that every coefficient $\mathcal{O}_{(r,s)}$ admits a covariant form which only involves geometric objects constructed from γ, $g_{(n)}$ and its covariant derivatives, a similar argument as in the proof above would establish that equation (37) is also sufficient for the spacetime to admit a Killing vector field in the case of analytic data with non-vanishing $\mathcal{O}$. It is hard to imagine that this is not the case, and in fact the result should follow from the expressions in [31], but the details need to be worked out. On the other hand, the necessity of (37) is true in general and the argument is totally analogous to the one presented above except that equations (40) and (41) contain also logarithmic terms. We will not discuss this case any further since for the rest of paper we shall focus on conformally flat $\mathscr{I}$ (hence $\mathcal{O} = 0$). We plan to come back to this open issue in a future work.

4 Characterization of generalized Kerr-de Sitter metrics

In this section, we will apply the results obtained in the previous sections to find a characterization of the generalized Kerr-de Sitter metrics [16]. We first prove that these metrics admit a smooth conformally flat $\mathscr{I}$. Then we combine with Theorem 2.3 to determine their initial data at $\mathscr{I}$, which is straightforwardly computable from equation (27). The data corresponding to Kerr-de Sitter in all dimensions are analytic. Therefore, as noted at the beginning of section 3, the identification of their data provide a characterization of the metric also in the case of n even. Hence, we perform the analysis simultaneously for n even and odd.

Like in the four dimensional case, the generalized Kerr-de Sitter metrics are $n+1$-dimensional Kerr-Schild type metrics. Namely, they admit the following form

$$\widetilde{g} = \widetilde{g}_{dS} + \widetilde{\mathcal{H}}\, \widetilde{k} \otimes \widetilde{k}$$

with $\widetilde{g}_{dS}$ the de Sitter metric, k is a null (w.r.t. to both $\widetilde{g}$ and $\widetilde{g}_{dS}$) field of 1-forms and $\widetilde{\mathcal{H}}$ is a smooth function. In order to unify the n odd and n even cases in one single expression, we define the following parameters

$$p := \left[\frac{n+1}{2}\right] - 1, \qquad q := \left[\frac{n}{2}\right],$$

where note, $p = q$ if n odd and $p + 1 = q$ if n even. The explicit expression of the Kerr-de Sitter metrics will be given using the so-called "spheroidal coordinates" $\{r, \alpha_i\}_{i=1}^{p+1}$ (see [16] for their detailed construction), with the redefinition $\rho := r^{-1}$. Strictly speaking, they do not quite define a coordinate system because the α_i functions are constrainted to satisfy

$$\sum_{i=1}^{p+1} \alpha_i^2 = 1.$$

However, it is safe to abuse the language and still call $\{\alpha_i\}$ coordinates. To complete $\{\rho, \alpha_i\}$ to full spacetime coordinates we include $\{\rho, t, \{\alpha_i\}_{i=1}^{p+1}, \{\phi_i\}_{i=1}^{q}\}$. The α_is and ϕ_is are related to polar and azimuthal angles of the sphere respectively and they take values in $0 \leq \alpha_i \leq 1$ and $0 \leq \phi_i < 2\pi$ for $i = 1, \cdots, q$ and (only when n odd) $-1 \leq \alpha_{p+1} \leq 1$. Associated to each ϕ_i there is one rotation parameter $a_i \in \mathbb{R}$. For notational reasons, it is useful to define a trivial parameter $a_{p+1} = 0$ in the case of n odd. The remaining ρ and t lie in $0 \leq \rho < \lambda^{1/2}$ and $t \in \mathbb{R}$. The domain of definition of ρ can be extended (across the Killing horizon) to $\rho > \lambda^{1/2}$, but this is unnecessary in this work since we are interested in regions near $\rho = 0$.

In addition, as we will work with the conformally extended metric $g = \rho^2 \widetilde{g}$, we directly write down the expresions of the following quantities, which admit a smooth extension to $\rho = 0$,

$$\hat{g} = \rho^2 \widetilde{g}_{dS}, \qquad \mathcal{H} = \rho^2 \widetilde{\mathcal{H}}, \qquad k_\alpha = \widetilde{k}_\alpha \tag{45}$$

and

$$g = \hat{g} + \mathcal{H}\ k \otimes k. \tag{46}$$

Notice that we specify the definition of k_α because the metrically associated vector field $k^\alpha = g^{\alpha\beta} k_\beta$ is no longer the same as $\widetilde{k}^\alpha = \widetilde{g}^{\alpha\beta}\widetilde{k}_\beta$. In order for the reader to compare with the original publication [16], we remark that the expressions given there are for the "physical" objects $\widetilde{g}_{dS}$, $\widetilde{\mathcal{H}}, \widetilde{k}$, using the coordinates $r := \rho^{-1}$ and denoting $\mu_i := \alpha_i$ instead.

Let us now introduce the functions

$$W := \sum_{i=1}^{p+1} \frac{\alpha_i^2}{1 + \lambda a_i^2} \qquad \Xi := \sum_{i=1}^{p+1} \frac{\alpha_i^2}{1 + \rho^2 a_i^2}, \tag{47}$$

Note that it is thanks to having introduced the spurious quantity $a_{p+1} \equiv 0$ that these expressions take a unified form in the n odd and n even cases. The explicit form of the objects in (45) in the case of generalized Kerr-de Sitter are

$$\begin{aligned}
\hat{g} &= -W(\rho^2 - \lambda)\mathrm{d}t^2 + \frac{\Xi}{\rho^2 - \lambda}\mathrm{d}\rho^2 + \delta_{p,q}\mathrm{d}\alpha_{p+1}^2 + \sum_{i=1}^{q} \frac{1 + \rho^2 a_i^2}{1 + \lambda a_i^2}\left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right) \\
&\quad + \frac{\lambda}{W(\rho^2 - \lambda)}\left(\sum_{i=1}^{p+1} \frac{\left(1 + \rho^2 a_i^2\right)\alpha_i \mathrm{d}\alpha_i}{1 + \lambda a_i^2}\right)^2, \\
k &= W\mathrm{d}t - \frac{\Xi}{\rho^2 - \lambda}\mathrm{d}\rho - \sum_{i=1}^{q} \frac{a_i \alpha_i^2}{1 + \lambda a_i^2}\mathrm{d}\phi_i, \\
\Pi &= \prod_{j=1}^{q}(1 + \rho^2 a_j^2), \quad \mathcal{H} = \frac{2M\rho^n}{\Pi\Xi}, \quad M \in \mathbb{R}.
\end{aligned}$$

The term $\delta_{p,q}$ only appears when $q = p$, i.e. when n is odd. In the case of even n, all terms multiplying $\delta_{p,q}$ simply go away.

The function $\mathcal{H} = O(\rho^n)$ and $k \otimes k = O(1)$. Therefore g decomposes as

$$g = \hat{g} + \rho^n q, \qquad \text{with} \qquad q = \frac{\mathcal{H}}{\rho^n} k \otimes k = O(1).$$

Let γ be the metric induced at $\Sigma = \{\rho = 0\}$ by g. By Lemma 2.2, we can define geodesic conformal factor Ω such that $\{\Omega = 0\} = \Sigma$ and which induces the same metric γ at Σ. Hence $\Omega = O(\rho)$ and therefore $\mathcal{H} = O(\Omega^n)$ and $q = O(1)$ (in Ω). So by Proposition 2.1 it follows that the generalized Kerr-de Sitter metrics in all dimensions admit a conformally flat $\mathscr{I}$. This can be also verified by explicit calculation. From (49), the induced metric at $\{\rho = 0\}$ has the following expression

$$\gamma = \lambda W \mathrm{d}t^2 + \delta_{p,q}\mathrm{d}\alpha_{p+1}^2 + \sum_{i=1}^{q} \frac{\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2}{1 + \lambda a_i^2} - \frac{1}{W}\left(\sum_{i=1}^{p+1} \frac{\alpha_i \mathrm{d}\alpha_i}{1 + \lambda a_i^2}\right)^2. \tag{52}$$

It is useful to define new coordinates

$$\hat{\alpha}_i^2 := \frac{1}{W}\frac{\alpha_i^2}{1 + \lambda a_i^2},$$

which from (47) are restricted to satisfy $\sum_{i=1}^{p+1} \hat{\alpha}_i^2 = 1$. Since also $\sum_{i=1}^{p+1} \alpha_i^2 = 1$, this allows us to express W (given in (47)) in terms of the hatted coordinates

$$W = \frac{1}{\sum_{i=1}^{p+1} 1 + \lambda \hat{\alpha}_i^2 a_i^2}. \tag{53}$$

A direct calculation shows that the metric (52) expressed with $\hat{\alpha}_i$s takes the form

$$\gamma = W\Big(\lambda \mathrm{d}t^2 + \delta_{p,q}\mathrm{d}\hat{\alpha}_{p+1}^2 + \sum_{i=1}^{q}\big(\mathrm{d}\hat{\alpha}_i^2 + \hat{\alpha}_i^2\mathrm{d}\hat{\phi}_i^2\big)\Big)\ |_{\sum_{i=1}^{p+1}\hat{\alpha}_i^2=1}\ . \tag{54}$$

A explicilty flat representative of the conformal class of γ can be obtained using the coordinates

$$x_i := e^{\sqrt{\lambda}t}\hat{\alpha}_i\cos\phi_i \qquad y_i := e^{\sqrt{\lambda}t}\hat{\alpha}_i\sin\phi_i, \quad i = 1,\cdots,q \tag{55}$$

together with $z := e^{\sqrt{\lambda}t}\hat{\alpha}_{p+1}$ if n odd, which are Cartesian for the following flat metric

$$\gamma_E := \frac{e^{2\sqrt{\lambda}t}}{W}\gamma = \delta_{p,q}\mathrm{d}z^2 + \sum_{i=1}^{q}\big(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\big). \tag{56}$$

This form will be used below to determine the conformal class of a conformal Killing vector ξ which we introduce next. Let us denote the projection of k onto $\mathscr{I}$ by

$$\xi_\alpha = \big(k_\alpha + (k_\beta u^\beta)u_\alpha\big)\ |_{\mathscr{I}}$$

with $u_\alpha = \nabla_\alpha\rho/|\nabla\rho|_g$ the unit timelike normal to $\mathscr{I}$. Explicitly

$$\xi = W\mathrm{d}t - \sum_{i=1}^{q}\frac{a_i\alpha_i^2}{1+\lambda a_i^2}\mathrm{d}\phi_i = W\big(\mathrm{d}t - \sum_{i=1}^{q}\hat{\alpha}_i^2 a_i\mathrm{d}\phi_i\big). \tag{57}$$

We view ξ as a covector in $\mathscr{I}$. Its metrically associated vector is, using (54),

$$\xi^\sharp = \frac{1}{\lambda}\partial_t - \sum_{i=1}^{q}a_i\partial_{\phi_i}, \tag{58}$$

which in Cartesian coordinates (55) of γ is

$$\xi^\sharp = \frac{1}{\sqrt{\lambda}}\widetilde{\xi^\sharp} - \sum_{i=1}^{q}a_i\eta_i \tag{59}$$

where we have introduced

$$\widetilde{\xi^\sharp} := \delta_{p,q}\partial_z + \sum_{i=1}^{q}x_i\partial_{x_i} + y_i\partial_{y_i}, \qquad \eta_i := x_i\partial_{y_i} - y_i\partial_{x_i}. \tag{60}$$

The vector $\widetilde{\xi^\sharp}$ is a homothety of γ_E and each η_i is a rotation of this metric. Consequently, $\xi^\sharp$ is a CKVF of γ.

The electric part of the rescaled Weyl tensor can be obtained at once from Lemma 2.4 using $\Omega = \rho$ and $m = n$, because by definition $(t\ |_{\mathscr{I}})_{\alpha\beta} = (\mathcal{H}/\rho^n)\ |_{\mathscr{I}}\ \xi_\alpha\xi_\beta$ and $\mathring{t}\ |_{\mathscr{I}}$ is its trace-free part. Note also that $H/\rho^n\ |_{\mathscr{I}} = 2M$. Thus

$$D = \rho^{2-n}C^\mu{}_{\alpha\nu\beta}\nabla_\mu\rho\nabla^\nu\rho\ |_{\mathscr{I}} = -\frac{1}{2}\lambda^2 n(n-2)\mathring{t}_{\alpha\beta}\ |_{\mathscr{I}} = -M\lambda^2 n(n-2)\left(\xi_\alpha\xi_\beta - \frac{|\xi|_\gamma^2}{n}\gamma_{\alpha\beta}\right).$$

Since, by equation (53) above,

$$|\xi|_\gamma^2 = W\left(\frac{1}{\lambda} + \sum_{i=1}^{q}a_i^2\hat{\alpha}_i^2\right) = \frac{1}{\lambda},$$

D can be cast as

$$D = \kappa D_\xi, \qquad \text{with} \quad \kappa := -\frac{Mn(n-2)}{\lambda^{\frac{n}{2}-1}}$$

and

$$D_\xi := \frac{1}{|\xi|_\gamma^{n+2}}\left(\xi_\alpha\xi_\beta - \frac{|\xi|_\gamma^2}{n}\gamma_{\alpha\beta}\right). \tag{61}$$

Summarizing, we have proven the following result

Proposition 4.1. *The asymptotic data corresponding to the $n+1$ dimensional generalized Kerr-de Sitter metrics is given by the class of conformally flat metrics and the class of TT tensors determined by* (61)*, where ξ is the field of 1-forms given by* (57) *when the metric γ is written in the coordinates where* (54) *holds.*

Now suppose that we let ξ to be any CKVF of γ. By direct calculation one shows that the corresponding D_ξ is still TT w.r.t. γ. The spacetimes corresponding to the class of data obtained in this way constitute a natural extension to arbitrary dimensions of the so-called *Kerr-de Sitter-like class* with conformally flat $\mathscr{I}$, first defined for $n=3$ in [21] and [22]. The details of this extended definition and its properties and structure will be given in a future work. What we want to prove now is that for data (Σ, γ, D_ξ) with γ conformally flat and ξ a CKVF of γ, only the conformal class of ξ (equivalently $\xi^\sharp$) matters. Then, by identifying the conformal class of (58) we will obtain a complete geometrical characterization of Kerr-de Sitter in all dimensions.

Lemma 4.1. *For any trasformation of the conformal group of $\mathscr{I} = (\Sigma, \gamma)$, i.e. $\phi \in \mathrm{Conf}(\mathscr{I})(\subset \mathrm{Diff}(\mathscr{I}))$ such that $\phi^\star(\gamma) = \omega^2\gamma$, the following equivalence of data holds*

$$(\Sigma, \gamma, D_{\phi_\star\xi}) \simeq (\Sigma, \phi^\star\gamma, \phi^\star(D_{\phi_\star\xi})) = (\Sigma, \omega^2\gamma, \omega^{2-n}D_\xi) \simeq (\Sigma, \gamma, D_\xi), \tag{62}$$

where the tensor $D_{\phi_\star\xi}$ is given by (61) *where $\phi_\star\xi$ is the one-form defined by $\gamma(\phi_\star\xi^\sharp, \cdot)$.*

Proof. The first equivalence in (62) is a consequence of the diffeomorphism equivalence of data and the last one a consequence of the conformal equivalence of data (cf. [21])[3], so we must verify the equality in the expression. On the one hand we have for every vector field $X \in T\Sigma$

$$(\phi^\star\phi_\star(\xi))(X) = (\phi_\star\xi)(\phi_\star X) = \gamma(\phi_\star\xi^\sharp, \phi_\star X) = \omega^2\gamma(\xi^\sharp, X) = \omega^2\xi(X)$$

that is $\phi^\star(\phi_\star(\xi)) = \omega^2\xi$. Moreover $|\phi_\star(\xi)|_\gamma = \sqrt{\gamma(\phi_\star\xi^\sharp, \phi_\star\xi^\sharp)} = \omega|\xi|_\gamma$. Thus

$$\phi^\star\left(D_{\phi_\star(\xi)}\right) = \frac{1}{|\phi_\star(\xi)|_\gamma^{n+2}}\left(\phi^\star(\phi_\star(\xi)_\flat \otimes \phi_\star(\xi)_\flat) - \frac{|\phi_\star(\xi)|^2_\gamma}{n}\phi^\star\gamma\right) = \omega^{-n+2}\frac{1}{|\xi|_\gamma^{n+2}}\left(\xi_\flat \otimes \xi_\flat - \frac{|\xi|^2_\gamma}{n}\gamma\right).$$

□

We now come back to Kerr-de Sitter and identify the conformal class of (58). Following the results in [23], a direct way to do that is to write $\xi^\sharp$ in any Cartesian coordinate system for any flat representative γ_E in the conformal class of metrics. One then associates to the explicit form of $\xi^\sharp$ in these coordinates a skew-symmetric endomorphism (equivalently a two-form) of the Minkowski spacetime $\mathbb{M}^{1,n+1}$. Let us denote this set SkewEnd($\mathbb{M}^{1,n+1}$) and we write $F(\xi^\sharp)$ for the skew-symmetric endomorphism associated to the CKVF $\xi^\sharp$. The classification of SkewEnd($\mathbb{M}^{1,n+1}$) up to $O(1,n+1)$ (i.e. the Lorentz group) transformations is equivalent to the classification of CKVFs up to conformal transformations, but the former is simpler because SkewEnd($\mathbb{M}^{1,n+1}$) are linear operators, while the CKVFs are vector fields (see [23] for additional details).

Consider Cartesian coordinates $\{X^\alpha\}_{A=1}^n$ for an n-dimensional Riemannian flat metric. Then an arbitrary CKVF is well-known (e.g. [33]) to be given by

$$\xi^\sharp = \left(\mathrm{b}^A + \nu X^A + (\mathrm{a}_B X^B)X^A - \frac{1}{2}(X_B X^B)\mathrm{a}^A - \omega^A{}_B X^B\right)\partial_{X^A}, \tag{63}$$

to which one associates [21], [23] the following skew-symmetric endomorphism, given in an orthonormal basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ of $\mathbb{M}^{1,n+1}$ with e_0 timelike,

$$F(\xi^\sharp) = \begin{pmatrix} 0 & -\nu & -\mathrm{a}^t + \mathrm{b}^t/2 \\ -\nu & 0 & -\mathrm{a}^t - \mathrm{b}^t/2 \\ -\mathrm{a} + \mathrm{b}/2 & \mathrm{a} + \mathrm{b}/2 & -\boldsymbol{\omega} \end{pmatrix}. \tag{64}$$

where $\mathrm{a}, \mathrm{b} \in \mathbb{R}^n$ are column vectors with components $\mathrm{a}^A, \mathrm{b}^A$ respectively, t stands for their transpose (row vector), $\nu \in \mathbb{R}$ and $\boldsymbol{\omega}$ is $n \times n$ real skew-symmetric matrix of components $(\delta_{AC}\omega^C{}_A =:$

[3] In [21] these equivalences are established in the case $n=3$ but the result extends easily to arbitrary dimension $n \geq 2$. We remark that our notation is slighly different for the TT tensor D_ξ, since the primery object ξ defining it is a covector in our case and a vector in [21]. This explains the differences on how objects are transformed. The choice in [21] is more natural to analyze conformal properties of the TT tensor, while the one here is more consistent with the covariant nature of D_ξ.

$)\omega_{AB} = -\omega_{BA}$. Understood as a map $F : \xi^\sharp \mapsto F(\xi^\sharp)$, F is a Lie algebra anti-homomorphism, i.e. $[F(\xi^\sharp), F(\xi'^\sharp)] = -F([\xi^\sharp, \xi'^\sharp])$. The $O(1, n+1)$ transformations on $F(\xi^\sharp)$ are translated into conformal tranformations of $\xi^\sharp$. That is, for every $\Lambda \in O(1, n+1)$, then $\Lambda \cdot F(\xi^\sharp) = F(\phi_{\Lambda\star}(\xi^\sharp))$ for a conformal transformation ϕ_Λ of γ_E, where "dot" denotes adjoint action, which in matrix notation corresponds simply to the multiplication of matrices $\Lambda \cdot F(\xi^\sharp) = \Lambda F(\xi^\sharp)\Lambda^{-1}$. As a consequence, the classification of $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$ up to $O(1, n+1)$ transformations is equivalent to the classification of CKVFs up to conformal transformations.

In [23], it is proven that the orbits of $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})/O(1, n+1)$ are characterized[4] by the eigenvalues of $-F(\xi^\sharp)^2$ together with the causal character of $\ker F(\xi^\sharp)$. The algorithm is as follows. Denote by $\mathcal{P}_{F^2}(-x)$ the characteristic polynomial of $-F(\xi^\sharp)^2$ and define

$$\mathcal{Q}_{F^2}(x) := (\mathcal{P}_{F^2}(-x))^{1/2} \quad (n \text{ even}), \qquad \mathcal{Q}_{F^2}(x) := \left(\frac{\mathcal{P}_{F^2}(-x)}{x}\right)^{1/2} \quad (n \text{ odd}).$$

From the properties of $F^2(\xi^\sharp)$, it follows [23] that $\mathcal{Q}_{F^2}(x)$ is a polynomial of degree $q+1$ with $q+1$ real roots counting multiplicity, with at most one of which negative. Then

Proposition 4.2. *Let* $\mathrm{Roots}\,(\mathcal{Q}_{F^2})$ *denote the set of roots of* $\mathcal{Q}_{F^2}(x)$ *repeated as many times as their multiplicity and sorted as follows*

a) If n odd, $\{\sigma; \mu_1^2, \cdots, \mu_p^2\} := \mathrm{Roots}\,(\mathcal{Q}_{F^2})$ sorted by $\sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2$ if $\ker F(\xi^\sharp)$ is timelike and $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq \sigma$ otherwise.

b) If n even, $\{-\mu_t^2, \mu_s^2; \mu_1^2, \cdots, \mu_p^2\} := \mathrm{Roots}\,(\mathcal{Q}_{F^2})$ sorted by $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$ if $\ker F(\xi^\sharp)$ is degenerate and $\mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq -\mu_t^2$ otherwise.

Then the parameters $\{\sigma; \mu_1^2, \cdots, \mu_p^2\}$ for n odd and $\{-\mu_t^2, \mu_s^2; \mu_1^2, \cdots, \mu_p^2\}$ for n even determine uniquely the class of $F(\xi^\sharp)$ up to $O(1, n+1)$ transformations and hence also the class of $\xi^\sharp$ up to conformal transformations.

We now apply these results to the Kerr-de Sitter case. We have already obtained a flat representative γ_E and have introduced corresponding Cartesian coordinates (56) . We have also obtained the explicit form of $\xi^\sharp$ in these coordinates, namely (59) and (60). Then, from Proposition 4.2 it is straightforward to identify the conformal class of $\xi^\sharp$. Denote the Cartesian coordinates in (55) by $\{X\}_{A=1}^n = \{z, \{x_i, y_i\}_{i=1}^q\}$ if n odd and $\{X\}_{A=1}^n = \{x_i, y_i\}_{i=1}^q$ if n even. From equations (59), (60) the parameters of $\xi^\sharp$ written as in (63) are $\nu = \lambda^{-1/2}$, $\mathrm{a}^A = \mathrm{b}^A = 0$ and $\omega_{AB} = 2a_i \delta^{2i}{}_{[A}\delta^{2i+1}{}_{B]}$ for n odd and $\omega_{AB} = 2a_i\delta^{2i-1}{}_{[A}\delta^{2i}{}_{B]}$ for n even. Thus, from equation (64) it is immediate

$$F(\xi^\sharp) = \begin{pmatrix} 0 & -\lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{pmatrix} \oplus (0) \bigoplus_{i=1}^{p} \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}, \qquad \text{if } n \text{ is odd}$$

$$F(\xi^\sharp) = \begin{pmatrix} 0 & -\lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{pmatrix} \bigoplus_{i=1}^{p+1} \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}, \qquad \text{if } n \text{ is even,}$$

where this block form is adapted to the following orthogonal decomposition of $\mathbb{M}^{1,n+1}$ as a sum of F-invariant subspaces

$$\mathbb{M}^{1,n+1} = \Pi_t \oplus \mathrm{span}\{e_2\} \bigoplus_{i=1}^{p} \Pi_i, \quad (n \text{ odd}), \qquad \mathbb{M}^{1,n+1} = \Pi_t \bigoplus_{i=1}^{p+1} \Pi_i, \quad (n \text{ even}),$$

where $\Pi_t = \mathrm{span}\{e_0, e_1\}$ for both cases and $\Pi_i = \mathrm{span}\{e_{2i+1}, e_{2i+2}\}$ for n odd and $\Pi_i = \mathrm{span}\{e_{2i}, e_{2i+1}\}$ for n even. Any timelike or null vector $v \in \mathbb{M}^{1,n+1}$ must have non-zero projection onto Π_t, so it may be written $v = v_t + v_s$, with $0 \neq v_t \in \Pi_t$, $v_s \in (\Pi_t)^\perp$. Hence $F(\xi^\sharp)(v) = F(\xi^\sharp)(v_t) + F(\xi^\sharp)(v_s)$, where from the block form it follows that $0 \neq F(\xi^\sharp)(v_t) \in \Pi_t$ and $F(\xi^\sharp)(v_s) \in (\Pi_t)^\perp$, thus $F(\xi^\sharp)(v) = F(\xi^\sharp)(v_t) + F(\xi^\sharp)(v_s) \neq 0$. Therefore, $\ker F(\xi^\sharp)$ is always spacelike or cero. It is straightforward to compute the polynomial $\mathcal{Q}_{F^2}(x)$

$$\mathcal{Q}_{F^2}(x) = (x + \lambda) \prod_{i=1}^{q} (x - a_i^2)$$

[4]Equivalent characterizations may be found in the literature (e.g. [21]) in terms of traces of even powers of $F(\xi^\sharp)$ and matrix rank.

where we may order the indices i, so that the rotation parameters a_i appear in decreasing order $a_1^2 \geq \cdots \geq a_q^2$. Hence, applying Proposition 4.2 we identify the parameters $\sigma := -\lambda^{-1}$ and $\mu_i^2 := a_i^2$ for n odd and $-\mu_t^2 := -\lambda^{-1}$, $\mu_s^2 := a_1^2$ and $\mu_i^2 := a_{i+1}^2$ for n even. Therefore:

Theorem 4.1. *Let $\widetilde{g}_{KdS}$ be a metric of the generalized Kerr-de Sitter family of metrics in all dimensions, namely given by* (46) *and* (49), (50), (51), *with cosmological constant λ and q rotation parameters a_i sorted by $a_1^2 \geq \cdots \geq a_q^2$. Then $\widetilde{g}_{KdS}$ is uniquely characterized by the class of initial data (Σ, γ, D_ξ), where γ is conformally flat and D_ξ is a TT tensor of γ of the form* (61)*, where $\xi^\sharp$ (such that $\gamma(\xi^\sharp, \cdot) = \xi$) is a CKVF of γ whose conformal class is uniquely determined according to Proposition 4.2 by the parameters $\{\sigma = -\lambda^{-1}, a_1^2, \cdots, a_p^2\}$ if n odd and $\{-a_t^2 = -\lambda^{-1}, a_s^2 = a_1^2; a_2^2, \cdots, a_{p+1}^2\}$ if n is even.*

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