Asymptotic behaviour of spacetimes with positive cosmological constant

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Abstract

In this thesis we study the asymptotic Cauchy problem of general relativity with positive cosmological constant in arbitrary $(n+1)$-dimensions. Our aim is to provide geometric characterizations of Kerr-de Sitter and related spacetimes by means of their initial data at conformally flat (n-dimensional) $\mathscr{I}$. In our setting, the conformal Killing vector fields (CKVFs) of $\mathscr{I}$ become very relevant because of their relation with the symmetries of the spacetime.

In the first part of the thesis, we study the CKVFs ξ of conformally flat n-metrics γ, as well as their equivalence classes $[\xi]$ up to conformal transformations of γ. We do that by analyzing in detail SkewEnd$(\mathbb{M}^{1,n+1})$, the skew-symmetric endomorphisms of the Minkowski space $\mathbb{M}^{1,n+1}$. The cases $n = 2, 3$ are worked out in special detail. A canonical form that fits every element in SkewEnd$(\mathbb{M}^{1,n+1})$ is obtained along with several applications. Of relevance for the study of asymptotic data is that it gives a canonical form for CKVFs which allows us to determine the conformal classes $[\xi]$ and study the quotient topology associated to these clases. In addition, the canonical form for CKVFs is applied to the $n = 3$ case to obtain a set of coordinates adapted to an arbitrary CKVF. With these coordinates we provide the set of asymptotic data which generate all conformally extendable spacetimes solving the $(\Lambda > 0)$-vacuum field equations and admitting two commuting symmetries, one of which axial. From this, a characterization of Kerr-de Sitter and related spacetimes follows. Our study provides in principle a good arena to test definitions of mass and angular momentum for positive cosmological constant.

In the second part of this thesis we focus in the asymptotic Cauchy problem in arbitrary dimensions. For this we use the Fefferman-Graham formalism. We carry out an study of the asymptotic initial data in this picture and extend an existing geometric characterization of them, in the conformally flat $\mathscr{I}$ case, to arbitrary signature and cosmological constant. We discuss the validity of this geometric characterization of data beyond the conformally flat $\mathscr{I}$ case. We provide a KID equation for asymptotic analytic data (which comprise Kerr-de Sitter). This equation being satisfyied by the data amounts to the existence of a Killing vector field in the corresponding spacetime. With the above results in hand we provide a geometric characterization of Kerr-de Sitter by means of its asymptotic initial data, which happen to be determined by the conformally flat class of metrics $[\gamma]$ and one particular conformal class of CKVFs $[\xi]$ of $[\gamma]$. These data admit a generalization, keeping $[\gamma]$ conformally flat, by allowing $[\xi]$ to be an arbitrary conformal class. This extends the so-called Kerr-de Sitter-like class with conformally flat $\mathscr{I}$, defined in previous works in four spacetime dimensions, to arbitrary dimensions. We study this class and prove that the corresponding spacetimes are contained in the set of $(\Lambda > 0)$-vacuum Kerr-Schild spacetimes, which share (conformally flat) $\mathscr{I}$ with their background metric (de Sitter). We name these Kerr-Schild-de Sitter spacetimes. The proof largely relies on our study of the space of classes of CKVFs and in particular on the properties of its quotient topology. In addition, we prove the converse inclusion, providing a full characterization of the Kerr-de Sitter-like class as the Kerr-Schild-de Sitter spacetimes.

Declaration of supervisor

Dr. D. Marc Mars Lloret, Catedrático de Física Teórica en el Departamento de Física Fundamental de la Universidad de Salamanca,

CERTIFICA:

Que el trabajo de investigación que se recoge en la siguiente memoria titulada *Comportamiento asintótico de espacio-tiempos con constante cosmológica positiva*, presentada por **D. Carlos Peón Nieto** para optar al Título de Doctor por la Universidad de Salamanca con la Mención de Doctorado Internacional, ha sido realizada en su totalidad bajo su dirección y autoriza su presentación.

En Salamanca, a 23 de septiembre de 2021

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List of publications

1) M. Mars and C. Peón-Nieto. Skew-symmetric endomorphisms in $\mathbb{M}^{1,3}$: a unified canonical form with applications to conformal geometry. *Classical and Quantum Gravity*, **38**:035005, 2020. DOI: 10.1088/1361-6382/abc18a.

2) M. Mars and C. Peón-Nieto. Skew-symmetric endomorphisms in $\mathbb{M}^{1,n}$: a unified canonical form with applications to conformal geometry. *Classical and Quantum Gravity*, **38**:125009, 2020. DOI: 10.1088/1361-6382/abf413

3) M. Mars and C. Peón-Nieto. Free data at spacelike $\mathscr{I}$ and characterization of Kerr-de Sitter in all dimensions, 2021. ArXiv:2105.06881v1 [gr-qc].

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Abbreviations

CKVF	**C**onformal **K**illing **V**ector **F**ield.
EFE	**E**instein **F**ield **E**quations.
FCFE	**F**riedrich **C**onformal **F**ield **E**quations.
FG	**F**efferman-**G**raham.
FGP	**F**efferman-**G**raham-**P**oincaré.
KID	**K**illing **I**nitial **D**ata.
TT	**T**raceless and **T**ransverse.

Chapter 1

Introduction

1.1 Context and motivation

Eversince its original formulation in 1915, the Einstein general theory of relativity has become the paradigm which governs the large and massive scales in nature. The astonishing phenomenology predicted in its final version [45], later observationally confirmed, largely contributed to its settlement. The original predictions were three: the precession of the perihelion in planetary motion, the gravitational redshit and the bending of light rays by effect of gravity. The precession of Mercury's perihelion had puzzled astrophysicists for decades, because no neat argument arose from Newton's laws, leading to rather cumbersome explanations such as postulating the existence of an intramercurian planet. However, in Einstein's theory, this precession appeared as a natural and accurate consequence of the equations. On the other hand, the bending of lightrays was soon observed in the celebrated Eddington and Dyson expedition in 1919 [42]. The gravitational redshift experiments took some more time to give concluding measurements, by Popper [126] in 1954, since the first ones by Adams [4] were considered too poor (see also [80, 81]). Since then, all these phenomena have been repeatedly observed. Indeed, the gravitational lensing, based in the bending of light rays when passing nearby massive spots in the universe, is today a useful effect for astronomical observations.

The theory delivered other exotic and controvesial predictions, such as the existence of black holes and the emision of gravitational waves, for which experimental confirmation had to be awaited until the new century. The first black hole solution was actually the first exact solution of the Einstein equations published soon after Einstein's theory by Schwarzschild [134]. For long time black holes were not considered as a serious physical prediction and their inherent singularities were regarded as a pathological consequence of the high symmetries of the model. This view, however, was proven wrong in both sides. In the formal aspect, the singularity theorems by Penrose [121] and Hawking and Penrose [76] showed that singularities are a stable feature of general relativity (see also the reviews [136, 137]). In the observational aspect, the extreme motion of stars measured at the center of our galaxy evidenced the presence of a black hole [68]. Moreover, with the Event Horizon Telescope array, the direct reconstruction of black hole shadows is possible and there are currently available images from data taken at the center of the galaxy M87 [44]. The processing of data taken from the center of our galaxy is now in progress and the results are expected soon. Therefore, the existence of black holes is today accepted by the vast majority of the general relativistic community.

The first theoretical approach to gravitational waves was carried by Einstein with its famous quadrupole formula. The later works by Bondi et al. [21], Sachs [130] and Newman and Penrose [109] gave the basic setting for a fully non-linear analysis of the gravitational waves, which largely relies on the asymptotic behaviour of the gravitational field. Subsequent works by Penrose in the 1960's [120, 122, 123] got deeper into the asymptotic analysis of general relativity, on which we will expand later. The technological challenges that the experimental measurements of gravitational waves entail delayed their first observation until 2017, where the LIGO experiment [1] confirmed the detection of the gravitational waves generated by the merge of two black holes. Note, however, that this detection is more than just another confirmation of Einstein's theory. It is claimed by the observational community that the surprisingly high number of events registered in the years following 2017 by the gravitational wave detectors LIGO Virgo and KAGRA is changing our understanding of the universe.

In view of the success of general relativity in explaining nature, there is no doubt that, within its range of applicability, it is the "correct" theory. The search for new exciting theories extending the general relativity, may lead one to believe that the theory is, in words of H. Friedrich, "essentially understood" [62], and that the formal study of general relativity is a matter of sharpening ideas. However, the simplicity of the Einstein equations is only apparent, namely,

$$Ric(\widetilde{g}) - \frac{Scal(\widetilde{g})}{2}\widetilde{g} + \Lambda\widetilde{g} = \frac{8\pi G}{c^4}T,$$

where $Ric(\widetilde{g})$ stands for the Ricci curvature tensor of the metric $\widetilde{g}$, $Scal(\widetilde{g})$ is the trace of $Ric(\widetilde{g})$, Λ the cosmological constant and T the stress-energy tensor, which accounts for presence of matter, radiation and other fields. This becomes obvious if one expands the tensor in terms of the metric components and its derivatives and casts the Einstein equations as a non-linear PDE problem. Just to make ourselves an idea, the Ricci tensor looks like

$$\begin{aligned}R_{\mu\nu} =&\frac{1}{2}\, g^{\rho\sigma}\partial_\nu\partial_\rho g_{\mu\sigma} + \frac{1}{2}\, g^{\rho\sigma}\partial_\mu\partial_\rho g_{\nu\sigma} - \frac{1}{2}\, g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} - \frac{1}{2}\, g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\\ &- \frac{1}{2}\,\partial_\rho g^{\rho\sigma}\,\partial_\nu g_{\mu\sigma} + \frac{1}{2}\,\partial_\rho g^{\rho\sigma}\,\partial_\mu g_{\nu\sigma} - \frac{1}{2}\,\partial_\rho g^{\rho\sigma}\,\partial_\sigma g_{\mu\nu} - \frac{1}{2}\,\partial_\nu g^{\rho\sigma}\,\partial_\mu g_{\rho\sigma}\\ &+ \frac{1}{4}\, g^{\kappa\lambda}\partial_\nu g_{\mu\kappa}\, g^{\rho\sigma}\partial_\lambda g_{\rho\sigma} + \frac{1}{4}\, g^{\kappa\lambda}\partial_\mu g_{\nu\kappa}\, g^{\rho\sigma}\partial_\lambda g_{\rho\sigma} - \frac{1}{4}\, g^{\kappa\lambda}\partial_\kappa g_{\mu\nu}\, g^{\rho\sigma}\partial_\lambda g_{\rho\sigma}\\ &- \frac{1}{4}\, g^{\kappa\lambda}\partial_\mu g_{\kappa\rho}\, g^{\rho\sigma}\partial_\nu g_{\lambda\sigma} - \frac{1}{2}\, g^{\kappa\lambda}\partial_\kappa g_{\mu\rho}\, g^{\rho\sigma}\partial_\sigma g_{\nu\lambda} + \frac{1}{2}\, g^{\kappa\lambda}\partial_\kappa g_{\mu\rho}\, g^{\rho\sigma}\partial_\lambda g_{\nu\sigma}.\end{aligned}$$

The unknown mathematical implications of these equations are still many and, perhaps, with a better understanding of them, even new phenomenology might be predicted. Therefore, the study of the formal aspects of the Einstein theory of general relativity is not purely a mathematical exercise, but also fundamental in physics. In this thesis, we shall address some of these mathematical problems, which will be described in the remainder of this introduction.

A spacetime is said to be globally hyperbolic if it contains a Cauchy surface, which is a spacelike hypersurface that is intersected exactly once by each inextendible timelike curve. Global hyperbolicity is a reasonable requirement for a physical spacetime. This is primarily because globally hyperbolic spacetimes are known to be uniquely determined by their initial configurations. Indeed, the Einstein equations admit a Cauchy problem which is longtime known to be well-posed by the landmark results of Y. Choquet-Bruhat [54] and Choquet-Bruhat and Geroch [30]. This allows, in particular, to extract interesting properties of the solutions without actually having to deal with the full complexity of the Einstein equations. This Cauchy problem splits the Einstein equations into constraint equations on an initial spacelike hypersurface[1] plus evolution equations, which propagate the fields (and the constraints). This is the classical initial value formulation and a set of initial data is by definition any solution of the constraint equations. Although certainly simpler that the full Einstein equations, they still pose a difficult problem in geometric analysis (see e.g. [85] and references therein). In addition, the solutions evolving from a set of initial data are local due to the intrinsic hyperbolicity of the evolution equations.

As mentioned above, the works by Bondi et al. [21], Sachs [130] and Newman and Penrose [109] were motivated by the fully non-linear study of the gravitational radiation. This led them to consider what in today's language would be called an *asymptotic characteristic initial value problem*. Following this track, the works by Roger Penrose [120, 122, 123] pioneered the use of conformal techniques in general relativity, enhancing the role that the conformal structure plays in the Einstein equations. He gave a precise definition of *asymptotic flatness* in terms of conformal extensions of the physical metric $\widetilde{g}$. Namely, given a smooth[2] manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$, a conformal extension of $(\widetilde{\mathcal{M}}, \widetilde{g})$ is a smooth

[1] It can be also cast as a characteristic initial value problem if the initial hypersurface is null (see [128]).

[2] We consider the smooth case for simplicity, but one could assume "sufficient differentiability" instead.

manifold $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M}$, whose interior can be identified with $\widetilde{\mathcal{M}} = \mathrm{Int}(\mathcal{M})$ and such that there exists a smooth function Ω in $\mathcal{M}$ which is positive in $\widetilde{\mathcal{M}}$, where it satisfies $g = \Omega^2\widetilde{g}$, and $\Omega \mid_{\partial\mathcal{M}} = 0$ and $\mathrm{d}\Omega \mid_{\partial\mathcal{M}} \neq 0$. The boundary equipped with its first fundamental form γ is denoted $\mathscr{I}$. This manifold, called *conformal infinity* or *null infinity*, gives a precise definition of the asymptotic region for the spacetime $(\widetilde{\mathcal{M}}, \widetilde{g})$. If the Ricci tensor of $\widetilde{g}$ satisfies $Ric(\widetilde{g}) = 0$ in a neighbourhood of $\mathscr{I}$ or, more generally, decays sufficently fast to zero at $\mathscr{I}$, then $\widetilde{g}$ is called asymptotically flat. The physical importance of asymptotically flat manifolds is that they are considered to model self-gravitating isolated systems, which are sufficiently far away from other systems so that one can ignore the influence of the latter except, possibly, for the effects of gravitational radiation.

When written in terms of the conformal metric g, the Einstein equations of $\tilde{g}$ are singular at $\mathscr{I}$. However, in a remarkable achievement H. Friedrich was able (by means of introducing carefully chosen variables) to rewrite the equations in spacetime dimension four as a system of geometric PDE that are regular at $\mathscr{I}$ (see the seminal works [57], [56] and the reviews [61], [63]). These equations allow to take into account the asymptotic behaviour of the spacetime by posing an "asymptotic PDE problem", on which we shall comment next. Furthermore, it should be mentioned that the conformal formulation of the Einstein equations have important consequences in the field of numerical relativity. We shall not discuss any of these here, as they are beyond the scope of this thesis, but we refer to [55] for a detailed review of the conformal field equations and their numerical aspects.

So far we have discussed classical results which historically have assumed zero cosmological constant. When it comes to determine the nature of the asymptotic PDE problem posed by the Friedrich equations, the sign of the cosmological constant has drastic consequences. This is because the Einstein equations determine the causal character of $\mathscr{I}$, which is null if $\Lambda = 0$; timelike if $\Lambda < 0$ and spacelike $\Lambda > 0$. The Λ zero and negative, are respectively a characteristic initial value problem and boundary value problem, for which existence and uniqueness is a hard and subtle issue. We shall briefly comment on these again in subsection 2.4.1, but let us now focus on the central case for this thesis, which is the positive Λ case.

From the physical point of view, it is noteworthy that the Supernova Cosmology Project have determined a universe with positive cosmological constant [125], recently confirmed again by the Planck collaboration [3]. Since then, the paradigm of cosmology assumes a positive cosmological constant, while the zero Λ case is still having an important relevance in mathematical relativity. However, in the recent years, the positive cosmological constant has increasingly caught the attention of several general relativists and many advances have been done in this direction. Just to quote some, on the general asymptotic framework [8, 11], on the gravitational radiation [12, 13, 51, 52, 132], the peeling property of the Weyl tensor [53] and on the definition of mass and momenta [23, 24, 41, 131, 143, 144].

The present thesis is yet another example.

From the formal side, it was also proven by Friedrich [58] that the Cauchy problem at $\mathscr{I}$ is always well-posed if the cosmological constant is positive. The well-posedness already gives a special interest to this problem. It is also noteworthy that associated to a conformal metric g solving the conformal Friedrich equations, there is a solution to the Einstein equations $\widetilde{g}$ which is "semiglobal" (i.e. the "physical" spacetime $\widetilde{g} = \Omega^{-2} g$ extends infinitely towards the future or past, depending on whether $\mathscr{I}$ is a final or an initial state). Morever, a remarkable simplification occurs in the constraint equations at $\mathscr{I}$, as opposed to the standard constraint equations of the classical initial value problem. The data at $\mathscr{I}$ consist of a Riemannian three-manifold (Σ, γ) which prescribes the (conformal) geometry of $\mathscr{I}$, together with a symmetric two-tensor D with vanishing trace an divergence, i.e. a transverse and traceless (TT) tensor. This tensor prescribes certain components of the suitably rescaled Weyl tensor at $\mathscr{I}$, known as the *electric part of the rescaled Weyl tensor*. Of course, since the result cannot depend on the conformal scaling of the physical metric, there is a large residual gauge freedom in the data, being all sets $(\Sigma, \omega^2\gamma, \omega^{-1}D)$ equivalent to (Σ, γ, D) for any smooth positive function ω of Σ.

As we shall discuss in more detail in subsection 2.4.1, the Friedrich conformal field equations are specially taylored to dimension four and do not appear to extend to higher dimensions. The basic problem is that there do not appear to be enough evolution equations that remain regular at $\mathscr{I}$ [61]. Actually, one of the fundamental objects in the conformal Friedrich equations is the rescaled Weyl tensor, which plays a central role in this thesis. Our analysis in Chapter 5 shows that in dimension higher than four this object is regular at $\mathscr{I}$ only in few particular cases. Thus, there are reasons to believe that any attempt to find a regular Cauchy problem well-posed at $\mathscr{I}$ based on this object will be unfruitful.

Before entering into the discussion of the mathematical aspects of the higher dimensional general relativity, it should be mentioned that there are also physical motivations in its study. These are, basically, that the modern theories aiming to concilliate general relativity with quantum mechanics, such as string theories or the AdS/CFT correspondence, seem to require more than four spacetime dimensions. We shall not discuss the physical aspects in any detail, as many of them lie beyond the classical formulations of general relativity, which is our interest here. We refer the interested reader to the reviews in string theory [106], AdS/CFT correspondence [83] and also in higher dimensional black hole [46].

As mentioned above, the higher dimensional Cauchy problem in general relativity requires a different approach than the one given by Friedrich to the four dimensional case. The formalism which eventually allowed for well-posedness results in appropriate circumstances is due to Fefferman and Graham, first given in the paper [48] and later extended into a monograph [50]. We review the basics of this formalism in Section 2.3, so we may just introduce here the very basic ideas in order to discuss the initial value problem of general relativity.

An important part of the Fefferman and Graham work focuses in the so called Poincaré metrics. Roughly speaking, these are asymptotically Einstein $(n+1)$-dimensional metrics, namely, conformally extendable metrics which satisfy the Einstein equations with non-zero cosmological constant (to a certain order) at (n-dimensional) $\mathscr{I}$. In the Fefferman-Graham formalism, their study is carried through an asymptotic formal series expansion, usually called Fefferman-Graham (FG) expansion, which is generated from the Einstein equations at $\mathscr{I}$. It should be noticed that the analysis by means of formal series expansions does not necessarily require the series to be convergent away from $\mathscr{I}$. This, however, sets a framework which allows to study asymptotic properties of Poincaré metrics and, as we shall next see, even prove some existence and uniqueness results if these metrics are Einstein also in a neighbourhood of $\mathscr{I}$. From now on, we shall use $n+1$ for the spacetime dimension and n for the dimension of $\mathscr{I}$.

The term "asymptotic expansion" means in this case that it is performed in terms of the conformal factor Ω "near" the boundary $\{\Omega = 0\}$. In the Fefferman and Graham setting a very particular conformal factor is employed, namely, the one whose gradient is geodesic with respect to the conformally extended metric $g = \Omega^2 \widetilde{g}$. The FG expansion associated to an asymptotically Einstein metric $\widetilde{g}$ is generated as follows. The first order coefficient is given by the boundary metric γ induced by g. Then, provided that the Einstein equations at $\mathscr{I}$ are satisfied to order m, the coefficients of an <u>even</u> power series expansion (directly obtained from derivatives of the metric in Ω) up to order m are recursively determined. However, a remarkable difference appears between the cases n even and n odd. If n is odd, one may keep generating even order terms to infinite order, by demanding that the Einstein equations are satisfied to infinite order at $\mathscr{I}$. If n is even, generically no power series expansion can be generated beyond the n-th order because of the presence of the so-called obstruction tensor $\mathcal{O}(\gamma)$, which is entirely determined by γ. One is then forced to introduce logarithmic terms, which spoil smoothness, but allows one to satisfy the Einstein equations to infinite order at $\mathscr{I}$. It is also remarkable that for both n even and odd, one can always introduce an undetermined smooth term $g_{(n)}\Omega^n$, with the only constraint that the trace and divergence of $g_{(n)}$ are determined by γ, being both zero if n is odd. The presence of this term does not destroy the Einstein asymptoticity, but of course, modifies the subsequent coefficients. Hence, the seed data which generate the FG expansion are a pair $(\gamma, g_{(n)})$.

Interestingly it is the obstruction tensor what allows Anderson [6] to find an asymptotic Cauchy problem for the Einstein equations in the n odd case. Although the core idea appears for the first time in [6], neither this paper nor [7], which attempts to give a detailed proof, are fully correct. The mistakes in those papers have recently been identified in [86] where a complete proof of the existence results has been provided. The idea in [6] relies on the fact that the obstruction tensor is conformally covariant and that it vanishes for all conformally Einstein metrics. Then, for $n+1$ even dimensional metrics $\widetilde{g}$, this tensor provides a differential equation $\mathcal{O}(\widetilde{g}) = 0$, which for Lorentzian conformally Einstein metrics, can be cast as a Cauchy problem at $\mathscr{I}$. Anderson (and the subsequent works mentioned above) proves that solutions of this Cauchy problem exist and are uniquely determined for every pair of symmetric two-tensors $(\gamma, g_{(n)})$, γ positive definite and $g_{(n)}$ traceless and transverse w.r.t. γ. A posteriori, γ determines the geometry of $\mathscr{I}$ and $g_{(n)}$ is n-th order coefficient of the asymptotic expansion of $\widetilde{g}$. Thus, Anderson's theorem associates a unique FG expansion, which recall a priori need not to be convergent, to a unique Einstein metric $\widetilde{g}$ in a neighbourhood of $\mathscr{I}$. This idea is not extendable to the n even case, for no obstruction tensor can be built out of $\widetilde{g}$ when $n+1$ is odd. In this case, however, a result by Kichenassamy [87] proves the convergence of the FG expansion in the case where the data are analytic, regardless of the parity of n. It should be noticed that in the n even case, the initial data $(\gamma, g_{(n)})$ also determine the geometry of $\mathscr{I}$ and the n-th order coefficient of the FG expansion, but $g_{(n)}$ has generically non-zero trace and divergence determined by γ (cf. Appendix A). In addition, just like in the four spacetime dimensional case, the initial data in these problems have a large conformal gauge freedom, namely, data $(\Sigma, \omega^2\gamma, \omega^{2-n} g_{(n)})$ (where Σ is the manifold on which γ is defined) are equivalent to $(\Sigma, \gamma, g_{(n)})$ for every smooth positive function ω of Σ.

An existence and uniqueness theorem can be used to characterize spacetimes by means of their Cauchy data. The situation is particularly interesting in the case of the asymptotic Cauchy problem for positive Λ, because of the simplicity of the data (specially if n odd), which potentially allows one to achieve classification results for spacetimes whose explicit form need not to be known. However, for this definition to be geometric, we must have a proper geometric characterization of the initial data, which for $n > 3$ is not straightforward. The original definition of the coefficient $g_{(n)}$ is not covariant, because the Fefferman-Graham expansion is constructed in a very particular set of coordinates, that is not in general easily obtainable. This issue will be addressed in Chapter 5, where we shall reformulate the initial data $(\Sigma, \gamma, g_{(n)})$, with (Σ, γ) locally conformally flat, as an equivalent set $(\Sigma, \gamma, \mathring{g}_{(n)})$, where $\mathring{g}_{(n)}$ is geometrically defined, up to a constant, as the electric part of the rescaled Weyl tensor at $\mathscr{I}$. This extends to the $\Lambda > 0$ case a previous result by Hollands-Ishibashi-Marolf [82] in the $\Lambda < 0$ case. Actually, this extension is straightforward if one takes into account general results [5],[139] relating the coefficients of the Fefferman-Graham expansion for opposite signs of Λ. A geometric reformulation of the initial data $(\Sigma, \gamma, g_{(n)})$ in the general case should be possible, but as we shall also discuss, it is not immediate to relate $g_{(n)}$ with the electric part of the rescaled Weyl tensor in general.

It should be remarked that geometric characterizations of spacetimes are important in general relativity because of the intrinsic diffeomorphism covariance of the theory.

Namely, for a physical spacetime $(\widetilde{\mathcal{M}}, \widetilde{g})$, i.e. with $\widetilde{g}$ satisfying the Einstein equations in $\widetilde{\mathcal{M}}$, the (also physical) manifold $(\widetilde{\mathcal{M}}, \phi^\star(\widetilde{g}))$ for every diffeomorphism ϕ of $\widetilde{\mathcal{M}}$, is physically equivalent to $(\widetilde{\mathcal{M}}, \widetilde{g})$. It is in general a very hard task to determine by inspection whether two metrics are diffeomorphic to each other and obtaining a geometric characterization may simplify this problem. Thus, geometric characterizations are also fundamental from a physical perspective.

In this context, it is worth highlighting the famous uniqueness theorems of stationary black holes. More specifically, the no-hair conjecture asserts, roughly speaking, that every stationary electrovacuum black hole solution is entirely characterized by its (suitably geometrically defined) mass, angular momentum and electric charge. The no-hair conjecture has been extensively studied in the zero cosmological constant setting and it is well-known (see e.g. [32, 102] and references therein) to be satisfied by static (i.e. Schwarzschild) and stationary axisymmetric configurations[3] (i.e. Kerr). The latter cases are of particular relevance because of the role that they are believed to play as the endpoint states of collapsing self-gravitating systems.

An alternative local characterization of the Kerr and Kerr-NUT metrics among spacetimes with one Killing vector field can be given in terms of the vanishing of the so-called Mars-Simon tensor [93, 138]. Remarkably, it has been shown [98] that in the non-zero cosmological constant case, the vanishing of the Mars-Simon tensor also characterizes the Kerr-NUT-(A)de Sitter metrics and related spacetimes. Recall that the latter generalize Kerr-NUT to the arbitrary cosmological constant setting, so they are also important from a physical perspective. Particularly, in the case of positive cosmological constant, the geometric characterizations of Kerr-de Sitter are interesting because this metric is expected to satisfy a uniqueness theorem among stationary, axisymmetric, $(\Lambda > 0)$-vacuum black hole spacetimes. We remark that a uniqueness theorem (in the sense of black holes uniqueness theorems) is a much more subtle result than simply a geometric characterization. Nevertheless, it is a step towards a possible uniqueness result in the future.

The results in [98] are used in [99, 100] to provide a characterization in terms of asymptotic initial data of Kerr-NUT-de Sitter metrics and related spacetimes, which altogether define the so-called *Kerr-de Sitter-like class*[4] (see also [66, 67] for a similar characterization of Kerr-de Sitter and Schwarzschild-de Sitter with spinorial techniques). An important part of this thesis is devoted to deepen into this characterization. Namely, we identify, in terms of asymptotic initial data, the Kerr-de Sitter-like class and Kerr-de Sitter family among the set of asymptotic initial data with $n = 3$ of all spacetimes with two symmetries, one of which axial. In addition, we obtain the asymptotic initial data

[3]The proof of the stationary case is not considered fully general, as one has to assume non-degenerate analytic horizons, which imply axisymmetry. Giving a general proof of this is still today a difficult open problem.

[4]We stress the difference between the Kerr-de Sitter family and Kerr-de Sitter-like class, the first being one of the multiple families included in the latter.

of Kerr-de Sitter in all dimensions, which allows us to extend to higher dimensions the definition of the Kerr-de Sitter-like class in the conformally flat $\mathscr{I}$ case. We will come back to this in subsection 1.2 below.

In any dimension, the initial data, that we denote generically (Σ, γ, D), must store all the information of the spacetime evolving from them. Specifically, the necessary and sufficient conditions for the existence of symmetries in the spacetime has been studied in four spacetime dimension. In the asymptotic Cauchy problem with positive Λ, this was determined by Paetz in [116] to be a neat geometric PDE involving γ, D and a conformal Killing vector field (CKVF) ξ of γ (cf. Theorem 2.35), known as the Killing initial data (KID) equation. The CKVF ξ is, a posteriori, the Killing vector field ζ of the spacetime, restricted to $\mathscr{I}$. It becomes natural to define the initial data for this case to be (Σ, γ, D, ξ). Apart from this $n = 3$ case, no previous results relating continuous local isometries to initial data at $\mathscr{I}$ were known in more dimensions. In this thesis we prove a higher dimensional result (cf. Theorem 5.18), analogous to the $n = 3$ one, restricted to the case of analytic metrics with zero obstruction tensor. The result is a natural generalization of the Theorem proven by Paetz.

The conformal Killing vector fields of a manifold (Σ, γ) define a Lie algebra $\mathrm{CKill}(\Sigma, \gamma)$ whose uniparametric group of diffeomorphisms are in general local conformal transformations of (Σ, γ), which we shall denote $\mathrm{ConfLoc}(\Sigma, \gamma)$. The fact that these conformal transformations are local raises certain difficulties, which we analyze in more detail in subsection 2.2.2, specially for the study of the quotient $\mathrm{CKill}(\Sigma, \gamma)/\mathrm{ConfLoc}(\Sigma, \gamma)$. The interest in the study of this quotient stems from the fact that the vector fields in $\mathrm{CKill}(\Sigma, \gamma)$ which lie in the same equivalence class in $\mathrm{CKill}(\Sigma, \gamma)/\mathrm{ConfLoc}(\Sigma, \gamma)$ actually generate the same symmetry (cf. Remark 2.37).

The issues with locality mentioned above appear because $\mathrm{ConfLoc}(\Sigma, \gamma)$ is actually given by the local action of an abstract Lie group G on Σ, whose algebra $\mathfrak{g}$ induce the set of conformal vector fields $\mathrm{CKill}(\Sigma, \gamma)$. Then, it should be possible to study the classes in $\mathrm{CKill}(\Sigma, \gamma)/\mathrm{ConfLoc}(\Sigma, \gamma)$ by means of the study of classes in $\mathfrak{g}/G$. In the case of locally conformally flat n-manifolds (Σ, γ), the Lie group G can be identified [100] with the orthochronous component of the Lorentz group $O^+(1, n+1)$, and $\mathfrak{g} = \mathfrak{o}(1, n+1)$ is well-known to admit a representation as the space of two-forms in $\mathbb{M}^{1,n+1}$, or equivalently, as skew-symmetric endomorphisms of Minkowski, $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$. This reason strongly motivates the study of $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$ in Chapters 3 and 4 of this thesis.

A typical way of studying quotients $\mathfrak{g}/G$ is by obtaining a *canonical form* (also normal form) which all elements in $\mathfrak{g}$ admit such that it is invariant under the adjoint action of the group G. In other words, a form shared by all elements in the orbits $[F]$ generated by adjoint action of the group on a given element $F \in \mathfrak{g}$, i.e. $F' \in [F]$ if and only if $F' = \Lambda \cdot F \cdot \Lambda^{-1}$ for some $\Lambda \in G$. This amounts to finding a unique respresentative for such orbits. We assume matrix representation of both $\mathfrak{g}$ and G and "dot" denotes usual multiplication of matrices. The most common example of a canonical form in this context is the well-known Jordan form, which represents the conjugacy classes of $GL(n, \mathbb{K})$ (where $\mathbb{K}$ is usually $\mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$). Besides this example, the problem of finding a canonical representative for the conjugacy classes of a Lie group has been adressed numerous times in the literature. The reader may find a list of canonical forms for algebras whose groups leave invariant a non-degenerate bilinear form in [39] (this includes symmetric, skew-symmetric and simplectic algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$) as well as the study of the affine orthogonal group (or Poincaré group) in [36] or [84]. Notice that these works deal, either directly or indirectly, with our case of interest $O(1, n)$ (and therefore its orthochronous component).

When giving a canonical form, it is usual to base it on criteria of irreducibility rather than uniformity (e.g. [36], [39], [84]). This is similar to what is done when the Darboux decomposition is applied to two-forms (i.e. elements of $\mathfrak{o}(1, n)$), for example in [100] or for the low dimensional case $n = 3$ (e.g. [74], [142]). As a consequence, all canonical forms found for the case of $\mathfrak{o}(1, n)$ require two different types of matrices to represent all orbits, one and only one fitting a given element. One of the results in this thesis gives unique matrix form which represents each element $F \in \mathfrak{o}(1, n)$, depending on a minimal number of parameters that allows one to easily determine its orbit under the adjoint action of $O^+(1, n)$. Indeed, these orbits coincide with those generated by the whole group $O(1, n)$. The unification of the canonical form is obviously achieved by loosing explicit irreducibility with respect to previous canonical forms. However, this canonical form will be proven to be fruitful by giving several applications, which we shall mention later in this introduction.

1.2 Aim of this thesis

The aim of this thesis is to study the asymptotic Cauchy problem of the Λ positive vaccum Einstein equations in all dimensions. Our intent is to provide characterizations of the Kerr-de Sitter family in terms of their asymptotic inital data, which may help understanding in what sense is this family of spacetimes special. Our point of departure is the characterization of the Kerr-de Sitter-like class, in the $n = 3$ case provided in [99, 100] by means of their asymptotic data at $\mathscr{I}$.

The asymptotic data for the Kerr-de Sitter-like class are of the form $(\Sigma, \gamma, \kappa D_\xi)$, where (Σ, γ) is a Riemannian three-manifold, κ a real (non-zero) constant and D_ξ a TT tensor of the form

$$D_\xi = \frac{1}{|\xi|_\gamma^5} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{3} \gamma \right) \tag{1.1}$$

with ξ a CKVF of γ and $\boldsymbol{\xi} := \gamma(\xi, \cdot)$. The TT tensor D_ξ has several remarkable properties. First, it is a very simple solution among all possible TT tensors. Second, it is easy to check that it satisfies the KID equation for ξ. Thus, the tensor D_ξ singles out a CKVF of $\mathscr{I}$ and a particular symmetry of the spacetime. Concerning the metric γ, besides the condition that it must admit a non-trivial CKVF (so that (1.1) makes sense) it is further restricted by the condition that its Cotton-York tensor is also of the form $\tilde{\kappa} D_\xi$ for $\tilde{\kappa} \in \mathbb{R}$. Recall that the Cotton-York tensor is defined only in three-dimensions and is constructed by dualization of the Cotton tensor in two of its indices. It is always a symmetric TT tensor, so taking the form (1.1) is admissible. The constant $\tilde{\kappa}$ is directly related to the so-called NUT charge of the spacetime and vanishes when the metric belongs to the Kerr-de Sitter family. $\tilde{\kappa} = 0$ is equivalent to γ being locally conformally flat because the Cotton-York tensor vanishes if and only if the metric is locally conformally flat. By conformal invariance of the asymptotic Cauchy problem, the data $(\Sigma, \gamma, \kappa D_\xi)$ happen to be uniquely determined by the conformal class $[\xi]$ of ξ, i.e. all CKVFs ξ' differening from ξ by a conformal diffeomorphism[5] ϕ of (Σ, γ). Therefore the study of conformal classes of CKVFs is indeed relevant in this thesis. We focus on the locally conformally flat γ case, because this one contains the Kerr-de Sitter family of metrics, but also because with the current techniques, conformal flatness of $\mathscr{I}$ is required for an analysis in higher dimensions.

Our first achivement is to give a classification in the $n = 3$ case, also in terms of their asymptotic data (Σ, γ, D), of all spacetimes admitting a smooth conformally flat $\mathscr{I}$, with at least two commuting symmetries (cf. Chapter 4). Our analysis assumes that one of these symmetries is axial, but removing this asumption gives raise to only a few extra cases, straightforwardly obtainable. The TT tensors D are obtained taking advantage of a canonical decomposition for CKVFs $\xi = \widetilde{\xi} + \eta$, inherent to the conformal class of ξ, where both $\widetilde{\xi}$, η are CKVFs, with η associated to an axial symmetry. By identifying the Kerr-de Sitter-like class (with conformally flat $\mathscr{I}$) within this set of data, we aim to shed some light on the role played by the CKVF ξ. For instance, the structure of the solution suggests a possible connection between the terms $\widetilde{\xi}$ and η with "mass" and "angular momentum" respectively.

However, our main aim is to extend this analysis to all dimensions. For that, a study of the Fefferman-Graham formalism and its asymptotic data is required. As mentioned above, the basic issue that we first address is how to provide a geometric definition of the asymptotic initial data in this picture. This can be done in the conformally flat $\mathscr{I}$ case, in terms of which we can calculate the initial data for the Kerr-de Sitter family of metrics in arbitrary dimensions (cf. [70]). We find these to be a locally conformally flat n-manifold (Σ, γ) and a TT tensor κD_ξ, with $\kappa \in \mathbb{R}$ and D_ξ of the form

$$D_\xi = \frac{1}{|\xi|_\gamma^{n+2}} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{n} \gamma \right). \tag{1.2}$$

This turns out to be a natural generalization of the $n = 3$ case (1.1). It is remarkable that the original metrics in [70] are constructed from heuristic arguments. Indeed, [70]

[5]The conformal diffeomorphism could be locally defined in an open neighbourhood $\mathcal{U} \subset \Sigma$. In such case the equivalence holds in $\mathcal{U} \cap \phi(\mathcal{U})$. A detailed discussion is given in Chapter 2.

contains no general proof of these metrics being Λ-vacuum solutions, which was given later in [75]. Our chacterization actually shows in which sense these metrics are a natural extension of Kerr-de Sitter in four spacetime dimensions. The TT tensor (1.2) shares the basic properties with (1.1), namely, it is a TT tensor for every CKVF ξ whose Cauchy development is only determined by the conformal class of ξ (keeping γ fixed to be locally conformally flat). This property will allow us to define the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ in more dimensions by simply allowing ξ to be an arbitrary CKVF. In order to demonstrate the expected connection between ξ and the symmetries of the Cauchy development of $(\Sigma, \gamma, \kappa D_\xi)$, we extend the KID equation to arbitrary dimensions in Chapter 5. We prove that this equation gives a necessary and sufficient condition for the Cauchy development of analytic data with zero obstruction tensor (if n is even) to admit a Killing vector field. These restrictions, however, are not a problem in our setup because the Kerr-de Sitter-like class is indeed analytic and moreover, conformal flatness of $\mathscr{I}$ implies the vanishing of the obstruction tensor for the even dimensional boundary metrics.

From our study in terms of initial data an interesting method to generate solutions of the Einstein equation follows. The idea is to use the well-posedness of the initial value problem to obtain limits of spacetimes from limits if their initial data. In Chapter 6 we apply this to the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$. In addition, it should be stressed that in order to tackle these questions, a considerable amount of mathematical tools are required, some of them already discussed. We shall describe them in more detail in the following section.

1.3 Contents

The study outlined in the previous section is organized as follows. We start in Chapter 2 by discussing in more detail the mathematical tools that we shall require in the following chapters. The basic concepts of conformal geometry and asymptotics are given in Section 2.2, with a review in subsection 2.2.1 on the n-sphere and its conformal transformations and its relation with the orthochronous Lorentz group. In addition, we provide a note on local conformal flatness in subsection 2.2.2, which applies for $n > 2$. The $n = 2$ has some particularities (cf. Remark 2.16) and it is addressed in more detail in Chapter 3. In section 2.3 we discuss the Fefferman-Graham formalism and review its two equivalent formulations: in terms of ambient metrics (cf. subsection 2.3.1) and of Poincaré metrics (cf. subsection 2.3.2). Related to this, we also include an Appendix A where we derive the fundamental equations of the Fefferman-Graham formalism for Poincaré metrics, which play a basic role in this thesis. Finally, in Section 2.4 the initial value problems of general relativity are reviewed, in four spacetime dimension in subsection 2.4.1 and in higher dimension in subsection 2.4.2.

Except for some results on the local conformal group in locally conformally flat spaces described in Chapter 2, the original results of this thesis start in Chapter 3. Firstly, in Section 3.1 we give a list of general useful properties of skew-symmetric endomorphisms, which shall also be required in Chapter 4. Then, Sections 3.2, 3.3 and 3.4 are devoted to the obtention and analysis of a canonical form for any given (non-zero) skew-symmetric endomorphism F of $\mathbb{M}^{1,3}$. The set of all skew-symemtric endomorphisms of $\mathbb{M}^{1,3}$ is denoted SkewEnd($\mathbb{M}^{1,3}$). The change of basis that yields the canonical form is not unique. This implies the existence of an invariance group, that we derive in Section 3.3. In Section 3.4 we analyze the generators of the invariance group and obtain a decomposition of the element F in terms of these. We also make a connection between this decomposition and the standard duality rotations for two-forms. In all these sections, the three-dimensional case is obtained and discussed as a corollary of the four-dimensional one.

The following Sections 3.5, 3.6, 3.7, 3.8 are devoted to the study of so-called global CKVs (GCKV) defined on Euclidean space $\mathbb{E}^2$, and which are directly related to global CKVFs on the sphere $\mathbb{S}^2$. We remark that these are a particular subset among all CKVFs of $\mathbb{E}^2$ (cf. Remark 3.14). Section 3.5 defines such vectors and in Section 3.6 we revisit the connection between them and the Lie algebra SkewEnd($\mathbb{M}^{1,3}$), already discussed in all dimensions in subsection 2.2.1. In Section 3.7 we apply all the results for the SkewEnd($\mathbb{M}^{1,3}$) algebra to the GCKVs of the sphere, namely, the obtention of a canonical form and its invariance group. As a useful consequence of the two viewpoints, we are able (Corollary 3.21) to obtain in a fully explicit form the change of basis that transforms any given F into its canonical form. Finally, Section 3.8 gives a set of coordinates adapted to an arbitrary GCKV ξ and a second orthogonal GCKV $\xi^{\perp}$, readily obtainable from ξ. The results concerning the canonical form of GCKV and the adapted coordinates are summarized in Theorem 3.23. Our last Section 3.9 gives two interesting applications for the previous results. First, given a GCKV ξ, Theorem 3.24 gives a list of all metrics, conformal to the metric of a 2-sphere, for which ξ is a Killing vector. Second, Theorem 3.25 gives an elegant solution of the TT tensors satisfying the KID equations in open sets of $\mathbb{E}^2$.

The analysis of Chapter 3 is extended to arbitrary dimension in Chapter 4. It is worth to remark that the low dimensional case deserves its own chapter because of the level of detail that it allows, hardly achievable in arbitrary dimension. In order to properly define the canonical form, in Section 4.1 we rederive a classification result for skew-symmetric endomorphisms (cf. Theorem 4.6), employing only elementary linear algebra methods. The results of this section are known (see e.g. [73], [89]), but the method is original and we believe more direct than other approaches in the literature. Section 4.1 leads to the definition of canonical form in Section 4.2. Section 4.3 deals with a particular type of skew-symmetric endomorphisms (the so-called *simple*, i.e. of minimal matrix rank), which will be useful in the analysis of CKVFs in the second part of the chapter. In Section 4.4 we work out some applications of our canonical form: identifying invariants which characterize the conjugacy classes of the orthochronous Lorentz group (cf. Theorem 4.22) and obtaining the topological structure of this quotient space (cf. Section 4.4.1). It is remarkable that we obtain sequences, contained in open domains of the quotient topology, whose limit points are non-unique. In other words, we prove by working out some particular limits, that the quotient topology is non-Hausdorff. This is not so surprising for such kind of quotients, but we will find interesting consequences of this fact in the Cauchy problem of general relativity in Chapter 6.

In Section 4.5 we use the homomorphism between $O^+(1, n+1)$ and Conf $(\mathbb{S}^n)$, described in Section 2.2.1, and apply the canonical form obtained for skew-symmetric endomorphisms to give a canonical form for CKVFs, together with a decomposed form (cf. Proposition 4.33) which is analogous to the one given for skew-symmetric endomorphisms in Theorem 4.6. It should be remarked that the canonical form of the CKVFs also determines their equivalence class under conformal transformations (cf. Theorem 4.35). In Section 4.6, we adapt coordinates to CKVFs in canonical form, first in the even dimensional case, from which the odd dimensional case is obtained as a consequence. These coordinates are analyzed in depth, obtaining the domain of definition as well as the form of a flat metric in adapted coordinates. The analysis is summarized in Theorem 4.45. Finally, in Section 4.7 we employ the adapted coordinates to find the most general class of data at spacelike $\mathscr{I}$ corresponding to spacetime dimension four, such that $\mathscr{I}$ is conformally flat and the $(\Lambda > 0)$-vacuum spacetime they generate admits at least two symmetries, one of which is axial. It is remarkable how easily these equations are solved with all the tools developed before. The solution is worked out in adapted coordinates, but the final form is diffeomorphism and conformally covariant (cf. Theorem 4.47). With this solution at hand, we are able to identify the Kerr-de Sitter family within (cf. Corollary 4.51).

In Chapter 5 we address the arbitrary dimensional asymptotic Cauchy problem in the Fefferman-Graham picture. We begin, in Section 5.1, by deriving two useful formulas for the Weyl tensor and its electric part (cf. Lemmas 5.2 and 5.4), which have several applications in the remainder of the thesis. We discuss the consequences of both formulas and we conclude that the electric part of the rescaled Weyl tensor is, generically, divergent at $\mathscr{I}$, while it is not if $\mathscr{I}$ is conformally flat, a case on which we focus next.

Some applications of Lemmas 5.2 and 5.4 are found readily in subsection 5.1.1. These include the FG expansion of all $(\Lambda \neq 0)$-vacuum Einstein metrics with constant curvature (i.e. locally isometric to de Sitter or anti-de Sitter if the signature is Lorentzian) obtained in Lemma 5.8 (see also Remark 5.9), and the decomposition in Proposition 5.11 for all metrics admitting a conformally flat $\mathscr{I}$. Another consequence of the formulae for the Weyl tensor is Theorem 5.14, also proven in subsection 5.1.1. This result establishes that a well-defined *free (TT) part* $\mathring{g}_{(n)}$ of the n-th order coefficient of the FG expansion coincides (up to a certain constant) with D, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ in the case when $\mathscr{I}$ is conformally flat and $n > 3$ (for $n = 3$ this is true in full generality). This theorem finds immediate application in the Cauchy problem of the Einstein equations at $\mathscr{I}$ with positive cosmological constant (cf. Corollary 5.17). In addition, exploring the necessary conditions for $\mathring{g}_{(n)}$ and D to coincide up to a constant, we come to the conclusion that conformal flatness of $\mathscr{I}$ is not only sufficient, but actually necessary as long as no purely magnetic Λ-vacuum spacetimes exists, in addition to the trivial case of constant curvature. Remarkably, the non-existence of the latter is a longstading and still open conjecture in general relativity (cf. Remark 5.10).

In Section 5.2 we derive the KID equation for analytic data at $\mathscr{I}$ for n odd or n even provided that the obstruction tensor vanishes (we indicate that the result should also hold when the obstruction tensor is non-zero, but this requires additional analysis). This equation is necessary and sufficient for the Cauchy development of the data at $\mathscr{I}$ to admit a Killing vector field. Our final Section 5.3 gives an interesting application of the previous results. Namely, we calculate the initial data of the Kerr-de Sitter family of metrics in all dimensions [70]. As discussed above, these data happen to be a natural extension of the $n = 3$ case studied in [100] of the form $(\Sigma, \gamma, \kappa D_\xi)$, where (Σ, γ) is a locally conformally flat manifold and κD_ξ is a TT tensor determined by a CKVF ξ of $\mathscr{I}$ of the form (1.2) and a constant $\kappa \in \mathbb{R}$. Like in the $n = 3$ case, the data turn out to be uniquely characterized by the conformal class of ξ (cf. Lemma 5.21). The characterization is completed by identifying the conformal class which defines the Kerr-de Sitter family with the results in Chapter 4.

The final Chapter 6 of this thesis is a non-trivial and interesting application of the previous results of this thesis. It turns out that the data of the form $(\Sigma, \gamma, \kappa D_\xi)$ provide a good set of initial data no matter which CKVF ξ one chooses. By previous results, the Cauchy development is uniquely determined by the conformal class $[\xi]$. The starting point in Section 6.1 is to define the spacetime corresponding to data $(\Sigma, \gamma, \kappa D_\xi)$, with (Σ, γ) locally conformally flat and ξ an arbitrary CKVF of γ as the *Kerr-de Sitter-like class* with conformally flat $\mathscr{I}$ (which for short we shall simply call Kerr-de Sitter-like class). One of the main results of this chapter (cf. Theorem 6.5) proves that the spacetimes in the Kerr-de Sitter-like class amount to all Kerr-Schild type metrics which solve the $(\Lambda > 0)$-vacuum field equations and which share a smooth (conformally flat) $\mathscr{I}$ with its background (i.e. de Sitter) metric. These are called *Kerr-Schild-de Sitter* spacetimes. It should be noted that "sharing a smooth conformally flat $\mathscr{I}$ with its background metric" is, in principle, more than simply admitting a smooth conformally flat $\mathscr{I}$ (cf. Remark 6.4). The other main result (cf. Theorem 6.6) constructs all the spacetime metrics in the Kerr-de Sitter-like class.

The sections in Chapter 6 give a proof of both theorems. In Section 6.2 we prove that the Kerr-Schild-de Sitter spacetimes are contained in the Kerr-de Sitter-like class by direct calculation of their initial data at $\mathscr{I}$. One easily finds that the initial data have the form $(\Sigma, \gamma, \kappa D_\xi)$, where D_ξ is determined by a vector field ξ. The subtle part of the proof is to show that ξ is a CKVF of γ, which we find as a consequence of the Λ-vacuum Kerr-Schild spacetimes being algebraically special. The inclusion of the Kerr-de Sitter-like class in the Kerr-Schild-de Sitter spacetimes is obtained by direct construction of the metrics in the class. This is achieved from limits of the Kerr-de Sitter family of metrics because of the following argument. As the data $(\Sigma, \gamma, \kappa D_\xi)$, with (Σ, γ) locally conformally flat, are only determined by the conformal classes of CKVFs $[\xi]$, the structure of this quotient space is directly inherited by the space of initial data of the form $(\Sigma, \gamma, \kappa D_\xi)$. Recall that this quotient was studied in subsection 4.4.1 in terms of skew-symmetric endomorphisms, which is a representation of the algebra of CKVFs. From well-posedness of the Cauchy problem, the limits of data must induce limits of spacetimes. More precisely, in the n even case, all spacetimes in the Kerr-de Sitter-like class are limits of the Kerr-de Sitter family with none of its rotation parameters vanishing. In the n odd case is similar, except that there is one exceptional case obtained by analytic extensions of the Kerr-de Sitter family. In any case, the limits and analytic extensions obtained are given in explicit Kerr-Schild form, proving that they are Kerr-Schild-de Sitter. From this last part, it is remarkable that the existence of such limits is neat at the level of initial data and follows from our analysis in subsection 4.4.1. However, it would be hard to guess at the level of spacetimes directly.

Chapter 2

Preliminaries

2.1 Conventions, definitions and identities

We start by listing the conventions, definitions and identities that we shall use in this thesis. Unless otherwise specified, the convenion of indices in the manifolds is as follows:

1. Greek indices $\alpha, \beta, \gamma, \cdots$ range from from 0 to n.
2. Lower case latin indices $i, j, k, \cdots$ range from 1 to n.
3. Upper case latin indices $I, J, K, \cdots$ range form 0 to $n+1$.

In some situations where several spaces arise, it will not be possible to respect this general convention. Any exception of the above rules will be clearly indicated.

The identities below are given for an N dimensional space, for which we use lower case latin indices $a, b, c, \cdots$. In the main text they will be adapted to the criterion above, depending on the case.

Our convention for the Riemann tensor is such that for any covector X_c

$$R^c{}_{adb} X_c = -\nabla_d \nabla_b X_a + \nabla_b \nabla_d X_a.$$

The covariant Riemann tensor is

$$R_{cadb} := g_{ce} R^e{}_{adb},$$

where the index is always lowered with its defining metric. Given a Riemann tensor, its Ricci tensor and Ricci scalar are, respectively,

$$R_{ab} := R^c{}_{acb}, \qquad R := R_{ab} g^{ab}.$$

Let $g^{(1)}$ and $g^{(2)}$ be two different metrics and let $\nabla^{(1)}, \nabla^{(2)}$ be their respective Levi-Civita connections. The difference of connections $S := \nabla^{(1)} - \nabla^{(2)}$ is the tensor given by

$$S^c{}_{ab} = \frac{1}{2}(g^{(1)\sharp})^{cd}(\nabla^{(2)}_a g^{(1)}{}_{bd} + \nabla^{(2)}_b g^{(1)}{}_{ad} - \nabla^{(2)}_d g^{(1)}{}_{ab}), \tag{2.1}$$

where for any metric g, we use $g^\sharp$ to denote its associated contravariant metric in index-free notation. When using indices, we will omit the $\sharp$ symbol and write simply $(g^\sharp)^{cd} = g^{cd}$. From this relation between the connections (2.1), a formula for the difference of Riemann and Ricci tensors follows (e.g. [146])

$$R^{(1)c}{}_{adb} - R^{(2)c}{}_{adb} = 2\nabla^{(2)}{}_{[d}S^c{}_{b]a} - 2S^e{}_{[d|a|}S^c{}_{b]e}, \tag{2.2}$$

$$R^{(1)}{}_{ab} - R^{(2)}{}_{ab} = 2\nabla^{(2)}{}_{[c}S^c{}_{b]a} - 2S^e{}_{[c|a|}S^c{}_{b]e}. \tag{2.3}$$

Expression (2.1) and identities (2.2) and (2.3) can be also written using derivatives $\nabla^{(1)}$ of $g^{(2)}$

$$S^c{}_{ab} = -\frac{1}{2}(g^{(2)\sharp})^{cd}(\nabla^{(1)}_a g^{(2)}{}_{bd} + \nabla^{(1)}_b g^{(2)}{}_{ad} - \nabla^{(1)}_d g^{(2)}{}_{ab}), \tag{2.4}$$

and

$$\begin{aligned} R^{(1)c}{}_{adb} - R^{(2)c}{}_{adb} &= 2\nabla^{(1)}{}_{[d}S^c{}_{b]a} + 2S^e{}_{[d|a|}S^c{}_{b]e}, \\ R^{(1)}{}_{ab} - R^{(2)}{}_{ab} &= 2\nabla^{(1)}{}_{[c}S^c{}_{b]a} + 2S^e{}_{[c|a|}S^c{}_{b]e}. \end{aligned} \tag{2.5}$$

We will often work with conformally related metrics $g = \Omega^2 \widetilde{g}$, where Ω is a sufficiently differentiable positive function. Particularizing (2.1) to $g^{(1)} = \widetilde{g}$ and $g^{(2)} = g$ and letting $\nabla = \nabla^{(2)}$ gives

$$S^c{}_{ba} = -\frac{1}{\Omega}\left(T_b\delta^c{}_a + T_a\delta^c{}_b - T^c g_{ba}\right), \qquad T_c := \nabla_c\Omega, \quad T^c := g^{cd}T_d. \tag{2.6}$$

Given its importance in this thesis, we compute explicitly the transformation of the Riemann and Ricci tensors for conformally related metrics. On the one hand we have

$$\begin{aligned} \nabla_{[d}S^c{}_{b]a} =& \frac{1}{\Omega^2}\left(T_{[d}T_{b]}\delta^c{}_a + T_{[d}\delta^c{}_{b]}T_a - T_{[d}g_{b]a}T^c\right) \\ &- \frac{1}{\Omega}\left(\nabla_{[d}T_{b]}\delta^c{}_a + \nabla_{[d}T_{|a|}\delta^c{}_{b]} - \nabla_{[d}T^c g_{|a|b]}\right) \\ =& \frac{1}{\Omega^2}\left(T_aT_{[d}\delta^c{}_{b]} - T^cT_{[d}g_{b]a}\right) - \frac{1}{\Omega}\left(\nabla_{[d}T_{|a|}\delta^c{}_{b]} - \nabla_{[d}T^c g_{|a|b]}\right) \end{aligned}$$

hence

$$\nabla_{[c}S^c{}_{b]a} = -\frac{1}{\Omega^2}(N-2)T_bT_a - \frac{T_cT^c}{\Omega^2}g_{ba} + \frac{N-2}{\Omega}\nabla_bT_a + \frac{1}{\Omega}g_{ba}\nabla_cT^c.$$

For the quadratic term, we get

$$
\begin{aligned}
S^e{}_{da}S^c{}_{be} &= \frac{1}{\Omega^2}(T_d\delta^e{}_a + T_a\delta^e{}_d - T^e g_{da})\,(T_b\delta^c{}_e + T_e\delta^c{}_b - T^c g_{eb})\\
&= \frac{1}{\Omega^2}(T_dT_b\delta^c{}_a + 2T_dT_a\delta^c{}_b + T_aT_b\delta^c{}_d + T^cT_bg_{da}\\
&\quad -T_dT^cg_{ab} - T_aT^cg_{db} - T_eT^eg_{da}\delta^c{}_b - T^cT_bg_{da})
\end{aligned}
$$

so

$$
S^e{}_{[d|a|}S^c{}_{b]e} = \frac{1}{\Omega^2}\left(T_{[d}\delta^c{}_{b]}T_a - T^cT_{[d}g_{b]a} - T_eT^eg_{a[d}\delta^c{}_{b]}\right)
$$

and

$$
S^e{}_{[c|a|}S^c{}_{b]e} = \frac{(N-2)}{\Omega^2}\left(-T_aT_b + T_eT^eg_{ab}\right).
$$

Replacing T by $\nabla\Omega$, (2.2) and (2.3) give

$$
\widetilde{R}^c{}_{adb} - R^c{}_{adb} = \frac{2}{\Omega}\left(\delta^c{}_{[d}\nabla_{b]}\nabla_a\Omega - g_{a[d}\nabla_{b]}\nabla^c\Omega\right) + 2g_{a[d}\delta^c{}_{b]}\frac{\nabla_e\Omega\nabla^e\Omega}{\Omega^2}, \tag{2.7}
$$

$$
\widetilde{R}_{ab} - R_{ab} = \frac{N-2}{\Omega}\nabla_a\nabla_b\Omega + g_{ab}\frac{\nabla_c\nabla^c\Omega}{\Omega} - g_{ab}\frac{N-1}{\Omega^2}\nabla_c\Omega\nabla^c\Omega, \tag{2.8}
$$

for two conformal metrics $g = \Omega^2\tilde{g}$. We can also calculate the relation between the Ricci scalars, taking trace in (2.8) with g

$$
\frac{\widetilde{R}}{\Omega^2} - R = \frac{2(N-1)}{\Omega}\nabla_c\nabla^c\Omega - \frac{N(N-1)}{\Omega^2}\nabla_c\Omega\nabla^c\Omega. \tag{2.9}
$$

The Weyl tensor, defined as follows,

$$
C^c{}_{adb} := R^c{}_{adb} - \frac{2}{N-2}(\delta^c{}_{[d}R_{b]a} - g_{a[d}R^c{}_{|b]}) + \frac{2R}{(N-1)(N-2)}\delta^c{}_{[d}g_{b]a}, \tag{2.10}
$$

is fundamental in conformal geometry. It can be also written in terms of the Schouten tensor,

$$
P_{ab} := \frac{1}{N-2}\left(R_{ab} - \frac{R}{2(N-1)}g_{ab}\right)
$$

as

$$
C^c{}_{adb} = R^c{}_{adb} + 2P_{a[d}\delta^c{}_{b]} + 2g_{a[d}P^c{}_{b]}. \tag{2.11}
$$

Both tensors are specially taylored for conformal geometry. The transformation law for the Schouten tensor of conformally related metrics g and $\widetilde{g}$ respectively is, from identities (2.8) and (2.9),

$$
\widetilde{P}_{ab} - P_{ab} = \frac{1}{\Omega}\nabla_a\nabla_b\Omega - \frac{1}{2\Omega^2}g_{ab}\nabla_c\Omega\nabla^c\Omega \tag{2.12}
$$

and the well-known invariance of the Weyl tensor also follows

$$\widetilde{C}^c{}_{adb} - C^c{}_{adb} = 0.$$

We shall also use index-free notation. For any vector fields X, Y, Z, X and field of one-forms ω, the Riemann tensor

$$Riem(\omega, Z, X, Y) := R^c{}_{adb}\omega_c Z^a X^d Y^b,$$

or in its covariant version

$$Riem(W, Z, X, Y) := R_{cadb} W^c Z^a X^d Y^b.$$

The Ricci tensor and Ricci scalar

$$Ric(X, Y) = R_{ab} X^a Y^b, \qquad Scal := R.$$

Also, the Weyl and Schouten tensors

$$Weyl(\omega, Z, X, Y) := W^c{}_{adb}\omega_c Z^a X^d Y^b, \quad Sch(X, Y) = P_{ab} X^a Y^b.$$

When it is necessary to specify the metric, we shall do so with parentheses, e.g. $Ric(g)$ denotes the Ricci tensor associated to g.

One important type of tensor that will be relevant in this thesis is the so-called traceless and transverse (TT) tensors. These are symmetric two-covariant tensors $\widetilde{D}$ on an N-manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$ with zero trace (traceless) and zero divergence (transverse):

$$\widetilde{g}^{ab}\widetilde{D}_{ab} = 0, \qquad \nabla_a(\widetilde{g}^{ab}\widetilde{D}_{bc}) = 0.$$

In index free notation $\mathrm{Tr}_\gamma(\widetilde{D})$ indicates the trace and $\mathrm{div}_{\widetilde{g}}(\widetilde{D})$ the divergence. The latter, is a well-behaved operation under conformal scalings $g = \Omega^2\widetilde{g}$, with Ω a smooth positive function of $\widetilde{\mathcal{M}}$. In the following lemma we recall two well-known conformal covariance results, that will be required later. The proof is added for completeness.

Lemma 2.1. *Let $\widetilde{g}$ and g be conformally related metrics $g = \Omega^2\widetilde{g}$ on a manifold $\widetilde{\mathcal{M}}$, with $\widetilde{\nabla}$, ∇ their respective Levi-Civita connections. Let $\widetilde{D}_{ab}$ be a symmetric two-covariant traceless tensor and $C^c{}_{adb}$ a tensor with the symmetries of the Weyl tensor. Then the following identities hold*

$$\Omega^{-N}\widetilde{\nabla}_a(\widetilde{g}^{ab}\widetilde{D}_{bc}) = \nabla_a(g^{ab}\Omega^{2-N}\widetilde{D}_{bc}),$$
$$\Omega^{3-N}\widetilde{\nabla}_c C^c{}_{adb} = \nabla_c(\Omega^{3-N}C^c{}_{adb}).$$

Proof. Consider the difference of connections tensor S in (2.6). Then

$$\begin{aligned}
\nabla_a(g^{ab}\Omega^{2-N}\widetilde{D}_{bc}) &= \widetilde{\nabla}_a(g^{ab}\Omega^{2-N}\widetilde{D}_{bc}) - S^a{}_{ad}g^{db}\Omega^{2-N}\widetilde{D}_{bc} + S^d{}_{ac}g^{ab}\Omega^{2-N}\widetilde{D}_{bd} \\
&= \widetilde{\nabla}_a(\widetilde{g}^{ab}\Omega^{-N}\widetilde{D}_{bc}) \\
&\quad + N\Omega^{1-N}T_d g^{db}\widetilde{D}_{bd} - \Omega^{1-N}(T_a\delta^d{}_c + T_c\delta^d{}_a - T^d g_{ac})g^{ab}\widetilde{D}_{bd} \\
&= -N\Omega^{-N-1}\widetilde{g}^{ab}T_a\widetilde{D}_{bc} + \Omega^{-N}\widetilde{\nabla}_a(\widetilde{g}^{ab}\widetilde{D}_{bc}) \\
&\quad + N\Omega^{-N-1}T_d\widetilde{g}^{db}\widetilde{D}_{bd} - \Omega^{1-N}(T^b\widetilde{D}_{bc} + T_c g^{db}\widetilde{D}_{bd} - T^d\widetilde{D}_{cd}) \\
&= \Omega^{-N}\widetilde{\nabla}_a(\widetilde{g}^{ab}\widetilde{D}_{bc}),
\end{aligned}$$

where for the last equality we have used the traceless property of $\widetilde{D}$. For the second equality, first expand

$$\nabla_c(\Omega^{3-N}C^c{}_{adb}) = (3-N)\Omega^{2-N}T_c C^c{}_{adb} + \Omega^{3-N}\nabla_c C^c{}_{adb} \tag{2.13}$$

with

$$\nabla_c C^c{}_{adb} = \widetilde{\nabla}_c C^c{}_{adb} - S^c{}_{cs}C^s{}_{adb} + S^s{}_{ca}C^c{}_{sdb} + S^s{}_{cd}C^c{}_{asb} + S^s{}_{cb}C^c{}_{ads}. \tag{2.14}$$

Expanding the above expression and taking into account the symmetries of C and that all its traces vanish, we get

$$\begin{aligned}
S^c{}_{cs}C^s{}_{adb} &= -\frac{1}{\Omega}\left(T_c\delta^c{}_s + T_s\delta^c{}_c - T^c g_{cs}\right)C^s{}_{adb} = -\frac{N}{\Omega}T_s C^s{}_{adb}, \\
S^s{}_{ca}C^c{}_{sdb} &= -\frac{1}{\Omega}\left(T_c\delta^s{}_a + T_a\delta^s{}_c - T^s g_{ca}\right)C^c{}_{sdb} = -\frac{2}{\Omega}T_c C^c{}_{adb}, \\
S^s{}_{cd}C^c{}_{asb} &= -\frac{1}{\Omega}\left(T_c\delta^s{}_d + T_d\delta^s{}_c - T^s g_{cd}\right)C^c{}_{asb} = -\frac{1}{\Omega}\left(T_c C^c{}_{adb} - T^s C_{dasb}\right), \\
S^s{}_{cb}C^c{}_{ads} &= -\frac{1}{\Omega}\left(T_c\delta^s{}_b + T_b\delta^s{}_c - T^s g_{cb}\right)C^c{}_{ads} = -\frac{1}{\Omega}\left(T_c C^c{}_{adb} - T^s C_{bads}\right).
\end{aligned}$$

Inserting into (2.14) yields

$$\begin{aligned}
\nabla_c C^c{}_{adb} &= \widetilde{\nabla}_c C^c{}_{adb} + \frac{N-4}{\Omega}T_s C^s{}_{adb} + \frac{T^s}{\Omega}\left(C_{sbda} + C_{sdab}\right) \\
&= \widetilde{\nabla}_c C^c{}_{adb} + \frac{N-3}{\Omega}T_s C^s{}_{adb}
\end{aligned} \tag{2.15}$$

where in the first equality we have rearranged indices of the four-covariant terms the last equality is a consequence of the first Bianchi identity

$$C_{sbda} + C_{sdab} = -C_{sabd}.$$

Now the second equality of the Lemma follows by inserting (2.15) into (2.13) □

2.1.1 Geometry of submanifolds

Consider an $(n+1)$-dimensional manifold $(\mathcal{M}, g)$ and a local foliation, whose leaves $\Sigma_\Omega = \{\Omega = const.\}$ are defined by a sufficiently differentiable function Ω. We denote g_Ω to the covariant projector (i.e. the projector with two low indices) onto the n-submanifolds Σ_Ω and we assume that the normal covector, given by $T_\alpha := \nabla_\alpha \Omega$, is nowhere null. Its normal unit is denoted $u_\alpha := \nabla_\alpha \Omega / |\nabla_\mu \Omega \nabla^\mu \Omega|^{1/2}$ and $\epsilon := u_\mu u^\mu$ determines the causal character of the foliation. All indices in $\mathcal{M}$ are moved with g. From the definition of the projector, we can write de decomposition

$$g = -\frac{\mathrm{d}\Omega^2}{\nu} + g_\Omega, \tag{2.16}$$

where $-\nu$ is the lapse function, $-\nu := \nabla_\mu \Omega \nabla^\mu \Omega$ and clearly $\mathrm{sign}(\nu) = -\epsilon$. We can construct **Gaussian-like coordinates** $\{\Omega, x^i\}$ adapted to the foliation, by taking coordinates $\{x^i\}$ of an initial leaf $\{\Omega = 0\}$ and propagating them as $T^\alpha \partial_\alpha (x^i) = 0$. Then, g_Ω has no terms in $\mathrm{d}\Omega$ and thus coincides with the metric induced in the leaves Σ_Ω. When T is geodesic, the Gaussian-like coordinates are actually Gaussian coordinates. We shall need the following explicit expressions in Gaussian-like coordinates.

Let us fix a leaf Σ_Ω and let $\nabla^{(\Omega)}$ be the Levi-Civita connection induced by g_Ω. By (2.16), the tangent-tangent components to the leaves of the metric g satisfy $g_{ij} = (g_\Omega)_{ij}$. Then, it follows that for any two vector fields $y, w \in T\Sigma_\Omega$, the tangent components of the covariant derivative $\nabla_y w$ satisfy

$$y^i \nabla_i w^j = y(w^j) + \Gamma^j_{ik} y^i w^k.$$

The fully tangent components of the Christoffel symbols satisfy, by decomposition (2.16),

$$\begin{aligned} \Gamma^j_{ik} &= \frac{1}{2} g^{j\mu} (\partial_i g_{k\mu} + \partial_k g_{i\mu} - \partial_\mu g_{ik}) \\ &= \frac{1}{2} g^{jl}_\Omega (\partial_i (g_\Omega)_{kl} + \partial_k (g_\Omega)_{il} - \partial_l (g_\Omega)_{ik}) = \Gamma^{(\Omega)}{}^j_{ik} \end{aligned}$$

where $\Gamma^{(\Omega)}{}^j_{ik}$ are the Christoffel symbols of the metric g_Ω. Thus

$$y^i \nabla_i w^j = y^i \nabla^{(\Omega)}_i w^j.$$

The same rule extends to all tangential derivatives of all tensors in Σ_Ω. The normal component of the $\nabla_y w$ can be written

$$y^i (\nabla_i w^\mu) u_\mu = -y^i w^j \nabla_i u_j =: -y^i w^j K_{ij}$$

The tangent components of $K := \nabla u$ define the *second fundamental form* of Σ_Ω. It is immediate to check that it is a symmetric tensor because

$$y^i w^j \nabla_i u_j = -u^\mu y^i \nabla_i w_\mu = -u^\mu (w^i \nabla_i y_\mu + [y,w]_\mu) = -u^\mu w^i \nabla_i y_\mu = w^i y^j \nabla_i u_j$$

where we have used that $[y,w] = \nabla_y w - \nabla_y w$ is tangential to Σ_Ω, hence orthogonal to u.

The second fundamental form can be also expressed in terms of the Lie derivative $\mathcal{L}_u g_\Omega$. It is easier to derive this in index-free notation:

$$\begin{aligned}
\mathcal{L}_u\left(g(y,w)\right) &= \mathcal{L}_u\left(g_\Omega(y,w)\right) = (\mathcal{L}_u g_\Omega)(y,w) + g_\Omega(\mathcal{L}_u y, w) + g_\Omega(y, \mathcal{L}_u w) \\
&= (\mathcal{L}_u g_\Omega)(y,w) + g_\Omega(\nabla_u y, w) + g_\Omega(y, \nabla_u w) - g_\Omega(\nabla_y u, w) - g_\Omega(y, \nabla_w u) \\
&= (\mathcal{L}_u g_\Omega)(y,w) + \underbrace{\nabla_u (g_\Omega(y,w))}_{=\mathcal{L}_u(g_\Omega(y,w))} - 2K(y,w),
\end{aligned}$$

from which it follows

$$K_{ij} = \frac{1}{2}(\mathcal{L}_u g_\Omega)_{ij} = \nabla_i u_j.$$

Summarizing, we have obtained the well-known Gauss formula (e.g. [40])

$$\nabla_y w = \nabla_y^{(\Omega)} w - \epsilon K(y,w) u.$$

From this, one can easily derive two fundamental formulas for the ambient curvature in terms of the geometry of the submanifolds. We do not include their derivation for the sake of brevity (see standard references, e.g. [40], [110]).

The first one is the *Gauss identity*, which relates the tangent components of the Riemann tensor of g with the Riemann tensor of g_Ω, denoted by ${R^{(\Omega)}}_{ijkl}$:

$$R_{ijkl} = {R^{(\Omega)}}_{ijkl} + \epsilon \left(K_{il} K_{jk} - K_{ik} K_{jl} \right). \tag{2.17}$$

The second is the *Codazzi identity* and gives the one-normal, three-tangential component of the Riemann tensor of g

$${R^\mu}_{jkl} u_\mu = (\nabla_k^{(\Omega)} K_{lj} - \nabla_l^{(\Omega)} K_{kj}), \tag{2.18}$$

where recall that $\nabla_k^{(\Omega)} K_{lj} = \nabla_k K_{lj}$ in Gaussian-like coordinates.

2.2 Basics on conformal geometry

In this section we review the basic tools on conformal geometry that we shall use in this thesis.

Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a Lorentzian $(n+1)$-manifold. The *causal structure* of a spacetime $(\widetilde{\mathcal{M}}, \widetilde{g})$ is the assignation of a null cone $\mathcal{N}_p$ at the tangent space of each point $p \in \widetilde{\mathcal{M}}$. This is of great physical relevance as it determines the causal character of curves in the manifold, which in turn establishes which two points are physically accesible from one another. The causal character of a submanifold is given by the signature of its first fundamental form γ. A submanifold is *timelike* if γ is Lorentzian; *spacelike* if it is positive definite and *null* if it is degenerate. For the latter we will also use *degenerate* submanifold.

The causal structure of a Lorentzian manifold is closely related to its conformal geometry. We start by giving basic definitions.

Definicin 2.2. Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a manifold. Then:

1. A metric g of $\widetilde{\mathcal{M}}$ is said to be **conformal** to $\widetilde{g}$ if there exists a smooth, positive function Ω of $\widetilde{\mathcal{M}}$ such that $g := \Omega^2 \widetilde{g}$.
2. The set of all conformal metrics $[\widetilde{g}]$ in $\widetilde{\mathcal{M}}$ is called **conformal class** of $\widetilde{g}$ and **conformal structure** of $(\widetilde{\mathcal{M}}, \widetilde{g})$.
3. A manifold $\widetilde{\mathcal{M}}$ equipped with a conformal structure $[\widetilde{g}]$ is called a **conformal manifold** $(\widetilde{\mathcal{M}}, [\widetilde{g}])$.

It is obvious that in the Lorentzian case a conformal structure determines a causal structure for the manifold. The converse statement is also true (e.g. [78]), namely, a causal structure on a spacetime $(\mathcal{M}, \widetilde{g})$ determines a unique conformal structure on $\widetilde{\mathcal{M}}$.

Analogously, one defines a conformal transformation between different manifolds equipped with metrics as follows:

Definicin 2.3. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two manifolds. A **conformal map** $\phi : \mathcal{M}_1 \to \mathcal{M}_2$, is a smooth map satisfying $\phi^\star(g_2) = \Omega^2 g_1$, for a smooth positive function Ω of $\mathcal{M}_1$. When $\mathcal{M}_1$ and $\mathcal{M}_2$ are of the same dimension, conformal maps are required to define a diffeomorphism between $\mathcal{M}_1$ and its image. When $\mathrm{Im}(\mathcal{M}_1) = \mathcal{M}_2$, ϕ is called a **conformal diffeomorphism**. If $(\mathcal{M}_1, g_1) = (\mathcal{M}_2, g_2)$, the set of conformal diffeomorphisms, denoted $\mathrm{Conf}(\mathcal{M}_1, g_1)$, is a group under composition called **conformal group** of $(\mathcal{M}_1, g_1)$.

The next notion we introduce is the conformal infinity and conformal extensions of metrics. Let us consider an $(n+1)$-dimensional manifold $\mathcal{M}$ with (n-dimensional) boundary $\partial\mathcal{M}$ and denote its interior $\widetilde{\mathcal{M}} := \mathrm{Int}(\mathcal{M})$. Let $\widetilde{g}$ be a smooth metric defined on $\widetilde{\mathcal{M}}$, but not necessarily at the boundary. We allow $\widetilde{g}$ to be pseudo-Riemannian of any signature.

Definicin 2.4. A pseudo-Riemannian manifold $\widetilde{g}$ in $\widetilde{\mathcal{M}}$ is said to be **conformally extendable** if there exists a manifold with boundary $\mathcal{M}$ such that $\widetilde{\mathcal{M}} := \mathrm{Int}(\mathcal{M})$ and a smooth function Ω of $\mathcal{M}$ positive in $\widetilde{\mathcal{M}}$ such that

$$\partial\mathcal{M} = \{\Omega = 0 \cap \mathrm{d}\Omega \neq 0\}, \qquad \text{and} \qquad g := \Omega^2 \widetilde{g},$$

is (at least) C^2-extendable to $\partial\mathcal{M}$. Then $(\mathcal{M}, g)$ is said to be a **conformal extension** of $(\widetilde{\mathcal{M}}, \widetilde{g})$.

The notion of conformal extension can be equivalently formulated by means of conformal diffeomorphisms. Namely, a manifold with boundary $(\mathcal{M}, g)$ is a conformal extension of $(\widetilde{\mathcal{M}}, \widetilde{g})$ if there exists a conformal diffeomorphism $\phi : \mathrm{Int}(\mathcal{M}) \rightarrow \widetilde{\mathcal{M}}$ such that Ω extends as a C^2 function to $\mathcal{M}$ and at $\partial\mathcal{M}$ it holds $\Omega = 0$ and $\mathrm{d}\Omega \neq 0$. The equivalence follows by the identification of $\widetilde{\mathcal{M}}$ with $\mathrm{Int}(\mathcal{M})$ by ϕ. We shall understand a conformal extension as given in Definition 2.4.

The requirements of differentiability of Definition 2.4 are minimal so that curvature tensors can be defined at $\partial\mathcal{M}$. In most cases we deal with smooth extensions, but it is useful to have a broader definition in order to accommodate the Fefferman-Graham formalism (cf. Section 2.3).

For a given conformal extension $(\mathcal{M}, g)$, the boundary geometry is given by its first fundamental form $\gamma := g|_{\partial\mathcal{M}}$ so that the manifold $\mathscr{I} := (\partial\mathcal{M}, \gamma)$ represents the asymptotic behaviour of $(\widetilde{\mathcal{M}}, \widetilde{g})$ and it is called "conformal infinity" or "null infinity". One can always scale g with a smooth positive function of $\mathcal{M}$ so that $g' = \omega^2 g$ induces a different first fundamental form $\gamma' = \omega^2|_{\mathscr{I}}\gamma$. Then, in order to define a notion of conformal infinity independent of the particular extension, one considers the manifold $\mathscr{I} := (\partial\mathcal{M}, [\gamma])$, where $[\gamma]$ is the class of bilinear forms obtained from γ by scaling with any smooth positive function. Obviously, when γ is non-degenerate, γ is a metric and $[\gamma]$ a conformal class of metrics in $\partial\mathcal{M}$.

So far, we have not imposed $\widetilde{g}$ nor g to satisfy any equations. In this thesis we will be interested in (physical) metrics $\widetilde{g}$ which satisfy the Λ-vacuum Einstein equations,

$$Ric(\widetilde{g}) - n\lambda\widetilde{g} = 0, \qquad \lambda := \frac{2\Lambda}{n-1}, \tag{2.19}$$

with particular emphasis in the case of positive cosmological constant Λ and Lorentzian signature. However, it is also interesting to weaken this condition. In particular, we will only impose (2.19) to be "asymptotically satisfied" to order m, for whatever sign of non-zero λ and signature (n_+, n_-). We will not in general consider vanishing cosmological constant. We say that (2.19) is asymptotically satisfied to order m if

$$Ric(\widetilde{g}) - n\lambda\widetilde{g} = O(\Omega^m), \qquad \lambda := \frac{2\Lambda}{n-1}, \tag{2.20}$$

where O and o are Landau's big O and little o. In particular, if $m \geq 1$, the equations and all their derivatives to order m vanish at $\Omega = 0$.

Definicin 2.5. A conformally extendable metric $\widetilde{g}$ is **asymptotically Einstein to order m** if it satisfies (2.20).

The weakest notion of asymptotically Einstein metrics that we shall deal with are the so-called [96] *asymptotically of constant curvature* (ACC).

Definicin 2.6. A metric $\widetilde{g}$ is **asymptotically of constant curvature (ACC)** if it asymptotically Einstein with $m = -1$.

Before justifying the name, let us remark that this is a generalization of the class of asymptotically hyperbolic metrics (e.g. [49, 72]), which correspond to the ACC case with negative λ and Riemannian signature.

Obviously, for ACC metrics, the Einstein tensor diverges at $\mathscr{I}$, but "relatively slowly", so one is still imposing interesting asymptotic conditions. From the transformation formula (2.8) for the Ricci tensors of conformal metrics $g = \Omega^2\widetilde{g}$, we have

$$\Omega R_{\alpha\beta} + (n-1)\nabla_\alpha\nabla_\beta\Omega + g_{\alpha\beta}\nabla_\mu\nabla^\mu\Omega = \Omega\left(\widetilde{R}_{\alpha\beta} + \widetilde{g}_{\alpha\beta} n \nabla_\mu\Omega\nabla^\mu\Omega\right), \tag{2.21}$$

where all indices are moved with the metric g. If $\widetilde{g}$ is ACC, we have

$$\begin{aligned} & \Omega R_{\alpha\beta} + (n-1)\nabla_\alpha\nabla_\beta\Omega + g_{\alpha\beta}\nabla_\mu\nabla^\mu\Omega = \Omega\left(\widetilde{g}_{\alpha\beta} n\left(\lambda + \nabla_\mu\Omega\nabla^\mu\Omega\right) + O(\Omega^{-1})\right) \\ \Longrightarrow \quad & \Omega\left(\widetilde{g}_{\alpha\beta} n\left(\lambda + \nabla_\mu\Omega\nabla^\mu\Omega\right)\right) = \Omega R_{\alpha\beta} + (n-1)\nabla_\alpha\nabla_\beta\Omega + g_{\alpha\beta}\nabla_\mu\nabla^\mu\Omega + O(1). \end{aligned}$$

By construction, the RHS extends to $\Omega = 0$, hence so it does the LHS,

$$\Omega\widetilde{g}_{\alpha\beta} n\left(\lambda + \nabla_\mu\Omega\nabla^\mu\Omega\right) = \Omega^{-1} g_{\alpha\beta} n\left(\lambda + \nabla_\mu\Omega\nabla^\mu\Omega\right),$$

which implies $\nabla_\mu\Omega\nabla^\mu\Omega|_{\mathscr{I}} = -\lambda$. On the other hand, if $\nabla_\mu\Omega\nabla^\mu\Omega|_{\mathscr{I}} = -\lambda$ holds, then it follows immediatelly from (2.21) that $\widetilde{g}$ must be ACC. Thus, we have proven,

Lemma 2.7. *A conformally extendable metric $\widetilde{g}$ is ACC if and only if for every conformal extension $g = \Omega^2\widetilde{g}$ it is satisfied*

$$g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\Big|_{\mathscr{I}} = -\lambda.$$

Using the relation (2.7) for the Riemann tensors of two conformal metrics $g = \Omega^2\widetilde{g}$ (with S given by (2.6)), and performing a computation to leading order in Ω, one readily obtains

$$\widetilde{R}_{\mu\alpha\nu\beta} = -g^{\sigma\lambda}\nabla_\sigma\Omega\nabla_\lambda\Omega(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu})\Omega^{-4} + O(\Omega^{-3}).$$

Hence, for every pair of linearly independent vectors X, Y of $\mathcal{M}$ spanning a non-null plane, the sectional curvature is given by

$$\mathcal{K}(X,Y) := \frac{\widetilde{Riem}(X,Y,X,Y)}{\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2} = -g^{\sigma\lambda}\nabla_\sigma\Omega\nabla_\lambda\Omega + O(\Omega).$$

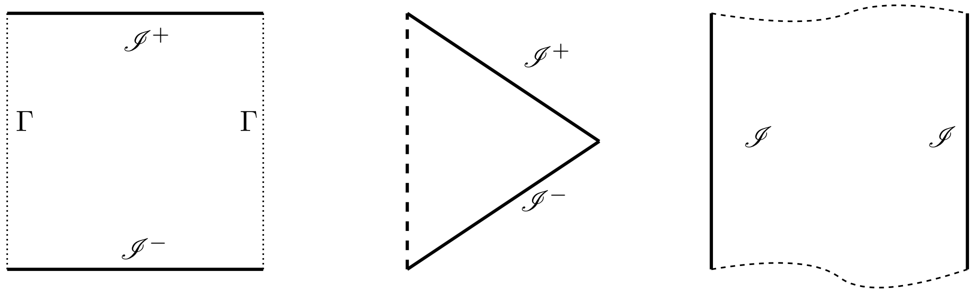

FIGURE 2.1: *Conformal diagrams of de Sitter, Minkowski and Anti-de Sitter*

Therefore, if $\widetilde{g}$ is ACC, the sectional curvature tends asymptotically to the constant value $-\lambda$. This justitifies the name "asymptotically of constant curvature" given to the metrics.

Imposing the relatively weak condition of a metric being ACC suffices to determine the causal character of $\mathscr{I}$, because its normal vector has norm given by $g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega = -\lambda$ (independently on the choice of conformal extension). Null infinity is therefore spacelike when $\lambda > 0$, null when $\lambda = 0$ and timelike when $\lambda < 0$. In particular, in the cases of positive or zero λ, $\mathscr{I}$ has generically two components (Figure 2.1) $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$. The (future) component $\mathscr{I}^+$ has empty intersection with the past of every point and the (past) component $\mathscr{I}^-$ has empty intersection with the future of every point. This does not happen in the case of negative λ because $\mathscr{I}$ is timelike (Figure 2.1). The causal character of $\mathscr{I}$ is particularly relevant for the asymptotic Cauchy problem of the Λ-vacuum Einstein metrics. Being Einstein is obviously stronger than being ACC, so all results proven in this section for ACC metrics also hold for that case too. In a similar manner, they also hold for intermediate notions of asymptoticity (weaker than Einstein and stronger than ACC) to be introduced in Section 2.3.

Note that in the non-zero λ cases, the first fundamental form γ at $\mathscr{I}$ is a non-degenerate bilinear form, thus a metric. From now one we restrict ourselves to $\lambda \neq 0$ and call γ the *boundary metric.*

With the notion of ACC metrics, we can already introduce a useful type of conformal extensions, called *geodesic*, as well as basic existence results.

Definicin 2.8. A conformal extension $(\mathcal{M}, g)$ of $(\widetilde{\mathcal{M}}, \widetilde{g})$ is **geodesic** if T is geodesic affinely parametrized w.r.t. g. Namely

$$T^\mu \nabla_\mu T_\nu = 0.$$

The necessary and sufficient condition for a conformal extension of an ACC metric to be geodesic is given in the next lemma.

Lemma 2.9. *Let $\widetilde{g}$ be an ACC metric. Then, a conformal extension $g = \Omega^2\widetilde{g}$ is geodesic if and only if*

$$\nabla_\alpha\Omega\nabla^\alpha\Omega = -\lambda.$$

Proof. The lemma follows from

$$\nabla^\beta\Omega\nabla_\beta\nabla_\alpha\Omega = \nabla^\beta\Omega\nabla_\alpha\nabla_\beta\Omega = \frac{1}{2}\nabla_\alpha\left(\nabla_\beta\Omega\nabla^\beta\Omega\right), \tag{2.22}$$

because if $\nabla_\alpha\Omega\nabla^\alpha\Omega = -\lambda$ the RHS of (2.22) vanishes and T is geodesic. Conversely, if T is geodesic then (2.22) is zero, so $\nabla_\alpha\Omega\nabla^\alpha\Omega$ is constant and, g being ACC, its value is everywhere equal to $-\lambda$. □

Another important result concerning geodesic conformal extensions is whether, for a given boundary metric γ, there exists one such conformal extension. The answer is that there always exists a unique geodesic conformal extension realizing a given γ. The proof (see also [72] for a similar argument) relies on the method of characteristics, which we describe briefly. For further details we refer to Chapter 3 of [47].

In a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M} = \Sigma$, consider a first order PDE Cauchy problem with inital data at Σ

$$F(x^\alpha; f, \nabla_\alpha f) = 0, \qquad f\mid_\Sigma = \phi, \tag{2.23}$$

where f is a scalar function. By the collar neighbourhood theorem [25], there exists a neighbourhood of $\partial\mathcal{M}$ which can be diffeomorphically mapped into a neighbourhood of $[0,\infty)\times\Sigma$. We consider coordinates $\{x^\alpha\} = \{\Omega, x^i\}$ adapted to this Cartesian product, where $\{x^i\}$ are coordinates on Σ, which is identified with $\{0\}\times\Sigma \subset [0,\infty)\times\Sigma$. Two functions $\{\phi, \psi_0\}$ of Σ are a set of *admissible initial data* whenever they satisfy the following *compatibility condition*[1]

$$F(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial\phi}{\partial x^1}, \cdots, \frac{\partial\phi}{\partial x^n}) = 0. \tag{2.24}$$

Denote by $\mathcal{D}_{\nabla_\alpha f}F$ the derivative of F w.r.t. $\nabla_\alpha f$ and let $V(x^\alpha; f, \nabla_\alpha f)$ be the vector of components $V^\alpha = \mathcal{D}_{\nabla_\alpha f}F$. Also, let T be the normal covector to Σ, i.e. $T_\alpha = \nabla_\alpha\Omega$. Then, for every set of admissible initial data, the Cauchy problem is said to be non-characteristic if

$$T\cdot V(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial\phi}{\partial x^1}, \cdots, \frac{\partial\phi}{\partial x^n}) \neq 0,$$

where $\cdot$ denotes the usual action of a covector on a vector. A non-characteristic Cauchy problem is known to be locally well-posed (e.g. [47]), i.e. that there exists a unique solution f of (2.23), satisfying $f|_\Sigma = \phi,\ \partial_0 f|_\Sigma = \psi_0$.

After this reminder, we can prove the next lemma.

[1] Compared to the the general setup of [47], the compatibility condition takes this simple form precisely because of the use of coordinates $\{\Omega, x^i\}$.

Lemma 2.10. *Let $\widetilde{g}$ be an ACC metric for $\lambda \neq 0$ with conformal infinity $(\Sigma, [\gamma])$. Then, for each representative $\gamma \in [\gamma]$, there exist a unique geodesic conformal extension $g = \Omega^2 \widetilde{g}$ which induces the metric γ at Σ.*

Proof. Consider a conformally extended metric g such that $g = \Omega^2 \widetilde{g}$ and $g \mid_{\Omega=0} = \gamma$. Let $\hat{g} \in [\widetilde{g}]$ be such that $\hat{g} = \omega^2 g$ with $\omega > 0$ and $\omega \mid_{\Omega=0} = 1$, so that $\hat{g}$ realizes the same boundary metric γ. Therefore $\hat{g} = \hat{\Omega}^2 \widetilde{g}$, with $\hat{\Omega} = \omega \Omega$, so by Lemma 2.9, we have to show that there exists a function ω such that $\hat{\Omega}$ satisfies (2.22) for the metric $\hat{g}$

$$\hat{g}^{\alpha\beta} \nabla_\alpha \hat{\Omega} \nabla_\beta \hat{\Omega} = \frac{g^{\alpha\beta}}{\omega^2} \nabla_\alpha (\omega \Omega) \nabla_\beta (\omega \Omega) = -\lambda.$$

Expanding the derivatives and defining $f := \log \omega$, this amounts to

$$g^{\alpha\beta} \left(2 \nabla_\alpha \Omega \nabla_\beta f + \Omega \nabla_\alpha f \nabla_\beta f \right) = \frac{-\lambda - g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega}{\Omega}. \tag{2.25}$$

The LHS of (2.25) is obviously regular at $\Omega = 0$. Also, since g is ACC

$$g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega \mid_{\Omega=0} = -\lambda,$$

thus the RHS tends to $-\partial_\Omega \left(g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega \right)$ at $\Omega = 0$, which has finite value at $\mathscr{I}$. Hence, we can pose a Cauchy problem at $\{\Omega = 0\}$, for which we must complete $\phi = \log \omega \mid_\Sigma = 0$ to admissible initial data for (2.25). These data must satisfy (2.24), thus ψ_0 is fixed to satisfy $2 g^{00} \psi_0 = -\partial_\Omega (g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega) \mid_\Sigma$. Observe that this is the unique possible set of admissible data once ϕ has been fixed. The vector field V has components $2 g^{\alpha\beta} (\nabla_\beta \Omega + \Omega \nabla_\beta u)$ and therefore

$$T \cdot V(x^0 = 0, x^i; \phi, \psi_0, \frac{\partial \phi}{\partial x^1}, \cdots, \frac{\partial \phi}{\partial x^n}) = 2 g^{\alpha\beta} \nabla_\alpha \Omega \nabla_\beta \Omega \mid_\Sigma = -2\lambda.$$

Hence the problem is non-characteristic if $\lambda \neq 0$. Existence and uniqueness follows the well-posedness result mentioned above. □

2.2.1 The conformal sphere $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$

We now introduce the construction of the conformal n-sphere as the projective cone in Minkowski $\mathbb{M}^{1,n+1}$ (see e.g. [133]). This procedure allows one to construct the conformal group of the sphere, $\mathrm{Conf}(\mathbb{S}^n)$ (cf. Definition 2.3), from the isotropies of $\mathbb{M}^{1,n+1}$. The conformal sphere, i.e. the n-sphere equipped with its conformal structure $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$, is specially relevant because of its relation with local conformal flatness (cf. subsection 2.2.2) as well as the Fefferman-Graham formalism (cf. Section 2.3).

Let us consider $\mathbb{M}^{1,n+1}$ endowed with Minkowskian coordinates $\{x^I\}_{I=0}^{n+1}$, so that the Minkowski metric is

$$g_L = \eta_{IJ} \mathrm{d}x^I \mathrm{d}x^J$$

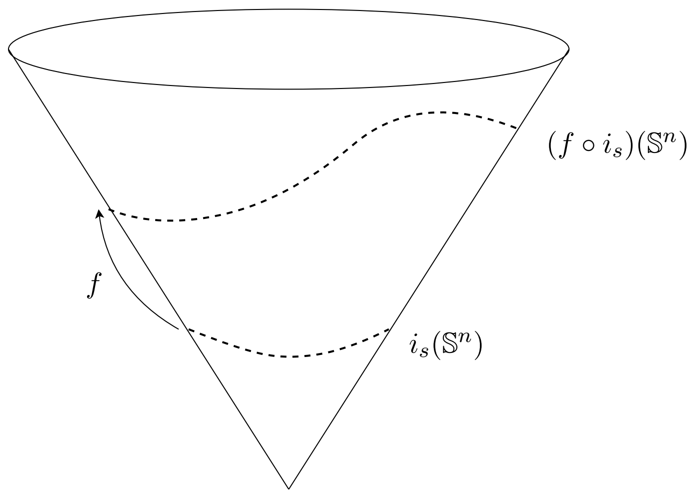

Figure 2.2: *A spherical section of the cone and a diffeomorphic arbitrary section.*

where η_{IJ} is $\eta_{00} = -1$ and $\eta_{II} = 1$, for $I = 1, \cdots, n+1$, and the rest of components are zero. Let us define the null cone $\mathcal{N} := \{x \in \mathbb{M}^{1,n+1} \mid g_L(x,x) = 0, \quad x \neq 0\}$. In $\mathcal{N}$, we define the equivalence relation $x \sim x'$ iff $x = \kappa x'$ for κ a non-zero real number. The projective cone is the quotient $\mathcal{N}/\sim$, which we canonically represent with the section $\{x^0 = 1\} \cap \mathcal{N}$. This section is identified with an n-sphere $\mathbb{S}^n$ via the isometric embedding $\iota_S : \mathbb{S}^n \hookrightarrow \mathcal{N}$, such that $\iota_S(\mathbb{S}^n) = \{x^0 = 1\} \cap \mathcal{N}$ and $\gamma_{\mathbb{S}^n} = \iota_S^\star(g_L)$ is the usual spherical metric. Now consider the scaling map $f : \mathbb{M}^{1,n+1} \to \mathbb{M}^{1,n+1}$, $x \mapsto f(x) = \omega(x)x$ for an arbitrary smooth positive function ω. Notice that $f(\iota_S(\mathbb{S}^n)) \subset \mathcal{N}$, so for each point $q \in \mathbb{S}^n$, $f(x_q) = \omega(x_q)x_q$, where $x_q = \iota_S(q)$ and we use Minkowskian coordinates in $\mathbb{M}^{1,n+1}$. This generates an arbitrary smooth section of the cone (Figure 2.2), which is also an n-dimensional submanifold. We may pullback now the metric by $\iota'_S := f \circ \iota_S$ and compare to the original spherical metric $\iota_S^\star(g_L)$. In order to simplify the notation, we do not specify in every step where each object is evaluated. Firstly

$$\begin{aligned}(f \circ \iota_S)^\star(g_L) =& \iota_S^\star(f^\star(g_L)) = \iota_S^\star(\eta_{IJ}\mathrm{d}(f(x))^I \mathrm{d}(f(x))^J) \\ =& \iota_S^\star\big(\eta_{IJ}x^I x^J (\mathrm{d}\omega)^2 \\ &+ \omega(x)\eta_{IJ}(x^I \mathrm{d}x^J \mathrm{d}\omega + x^J \mathrm{d}\omega \mathrm{d}x^I) + \omega(x)^2 \eta_{IJ}\mathrm{d}x^I \mathrm{d}x^J\big),\end{aligned}$$

where note that the "cross terms" can be written

$$\eta_{IJ}(x^I \mathrm{d}x^J \mathrm{d}\omega + x^J \mathrm{d}\omega \mathrm{d}x^I) = \frac{1}{2}(\mathrm{d}(\eta_{IJ}x^I x^J)\mathrm{d}\omega + \mathrm{d}\omega \mathrm{d}(\eta_{IJ}x^I x^J)).$$

Then, since $\iota_S^\star(\eta_{IJ}x^I x^J)_q = (\eta_{IJ}x_q^I x_q^J) = 0$ and $\iota_S^\star\big(\mathrm{d}(\eta_{IJ}x^I x^J)\big)_q = \mathrm{d}(\eta_{IJ}x_q^I x_q^J) = 0$ (because $\eta_{IJ}x_q^I x_q^J$ is constant equal to zero along $\mathbb{S}^n$), it follows

$$(f \circ \iota_S)^\star(g_L)_q = \iota_S^\star\left(\omega^2 \eta_{IJ}\mathrm{d}x^I \mathrm{d}x^J\right)_q = \omega^2(x_q)\iota_S^\star(g_L)_q.$$

Thus, the pullback metric at $\mathbb{S}^n$ is the original spherical metric scaled by ω^2. That is,

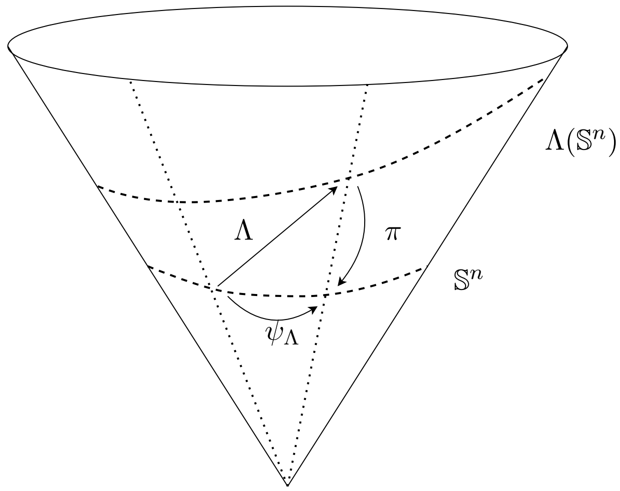

FIGURE 2.3: *Two spherical sections of the cone. The composition of transformations $\pi \circ \Lambda$ defines a conformal tranformation of the sphere ψ_Λ.*

by scaling points of $\mathbb{S}^n$ along the generators of the cone we generate new sections which are conformal to $\mathbb{S}^n$. In other words, the projectivization of the null cone amouts to the n-sphere equipped with its conformal structure $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$.

The conformal sphere constructed as the projective cone in $\mathbb{M}^{1,n+1}$ can be used to generate the set of all conformal diffeomorphisms of $\mathbb{S}^n$. Firstly, the Lorentz group $O(1,n+1)$ acts by isometries on $\mathcal{N}$. Thus, the action of an element $\Lambda \in O(1,n+1)$ on $\mathbb{S}^n$ generates a new section $\Lambda(\mathbb{S}^n)$ of the null cone, which must also be spherical. Therefore, for each $\Lambda \in O(1,n+1)$ there corresponds one transformation ψ_Λ of the conformal group of diffeomorphisms of $\mathbb{S}^n$, $\mathrm{Conf}(\mathbb{S}^n)$, which assigns to each $x \in \mathbb{S}^n$ the point $\psi_\Lambda(x) \in \mathbb{S}^n$ given by $\psi_\Lambda(x) := (\pi \circ \Lambda)(x)$, where π is the projection $\pi : \mathcal{N} \to \mathbb{S}^n$ (Figure 2.3). Conversely [133], for each $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ one can find exactly two transformations $O(1,n+1)$, Λ_+ and Λ_- such that $\psi = \pi \circ \Lambda_\pm$. One of these transformations Λ_+ preserves the time orientation, while Λ_- reverses it. Hence, the conformal group $\mathrm{Conf}(\mathbb{S}^n)$ is in one-to-one correspondence with the orthochronous component of the Lorentz group $O^+(1,n+1)$. Moreover, since the action of $O^+(1,n+1)$ is well-defined on rays on $\mathcal{N}$, the correspondence is an homomorphism of groups, because $\psi_\Lambda \circ \psi_{\Lambda'} = \pi \circ \Lambda \circ \pi \circ \Lambda' = \pi \circ \Lambda \circ \Lambda' = \psi_{\Lambda \circ \Lambda'}$, where note that the second equality holds precisely because Λ has a well-defined action on the rays of the cone.

For calculations, it is often useful to give a representation of $\mathrm{Conf}(\mathbb{S}^n)$ in the set of conformal transformations of the Euclidean space $\mathbb{E}^n$. Observe that we do not use the word "group" because, as we will see next, it is not a group globally acting on $\mathbb{E}^n$ (its action is only local in a precise sense). This set is denoted $\mathrm{ConfLoc}(\mathbb{E}^n)$ and we will later give an abstract general definition in subsection 2.2.2. The n-sphere $\mathbb{S}^n$, embedded as above in $\mathbb{M}^{1,n+1}$, may be projected into an n-dimensional spacelike plane of $\mathbb{E}^n \subset \{x^0 = 1\} \subset \mathbb{M}^{1,n+1}$ via $St_N : \mathbb{S}^n \to \mathbb{E}^n$, the stereographic projection, defined w.r.t. to a pole $N \in \mathbb{S}^n$ and at a signed distance d from $\mathbb{S}^n$ to $\mathbb{E}^n$ (cf. Figure 2.4). The relation between the metric $\gamma_{\mathbb{E}^n}$ of $\mathbb{E}^n$ and $\gamma_{\mathbb{S}^n}$ is well-known to be conformal (e.g. [133]). Hence, for each conformal diffeomorphism $\psi_\Lambda \in \mathrm{Conf}(\mathbb{S}^n)$ the maps of the form $\phi_\Lambda := St_N \circ \psi_\Lambda \circ (St_N)^{-1}$ are conformal transformations of $\mathbb{E}^n$. In the particular cases where $\psi_\Lambda(N) = N$, ϕ_Λ is an affine transformation [133], thus a (global) conformal diffeomorphism of $\mathbb{E}^n$. In any other case, ϕ_Λ gives a conformal diffeomorphism of $\mathbb{E}^n \backslash \{p_1, p_2\}$, where the points $p_1, p_2 \in \mathbb{E}^n$, satisfy $N = \psi_\Lambda(St_N^{-1}(p_1))$ and $N = \psi_\Lambda^{-1}(St_N^{-1}(p_2))$. Since the points p_1, p_2 depend on the particular transformation ψ_Λ one must proceed carefully with the maps ϕ_Λ, because the domain where they are well-defined changes under composition. However, one can easily see that, away from conflictive points, it holds

$$\phi_\Lambda \circ \phi_{\Lambda'} = \psi_\Lambda \circ \psi_{\Lambda'} = \psi_{\Lambda \circ \Lambda'} = \phi_{\Lambda \circ \Lambda'},$$

so the composition preserves the group law. The set of conformal transformations of $\mathbb{E}^n$ obtained from $\mathrm{Conf}(\mathbb{S}^n)$ as just described is denoted $\mathrm{ConfLoc}(\mathbb{E}^n)$.

Notice that there is a certain freedom in the above construction, such as the choice of section of the cone to represent the projectivization, as well as the pole N and distance d in the definition of St_N. We now see how this freedom can be absorbed in the choice of coordinates and flat metric.

Consider Minkowskian coordinates $\{x^I\}$ of $\mathbb{M}^{1,n+1}$ and let us pick the section of the cone $\mathbb{S}^n = \mathcal{N} \cap \{x^0 = 1\}$. Any other spherical section of the future cone $\mathbb{S}'^n$ is related to $\mathbb{S}^n$ by a transformation $\Lambda \in O^+(1, n+1)$, i.e. $\mathbb{S}'^n = \Lambda(\mathbb{S}^n)$. In the coordinates defined by $x'^I = (\Lambda^{-1})^I{}_J x^J$, the section $\mathbb{S}'^n$ looks the same as $\mathbb{S}^n$ in coordinates $\{x^I\}$, that is $\mathbb{S}'^n = \mathcal{N} \cap \{x'^0 = 1\}$. Hence, for any representative we can assume Minkowskian coordinates of $\mathbb{M}^{1,n+1}$ into which $\mathbb{S}^n = \mathcal{N} \cap \{x^0 = 1\}$. In a similar way, any two possible poles N and N' in $\mathbb{S}^n$ are related by an $SO(n)$ transformation. So the same idea applies and we may by default select Minkowskian coordinates of $\mathbb{M}^{1,n+1}$ into which N is given by $x^0 = -x^1 = 1$ and the rest of components are zero. Finally, let two Euclidean n-planes $\mathbb{E}^n$ and $\mathbb{E}'^n$, both lying in the hyperplane $\{x^0 = 1\}$, at respective signed distances d and d' of N (neither equal lo zero) and equipped with metrics (induced by g_L) $\gamma_{\mathbb{E}^n}$ and $\gamma_{\mathbb{E}'^n}$ respectively. It is immediate that $\gamma_{\mathbb{E}^n}$ and $\gamma_{\mathbb{E}'^n}$ must be homothetic to each other (cf. Figure 2.4). Hence, for a given flat metric $\gamma_{\mathbb{E}^n}$, d fixes a scale $s^2 \gamma_{\mathbb{E}^n}, s^2 \in \mathbb{R}$. So if we allow $\gamma_{\mathbb{E}^n}$ to be scaled by a constant[2], the distance d may be set $d = 2$.

Summarizing, w.l.o.g. we may consider Minkowskian coordinates $\{x^I\}$ of $\mathbb{M}^{1,n+1}$ such that $\mathbb{S}^n = \{x^0 = 1\} \cap \mathcal{N}$ and $N = (1, -1, 0, \cdots, 0)$. In addition, we may also fix $d = 2$ by setting an adequate flat metric $\gamma_{\mathbb{E}^n}$, so we consider $\mathbb{E}^n = \{x^0 = x^1 = 1, y^A := x^{A+1}\}$, where $\{y^A\}$, for $A = 1, \cdots, n$, are Cartesian coordinates of $\gamma_{\mathbb{E}^n}$ inherited from $\mathbb{M}^{1,n+1}$. For some applications, we may be given a flat space with a fixed flat metric $(\mathbb{E}^n, \gamma_{\mathbb{E}^n})$ and Cartesian coordinates $\{y^A\}$. Then, we shall embed $(\mathbb{E}^n, \gamma_{\mathbb{E}^n})$ in $\mathbb{M}^{1,n+1}$ as the n-submanifold $\mathbb{E}^n = \{x^0 = x^1 = 1, y^A = x^{A+1}\}$ so the above conventions hold.

[2]This is actually innocuous in problems with conformal equivalence.

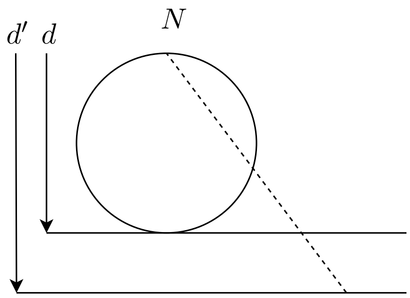

FIGURE 2.4: *Stereographic projection w.r.t. to pole N and distance d. The arrow indicates growing positive distance. For difference distances d and d', the planes are homothetic when vertically identified.*

With the above choices we have constructed the map

$$\begin{aligned} \phi : O^+(1,n+1) &\longrightarrow \mathrm{ConfLoc}(\mathbb{E}^n), \\ \Lambda &\longmapsto \phi_\Lambda. \end{aligned}$$

The differential of this map transforms the respective Lie algebras of the groups. The Lie algebra of $O^+(1,n+1)$ consists of the set of two-forms of Minkowski, or equivalently, the set of skew-symmetric endomorphisms, to which an important part of this thesis is devoted. This set will be denoted $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$. On the other hand, the Lie algebra of $\mathrm{ConfLoc}(\mathbb{E}^n)$ is [133] the set of conformal Killing vector fields (CKVFs) of the metric $\gamma_{\mathbb{E}^n}$. These are vector fields ξ satisfying

$$\mathcal{L}_\xi \gamma_{\mathbb{E}^n} = \frac{2}{n}(\mathrm{div}_{\gamma_{\mathbb{E}^n}} \xi)\gamma_{\mathbb{E}^n},$$

that we shall denote $\mathrm{CKill}(\mathbb{E}^n)$. We next give, for later use, the explicit form of the differential map

$$\begin{aligned} \phi_\star : \mathrm{SkewEnd}(\mathbb{M}^{1,n+1}) &\longrightarrow \mathrm{CKill}(\mathbb{E}^n), \\ F &\longmapsto \phi_\star(F) =: \xi. \end{aligned}$$

In order to emphasize the fact that a CKVF is the image by $\phi_\star$ of a skew-symmetric endomorphism F, we write ξ_F. Conversely, if F is the preimage by $\phi_\star$ of a CKVF ξ, we shall use the notation $F(\xi)$. Besides, recall that ϕ preserves the group law, i.e. $\phi_\Lambda \circ \phi_{\Lambda'} = \phi_{\Lambda\circ\Lambda'}$, so in this sense it is a morphism of groups. As a consequence of this (cf. Theorem 2.11), the action of $O^+(1,n+1)$ on $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$, also maps to the action of $\mathrm{ConfLoc}(\mathbb{E}^n)$ on $\mathrm{CKill}(\mathbb{E}^n)$. Similarly, the differential $\phi_\star$ is a Lie algebra (anti)homomorphism[3].

Theorem 2.11 below gives the explicit form of the differential $\phi_\star$, as given in [100]. In this reference, the Killing vector fields of Minkowski are first mapped into the Poincaré disk model of the $(n+1)$-dimensional hyperboloid of future unit timelike vectors. The conformal infinity of the $(n+1)$-dimensional Poincaré disk $\mathbb{D}^{n+1}$ is $\mathbb{S}^n$, and the Killing vector fields extend to CKVFs of $\mathbb{S}^n$. The disk $\mathbb{D}^{n+1}$ is then mapped to the hyperbolic half plane model via an inversion map, in such a way that the boundary $\mathbb{S}^n$ maps to the boundary $\mathbb{E}^n$ by a stereographic projection. In this way, CKVFs of $\mathbb{E}^n$ are obtained from CKVFs of $\mathbb{S}^n$. The whole procedure, with the various choices made in [100], is equivalent to the construction described above, including the choices for the stereographic projection and representative of the projective cone.

[3]Observe that by switching the sign in the CKVFs $\xi'_F := -\xi_F$ the antihomorphism becomes an homomorphism $[\xi'_F, \xi'_G] = [\xi_F, \xi_G] = -\xi_{[F,G]} = \xi'_{[F,G]}$. We keep the sign which gives the usual form of a CKVF.

The matrix in Theorem 2.11 is an endomorphsim with entries $F^I{}_J$, where the upper index ^{I} stands for row and the lower index $_J$ stands for column. Acting on vectors $v = v^I e_J$ of $\mathbb{M}^{1,n+1}$ gives

$$F(v) = F^I{}_J v^J e_I.$$

Teorema 2.11. *[[100]] Let $\mathbb{M}^{1,n+1}$ be endowed with Minkowskian coordinates $\{x^I\}$ and consider any element $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$ written in the basis $\{\partial_{x^I}\}$ in the form*

$$F = \begin{pmatrix} 0 & -\nu & -\mathrm{a}^t + \mathrm{b}^t/2 \\ -\nu & 0 & -\mathrm{a}^t - \mathrm{b}^t/2 \\ -\mathrm{a} + \mathrm{b}/2 & \mathrm{a} + \mathrm{b}/2 & -\boldsymbol{\omega} \end{pmatrix}, \tag{2.26}$$

where $\mathrm{a}, \mathrm{b} \in \mathbb{R}^n$ are column vectors, ^{t} stands for the transpose and $\boldsymbol{\omega}$ is a skew-symmetric $n \times n$ matrix ($\boldsymbol{\omega} = -\boldsymbol{\omega}^t$). Then, in the Cartesian coordinates $\{y^A\}$ of $\mathbb{E}^n$ defined by the embedding $i : \mathbb{E}^n \hookrightarrow \mathbb{M}^{1,n+1}$, $i(\mathbb{E}^n) = \{x^0 = x^1 = 1, x^{A+1} = y^A\}$, the image by $\phi_\star$ of F is the CKVF

$$\xi_F = \left(\mathrm{a}^A + \nu y^A + (\mathrm{a}_B y^B) y^A - \frac{1}{2}(y_B y^B)\mathrm{a}^A - \omega^A{}_B y^B \right) \partial_{y^A}. \tag{2.27}$$

Moreover, $\xi_{Ad_\Lambda(F)} = \phi_{\Lambda\star}(\xi_F)$, where $Ad_\Lambda(F) := \Lambda \cdot F \cdot \Lambda^{-1}$, for every $\Lambda \in O^+(1, n+1)$ and $\phi_\star$ is a Lie algebra (anti)homomorphism, i.e. $[\xi_F, \xi_G] = -\xi_{[F,G]}$.

Observacin 2.12. *For later use, we write explicitly the parameters of the vector field $\nu, \mathrm{a}^A, \mathrm{a}^A, \omega^A{}_B$ in terms of the entries $F^I{}_J$ of the endomorphism F:*

$$\begin{aligned} \nu &= -F^0{}_1, & \mathrm{a}_A &= -\frac{1}{2}\left(F^0{}_{A+1} + F^1{}_{A+1}\right), \\ \mathrm{b}_A &= \frac{1}{2}\left(F^0{}_{A+1} - F^1{}_{A+1}\right), & \omega^A{}_B &= -F^{A+1}{}_{B+1}. \end{aligned} \tag{2.28}$$

where capital Latin indices are lowered with the Kronnecker δ_{AB}.

2.2.2 Local conformal transformations and local conformal flatness

The conformal diffeomorphisms of a manifold (Σ, γ) need not to be globally defined, as we have seen in the case of $\mathbb{E}^n$ (cf. subsection 2.2.2). This raises a difficulty for establishing conformal equivalences of global objects, such as global vector fields, because if ϕ is only defined in an open neighbourhood $\phi : \mathcal{U} \to \Sigma$, any conformal relation between vector fields must be restricted to $\mathcal{U}$ and $\phi(\mathcal{U})$. In the particular case of locally conformally flat manifolds we can use the conformal sphere as a reference to make these relations global.

Following [148], we define:

Definicin 2.13. A Riemannian n-manifold (Σ, γ) is **locally conformally flat** if there exists an open cover $\{\mathcal{V}_a\}$ of Σ and a collection of conformal maps $\{\chi_a\}$ from $\mathcal{V}_a$ to the n-sphere, $\chi_a : \mathcal{V}_a \to \mathbb{S}^n$. The set of pairs $\{\mathcal{V}_a, \chi_a\}$ is called a **conformal cover**.

A conformal cover $\{\mathcal{V}_a, \chi_a\}$ is said to be **maximal** if every possible conformal map $\chi_b : \mathcal{V}_b \to \mathbb{S}^n$ of a domain $\mathcal{V}_b \subset \Sigma$ is contained in $\{\mathcal{V}_a, \chi_a\}$.

Observe that a maximal conformal cover $\{\mathcal{V}_a, \chi_a\}$ of (Σ, γ) can always be constructed as the union of every conformal cover. It is also clear that the maximal cover is unique.

We next prove that the maximal conformal cover provides a cover of the sphere:

Lemma 2.14. *Given the maximal conformal cover $\{\mathcal{V}_a, \chi_a\}$ of a locally conformally flat manifold (Σ, γ), the images $\{\mathcal{W}_a := \mathcal{V}_a(\chi_a)\}$ are a cover of $\mathbb{S}^n$.*

Proof. The group of diffeomorphisms $\mathrm{Conf}(\mathbb{S}^n)$ acts transitively on the sphere (note that it contains $SO(n)$). As a consequence, given any $(\mathcal{V}_b, \chi_b) \in \{\mathcal{V}_a, \chi_a\}$ the set of all neighourboods $(\psi \circ \chi_b)(\mathcal{V}_b)$ generated with every $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ covers $\mathbb{S}^n$. Now, since every $\chi'_b := \psi \circ \chi_b$ is a conformal map from $\mathcal{V}_b$ to $\mathbb{S}^n$, it must be contained in the maximal cover and the lemma follows. □

From now on, we shall assume that every locally conformally flat manifold (Σ, γ) is endowed with its maximal conformal cover. Next, we define the local conformal transformations of (Σ, γ) as follows

Definicin 2.15. A map $\phi : \mathcal{U} \to \Sigma$, where $\mathcal{U} \subset \Sigma$ is an open set, is called a **local diffeomorphism of** Σ if ϕ is a diffeomorphism of $\mathcal{U}$ onto its image. The set **ConfLoc($\boldsymbol{\Sigma}, \boldsymbol{\gamma}$)** is the set of local diffeomorphisms such that $\phi^\star(\gamma|_{\phi(\mathcal{U})}) = \omega^2 \gamma|_{\mathcal{U}}$, for a positive smooth function on $\mathcal{U}$.

Observacin 2.16. *In the following discussion, global extendability of the conformal transformations and CKVFs of the n-sphere will be key. This property is true for every conformal transformation and CKVF of $\mathbb{S}^n$ and dimension $n > 2$ [20]. For $n = 2$, $\mathbb{S}^2$ admits non-global conformal transformations, as an indirect consequence of its complex structure (cf. Remark 3.14). Nevertheless, note that all the global transformations $\mathrm{Conf}(\mathbb{S}^2)$ are also generated from the orthochronous Lorentz group $O^+(1,3)$ by the procedure explained in subsection 2.2.1 (e.g. [133]).*

In a locally conformally flat 2-manifold (Σ, γ), the non-global conformal transformations of $\mathbb{S}^2$ as well as the global conformal transformations $\mathrm{Conf}(\mathbb{S}^2)$, induce transformations of $\mathrm{ConfLoc}(\Sigma, \gamma)$ which are not a priori distinguishable. To avoid this difficulty, we shall restrict ourselves to the $n > 2$ case in this subsection. The $n = 2$ case will be described in detail in Chapter 3.

Let (Σ, γ) be a locally conformally flat manifold. We want to establish a relationship between $\mathrm{ConfLoc}(\Sigma, \gamma)$ and $\mathrm{Conf}(\mathbb{S}^n)$. We start by showing that to each transformation $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ one can associate maps $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$. Choose a conformal map $\chi_b : \mathcal{V}_b \to \mathbb{S}^n$. As a consequence of Lemma 2.14 and restricting $\mathcal{V}_b$ if necessary, the image

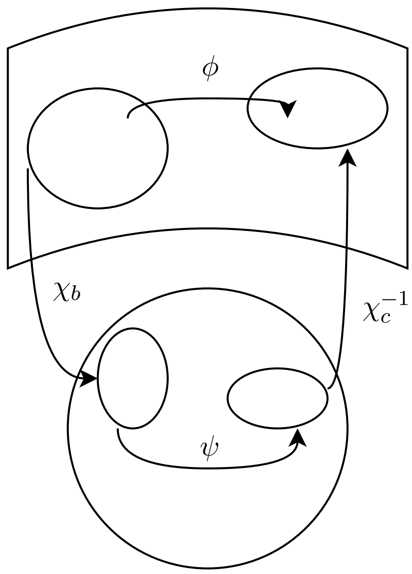

FIGURE 2.5: *Relation between elements* $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ *and* $\psi \in \mathrm{Conf}(\mathbb{S}^n)$.

$\psi(\chi_b(\mathcal{V}_b))$ lies in the image of some map χ_c in the maximal cover. Then $\phi := \chi_c^{-1} \circ \psi \circ \chi_b$ is clearly an element of $\mathrm{ConfLoc}(\Sigma, \gamma)$ (Figure 2.5). One can construct as many elements of $\mathrm{ConfLoc}(\Sigma, \gamma)$ as conformal maps χ_c exist in the maximal cover satisfying the required condition. Also, observe that the transitivity property of $\mathrm{Conf}(\mathbb{S}^n)$ induces a transitivity property in $\mathrm{ConfLoc}(\Sigma, \gamma)$ in the sense that the map ϕ can always be constructed so that $\phi(p) = q$ for any two given points $p, q \in \Sigma$. Indeed, such ϕ can be constructed from any $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ satsifying $\psi(\chi_b(p)) = \chi_c(q)$.

Conversely, to each $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ defined in a neighbourhood $\mathcal{U} \subset \Sigma$, one can locally associate a map ψ. Let $(\mathcal{V}_b, \chi_b)$ and $(\mathcal{V}_c, \chi_c)$ belong to the maximal conformal cover $\{\mathcal{V}_a, \chi_a\}$ of Σ and satisfy that the intersections $\mathcal{U} \cap \mathcal{V}_b$ and $\phi(\mathcal{U}) \cap \mathcal{V}_c$ are non-empty. The map $\psi := \chi_c \circ \phi \circ \chi_b^{-1}$ is well-defined on $\chi_b(\mathcal{U} \cap \mathcal{V}_b) \subset \mathbb{S}^n$ and it is obviously a conformal map. It is a fundamental property of the conformal group of the sphere [133], that there always exists a unique element $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ extending the previous map to the whole sphere. As before, the assingment of a given element $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ to an element of $\mathrm{Conf}(\mathbb{S}^n)$ is highly non-unique. Thus, there is no one-to-one correspondence between $\mathrm{ConfLoc}(\Sigma, \gamma)$ and $\mathrm{Conf}(\mathbb{S}^n)$. However, as we show next this correspondence provides a useful notion of conformal class for (local) conformal vector fields in (Σ, γ).

Before doing this, let us relate $\mathrm{ConfLoc}(\mathbb{E}^n)$, as constructed in subsection 2.2.1, with the abstract definition of ConfLoc. Recall that a map $\phi \in \mathrm{ConfLoc}(\mathbb{E}^n)$, constructed from a $\psi \in \mathrm{Conf}(\mathbb{S}^n)$, defines a diffeomorphism in $\mathbb{E}^n$ minus two points, which correspond with the preimage and the image of north pole N by ψ. When $\psi(N) \neq N$, the map ϕ is a so-called Möbius transformation [20] and takes the explicit form

$$\phi(y) = K \frac{R(y - p_1)}{|y - p_1|^2} + p_2, \tag{2.29}$$

where $K \in \mathbb{R}^+$, R is a rotation and p_1, p_2 are the points in $\mathbb{E}^n$ associated to ϕ as described above. This defines a map $\mathbb{E}^n \backslash \{p_1\} \to \mathbb{E}^n \backslash \{p_2\}$. When $\psi(N) = N$, ϕ is an affine transformation of $\mathbb{E}^n$, hence a global diffeomorphism. Given an open set $\mathcal{U} \subset \mathbb{E}^n$ the elements of $\mathrm{ConfLoc}(\Sigma, \gamma_{\mathbb{E}^n})$ whose domain is $\mathcal{U}$ are precisely the collection of Möbius transformations (2.29) satisfying $p_1, p_2 \in \mathbb{E}^n \setminus \mathcal{U}$, together with the set of all affine transformations.

We have now the necessary tools to define the notion of conformal class of CKVFs. We define:

Definicin 2.17. Let ξ be a CKVF of a Riemannian manifold (Σ, γ). The **conformal class** of ξ is the set of all CKVFs ξ' defined in some non-empty open neighbourhood $\mathcal{U}$ and generated by an element $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ whose domain is $\mathcal{U}$. Specifically, it consists of all fields $\phi_\star(\xi|_{\mathcal{U}}) = \xi'|_{\phi(\mathcal{U})}$. A conformal class is said to be **global** if $\mathcal{U} = \phi(\mathcal{U}) = \Sigma$.

This definition is local, and nothing guarantees that ξ' can be extended to a global CKVF in Σ. However, when Σ is locally conformally flat, we can show that there is a precise sense in which this local conformal class can be put in a one-to-one correspondance with a global conformal class in the sphere. We do this next.

Let (Σ, γ) be a locally conformally flat manifold and a global CKVF ξ. Let ξ' be an element of the conformal class of ξ and let $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ be the map relating them, defined in a neighbourhood $\mathcal{U} \subset \Sigma$. Let also $(\mathcal{V}_b, \chi_b)$ and $(\mathcal{V}_c, \chi_c)$ be pairs in the maximal conformal cover of (Σ, γ) with non-empty intersections $\mathcal{U} \cap \mathcal{V}_b$ and $\phi(\mathcal{U}) \cap \mathcal{V}_c$. Denote their images as $\mathcal{W}_b = \chi_b(\mathcal{U} \cap \mathcal{V}_b)$ and $\mathcal{W}_c = \chi_c(\phi(\mathcal{U}) \cap \mathcal{V}_c)$. One can locally assign CKVFs of $\mathbb{S}^n$ in $\mathcal{W}_b$ and $\mathcal{W}_c$, through the maps χ_b and χ_c, i.e. $\zeta := \chi_{b\star}(\xi)$ and $\zeta' := \chi_{c\star}(\xi')$. The sphere being simply connected, it follows easily that ζ, ζ' extend uniquely to global CKVFs in the sphere (as each one of them is the generator of a unique $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ [133]). The vector fields ζ, ζ' are locally related by the map $\psi := \chi_c \circ \phi \circ \chi_b^{-1}$, which obviously satisfies $\psi \in \mathrm{Conf}(\mathbb{S}^n)$ and we have already mentioned that ψ extends to an element in $\mathrm{Conf}(\mathbb{S}^n)$. The relation $\psi_\star(\zeta) = \zeta'$ is global because $\psi_\star(\zeta)$ is a CKVF that equals ζ' in $\mathcal{W}_c$, so it must equal ζ' everywhere, by the uniqueness of extensions of CKVFs on the sphere.

The vector field ζ, associated to a given global CKVF ξ of (Σ, γ), depends on the element $(\mathcal{V}_b, \chi_b)$ of the maximal cover used to define it. However, let $(\mathcal{V}_b, \chi_b)$ and $(\mathcal{V}_c, \chi_c)$ in the maximal cover have domains with non-empty intersection, i.e. $\mathcal{V}_b \cap \mathcal{V}_c \neq \emptyset$. In $\mathbb{S}^n$, define the CKVFs of $\zeta_b := \chi_{b\star}(\xi)$ and $\zeta_c := \chi_{c\star}(\xi)$. Then, the map $\psi := \chi_c \circ \chi_b^{-1}$ restricted to $\chi_b(\mathcal{V}_b \cap \mathcal{V}_c)$ satisfies $\psi_\star(\zeta_b) = \zeta_c$. But ψ extends to a global map in $\mathrm{Conf}(\mathbb{S}^n)$ and, by the argument above, this relation also extends globally to $\mathbb{S}^n$. Therefore, the vector fields ζ_b, ζ_c associated to ξ are in the same global conformal class of $\mathbb{S}^n$ if $\mathcal{V}_b$ and $\mathcal{V}_c$ intersect. Moreover, if Σ is connected, this is true even if $\mathcal{V}_b \cap \mathcal{V}_c = \emptyset$, because $\mathcal{V}_b$ and $\mathcal{V}_c$ can be joined through a finite sequence of neighbourhoods[4] $\{\mathcal{V}_k\}_{i=1}^K$ in the maximal cover $\{\mathcal{V}_k, \chi_k\}_{k=1}^K \subset \{\mathcal{V}_a, \chi_a\}$ such that $\mathcal{V}_k \cap \mathcal{V}_{k+1} \neq \emptyset$ and $\mathcal{V}_1 = \mathcal{V}_b$ and $\mathcal{V}_K = \mathcal{V}_c$. In $\mathcal{V}_k \cap \mathcal{V}_{k+1}$, the map $\psi_k = \chi_{k+1}^{-1} \circ \chi_k$ establishes a conformal map. All such maps, extended globally in $\mathbb{S}^n$ and combined $\psi := \psi_1 \circ \cdots \circ \psi_{K-1}$, determine a conformal relation $\zeta_b = \psi_\star(\zeta_c)$.

Thus, the above discussion shows that all CKVFs in the conformal class of ξ and ξ' of a connected, locally conformally flat manifold Σ, determine a unique global conformal class of CKVF in $\mathbb{S}^n$. The converse is also true because of the following argument. Let $(\mathcal{V}_b, \chi_b)$ belong to the maximal conformal cover and consider $\zeta = \chi_{b\star}(\xi)$ and $\zeta' = \psi_\star(\zeta)$ for any $\psi \in \mathrm{Conf}(\mathbb{S}^n)$. Then, as a consequence of Lemma 2.14, there exists a pair $(\mathcal{V}_c, \chi_c)$ in the maximal conformal cover such that $\chi_c(\mathcal{V}_c) \cap \psi(\chi_b(\mathcal{V}_b)) \neq \emptyset$. Hence, in $\chi_c(\mathcal{V}_c) \cap \psi(\chi_b(\mathcal{V}_b))$ the vector field ζ' induces, via χ_c^{-1}, a CKVF ξ' of γ. By construction, the map $\phi := \chi_c^{-1} \circ \psi \circ \chi_b$ belongs to $\mathrm{ConfLoc}(\Sigma, \gamma)$ and satisfies $\phi_\star(\xi) = \xi'$ on a non-empty domain. Thus, ξ' is in the conformal class of ξ. Summarizing

Proposicin 2.18. *Let (Σ, γ) be a Riemannian, connected and locally conformally flat n-manifold with $n > 2$. Then, the conformal classes of CKVF in (Σ, γ) as given in Definition 2.17 are in one-to-one correspondence with global conformal classes of CKVFs of $\mathbb{S}^n$.*

2.3 Fefferman-Graham expansion

The results in this thesis concerning general relativity in dimensions higher than four are based on the Fefferman and Graham formalism (see the seminal paper [48], later expanded into the monograph [50]). This framework was originally intended for the obtention of conformal invariant quantities (specially scalars) for a given conformal structure $(\Sigma, [\gamma])$ of dimension n and signature (n_+, n_-). From $(\Sigma, [\gamma])$, two constructions emerge which are in a precise sense equivalent. The first one, which is actually the main object of study by Fefferman and Graham, are the ambient metrics $g_{\mathcal{A}}$. These are $(n+2)$-dimensional metrics of signature $(n_+ + 1, n_- + 1)$ living in a so-called ambient space $\mathcal{G}$. The space $\mathcal{G}$ contains a hypersurface $\mathcal{N}$, whose projectivization yields $(\Sigma, [\gamma])$. The second construction are the Poincaré metrics, which are asymptotically Einstein metrics of dimension $n+1$ and signature $(n_+ + 1, n_-)$ or $(n_+, n_- + 1)$, conformally extendable with $\mathscr{I} = (\Sigma, [\gamma])$.

The model example for ambient and Poincaré metrics is the conformal n-sphere $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$, obtained as the projectivization of the null cone $\mathcal{N}$ in Minkowski space $\mathbb{M}^{1,n+1}$ (cf. subsection 2.2.1) or, as we show next, the conformal infinity of the Riemannian hyperboloid

[4]Connected manifolds are path connected so there exists a continuous curve $\alpha : [0,1] \to \Sigma$ joining a point $p \in \mathcal{V}_b$ with a point $q \in \mathcal{V}_c$. The set of points $\alpha([0,1])$ is compact, so from any cover one can extract a finite subcover. It suffices to start with the full cover $\{\mathcal{V}_a\}$ associated to the maximal conformal cover, extract a finite subcover and, in necessary, supplement with $\mathcal{V}_b$, $\mathcal{V}_c$ to fulfill all the properties that we require.

and of the Lorentzian hyperboloid (i.e. de Sitter space). The Minkowski spacetime of dimension $n+2$ is an ambient space for $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$ and each $(n+1)$-dimensional hyperboloid is a Poincaré metric for $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$.

Let us consider the hyperboloids $\mathcal{H}_\lambda := \{x \in \mathbb{M}^{1,n+1} \mid g_L(x,x) = \lambda\}$, for λ a non-zero real constant. The sign of λ determines whether $\mathcal{H}_\lambda$ is a one-sheeted hyperboloid, i.e. de Sitter space ($\lambda > 0$), or a two-sheeted hyperboloid ($\lambda < 0$). In the later case we restrict ourselves to the connected component with $\{x^0 > 0\}$. We first focus on the $\lambda > 0$ and the $\lambda < 0$ is indicated at the end. Parametrizing $\mathcal{H}_\lambda$ with the set of functions $\{t, \{\alpha_i\}_{i=1}^{n+1}\}$ satisfying,

$$x^0 = \lambda^{1/2}\sinh(\lambda^{-1/2}t), \qquad x^i = \lambda^{1/2}\cosh(\lambda^{-1/2}t)\alpha_i, \quad \text{with} \quad \sum_{i=1}^{n+1}\alpha_i^2 = 1$$

the induced metric on $\mathcal{H}_\lambda$ takes the form

$$g_L|_{\mathcal{H}_\lambda} = -\mathrm{d}t^2 + \lambda\cosh^2(\lambda^{-1/2}t)\gamma_{\mathbb{S}^n},$$

where $\gamma_{\mathbb{S}^n} := \delta_{ij}\mathrm{d}\alpha^i\mathrm{d}\alpha^j|_{\sum_{i=1}^{n+1}\alpha_i^2=1}$ is a spherical n-metric. On the domain $\{t > 0\}$ let us introduce $\Omega := (\cosh(\lambda^{-1/2}t))^{-1}$ so that $\mathrm{d}t = -\sqrt{\lambda}\Omega^{-1}(1-\Omega^2)^{-1/2}\mathrm{d}\Omega$ and we obtain the following metric conformal to $g_L|_{\mathcal{H}_\lambda}$

$$\Omega^2\, g_L|_{\mathcal{H}_\lambda} = -\frac{\lambda}{1-\Omega^2}\mathrm{d}\Omega^2 + \lambda\gamma_{\mathbb{S}^n}.$$

This metric is obviously extendable to $\{\Omega = 0\}$ and we recover the well-known fact that $\mathscr{I} = (\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$ for de Sitter spacetime. The $\lambda < 0$ case is analogous in terms of the parametrization $x^0 := |\lambda|^{1/2}\cosh(|\lambda|^{-1/2}t), \quad x^i := |\lambda|^{1/2}\sinh(|\lambda|^{-1/2}t)\alpha_i$ where also $\sum_{i=1}^{n+1}\alpha_i^2 = 1$.

The Fefferman and Graham construction extends the above example from the n-sphere to general conformal manifolds. Generically, both ambient and Poincaré metrics are solutions of certain PDEs whose initial data are determined by the conformal structure. Namely, ambient metrics are asked to be Ricci flat at a null hypersurface $\mathcal{N}$, analogue to the null cone in the $(\mathbb{S}^n, [\gamma_{\mathbb{S}^n}])$ example; while Poincaré metrics are asked to be asymptotically Einstein with non-zero cosmological constant. Whether these PDEs propagate away from the initial surface is not required in the Fefferman and Graham construction, so their analysis remains formal, i.e. they find formal (non-necessarily convergent) series solving the PDEs at the initial hypersurface. It is important to stress that for the construction of conformally invariant scalars, the formal solutions are sufficient [50]. We are, however, interested in the asymptotic initial value problem of general relativity, which also uses the Fefferman and Graham formalism [5–7, 86, 87, 129], and it is more focused on Poincaré metrics. Nevertheless, for the sake of completeness, we outline first the main idea of the ambient construction, which is in the base of the Fefferman and Graham formalism.

2.3.1 Ambient metrics

The idea of the ambient space is to lift the conformal structure $(\Sigma, [\gamma])$ into an $\mathbb{R}^+$-bundle $\mathcal{N}$, called the metric bundle, in such a way that each section $(t(x)^2\gamma, x) \in \mathcal{N}$ gives a conformal representative of $(\Sigma, [\gamma])$. Then, one embeds $\mathcal{N}$ into a neighbourhood of $\mathcal{G} = \mathcal{N} \times \mathbb{R}$ identifying $\mathcal{N}$ with $\mathcal{N} \times \{0\}$. The ambient metric may extend from $\mathcal{N}$ in a non-unique way, but this freedom can be dealt with by constructing a certain equivalence class of metrics provided they are all Ricci flat to a certain order at $\mathcal{N}$.

In more detail, the ambient construction is as follows. One starts with a conformal manifold $(\Sigma, [\gamma])$ and chooses a representative $\gamma \in [\gamma]$. The manifold (Σ, γ) is the base of the metric bundle $\mathcal{N}$, which consists of all pairs (h, x) with $x \in \Sigma$ and $h = s^2\gamma$ for some $s \in \mathbb{R}^+$. The bundle $\mathcal{N}$ is endowed with a dilation operator δ_s which scales the first term in the pair $\delta_s(h, t) = (s^2h, t)$ for all $s \in \mathbb{R}^+$. Thus, $\mathcal{N}$ is an $\mathbb{R}^+$-bundle . Associated to each representative γ, the exists a trivialization $\mathcal{N} \simeq \mathbb{R}^+ \times \Sigma$ whose points (t, x) are associated to pairs $(t^2\gamma, x)$ and thus, its sections $(t(x)^2, x)$ correspond to conformal representatives $(t(x)^2\gamma, x)$ of $(\Sigma, [\gamma])$. In the remainder, we shall work assuming a trivialization, namely, identifying $\mathcal{N}$ with $\mathbb{R}^+ \times \Sigma$.

We consider local coordinates $\{x^i\}_{i=1}^n$ of Σ which endow $\mathcal{N}$ with coordinates $\{t, x^i\}$. In these coordinates, the bundle projection map is $\pi : (t, x) \mapsto x$, the dilations $\delta_s(t, x) \mapsto (st, x)$, $\forall s \in \mathbb{R}$, and the infinitesimal generator of δ_s is $\tau = t\partial_t$. Moreover, at each point of $T\mathcal{N}$, a symmetric two-tensor g_0 is defined by $g_0(X, Y) := t^2\gamma(\pi_\star X, \pi_\star Y)$, $\forall X, Y \in T_{(t,x)}\mathcal{N}$. In coordinates $\{t, x^i\}$ it reads

$$g_0 = t^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

Now consider the embedding $\iota : \mathcal{N} \hookrightarrow \mathcal{N} \times \mathbb{R}$, where $\iota(\mathcal{N}) = \mathcal{N} \times \{0\}$, whose points we denote by $(z, \rho) \in \mathcal{N} \times \mathbb{R}$. The action of the dilations δ_s extends to $\mathcal{N} \times \mathbb{R}$ by leaving the second factor invariant and acting on the first factor as already defined. The local coordinates $\{t, x^i\}$ of $\mathcal{N}$ extend to local coordinates $\{t, x_i, \rho\}$ of $\mathcal{N} \times \mathbb{R}$. In $\mathcal{N} \times \mathbb{R}$ we define $\mathcal{G}$, a neighbourhood of $\mathcal{N} \times \{0\}$ which is dilation invariant. In addition $\mathcal{G}$ is such that, for every $z \in \mathcal{N}$, the set of all $\rho \in \mathbb{R}$ for which $(z, \rho) \in \mathcal{G}$ is an open interval I_z. In $\mathcal{G}$, we will define a metric $g_{\mathcal{A}}$ for which $\nabla\rho$ is a geodesic vector.

We next give the conditions which make $\mathcal{G}$ an *ambient space* and $g_{\mathcal{A}}$ an *ambient metric*. The definition that we give corresponds, in the original work [50], to a special case called *straight* ambient metric in *normal form*. We emphasize that assuming normal form does not entail any loss of generality because Fefferman and Graham also show that, up to a certain equivalence relation, every ambient metric admits a normal form and it is straight in a sufficiently small neighbourhood of $\mathcal{N} \times \{0\}$.

Before giving the precise definition, we introduce a notion of decay for symmetric tensors on $\mathcal{G}$.

Definicin 2.19. A symmetric two-tensor S of $\mathcal{G}$ is said to be $O^+(\rho^m)$ if it is $O(\rho^m)$ and moreover, at each point $z \in \mathcal{N}$, $\iota^\star(\rho^{-m}S)$ is of the form $\pi^\star(\sigma)$, for a symmetric two-tensor σ of $\mathcal{N}$, which may depend on t and x.

Now we define:

Definicin 2.20. Let $(\Sigma, [\gamma])$ be a conformal n-manifold of signature (n_+, n_-) and fix a representative $\gamma \in [\gamma]$. An **ambient space** $(\mathcal{G}, g_{\mathcal{A}})$ for (Σ, γ) is an $(n+2)$-dimensional manifold of signature $(n_+ + 1, n_- + 1)$ such that, in the local coordinates $\{t, x^i, \rho\}$, takes the form

$$g_{\mathcal{A}} = 2\rho \mathrm{d}t^2 + 2t\mathrm{d}t\mathrm{d}\rho + t^2 g_\rho(x), \tag{2.30}$$

where $g_\rho(x) = g_\rho(x)_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is a 1-parameter family of n-metrics, with parameter ρ, such that $g_{\rho=0} = g_0$. Moreover:

1. If $n = 2$ or $n \geq 3$ and it is odd, $Ric(g_{\mathcal{A}})$ vanishes to infinite order at every point of $\mathcal{N} \times \{0\}$.
2. If $n \geq 4$ and it is even, $Ric(g_{\mathcal{A}}) = O^+(\rho^{n/2-1})$.

Observe that (2.30) is already decomposed in a 2×2 block with terms $\mathrm{d}\rho$ and $\mathrm{d}t$, plus an $n \times n$ block with only terms $\mathrm{d}x^i$. Using this structure, it is an immediate calculation that

$$\nabla_{\partial_\rho}\partial_\rho = \Gamma^I_{\rho\rho}\partial_I = 0,$$
$$\nabla_{t\partial_t}(t\partial_t) = t\partial_t + t^2\Gamma^I_{tt}\partial_I = t\partial_t,$$

thus $\nabla\rho$ and τ are both geodesic with the first affinely parametrized.

The main existence result for ambient metrics is as follows.

Teorema 2.21 (Fefferman-Graham [48, 50])**.** *Let $(\Sigma, [\gamma])$ be a smooth conformal manifold of dimension $n \geq 2$ and let $\gamma \in [\gamma]$ be a representative of the conformal class of metrics. Then*

a) There exists an ambient space $(\mathcal{G}, g_{\mathcal{A}})$ for (Σ, γ).

b) Let $(\mathcal{G}_1, g_{\mathcal{A}1})$ and $(\mathcal{G}_2, g_{\mathcal{A}2})$ be two ambient spaces for (Σ, γ). Then, if n is odd $g_{\mathcal{A}1} - g_{\mathcal{A}2}$ vanishes to infinite order at $\mathcal{N} \times \{0\}$. If n is even $g_{\mathcal{A}1} - g_{\mathcal{A}2} = O^+(\rho^{n/2})$.

Note that statement $b)$ of Theorem 2.21 says that, for n odd, the metric γ uniquely determines the ambient metric $g_{\mathcal{A}}$ to infinite order at $\mathcal{N} \times \{0\}$, while for n even, it only determines it up to order $n/2$. This is a consequence of the mechanism by which these metrics are generated. This is outlined here for ambient metrics, but in Appendix A we will give a more precise derivation of the formulae in the equivalent setting of Poincaré metrics.

Roughly speaking, the equation $Ric(g_{\mathcal{A}}) = 0$ at $\mathcal{N} \times \{0\}$ is used to generate recursively the terms of the formal expansion of g_ρ, with initial condition $g_{\rho=0} = \gamma$. The m-th order coefficient in the expansion satisfies the following equation, which arises from taking derivatives of the equation $Ric(g_{\mathcal{A}}) = 0$ and evaluating at $\mathcal{N}$

$$\left(\frac{n}{2} - m\right) \partial_\rho^m g_\rho\Big|_{\rho=0} = \mathcal{RHS}(g_\rho, \partial_\rho^{m'<m} g_\rho)\Big|_{\rho=0} \tag{2.31}$$

where $\mathcal{RHS}(g_\rho, \partial_\rho^{m'<m} g_\rho)$ indicates that the RHS of (2.31) depends on g_ρ (and its inverse g_ρ^{-1}) and derivatives in ρ of order lower than m. There also appear derivatives of them in the variables x^i up to second order. If n is odd or $n = 2$, one can impose $Ric(g_{\mathcal{A}}) = 0$ to infinite order at $\mathcal{N} \times \{0\}$ and the generation of polynomial terms in the expansion goes on to infinite order. The situation is more subtle if n is even and larger than two. In this case, the equation $Ric(g_{\mathcal{A}}) = 0$ to the order $\rho^{n/2}$ at $\mathcal{N} \times \{0\}$, cannot be satisfied by a formal power series expansion, because the factor in the LHS of (2.31) vanishes identically. One says then that the existence of the power series is obstructed by the presence of the so-called *obstruction tensor* $\mathcal{O}$, which is essentially given by the RHS of (2.31) with $m = n/2$. If this tensor, which entirely depends on γ, is non-zero, one must include logarithmic terms in the expansion in order to satisfy the equations (2.31) at order $\rho^{n/2}$. The logarithmic terms spoil smoothness, but allow one to keep the expansion so that $Ric(g_{\mathcal{A}}) = 0$ holds to infinite order at $\mathcal{N} \times \{0\}$. Hence, smooth solutions are determined by the metric γ only up to order $\rho^{n/2}$.

The obstruction tensor has interesting properties, which we give in the next theorem. As a tensor determined by a metric γ, we shall denote its explicit dependence by $\mathcal{O}(\gamma)$ if necessary.

Teorema 2.22 (Fefferman-Graham [50]). *Let $n \geq 4$ be even. The obstruction tensor $\mathcal{O}_{ij}$ depends only on the geometry of (Σ, γ) and*

1. *It admits a covariant expression in terms of γ, the contravariant metric $\gamma^\sharp$, $Ric(\gamma)$ and its covariant derivatives.*

2. *$\mathcal{O}_{ij}$ is traceless and divergence-free.*

3. *$\mathcal{O}_{ij}$ is conformally covariant of weight $n - 2$, i.e. $\mathcal{O}(\omega^2\gamma) = \omega^{2-n}\mathcal{O}(\gamma)$ for every smooth positive function of Σ.*

4. *If γ is conformally Einstein, then $\mathcal{O}_{ij} = 0$. In particular if γ is locally conformally flat $\mathcal{O}_{ij} = 0$.*

Coming back to the series expansion, if $n \geq 4$ is even we already indicated that one can continue generating terms in the expansion so $Ric(g_{\mathcal{A}}) = 0$ to infinite order at $\mathcal{N} \times \{0\}$

provided smoothness is dropped and only finite differentiability up to order $n/2$ at $\rho = 0$ is required. The metric is no longer a formal power series expansion, but it is called *polyhomogeneous expansion*, because of the presence of logarithmic terms. Moreover, one can freely prescribe a TT tensor which appears multiplying a power $\rho^{n/2}$ without logarithms in the expansion. This is possible precisely because the equation $Ric(g_{\mathcal{A}}) = 0$ (cf. (2.31)) does not determine this term. On the other hand, its trace and divergence cannot be prescribed because they are determined by geometric identities, also coming from the vanishing of $Ric(g_{\mathcal{A}})$ at $\mathcal{N}$.

If is n odd, something similar happens in the non-smooth case, in the sense that an undetermined TT tensor multiplying the half-integer power $\rho^{n/2}$ can also be prescribed. The presence of such a term clearly means that the ambient metric is no longer smooth. However, the situation is different in the case of Poincaré metrics where a corresponding free term can be added without spoiling differentiability of the metric to all orders. Indeed, this undertermined term plays a key role in the free data at null infinity in the context of Poincaré metrics, as we will see in Appendix A.

Because of their different behaviour, we state the existence results for generalized ambient metrics into separate Theorems for n even and n odd.

Teorema 2.23 (Fefferman-Graham [50])**.** *Let $(\Sigma, [\gamma])$ be a conformal n-manifold with n odd, and let $\gamma \in [\gamma]$ be a representative of the conformal class. Let also h be a symmetric TT tensor of $(\Sigma, [\gamma])$. Then, there exists a generalized ambient metric $g_{\mathcal{A}}$ for γ*

$$(g_{\mathcal{A}})_{IJ} = \psi^{(0)}_{IJ} + \psi^{(1)}_{IJ} |\rho|^{n/2}$$

where $\psi^{(0)}$ and $\psi^{(1)}$ extend smoothly to $\rho = 0$ and $\psi_{ij}|_{\rho=0} = t^2 h_{ij}$. These conditions uniquely determine the coefficients of the Taylor expansions at $\rho = 0$ of $\psi^{(0)}$ and $\psi^{(1)}$ to infinite order.

Teorema 2.24 (Fefferman-Graham [50])**.** *Let $(\Sigma, [\gamma])$ be a conformal n-manifold with n even, $\gamma \in [\gamma]$ be a representative of the conformal class and h a symmetric two-tensor on Σ. Then, there exist a one-form $\mathfrak{b}(\gamma)$ and a scalar $\mathfrak{a}(\gamma)$, both covariantly determined by γ, such that if $\mathrm{Tr}_{\gamma} h = \mathfrak{a}(\gamma)$ and $\mathrm{div}_{\gamma} h = \mathfrak{b}(\gamma)$, then there exists a generalized ambient metric $g_{\mathcal{A}}$ for γ*

$$(g_{\mathcal{A}})_{IJ} = \sum_{N=0}^{\infty} \psi^{(N)}_{IJ} (\rho^{n/2} \log|\rho|)^N,$$

such that every $\psi^{(N)}$ extend smoothly to $\rho = 0$ and $\partial_\rho^{n/2} (\psi)^{(0)}_{ij} = (n/2)!\ t^2 h_{ij}$ at $\rho = 0$. These conditions uniquely determine the Taylor expansions at $\rho = 0$ of $\psi^{(N)}_{IJ}$ to infinite order. Moreover $g_{\mathcal{A}}$ is smooth if and only if the obstruction tensor vanishes. If $n = 2$, then the obstruction tensor is identically zero, $\mathfrak{b}(\gamma) = \frac{1}{2\lambda} \mathrm{d}(\mathrm{Scal}(\gamma))$ and $\mathfrak{a} = \frac{1}{\lambda} \mathrm{Scal}(\gamma)$.

2.3.2 Poincaré metrics

We now review the main ideas on the Poincaré (smooth) metrics and generalized Poincaré metrics, which as in the case of ambient metrics are not necessarily smooth if n is even. We denote both such metrics $\widetilde{g}$. The construction arises by imposing certain asymptotic behaviour which makes $\widetilde{g}$ asymptotically Einstein to certain order with $\mathscr{I}$ being a prescribed conformal manifold $(\Sigma, [\gamma])$.

Let $(\Sigma, [\gamma])$ be a conformal n-manifold of signature (n_+, n_-). A Poincaré metric $\widetilde{g}$ is a smooth $(n+1)$-metric of signature $(n_+ + 1, n_-)$ or $(n_+, n_- + 1)$ defined in the interior of a manifold $\widetilde{\mathcal{M}} := \mathrm{Int}(\mathcal{M})$ with boundary $\Sigma = \partial\mathcal{M}$. The manifold $\mathcal{M}$ is a neighbourhood of $[0, \infty) \times \Sigma$, where $\widetilde{g}$ admits a smooth conformal extension $g = \Omega^2 \widetilde{g}$ with prescribed conformal infinity $\mathscr{I} := (\Sigma, [\gamma])$. In $\mathcal{M}$, Ω is a defining function of $\{0\} \times \Sigma \subset [0, \infty) \times \Sigma$ and we understand Σ in $\mathcal{M}$ as the image of the embedding $i : \Sigma \hookrightarrow \Sigma \times [0, \infty)$ such that $i(\Sigma) = \Sigma \times \{0\}$. Moreover, Poincaré metrics are asymptotically Einstein, with decay rate depending on the parity of n. For the n even case, we adapt the Definition 2.19 of O^+ to the case of Poincaré metrics

Definicin 2.25. A symmetric 2-tensor field S of $\mathcal{M}$ is $O^+(\Omega^m)$ if $S = O(\Omega^m)$ and $\mathrm{Tr}_\gamma i^\star(\Omega^{-m} S|_\Sigma) = 0$.

With this notation we can give the formal definitions [48], [50]:

Definicin 2.26. Let $(\Sigma, [\gamma])$ be a conformal n-manifold of signature (n_+, n_-) and λ a positive (resp. negative) real constant. A Poincaré metric for $(\Sigma, [\gamma])$, is a smooth metric $\widetilde{g}$ of signature $(n_+ + 1, n_-)$ (resp. $(n_+, n_- + 1)$) admitting a smooth conformal extension such that $\mathscr{I} = (\Sigma, [\gamma])$ and

1. If $n = 2$ or $n \geq 3$ and odd, $Ric(\widetilde{g}) - \lambda n \widetilde{g}$ vanishes to infinite order at Σ.

2. If $n \geq 4$ and even, $Ric(\widetilde{g}) - \lambda n \widetilde{g}$ is $O^+(\Omega^{n-2})$.

A Poincaré metric is by definition asymptotically Einstein, so in particular it is ACC. Hence, by Lemma 2.10, for each boundary metric $\gamma \in [\gamma]$ there exists a geodesic conformal extension $g = \Omega^2 \widetilde{g}$. We now define

Definicin 2.27. An ACC metric is said to be in **normal form** w.r.t. a boundary metric γ if

$$\widetilde{g} = \frac{1}{\Omega^2}\left(-\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega\right), \qquad g = \left(-\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega\right), \tag{2.32}$$

where g_Ω is the metric induced on the leaves Σ_Ω by g and $g_\Omega \mid_{\Omega=0} = \gamma$.

Associated to each geodesic extension, there exists a set of Gaussian coordinates $\{\Omega, x^i\}$ in which g and $\widetilde{g}$ are in normal form. These coordinates are most adequate for working with Poincaré metrics and will be assumed from now on unless otherwise specified.

For a given ambient metric $g_{\mathcal{A}}$, one can asign a Poincaré metric in the region $\rho < 0$ and another Poincaré metric in the region $\rho > 0$. They are obtained by restricting $g_{\mathcal{A}}$ onto the "hyperboloids" $\mathcal{H}_\lambda := \mathcal{G} \cap \{|\tau|^2 = \lambda^{-1}\}$, for a real non-zero constant $\lambda^{-1} \in \mathbb{R}$ with $\mathrm{sign}(\lambda) := \mathrm{sign}(\rho)$. Writing $g_{\mathcal{A}}$ as in (2.30), and changing the local coordinates $\{t, x^i, \rho\}$ to $\{s, x^i, \Omega\}$ by setting $\rho =: \Omega^2/(2\lambda)$ and $s := \Omega t$, with $\Omega > 0$, we obtain $|\tau|^2 = \lambda^{-1}s^2$ thus $\mathcal{H}_\lambda = \mathcal{G} \cap \{s = 1\}$. Note that the constant λ dissapears from the definition of $\mathcal{H}_\lambda$ because the coordinates $\{\Omega, s\}$ do not cover both regions $\rho > 0$ and $\rho < 0$ simultaneously. In terms of the new coordinates the ambient metric is

$$g_{\mathcal{A}} = \frac{\mathrm{d}s^2}{\lambda} + \frac{s^2}{\Omega^2}\left(-\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega\right) = \frac{\mathrm{d}s^2}{\lambda} + s^2\widetilde{g} \quad \text{with} \quad \widetilde{g} := \frac{1}{\Omega^2}\left(-\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega\right) \tag{2.33}$$

where g_Ω is the one-parameter family of metrics g_ρ reparametrized by $\rho = \Omega^2/(2\lambda)$. The Ricci flatness of $Ric(g_{\mathcal{A}})$ at $\mathcal{N} \times \{0\}$ translates [50] into the Einstein asymptoticity for $\widetilde{g}$, according to Defintion 2.26. In other words, $\widetilde{g}$ is a Poincaré metric in normal form for γ.

Regardless of the value of λ, the metric on each hypersurface $\mathcal{H}_\lambda$ is a Poincaré metric in normal form, inducing the same initial data at $\mathscr{I}$, which actually is the projectivization of $\mathcal{N}$, i.e. $(\Sigma, [\gamma])$. This is interesting from the point of view of the asymptotic Cauchy problem of GR, to which section 2.4 is devoted. The Poincaré metric in the interior region $\rho < 0$ gives an asymptotic solution of the $\Lambda < 0$ vacuum Einstein equations with prescribed boundary data at $\mathscr{I}$, while the one in the exterior region $\rho > 0$ gives an asymptotic solution the $\Lambda > 0$ vacuum Einstein equations propagating the initial data at $\mathscr{I}$. Thus, both Poincaré metrics are in a certain sense analogue, but not equal (in particular, their signatures are different). To understand this correspondence it is useful to bear in mind the example of the conformal n-sphere in subsection 2.2.1. As we are interested in the $\Lambda > 0$ case, we restrict ourselves to such values. Instead, the original publication [50] concentrates on the the $\Lambda < 0$ case, so one must be careful when comparing the corresponding expressions.

Another important observation is that it is common in the literature (e.g. [5, 6, 50]) to fix the parameter λ to 1. From a geometrical point of view, this means choosing the hypersurface $\mathcal{H}_{\lambda=1}$ to define the Poincaré metric. This certainly simplifies the calculations. However, from a physical point of view it is desirable to keep track of the role that the cosmological constant plays in the expressions. Another benefit is that it allows one to make consistency checks on the calculations. This is because the expression obtained by setting $\lambda = 1$ in a formula with general λ must agree with the expression obtained by scaling the conformal factor by $\lambda^{1/2}$ in the same formula, as follows from the next argument. Given a Poincaré metric $\widetilde{g}$ with constant λ it follows that $\widetilde{g}_\lambda := \lambda\widetilde{g}$ is a Poincaré metric with $\lambda = 1$, as we show next. Firstly, $Ric(\widetilde{g}_\lambda) - n\widetilde{g}_\lambda = Ric(\widetilde{g}) - n\lambda\widetilde{g}$ as a consequence of the invariance of the Ricci tensor under scaling of the metric. Secondly, fix a conformal extension of the original metric so that $g = \Omega^2\widetilde{g} = \lambda^{-1}\Omega^2\widetilde{g}_\lambda$. This means that the conformal factor $\Omega_\lambda = \lambda^{-1/2}\Omega$ defines a conformal extension of $\widetilde{g}_\lambda$. Both things together show that $\widetilde{g}_\lambda$ is a Poincaré metric with $\lambda = 1$ according to Definition 2.26. It follows at once from the expression of $\widetilde{g}$ in (2.33) that the expressions with $\lambda = 1$ can be generalized to an arbitrary value of λ upon scaling the conformal factor by $\lambda^{-1/2}$ (and vice versa).

The Poincaré metrics are recursively generated from the asymptotic Einstein equations, just like ambient metrics are generated from Ricci flatness at $\mathcal{N} \times \{0\}$. Indeed, the equations for g_ρ obtained from $Ric(g_{\mathcal{A}}) = 0$ at $\mathcal{N} \times \{0\}$ are, after redefining $\rho = \Omega^2/2\lambda$, exactly the same as those obtained for g_Ω from $Ric(\widetilde{g}) - \lambda\widetilde{g} = 0$ at $\{\Omega = 0\}$. In the n odd or $n = 2$ cases, the equations for ambient metrics generate a smooth metric to infinite order, uniquely determined at $\rho = 0$ by the metric γ. The corresponding Poincaré metric is smooth and even in Ω to infinite order. For n even case, the smooth power expansion of the ambient metrics is obstructed at the order $\rho^{n/2}$. Consequently, so are the Poincaré metrics at order n. Moreover, the expansion of the smooth Poincaré metric is also even in all orders previous to the n.

The asymptotic expansions for g_ρ in Theorems 2.23 and 2.24 are translated into an asymptotic expansion for g_Ω, which will be called Fefferman-Graham (FG) expansion. Explicitly, this has the form

$$g_\Omega \sim \sum_{s=0}^{(n-1)/2} g_{(2s)}\Omega^{2s} + \sum_{s=n}^{\infty} g_{(s)}\Omega^{s}, \qquad \text{if } n \text{ is odd}, \tag{2.34}$$

$$g_\Omega \sim \sum_{s=0}^{\infty} g_{(2s)}\Omega^{2s} + \sum_{s=n/2}^{\infty} \sum_{t=1}^{m_t} \mathcal{O}_{(s,t)}\Omega^{2s}(\log\Omega)^t, \qquad \text{if } n \text{ is even}, \tag{2.35}$$

where $m_s \leq 2s - n + 1$ in an integer for each s, the coefficients $g_{(s)}$ are objects defined at $\mathscr{I}$ and extended to Σ as independent of Ω. The first logartighmit term involves $\mathcal{O}_{(n/2,1)}$ which is precisely the obstruction tensor $\mathcal{O}$ in Theorem 2.22. These expansions can be generated independently of the ambient construction and we devote Appendix A to doing so in the particular case of vanishing obstruction tensor. In the rest of this subsection, we describe the main properties of the FG expansions and provide existence results analogous to Theorems 2.23 and 2.24.

The n odd case (2.34) contains even powers of Ω up to order $2s < n$. They are recursively generated only from the zero-th order, i.e. the boundary metric γ. The n-th order term $g_{(n)}$ is a symmetric TT tensor and independent of previous coefficients. Terms of order $s > n$ may be even or odd and are generated exclusively from $(\gamma, g_{(n)})$. Thus, a unique FG expansion arises from any pair of tensors $(\gamma, g_{(n)})$, with γ symmetric of signature (n_+, n_-) and $g_{(n)}$ TT w.r.t. γ. Observe that, unlike in the case of ambient metrics, the n-th order (which corresponds to term $\rho^{n/2}$ in the ambient metric) and subsequent odd order terms, do not entail any loss of smoothness. The convergence of such series in a neighbourhood of $\mathscr{I}$ or existence and uniqueness of non-analytic solutions will be addressed in section 2.4.

In the n even case, the expansion (2.35) is even in all powers of Ω and also contains logarithmic terms. Up to order $2s < n$, all terms are generated by the boundary metric γ. At the order n, two coefficients appear: $g_{(n)}$, which multiplies solely Ω^n, and $\mathcal{O} = O_{(n/2,1)}$ which multiplies the first logarithmic term $\Omega^n \log \Omega$. The obstruction tensor $\mathcal{O}$ as well as the trace and divergence of $g_{(n)}$, are determined exlusively by γ. One can always add to $g_{(n)}$ an arbitrary TT term $\mathring{g}_{(n)}$, so that the asymptotic Einstein equations are still satisfied. Hence, in this case the boundary metric γ and the n-th order coefficient $g_{(n)}$ with trace and divergence fixed by γ, determine a unique FG expansion. If the obstruction tensor does not vanish, the logarithmic term makes the series non-smooth. Therefore, these metrics do not fulfill all the requirements for being Poincaré metrics. We shall call them *Fefferman-Graham-Poincaré* (FGP) metrics. Specifically:

Definicin 2.28. In $n+1$ dimensions, with n even or odd, a **Fefferman-Graham-Poincaré (FGP)** metric is a metric satisfying the Einstein equations to infinite order at $\mathscr{I}$.

In other words, a FGP is an asymptotically Einstein metric (cf. Definition 2.5) to infinite order. The convergence of the asymptotic series of FGP metrics with n even will be addressed in section 2.4. Unfortunately, there are no yet existence and uniqueness results for the non-analytic case with n even.

We now summarize the above discussion in a lemma for future referrence. All the properties in Lemma 2.29 which refer to the zero obstruction case are proven in Appendix A.

Lemma 2.29 (Properties of the FG expansion)**.**

1. *Each coefficient $g_{(s)}$ with $0 < s < n$ depends on previous order coefficients up to order $g_{(s-2)}$ and tangential derivatives of them up to second order. This is also true for $n < s$ if n odd or n even with $\mathcal{O} = 0$. If n is even and $\mathcal{O} \neq 0$, the terms $g_{(s)}$ and $\mathcal{O}_{(s,t)}$ with $n < s$ depend on previous terms up to order $g_{(s-2)}$ and $\mathcal{O}_{(s-2,t)}$.*
2. *Up to order n, both expansions (2.34), (2.35) are even and all terms $g_{(s)}$ with $s < n$ or $s = n+1$ (but not $s = n$) are solely generated from γ. If n is even, $\mathcal{O}$ is also generated from γ.*
3. *The n-th order coefficient $g_{(n)}$ is independent on previous terms except for*

$$\mathrm{Tr}_\gamma g_{(n)} = \mathfrak{a}, \qquad \mathrm{div}_\gamma g_{(n)} = \mathfrak{b},$$

where $\mathfrak{a} = 0$, $\mathfrak{b} = 0$ for n odd and $\mathfrak{a}$ is a scalar and $\mathfrak{b}$ a one-form determined by γ for n even.

So far we have not discussed existence results of FGP metrics. We emphazise that these existence results are *not* of Einstein metrics, since the equations are only satisfied asymptotically at infinite order. We conclude this section with an existence result due to Fefferman and Graham.

Teorema 2.30 (Fefferman-Graham [50]). *Let (Σ, γ) be a pseudo-Riemannian manifold of signature (n_+, n_-) and let h be a symmetric two-tensor of Σ.*

- *If $n = 2$ and if $\mathrm{div}_\gamma h = \frac{1}{2\lambda}\mathrm{d}(\mathrm{Scal}(\gamma))$ and $\mathrm{Tr}_\gamma h = \frac{1}{\lambda}\mathrm{Scal}(\gamma)$, there exists an even (i.e. with only non-zero coefficients of even order) Poincaré metric g in normal form w.r.t γ which admits an expansion of the form (2.35) (with $\mathcal{O}_{(r,s)} = 0$) and $g_{(2)} = h$.*
- *If $n \geq 3$ is odd and if $\mathrm{div}_\gamma h = 0$ and $\mathrm{Tr}_\gamma h = 0$, there exists a Poincaré metric g in normal form w.r.t γ, which admits an expansion of the form (2.34) such that $g_{(n)} = h$ (in particular, trace-free).*
- *If $n \geq 4$ is even, there exist a one-form $\mathfrak{b}(\gamma)$ and a scalar $\mathfrak{a}(\gamma)$, both covariantly determined by γ, in such a way that if $\mathrm{div}_\gamma h = \mathfrak{b}(\gamma)$ and $\mathrm{Tr}_\gamma(h) = \mathfrak{a}(\gamma)$, then there exists a FGP in normal form w.r.t. γ which admits an expansion of the form (2.35) such that $g_{(n)} = h$. The solution is smooth if and only if the obstruction tensor of γ vanishes.*

2.4 Asymptotic initial value problems in GR

In this section we review the asymptotic Cauchy problem in general relativity. In subsection 2.4.1 we construct the *Friedrich's conformal field equations* (FCFE) [56, 57]. Although these can be formulated in any dimension, they are adequate for initial value problems only in four spacetime dimensions. In subsection 2.4.2 we give higher dimensional results which, as advanced in section 2.3, are based in the Fefferman and Graham formalism. We focus on well-possedness results for the asymptotic initial value problem.

As already mentioned, we make special emphasis in the case of Lorentzian metrics solving $\Lambda > 0$ vacuum Einstein's equations (2.19). For other values of the cosmological constant one can pose other asymptotic problems, such as initial-boundary value problems ($\Lambda < 0$) or characteristic initial value problems ($\Lambda = 0$). We shall not discuss them in any detail here and will just mention some fundamental results and references.

2.4.1 Friedrich's Conformal Field Equations

For the derivation in this section we follow [63]. Consider an $(n+1)$-dimensional Lorentzian metric g with $n \geq 2$. Using the Bianchi identity $\nabla_{[\sigma} R^{\mu}{}_{|\alpha|\nu\beta]} = 0$, it follows from (2.11)

$$\nabla_{[\sigma} C^{\mu}{}_{|\alpha|\nu\beta]} = 2\nabla_{[\sigma}\left(P_{|\alpha|\nu}\delta^{\mu}{}_{\beta]}\right) + 2\nabla_{[\sigma}\left(g_{|\alpha|\nu}P^{\mu}{}_{\beta]}\right). \tag{2.36}$$

Contracting the indices σ and μ we are left with

$$\nabla_\mu C^\mu{}_{\alpha\nu\beta} = 2(n-2)\nabla_{[\nu}P_{\beta]\alpha} - 2g_{\alpha\beta}\nabla_{[\mu}P^\mu{}_{\nu]} + 2g_{\alpha\nu}\nabla_{[\mu}P^\mu{}_{\beta]}. \tag{2.37}$$

Since C is traceless, contraction of the indices α and β fields

$$0 = \nabla_{[\nu}P^\mu{}_{\mu]}, \tag{2.38}$$

which is just a rewriting of the divergence-free property of the Einstein tensor. Hence

$$\nabla_\mu C^\mu{}_{\alpha\nu\beta} = 2(n-2)\nabla_{[\nu}P_{\beta]\alpha}. \tag{2.39}$$

Now let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be an $(n+1)$-Lorentzian Einstein manifold and let $(\mathcal{M}, g)$ be a conformal extension $g = \Omega^2\widetilde{g}$. As usual, let ∇, $\widetilde{\nabla}$ denote their respective Levi-Civita connections. By conformal invariance, we can replace the Weyl tensor $\widetilde{C}^\mu{}_{\alpha\nu\beta}$ of $\widetilde{g}$ by $C^\mu{}_{\alpha\nu\beta}$. Equation (2.37) for $\widetilde{g}$ is

$$\widetilde{\nabla}_\mu C^\mu{}_{\alpha\nu\beta} = 0. \tag{2.40}$$

The divergence free property of a tensor with the symmetries of the Weyl tensor is a conformally covariant property (cf. Lemma 2.1). Specifically, the following identity holds

$$\nabla_\mu(\Omega^{2-n}C^\mu{}_{\alpha\nu\beta}) = \Omega^{2-n}\widetilde{\nabla}_\mu C^\mu{}_{\alpha\nu\beta} = 0.$$

Hence, the rescaled Weyl tensor defined by

$$c^\mu{}_{\alpha\nu\beta} := \Omega^{2-n}C^\mu{}_{\alpha\nu\beta} \tag{2.41}$$

satisfies the following *Bianchi equation*

$$\nabla_\mu c^\mu{}_{\alpha\nu\beta} = 0. \tag{2.42}$$

The tensor c is in a certain sense the fundamental object in the FCFE. Equation (2.11) can obviously be rewritten as

$$\Omega^{n-2}c^\mu{}_{\alpha\nu\beta} = R^\mu{}_{\alpha\nu\beta} + 2P_{\alpha[\nu}\delta^\mu{}_{\beta]} + 2g_{\alpha[\nu}P^\mu{}_{\beta]}. \tag{2.43}$$

In terms of c and after using (2.42), expression (2.39) takes the form,

$$\Omega^{n-3}(\nabla_\mu\Omega)c^\mu{}_{\alpha\nu\beta} = 2\nabla_{[\nu}P_{\beta]\alpha}. \tag{2.44}$$

In order to obtain equations for the conformal factor we use the transformation law (2.12). Firstly, observe that the Einstein equations for $\widetilde{g}$ are

$$\widetilde{P}_{\alpha\beta} = \frac{\lambda}{2}\widetilde{g}_{\alpha\beta}, \tag{2.45}$$

and its trace gives $\widetilde{P} := \widetilde{g}^{\alpha\beta}P_{\alpha\beta} = (n+1)\lambda/2$. Taking trace with $g^\sharp$ in (2.12) gives

$$\widetilde{P} = \Omega^2 P + \Omega\nabla_\mu\nabla^\mu\Omega - \frac{n+1}{2}\nabla_\mu\Omega\nabla^\mu\Omega. \tag{2.46}$$

Substuting $\lambda/2 = \widetilde{P}/(n+1)$ and using (2.46), the Einstein equation (2.45) can be written in the form

$$\widetilde{P}_{\alpha\beta} = \left(\frac{P}{n+1} + \frac{1}{n+1}\frac{\nabla_\mu\nabla^\mu\Omega}{\Omega} - \frac{\nabla_\mu\Omega\nabla^\mu\Omega}{2\Omega^2}\right)g_{\alpha\beta}$$

which has the advantage that λ is no longer explicit and it only involves geometric quantities associated to g. Inserting this into (2.12) yields the equation for the conformal factor in the FCFE, namely

$$\nabla_\beta\nabla_\alpha\Omega = -\Omega P_{\beta\alpha} + s g_{\beta\alpha}, \tag{2.47}$$

where

$$s := \frac{1}{n+1}\nabla_\mu\nabla^\mu\Omega + \frac{1}{n+1}P\Omega. \tag{2.48}$$

Observe that this equation makes no reference to the value of λ. Taking a derivative ∇_ν in (2.47), commuting ∇_ν and ∇_β in the LHS and contracting with $g^{\nu\alpha}$ one obtain

$$\nabla_\beta\nabla_\nu\nabla^\nu\Omega + R^\mu{}_\beta\nabla_\mu\Omega = -(\nabla_\nu\Omega)P^\nu{}_\beta - \Omega\nabla_\nu P^\nu{}_\beta + \nabla_\beta s. \tag{2.49}$$

The Ricci tensor can be written in terms of the Schouten tensor as

$$R^\mu{}_\beta = P\delta^\mu{}_\beta + (n-1)P^\mu{}_\beta. \tag{2.50}$$

Using (2.48), (2.50) and (2.38) in (2.49) gives another of the FCFE, namely

$$\nabla_\beta s = -(\nabla_\nu\Omega)P^\nu{}_\beta. \tag{2.51}$$

The last equation in the FCFE is for the cosmological constant. It is obtained directly from $\lambda = 2\widetilde{P}/(n+1)$, replacing $\widetilde{P}$ from and using the definition of s in (2.48). The result is

$$\lambda = 2\Omega s - \nabla_\mu\nabla^\mu\Omega. \tag{2.52}$$

Observe that (2.47) and (2.51) imply that the RHS of (2.52) is constant, so this equation only needs to hold at one point.

Definicin 2.31. The **Friedrich conformal field equations (FCFE)** is the system of equations (2.42), (2.43), (2.44), (2.47), (2.51) and (2.52) for the unknowns $\{g_{\alpha\beta}, \Omega, s, P_{\alpha\beta}, c^\mu{}_{\alpha\nu\beta}\}$.

A remarkable feature of the FCFE, is that they extend regularly to $\mathscr{I}$. This allows one to pose a Cauchy problem at $\mathscr{I}$. Another important property is that the conformal factor Ω, despite being one of the variables of the FCFE, possesses a large gauge freedom. This is obvious if one observes that the FCFE are satisfied for every metric g and conformal factor Ω related to an Einstein metric $\widetilde{g}$ by $g = \Omega^2\widetilde{g}$. On the one hand, this allows one to choose the best conformal gauge in order to manipulate the equations. It also implies that the initial data have a large residual conformal gauge freedom (cf. Theorem 2.34). Before getting deeper into these facts, we shall make an observation concerning the dimensions in which the FCFE provides a well-posed system of equations.

As it is well-known (e.g. [63]), that from the full Bianchi identity (2.36) for the physical metric $\widetilde{g}$

$$\widetilde{\nabla}_\sigma C^\mu{}_{\alpha\nu\beta} + \widetilde{\nabla}_\nu C^\mu{}_{\alpha\beta\sigma} + \widetilde{\nabla}_\beta C^\mu{}_{\alpha\sigma\nu} = 0,$$

one can extract a subsystem of hyperbolic equations. Moreover [63], in the four dimensional case, i.e. $n = 3$, the full Bianchi identity is equivalent to the contracted one (2.40). Thus, equation (2.42) implies hyperbolic equations for c for $n = 3$. In the higher dimensional case this is no longer true, as the contracted Bianchi identity can no longer provide sufficient evolution equations for the rescaled Weyl tensor. One could attempt finding a suitable set of evolution equations using the full Bianchi identity. This, however, cannot be done in such a way that the system remains regular at $\mathscr{I}$ [63]. Hence, for the higher dimensional problem one must find a different system of equations.

We now come back to the initial value problem of the FCFE at $\mathscr{I}$. As discussed in section 2.2, the causal character of $\mathscr{I}$ is determined by the sign of the cosmological constant. This, in turn, determines the nature of the asymptotic PDE problem. We start by describing very briefly the $\Lambda = 0$ and $\Lambda < 0$ cases and the concentrate on the $\Lambda > 0$ case

In the $\Lambda = 0$ case $\mathscr{I}$ is degenerate and the initial value problem is characteristic. Roughly speaking, this means that one of the natural directions of propagation of the initial data (determined by the equations) is parallel to the initial hypersurface. In order to obtain a well-posed problem (see [128] and also [90]), one must suplement the data at $\mathscr{I}^-$ with data on an outgoing null hypersurface $\mathscr{K}$, which intersects $\mathscr{I}^-$ in a 2-dimensional spacelike surface (Figure 2.6). In the negative Λ case $\mathscr{I}$ is timelike, so instead of an initial surface it defines a boundary. One can consider an initial-boundary problem, setting data on a hypersurface Σ which intersects $\mathscr{I}$ (Figure 2.6), where a boundary condition is imposed. In [59], a geometric uniqueness result is achieved for a certain class of boundary conditions. In [64] the problem with general boundary conditions is considered.

In the positive Λ case $\mathscr{I}$ is spacelike and the problem turns out to be well-posed [58]. The system admits a reduction to a symmetric hyperbolic system of evolution equations (see [60]), which propagate the constraints. Then, the local existence and uniqueness of a spacetime evolving (Figure 2.6) from data at $\mathscr{I}^-$ is guaranteed by standard theorems (e.g. [146]). Existence and uniqueness can be used for local characterization of spacetimes, but we must restrict the spacetime to the patch evolving from $\mathscr{I}^-$. In this patch

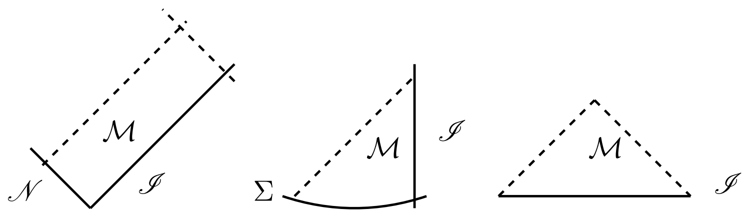

FIGURE 2.6: *Asymptotic data problems from left to right: characteristic initial value problem, initial-boundary value problem and inital value problem.*

$\mathscr{I} = \mathscr{I}^-$ (Figure 2.6), and for this reason, we will not explicitly distinguish different components of $\mathscr{I}$ in this context.

In order to give the freely specifiable data for this problem, we must first define the electric part of the rescaled Weyl tensor (2.41) w.r.t. to the normal u:

Definicin 2.32. The **electric part** of the Weyl tensor C and of the rescaled Weyl tensor c w.r.t. to a unit vector u are, respectively

$$(C_\perp)_{\alpha\beta} := C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu, \qquad (c_\perp)_{\alpha\beta} := c^\mu{}_{\alpha\nu\beta} u_\mu u^\nu.$$

If u is normal to $\mathscr{I}$ the **$\mathscr{I}$-electric part of the rescaled Weyl tensor** is

$$D_{\alpha\beta} := (c_\perp)_{\alpha\beta}|_{\mathscr{I}}. \tag{2.53}$$

Observe that D is tangent to $\mathscr{I}$.

Each set of initial data is (Σ, γ, D), with (Σ, γ) a Riemannian manifold which prescribes the geometry of $\mathscr{I}$, and D a TT tensor which prescribes the $\mathscr{I}$-electric part of the rescaled Weyl tensor. Then, we define

Definicin 2.33. An **asymptotic data set** for $n = 3$ is the triad (Σ, γ, D), where (Σ, γ) is a Riemannian 3-manifold and D is a symmetric two-tensor, TT w.r.t. γ.

As mentioned after Definition 2.31, the freely specifiable data at $\mathscr{I}$ possesses a large gauge group arising from the conformal freedom in the FCFE. For every smooth positive function ω of Σ, the following equivalence of data holds

$$(\Sigma, \gamma, D) \simeq (\Sigma, \omega^2\gamma, \omega^{2-n} D), \tag{2.54}$$

which we write for arbitrary n for later use, even though at this point n has been fixed to $n = 3$. In other words, the equivalence (2.54) defines the class $([\gamma], [D])$ in Σ, whose elements are given by the following conformal behaviour. If $\gamma \in [\gamma]$ and $D \in [D]$ are two representatives, with D being TT w.r.t. γ, then any other pair $\gamma' \in [\gamma]$ and $D' \in [D]$ must be

$$\gamma' = \omega^2\gamma, \qquad D' = \omega^{2-n} D,$$

for ω a smooth, positive function of Σ. The fact that D is TT w.r.t γ if and only if D' is TT w.r.t γ' is a direct consequence of Lemma 2.1.

We can now summarize the above discussion in the following theorem:

Teorema 2.34 (Friedrich [58]). *Let (Σ, γ, D) an asymptotic initial data set for $n = 3$. Then, there exists a unique, maximal, globally hyperbolic solution $(\widetilde{\mathcal{M}}, \widetilde{g})$ of (2.19), admitting a smooth conformal extension $g = \Omega^2 \widetilde{g}$, which induces the boundary metric γ and the $\mathscr{I}$-electric part of rescaled Weyl tensor (2.53) coincides with D. Each representative in the class $(\Sigma, [\gamma], [D])$ determines the same physical metric $\widetilde{g}$, but different conformal extensions.*

Since the initial data determine a unique spacetime, they must store all the information within them. Sometimes this information can be made very explicit. One example is determining necessary and sufficient conditions that the initial data set must satisfy in order for the evolving spacetime to have a Killing vector field. These are the so-called Killing Initial Data (KID) equations. Originally formulated for the Cauchy problem of the Einstein equations (see [17] and [31]), the same idea can be applied for the asymptotic Cauchy problem of the FCFE (see [115] for the characteristic case and [28] for the initial-boundary problem). We are specially interested in the spacelike $\mathscr{I}$ case [116]. In this case, the KID equations are particularly simple. They are given by a unique geometric formula which depends on a CKVF ξ of γ which, a posteriori, is the restriction at $\mathscr{I}$ of the Killing vector field ζ that the spacetime admits. As one could expect, the KID equations are conformally well-behaved.

Teorema 2.35 (Paetz [116]). *The spacetime corresponding to an asymptotic data set (Σ, γ, D) for $n = 3$ admits a Killing vector field if and only if there exists a CKVF ξ of γ such that*

$$\mathcal{L}_\xi D + \frac{1}{3}(\mathrm{div}_\gamma \xi) D = 0. \tag{2.55}$$

The same KID equation (2.55) is satisfied for any two representatives γ', D' of the classes $[\gamma], [D]$.

We also define

Definicin 2.36. An **asymptotic Killing initial data set** (asymptotic KID) for (Σ, γ, D, ξ) is an asymptotic initial data set (Σ, γ, D) with a CKVF ξ satisfying the KID equation.

Observacin 2.37. *It is interesting to notice that the diffeomorphism equivalence of data implies that, for any diffeomorphism ϕ, the asymptotic KID $(\Sigma, \phi^\star(\gamma), \phi^\star(D), \phi_\star^{-1}(\xi))$ is equivalent to (Σ, γ, D, ξ). Let now $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$ be defined in an open subset $\mathcal{U} \subset \Sigma$ (cf. Definition 2.15) and assume that $\mathcal{U}' = \phi(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. Then, in $\mathcal{U}'$, the data $(\Sigma, \phi^\star(\gamma), \phi^\star(D), \phi_\star^{-1}(\xi)) = (\Sigma, \omega^2\gamma, \omega^{2-n} D, \phi_\star^{-1}(\xi))$ is, using the conformal gauge freedom, equivalent to $(\Sigma, \gamma, D, \phi_\star^{-1}(\xi))$. That is, the conformal class of CKVFs as given* *in Definition 2.17 establishes an equivalence between CKVFs (restricted to $\mathcal{U}'$) of the data generating the same symmetry.*

In order to either select the desired conformal representative (of ξ and/or for γ) we shall often use this equivalence through local diffeomorphisms in $\mathrm{ConfLoc}(\Sigma,\gamma)$*. For this to make sense, we must restrict to the open subset $\mathcal{U}'$ of Σ where this equivalence holds. When this happens, we implicitly restrict $\Sigma = \mathcal{U}'$.*

2.4.2 Higher dimensional results

The higher dimensional asymptotic Cauchy problem of general relativity relies in the Fefferman and Graham formalism, already introduced in Section 2.3. In this subsection we describe the existence and uniqueness results that we shall require.

In subsection 2.3.2, we have explained how to associate a formal power or polyhomogeneous series to a FPG metric $\widetilde{g}$. The metric induces data γ and $g_{(n)}$, from which a unique FG series arises. In particular, this is also true for Einstein metrics. Recall that the trace and divergence of $g_{(n)}$ are given by a scalar $\mathfrak{a}(\gamma)$ and a one-form $\mathfrak{b}(\gamma)$ determined by γ (cf. Lemma A.7). The converse is more delicate, namely, whether for data $(\gamma, g_{(n)})$ (equivalently, for a FG expansion), there exists a unique conformally extendable Einstein metric $\widetilde{g}$ realizing these data at $\mathscr{I}$. The question is answered affirmatively for arbitrary n if the initial data are analytic [48, 50, 87] and in the general case if n is odd [6, 7, 86, 129].

If $(\gamma, g_{(n)})$ are in the analytic class, standard convergence results for Fuchsian problems [14] can be applied to establish convergence of the FG expansion when the obstruction tensor vanishes [48, 50]. These results hold regardless of the signature of γ. In the case of non-vanishing obstruction tensor, Kichenassamy [87] has established convergence in a neighbourhood of $\mathscr{I}$ (see also [129]). This result is proven under the assumption of the boundary metric γ being Riemannian.

Hence, analyticity of the data is sufficient to prove convergence of the FG series expansion, irrespectively of whether the obstruction tensor is zero or not. When the obstruction tensor is zero, the metric $\widetilde{g}$ defined by the (convergent) power series is analytic (in the sense that the Taylor series converges to the metric). Therefore, so are all its derivatives and in particular, the tensor $Ric(\widetilde{g}) - \lambda \widetilde{g}$. This tensor vanishes to infinite order at $\mathscr{I}$ and analiticity implies that it vanishes in a neighbourhood of $\mathscr{I}$, i.e. $\widetilde{g}$ is Einstein in that neighbourhood. When the obstruction tensor is non-zero, the propagation of the Einstein equations away from $\mathscr{I}$ is not so immediate, but can also be proven[5] [87]. Observe that although the conformal metric in this case is no longer smooth at $\mathscr{I}$, the form of FG expansion (2.35) shows that the geodesic extension $g = \Omega^2 \widetilde{g}$ still extends

[5]Reference [87] actually proves Ricci flatness of the ambient metric in a neighbourhood of $\mathcal{N}$, but this condition is known to be equivalent [50] to the associated Poincaré metric being Einstein in a neighbourhood of $\mathscr{I}$.

as a C^n metric to $\mathscr{I}$. So sufficient differentiability is always granted to define curvature tensors at $\mathscr{I}$ of the conformal metric.

Therefore, in the analytic case we can bypass the difficulty of finding an equivalent system to the Einstein equations which admits a well-posed asymptotic initial value problem (see discussion below Definition 2.31). However, analytic data are restrictive and it is clearly of interest to see if such equivalent system of equations can be found. In [6] one such system is found when n is odd, thus extending the existence and uniqueness results beyond the analytic case. Anderson proposes to use the obstruction tensor as a replacement of the Einstein equations. The obstruction tensor is known to vanish for metrics conformal to an Einstein metric (cf. Theorem 2.22). Thus, denoting $\mathcal{O}(\widetilde{g})$ the obstruction tensor of $\widetilde{g}$, Anderson proposed studying well-posedness of the equation

$$\mathcal{O}(\widetilde{g}) = 0. \tag{2.56}$$

Using the conformal gauge freedom and harmonic coordinates (i.e. satisying $\Box x^\alpha = 0$) and assuming that $\widetilde{g}$ is Lorentzian, the author is able to reduce (2.56) to a hyperbolic system of equations. The Cauchy problem for this equation admits many solutions which are not conformal to an Einstein metric, but constraining appropriately the initial data one can show that the solution is conformally Einstein. Moreover, by conformal covariance there is no problem in setting up initial data directly at $\mathscr{I}$. Recall that, as mentioned in the introduction, the first two proofs of Andeson's theorem in [6] and [7] are not fully correct, and they have been recently amended in [86].

In both the analytic case and in Anderson's approach, the final outcome is that one can associate an Einstein metric $\widetilde{g}$ to each FG expansions. Thus, the free data $(\Sigma, \gamma, \widehat{D})$ must generate the corresponding FG expansion. Namely, the Riemannian metric γ prescribes the geometry of $\mathscr{I}$, i.e. the zero-th order coefficient $g_{(0)}$, while the tensor $\widehat{D}$ prescribes the n-th order coefficient in the FG expansion $g_{(n)}$. For $n > 3$, the relation of the latter with the electric part of the rescaled Weyl tensor at $\mathscr{I}$ is limited to few cases, as we will see in Chapter 5. Observe that for n odd $\widehat{D}$ is always TT w.r.t. γ while for n even $\widehat{D}$ has trace and divergence determined by γ, by the scalar $\mathfrak{a}(\gamma)$ and the one-form $\mathfrak{b}(\gamma)$ given in Appendix (A) (cf. Lemma A.7). Thus, we generalize the Definition 2.33 to arbitrary n

Definicin 2.38. An **asymptotic data set** is the triad $(\Sigma, \gamma, \widehat{D})$, where (Σ, γ) is a Riemannian n-manifold and $\widehat{D}$ a symmetric two-tensor whose trace and divergence are determined by γ through the scalar $\mathfrak{a}(\gamma)$ and the one-form $\mathfrak{b}(\gamma)$ in Lemma A.7. In particular for n odd $\widehat{D}$ is a TT tensor w.r.t. γ.

If n is odd, the conformal equivalence of data is the same as in the $n = 3$ case, in the sense that conformal class of data $(\Sigma, [\gamma], [\widehat{D}])$ defined by (2.54) determine the same physical solution (cf. [6]). If n is even, this is more subtle because $\widehat{D}$ has generically non-zero trace and divergence. First, we remark that the constraint equations $\mathrm{div}_\gamma \widehat{D} =$

$\mathfrak{b}(\gamma)$, $\mathrm{Tr}_\gamma \widehat{D} = \mathfrak{a}(\gamma)$ can always be locally solved[6] for any metric γ (cf. [50]). In order to determine the class of equivalent data $(\Sigma, [\gamma], [\widehat{D}])$, one fixes a representative of the class of metrics γ and $\widehat{D}_0$ a solution to the constraint equations $\mathrm{div}_\gamma \widehat{D}_0 = \mathfrak{b}(\gamma)$, $\mathrm{Tr}_\gamma \widehat{D}_0 = \mathfrak{a}(\gamma)$. Whenever one knows that the data $(\Sigma, \gamma, \widehat{D}_0)$ correspond to a unique Einstein metric $\widetilde{g}_0$, a conformal change of boundary metric $\gamma' = \omega^2\gamma$ determines a unique geodesic conformal factor Ω' for which $\Omega'^2\widetilde{g}_0$ induces a new n-th order coefficient $g'_{(n)} =: \widehat{D}'_0$. The tensor $\widehat{D}'_0$ satisfies the constraint equations $\mathrm{div}_{\gamma'} \widehat{D}'_0 = \mathfrak{b}(\gamma')$, $\mathrm{Tr}_{\gamma'} \widehat{D}'_0 = \mathfrak{a}(\gamma')$ and it is computable in terms of ω, γ and $\widehat{D}_0$. This defines the conformal class $[\widehat{D}_0]$, although the explicit transformation formula is hard to be given with generality. If one can select "canonical" background data $(\Sigma, \gamma, \widehat{D}_0)$ for which the conformal class $[\widehat{D}_0]$ is computable, then one could define a free TT part $D := \widehat{D} - \widehat{D}_0$ for every tensor $\widehat{D}$ satisfying the constraint equations. Notice that the fact that D is TT is immediate as the trace and divergence of $\widehat{D}$ and $\widehat{D}_0$ must be equal because they are determined exclusively by γ. Also, notice that knowning $(\Sigma, \gamma, \widehat{D})$ implies knowing (Σ, γ, D) and viceversa. Then, one expects that the free part D behaves as $\omega^{2-n}D$ under conformal scalings of the boundary metric. This is what we achieve in the conformally flat $\mathscr{I}$ case, by selecting de Sitter as reference spacetime (see Definition 5.12). Moreover, in this case we prove that the free part coincides, up to a constant factor, with the electric part of the rescaled Weyl tensor at $\mathscr{I}$ (cf. Theorem 5.14), so the conformal tranformation formula is immediate. For this reason, our results do not require a general explicit tranformation formula for the equivalence classes of data, although it would be interesting to study and clarify this issue.

Teorema 2.39. *Let $(\Sigma, \gamma, \widehat{D})$ be an asymptotic initial data set. If n is odd [6, 7, 86] or if $\gamma, \widehat{D}$ are analytic [48, 50, 87], then there exists a unique metric $\widetilde{g}$, which solves (2.19) and admits a conformal extension such that the boundary metric can be identified with (Σ, γ) and $\widehat{D}$ prescribes $g_{(n)}$, the n-th order coefficient of its FG expansion. Each representative in the class $(\Sigma, [\gamma], [\widehat{D}])$ determines the same physical metric $\widetilde{g}$, but different conformal extensions.*

In this thesis we shall discuss the extension of Theorem 2.35 to higher dimensional cases. Namely, what are the necessary and sufficient conditions for n-dimensional asymptotic KID to generate a symmetry. For the cases studied, the result is a natural extension of formula (2.55) (cf. Theorem 5.18), namely

$$\mathcal{L}_\xi g_{(n)} + \frac{n-2}{n}\mathrm{div}_\gamma(\xi) g_{(n)} = 0.$$

We remark that the definition of asymptotic KID in arbitrary dimension is completely analogous to Definition 2.36, but with n-dimensional asymptotic data (cf. Definition 2.38 below) and the n-dimensional KID equation. For asymptotic KID in higher dimension, the equivalence of data in Remark 2.37 under local conformal transformations $\mathrm{ConfLoc}(\Sigma, \gamma)$ also holds when restricted to suitable domains.

[6]The existence of global solutions is a harder problem which is not always guaranteed [50].

Chapter 3

Skew-symmetric endomorphisms of $\mathbb{M}^{1,2}$ and $\mathbb{M}^{1,3}$ & CKVFs of $\mathbb{S}^2$

In this Chapter we deal with skew-symmetric endomorphisms of four and three dimensional Lorentzian vector spaces, which shall be often identified with $\mathbb{M}^{1,2}$ and $\mathbb{M}^{1,3}$ respectively, as well as CKVFs of the 2-sphere $\mathbb{S}^2$. The contents of this Chapter have been published in [94].

We start, in Section 3.1, by proving several basic properties of skew-symmetric endomorphisms of Lorentzian vector spaces. This Section is worked out in arbitrary dimension, because some of the properties will be useful in Chapter 4, where we shall extend many of the results in this Chapter to higher dimension. In Section 3.2 we obtain a canonical form for every skew-symmetric endomorphism of $\mathbb{M}^{1,2}$ and $\mathbb{M}^{1,3}$, i.e. a unique matrix form, depending on an optimal number of parameters, to which every single element can be brought to. The orthogonal unit vector bases realizing this canonical form are non-unique, which means that there exists a group of invariance, which we analyze in depth in Section 3.3 and whose generators are obtained in Section 3.4.

The analysis of skew-symmetric endomorphisms is then carried into the set of global CKVFs of the Riemann 2-sphere in Sections 3.6 and 3.7, for which complex coordinates are specially suited. We remark that $\mathbb{S}^2$ is particular in that not all CKVFs are global, as observed previously in subsection 2.2.2 (cf. Remark 2.16) and here in Section 3.5 (cf. Remark 3.14). A canonical form for global CKVFs, based in the canonical form of Section 3.2, is given in Section 3.7. This canonical form is used in Section 3.8 to obtain adapted coordinates which fit every global CKVF ξ. The adapted coordinates finds an application in subsection 3.9.1, where we find a class of metrics for which ξ is a Killing vector. Finally, in subsection 3.9.2 we obtain a class of TT tensors such that an arbitrary (non-necessarily global) CKVF satisfies the KID equations in two dimensions.

We note that Chapter 4 addresses similar issues but in arbitrary dimension. It is worth to remark that the low dimensional case deserves a separate analysis because some of the core results can be given with a level of detail difficult to reach in arbitrary dimension. For example, the explicit transformation which takes any skew-symmetric endomorphism and a CKVF to its canonical form is calculated. Similarly, the corresponding invariance group is also explicitly calculated. These two are examples of results which would be difficult to obtain in arbitrary dimension.

In this chapter, capital Latin indices I, J, K, take values in $0, 1, 2, 3$ and lower case Latin indices $i, j, k = 0, 1, 2$.

3.1 Basic properties of skew-symmetric endomorphism of $\mathbb{M}^{1,d-1}$

Let V be a d-dimensional vector space endowed with a Lorentzian metric g of signature $\{-, +, \cdots, +\}$. The vector space V shall often be identified with Minkowski spacetime $\mathbb{M}^{1,d-1}$ and the scalar product with g is denoted by $\langle\, ,\, \rangle$.

Definicin 3.1. An endomorphism $F : V \longrightarrow V$ is **skew-symmetric** when it satisfies

$$\langle x, F(y)\rangle = -\langle F(x), y\rangle \qquad \forall x, y \in V.$$

We denote this set by $\mathbf{SkewEnd}(\boldsymbol{V}) \subset \mathrm{End}\,(V)$.

We take, by definition, that eigenvectors of an endomorphism are always non-zero. In our convention all vectors with vanishing norm are null (in particular, the zero vector is null). We denote $\ker F$ and $\mathrm{Im}\ F$, respectively, to the kernel and image of $F \in \mathrm{End}\,(V)$.

The first result that we state and prove is a compendium of the basic properties of $\mathrm{SkewEnd}(V)$ that we shall often use.

Lemma 3.2. *[Basic facts about skew-symmetric endomorphisms] Let F be a skew-symmetric endomorphism in a Lorentzian vector space V of dimension d. Then*

1) $\forall w \in V$, $F(w)$ is perpendicular to w, i.e. $\langle F(w), w\rangle = 0$.

2) Im $F \subset (\ker F)^{\perp}$ and $\ker F \subset (\mathrm{Im}\ F)^{\perp}$.

3) If $w \in \ker F \cap \mathrm{Im}\ F$ then w is null.

4) If $w \in V$ is a non-null eigenvector of F, then its eigenvalue is zero.

5) If w is an eigenvector of F with zero eigenvalue, then all vectors in $\mathrm{Im}\ F$ are orthogonal to w, i.e. $\mathrm{Im}\ F \subset w^{\perp}$.

6) The non-zero eigenvalues of F are either real or purely imaginary.

7) If F restricts to a subspace $U \subset V$ (i.e. $F(U) \subset U$), then it also restricts to $U^{\perp}$.

8) $\dim \mathrm{Im}\ F$ *is always even. Equivalently,* $\dim \ker\ F$ *has the parity of* d.

Proof. Property 1) is immediate by definition of skew-symmetry $\langle w, F(w)\rangle = -\langle F(w), w\rangle$.

For 2), let $v \in \ker F$ and w be of the form $w = F(u)$ for some $u \in V$, then

$$\langle w, v\rangle = \langle F(u), v\rangle = -\langle u, F(v)\rangle = 0$$

the last equality following because $F(v) = 0$. As a consequence of 2) it follows 3), because w belongs both to $\ker F$ and to its orthogonal, so in particular it must be orthogonal to itself, hence null.

Property 4) is immediate from

$$0 = \langle w, F(w)\rangle = \lambda\langle w, w\rangle$$

so if w is non-null, its eigenvalue λ must be zero. Eigenvectors with zero eigenvalue may be both null and non-null.

Property 5) is a corollary of 2) because by hypothesis $w \in \ker F$ so

$$\mathrm{Im}\ F \subset (\ker F)^{\perp} \subset w^{\perp}$$

the last inclusion being a consequence of the general fact $U_1 \subset U_2 \Longrightarrow U_2^{\perp} \subset U_1^{\perp}$.

To prove 6), let λ be a non-zero (possibly complex) eigenvalue and w its associated eigenvector in $V_{\mathbb{C}}$, the complexification of V. Since F is real, the complex conjugate $\lambda^{\star} \in \mathbb{C}$ is an eigenvalue with eigenvector $w^{\star} \in V_{\mathbb{C}}$, so

$$\langle F(w), w^{\star}\rangle = \lambda \langle w, w^{\star}\rangle = -\lambda^{\star} \langle w, w^{\star}\rangle .$$

Thus, either λ is purely imaginary or, if not, $w, w^{\star}$ must be orthogonal and by 4) both must be also null. Then, denoting $w = u + iv$ for $u, v \in V$, the nullity condition implies $\langle u, v\rangle = 0$ and $\langle u, u\rangle = \langle v, v\rangle$ and orthogonality $w^{\star}$ implies $\langle u, u\rangle = -\langle v, v\rangle$. Hence u, v are null and proportional, i.e. $u = av$ for some $a \in \mathbb{R}$, in consequence $w = (a + i)v$. Therefore, $v \in V$ is a real null eigenvector and its corresponding eigenvalue λ must be real.

Property 7) is true because for any u in a F-invariant subspace U and $w \in U^{\perp}$

$$0 = \langle F(u), w\rangle = -\langle u, F(w)\rangle .$$

Finally, for 8), consider the 2-form $\boldsymbol{F}$ assigned to every $F \in \mathrm{SkewEnd}(V)$ by the standard relation

$$\boldsymbol{F}(e, e') = \left\langle e, F(e')\right\rangle , \qquad \forall e, e' \in V. \tag{3.1}$$

The matrix representing $\boldsymbol{F}$ is skew in the usual sense, hence the dimension of Im $\boldsymbol{F} \subset V^\star$ (the dual of V) is the rank of that matrix, which is known to be even (see e.g. [65]) and clearly $\dim \mathrm{Im}\ \boldsymbol{F} = \dim \mathrm{Im}\ F$. □

An important part of the results that we will prove rely on the existence of F-invariant two-dimensional non-null spaces. The next two lemmas give their basic properties.

Lemma 3.3. *Let $F \in \mathrm{SkewEnd}(V)$. Then F has a F-invariant spacelike plane Π_s if and only if*

$$F(u) = \mu v, \qquad F(v) = -\mu u, \tag{3.2}$$

for $\Pi_s = \mathrm{span}\{u, v\}$ with $u, v \in V$ spacelike, orthogonal, unit and $\mu \in \mathbb{R}$. Moreover, (3.2) is satisfied for $\mu \neq 0$ if and only if $\pm i\mu$ are eigenvalues of F with (null) eigenvectors $u \pm iv$,

$$F(u + iv) = -i\mu(u + iv), \qquad F(u - iv) = i\mu(u - iv), \tag{3.3}$$

for $u, v \in V$ spacelike, orthogonal with the same square norm.

Proof. If (3.2) is satisfied for $u, v \in V$ spacelike, orthogonal, unit, then $\Pi_s = \mathrm{span}\{u, v\}$ is obviously F-invariant spacelike. On the other hand, if Π_s is F-invariant, then it must hold that

$$F(u) = a_1 u + a_2 v, \qquad F(v) = b_1 u + b_2 v, \qquad a_1, a_2, b_1, b_2 \in \mathbb{R},$$

for a pair of orthogonal, unit, spacelike vectors u, v spanning Π_s. Using skew-symmetry and the orthogonality and unitarity of u, v, the constants are readily determined: $a_2 = b_2 = 0$ and $a_2 = -b_1 =: \mu$, which implies (3.2). This proves the first part of the lemma.

For the second part, it is immediate that if (3.2) holds with $\mu \neq 0$, then $\pm i\mu$ are eigenvalues of F with respective eigenvectors $u \pm iv$. The orthogonality of u, v follows from $\langle F(u), u \rangle = 0 = \mu \langle v, u \rangle$ and the equality of norm from skew-symmetry $\langle F(u), v \rangle = -\langle u, F(v) \rangle \Longrightarrow \mu \langle v, v \rangle = \mu \langle u, u \rangle$. Assume now that F has an eigenvalue $i\mu \neq 0$ with (necessarily null) eigenvector $w = u + iv$, for $u, v \in V$. Since F is real, neither u nor v can be zero. From the nullity property $\langle w, w \rangle = 0$, it follows that $\langle u, u \rangle - \langle v, v \rangle = 0$ and $\langle u, v \rangle = 0$. Hence, u, v are orthogonal with the same norm, so they are either null and proportional, which can be discarded because it would imply that u (and v) is a real eigenvector with complex eigenvalue; or otherwise u, v are spacelike, thus the lemma follows.

□

There is an analogous result for F-invariant timelike planes:

Lemma 3.4. *Let $F \in \mathrm{SkewEnd}(V)$. Then F has a F-invariant timelike plane Π_t if and only if*

$$F(e) = \mu v, \qquad F(v) = \mu e, \tag{3.4}$$

for $\Pi_t = \text{span}\{e, v\}$ *with* $e, v \in V$ *for* e *timelike unit orthogonal to* v *spacelike, unit and* $\mu \in \mathbb{R}$*. Moreover,* (3.2) *is satisfied for* $\mu \neq 0$ *if and only if* $\pm\mu$ *are eigenvalues of* F *with (null) eigenvectors* $e \pm v$*,*

$$F(e + v) = \mu(e + v), \qquad F(e - v) = -\mu(e - v). \tag{3.5}$$

for $e, v \in V$ *orthogonal, timelike and spacelike respectively with opposite square norm.*

Proof. For the first claim, repeat the first part of the proof of Lemma 3.3 assuming $u = e$ timelike.

For the second claim, assume (3.2) is satisfied with $\mu \neq 0$. Then it is immediate that $\langle F(e), e\rangle = 0 = \mu \langle v, e\rangle$, hence e, v are orthogonal and by skew-symmetry $\langle F(e), v\rangle = -\langle e, F(v)\rangle \implies \mu \langle v, v\rangle = -\mu \langle e, e\rangle$, i.e. must have opposite square norm. Conversely, let $\pm\mu \neq 0$ be a pair of eigenvalues with respective null eigenvectors $q_\pm$, that w.l.o.g can be chosen future directed. Then $e := q_+ + q_-$ and $v := q_+ - q_-$ are orthogonal, with opposite square norm $\langle e, e\rangle = 2 \langle q_+, q_-\rangle = -\langle v, v\rangle < 0$, and they satisfy (3.4). □

With these results we can now define:

Definicin 3.5. An F-invariant spacelike or timelike plane is called a **eigenplane** and the parameter μ in equations (3.2) and (3.4) is its associated **eigenvalue**.

Observacin 3.6. *Notice that a simple change of order in the vectors spanning a timelike or spacelike eigenplane switches the sign of the eigenvalue* μ*. Thus, unless otherwise stated, we will consider the* **eigenvalues of eigenplanes** *(both spacelike and timelike)* **non-negative** *by default.*

3.2 Canonical form of skew-symmetric endomorphisms in $\mathbb{M}^{1,3}$

The first step towards our canonical form for F is the following classification result, which relies on the properties described above.

Lemma 3.7 (Classification of $\text{SkewEnd}(\mathbb{M}^{1,3})$)**.** *Let* $F \in \text{SkewEnd}(V)$ *in a Lorentzian vector space* (V, g) *of dimension four. If* $F \neq 0$ *then one of the following exclusive possibilities hold:*

a) *F has a spacelike eigenvector orthogonal to a null eigenvector, both with vanishing eigenvalue.*

b) *F has a spacelike eigenplane (as well as a timelike orthogonal eigenplane).*

Proof. Since F is not identically zero, $\dim\ker F$ only can be either 2 or 0. Consider first $\dim\ker F = 0$ and let us prove that $b)$ must happen. We show this by proving that equations (3.3) and (3.5) must be satisfied. Since $\ker F = \{0\}$, F can only have non-zero eigenvalues, and we already know that they are either real or purely imaginary. The existence of a purely imaginary one leads to equations (3.3), which in turn implies (3.5). Suppose now that all eigenvalues are real non-zero. If there exist two different real eigenvalues μ, μ' their respective eigenvectors w, w' (which recall are null) must satisfy

$$\langle F(w), w' \rangle = \mu \langle w, w' \rangle = -\mu' \langle w, w' \rangle .$$

The product $\langle w, w' \rangle$ cannot be zero, as otherwise w, w' would be proportional and the eigenvalues μ and μ' would be the same. Thus, $\mu = -\mu'$, and hence (3.5), and also (3.3), hold. The remaining case is when all eigenvalues are equal, i.e. the characteristic polynomial is $p_F = (F - I\mu)^4$. By the Cayley-Hamilton theorem $\langle p_F(u), v \rangle = 0$, $\forall u, v \in V$. In particular, $\langle p_F(u), v \rangle = \langle p_F(v), u \rangle, \forall u, v \in V$. By skew-symmetry the even powers on each side cancel out and we are left with

$$\begin{aligned} -4\mu \langle F^3(u), v \rangle - 4\mu^3 \langle F(u), v \rangle &= -4\mu \langle F^3(v), u \rangle - 4\mu^3 \langle F(v), u \rangle \\ &= 4\mu \langle F^3(u), v \rangle + 4\mu^3 \langle F(u), v \rangle , \qquad \forall u, v \in V. \end{aligned}$$

Since we are in the case $\mu \in \mathbb{R} \setminus \{0\}$ we conclude that $F(F^2 + \mu^2) = 0$, and since F is invertible ($\ker F = \{0\}$) also $F^2 + \mu^2 = 0$. But this means that F admits a complex eigenvalue, which is a contradiction, and we have exhausted all possible cases with $\dim\ker F = 0$.

Now let $\dim\ker F = 2$. According to the causal character of $\ker F$, either $\ker F$ is null, and we are in case $a)$ of the lemma or $\ker F$ is non-degenerate, and we are in case $b)$. The fact that cases $a)$ and $b)$ are mutually exclusive is obvious. □

The classification in Lemma 3.7 contains two possible cases. It is common to use this result to find simple forms for each case, for example, in case $a)$ by including in the basis two orthogonal vectors $k, e \in \ker F$; or in case $b)$, by combining bases in the orthogonal and timelike eigenplanes, so that F is explicitly a direct sum of two 2-dimensional endomorphisms. In the following Proposition we find a canonical form which includes cases $a)$ and $b)$ simultaneously, and which depends on two parameters only.

Proposicin 3.8. *For every non-zero $F \in \mathrm{SkewEnd}(V)$, with (V, g) a four-dimensional Lorentzian vector space with a choice of time orientation , there exists an orthonormal unit basis $B := \{e_0, e_1, e_2, e_3\}$, with e_0 timelike future directed such that*

$$\begin{pmatrix} F(e_0) \\ F(e_1) \\ F(e_2) \\ F(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 + \frac{\sigma}{4} & \frac{\tau}{4} \\ 0 & 0 & 1 + \frac{\sigma}{4} & \frac{\tau}{4} \\ -1 + \frac{\sigma}{4} & -1 - \frac{\sigma}{4} & 0 & 0 \\ \frac{\tau}{4} & -\frac{\tau}{4} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \qquad \sigma, \tau \in \mathbb{R}, \tag{3.6}$$

where $\sigma := -\frac{1}{2}\mathrm{Tr}F^2$ *and* $\tau^2 := -4\det F$*, with* $\tau \geq 0$*. Moreover, if* $\tau = 0$ *the vector* e_3 *can be taken to be any spacelike unit vector lying in the kernel of* F*.*

Proof. By Lemma 3.7 there exist two possible cases. We start proving the proposition assuming that we are in case a). Let $\mathrm{span}\{k, e\} = \ker F$, with $k, e \in V$ a pair of orthogonal null and spacelike unit vectors respectively. We can complete them to a semi-null basis $B = \{k, l, w, e\}$, i.e. such that $\langle k, l\rangle = -2$, $\langle w, w\rangle = \langle e, e\rangle = 1$ and the rest of scalar products all zero. Using these orthogonality relations and skew-symmetry of F we can calculate:

$$F(k) = 0, \qquad F(l) = aw, \qquad F(w) = \frac{a}{2}k, \qquad F(e) = 0,$$

for a constant $a \in \mathbb{R} \setminus \{0\}$. Redefine a new basis $\{l', k', w', e'\}$, with $k' := \frac{\epsilon a}{2}k$, $l' := \frac{2\epsilon}{a}l$, $w' := -\epsilon w$, $e' := e$, where $\epsilon^2 = 1$ is chosen so that k', l' are future directed. Then

$$F(k') = 0, \qquad F(l') = -2w', \qquad F(w') = -k', \qquad F(e') = 0,$$

which in the orthonormal basis $B = \{e_0, e_1, e_2, e_3\}$ given by $k' = e_0 + e_1$, $l' = e_0 - e_1, w' = e_2, e' = e_3$ is

$$F(e_0) = -e_2, \qquad F(e_1) = e_2, \qquad F(e_2) = -e_0 - e_1, \qquad F(e_3) = 0.$$

This corresponds to expression (3.6) with $\sigma = \tau = 0$.

It remains to prove the proposition for case b). In this case, there exist timelike and spacelike eigenplanes, $\Pi_t = \mathrm{span}\{e'_0, e'_1\}$ and $\Pi_s = \mathrm{span}\{e'_2, e'_3\}$ respectively, i.e. fulfilling equations (3.2) and (3.4) for respective eigenvalues μ_0 and μ_1, such that at most one of them vanishes. We can take the bases of Π_t, Π_s so that that $B' := \{e'_0, e'_1, e'_2, e'_3\}$ is an orthonormal basis of V, with e_0 past directed and the eigenvalues μ_0 and μ_1 are positive or (at most one) zero. Then, the following change of basis is well-defined:

$$\begin{aligned} e_0 &= \frac{-1}{\sqrt{\mu_0^2+\mu_1^2}}\left[\left(1 + \frac{\mu_0^2+\mu_1^2}{4}\right)e'_0 + \left(1 - \frac{\mu_0^2+\mu_1^2}{4}\right)e'_2\right], \qquad & e_2 &= \frac{1}{\sqrt{\mu_0^2+\mu_1^2}}\left(\mu_0 e'_1 + \mu_1 e'_3\right), \\ e_1 &= \frac{1}{\sqrt{\mu_0^2+\mu_1^2}}\left[\left(1 - \frac{\mu_0^2+\mu_1^2}{4}\right)e'_0 + \left(1 + \frac{\mu_0^2+\mu_1^2}{4}\right)e'_2\right], \qquad & e_3 &= \frac{1}{\sqrt{\mu_0^2+\mu_1^2}}\left(-\mu_1 e'_1 + \mu_0 e'_3\right). \end{aligned} \tag{3.7}$$

One checks by explicit computation that $B := \{e_0, e_1, e_2, e_3\}$ is an orthonormal basis, with e_0 timelike and future directed (because $\langle e_0, e'_0\rangle > 0$). It is also a matter of direct calculation to see that

$$\begin{aligned} F(e_0) &= \left(-1 + \tfrac{\sigma}{4}\right)e_2 + \tfrac{\tau}{4}e_3, \qquad & F(e_1) &= \left(1 + \tfrac{\sigma}{4}\right)e_2 + \tfrac{\tau}{4}e_3, \\ F(e_2) &= \left(-1 + \tfrac{\sigma}{4}\right)e_0 - \left(1 + \tfrac{\sigma}{4}\right)e_1, \qquad & F(e_3) &= \tfrac{\tau}{4}\left(e_0 - e_1\right), \end{aligned}$$

where the parameters $\sigma, \tau \in \mathbb{R}$ are $\sigma = \mu_1^2 - \mu_0^2$ and $\tau = 2\mu_0\mu_1 \geq 0$. This corresponds to (3.6) with at most one of the parameters σ, τ vanishing.

To show the last statement, a simple computation shows that (when $\tau = 0$) the kernel of F is given by

$$\ker F = \left\{ a\left(1+\frac{\sigma}{4}\right)e_0 + a\left(1-\frac{\sigma}{4}\right)e_1 + be_3, \qquad a, b \in \mathbb{R} \right\}.$$

The subset of spacelike unit vectors in $\ker F$ is given by $1 + a^2\sigma > 0$ and $b = \epsilon\sqrt{1+a^2\sigma}$, $\epsilon = \pm 1$. We introduce the four vectors

$$\begin{aligned}
e'_0 &= \left(\frac{b+\epsilon}{2} + \left(1+\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}\right)e_0 + \left(1-\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}e_1 + a\left(1+\frac{\sigma}{4}\right)e_3,\\
e'_1 &= -\left(1-\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}e_0 + \left(\frac{b+\epsilon}{2} - \left(1+\frac{\sigma^2}{16}\right)\frac{b-\epsilon}{\sigma}\right)e_1 + a\left(-1+\frac{\sigma}{4}\right)e_3,\\
e'_2 &= \epsilon e_2,\\
e'_3 &= a\left(1+\frac{\sigma}{4}\right)e_0 + a\left(1-\frac{\sigma}{4}\right)e_1 + be_3,
\end{aligned}$$

and observe that they are well-defined for all values of σ, including zero. A straightforward computation shows that this is an orthonormal basis, and that (3.6) holds with $\tau = 0$. The last statement of the Proposition follows. □

Obtaining a canonical form in the three-dimensional case is much easier, the main reason being that any two-form in three-dimensions is simple, i.e. $\boldsymbol{F} \wedge \boldsymbol{F} = 0$ or, in other words, that $\boldsymbol{F}$ is of rank one in the sense of Darboux [141]. So, the reader may wonder why it has not been treated before. The reason is that we can obtain the three dimensional case as a direct corollary of the four-dimensional one. The construction is as follows. Let $F \in \mathrm{SkewEnd}(V)$ with V Lorentzian three-dimensional. From F we may define an auxiliary skew-symmetric endomorphism $\widehat{F}$ defined on $V \oplus \mathbb{E}_1$ endowed with the product metric ($\mathbb{E}_1$ is the one-dimensional Euclidean space). It is obvious that this space is a Lorentzian four-dimensional vector space. We denote by E_3 a unit vector in $\mathbb{E}_1$ and define $\widehat{F}$ simply by $\widehat{F}(u + aE_3) = F(u) + 0$, for all $u \in V$ and $a \in \mathbb{R}$ (we will identify $u \in V$ with $u + 0 \in V \oplus \mathbb{E}_1$ from now on). It is immediate to check that $\widehat{F}$ is skew-symmetric. Moreover, it has $\tau = 0$, by construction. Then, the following Corollary is immediate:

Corolario 3.9. *For every non-zero $F \in \mathrm{SkewEnd}(V)$, with (V, g) a Lorentzian three-dimensional vector space with a choice of time orientation, there exists an orthonormal basis $B := \{e_0, e_1, e_2\}$, with e_0 timelike future directed such that*

$$\begin{pmatrix} F(e_0)\\ F(e_1)\\ F(e_2)\end{pmatrix} = \begin{pmatrix} 0 & 0 & -1+\frac{\sigma}{4}\\ 0 & 0 & -1-\frac{\sigma}{4}\\ -1+\frac{\sigma}{4} & 1+\frac{\sigma}{4} & 0\end{pmatrix}\begin{pmatrix} e_0\\ e_1\\ e_2\end{pmatrix}, \qquad \sigma := -\frac{1}{2}\mathrm{Tr}\left(F^2\right) \in \mathbb{R}. \quad (3.8)$$

Proof. By the last statement of Proposition 3.8, the canonical basis $B = \{e_0, e_1, e_2, e_3\}$ of $\widehat{F}$ can be taken with $e_3 = E_3$, which means that $\{e_0, e_1, e_2\}$ is a basis of V. □

We may now properly define the canonical form of an element of $\mathrm{SkewEnd}(\mathbb{M}^{1,3})$ and $\mathrm{SkewEnd}(\mathbb{M}^{1,2})$

Definicin 3.10. An element F of $\mathrm{SkewEnd}(\mathbb{M}^{1,3})$ or $\mathrm{SkewEnd}(\mathbb{M}^{1,2})$ is said to be in **canonical form** if it takes the forms, respectively, of equations (3.6) or (3.8), in an orthonormal basis B, called **canonical basis**.

From the canonical form (3.6), we can recover (cf. Remark 3.11 below) the classification in Lemma 3.7 in terms of the parameters σ, τ. In a similar way, a classification result for three-dimensional skew-symmetric endomorphisms (cf. Remark 3.12 below) follows from the canonical form (3.8).

Observacin 3.11. *Let $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,3})$ be in canonical form. Then only two exclusive possibilities arise:*

1. *If either σ or τ do not vanish, F has a timelike eigenplane and an orthogonal spacelike eigenplane with respective eigenvalues*

$$\mu_t := \sqrt{(-\sigma+\rho)/2} \quad \textit{and} \quad \mu_s := \sqrt{(\sigma+\rho)/2} \quad \textit{for} \quad \rho := \sqrt{\sigma^2+\tau^2} \geq 0. \tag{3.9}$$

   The inverse relation between μ_t, μ_s and σ, τ is $\sigma = \mu_s^2 - \mu_t^2$ and $\tau = 2\mu_t\mu_s$.

2. *Otherwise, $\sigma = \tau = 0$ if and only if $\ker F$ is degenerate two-dimensional. Equivalently, F has a null eigenvector orthogonal to a spacelike eigenvector both with vanishing eigenvalue.*

One can easily check that when $\tau = 0$, the sign of σ determines the causal character of $\ker F$, *namely $\sigma < 0$ if $\ker F$ is spacelike, $\sigma = 0$ if $\ker F$ is degenerate and $\sigma > 0$ if $\ker F$ is timelike. Obviously, $\tau \neq 0$ implies $\ker F = \{0\}$. The characteristic polynomial of F is directly calculated from* (3.6)

$$\mathcal{P}_F(x) = (x^2 - \mu_t^2)(x^2 + \mu_s^2).$$

Observacin 3.12. *Let $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,2})$ be in canonical form. One can see by direct calculation that $q := (1+\sigma/4)e_0 + (1-\sigma/4)e_1$ generates $\ker F$ and furthermore $\langle q, q\rangle = -\sigma$. Hence, the sign of σ determines the causal character of $\ker F$, namely it is spacelike if $\sigma < 0$, degenerate if $\sigma = 0$ and timelike if $\sigma > 0$. Moreover, when $\sigma \neq 0$, F has an eigenplane with opposite causal character than q and eigenvalue $\sqrt{|\sigma|}$. The characteristic polynomial of F reads*

$$\mathcal{P}_F(x) = x(x^2 + \sigma).$$

At this point, it is convenient to comment on the relation between our results and previous canonical forms of skew-symmetric endomorphisms. It is standard in the literature to work with two-forms of $\mathbb{M}^{1,3}$, also called bivectors, instead of skew-symmetric endomorphisms. The usual classification of two-forms in $\mathbb{M}^{1,3}$ (which can be found in e.g. [73] and [142]) reduces to two cases with their respective canonical forms, namely

$$\boldsymbol{F} = a\boldsymbol{e} \wedge \boldsymbol{w} + b\boldsymbol{u} \wedge \boldsymbol{v}, \qquad \boldsymbol{F} = \boldsymbol{k} \wedge \boldsymbol{v}, \qquad a, b \in \mathbb{R} \tag{3.10}$$

where $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}$ are spacelike, unit and orthogonal to each other, $\boldsymbol{e}$ is unit and orthogonal to all of them and $\boldsymbol{k}$ is null and orthogonal to $\boldsymbol{v}$. Our main improvement is that we no longer need to distinguish two cases and we are able to cover every case with one single canonical form. The first of the canonical forms in (3.10) obviously corresponds to a skew-symmetric endomorphism which admits a timelike eigenplane with eigenvalue a and a spacelike eigenplane with eigenvalue b. These endomorphisms correspond to a canonical form (3.6) in which at least one of the parameters σ, τ is not zero (cf. Remark 3.11). From (3.10) it follows easily that a, b are directly related to the eigenvalues of F, specifically it holds $|a| = \mu_t$ and $|b| = \mu_s$. The second canonical form in (3.10) corresponds with a skew-symmetric endomorphism that has a null eigenvector orthogonal to a spacelike eigenvector, both with zero eigenvalue, which in our canonical form is $\sigma = \tau = 0$ (cf. Remark 3.11). We also remark that our result is valid only for real skew-symmetric endomorphisms, because it relies on Lemma 3.7.

The three dimensional case is always simple (i.e. a bivector) and thus can be written as product of two one-forms, whose causal character will determine the classification. Here we have treated this case as a corollary of the four-dimensional one. This approach will be useful in our extension of the classification results to the higher dimensional case in Chapter 4.

3.3 Group of invariance of the canonical form

In this section F is a non-zero skew-symmetric endomorphism in a four-dimensional vector space, and $B = \{e_0, e_1, e_2, e_3\}$ is a canonical basis, i.e. one where e_0 is future directed and (3.6) holds. It is useful to introduce the semi-null basis $\{\ell, k, e_2, e_3\}$ defined by $\ell = e_0 + e_1$, $k = e_0 - e_1$. In this basis the endomorphism F takes the form

$$F(\ell) = \frac{\sigma}{2} e_2 + \frac{\tau}{2} e_3, \quad F(k) = -2e_2, \quad F(e_2) = -\ell + \frac{\sigma}{4} k, \quad F(e_3) = \frac{\tau}{4} k. \tag{3.11}$$

We are interested in finding the most general orthochronous Lorentz transformation which transforms B into a basis $B' = \{e'_0, e'_1, e'_2, e'_3\}$ in which F takes the same form. In terms of the corresponding semi-null basis $\{\ell', k', e'_2, e'_3\}$ we must impose (3.11) with primed vectors. We start with the following lemma:

Lemma 3.13. *Let F be skew-symmetric and $\{\ell, k, e_2, e_3\}$ be a semi-null basis that satisfies*

$$F(k) = -2e_2, \qquad F(e_2) = -\ell + \frac{\sigma}{4}k \tag{3.12}$$

and

$$\langle F(\ell), F(\ell)\rangle = \frac{\sigma^2+\tau^2}{4}. \tag{3.13}$$

Then either the semi-null basis $\{\ell, k, e_2, e_3\}$ or $\{\ell, k, e_2, -e_3\}$ fulfils (3.11), and both do whenever $\tau = 0$.

Proof. Skew-symmetry imposes $F(\ell)$ and $F(e_3)$ to satisfy

$$F(\ell) = \frac{\sigma}{2}e_2 + \frac{q}{2}e_3, \qquad F(e_3) = \frac{q}{4}k', \qquad q \in \mathbb{R}.$$

Condition (3.13) imposes $q^2 = \tau^2$. Thus $q = \pm\tau$. Since reflecting e_3 replaces q by $-q$, either the basis $\{\ell, k, e_2, e_3\}$ or the basis $\{\ell, k, e_2, -e_3\}$ satisfies (3.11) with $\tau \geq 0$ (and both do in case $\tau = 0$). □

Thus, to understand the group of invariance of (3.11) it suffices to impose (3.12)-(3.13) for $\{\ell', k', e_2'\}$. Let us decompose k' in the original basis as

$$k' = Ak + B\ell + c_2e_2 + c_3e_3. \tag{3.14}$$

Observe that $A, B \geq 0$ as a consequence of k' being future directed. Let us introduce two vectors e_2' and ℓ' so that (3.12) are satisfied, namely

$$e_2' := -\frac{1}{2}F(k') = \left(A - \frac{B\sigma}{4}\right)e_2 - \frac{B\tau}{4}e_3 + \frac{c_2}{2}\ell - \frac{1}{8}\left(\sigma c_2 + \tau c_3\right)k, \tag{3.15}$$

$$\begin{aligned}\ell' &:= \frac{\sigma}{4}k' - F(e_2') \\ &= \frac{B\left(\sigma^2+\tau^2\right)}{16}k + A\ell - \frac{1}{4}\left(\sigma c_2 + \tau c_3\right)e_2 + \frac{1}{4}\left(\sigma c_3 - \tau c_2\right)e_3. \end{aligned}\tag{3.16}$$

The conditions of k' being null, future directed and e_2' spacelike and unit are easily found to be equivalent to

$$-4AB + ||c||^2 = 0, \qquad A, B \geq 0, \tag{3.17}$$

$$A^2 + \frac{\sigma^2+\tau^2}{16}B^2 + \frac{\sigma}{8}\left(c_2^2 - c_3^2\right) + \frac{\tau}{4}c_2c_3 = 1, \tag{3.18}$$

where we have set $||c||^2 = c_2^2 + c_3^2$. Under (3.17)-(3.18) one easily checks that the conditions $\langle e_2', k'\rangle = 0$, $\langle e_2', \ell'\rangle = 0$, $\langle \ell', \ell'\rangle = 0$ and $\langle \ell', k'\rangle = -2$ are all identically satisfied. Thus, $\{\ell', k', e_2'\}$ defines a timelike hyperplane and we can introduce e_3' as one of its two unit normals. By construction, the semi-null basis $\{\ell', k', e_2', e_3'\}$ satisfies

(3.12). By Lemma 3.13, this basis or the one defined with the reversed e_3' will be a canonical basis of F if and only if (3.13) is satisfied. By skew-symmetry, this condition is equivalent to

$$\langle \ell', F^2(\ell') \rangle + \frac{\sigma^2 + \tau^2}{4} = 0. \tag{3.19}$$

Directly from (3.11) we compute

$$F^2(\ell) = -\frac{\sigma}{2}\ell + \frac{\sigma^2+\tau^2}{8}k, \qquad F^2(k) = 2\ell - \frac{\sigma}{2}k,$$
$$F^2(e_2) = -\sigma e_2 - \frac{\tau}{2}e_3, \qquad F^2(e_3) = -\frac{\tau}{2}e_2,$$

from where it follows

$$F^2(\ell') = \frac{1}{2}\left(\frac{(\sigma^2+\tau^2)B}{4} - \sigma A\right)\ell + \frac{\sigma^2+\tau^2}{8}\left(A - \frac{1}{4}\sigma B\right)k$$
$$+ \frac{\left(2\sigma^2+\tau^2\right)c_2 + \sigma\tau c_3}{8}e_2 + \frac{\tau(\sigma c_2 + \tau c_3)}{8}e_3.$$

One easily checks that (3.19) is identically satisfied when (3.17)-(3.18) hold. Thus, it only remains to solve this algebraic system. To that aim, it is convenient to introduce $Q \geq 0$ and an angle $\theta \in [0, \frac{\pi}{2}]$ defined by

$$\sigma = Q\cos(2\theta), \qquad \tau = Q\sin(2\theta). \tag{3.20}$$

When $\sigma^2 + \tau^2 > 0$, $\{Q, \theta\}$ are uniquely defined. When $\sigma = \tau = 0$, then $Q = 0$ and θ can take any value. Define also λ_2, λ_3 by

$$c_2 = 2\lambda_2\cos\theta - 2\lambda_3\sin\theta, \qquad c_3 = 2\lambda_2\sin\theta + 2\lambda_3\cos\theta.$$

In terms of the new variables, equations (3.17)-(3.18) become (with obvious meaning for $||\lambda||^2$)

$$AB - ||\lambda||^2 = 0, \qquad 16A^2 + Q^2B^2 + 8Q\left(\lambda_2^2 - \lambda_3^2\right) - 16 = 0, \qquad A, B \geq 0.$$

When $Q = 0$, the solution is clearly $A = 1, B = ||\lambda||^2$, with unrestricted λ_2, λ_3. When $Q > 0$, we may multiply the first equation by Q and find the equivalent problem

$$(4A + QB)^2 = 16(1 + Q\lambda_3^2), \qquad (4A - QB)^2 = 16(1 - Q\lambda_2^2), \qquad A, B \geq 0.$$

This system is solvable if and only if

$$|\lambda_2| \leq \frac{1}{\sqrt{Q}} \tag{3.21}$$

and the solution is given by

$$A = \frac{1}{2}\left(\sqrt{1+Q\lambda_3^2} + \epsilon\sqrt{1-Q\lambda_2^2}\right), \quad B = \frac{2}{Q}\left(\sqrt{1+Q\lambda_3^2} - \epsilon\sqrt{1-Q\lambda_2^2}\right), \tag{3.22}$$

where $\epsilon = \pm 1$. Observe that the branches $\epsilon = 1$ and $\epsilon = -1$ are connected to each other across the set $|\lambda_2| = 1/\sqrt{Q}$. Note also that the case $Q = 0$ is included as a limit $Q \to 0$ in the branch $\epsilon = 1$ (and then the bound (3.21) becomes vacuous, in accordance with the unrestricted values of $\{\lambda_2, \lambda_3\}$ when $Q = 0$). We can now write down explicitly the vectors ℓ', k', e_2' defined in (3.14), (3.15) and (3.16). It is useful to introduce the two spacelike, orthogonal and unit vectors

$$u_2 = \cos\theta\, e_2 + \sin\theta\, e_3, \qquad u_3 = -\sin\theta\, e_2 + \cos\theta\, e_3$$

which simplify the expression to

$$\begin{aligned}
\ell' &= \frac{Q^2}{16} Bk + A\ell + \frac{Q}{2}\left(-\lambda_2 u_2 + \lambda_3 u_3\right), \\
k' &= Ak + B\ell + 2\lambda_2 u_2 + 2\lambda_3 u_3, \\
e_2' &= \left(\lambda_2 \cos\theta - \lambda_3 \sin\theta\right)\ell - \frac{Q}{4}\left(\lambda_2\cos\theta + \lambda_3 \sin\theta\right) k \\
&\quad + \epsilon\cos\theta\sqrt{1-Q\lambda_2^2}\, u_2 - \sin\theta\sqrt{1+Q\lambda_3^2}\, u_3,
\end{aligned}$$

where A, B must be understood as given by (3.22) (including the limiting case $Q = 0$). The fourth vector e_3' is unit and orthogonal to all of them. The following pair of vectors satisfy these properties (and of course there are no others),

$$\begin{aligned}
e_3' = \widehat{\epsilon}\Big(&\left(\lambda_3\cos\theta + \lambda_2 \sin\theta\right)\ell + \frac{Q}{4}\left(\lambda_3\cos\theta - \lambda_2\sin\theta\right) k \\
&+ \epsilon \sin\theta\sqrt{1-Q\lambda_2^2}\, u_2 + \cos\theta\sqrt{1+Q\lambda_3^2}\, u_3\Big)
\end{aligned} \tag{3.23}$$

where $\widehat{\epsilon} = \pm 1$. It is also straightforward to check that $F(e_3') = \widehat{\epsilon}(\tau/4)k'$. Thus, if $\tau \neq 0$, we must choose $\widehat{\epsilon} = 1$ while in the case $\tau = 0$ both signs are possible (in accordance with Lemma 3.13). Summarizing, the most general orthochronous Lorentz transformation that transforms a canonical semi-null basis of F into another one is given by

$$\begin{pmatrix} \ell' \\ k' \\ e_2' \\ \widehat{\epsilon} e_3' \end{pmatrix} = \begin{pmatrix}
\frac{1}{2}\left(\sqrt{1+Q\lambda_3^2} + \epsilon\sqrt{1-Q\lambda_2^2}\right) & \frac{Q}{8}\left(\sqrt{1+Q\lambda_3^2} - \epsilon\sqrt{1-Q\lambda_2^2}\right) & -Q\lambda_2/2 & Q\lambda_3/2 \\
\frac{2}{Q}\left(\sqrt{1+Q\lambda_3^2} - \epsilon\sqrt{1-Q\lambda_2^2}\right) & \frac{1}{2}\left(\sqrt{1+Q\lambda_3^2} + \epsilon\sqrt{1-Q\lambda_2^2}\right) & 2\lambda_2 & 2\lambda_3 \\
\lambda_2\cos\theta - \lambda_3\sin\theta & -Q(\lambda_2\cos\theta + \lambda_3\sin\theta)/4 & \epsilon\cos\theta\sqrt{1-Q\lambda_2^2} & -\sin\theta\sqrt{1+Q\lambda_3^2} \\
\lambda_3\cos\theta + \lambda_2\sin\theta & Q(\lambda_3\cos\theta - \lambda_2\sin\theta)/4 & \epsilon\sin\theta\sqrt{1-Q\lambda_2^2} & \cos\theta\sqrt{1+Q\lambda_3^2}
\end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix} := \mathcal{T}_F(\lambda_2, \lambda_3, \epsilon) \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix}$$

where $\widehat{\epsilon} = 1$, unless $\tau = 0$ in which case $\widehat{\epsilon} = \pm 1$. Concerning the global structure of the group, recall that λ_3 takes any value in the real line, while $|\lambda_2| \leq 1/\sqrt{Q}$. We have already mentioned that as long as $Q \neq 0$, the two branches $\epsilon = \pm 1$ are connected to each other through the values $|\lambda_2| = 1/\sqrt{Q}$. The topology of the group is therefore $\mathbb{R} \times \mathbb{S}^1$ (hence connected) when $Q \neq 0$ and $\tau \neq 0$. When $Q \neq 0$, $\tau = 0$ the group has two connected components (one corresponding to each value of $\widehat{\epsilon}$) each one with the topology of $\mathbb{R} \times \mathbb{S}^1$. Finally, when $Q = 0$, the group has two connected components (again one for each value of $\widehat{\epsilon}$) and the topology of each component is $\mathbb{R}^2$. By construction all elements of the group (in all cases) are orthochronous Lorentz transformations. Moreover, it is immediate to check that the determinant of $\mathcal{T}_F(\lambda_2, \lambda_3, \epsilon)$ is one for all values of $\lambda_2, \lambda_3, \epsilon$. Thus, all elements with $\widehat{\epsilon} = 1$ preserve orientation, while the elements with $\widehat{\epsilon} = -1$ reverse orientation.

3.3.1 Invariance group in the three-dimensional case

We have found before that for any non-zero skew-symmetric endomorphism F in $\mathbb{M}^{1,2}$ there exists an orthonormal, future directed basis $B_3 = \{e_0, e_1, e_2\}$ where F takes the canonical form (3.8). As in the previous case it is natural to ask what is the group of invariance of F, i.e. the most general orthochronous Lorentz transformation which transforms B into a basis where F takes the same form. From F, recall the auxiliary skew-symmetric endomorphism $\widehat{F}$ defined on $\mathbb{M}^{1,2} \oplus \mathbb{E}_1$ that was introduced before Corollary 3.9, that is, the endomorphism that acts as $\widehat{F}(u + ae_3) = F(u) + 0$, for all $u \in \mathbb{M}^{1,2}$ and $a \in \mathbb{R}$ where $\mathbb{E}_1 = \text{span}\{E_3\}$, with E_3 unit. Moreover, the basis $B := \{e_0, e_1, e_2, e_3 = E_3\}$ is canonical for $\widehat{F}$ in the sense of Definition 3.10 and in addition $\tau = 0$. It is clear that there exists a bijection between the set of orthonormal, future directed bases $B'_3 = \{e'_0, e'_1, e'_2\}$ where F takes its canonical form and the set of future directed orthonormal bases B' in $\mathbb{M}^{1,2} \oplus \mathbb{E}_1$ where $\widehat{F}$ takes its canonical form and the last element of B' is E_3. Thus, in order to determine the group of invariance of F it suffices to study the subgroup of invariance of $\widehat{F}$ which preserves the vector e_3. Since $\tau = 0$ we must impose

$$B = Q \sin(2\theta) = 2Q \cos\theta \sin\theta = 0$$

and three separate cases arise: (case 1) when $Q \neq 0, \theta = 0$, (case 2) when $Q = 0$ and (case 3) when $Q \neq 0, \theta = \pi/2$. Equivalently, cases 1, 2 and 3 correspond respectively to $\sigma > 0$, $\sigma = 0$ and $\sigma < 0$. Recall also that when $Q = 0$ we may choose any value of $\theta \in [0, \pi/2]$ w.l.o.g. We choose $\theta = 0$ in this case. Recall also that the case $Q = 0$ is recovered as a limit $Q \to 0$ after setting $\epsilon = 1$.

We only need to impose the condition $e'_3 = e_3$ in each case. Directly from (3.23) one finds (we also use that $Q = |\sigma|$)

$$
\begin{aligned}
e'_3 &= \widehat{\epsilon}\left(\lambda_3 \ell + \frac{|\sigma|}{4}\lambda_3 k + \sqrt{1+|\sigma|\lambda_3^2}\, e_3\right) && \text{Case 1} \\
e'_3 &= \widehat{\epsilon}\left(\lambda_3 \ell + e_3\right) && \text{Case 2} \\
e'_3 &= \widehat{\epsilon}\left(\lambda_2 \ell - \frac{|\sigma|}{4}\lambda_2 k + \epsilon\sqrt{1-|\sigma|\lambda_2^2}\right) e_3, && \text{Case 3}
\end{aligned}
$$

Thus, cases 1 and 2 require $\widehat{\epsilon} = 1, \lambda_3 = 0$ and in case 3 we must set $\widehat{\epsilon} = \epsilon, \lambda_2 = 0$. Inserting these values in the group of invariance of $\hat{F}$ one finds the most general orthochronous Lorentz transformation that preserves the form of F. We express the result in the canonically associated semi-null bases $\ell = e_0 + e_1, k = e_0 - e_1, e_2 = e_2$. Renaming λ_2, λ_3 as λ, the three cases can be written in the following form

$$
\begin{pmatrix} \ell' \\ k' \\ e'_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(1+\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & \frac{|\sigma|}{8}\left(1-\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & -\frac{|\sigma|\lambda}{2} \\ \frac{2}{|\sigma|}\left(1-\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & \frac{1}{2}\left(1+\epsilon\sqrt{1-|\sigma|\lambda^2}\right) & 2\lambda \\ \lambda & -\frac{|\sigma|\lambda}{4} & \epsilon\sqrt{1-|\sigma|\lambda^2} \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \end{pmatrix} \qquad \sigma \geq 0
$$

$$
\begin{pmatrix} \ell' \\ k' \\ e'_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\epsilon+\sqrt{1+|\sigma|\lambda^2}\right) & \frac{|\sigma|}{8}\left(\sqrt{1+|\sigma|\lambda^2}-\epsilon\right) & -\frac{|\sigma|\lambda}{2} \\ \frac{2}{|\sigma|}\left(\sqrt{1+|\sigma|\lambda^2}-\epsilon\right) & \frac{1}{2}\left(\sqrt{1+|\sigma|\lambda^2}+\epsilon\right) & -2\lambda \\ -\lambda & -\frac{|\sigma|\lambda}{4} & \sqrt{1+|\sigma|\lambda^2} \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \end{pmatrix} \qquad \sigma < 0
$$

with the understanding that the case $\sigma = 0$ is obtained from the first expression by setting $\epsilon = 1$ and then performing the limit $\sigma \to 0$.

When $\sigma > 0$, the parameter λ is restricted to $|\lambda| \leq 1/|\sigma|$ and the two branches $\epsilon = 1$ and $\epsilon = -1$ are connected through $|\lambda| = |\sigma|$. The group is connected and has topology $\mathbb{S}^1$. As an immediate consequence all the elements in the group are not only orthochronous Lorentz transformations (by construction) but also orientation preserving, as they are all connected to the identity. This can also be checked by computing the determinant of its matrix representation, which is one irrespectively of the value of λ and ϵ. When $\sigma = 0$ the parameter λ takes values in the real line and the group has $\mathbb{R}$-topology. Again all its elements are orientation preserving. In fact, in this case the group is simply the set of null rotations preserving ℓ. Finally, in the case $\sigma < 0$, λ also takes values in the real line and the group has two connected components (corresponding to the two values of ϵ). Each component has topology $\mathbb{R}$. The determinant of the matrix representation is now ϵ, so the Lorentz transformations with $\epsilon = 1$ preserve orientation (and define the connected component to the identity) while $\epsilon = -1$ reverse orientation.

3.4 Generators of the invariance group

Returning to the four dimensional case, the identity element $\boldsymbol{e}$ of the group of invariance corresponds to $\lambda_2 = \lambda_3 = 0$ and $\epsilon = \widehat{\epsilon} = 1$. We may compute the Lie algebra that generates it by taking derivatives of the group transformation with respect to λ_2 and λ_3 respectively and evaluating at $\boldsymbol{e}$. This defines two skew-symmetric endomorphisms

$$h_2 := \left.\frac{\partial \mathcal{T}_F(\lambda_2, \lambda_3, \epsilon)}{\partial \lambda_2}\right|_{\boldsymbol{e}}, \qquad h_3 := \left.\frac{\partial \mathcal{T}_F(\lambda_2, \lambda_3, \epsilon)}{\partial \lambda_3}\right|_{\boldsymbol{e}}.$$

It is immediate to obtain their explicit expression

$$\begin{pmatrix} h_2(\ell) \\ h_2(k) \\ h_2(e_2) \\ h_2(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{Q}{2}\cos\theta & -\frac{Q}{2}\sin\theta \\ 0 & 0 & 2\cos\theta & 2\sin\theta \\ \cos\theta & -\frac{Q}{4}\cos\theta & 0 & 0 \\ \sin\theta & -\frac{Q}{4}\sin\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix},$$

$$\begin{pmatrix} h_3(\ell) \\ h_3(k) \\ h_3(e_2) \\ h_3(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{Q}{2}\sin\theta & \frac{Q}{2}\cos\theta \\ 0 & 0 & -2\sin\theta & 2\cos\theta \\ -\sin\theta & -\frac{Q}{4}\sin\theta & 0 & 0 \\ \cos\theta & \frac{Q}{4}\cos\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell \\ k \\ e_2 \\ e_3 \end{pmatrix}.$$

Note that any skew-symmetric endomorphism G that commutes with F generates a one-parameter subgroup of Lorentz transformations that leaves the form of F invariant. It follows that this uniparametric group is necessarily a subgroup of the full invariance group of F. Hence G must belong to the Lie algebra generated by h_2 and h_3. Conversely, h_2, h_3 (and any linear combination thereof) defines a skew-symmetric endomorphism that commutes with F. In other words, $\mathcal{C}_F := \text{span}\{h_2, h_3\}$ defines the Lie subalgebra of $so(1,3)$ formed by the elements that commute with F. This Lie subalgebra is called the *centralizer* of F (e.g. [89]) and, as we have just shown, it is two-dimensional for any non-zero F. An easy computation shows that $[h_2, h_3] = 0$, so the centralizer of F is an Abelian Lie algebra. With these properties, it is not difficult to obtain the exponentiated form of the group elements. Define the two C^1 functions $t_\epsilon(s), t_3(s)$ (prime denotes derivative with respect to s)

$$t'_\epsilon = \epsilon\sqrt{1 - Qt_\epsilon^2}, \qquad t_\epsilon(s=0) = 0,$$
$$t'_3 = \sqrt{1 + Qt_3^2}, \qquad t_3(s=0) = 0,$$

and set

$$\mathcal{T}_\epsilon(s) := \begin{pmatrix} \frac{1}{2}(1+t'_\epsilon) & \frac{Q}{8}(1-t'_\epsilon) & -\frac{Q}{2}\cos\theta t_\epsilon & -\frac{Q}{2}\sin\theta t_\epsilon \\ \frac{2}{Q}(1-t'_\epsilon) & \frac{1}{2}(1+t'_\epsilon) & 2\cos\theta t_\epsilon & 2\sin\theta t_\epsilon \\ \cos\theta t_\epsilon & -\frac{Q}{4}\cos\theta t_\epsilon & \cos^2\theta t'_\epsilon + \sin^2\theta & \sin\theta\cos\theta\,(t'_\epsilon - 1) \\ \sin\theta t_\epsilon & -\frac{Q}{4}\sin\theta t_\epsilon & \sin\theta\cos\theta\,(t'_\epsilon - 1) & \sin^2\theta t'_\epsilon + \cos^2\theta \end{pmatrix}$$

$$\mathcal{T}_3(s) := \begin{pmatrix} \frac{1}{2}(1+t'_3) & \frac{Q}{8}(t'_3-1) & -\frac{Q}{2}\sin\theta t_3 & \frac{Q}{2}\cos\theta t_3 \\ \frac{2}{Q}(t'_3-1) & \frac{1}{2}(1+t'_3) & -2\sin\theta t_3 & 2\cos\theta t_3 \\ -\sin\theta t_3 & -\frac{Q}{4}\sin\theta t_3 & \cos^2\theta + t'_3\sin^2\theta & \sin\theta\cos\theta\,(1-t'_3) \\ \cos\theta t_3 & \frac{Q}{4}\cos\theta t_3 & \sin\theta\cos\theta\,(1-t'_3) & \sin^2\theta + \cos^2\theta t'_3 \end{pmatrix}$$

(in the right-hand sides t_ϵ, t'_ϵ etc. are to be understood evaluated at s). By direct computation one checks that (Id stands for the 4×4 identity matrix)

$$\begin{aligned} &\frac{d\mathcal{T}_\epsilon}{ds} = h_2 \mathcal{T}_\epsilon, \qquad \mathcal{T}_{\epsilon=1}(s=0) = \mathrm{Id}, \\ &\frac{d\mathcal{T}_3}{ds} = h_3 \mathcal{T}_3, \qquad \mathcal{T}_3(s=0) = \mathrm{Id}, \\ &\mathcal{T}_F(\lambda_2, \lambda_3, \epsilon)|_{\lambda_2 = t_\epsilon(s_1), \lambda_3 = t_3(s_2)} = \mathcal{T}_\epsilon(s_1)\mathcal{T}_3(s_2) = \mathcal{T}_3(s_2)\mathcal{T}_\epsilon(s_1). \end{aligned}$$

This shows in particular that $\mathcal{T}_{\epsilon=1}(s) = \exp(sh_2)$ and $\mathcal{T}_3(s) = \exp(sh_3)$. Observe also that (in agreement with a previous discussion), when $Q \neq 0$ the branch $\mathcal{T}_{\epsilon=-1}$ is connected to the branch $\mathcal{T}_{\epsilon=1}$ because in this case

$$\begin{aligned} t_{\epsilon=1}(s) &= \frac{\sin(\sqrt{Q}s)}{\sqrt{Q}}, \qquad & s \in \left[-\frac{\pi}{2\sqrt{Q}}, \frac{\pi}{2\sqrt{Q}}\right], \\ t_{\epsilon=-1}(s) &= -\frac{\sin(\sqrt{Q}s)}{\sqrt{Q}}, \qquad & s \in \left[-\frac{\pi}{2\sqrt{Q}}, \frac{\pi}{2\sqrt{Q}}\right], \end{aligned}$$

so that $s = \pm\pi/(2\sqrt{Q})$ in the first branch is smoothly connected to $s = \mp\pi/(2\sqrt{Q})$ in the second branch.

From the matrix representation of h_2 and h_3 it is obvious (the last two columns are linearly dependent) that $\det(h_2) = \det(h_3) = 0$ so both h_2, h_3 are simple, i.e. of matrix rank two. Moreover,

$$-\mathrm{tr}\,(h_2 \circ h_2) = \mathrm{tr}\,(h_3 \circ h_3) = 2Q \tag{3.24}$$

and $\mathrm{tr}\,(h_2 \circ h_3) = 0$. Given that F commutes with itself, i.e. $F \in \mathcal{C}_F$, it must be a linear combination of h_2 and h_3. Indeed, it is immediate to check that

$$F = -\cos\theta h_2 + \sin\theta h_3. \tag{3.25}$$

This expression suggests that the connection between F and the basis $\{h_2, h_3\}$ is via a duality rotation. To show that this is indeed the case, we define the one-forms $\{\boldsymbol{\ell}, \boldsymbol{k}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ metrically associated to the semi-null basis $\{\ell, k, e_2, e_3\}$. Also, for any skew-symmetric endomorphism F, we associate the two-form $\boldsymbol{F}$ by the standard relation (3.1). It is straightforward to find the explicit forms of $\boldsymbol{h_2}$ and $\boldsymbol{h_3}$ to be[1]

$$\boldsymbol{h_2} = \left(\boldsymbol{\ell} - \frac{Q}{4}\boldsymbol{k}\right) \wedge (\cos\theta \boldsymbol{e_2} + \sin\theta \boldsymbol{e_3}),$$
$$\boldsymbol{h_3} = \left(\boldsymbol{\ell} + \frac{Q}{4}\boldsymbol{k}\right) \wedge (-\sin\theta \boldsymbol{e_2} + \cos\theta \boldsymbol{e_3}). \tag{3.26}$$

Duality rotations of a two-form are defined in terms of the Hodge-dual operator, which in turn depends in a choice of orientation in the vector space. To keep the comparison fully general, we let $\kappa = +1$ ($\kappa = -1$) when the orientation in $\mathbb{M}^{1,3}$ is such that the basis $\{\ell, k, e_2, e_3\}$ is positively (negatively) oriented. Equivalently, if $\boldsymbol{\eta}$ is the volume form that defines the orientation, κ is given by

$$\boldsymbol{\eta}(\ell, k, e_2, e_3) = 2\kappa. \tag{3.27}$$

Let $\boldsymbol{G}^{\star}$ denote the Hodge dual[2] associated to $\boldsymbol{G}$. It is then immediate to check that

$$\boldsymbol{h_2^{\star}} = \kappa \boldsymbol{h_3}.$$

Defining $\boldsymbol{f} := -\boldsymbol{h_2}$ and $\mu := -\kappa\theta$, we may rewrite (3.25) as

$$\boldsymbol{F} = \cos\mu \boldsymbol{f} + \sin\mu \boldsymbol{f}^{\star} \tag{3.28}$$

which indeed shows that $\boldsymbol{F}$ is obtained from the simple form $\boldsymbol{f}$ by a duality rotation of angle μ. Notice that $f_{\alpha\beta}f^{\alpha\beta} = 2Q \geq 0$ (by (3.24)). For later use, we observe that the most general linear combination $\boldsymbol{f} = a_0\boldsymbol{h_2} + b_0\boldsymbol{h_3}$ that defines a simple 2-form such that $f_{\alpha\beta}f^{\alpha\beta} \geq 0$ and (3.28) holds for some value of μ is:

$$\begin{aligned} Q = 0: \quad & \boldsymbol{f} = -\cos(\theta + \kappa\mu)\boldsymbol{h_2} + \sin(\theta + \kappa\mu)\boldsymbol{h_3}, && \mu \in \mathbb{R} \\ Q > 0: \quad & \boldsymbol{f} = -\cos(n\pi)\boldsymbol{h_2}, \quad \mu = -\kappa\theta + n\pi, && n \in \mathbb{N}. \end{aligned} \tag{3.29}$$

This can be proved easily from the explicit expressions of $\boldsymbol{h_2}, \boldsymbol{h_3}$ and the fact that they are linearly independent simple 2-forms.

One may wonder whether this connection with duality rotations could have been used as the starting point to obtain in an easy and natural way the canonical form of F. We will argue that this alternative approach, although possible, it is far from obvious and cannot be regarded as natural.

We fix a skew-symmetric endomorphism F in a a four-dimensional vector space with a Lorentzian metric, and let $\boldsymbol{F}$ be the metrically associated 2-form. Define as before $\sigma := -\frac{1}{2}\text{trace}\left(F^2\right)$ and $\tau^2 = -4\det(F)$, $\tau > 0$ where the determinant is taken for any

[1]Our convention for the exterior product is $\boldsymbol{u} \wedge \boldsymbol{v} := \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}$.
[2]In abstract index notation $\boldsymbol{G}^{\star}_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\boldsymbol{G}^{\mu\nu}$.

matrix representation of F in an orthonormal basis. The invariants σ and τ are directly related to the two algebraic invariants of $\boldsymbol{F}$ as

$$\sigma = \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}, \qquad \tau = \frac{1}{2} \mathrm{abs} \left(F_{\alpha\beta} F^{\star\alpha\beta} \right). \tag{3.30}$$

The first one follows trivially from the definition of σ. The second is a well-known algebraic identity that can be found e.g. in [92]. Given F, a duality rotation of angle $-\mu$ defines the 2-form $\overset{\mu}{\boldsymbol{F}}$ as [127], [105],

$$\overset{\mu}{\boldsymbol{F}} := \cos\mu \boldsymbol{F} - \sin\mu \boldsymbol{F}^{*}. \tag{3.31}$$

A simple computation shows that $\overset{\mu}{\boldsymbol{F}}$ is simple (i.e. $\overset{\mu}{F}_{\alpha\beta} \overset{\mu}{F}{}^{\star\alpha\beta} = 0$) and satisfies $\overset{\mu}{F}{}_{\alpha\beta} \overset{\mu}{F}{}^{\alpha\beta} \geq$ 0 if and only if (cf. [105])

$$\begin{aligned} \sigma \sin(2\mu) + \widehat{\kappa}\tau\cos(2\mu) &= 0, \\ \sigma\cos(2\mu) - \widehat{\kappa}\tau\sin(2\mu) &\geq 0, \end{aligned} \tag{3.32}$$

where $\widehat{\kappa}$ is the sign defined by $\frac{1}{2} F_{\alpha\beta} F^{\star\alpha\beta} = \widehat{\kappa}\tau$ (when $\tau = 0$, $\widehat{\kappa}$ can take any value $\widehat{\kappa} = \pm 1$). Inserting (3.20) we find that whenever $Q = 0$ all values of μ solve (3.32) (which reflects the fact that $\boldsymbol{F}$ is null, and so are all its duality rotated 2-forms). When $Q \neq 0$, the solutions of (3.32) are $\mu = -\widehat{\kappa}\theta + n\pi$, $n \in \mathbb{N}$. Thus, we recover the expression in (3.29) provided we can ensure that $\widehat{\kappa} = \kappa$. Note that the sign of $F_{\alpha\beta} F^{\star\alpha\beta}$ only depends on F and the choice of orientation. It is a matter of direct checking that F as given in (3.11) with the choice of orientation where (3.27) holds satisfies $F_{\alpha\beta} F^{\star\alpha\beta} = 2\kappa\tau$, so that indeed $\widehat{\kappa} = \kappa$ follows (unless $\tau = 0$, of course, in which case $\widehat{\kappa} = \pm 1$).

We can now show how the canonical basis can be constructed from F using a duality rotation approach. Fixed an orientation on the vector space (i.e. a choice of volume form $\boldsymbol{\eta}$, and its associated Hodge dual) define σ and τ as in (3.30). Let $\widehat{\kappa} \in \{-1, 1\}$ be such that $2\widehat{\kappa} = F_{\alpha\beta} F^{\star\alpha\beta}$ (if $\tau = 0$, we allow any sign for $\widehat{\kappa}$). Introduce θ so that (3.20) holds with $\theta \in [0, \pi/2]$ (if $\sigma = \tau = 0$ then θ can take any value in this interval). Define then $\mu = -\widehat{\kappa}\theta$ and construct $\overset{\mu}{\boldsymbol{F}}$ by (3.31). We let $\boldsymbol{h_2} := -\overset{\mu}{\boldsymbol{F}}$. Since this 2-form is simple, there exist two linearly independent vectors a, b such that $\boldsymbol{h_2} = \boldsymbol{a} \wedge \boldsymbol{b}$. These vectors are obviously not unique, but certainly at least one of them must be spacelike. It can also be taken unit. We let $E_2 := b$ have this property. Exploiting the freedom $a \to a + sE_2$, $s \in \mathbb{R}$ we may take a perpendicular to E_2. By construction $(h_2)_{\alpha\beta} (h_2)^{\alpha\beta} \geq$ 0 (recall (3.32)) which is equivalent to $\langle a, a \rangle \geq 0$, i.e. a is spacelike or null. Let $Q \geq 0$ be defined by $Q = \langle a, a \rangle$. It is clear that there exists a timelike plane Π containing a and orthogonal to E_2 (this plane is obviously non-unique). Fixed Π, it is easy to show that there exists a future directed a null basis $\{\ell, k\}$ on Π satisfying $\langle \ell, k \rangle = -2$ and such that $\boldsymbol{a} = \boldsymbol{\ell} - (1/4)Q\boldsymbol{k}$. Finally, consider the timelike hyperplane defined by $\mathrm{span}\{\ell, k, E_2\}$ and select the unique unit normal E_3 to this hyperplane satisfying the orientation requirement (cf. (3.27))

$$\eta(\ell, k, E_2, E_3) = 2\widehat{\kappa}.$$

So far, from a non-zero F we have constructed a (collection of) semi-null basis $\{\ell, k, E_2, E_3\}$ in quite a natural way. Observe that when $\sigma = \tau = 0$, the angle θ is arbitrary, so the semi-null basis has extra additional freedom in this case. What appears to be hard to guess from this construction is that instead of $\{E_2, E_3\}$ we should introduce $\{e_2, e_3\}$ by means of the θ-dependent rotation (cf. (3.26))

$$E_2 = \cos\theta e_2 + \sin\theta e_3, \qquad E_3 = -\sin\theta e_2 + \cos\theta e_3. \tag{3.33}$$

It is by using this transformation that the form of F in the basis $\{\ell, k, e_2, e_3\}$ takes a form that depends only on the invariants σ, τ. It is remarkable that the θ-freedom inherent to the case $\sigma = \tau = 0$ (i.e. when F is null) drops out after performing the rotation (3.33), and we get a canonical form that covers all cases and depends only on σ and τ, irrespectively of which values these invariants may take.

3.5 Global conformal Killing vectors on the plane

In the following sections we connect our previous results with the Lie algebra of conformal Killing vector fields of the sphere and the group of motions they generate, i.e. the Möbius group. In our analysis, it is useful to employ the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Although we will rederive some of the results we need here, we refer the reader to [108] and [135] for more details about the Möbius transformations on the Riemann sphere. Some of the contents may also be found in other more general references such as [124] and [133]. Regarding Lie groups and Lie algebras, most of the results we will employ can be found in introductory level textbooks such as [73], but other references [71], [89] are also appropriate.

Consider the Euclidean plane $\mathbb{E}^2 = (\mathbb{R}^2, g_E)$ and select Cartesian coordinates $\{x, y\}$. Recall that the set of CKVFs on $\mathbb{E}^2$ is given by

$$\xi = U(x,y)\partial_x + V(x,y)\partial_y$$

where U, V satisfy the Cauchy-Riemann conditions $\partial_x U = \partial_y V$, $\partial_y U = -\partial_x U$. These vector fields satisfy

$$\mathcal{L}_\xi g_E = 2\left(\partial_x U + \partial_y V\right) g_E. \tag{3.34}$$

Observacin 3.14. *The space of CKVFs is in this case infinite dimensional, as it is obvious that every analytic complex function defines a solution of the conformal Killing* *equation* (3.34)*. As we have discussed in Remark 2.16, the uniparametric group of diffeomorphisms associated to a generic CKVF of* $\mathbb{E}^2$ *induces a conformal transformation in the sphere* $\mathbb{S}^2$ *which, in many cases, is a local conformal transformation which does not admit a global extension in* $\mathbb{S}^2$ *(e.g. [133] and [20]). Namely, in the terminology of Section 2.2.1, it is an element of* $\mathrm{ConfLoc}(\mathbb{S}^2)$ *and not of* $\mathrm{Conf}(\mathbb{S}^n)$*. As seen in subsection 2.2.2,* $\mathrm{ConfLoc}(\mathbb{S}^2)$ *is not a group. We are interested here in the global conformal diffeomorphisms* $\mathrm{Conf}(\mathbb{S}^2)$*, which as we will discuss below, correspond to Möbius transformations. Thus, we shall restrict our discussion to the CKVFs of* $\mathbb{E}^2$ *whose associated transformations are global in the sphere.*

We emphasize that the case $n = 2$ *is very special in that there exists conformal transformations of* $\mathbb{S}^2$ *which are not global. This does not happen for* $n > 2$*, where all conformal transformations are global [20]. This can be seen as an indirect consequence of* $\mathbb{S}^2$ *admitting a complex structure, which enlarges the number of solutions of* (3.34)*. This is a unique feature of* $\mathbb{S}^2$ *[22].*

Therefore, taking Remark 3.14 into account, we consider the one-point compactification of $\mathbb{E}^2$ into the Riemann sphere $\mathbb{S}^2$. It is standard (e.g. [133]) that the set of conformal Killing vectors that extend smoothly to $\mathbb{S}^2$ is given by the subset of CKVFs for which U and V are polynomials of degree at most two. In what follows, we shall restrict our discussion to this set.

Definicin 3.15. The set of conformal Killing vectors which extend globally to $\mathbb{S}^2$ are called **global conformal Killing vectors (GCKV)**.

Thus, the set of GCKV is parametrized by six real constants $\{\mathrm{b}_x, \mathrm{b}_y, \nu, \omega, \mathrm{a}_x, \mathrm{a}_y\}$ and take the form

$$\begin{aligned}\xi =& \left(\mathrm{b}_x + \nu x - \omega y + \frac{1}{2}\mathrm{a}_x\left(x^2 - y^2\right) + \mathrm{a}_y xy\right)\partial_x \\ &+ \left(\mathrm{b}_y + \nu y + \omega x + \frac{1}{2}\mathrm{a}_y\left(y^2 - x^2\right) + \mathrm{a}_x xy\right)\partial_x \\ &= \xi(\mathrm{a}_x, \mathrm{b}_x, \nu, \omega, \mathrm{b}_x, \mathrm{b}_y)\end{aligned} \tag{3.35}$$

It is clear that the use of complex coordinates is advantageous in this context. For reasons that will be clear later, it is convenient for us to introduce the complex coordinate $z = \frac{1}{2}(x - iy)$. In terms of z, the set of CKVFs is given by $\xi = f\partial_z + \overline{f}\partial_{\overline{z}}$ (recall that bar denotes complex conjugation) where f is a holomorphic function of z, while U, V are defined by $2f = U - iV$. The set of GCKV is parametrized by three complex constants $\{\mu_0, \mu_1, \mu_2\}$ as

$$\xi = \left(\mu_0 + \mu_1 z + \frac{1}{2}\mu_2 z^2\right)\partial_z + \left(\overline{\mu_0} + \overline{\mu_1 z} + \frac{1}{2}\overline{\mu_2 z^2}\right)\partial_{\overline{z}}. \tag{3.36}$$

The relationship between the two sets of parameters is immediately checked to be (we emphasize that this specific form depends on our choice of complex coordinate z)

$$\mu_0 = \frac{1}{2}\left(\mathrm{b}_x - i\mathrm{b}_y\right), \quad \mu_1 = \nu - i\omega, \quad \mu_2 = 2\left(\mathrm{a}_x + i\mathrm{a}_y\right). \tag{3.37}$$

We denote the GCKV with parameters $\mu := (\mu_0, \mu_1, \mu_2)$ as $\xi_{\{\mu\}}$. We shall need the following lemma concerning orthogonal and commuting GCKV. The result should be known but we did not find an appropriate reference.

Lemma 3.16. *Let $\xi_{\{\mu\}}, \xi_{\{\sigma\}}$ be global conformal Killing vector fields on $\mathbb{E}^2$ with corresponding parameters $\mu = \{\mu_0, \mu_1, \mu_2\}, \sigma = \{\sigma_0, \sigma_1, \sigma_2\}$. Assume that $\xi_{\{\mu\}}$ is not the zero vector field. Then*

1. *$\xi_{\{\sigma\}}$ is everywhere perpendicular to $\xi_{\{\mu\}}$ if and only if $\sigma = i\, r\, \mu$ with $r \in \mathbb{R}$.*
2. *$\xi_{\{\sigma\}}$ commutes with $\xi_{\{\mu\}}$ if and only if $\sigma = c\mu$ with $c \in \mathbb{C}$.*

Moreover, $\xi_{c\mu}$ has Euclidean norm

$$g_E(\xi_{\{c\mu\}}, \xi_{\{c\mu\}})|_p = |c|^2 g_E(\xi_{\{\mu\}}, \xi_{\{\mu\}})|_p, \qquad \forall p \in \mathbb{E}^2.$$

Proof. Let $f_\mu = \mu_0 + \mu_1 z + \frac{1}{2}\mu_2 z^2$ so that $\xi_{\{\mu\}} = f_\mu \partial_z + \overline{f_\mu}\partial_{\overline{z}}$ and define f_σ correspondingly. The Euclidean metric is $g_E = 4dzd\overline{z}$, so

$$g_E(\xi_{\{\mu\}}, \xi_{\{\mu\}})|_p = 2\left(f_\mu \overline{f_\sigma} + \overline{f_\mu} f_\sigma\right)|_{z(p)}. \tag{3.38}$$

The condition of orthogonality is equivalent to $f_\mu \overline{f_\sigma} + \overline{f_\mu} f_\sigma = 0$. This is a polynomial in $\{z, \overline{z}\}$, so its vanishing is equivalent to the vanishing of all its coefficients. Expanding, we find

$$\mu_0\overline{\sigma_0} + \overline{\mu_0}\sigma_0 = 0, \qquad \mu_1\overline{\sigma_1} + \overline{\mu_1}\sigma_1 = 0, \qquad \mu_2\overline{\sigma_2} + \overline{\mu_2}\sigma_2 = 0, \tag{3.39}$$

$$\mu_1\overline{\sigma_0} + \overline{\mu_0}\sigma_1 = 0, \qquad \mu_2\overline{\sigma_0} + \overline{\mu_0}\sigma_2 = 0, \qquad \mu_2\overline{\sigma_1} + \overline{\mu_1}\sigma_2 = 0. \tag{3.40}$$

Equations (3.39) are equivalent to the existence of three real numbers $\{q_1, q_2, q_3\}$ such that $\mu_a\overline{\sigma_a} = iq_a$, $a = 0, 1, 2$. Multiplying the equations in (3.40) respectively by $\mu_0\overline{\mu_1}$, $\mu_0\overline{\mu_2}$ and $\mu_1\overline{\mu_2}$ one finds

$$\begin{aligned} & q_0|\mu_1|^2 - q_1|\mu_0|^2 = 0, \quad q_0|\mu_2|^2 - q_2|\mu_0|^2 = 0, \quad q_1|\mu_2|^2 - q_2|\mu_1|^2 = 0 \\ \Longleftrightarrow \quad & (q_0, q_1, q_2) \times \left(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2\right) = (0, 0, 0), \end{aligned}$$

where $\times$ stands for the standard cross product. Since $(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2) \neq (0, 0, 0)$ (from our assumption that $\xi_{\{\mu\}}$ is not identically zero) there exists a real number r such that $(q_0, q_1, q_2) = -r(|\mu_0|^2, |\mu_1|^2, |\mu_2|^2)$. Thus $\mu_a\overline{\sigma_a} = -ir|\mu_a|^2$. Fix $a \in \{0, 1, 2\}$. If $\mu_a \neq 0$, it follows that $\overline{\sigma_a} = -ir\overline{\mu_a}$. If, instead, $\mu_a = 0$ then it follows from (3.40) (since at least one of the μ's is not zero) that $\sigma_a = 0$. In either case we have $\sigma_a = ir\mu_a$. This proves point 1. in the lemma.

For point 2. we compute the Lie bracket and find

$$[\xi_{\{\mu\}}, \xi_{\{\sigma\}}] = \left(f_\mu \frac{df_\sigma}{dz} - f_\sigma \frac{df_\mu}{dz} \right) \partial_z + \left(\overline{f_\mu} \frac{d\overline{f_\sigma}}{d\overline{z}} - \overline{f_\sigma} \frac{d\overline{f_\mu}}{d\overline{z}} \right) \partial_{\overline{z}}.$$

The two vectors commute iff

$$\begin{aligned} f_\mu \frac{df_\sigma}{dz} - f_\sigma \frac{df_\mu}{dz} &= \mu_0\sigma_1 - \mu_1\sigma_0 + (\mu_0\sigma_2 - \mu_2\sigma_0)\, z + \frac{1}{2}(\mu_1\sigma_2 - \mu_2\sigma_2)\, z^2 = 0 \\ \iff \qquad & (\sigma_0, \sigma_1, \sigma_2) \propto (\mu_0, \mu_1, \mu_2), \end{aligned}$$

and point 2. is proved. The last claim of the lemma follows from (3.38) and the linearity $f_{c\mu} = cf_\mu$. □

An immediate corollary of this result is that the set of GCKV that commute with a given GCKV $\xi_{\{\mu\}}$ is two-dimensional and generated by $\xi_{\{\mu\}}$ and $\xi^\perp_{\{\mu\}} := \xi_{\{-i\mu\}}$.

Recall that a Möbius transformation is a diffeomorphism of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ of the form

$$\begin{aligned} \chi^{\mathbb{A}} : \mathbb{C} \cup \{\infty\} &\longrightarrow \mathbb{C} \cup \{\infty\} \\ z &\longrightarrow \chi^{\mathbb{A}}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \mathbb{A} := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \end{aligned} \tag{3.41}$$

The set of Möbius transformations forms a group under composition, which we denote by Moeb, and the map $\chi : SL(2, \mathbb{C}) \longrightarrow \text{Moeb}$ defined by $\chi(\mathbb{A}) = \chi^{\mathbb{A}}$ is a group morphism. The kernel of this morphism is $K := \{\mathbb{I}_2, -\mathbb{I}_2\}$ and in fact χ descends to an isomorphism between $PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/K$ and Moeb. In geometric terms, the Möbius group corresponds to the set of orientation-preserving conformal diffeomorphisms of the standard sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$ (recall that a diffeomorphism $\chi := \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is conformal if $\chi^\star(g_{\mathbb{S}^2}) = \Omega^2 g_{\mathbb{S}^2}$ for some $\Omega \in C^\infty(\mathbb{S}^2, \mathbb{R}^+)$). The Möbius group thus transforms conformal Killing vectors of $\mathbb{S}^2$ into themselves, and, hence it also transforms global GCKV of $\mathbb{E}^2$ into themselves. In other words, given a GCKV $\xi_{\{\mu\}}$, the vector field $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}})$ is also a GCKV[3]. Let $\mu' := (\mu'_0, \mu'_1, \mu'_2)$ be the set of parameters of $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}}) =: \xi_{\{\mu'\}}$. A

[3]Note that $\chi^{\mathbb{A}}$ has singularities as a map from $\mathbb{E}^2$ into $\mathbb{E}^2$, but $\chi^{\mathbb{A}}_\star(\xi_{\{\mu\}})$ extends smoothly to all $\mathbb{E}^2$, and in fact to the whole Riemann sphere. Again this is standard and well-understood, so we will abuse the notation and write $\chi^A_\star$ as if the map χ^A were well-defined everywhere on $\mathbb{E}^2$

straightforward computations shows that

$$\begin{pmatrix} \mu_0' \\ \mu_1' \\ \mu_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha^2 & -\alpha\beta & \frac{1}{2}\beta^2 \\ -2\alpha\gamma & \alpha\delta+\beta\gamma & -\beta\delta \\ 2\gamma^2 & -2\gamma\delta & \delta^2 \end{pmatrix}}_{:=\mathbb{Q}_{\mathbb{A}}} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{pmatrix}. \tag{3.42}$$

The determinant of this matrix is one, so $\mathbb{Q}_{\mathbb{A}} \in SL(3,\mathbb{C})$. As a consequence of $\chi^{\mathbb{A}_1} \circ \chi^{\mathbb{A}_2} = \chi^{\mathbb{A}_1 \cdot \mathbb{A}_2}$ (where $\cdot$ denotes product of matrices), it follows that the map $\mathbb{Q} : SL(2,\mathbb{C}) \longrightarrow SL(3,\mathbb{C})$ defined by $\mathbb{Q}(\mathbb{A}) = \mathbb{Q}_{\mathbb{A}}$ is a morphism of groups, i.e. $\mathbb{Q}_{\mathbb{A}_1} \cdot \mathbb{Q}_{\mathbb{A}_2} = \mathbb{Q}_{\mathbb{A}_1 \cdot \mathbb{A}_2}$. This property can also be confirmed by explicit computation. In particular $\mathbb{Q}$ defines a representation of the group $SL(2,\mathbb{C})$ on $\mathbb{C}^3$. It is easy to show that this representation is actually isomorphic to the adjoint representation. Recall that for matrix Lie group G (i.e. a Lie subgroup of $GL(n,\mathbb{C})$), the adjoint representation Ad takes the explicit form (e.g. [73])

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ g &\longrightarrow \mathrm{Ad}(g) := \mathrm{Ad}_g : \begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{g} \\ X & \to & gXg^{-1} \end{array} \end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathrm{Aut}(\mathfrak{g})$ is the set of automorphisms of $\mathfrak{g}$. The isomorphism between $\mathbb{Q}$ and Ad is as follows. Let us choose the basis of $sl(2,\mathbb{C})$ given by

$$w^0 := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad w^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad w^2 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and define the vector space isomorphism $h : \mathbb{C}^3 \to sl(2,\mathbb{C})$ defined by $h(\mu_0,\mu_1,\mu_2) = \mu_a w^a$ $(a, b, \cdots = 0, 1, 2)$. One then checks easily by explicit computation that $h^{-1} \circ \mathrm{Ad}_g \circ h = \mathbb{Q}(g)$, for all $g \in SL(2,\mathbb{C})$.

Recall that the Killing form of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear map on $\mathfrak{g}$ defined by $B(\mathfrak{a}_1,\mathfrak{a}_2) := \mathrm{Tr}\,(\mathrm{ad}(\mathfrak{a}_1) \circ \mathrm{ad}(\mathfrak{a}_2))$ where $ad(\mathfrak{a})$, $\mathfrak{a} \in \mathfrak{g}$ is the adjoint endomorphism $ad(\mathfrak{a}) : \mathfrak{g} \to \mathfrak{g}$ defined by $ad(\mathfrak{a})(\mathfrak{b}) := [\mathfrak{a},\mathfrak{b}]$. The Lie algebra $sl(2,\mathbb{C})$ is semi-simple, so its Killing form is non-degenerate (e.g. [89]). The explicit form in the basis $\{w_0, w_1, w_2\}$ is given by

$$B(\mu_a w^a, \sigma_a w^a) = 8\,(\mu_1\sigma_1 - \mu_0\sigma_2 - \mu_2\sigma_0)\,.$$

A fundamental property of the Killing form is that it is invariant under automorphisms (see e.g. [29]), so in particular under the adjoint representation $B(\mathrm{Ad}_g(\mathfrak{a}), \mathrm{Ad}_g(\mathfrak{b})) = B(\mathfrak{a},\mathfrak{b})$ for all $g \in G$. Given $\{\mu\}$ we define two real quantities $\sigma_{\{\mu\})}$, $\tau_{\{\mu\}}$ by

$$\sigma_{\{\mu\}} - i\tau_{\{\mu\}} := 2\mu_0\mu_2 - \mu_1^2.$$

As a consequence of the discussion above, the quantities $\sigma_{\{\mu\}}$, $\tau_{\{\mu\}}$ associated to a GCKV $\xi_{\{\mu\}}$ are invariant under Möbius transformations. We have now all necessary ingredients to determine the set of Möbius transformations that transform a GCKV into its canonical form. Before doing so, however, we particularize some of the aspects of the CKVFs discussed in Section 2.2.1 to the case at hand of $\mathbb{S}^2$.

3.6 GCKV and skew-symmetric endomorphisms

We now give the explicit form of the isomorphism between the group $\mathrm{Conf}(\mathbb{S}^2)$ (represented as Möbius transformations on $\mathbb{R}^2$) and orthochronous component of the Lorentz group $O^+(1,3)$. Let $\mathbb{M}^{1,3}$ be endowed an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ with associated Cartesian coordinates $\{X^0, X^2, X^2, X^3\}$. Recall that in Section 2.2.1 (we particularize here to dimension $n = 2$), the conformal Euclidean plane was constructed based on certain choices, in particular, fixing a representative of the conformal sphere $\mathbb{S}^2 = \{X^0 = 1 = (X^1)^2 + (X^2)^2 + (X^3)^2\}$ and constructing the stereographic projection St_N w.r.t. to the pole $N = (1, -1, 0, 0)$ onto the plane $\Pi_N = \{X^0 = X^1 = 1, x := X^2, y := X^3\}$, which we identify with $\mathbb{E}^2$. With these choices, the explicit map between the set of skew-symmetric endomorphisms $\mathrm{SkewEnd}(\mathbb{M}^{1,3})$ and the set of CKVF on $\mathbb{E}^2$ is

$$\phi_\star : \mathrm{SkewEnd}(\mathbb{M}^{1,3}) \longrightarrow \mathrm{CKill}(\mathbb{E}^2)$$
$$F = \begin{pmatrix} 0 & -\nu & -\mathrm{a}_x + \frac{\mathrm{b}_x}{2} & -\mathrm{a}_y + \frac{\mathrm{b}_y}{2} \\ -\nu & 0 & -\mathrm{a}_x - \frac{\mathrm{b}_x}{2} & -\mathrm{a}_y - \frac{\mathrm{b}_y}{2} \\ -\mathrm{a}_x + \frac{\mathrm{b}_x}{2} & \mathrm{a}_x + \frac{\mathrm{b}_x}{2} & 0 & -\omega \\ -\mathrm{a}_y + \frac{\mathrm{b}_y}{2} & \mathrm{a}_y + \frac{\mathrm{b}_y}{2} & \omega & 0 \end{pmatrix} \longrightarrow \xi_F, \tag{3.43}$$

where ξ_F is given by (3.35) and we shall explicitly denote the dependence of ξ_F on the parameters $\{\mathrm{b}_x, \mathrm{b}_y, \nu, \omega, \mathrm{a}_x, \mathrm{a}_y\}$ by

$$\xi_F := \xi(\mathrm{b}_x, \mathrm{b}_y, \nu, \omega, \mathrm{a}_x, \mathrm{a}_y).$$

Also recall, that given an (active) orthochronous Lorentz transformation $\Lambda(e_\mu) = \Lambda^\nu{}_\mu e_\nu$, we may consider the skew-symmetric endomorphism $F_\Lambda := \Lambda \circ F \circ \Lambda^{-1}$. The construction above guarantees that

$$\xi_{F_\Lambda} = \Xi^\Lambda_\star(\xi_F)$$

where Ξ^Λ is the conformal diffeomorphism associated to the Lorentz transformation Λ. Let us restrict from now on to proper (i.e. orthochronous and orientation preserving) Lorentz transformations $SO^+(1,3)$. Thus, Ξ^Λ is an orientation preserving conformal diffeomorphism, and having fixed the coordinate system $\{x, y\} \in \mathbb{R}^2$, as well as $z = \frac{1}{2}(x - iy)$, Ξ^Λ is a Möbius transformation. Thus there exists a pair $\pm\mathbb{A} \in SL(2, \mathbb{C})$

such that $\chi^{\pm\mathbb{A}(\Lambda)} = \Xi^{\Lambda}$. We are interested in determining the explicit form of $\mathbb{A}(\Lambda)$ (actually of its inverse map $\Lambda(\mathbb{A})$). Having also fixed a future directed orthonormal basis $\{e_0, e_1, e_2, e_3\}$, we may represent a proper Lorentz transformation as an element of $SO^+(1,3)$ (the connected component of the identity of $SO(1,3)$). The aim is, thus, to determine the map $\mathcal{O} : SL(2,\mathbb{C}) \to SO^+(1,3)$ satisfying $\Xi^{\mathcal{O}(\mathbb{A})} = \chi^{\mathbb{A}}$. Of course, this map depends on the choices we have made concerning the pole N and plane Π_N to perform the stereographic projection.

As discussed at length in many references, (see e.g. [124], pp. 8-24), when the position vector of the north pole N' is chosen to be e_3, the plane is selected to be $\Pi'_{N'} = \{X^0 = 1, X^3 = 0\}$ and the complex coordinate z' in this plane is taken as $z' = X^1 + iX^2$, the corresponding map $\mathcal{O}'$ is (we parametrize $\mathbb{A}$ is in (3.41))

$$\mathcal{O}'(\mathbb{A}) = \frac{1}{2}\begin{pmatrix} \alpha\overline{\alpha}+\beta\overline{\beta}+\gamma\overline{\gamma}+\delta\overline{\delta} & \alpha\overline{\beta}+\beta\overline{\alpha}+\gamma\overline{\delta}+\delta\overline{\gamma}) & i(\alpha\overline{\beta}-\beta\overline{\alpha}+\gamma\overline{\delta}-\delta\overline{\gamma}) & \alpha\overline{\alpha}-\beta\overline{\beta}+\gamma\overline{\gamma}-\delta\overline{\delta} \\ \alpha\overline{\gamma}+\beta\overline{\delta}+\gamma\overline{\alpha}+\delta\overline{\beta} & \alpha\overline{\delta}+\beta\overline{\gamma}+\gamma\overline{\beta}+\delta\overline{\alpha} & i(\alpha\overline{\delta}-\beta\overline{\gamma}+\gamma\overline{\beta}-\delta\overline{\alpha}) & \alpha\overline{\gamma}-\beta\overline{\delta}+\gamma\overline{\alpha}-\delta\overline{\beta} \\ i(-\alpha\overline{\gamma}-\beta\overline{\delta}+\gamma\overline{\alpha}+\delta\overline{\beta}) & i(-\alpha\overline{\delta}-\beta\overline{\gamma}+\gamma\overline{\beta}+\delta\overline{\alpha}) & \alpha\overline{\delta}-\beta\overline{\gamma}-\gamma\overline{\beta}+\delta\overline{\alpha} & i(-\alpha\overline{\gamma}+\beta\overline{\delta}+\gamma\overline{\alpha}-\delta\overline{\beta}) \\ \alpha\overline{\alpha}+\beta\overline{\beta}-\gamma\overline{\gamma}-\delta\overline{\delta} & \alpha\overline{\beta}+\beta\overline{\alpha}-\gamma\overline{\delta}-\delta\overline{\gamma} & i(\alpha\overline{\beta}-\beta\overline{\alpha}-\gamma\overline{\delta}+\delta\overline{\gamma}) & \alpha\overline{\alpha}-\beta\overline{\beta}-\gamma\overline{\gamma}+\delta\overline{\delta} \end{pmatrix}$$

We may take advantage of this fact to determine our $\mathcal{O}(\mathbb{A})$. To do that we simply need to relate the action of the Möbius group in the plane $\Pi_N := \{X^0 = X^1 = 1\}$ (in the coordinate z) with the corresponding action on the plane $\Pi'_{N'} := \{X^0 = 1, X^3 = 0\}$ in the coordinate z'. At this point we can explain the reason why we have chosen $z = \frac{1}{2}(x - iy)$. The reason for the factor 2 comes from the fact that the plane Π_N lies at distance $d = 2$ from the point of stereographic projection, while the plane $\Pi'_{N'}$ lies at distance $d = 1$ of its corresponding stereographic point. The sign is introduced because the basis $\{-e_1, e_2, e_3\}$ (with respect to which the point N and the coordinates $\{x, y\}$ are defined) has opposite orientation than the basis $\{e_3, e_1, e_2\}$ with respect to which the point N' and the coordinates $\{X^1, X^2\}$ are built. By introducing a minus sign in z we make sure that the transformation ψ of $\mathbb{S}^2$ defined by $\{z(p) = z'(\chi(p))\}$ is orientation preserving (where $z(p)$ and $z'(p)$ stand for the two respective stereographic projections of $\mathbb{S}^2$ onto $\mathbb{C}^2 \cup \{\infty\}$). Now, a straightforward computation shows that an orientation preserving conformal diffeomorphism $\chi : \mathbb{S}^2 \to \mathbb{S}^2$ which in the plane Π_N takes the form

$$z(\chi(p)) = \frac{\alpha z(p) + \beta}{\gamma z(p) + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \qquad p \in \mathbb{S}^2,$$

has the following form in the $\Pi'_{N'}$ plane

$$z'(\chi(p)) = \frac{\alpha' z'(p) + \beta'}{\gamma' z'(p) + \delta'},$$

where

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} U, \qquad U := \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix}.$$

Since the map $\mathcal{O}'$ is a morphism of groups, it follows that the Lorentz transformation $\mathcal{O}(\mathbb{A})$ is given by

$$\mathcal{O}(\mathbb{A}) = \mathcal{O}'(\mathbb{A}') = \mathcal{O}'(U)^{-1}\mathcal{O}'(\mathbb{A})\mathcal{O}'(U)$$

The $SO^{+}(1,3)$ Lorentz matrix $\mathcal{O}'(U)$ is the rotation

$$\mathcal{O}'(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and we conclude that the Lorentz transformation $\mathcal{O}(\mathbb{A})$ takes the explicit form

$$\mathcal{O}(\mathbb{A}) = \frac{1}{2}\begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & -\alpha\overline{\alpha} + \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\beta} + \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma} & i(-\alpha\overline{\beta} + \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) \\ -\alpha\overline{\alpha} - \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\alpha} - \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} & -\alpha\overline{\beta} - \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma} & i(\alpha\overline{\beta} - \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) \\ \alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} + \delta\overline{\beta} & -\alpha\overline{\gamma} + \beta\overline{\delta} - \gamma\overline{\alpha} + \delta\overline{\beta} & \alpha\overline{\delta} + \beta\overline{\gamma} + \gamma\overline{\beta} + \delta\overline{\alpha} & i(-\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha}) \\ i(\alpha\overline{\gamma} + \beta\overline{\delta} - \gamma\overline{\alpha} - \delta\overline{\beta}) & i(-\alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} - \delta\overline{\beta}) & i(\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} - \delta\overline{\alpha}) & \alpha\overline{\delta} - \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha} \end{pmatrix} \quad (3$$

(to avoid ambiguities, recall that the Lorentz transformation defined by this matrix is $\Lambda(e_I) = \Lambda^J{}_I e_J$ with $\Lambda^J{}_I$ the row J and column I).

3.7 Canonical form of the GCKV

We start with a definition motivated by the canonical form of skew-symmetric endomorphisms discussed in Section 3.2.

Definicin 3.17. Let $\mathbb{E}^2$ be Euclidean space and $\{x, y\}$ a Cartesian coordinate system. A GCKV ξ is called **canonical** with respect to $\{x, y\}$ if it has the form

$$\xi = (\mu_0 + z^2)\partial_z + \left(\overline{\mu_0} + \overline{z}^2\right)\partial_{\overline{z}}, \qquad z := \frac{1}{2}(x - iy), \quad \mu_0 \in \mathbb{C}.$$

Equivalently, a GCKV is canonical with respect to $\{x, y\}$ whenever its corresponding form (3.36) has $\mu_1 = 0$ and $\mu_2 = 2$. We next characterize the class of Möbius transformations $\chi^{\mathbb{A}}$ which send a given GCKV into its canonical form.

Proposicin 3.18. *Let $\{x, y\}$ be a Cartesian coordinate system in $\mathbb{E}^2$. Let ξ be a non-trivial GCKV and define the complex constants $\{\mu_0, \mu_1, \mu_2\}$ such that $\xi = \xi_{\{\mu\}}$ when expressed in the complex coordinate $z = (x - iy)/2$ and its complex conjugate. Then $\chi^{\mathbb{A}} \in$ Moeb has the property that $\chi^{\mathbb{A}}_{\star}(\xi)$ is written in canonical form with respect to*

$\{x, y\}$ *if and only if*

$$\mathbb{A} = \begin{pmatrix} \frac{1}{2}\left(\delta\mu_2 - \gamma\mu_1\right) & \frac{1}{2}\delta\mu_1 - \gamma\mu_0 \\ \gamma & \delta \end{pmatrix}, \quad \frac{1}{2}\delta^2\mu_2 - \gamma\delta\mu_1 + \gamma^2\mu_0 = 1. \tag{3.45}$$

Moreover, for any such $\mathbb{A}$*, it holds*

$$\chi_\star^{\mathbb{A}}(\xi) = \left(\frac{1}{4}\left(\sigma_{\{\mu\}} - i\tau_{\{\mu\}}\right) + z^2\right)\partial_z + \left(\frac{1}{4}\left(\sigma_{\{\mu\}} + i\tau_{\{\mu\}}\right) + \bar{z}^2\right)\partial_{\bar{z}}.$$

Proof. From (3.42) and the fact that the canonical form has $\mu_1' = 0$ and $\mu_2' = 2$, we need to find the most general $\alpha, \beta, \gamma, \delta$ subject to $\alpha\delta - \beta\gamma = 1$ such that

$$-2\alpha\gamma\mu_0 + (\alpha\delta + \beta\gamma)\,\mu_1 - \beta\delta\mu_2 = 0, \tag{3.46}$$

$$2\gamma^2\mu_0 - 2\gamma\delta\mu_1 + \delta^2\mu_2 = 2. \tag{3.47}$$

The first can be written, using the determinant condition $\alpha\delta - \beta\gamma = 1$, as $-2\alpha\gamma\mu_0 + (1 + 2\beta\gamma)\mu_1 - \beta\delta\mu_2 = 0$. Multiplying by δ yields

$$\begin{aligned} 0 &= -2\alpha\delta\gamma\mu_0 + \delta\mu_1 + \beta\left(2\gamma\delta\mu_1 - \delta^2\mu_2\right) = -2\alpha\delta\gamma\mu_0 + \delta\mu_1 + \beta\left(2\gamma^2\mu_0 - 2\right) \\ &= -2\gamma\mu_0 + \delta\mu_1 - 2\beta \qquad \Longrightarrow \qquad \beta = \frac{1}{2}\delta\mu_1 - \gamma\mu_0, \end{aligned} \tag{3.48}$$

where in the second equality we used (3.47) and in the third one we inserted the determinant condition. To determine α we compute

$$\begin{aligned} & \alpha\delta = 1 + \beta\gamma = 1 + \frac{1}{2}\gamma\delta\mu_1 - \gamma^2\mu_0 = \frac{1}{2}\delta\left(\delta\mu_2 - \gamma\mu_1\right) \\ \Longrightarrow \quad & \delta\left(\alpha + \frac{1}{2}\gamma\mu_1 - \frac{1}{2}\delta\mu_2\right) = 0, \end{aligned}$$

where in the third equality we used (3.47) to replace $\gamma^2\mu_0$. If $\delta \neq 0$ we conclude that $\alpha = (1/2)(\gamma\mu_1 - \delta\mu_2)$, and the form of $\mathbb{A}$ is necessarily as given in (3.45). If, on the other hand, $\delta = 0$, then the determinant condition forces $\gamma \neq 0$. Thus, equation (3.46) gives $-2\alpha\mu_0' + \beta\mu_1 = 0$, which after using (3.48) implies $\alpha = -(1/2)\gamma\mu_1$, so (3.45) also follows. This proves the "only if" part of the statement. For the "if" part one simply checks that β and α obtained above indeed satisfy (3.46)-(3.47), as soon as γ, δ satisfy the determinant condition given in (3.45).

The second part of the Proposition is immediate form the fact that $2\mu_0\mu_2 - \mu_1^2$ is invariant under (3.42). Thus, $\chi_\star^{\mathbb{A}}(\xi)$ has μ_0' satisfying

$$4\mu_0' = 2\mu_0'\mu_2' - {\mu_1'}^2 = 2\mu_0\mu_2 - \mu_1^2 = \sigma_{\{\mu\}} - i\tau_{\{\mu\}}. \tag{3.49}$$

□

Corolario 3.19. *The subgroup of $SL(2,\mathbb{C})$ that leaves invariant a GCKV field in canonical form with parameter μ_0 is given by*

$$\mathbb{A}_{\mu_0} = \left\{ \begin{pmatrix} \delta & -\gamma\mu_0 \\ \gamma & \delta \end{pmatrix}, \quad \delta^2 + \mu_0\gamma^2 = 1 \right\}.$$

Proof. Insert $\mu_1 = 0$ and $\mu_2 = 2$ into (3.45).

Corolario 3.20. *Given any GCKV ξ as in Proposition 3.18, the set of elements $\mathbb{A} \in SL(2,\mathbb{C})$ such that $\chi_\star^{\mathbb{A}}(\xi)$ takes the canonical form is*

$$\mathbb{A}_{\frac{1}{4}(\sigma_{\{\mu\}} - i\tau_{\{\mu\}})} \cdot \mathbb{A}_0$$

where $\mathbb{A}_0$ is any element of $SL(2,\mathbb{C})$ satisfying (3.45).

Proof. Fix $\mathbb{A}_0$ satisfying (3.45). Any other element $\mathbb{A}_1$ will satisfy (3.45) if and only if $\mathbb{A}_1 \cdot \mathbb{A}_0^{-1}$ leaves invariant the column vector $(\mu_0', 0, 2)$, $4\mu_0' := \sigma_{\{\mu\}} - i\tau_{\{\mu\}}$, i.e. if and only if $\mathbb{A}_1 \cdot \mathbb{A}_0^{-1} \in \mathbb{A}_{\mu_0'}$ (cf. Corollary 3.19). Thus $\mathbb{A}_1 = \mathbb{A}_{\mu_0'} \cdot \mathbb{A}_0$ and the corollary is immediate by Proposition 3.18. □

Corolario 3.21. *Let F be a non-zero skew-symmetric endomorphism in $\mathbb{M}^{1,3}$ and let the matrix (F) be defined by $F(e_I) = F^J{}_I e_J$ where $\{e_I\}_{I=0,1,2,3}$ is an orthonormal basis. Define $\{\mathrm{b}_x, \mathrm{b}_y, \nu, \omega, \mathrm{a}_x, \mathrm{a}_y\}$ so that (F) reads as in (3.43). Define μ_0, μ_1, μ_2 by means of (3.37) and let $\Lambda := \mathcal{O}(\mathbb{A})$, where $\mathbb{A}$ is any of the matrices defined in Proposition 3.18. Then, in the basis $e'_J := \Lambda^I{}_J e_I$, the endomorphism F takes the canonical form (3.6) with $\sigma - i\tau = 2\mu_0\mu_2 - \mu_1^2$.*

In Proposition 3.8 we showed the existence of the canonical form of $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,3})$, and this motivated the Definition 3.17 of canonical form of GCKVs. However, it is only in Corollary 3.21 that we have been able to (easily) find the explicit change of basis that takes F to its canonical form. This is possible because we are dealing with low dimensions and the GCKVs take a very simple expression in complex coordinates of the Riemann sphere, but this is a much more difficult problem in higher dimensions.

We can however easily derive the three-dimensional case as a simple consequence. For that we consider, as usual, the extension $\widehat{F} \in \mathrm{SkewEnd}(\mathbb{M}^{1,3})$ of $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,2})$ described before Corollary 3.9. In the basis $\{e_0, e_1, e_2, e_3 := E_3\}$, $\widehat{F}$ has $\mathrm{a}_y = \mathrm{b}_y = \omega = 0$, so the quantities μ_0, μ_1, μ_2 defined in (3.37) are real. In order to apply Corollary 3.21 to find the change of orthonormal basis $\{e_0, e_1, e_2\}$ that brings F into its canonical form we simply need to impose that $e'_3 = e_3$, which amounts to $\Lambda^0{}_3 = \Lambda^1{}_3 = \Lambda^2{}_3 = 0$ and $\Lambda^0{}_3 = 1$. It is easy to show (recall that α, β are expressed in terms of γ, δ in the matrix $\mathbb{A}$ of Corollary 3.21) that the general solution to the first three equations is $\gamma\bar{\delta} = \bar{\gamma}\delta$.

The condition $\Lambda^0{}_3 = 1$ is then

$$\frac{1}{2}\delta\bar{\delta}\mu_2 - \gamma\bar{\delta}\mu_1 + \gamma\bar{\gamma}\mu_0 = 1.$$

Multiplying by δ and using the determinant condition in (3.45) implies $\delta = \bar{\delta}$, while multiplying by γ gives $\gamma = \bar{\gamma}$, and then $\Lambda^0{}_3 = 1$ is just identical to the determinant condition so no more consequences can be extracted. Thus all parameters $\alpha, \beta, \gamma, \delta$ are real. Summarizing:

Corolario 3.22. *Let F be a non-zero skew-symmetric endomorphism of $\mathbb{M}^{1,2}$ and the matrix (F) be defined by $F(e_i) = F^j{}_i e_j$ where $\{e_i\}_{i=0,1,2}$ is an orthonormal basis. Define $\mu_0 := (F^1{}_3 - F^2{}_3)/2,\ \mu_1 := -F^1{}_2,\ \mu_2 := -(F^1{}_3 + F^2{}_3)$. For any pair of real numbers γ, δ satisfying $\delta^2\mu_2 - 2\gamma\delta\mu_1 + 2\gamma^2\mu_0 = 2$, let $\alpha := (\delta\mu_2 - \gamma\mu_1)/2$ and $\beta := \delta\mu_1/2 - \gamma\mu_0$. Then, in the basis $e'_i := \Lambda^j{}_i e_j$, with*

$$\Lambda := \begin{pmatrix} \frac{1}{2}\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right) & \frac{1}{2}\left(-\alpha^2+\beta^2-\gamma^2+\delta^2\right) & \alpha\beta+\gamma\delta \\ \frac{1}{2}\left(-\alpha^2-\beta^2+\gamma^2+\delta^2\right) & \frac{1}{2}\left(\alpha^2-\beta^2-\gamma^2+\delta^2\right) & -\alpha\beta+\gamma\delta \\ \alpha\gamma+\beta\delta & -\alpha\gamma+\beta\delta & \alpha\delta+\beta\gamma \end{pmatrix},$$

the endomorphism F takes the canonical form (3.8) with $\sigma = 2\mu_0\mu_2 - \mu_1^2$.

3.8 Adapted coordinates to a GKCV

So far we have explored the action of the Möbius group on a GCKV and have found that for any such vector, there exists a set of transformations that brings it into a canonical form. The perspective so far has been active. We now change the point of view and exploit the previous results to find coordinate systems in (appropriate subsets of) $\mathbb{E}^2$ that rectify a given (and fixed) GKCV ξ.

Consider $\mathbb{E}^2$ and fix a non-trivial GCKV field ξ. Let us select a Cartesian coordinate system $\{x, y\}$ and define, as before $z = (1/2)(x - iy)$ and $\overline{z} = (1/2)(x + iy)$. When expressed in the $\{z, \overline{z}\}$ coordinate system ξ will be $\xi = \xi_{\{\mu\}}$ for some triple of complex numbers $\{\mu\} = \{\mu_0, \mu_1, \mu_2\}$. We now view the Möbius transformation as a change of coordinates. Specifically, given $\alpha, \beta, \gamma, \delta$ complex constants satisfying $\alpha\delta - \beta\gamma = 1$, the quantity

$$\omega = \frac{\alpha z + \beta}{\gamma z + \delta}$$

and its complex conjugate $\overline{\omega}$ define a coordinate system on $\mathbb{R}^2 \setminus \{\gamma z + \delta = 0\}$. The inverse of this coordinate transformation is, obviously,

$$z = \frac{\delta\omega - \beta}{-\gamma\omega + \alpha}. \tag{3.50}$$

It is well-known that transformations of a manifold can be dually seen as coordinate changes in suitable restricted coordinate patches. We will refer to (3.50) as a Möbius coordinate change. With this point of view, we may express ξ in the coordinate system $\{\omega, \overline{\omega}\}$ and the duality above implies that ξ takes the form

$$\xi = \left(\mu'_0 + \mu'_1\omega + \frac{1}{2}\mu'_2\omega^2\right)\partial_\omega + \left(\overline{\mu'_0} + \overline{\mu'_1}\,\overline{\omega} + \frac{1}{2}\overline{\mu'_2}\,\overline{\omega}^2\right)\partial_{\overline{\omega}}$$

with $\{\mu'_0, \mu'_1, \mu'_2\}$ given by (3.42) (this can also be checked by direct computation).

We may now take $\{\alpha, \beta, \gamma, \delta\}$ so that corresponding matrix $\mathbb{A}$ satisfies (3.45). It follows that ξ takes the canonical form

$$\xi := \left(\frac{1}{4}\left(\sigma_{\{\mu\}} - i\tau_{\{\mu\}}\right) + \omega^2\right)\partial_\omega + \left(\frac{1}{4}\left(\sigma_{\{\mu\}} + i\tau_{\{\mu\}}\right) + \overline{\omega}^2\right)\partial_{\overline{\omega}}.$$

By Lemma 3.16, the vector $\xi^\perp$ defined by $\xi^\perp := \xi_{\{i\,\mu\}}$ is a GCKV orthogonal to ξ everywhere, with the same pointwise norm as ξ and satisfying $[\xi, \xi^\perp] = 0$. In particular ξ and $\xi^\perp$ are linearly independent except at points where both vanish identically. As a consequence, it makes sense to tackle the problem of finding coordinates that rectify ξ by trying to determine a coordinate system $\{v_1, v_2\}$ (on a suitable subset of $\mathbb{R}^2$) such that

$$\xi = \partial_{v_1}, \qquad \xi^\perp = \partial_{v_2}.$$

Assume that we have already transformed into the coordinates $\{\omega, \overline{\omega}\}$ where ξ (and also $\xi^\perp$) take their canonical forms

$$\xi = \left(\frac{1}{4}Qe^{-2i\theta} + \omega^2\right)\partial_\omega + \text{c.c}, \qquad \xi^\perp = \left(\frac{i}{4}Qe^{-2i\theta} + i\omega^2\right)\partial_\omega + \text{c.c} \tag{3.51}$$

where we have defined the real constants $Q \geq 0$ and $\theta \in [0, \pi)$ by

$$\sigma_{\{\mu\}} - i\tau_{\{\mu\}} = Qe^{-2i\theta} \tag{3.52}$$

and where c.c. stands for complex conjugate of the previous term. We are seeking a coordinate system $\{\zeta, \overline{\zeta}\}$ defined by

$$\zeta := \frac{1}{2}\left(v_1 + iv_2\right)$$

such that

$$\xi - i\xi^\perp = \partial_\zeta$$

(this is because $\partial_\zeta = \partial_{v_1} - i\partial_{v_2}$). Since $\xi - i\xi^\perp = 2\left(\frac{1}{4}Qe^{-2i\theta} + \omega^2\right)\partial_\omega$ the coordinate change musty satisfy the ODE

$$\frac{d\zeta}{d\omega} = \frac{1}{2\omega^2 + \frac{Q}{2}e^{-2i\theta}}.$$

This equation can be integrated immediately. The result is

$$\zeta(\omega) = \zeta_0 + \frac{-ie^{i\theta}}{2\sqrt{Q}} \ln\left(\frac{\omega - i\frac{\sqrt{Q}}{2}e^{-i\theta}}{\omega + i\frac{\sqrt{Q}}{2}e^{-i\theta}}\right) \qquad \Longleftrightarrow$$

$$\omega(\zeta;\zeta_0) = \frac{i\sqrt{Q}e^{-i\theta}}{2}\frac{1 + e^{2i\sqrt{Q}e^{-i\theta}(\zeta-\zeta_0)}}{1 - e^{2i\sqrt{Q}e^{-i\theta}(\zeta-\zeta_0)}}, \tag{3.53}$$

where ζ_0 is an arbitrary complex constant. These expressions include the case $Q = 0$ as a limit. Explicitly

$$\zeta - \zeta_0 = -\frac{1}{2\omega} \qquad \Longleftrightarrow \qquad \omega = -\frac{1}{2(\zeta - \zeta_0)}. \tag{3.54}$$

Since the logarithm is a multivalued complex function, one needs to be careful concerning the domain and range of this coordinate change. In the $\{\omega, \overline{\omega}\}$ plane, the vector field ξ vanishes at the two points (cf. (3.51)) $\omega = \pm i\frac{\sqrt{Q}}{2}e^{-i\theta}$ (which degenerate to the point at the origin when $Q = 0$). It is clear that neither of these points will be covered by the $\{\zeta, \overline{\zeta}\}$ coordinate system. The case $Q = 0$ is very simple because, from (3.54), it is clear that the $\{\zeta, \zeta\}$ coordinate system covers the whole $\{\omega, \overline{\omega}\}$ plane except the origin. Since the point at infinity in the ω-plane is sent to the point ζ_0 in the ζ-plane we conclude that the $\{\zeta, \overline{\zeta}\}$ coordinate covers the whole Riemann sphere except the single point where ξ vanishes.

When $Q \neq 0$, the situation is more interesting. The reason in the multivaluedness of the logarithm. This suggests that the coordinate change may in fact define a larger manifold that covers the original one. In order to discuss this, let is introduce the auxiliary function

$$\mathfrak{z} := \frac{\omega - i\frac{\sqrt{Q}}{2}e^{-i\theta}}{\omega + i\frac{\sqrt{Q}}{2}e^{-i\theta}}.$$

This is a Möbius transformation, so it maps diffeomorphically $\mathbb{C} \cup \{\infty\}$ onto itself. The two zeroes of ξ are mapped respectively to the origin and infinity in the $\mathfrak{z}$ variable. Since (3.53) can be written as $\zeta - \zeta_0 = -ie^{i\theta}\ln(\mathfrak{z})/(2\sqrt{Q})$ and $\ln(\mathfrak{z}) = \ln|\mathfrak{z}| + i(\arg(\mathfrak{z}) + 2\pi m), m \in \mathbb{N}$, a single value of $\mathfrak{z}$ may be mapped to an infinite number of points depending on the branch of logarithm one takes. One may decide to restrict the $\{\zeta, \overline{\zeta}\}$-domain to be the band $B := \{\zeta \in \mathbb{C} : \mathrm{Im}(2i\sqrt{Q}e^{-i\theta}(\zeta - \zeta_0)) \in (0, 2\pi)\}$ and then the coordinate change $\zeta(\mathfrak{z})$ defines a diffeomorphism between $\mathbb{C} \setminus \{\mathfrak{z} = (r, 0), r \geq 0\}$ into B. Let $\partial_1 B$ be the connected component of ∂B defined by $\mathrm{Im}(2i\sqrt{Q}e^{-i\theta}(\zeta - \zeta_0)) = 0$ and $\partial_2 B$ the other component $\partial_2 B := \{\mathrm{Im}(2i\sqrt{Q}e^{-i\theta}(\zeta - \zeta_0)) = 2\pi\}$, then the semi-line

$\{\mathfrak{z} = r\}$, with r real and positive and $\arg(\mathfrak{z}) \in \{0, 2\pi\}$, is mapped to the respective points $\zeta_1(r) = -ie^{i\theta}\ln(r)/(2\sqrt{Q}) \in \partial_1 B$ and $\zeta_2(r) = -ie^{i\theta}\ln(r)/(2\sqrt{Q}) + \pi e^{i\theta}/\sqrt{Q} \in \partial_2 B$. This shows that these two boundaries are identified by means of the translation defined by the shift

$$\zeta_t := \pi e^{i\theta}/\sqrt{Q}.$$

The topology of the resulting manifold is $\mathbb{R} \times \mathbb{S}^1$. This is in agreement with the fact that ξ vanishes at precisely two points of the Riemann sphere, and the complement of two points on a sphere is indeed a cylinder. The alternative is to let ζ take values in all $\mathbb{C}$ and consider the inverse map

$$\mathfrak{z}(\zeta) := e^{2i\sqrt{Q}e^{-i\theta}(\zeta-\zeta_0)}.$$

It is clear that this defines an infinite covering of the $\mathfrak{z}$-punctured complex plane $\mathbb{C} \setminus \{0\}$. As described above, the fundamental domain of this covering is the (open) band B limited by the lines (see Figure 3.1, where we have set $\zeta_0 = 0$ for definiteness)

$$\begin{aligned} \zeta_1(s) &= \zeta_0 + \frac{-ie^{i\theta}s}{2\sqrt{Q}}, & s \in \mathbb{R}, \\ \zeta_2(s) &= \zeta_0 + \frac{-ie^{i\theta}s}{2\sqrt{Q}} + \zeta_t, & s \in \mathbb{R}. \end{aligned}$$

The ζ-complex plane therefore corresponds to the complete unwrapping of the cylinder, i.e. to its universal covering. In the $\{\zeta, \overline{\zeta}\}$ coordinate system we have

$$\xi = \frac{1}{2}\left(\partial_\zeta + \partial_{\overline{\zeta}}\right), \qquad \xi^\perp = \frac{i}{2}\left(\partial_\zeta - \partial_{\overline{\zeta}}\right),$$

so ξ points along the real axis and $\xi^\perp$ into the imaginary axis. The angle of the boundaries $\partial_1 B$ (and $\partial_2 B$) with the real axis is $\frac{\pi}{2} + \theta$. For generic values of θ it follows that the integral lines of ξ descend to the quotient $\overline{B}$ (with the boundaries identified as above) as open lines that asymptote to the two points at infinity along the band (as in Figure 3.2). Observe that these two asymptotic values correspond to $\mathfrak{z} = 0$ or $\mathfrak{z} = \infty$, which correspond to the two zeros of ξ. Thus, the integral lines of ξ start asymptotically at one of its zeros and approaches asymptotically the other zero. Along the way, the integral lines circle each zero an infinite number of times (because the projection to the lines parallel to the real axis descend to the quotient in such a way that they intersect the boundaries of B an infinite number of times). The only exception to this behaviour is when $\theta = \frac{\pi}{2}$ or when $\theta = 0$ (recall that by construction $\theta \in [0, \pi)$). In the former case, the integral lines of ξ, never leave the fundamental domain. This means that the curves asymptote to the two zeros of ξ and they never encircle them along the way. The case $\theta = 0$ corresponds to the situation when the projection of the integral lines of ξ define closed curves on $\overline{B}$ with the boundaries identified . This is the situation when the

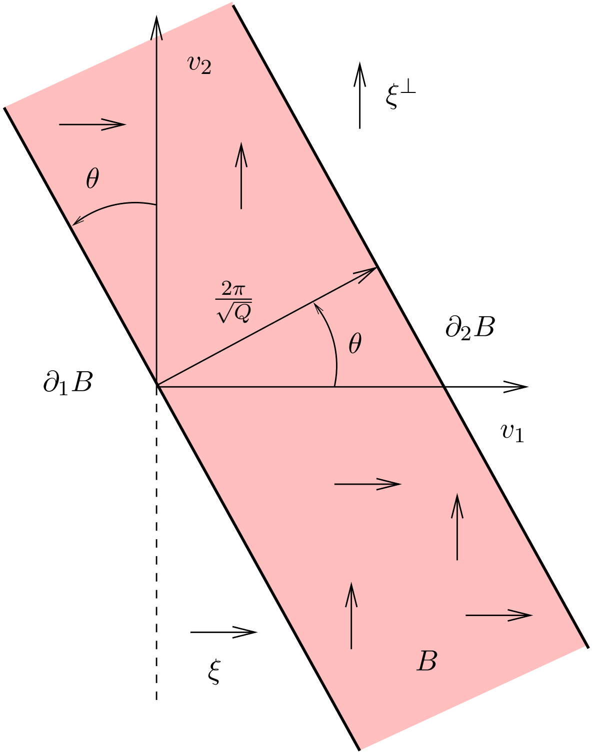

FIGURE 3.1: *Domain of the complex coordinate $\zeta = \frac{1}{2}(v_1 + iv_2)$ adapted to $\xi = \partial_{v_1}$ and $\xi^{\perp} = \partial_{v_2}$. The parameters Q and θ determine the width and tilt of the band respectively. The factor two in the distance between the boundaries arises because $\zeta = \frac{1}{2}(v_1 + iv_2)$.*

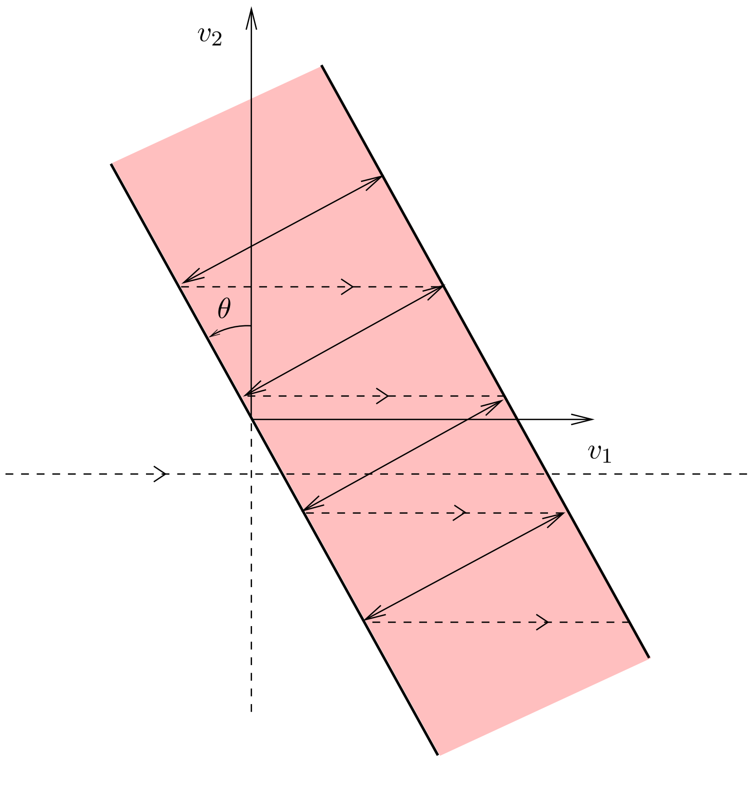

FIGURE 3.2: *Integral lines of ξ (dashed line). The points joint by arrows are identified.*

integral curves of ξ in the original $\{\omega, \overline{\omega}\}$ plane are topological circles (which degenerate to points at the zeroes of ξ).

It is interesting to see how the limit $Q = 0$ is recovered in this setting. The translation vector that identifies points in the boundary $\partial_1 B$ with points in the boundary $\partial_2 B$ diverges as $Q \to 0$. Thus, the band B becomes larger and larger until it covers the whole ζ-plane in the limit. On other words, the ζ-coordinate is no longer a covering of the original ω-coordinate. In the limit, ξ vanishes at only one point in the ω-plane (the origin) which is sent to infinity in the ζ-coordinates. It is by the process of the band B becoming wider and wider that the limits at infinity along the band, which correspond to two points for any non-zero value of Q, merge into a single point when $Q = 0$. The process also explains in which sense the parameter θ, which measures the inclination of the band B becomes irrelevant in the limit $Q = 0$, in agreement with the fact that (3.52) lets θ take any value when $\sigma_{\{\mu\}} - i\tau_{\{\mu\}}$ (and hence also Q) vanishes.

In all the expressions above we have maintained the additive integration constant ζ_0, instead of setting it to zero as the simplest choice. The reason is that ζ_0 can be directly connected with the freedom one has in performing the coordinate change (3.50) that brings ξ into its canonical form. To understand this we simply note that, from (3.53) one can check that the following identity holds

$$\omega(\zeta;\zeta_0) = \frac{\cos\left(\sqrt{Q}e^{-i\theta}\zeta_0\right)\omega(\zeta;0) - \frac{\sqrt{Q}}{2}e^{-i\theta}\sin\left(\sqrt{Q}e^{-i\theta}\zeta_0\right)}{\frac{2}{\sqrt{Q}}e^{i\theta}\sin\left(\sqrt{Q}e^{-i\theta}\zeta_0\right)\omega(\zeta;0) + \cos\left(\sqrt{Q}e^{-i\theta}\zeta_0\right)}.$$

Thus, the relation between $\omega(\zeta;0)$ and $\omega(\zeta;\zeta_0)$ is a Möbius transformation defined by the matrix

$$\begin{pmatrix} \cos\left(\sqrt{Q}e^{-i\theta}\zeta_0\right) & -\frac{\sqrt{Q}}{2}e^{-i\theta}\sin\left(\sqrt{Q}e^{-i\theta}\zeta_0\right) \\ \frac{2}{\sqrt{Q}}e^{i\theta}\sin\left(\sqrt{Q}e^{-i\theta}\zeta_0\right) & \cos\left(\sqrt{Q}e^{-i\theta}\zeta_0\right) \end{pmatrix}.$$

It is immediate to check that, letting ζ_0 take any value, one runs along the full subgroup $\mathbb{A}_{\frac{1}{4}Qe^{-2i\theta}}$ defined in Corollary 3.19. Thus, by Corollary 3.20, the freedom in performing the coordinate change (3.50) that transforms ξ into its canonical form can be absorbed into the additive constant ζ_0, and vice-versa. Having understood this, we will set $\zeta_0 = 0$ from now on.

So far we have considered ξ without referring to any specific metric. We now endow $\mathbb{R}^2$ coordinated by $\{x, y\}$ (or $\{z, \overline{z}\}$) with the following class of metrics. Let $u := \{u_0, u_1, u_2, u_3\} \in \mathbb{R}^4$, $u \neq 0$, and define

$$\begin{aligned} g_u &:= \frac{1}{\Omega_u^2}\left(dx^2 + dy^2\right) = \frac{1}{\Omega_u^2}4dzd\overline{z}, \qquad (3.55) \\ \Omega_u &:= u_0 + u_1 + u_2 x + u_3 y + \frac{1}{4}(u_0 - u_1)(x^2 + y^2) \\ &= u_0(1 + z\overline{z}) + u_1(1 - z\overline{z}) + u_2(z + \overline{z}) + u_3 i(z - \overline{z}). \end{aligned}$$

The Gauss curvature of g_u is $\kappa_u := u_0^2 - u_1^2 - u_2^2 - u_3^2$. Since $g_{-u} = g_u$, there is a sign freedom in u that we must keep in mind. When $\kappa_u \geq 0$, then it must be that $u_0 \neq 0$ and the sign freedom may be fixed by the requirement $u_0 > 0$. However, this is no longer possible when $\kappa_u < 0$.

Observe that $g_{\{u_0=\frac{1}{2}, u_1=\frac{1}{2}, u_2=0, u_3=0\}} = g_E := 4dzd\overline{z}$. Under a Möbius coordinate change (3.50), the metric g_u takes the form

$$
\begin{aligned}
g_u &= \frac{1}{\Omega_{u'}^2} 4d\omega d\overline{\omega}, \\
\Omega_{u'} &= u_0'(1+\omega\overline{\omega}) + u_1'(1-\omega\overline{\omega}) + u_2'(\omega+\overline{\omega}) + u_3' i(\omega-\overline{\omega}),
\end{aligned}
$$

where the constants $u' := \{u_0', u_1', u_2', u_3'\}$ are obtained from $u = \{u_0, u_1, u_2, u_3\}$ by the transformation

$$
\epsilon \begin{pmatrix} u_0' \\ u_1' \\ u_2' \\ u_3' \end{pmatrix} = \underbrace{\frac{1}{2} \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta} & \alpha\overline{\alpha} - \beta\overline{\beta} + \gamma\overline{\gamma} - \delta\overline{\delta} & -\alpha\overline{\beta} - \beta\overline{\alpha} - \gamma\overline{\delta} - \delta\overline{\gamma} & i(\alpha\overline{\beta} - \beta\overline{\alpha} + \gamma\overline{\delta} - \delta\overline{\gamma}) \\ \alpha\overline{\alpha} + \beta\overline{\beta} - \gamma\overline{\gamma} - \delta\overline{\delta} & \alpha\overline{\alpha} - \beta\overline{\beta} - \gamma\overline{\gamma} + \delta\overline{\delta} & -\alpha\overline{\beta} - \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma} & i(\alpha\overline{\beta} - \beta\overline{\alpha} - \gamma\overline{\delta} + \delta\overline{\gamma}) \\ -(\alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} + +\delta\overline{\beta}) & -\alpha\overline{\gamma} + \beta\overline{\delta} - \gamma\overline{\alpha} + \delta\overline{\beta} & \alpha\overline{\delta} + \beta\overline{\gamma} + \gamma\overline{\beta} + \delta\overline{\alpha} & i(-\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha}) \\ i(-\alpha\overline{\gamma} - \beta\overline{\delta} + \gamma\overline{\alpha} + \delta\overline{\beta}) & i(-\alpha\overline{\gamma} + \beta\overline{\delta} + \gamma\overline{\alpha} - \delta\overline{\beta}) & i(\alpha\overline{\delta} + \beta\overline{\gamma} - \gamma\overline{\beta} - \delta\overline{\alpha}) & \alpha\overline{\delta} - \beta\overline{\gamma} - \gamma\overline{\beta} + \delta\overline{\alpha} \end{pmatrix}}_{=\Lambda_{(\alpha,\beta,\gamma,\delta)}} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}
$$

where $\epsilon := \pm 1$. This sign reflects the impossibility (in general) of choosing between u and $-u$. One can check that $\Lambda_{(\alpha,\beta,\gamma,\delta)} = \mathcal{O}(\mathbb{A}^{-1})^T$ (3.44) where $\mathbb{A}$ is as in (3.41) and ^{T} denotes transpose. It follows that $\Lambda(\alpha,\beta,\gamma,\delta)$ defines a morphism of groups between $SL(2,\mathbb{C})$ and $SO^+(1,3)$ and that u transforms as the components of a covector in the Minkowski spacetime. Also observe that when u is timelike or null (i.e. $\kappa_u \geq 0$), the choice $u_0, u_0' > 0$ selects $\epsilon = 1$.

In order to express the metric in the coordinates $\{v_2, v_2\}$ we need to compute the functions $\omega\overline{\omega}$, $\omega + \overline{\omega}$ and $i(\omega - \overline{\omega})$ in terms of these variables. For notational simplicity we introduce the auxiliary quantities

$$
b_1 := v_1 \cos\theta + v_2 \sin\theta, \qquad b_2 := v_2 \cos\theta - v_1 \sin\theta. \tag{3.56}
$$

From (3.53) with $\zeta_0 = 0$, a straightforward computation that uses basic trigonometry yields

$$
\begin{aligned}
\omega\overline{\omega} &= \frac{Q\left(\cosh\left(\sqrt{Q}b_2\right) + \cos\left(\sqrt{Q}b_1\right)\right)}{4\left(\cosh\left(\sqrt{Q}b_2\right) - \cos\left(\sqrt{Q}b_1\right)\right)}, \\
\omega + \overline{\omega} &= \frac{\sqrt{Q}\sin\theta\sinh\left(\sqrt{Q}b_2\right) - \sqrt{Q}\cos\theta\sin\left(\sqrt{Q}b_1\right)}{\cosh\left(\sqrt{Q}b_2\right) - \cos\left(\sqrt{Q}b_1\right)}, \\
i\left(\omega - \overline{\omega}\right) &= -\frac{\sqrt{Q}\cos\theta\sinh\left(\sqrt{Q}b_2\right) + \sqrt{Q}\sin\theta\sin\left(\sqrt{Q}b_1\right)}{\cosh\left(\sqrt{Q}b_2\right) - \cos\left(\sqrt{Q}b_1\right)}.
\end{aligned}
$$

Since $d\omega = \frac{d\omega}{d\zeta}d\zeta = 2(\omega^2 + \frac{Q}{4}e^{-2i\theta})d\zeta$, determining the line-element $d\omega d\overline{\omega}$ requires expressing $|\omega^2 + Q/4e^{-2i\theta}|^2$ in terms of $\{v_1, v_2\}$. The result is obtained by a direct computation,

$$4\left(\omega^2 + \frac{Q}{4}e^{-2i\theta}\right)\left(\overline{\omega}^2 + \frac{Q}{4}e^{2i\theta}\right) = \frac{Q^2}{\left(\cosh(\sqrt{Q}b_2) - \cos(\sqrt{Q}b_1)\right)^2}.$$

Let us introduce the functions

$$\begin{aligned}
f_+(v_1, v_2) &:= \frac{1}{4}\left(\cosh(\sqrt{Q}b_2) + \cos(\sqrt{Q}b_1)\right), \\
f_-(v_1, v_2) &:= \frac{1}{Q}\left(\cosh(\sqrt{Q}b_2) - \cos(\sqrt{Q}b_1)\right), \\
f_2(v_1, v_2) &:= \frac{1}{\sqrt{Q}}\left(\sin\theta\sinh(\sqrt{Q}b_2) - \cos\theta\sin(\sqrt{Q}b_1)\right), \\
f_3(v_1, v_2) &:= \frac{-1}{\sqrt{Q}}\left(\cos\theta\sinh(\sqrt{Q}b_2) + \sin\theta\sin(\sqrt{Q}b_1)\right),
\end{aligned} \tag{3.57}$$

so that we may express

$$\omega\overline{\omega} = \frac{f_+}{f_-}, \qquad \omega + \overline{\omega} = \frac{f_2}{f_-}, \qquad i\left(\omega - \overline{\omega}\right) = \frac{f_3}{f_-}.$$

All these function admit smooth limits at $Q \to 0$, with corresponding expressions

$$\begin{aligned}
f_+(v_1, v_2) &= \frac{1}{2} \\
f_2(v_1, v_2) &= -v_1 \\
f_3(v_2, v_2) &= -v_2 \\
f_-(v_1, v_2) &= \frac{1}{2}\left(v_1^2 + v_2^2\right).
\end{aligned}$$

For $Q \neq 0$, the functions $\{f_+, f_-, f_2, f_3\}$ are all periodic in the variable b_1 with periodicity $2\pi/\sqrt{Q}$. This corresponds to the fact that the ζ-plane is a covering of the ω-plane, with the identification defined by the translation ζ_t.

Thus, in the adapted coordinates $\{v_1, v_2\}$ where $\xi = \partial_{v_1}$ and $\xi^\perp = \partial_{v_2}$, the metric $g_0 := 4d\omega d\overline{\omega}$ takes the form

$$g_0 = \frac{4}{f_-^2}d\zeta d\overline{\zeta} = \frac{Q^2}{\left(\cosh\left(\sqrt{Q}b_2\right) - \cos\left(\sqrt{Q}b_1\right)\right)^2}\left(dv_1^2 + dv_2^2\right).$$

Hence, the metric g_u becomes

$$\begin{aligned}
g_u &= \frac{1}{\left((u_0' - u_1')f_+ + (u_0' + u_1')f_- + u_2'f_2 + u_3'f_3\right)^2}\left(dv_1^2 + dv_2^2\right) \\
&:= \frac{1}{\widehat{\Omega}^2(v_1, v_2)}\left(dv_1^2 + dv_2^2\right).
\end{aligned} \tag{3.58}$$

We may now summarize the results obtained so far concerning GCKV.

Teorema 3.23. *Let $\mathbb{E}_2$ be the Euclidean plane and $\{x,y\}$ be Cartesian coordinates. Let ξ be a GCKV in this space and define the complex constants $\{\mu_0,\mu_1,\mu_2\}$ by means of the expression of ξ given by (3.36) in the complex coordinates $z=\frac{1}{2}(x-iy)$, $\overline{z}=\frac{1}{2}(x-iy)$. Define*

$$\alpha=\frac{1}{2}\left(\delta\mu_2-\gamma\mu_1\right), \qquad \beta=\frac{1}{2}\delta\mu_1-\gamma\mu_0,$$

where γ and δ are any pair of complex constants satisfying

$$\frac{1}{2}\delta^2\mu_2-\gamma\delta\mu_1+\gamma^2\mu_0=1.$$

Then ξ takes its canonical form (cf. Proposition 3.18)

$$\xi=\left(\mu_0'+\omega^2\right)\partial_\omega+\left(\overline{\mu_0'}+\overline{\omega}^2\right)\partial_{\overline{\omega}}, \qquad 4\mu_0':=2\mu_0\mu_2-\mu_1^2,$$

in the coordinate system $\{\omega,\overline{\omega}\}$ defined by $\omega=(\alpha z+\beta)/(\gamma z+\delta)$. Any other coordinate system $\{\omega',\overline{\omega}'\}$ where ξ is in canonical form is related to $\{\omega,\overline{\omega}\}$ by (cf. Corollary 3.19)

$$\omega'=\frac{\delta'\omega-\gamma'\mu_0'}{\gamma'\omega+\delta'}, \qquad \delta'^2+\mu_0'\gamma'^2=1.$$

In addition, the real coordinates $\{v_1,v_2\}$ defined by $\zeta:=v_1+iv_2$ together with (3.53) and $4\mu_0':=\sigma_{\{\mu\}}-i\tau_{\{\mu\}}=Qe^{-2i\theta}$ are adapted to ξ and $\xi^\perp:=\xi_{\{i\mu\}}$ (cf. Lemma 3.16), namely $\xi=\partial_{v_1}$ and $\xi^\perp=\partial_{v_2}$. Moreover, the class of metrics (3.55) is written in adapted coordinates as (3.58).

We mentioned above that the freedom in the coordinate change that brings ξ into its canonical form can be translated into the freedom of a constant shift in the coordinates $\{v_1,v_2\}$. Given $\{\tilde{v}_1,\tilde{v}_2\}$ let $\tilde{b}_1$ and $\tilde{b}_2$ by defined exactly by the same expression as (3.56) but with $\{v_1,v_2\}$ replaced by $\{\tilde{v}_1,\tilde{v}_2\}$. Similarly, we introduce four functions $\{\tilde{f}_+(\tilde{v}_1,\tilde{v}_2)$, $\tilde{f}_-(\tilde{v}_1,\tilde{v}_2)$, $\tilde{f}_2(\tilde{v}_1,\tilde{v}_2)$, $\tilde{f}_3(\tilde{v}_1,\tilde{v}_2)\}$ by the same definition as (3.57), with $\{b_1,b_2\}$ replaced by $\{\tilde{b}_1,\tilde{b}_2\}$. Let us now consider the coordinate change

$$\left\{\begin{array}{l} v_1=\tilde{v}_1-\cos\theta\ell_1+\sin\theta\ell_2 \\ v_2=\tilde{v}_2-\sin\theta\ell_1-\cos\theta\ell_2 \end{array}\right. \tag{3.59}$$

where ℓ_1 and ℓ_2 are constants. Then $b_1=\tilde{b}_1-\ell_1$ and $b_2=\tilde{b}_2-\ell_2$ and we may relate the functions $\{f\}$ written in terms of $\{\tilde{v}_1,\tilde{v}_2\}$ with the functions $\{\tilde{f}\}$. The result is

$$\begin{pmatrix} 2f_+ \\ 2f_- \\ f_2 \\ f_3 \end{pmatrix}_{\tilde{v}_1,\tilde{v}_2} = \begin{pmatrix} \frac{1}{2}(\mathrm{Coh}+\mathrm{Co}) & \frac{Q}{8}(\mathrm{Coh}-\mathrm{Co}) & -\frac{\sqrt{Q}}{2}\mathrm{Si} & \frac{\sqrt{Q}}{2}\mathrm{Sih} \\ \frac{2}{Q}(\mathrm{Coh}-\mathrm{Co}) & \frac{1}{2}(\mathrm{Coh}+\mathrm{Co}) & \frac{2}{\sqrt{Q}}\mathrm{Si} & \frac{2}{\sqrt{Q}}\mathrm{Sih} \\ \frac{1}{\sqrt{Q}}(\cos\theta\mathrm{Si}-\sin\theta\mathrm{Sih}) & -\frac{\sqrt{Q}}{4}(\cos\theta\mathrm{Si}+\sin\theta\mathrm{Sih}) & \cos\theta\mathrm{Co} & -\sin\theta\mathrm{Coh} \\ \frac{1}{\sqrt{Q}}(\cos\theta\mathrm{Sih}+\sin\theta\mathrm{Si}) & \frac{\sqrt{Q}}{4}(\cos\theta\mathrm{Sih}-\sin\theta\mathrm{Si}) & \sin\theta\mathrm{Co} & \cos\theta\mathrm{Coh} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2\tilde{f}_+ \\ 2\tilde{f}_- \\ \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} := W(\ell_1,\ell_2) \begin{pmatrix} 2\tilde{f}_+ \\ 2\tilde{f}_- \\ \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \tag{3.60}$$

where for notational simplicity we have introduced $\mathrm{Co} = \cos(\sqrt{Q}\ell_1)$, $\mathrm{Coh} = \cosh(\sqrt{Q}\ell_2)$, $\mathrm{Si} = \sin(\sqrt{Q}\ell_1)$, $\mathrm{Sih} = \sinh(\sqrt{Q}\ell_2)$. If we compare $W(\ell_1,\ell_2)$ and $\mathcal{T}(\lambda_2,\lambda_3,\epsilon)$ we see that the matrices are identical after setting

$$\lambda_2 = \frac{1}{\sqrt{Q}}\sin(\sqrt{Q}\ell_1), \quad \lambda_3 = \frac{1}{\sqrt{Q}}\sinh(\sqrt{Q}\ell_2), \quad \epsilon\sqrt{1-Q\lambda_2^2} = \cos(\sqrt{Q}\ell_1). \tag{3.61}$$

Of course this does not happen by chance. We have seen before that the shift in ζ corresponds to the subgroup of Möbius transformation that leaves the canonical form of ξ invariant. By the relationship between GCKV and skew-symmetric endomorphism in $\mathbb{M}^{1,3}$ described in Section 3.6 (se also subsection 2.2.1), this Möbius subgroup corresponds to the set of orthochronous Lorentz transformations that leave the skew-symmetric endomorphism invariant, and this is precisely the group $\{\mathcal{T}(\lambda_2,\lambda_3,\epsilon)\}$. With the choice we have made of the shift constants (3.59), the relationship between the parameters $\{\ell_1,\ell_2\}$ and $\{\lambda_2,\lambda_3\}$ take the remarkably simple form given by (3.61). Note that the map $(\ell_1,\ell_2) \to (\lambda_2,\lambda_3,\epsilon)$ is again a covering. If we let ℓ_2 be periodic with periodicity $\frac{2\pi}{\sqrt{Q}}$, the map is a bijection. Observe that, to make the comparison work, we have inserted a factor 2 in front of $f_\pm$ in the column vector (3.60). The reason is easy to understand. The constants $\{u'_0,u'_1,u'_2,u'_3\}$ in the conformal factor $\widehat{\Omega}$ in the metric g_u define a Lorentz covector of length $-{u'_0}^2+{u'_1}^2+{u'_2}^2+{u'_3}^2 = -(u'_0+u'_1)(u'_0-u'_1)+{u'_2}^2+{u'_3}^2$. This means that, viewed as vectors in a Lorentz space, the basis $\{f_+,f_-,f_2,f_3\}$ is semi-null, but with scalar product $\langle f_+,f_-\rangle = \frac{1}{2}$ However, the transformation law $\mathcal{T}(\lambda_2,\lambda_3,\epsilon)$ was written in a semi-null basis $\{\ell,k,e_2,e_3\}$ with normalization $\langle \ell,k\rangle = -2$, which is precisely the normalization of the basis $\{2f_+,2f_-,f_2,f_3\}$.

Having obtained the transformation law for $\{f_+,f_-,f_2,f_3\}$ it follows immediately that under the coordinate transformation (3.59), the metric g_u becomes

$$g_u = \frac{1}{\left((\tilde{u}_0-\tilde{u}_1)\tilde{f}_+ + (\tilde{u}_0+\tilde{u}_1)\tilde{f}_- + \tilde{u}_2\tilde{f}_1 + \tilde{u}_3\tilde{f}_2\right)^2}\left(d\tilde{v}_1^2 + d\tilde{v}_2^2\right)$$

where the constants $\{\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ are given by

$$\begin{pmatrix} \frac{1}{2}(\tilde{u}_0 - \tilde{u}_1) \\ \frac{1}{2}(\tilde{u}_0 + \tilde{u}_1) \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} = \epsilon (W(\ell_1, \ell_2))^T \begin{pmatrix} \frac{1}{2}(u'_0 - u'_1) \\ \frac{1}{2}(u'_0 + u'_1) \\ u'_2 \\ u'_3 \end{pmatrix}$$

(the reason for the sign ϵ is the same as discussed before).

3.9 Applications

3.9.1 Killing vectors of g_u

Our aim is to determine under which conditions ξ is a Killing vector of the metric g_u. We will address the question by analyzing the situation in the adapted coordinates. Since $\xi = \partial_{v_1}$, ξ will be a Killing vector of g_u if and only if the function $\widehat{\Omega}$ satisfies $\partial_{v_1}\widehat{\Omega} = 0$. It is straightforward to check that

$$\begin{aligned} \partial_{v_1} f_+ &= \frac{Q}{4}\left(\cos(2\theta) f_2 + \sin(2\theta) f_3\right), \\ \partial_{v_1} f_- &= -f_2, \\ \partial_{v_1} f_2 &= -2f_+ + \frac{Q}{2}\cos(2\theta) f_-, \\ \partial_{v_1} f_3 &= \frac{Q}{2}\sin(2\theta) f_-, \end{aligned}$$

which imply

$$\begin{aligned} \partial_{v_1}\widehat{\Omega} = &- 2u'_2 f_+ + \frac{Q}{2}\left(\cos(2\theta)u'_2 + \sin(2\theta)u'_3\right) f_- \\ &+ \left(\frac{Q}{2}\cos(2\theta)u_- - 2u'_+\right) f_2 + \frac{Q}{2}\sin(2\theta)u'_- f_3, \end{aligned}$$

where we have set $u'_\pm := \frac{1}{2}(u'_0 \pm u'_1)$. The functions $\{f_+, f_-, f_2, f_3\}$ are linearly independent, so this derivative will vanish if and only if each coefficient vanishes. If $Q\sin(2\theta) \neq 0$, it is immediate that the only solution is $u'_+ = u'_- = u'_2 = u'_3 = 0$, which is not possible for a metric g_u. Thus, a necessary condition for ξ to be a Killing vector of (any) g_u is that the invariant (see (3.52)) $\sigma_{\{\mu\}} - i\tau_{\{\mu\}}$ be real (i.e. $\tau_{\{\mu\}} = 0$). When $Q \neq 0$, the condition $\sin(2\theta) = 0$ is $\theta \in \{0, \frac{\pi}{2}\}$ (recall that $\theta \in [0, \pi)$ by construction). To cover all cases at once we set $\cos\theta = \hat{\epsilon}$ and $\sin\theta = 1 - \hat{\epsilon}$, with $\hat{\epsilon}^2 = \hat{\epsilon}$. Then $\cos(2\theta) = 2\hat{\epsilon} - 1$ (this choice is also valid when $Q = 0$ because θ can be fixed to any value). Then

$$\partial_{v_1}\widehat{\Omega} = 0 \quad \Longleftrightarrow \quad (u'_-, u'_+, u'_2, u'_3) = s_1 \underbrace{\left(1, \frac{Q}{4}(2\hat{\epsilon} - 1), 0, 0\right)}_{w_1} + s_2 \underbrace{(0, 0, 0, 1)}_{w_2}, \quad s_1, s_2 \in \mathbb{R}.$$

The Lorentzian norm of this vector is $-4u'_+u'_- + {u'_2}^2 + {u'_3}^2 = -(2\hat{\epsilon}-1)Qs_1^2 + s_2^2$. Under the constant shift given by ℓ_1, ℓ_2, the two-dimensional vector space spanned by w_1 and w_2 remains invariant, and the vector $s_1w_1 + s_2w_2$ transforms to $\tilde{s}_1w_2 + \tilde{s}_2w_3$ with

$$\begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{pmatrix} = \epsilon \begin{pmatrix} \hat{\epsilon}\cosh(\sqrt{Q}\ell_2) + \cos(\sqrt{Q}\ell_1)(1-\hat{\epsilon}) & \frac{1}{\sqrt{Q}}\left(\sinh(\sqrt{Q}\ell_2)\hat{\epsilon} + \sin(\sqrt{Q}\ell_1)(1-\hat{\epsilon})\right) \\ \sqrt{Q}\left(\sinh(\sqrt{Q}\ell_2)\hat{\epsilon} - \sin(\sqrt{Q}\ell_1)(1-\hat{\epsilon})\right) & \hat{\epsilon}\cosh(\sqrt{Q}\ell_2) + \cos(\sqrt{Q}\ell_1)(1-\hat{\epsilon}) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

This transformation leaves the norm $-(2\hat{\epsilon}-1)Qs_1^2+s_2^2$ invariant (as it must) and defines a group which is one-dimensional when $Q \neq 0$ and two-dimensional when $Q = 0$. Thus, when transforming the vector u into the original coordinate system $\{z, \overline{z}\}$ we may ignore the action of the invariance group that leaves the canonical form of ξ invariant provided we let u take all non-zero values in the vector space $\mathrm{span}\{w_1, w_2\}$. We may summarize the result in the following theorem.

Teorema 3.24. *Given a non-identically zero GCKV ξ in two-dimensional Euclidean space and let $\{\mu\} := \{\mu_0, \mu_1, \mu_2\}$ be the set of parameters such that $\xi = \xi_{\{\mu\}}$ in the coordinate system $\{z, \overline{z}\}$. Let $U \subset \mathbb{R}^4 \setminus \{0\}$ be defined by the property that for all $u \in U$, ξ is a Killing vector of the metric g_u (defined in (3.55)). Then*

- *If $2\mu_0\mu_2 - \mu_1^2 \notin \mathbb{R}$ then $U = \emptyset$.*
- *If $2\mu_0\mu_2 - \mu_1^2 \in \mathbb{R}$, let δ, γ be any pair of complex numbers satisfying*

$$\frac{1}{2}\delta^2\mu_2 - \gamma\delta\mu_1 + \gamma^2\mu_0 = 1$$

and set $\alpha = \frac{1}{2}(\delta\mu_2 - \gamma\mu_1)$ and $\beta = \frac{1}{2}\delta\mu_1 - \gamma\mu_0$. Then $u \in U$ if and only if

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathcal{O}(\mathbb{A})^T \begin{pmatrix} s_1\left(\frac{1}{4}(2\mu_0\mu_2 - \mu_1^2) + 1\right) \\ s_1\left(\frac{1}{4}(2\mu_0\mu_2 - \mu_1^2) - 1\right) \\ 0 \\ s_2 \end{pmatrix}$$

where $(s_1, s_2) \in \mathbb{R}^2 \setminus \{0\}$, $\mathbb{A}$ is the matrix (3.41) and $\mathcal{O}(\mathbb{A})$ was defined in (3.44).

Moreover, such g_u has constant curvature κ_u given by

$$\kappa_u = s_1^2(2\mu_0\mu_2 - \mu_1^2) - s_2^2.$$

Proof. We only need to check that $w_1 = (1, \frac{1}{4}(2\mu_0\mu_2 - \mu_1^2), 0, 0)$, This is an immediate consequence of the definitions (3.52) and (3.49), which in the case $\cos\theta = \hat{\epsilon}$ and $\sin\theta = 1 - \hat{\epsilon}$ imply

$$Q\,(2\hat{\epsilon} - 1) = 2\mu_0\mu_2 - \mu_1^2.$$

□

One may wonder why this problem has no been addressed in the original coordinate system $\{z, \overline{z}\}$. The Lie derivative of a metric $g_\Psi := 4\Psi^{-2}dzd\overline{z}$ along $\xi_{\{\mu\}}$ (given by (3.36)) is

$$\mathcal{L}_{\xi_{\{\mu\}}} g_\Psi = \left(-2\xi_{\{\mu\}}(\Psi) + \Psi\left(\mu_1 + \overline{\mu_1} + \mu_2 z + \overline{\mu_2 z}\right)\right) g_\Psi.$$

Thus $\xi_{\{\mu\}}$ is a Killing vector of g_u if and only if

$$-2\xi_{\{\mu\}}(\Omega_u) + \Omega_u\left(\mu_1 + \overline{\mu_1} + \mu_2 z + \overline{\mu_2 z}\right) = 0.$$

The computation gives a polynomial in $\{z, \overline{z}\}$ of degree two. Equating each coefficient to zero, one finds that the conditions that need to be satisfied can be written in the form

$$\begin{pmatrix} 0 & -\nu & -\mathrm{a}_x + \frac{\mathrm{b}_x}{2} & -\mathrm{a}_y + \frac{\mathrm{b}_y}{2} \\ -\nu & 0 & -\mathrm{a}_x - \frac{\mathrm{b}_x}{2} & -\mathrm{a}_y - \frac{\mathrm{b}_y}{2} \\ -\mathrm{a}_x + \frac{\mathrm{b}_x}{2} & \mathrm{a}_x + \frac{\mathrm{b}_x}{2} & 0 & -\omega \\ -\mathrm{a}_y + \frac{\mathrm{b}_y}{2} & \mathrm{a}_y + \frac{\mathrm{b}_y}{2} & \omega & 0 \end{pmatrix} \begin{pmatrix} -u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.62}$$

where we have expressed $\{\mu\}$ in terms of its real and imaginary parts by means of (3.37). Recalling the relationship between GCKV ξ and skew-symmetric endomorphisms F_ξ we conclude that $\xi_{\{\mu\}}$ is a Killing vector of g_u if and only if the non-zero Lorentz vector $(-u_0, u_1, u_2, u_3)$ lies in the kernel of F_ξ (observe that this vector is obtained from the covector u by raising indices with the Minkowski metric). Being skew-symmetric and not identically zero, F_ξ can only have rank two or four, so in order to admit a non-trivial kernel, the rank must be two. This corresponds to the condition $\tau_{\{\mu\}} = 0 \iff \mathrm{Im}(2\mu_0\mu_2 - \mu_1^2) = 0$. So, the kernel is two-dimensional, which recovers the statement in Theorem 3.24 that the set $U \cup \{0\}$ is a two-dimensional vector space. Thus, the problem becomes geometrically very neat in the original coordinate system. However, in Theorem 3.24 we have been able to determine explicitly the vector subspace $U \cup \{0\}$ (equivalently the kernel of F_ξ, after index raising) in a way that covers all cases at once. It is not so clear how to achieve the same by a direct attempt of solving (3.62) in such a way that the solution covers all possible values of $\{\mathrm{b}_x, \mathrm{b}_y, \nu, \omega, \mathrm{a}_x, \mathrm{a}_y\}$ under the restriction $\mathrm{b}_x\mathrm{a}_y - \mathrm{b}_y\mathrm{a}_x + \nu\omega = 0$ (namely $\mathrm{Im}(2\mu_0\mu_2 - \mu_1^2) = 0$).

The issue addressed in Theorem 3.24 is to determine for which metrics g_u a given GCKV is Killing. A complementary problem is to fix g_u and determine all GCKV which are Killings of g_u. This problem may be approached in the language of skew-symmetric endomorphisms. A skew-symmetric endomorphism F in $\mathbb{M}^{1,3}$ of rank two is necessarily of the form $F = q_1 \otimes \boldsymbol{q_2} - q_2 \otimes \boldsymbol{q_1}$ where q_1 and q_2 are linearly independent Lorentz vectors and recall that boldface denote the metrically related one-form. A vector u lies in the kernel of F if and only if it is orthogonal to q_1 and q_2. Thus, the set of Killing vectors of g_u is obtained from all skew-symmetric endomorphisms

$$F_{u^\perp} := \{F = q_1 \otimes \boldsymbol{q_2} - q_2 \otimes \boldsymbol{q_1}; \qquad \mathrm{span}\{q_1, q_2\} = u^\perp\}.$$

where $u^\perp$ stands for the set of vectors in the kernel of the covector (u_0, u_1, u_2, u_3). We do not attempt to find an explicitly parametrization of all Killing vectors of g_u that covers at once all possible choices of u (this problem does not appear to be simple either in terms of endomorphisms, or by using canonical forms of ξ).

3.9.2 Transverse and traceless and Lie constant tensors on $\mathbb{E}^2$

As discussed in Chapter 2, the transverse and traceless symmetric 2-covariant tensors, namely, tensors $D_{\alpha\beta} = D_{\beta\alpha}$ satisfying (indices are raised with a metric g and ∇ is the corresponding Levi-Civita connection)

$$\nabla_\alpha D^{\alpha\beta} = 0 \quad \text{(transverse)}, \qquad D^\alpha{}_\alpha = 0 \quad \text{(traceless)}$$

play a prominent role in General Relativity, in several circumstances. For example, they are fundamental for the construction of initial data in spacelike slices with prescribed regularity at spacelike infinity [37] or black hole initial data [16]. Of particular interest for us, is the free data at null infinity for Λ-vacuum spacetimes with positive cosmological constant (cf. Section 2.4 and references therein). In this setup, an interesting subclass that arises when the spacetime admits Killing vectors is the subclass of TT tensors which satisfy the KID equation [116] (cf. Theorem 2.35). In dimension n, this equation is (cf. Section 5.2)

$$\mathcal{L}_\xi D_{\alpha\beta} + \frac{n-2}{n}(\mathrm{div}_g \xi) D_{\alpha\beta} = 0$$

where ξ is a conformal Killing vector of g and $\mathcal{L}_\xi$, $\mathrm{div}_g\xi$ stand respectively for the Lie derivative along ξ and the divergence of ξ with respect to g. In dimension $n = 2$ the general solution of (local) TT tensors satisfying the KID equation can be explicitly solved. Although this dimension is not particularly interesting from a physical point of view, there are several motivations for presenting the result. Firstly, dimensional reduction is a useful tool in many geometric problems, so it is not unlikely that the case of dimension two may find applications in higher dimensions. Also, the $n = 2$ case may serve as a toy model to address the (much more difficult) problem in higher dimensions. In addition, the solution we find turns out to admit an interesting generalization in arbitrary dimension (cf. Section 4.7). And lastly, it is remarkable, that the problem is so simple in dimension $n = 2$ that its general solution can be explicitly given.

A key property of the TT conditions and of the KID equations is their conformal covariance (cf. Lemma 2.1). Also, if D satisfies the KID equation for g, then $\Omega^{2-n}D$ also satisfies the KID equation for $\Omega^2 g$. In dimension $n = 2$ one actually has conformal invariance. Since all two-dimensional metrics are locally conformal to the flat metric, and we are interested in solving the (more general) local problem, we may assume that $g = 4dzd\overline{z}$. As already mentioned, a vector field ξ is conformal of this metric if and only if $\xi = f(z)\partial_z + \overline{f}(\overline{z})\partial_{\overline{z}}$. We expand $D = D_{zz}dz^2 + D_{\overline{z}\overline{z}}d\overline{z}^2 + 2D_{z\overline{z}}dzd\overline{z}$. The condition of being traceless is $D_{z\overline{z}} = 0$ and D real requires $D_{\overline{z}\overline{z}} = D_{zz}$, With these restrictions, the transverse equations take the following explicit and simple form

$$\partial_z D_{\overline{z}\overline{z}} = 0, \qquad \partial_{\overline{z}} D_{zz} = 0,$$

so D_{zz} is a holomorphic function of z. Imposing transverse and traceless as well as the reality condition, the KID equations read

$$f\frac{D_{zz}}{dz} + 2D_{zz}\frac{df}{dz} = 0,$$

which integrates to $D_{zz} = \frac{q}{f^2}, q \in \mathbb{C}$. Writing $q = q_1 + iq_2$, with real q_1, q_2, we conclude that the most general (real) TT tensor that satisfies the KID equation is a linear combination of (we add the factor 4 for convenience)

$$D_1 := \frac{1}{4}\left(\frac{1}{f^2}dz^2 + \frac{1}{\overline{f}^2}d\overline{z}^2\right), \qquad D_2 = \frac{i}{4}\left(\frac{1}{\overline{f}^2}d\overline{z}^2 - \frac{1}{f^2}dz^2\right).$$

These expressions are valid in the coordinate system $\{z, \overline{z}\}$. We are interested in covariant expressions that are valid in any coordinate system, and are explicitly invariant under conformal transformations. To achieve this, we introduce the vector field

$$\xi^{\perp} := i\left(f\partial_z - \overline{f}\partial_{\overline{z}}\right). \tag{3.63}$$

This is everywhere orthogonal to ξ and has the same norm at every point. If the zeros of ξ do not separate the manifold, these two properties define $\xi^{\perp}$ in terms of ξ uniquely except for a global sign. If the zeroes of ξ separate the manifold, $\xi^{\perp}$ is still uniquely defined (up to a sign) if one adds the condition that $\xi^{\perp}$ is a conformal Killing vector of g (which (3.63) clearly is). Thus, we may speak of $\xi^{\perp}$ unambiguously (up to global sign), once ξ has been fixed. Next we note that, in the $\{z, \overline{z}\}$ coordinate system and with respect to the metric $g_E := 4dzd\overline{z}$ we have

$$\boldsymbol{\xi} = 2fd\overline{z} + 2\overline{f}d\overline{z}, \qquad |\xi|^2_{g_E} := g_E(\xi,\xi) = 4f\overline{f},$$
$$\boldsymbol{\xi}^{\perp} = 2ifd\overline{z} - 2i\overline{f}dz, \qquad |\xi^{\perp}|^2_{g_E} = 4f\overline{f},$$

and then we may write

$$D_1 = \frac{1}{|\xi|^4_{g_E}}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi} - \frac{1}{2}|\xi|^2_{g_E}g_E\right),$$
$$D_2 = \frac{1}{2|\xi|^4_{g_E}}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi}^{\perp} + \boldsymbol{\xi}^{\perp}\otimes\boldsymbol{\xi}\right).$$

These expressions are obviously coordinate independent and also conformally invariant. Thus, D_1 and D_2 take this form also for the original metric g. Notice that at the fixed points of ξ, i.e. those points where ξ vanishes, the general solution $D = c_1 D_1 + c_2 D_2$ for $c_1, c_2 \in \mathbb{R}$ diverges unless $c_1 = c_2 = 0$. This follows from the fact that the square norm of D is

$$D_{\alpha\beta} D^{\alpha\beta} = \frac{1}{2|\xi|^4_{g_E}} (c_1^2 + c_2^2),$$

which is regular at the fixed points of ξ only if $c_1 = c_2 = 0$. Summarizing, we have proved the following theorem.

Teorema 3.25. *Let (M, g) be a two-dimensional Riemannian manifold and ξ a conformal Killing vector of g. Let D be a (real) transverse and traceless symmetric, 2-covariant tensor that satisfies the KID equation with respect to ξ. Then D is a linear combination (with constants) of*

$$D_\xi := \frac{1}{|\xi|^4_g} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{1}{2} |\xi|^2_g g \right),$$
$$D_{\xi,\xi^\perp} := \frac{1}{2|\xi|^2_g |\xi^\perp|^2_g} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi}^\perp + \boldsymbol{\xi}^\perp \otimes \boldsymbol{\xi} \right),$$

where $\xi^\perp$ is defined as described above and $\boldsymbol{\xi} := g(\xi, \cdot)$, $\boldsymbol{\xi}^\perp := g(\xi^\perp, \cdot)$. Moreover, the only solution regular at any of the fixed points of ξ is the zero tensor.

We note that Theorem 3.25 has found interesting applications for gravitational radiation at null infinity in [51].

Chapter 4

Skew-symmetric endomorphisms of $\mathbb{M}^{1,n+1}$& CKVFs of $\mathbb{S}^n$

In this Chapter we deal with skew-symmetric endomorphisms of Lorentzian vector spaces of arbitrary dimensions, which we identify with $\mathbb{M}^{1,n+1}$, and its relation with CKVFs of the n-sphere $\mathbb{S}^n$. The contents are essentially a generalization to arbitrary dimension of many of the results in Chapter 3. They have been published in [95].

In Section 4.1 we rederive a known classification result (e.g. [39]) for skew-symmetric endomorphisms of d-dimensional Lorentzian vector spaces SkewEnd($\mathbb{M}^{1,d-1}$). Based on this and with the results of Section 3.2, we give a canonical form in Section 4.2 for each element in SkewEnd($\mathbb{M}^{1,d-1}$) depending on a minimal number of parameters. In Section 4.4, we show that this canonical form is shared by every pair of elements in SkewEnd($\mathbb{M}^{1,d-1}$) differing by an orthochronous Lorentz transformation, i.e. it defines the orbits of the orthochronous Lorentz group $O^+(1,d-1)$ under the adjoint action on its algebra. Using this form, we obtain a useful set of limits in the quotient topology of SkewEnd($\mathbb{M}^{1,d-1}$)$/O^+(1,d.1)$, which will find application in Chapter 6 for the analysis of asymptotic intial data.

In the subsequent Sections, we apply the above results to the set of CKVFs of $\mathbb{S}^n$ (with $n > 2$). From the relations between SkewEnd($\mathbb{M}^{1,n+1}$) and the CKVFs of the sphere $\mathbb{S}^n$ given in subsection 2.2.1, a canonical form for CKVFs follows immediately in Section 4.5. This form is used in Section 4.6 to find adapted coordinates to an arbitrary CKVF that covers all cases at the same time. We do the calculation for even n and obtain the case of odd n as a consequence. With these coordinates at hand, in Section 4.7 we obtain a wide class of TT-tensors for $n = 3$ solving the KID equations for two commuting CKVFs, one of which is axial. The commuting CKVFs are obtained taking advantage of the structure of the canonical form obtained in Section 4.5. These tensors provide Cauchy data at conformally flat null infinity $\mathscr{I}$. Specifically, this class of data is characterized for generating $\Lambda > 0$-vacuum spacetimes with two-symmetries, one of which axial, admitting a conformally flat $\mathscr{I}$. The class of data is infinite dimensional, depending on two arbitrary functions of one variable as well as a number of constants. Moreover, it contains the data for the Kerr-de Sitter spacetime, which we explicitly identify within.

4.1 Classification of skew-symmetric endomorphisms

Let V be a d dimensional Lorentzian vector space. The first step towards the definition of canonical form of skew-symmetric endomorphisms of V in any dimension is the classification result proven in this Section. The strategy is the decomposition of an arbitray element $F \in \mathrm{SkewEnd}(V)$ into orthogonal sum of spacelike and timelike eigenplanes (cf. Definition 3.5). The first question we address here is under which conditions such a plane exists (cf. Proposition 4.5). We start with some preliminary results.

Lemma 4.1. *Let V be a Lorentzian vector space and $F \in \mathrm{SkewEnd}(V)$. Then there exist two vectors $w, v \in V$, with $w \neq 0$, such that one of the three following exclusive possibilities hold*

(i) w is a null eigenvector of F.

(ii) w is a non-null eigenvector (with zero eigenvalue).

(iii) $w =: u, v$ are orthogonal, spacelike and with the same norm, and define an eigenplane of F with non-zero eigenvalue, i.e.

$$F(u) = \mu v, \qquad F(v) = -\mu u, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

If, instead, V is Riemannian, only cases (ii) and (iii) can arise.

Proof. From the Jordan block decomposition theorem we know that there is at least one, possibly complex, eigenvalue $s_1 + is_2$ with eigenvector $w + iv$, that is, $F(w + iv) = (s_1 + is_2)(w + iv)$, or equivalently:

$$F(w) = s_1 w - s_2 v, \tag{4.1}$$

$$F(v) = s_2 w + s_1 v. \tag{4.2}$$

This system is invariant under the interchange $(w, v) \to (-v, w)$, so without loss of generality we may assume $w \neq 0$. The respective scalar products of (4.1) and (4.2) with w, v yield

$$\left.\begin{array}{ll} s_1 \langle w, w \rangle - s_2 \langle w, v \rangle & = 0 \\ s_1 \langle v, v \rangle + s_2 \langle w, v \rangle & = 0 \end{array}\right\} \qquad \Longleftrightarrow \qquad \begin{pmatrix} \langle w, w \rangle & -\langle w, v \rangle \\ \langle v, v \rangle & \langle w, v \rangle \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

Observe that if $s_1 + is_2 \neq 0$ the determinant of the matrix must vanish. i.e.

$$\langle w, v\rangle \left(\langle w, w\rangle + \langle v, v\rangle\right) = 0.$$

Hence, we can distinguish the following possibilities:

(a) $s_1 = s_2 = 0$. Then w is an eigenvector of F with vanishing eigenvalue so we fall into cases (i) or (ii).

(b) $s_1 + is_2 \neq 0$. From $\langle w, v\rangle \left(\langle w, w\rangle + \langle v, v\rangle\right) = 0$ we distinguish two cases:

(b.1) $\langle w, v\rangle = 0$. If $s_1 \neq 0$ then (4.3) forces w and v to be both null and, being also orthogonal to each other, there is $a \in \mathbb{R}$ such that $v = aw$ and we fall into case (i). So, we can assume $s_1 = 0$ (and then $s_2 \neq 0$). Let $\mu := -s_2$ and $u := w$, thus (iii) follows from equations (4.1), (4.2) and Lemma 3.3.

(b.2) $\langle w, v\rangle \neq 0$. Then $\langle w, w\rangle = -\langle v, v\rangle$ and the matrix problem (4.3) reduces to

$$s_1 \langle w, w\rangle - s_2 \langle w, v\rangle = 0.$$

In addition, (4.1) and (4.2) imply

$$\langle F(w), v\rangle = s_1 \langle w, v\rangle - s_2 \langle v, v\rangle = s_1 \langle w, v\rangle + s_2 \langle w, w\rangle = \langle F(v), w\rangle .$$

But skew-symmetry requires $\langle F(w), v\rangle = -\langle F(v), w\rangle$, so $\langle F(v), w\rangle = 0$ and we conclude

$$s_1 \langle w, v\rangle + s_2 \langle w, w\rangle = 0.$$

Combining with (4.1) yields

$$\begin{pmatrix} \langle w, w\rangle & -\langle w, v\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is non-zero which yields a contradiction with $s_1+is_2 \neq 0$. So this case is empty.

To conclude the proof, we must consider the case when the vector space V is Riemannian. The proof is identical except from the fact that all cases involving null vectors are imposible from the start.

□

Observacin 4.2. *One may wonder why the lemma includes the possibility of having a spacelike eigenplane (case (iii)), but not a timelike eigenplane. The reason is that invariant timelike planes, which are indeed possible, fall into case (i) by Lemma 3.4, because $e \pm v$ are null eigenvectors.*

In the case of Riemmanian signature, Lemma 4.1 can be reduced to the following single statement:

Corolario 4.3. *Let V be Riemannian of dimension d and $F \in \mathrm{SkewEnd}(V)$. If $d = 1$ then $F = 0$ and if $d \geq 2$ then there exist two orthogonal and unit vectors u, v satisfying*

$$F(u) = \mu v, \qquad F(v) = -\mu u, \qquad \mu \in \mathbb{R} \tag{4.4}$$

Proof. The case $d = 1$ is trivial, so let us assume $d \geq 2$. By the last statement of Lemma 4.1 either there exists an eigenvector w with zero eigenvalue or the pair $\{u, v\}$ claimed in the corollary exists. In the former case, we consider the vector subspace $w^{\perp}$. Its dimension is at least one and F restricts to this space so again either the pair $\{u, v\}$ exists or there is $e \in w^{\perp}$ satisfying $F(e) = 0$. But then $\{w, e\}$ are orthogonal and non-zero. Normalizing we find a pair $\{u, v\}$ that satisfies (4.4) with $\mu = 0$, □

Lemma 4.1 lists a set of cases, one of which must always occur. However, we now show that, if the dimension is sufficiently high, case (i) of that lemma implies one of the other two:

Lemma 4.4. *Let $F \in \mathrm{SkewEnd}(V)$, with V Lorentzian of dimension at least four. If F has a null eigenvector, then it also has either a spacelike eigenvector or a spacelike eigenplane.*

Proof. Let $k \in V$ be a null eigenvector of F. The space $A := k^{\perp} \subset V$ is a null hyperplane and F restricts to A. On this space we define the standard equivalence relation $v_0 \sim v_1$ iff $v_0 - v_1 = ak$, $a \in \mathbb{R}$. The quotient $A/\sim$ (which has dimension at least two) inherits a positive definite metric $\overline{g}$ and F also descends to the quotient. More precisely, if we denote the equivalence class of any $v \in A$ by $\overline{v}$, then for any $\overline{v} \in A/\sim$ and any $v \in \overline{v}$ the expression $\overline{F}(\overline{v}) = \overline{F(v)}$ is well-defined (i.e. independent of the choice of representative v) and hence defines an endomorphism $\overline{F}$ of $A/\sim$ which, moreover, satisfies

$$\langle \overline{F}(\overline{v_1}), \overline{v_2} \rangle_{\overline{g}} = -\langle \overline{v_1}, \overline{F}(\overline{v_2}) \rangle_{\overline{g}}.$$

In other words $\overline{F}$ is a skew-symmetric endomorphism in the Riemannian vector space $A/\sim$. By Corollary 4.3 (here we use that the dimension of $A/\sim$ is at least two) there exists a pair of orthogonal and $\overline{g}$-unit vectors $\{\overline{e_1}, \overline{e_2}\}$ satisfying

$$\overline{F}(\overline{e_1}) = a\,\overline{e_2}, \qquad \overline{F}(\overline{e_2}) = -a\,\overline{e_1}, \qquad a \in \mathbb{R}.$$

Select representatives $e_1 \in \overline{e_1}$ and $e_2 \in \overline{e_2}$. In terms of F, the condition (4.1) and the fact that k is eigenvector require the existence of constants σ, a, λ_1 and λ_2 such that

$$F(k) = \sigma k, \qquad F(e_1) = ae_2 + \lambda_1 k, \qquad F(e_2) = -ae_1 + \lambda_2 k.$$

Whenever $a^2+\sigma^2\neq 0$ the vectors

$$u := e_1 - \frac{1}{a^2+\sigma^2}\left(a\lambda_2+\sigma\lambda_1\right)k, \qquad v := e_2 + \frac{1}{a^2+\sigma^2}\left(a\lambda_1-\sigma\lambda_2\right)k$$

satisfy $F(u)=av$ and $F(v)=-au$. Since u and v are spacelike, unit and orthogonal to each other the claim of the proposition follows (with $\mu=a$). If $\sigma=a=0$, then either $\lambda_1=\lambda_2=0$ and then $\{e_1,e_2\}$ are directly the vectors $\{u,v\}$ claimed in the proposition (with $\mu=0$), or at least one of the λs (say λ_2) is not zero. Then $e:=e_1-\frac{\lambda_1}{\lambda_2}e_2$ is a spacelike eigenvector of F. □

Now we have all the ingredients to show one of the main results of this section, that will eventually allow us to classify skew-symmetric endomorphisms of Lorentzian vector spaces.

Proposicin 4.5. *Let V be a Lorentzian vector space of dimension at least five and $F\in\mathrm{SkewEnd}(V)$. Then, there exists a spacelike eigenplane.*

Proof. We examine each one of the three possibilities described in Lemma 4.1. Case (iii) yields the result trivially, so we can assume that F has an eigenvector w.

If we are in case (ii), the vector w is either spacelike or timelike. If it is timelike we consider the Riemannian space $w^\perp$ where F restricts. We may apply Corollary 4.3 (note that $w^\perp$ has dimension at least four) and conclude that the vectors $\{u,v\}$ exist. So it remains to consider the case when x is spacelike and F admits no timelike eigenvectors. We restrict to $w^\perp$ which is Lorentzian and of dimension at least four. Applying again Lemma 4.1, either there exists a spacelike eigenplane, or a second eigenvector $y\in w^\perp$, which can only be spacelike or null. If y is spacelike, $\{u:=w,\ v:=y\}$ span a spacelike eigenplane with $\mu=0$. If y is null, we may apply Lemma 4.4 to $F\mid_{w^\perp}$ to conclude that either a spacelike eigenplane exists, or there is a spacelike eigenvector $e\in w^\perp$, so the pair $\{u:=e, v:=w\}$ satisfies (3.2) with $\mu=0$. This concludes the proof of case (ii).

In case (i), i.e. when there is a null eigenvector w we can apply Lemma 4.4 and conclude that either $\{u,v\}$ exist, or there is a spacelike eigenvector $e\in V$, in which case we are into case (ii), already solved. This completes the proof.

□

We have now all the necessary ingredients to give a complete classification of skew-symmetric endomorphisms of Lorentzian vector spaces. In the next result, we identify Lorentzian (sub)spaces of d-dimension with the Minkowski space $\mathbb{M}^{1,d-1}$. Also, for any real number $x\in\mathbb{R}$, $[x]\in\mathbb{Z}$ denotes its integer part.

Teorema 4.6 (Classification of skew-symmetric endomorphisms in Lorentzian spaces)**.** *Let $F\in\mathrm{SkewEnd}(V)$ with V Lorentzian of dimension $d>2$. Then V has a set of*

$[\frac{d-1}{2}]-1$ mutually orthogonal spacelike eigenplanes $\{\Pi_i\}$, $i=1,\cdots,[\frac{d-1}{2}]-1$, so that V admits one of the following decompositions into direct sum of F-invariant subspaces:

a) If d even $V=\mathbb{M}^{1,3}\oplus\Pi_{\frac{d-4}{2}}\oplus\cdots\oplus\Pi_1$ and either $F\mid_{\mathbb{M}^{1,3}}=0$ or otherwise one of the following cases holds:

a.1) $F\mid_{\mathbb{M}^{1,3}}$ has a spacelike eigenvector e orthogonal to a null eigenvector with vanishing eigenvalue and then $\mathbb{M}^{1,3}=\mathbb{M}^{1,2}\oplus\mathrm{span}\{e\}$.

a.2) $F\mid_{\mathbb{M}^{1,3}}$ has a spacelike eigenplane $\Pi_{\frac{d-2}{2}}$ (as well as a timelike eigenplane $\mathbb{M}^{1,1}$ orthogonal to $\Pi_{\frac{d-2}{2}}$) and then $\mathbb{M}^{1,3}=\mathbb{M}^{1,1}\oplus\Pi_{\frac{d-2}{2}}$.

b) If d odd $V=\mathbb{M}^{1,2}\oplus\Pi_{\frac{d-3}{2}}\oplus\cdots\oplus\Pi_1$ and either $F\mid_{\mathbb{M}^{1,2}}=0$ or otherwise one of the following cases holds:

b.1) $F\mid_{\mathbb{M}^{1,2}}$ has a spacelike eigenvector e and then $\mathbb{M}^{1,2}=\mathbb{M}^{1,1}\oplus\mathrm{span}\{e\}$.

b.2) $F\mid_{\mathbb{M}^{1,2}}$ timelike eigenvector t and then $\mathbb{M}^{1,3}=\mathrm{span}\{t\}\oplus\Pi_{\frac{d-1}{2}}$.

b.3) $F\mid_{\mathbb{M}^{1,2}}$ has a null eigenvector with vanishing eigenvalue.

Proof. The proof is a simple combination of the previous results. First, if $d\geq 5$, we can apply Proposition 4.5 to obtain the first spacelike eigenplane Π_1. Then $\Pi_1^{\perp}$ is Lorentzian of dimension $d-2$. If $d-2\geq 5$, we can apply again Proposition 4.5 to obtain a second eigenplane Π_2. Continuing with this process, depending on d, two things can happen:

a) If d even, we get $\frac{d-4}{2}$ $\left(=[\frac{d-1}{2}]-1\right)$ spacelike eigenplanes, until we eventually reach a Lorentzian vector subspace of dimension four, $\mathbb{M}^{1,3}$, where Proposition 4.5 cannot be applied. In $\mathbb{M}^{1,3}$, either $F\mid_{\mathbb{M}^{1,3}}=0$ or otherwise cases $a.1$) and $a.2$) follow from Remark 3.11, cases 2 and 1 respectively.

b) If d odd, we get $\frac{d-3}{2}$ $\left(=[\frac{d-1}{2}]-1\right)$ spacelike eigenplanes, until we reach a Lorentzian vector subspace of dimension three, $\mathbb{M}^{1,2}$. In $\mathbb{M}^{1,2}$, either $F\mid_{\mathbb{M}^{1,2}}=0$ or by Remark 3.12 there exists a unique eigenvector σ with vanishing eigenvalue. If q null, case $b.3$) follows. If it is spacelike $e:=q$, F restricts to $e^{\perp}=\mathbb{M}^{1,1}\subset\mathbb{M}^{1,2}$ and $b.1$) follows. If q timelike, the same argument applies with $t:=q$ and $t^{\perp}\subset\mathbb{M}^{1,2}$ defines the remaining spacelike plane $\Pi_{\frac{d-1}{2}}$.

□

4.2 Canonical form for skew-symmetric endomorphisms

Our aim here is to extend the results in Proposition 3.8 and Corollary 3.9 to arbitrary dimensions. To do that, we will employ the classification Theorem 4.6 derived in Section 4.1, from which it immediately follows a decomposition of any $F\in\mathrm{SkewEnd}(V)$

into direct sum of skew-symmetric endomorphisms of the subspaces that F restricts to, namely

$$F = F|_{\mathbb{M}^{1,3}} \bigoplus_{i=1}^{[\frac{d-1}{2}]-1} F|_{\Pi_i} \qquad \text{if } d \text{ even}, \tag{4.5}$$

$$F = F|_{\mathbb{M}^{1,2}} \bigoplus_{i=1}^{[\frac{d-1}{2}]-1} F|_{\Pi_i} \qquad \text{if } d \text{ odd}, \tag{4.6}$$

where Π_i are spacelike eigenplanes. In what follows, we will denote

$$p := [(d-1)/2] - 1.$$

Notice that the blocks $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ may also admit different subdecompositions depending on the case, but our purpose is to remain as unified as possible, so we leave this part unaltered. It will be convenient for the remainder to give a name to the decompositions (4.5) and (4.6):

Definicin 4.7. Let $F \in \mathrm{SkewEnd}(V)$ non-zero for V Lorentzian d-dimensional. Then, a decomposition of the form (4.5) or (4.6) is called **block form** of F. A basis that realizes a block form is called **block form basis**.

Writing F in block form form allows us to work with F as a sum of skew-symmetric endomorphisms of riemmanian two-planes plus one skew-symmetric endomorphism of a three or four dimensional Lorentzian vector space. For the latter we will employ the canonical forms in Proposition 3.8 and Corollary 3.9, and for the former, it is immediate that in every (suitably oriented) orthonormal basis of Π_i

$$F\,|_{\Pi_i} = \begin{pmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{pmatrix}, \qquad 0 \leq \mu_i \in \mathbb{R}. \tag{4.7}$$

Having defined a canonical form for four, three and two dimensional endomorphisms (i.e. matrices (3.6), (3.8) and (4.7) respectively), the idea is to extend this result to arbitrary dimensions finding a systematic way to construct a block form (4.5), (4.6) such that each of the blocks are in canonical form. This is not immediate, firstly, because the block form does not require the blocks $F|_{\mathbb{M}^{1,3}}$ or $F|_{\mathbb{M}^{1,2}}$ to be non-zero and secondly, because, unlike in the four and three dimensional cases, the parameters σ, τ of the four and three dimensional blocks cannot be invariantly defined as, for example, traces of F^2 or determinant of F. The first of these concerns is easily solved by suitably choosing a block form:

Lemma 4.8. *Let $F \in \mathrm{SkewEnd}(V)$ be non-zero for V Lorentzian of dimension d. Then there exists a block form (4.5) and (4.6) such that $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ are non-zero and they either contain no spacelike eigenplanes or they contain one with largest eigenvalue* *(among all spacelike eigenplanes of F). In addition, the rest of spacelike eigenplanes Π_i are sorted by decreasing value of μ_i^2, i.e. $\mu_1^2 \geq \mu_2^2 \geq \cdots \geq \mu_p^2$.*

Proof. If $\ker F$ is degenerate, it must correspond with cases $a.1$) (d even) or $b.3$) (d odd) of Theorem 4.6. Hence, in any block form the blocks $F|_{\mathbb{M}^{1,3}}$ and $F|_{\mathbb{M}^{1,2}}$ are non-zero and they do not contain any spacelike eigenplane, as claimed in the lemma. So let us assume that $\ker F$ is non-degenerate or zero, which discards cases $a.1$) and $b.3$) of Theorem 4.6. In all possible cases, any block form admits the following splitting in

$$F|_{\mathbb{M}^{1,3}} = F|_{\Pi_t} \oplus F|_{\Pi_s}, \qquad F|_{\mathbb{M}^{1,2}} = F|_{\mathrm{span}\{v\}} \oplus F|_{v^\perp}, \tag{4.8}$$

with Π_s, Π_t spacelike and timelike eigenplanes with (possibly zero) respective eigenvalues μ_s and μ_t, v a timelike or spacelike eigenvector (in $\ker F$) and $v^\perp \subset \mathbb{M}^{1,2}$ an eigenplane with opposite causal character than v. If v is spacelike, then either $F|_{v^\perp}$ is non-zero, in which case $F|_{\mathbb{M}^{1,2}} \neq 0$ and clearly contains no spacelike eigenplanes (which is one of the possibilities in the lemma), or $F|_{v^\perp} = 0$ and then $F|_{\mathbb{M}^{1,2}} = 0$, so we can rearrange the decomposition (4.8) using some timelike vector $v' \in v^\perp$ instead of v, i.e. $F|_{\mathbb{M}^{1,2}} = F|_{\mathrm{span}\{v'\}} \oplus F|_{v'^\perp}$. Hence, in the case of d odd, we may assume that v is timelike and $v^\perp \subset \mathbb{M}^{1,2}$ is a spacelike eigenplane. Let Π_μ be a spacelike eigenplane of F with largest eigenvalue μ among Π_s (d even) or $v^\perp$ (d odd) and $\Pi_1, \cdots, \Pi_p$. Then, switching $F|_{\Pi_s}$ or $F|_{\Pi_{v^\perp}}$ by $F|_{\Pi_\mu}$ we construct

$$\hat{F}\,|_{\mathbb{M}^{1,3}} := F|_{\Pi_t} \oplus F|_{\Pi_\mu}, \qquad \hat{F}\,|_{\mathbb{M}^{1,2}} := F|_{\mathrm{span}\{v\}} \oplus F|_{\Pi_\mu}.$$

The resulting matrix is still in block form and has non-zero blocks $\hat{F}\,|_{\mathbb{M}^{1,3}}$, $\hat{F}\,|_{\mathbb{M}^{1,2}}$ containing a spacelike eigenplane with largest eigenvalue, which is the other possibility in the lemma. The last claim follows by simply rearranging the remaining spacelike eigenplanes Π_i by decreasing order of μ_i^2. □

With a skew-symmetric endomorphism F in the block form given in Lemma 4.8 we can take each one of the blocks to its respective canonical form. Let us denote $F_{\sigma\tau} := F|_{\mathbb{M}^{1,3}}$ (if d even), $F_\sigma := F|_{\mathbb{M}^{1,2}}$ (if d odd) and $F_{\mu_i} := F|_{\Pi_i}$ when written in the canonical forms (3.6), (3.8) and (4.7) respectively. Consequently

$$F = F_{\sigma\tau} \bigoplus_{i=1}^{p} F_{\mu_i} \quad (d \text{ even}), \qquad F = F_{\sigma} \bigoplus_{i=1}^{p} F_{\mu_i} \quad (d \text{ odd}), \tag{4.9}$$

where, notice, each of the blocks is written in an orthonormal basis of the corresponding subspace, which moreover is future directed if the subspace is Lorentzian, i.e. $\mathbb{M}^{1,3}$ or $\mathbb{M}^{1,2}$ (cf. Proposition 3.8 and Corollary 3.9). Hence, the form given in (4.9) corresponds to a future directed, orthonormal basis of $\mathbb{M}^{1,d-1}$.

Our aim now is to give an invariant definition of σ, τ, μ_i. A possible way to do this is through the eigenvalues of F^2. One may wonder why not to use directly the eigenvalues of F. One reason is that since we are interested in real Lorentzian vector spaces V (although, for practical reasons, we may rely on the complexification $V_{\mathbb{C}}$ for some proofs), it is more consistent to give our canonical form in terms of real quantities, while the eigenvalues of F may be complex. In addition, the canonical form will require to sort them in some way, for which using real numbers is better suited.

The characteristic polynomial of F is known (e.g. [100]) to possess the following parity:

$$\mathcal{P}_F(x) = (-1)^d \mathcal{P}_F(-x). \tag{4.10}$$

Thus, a simple calculation relates the characteristic polynomials of F and F^2

$$\begin{aligned} \mathcal{P}_{F^2}(x) &= \det(x\, \mathrm{Id}_d - F^2) = \det\left(\sqrt{x}\, \mathrm{Id}_d - F\right) \det\left(\sqrt{x}\, \mathrm{Id}_d + F\right) \\ &= (-1)^d \mathcal{P}_F(\sqrt{x}) \mathcal{P}_F(-\sqrt{x}) = \left(\mathcal{P}_F(\sqrt{x})\right)^2, \end{aligned} \tag{4.11}$$

$\sqrt{x}$ being any of the square roots of x in $\mathbb{C}$ and Id_d the $d \times d$ identity matrix. We can extract some conclusions from (4.11):

Lemma 4.9. *Let $F \in \mathrm{SkewEnd}(V)$ for V Lorentzian of dimension d. Then the non-zero eigenvalues of F^2 have even multiplicity m_a and the zero eigenvalue has multilplicity m_0 with the parity of d. In addition, F possesses p_a (resp. exaclty one) spacelike (resp. timelike) eigenplanes with eigenvalue $\mu \neq 0$ if and only if F^2 has a negative (resp. positive) non-zero eigenvalue $-\mu^2$ (resp. μ^2) with multiplicity $m_a := 2p_a$ (resp. exactly two).*

Proof. It is an immediate consequence of equation (4.11) that non-zero eigenvalues of F^2 must have even multiplicity m_a. Moreover, since the sum of all multiplicites adds up to the dimension d, the multiplicity of the zero m_0 has the parity of d.

Combining Lemma 3.3 and equation (4.11), F has a spacelike eigenplane Π with non-zero eigenvalue μ if and only if F^2 has a negative double[1] eigenvalue $-\mu^2$. If $d \leq 4$, there cannot be any other spacelike eigenplanes in $\Pi^\perp$, so applying the same argument to $F|_{\Pi^\perp} \in \mathrm{SkewEnd}(\Pi^\perp)$, the multiplicity m_a of $-\mu^2$ must be $m_a = 2$. If $d > 4$ and $m_a \geq 4$, then $-\mu^2$ is an eigenvalue of $(F|_{\Pi^\perp})^2$ with multiplicity $m_a - 2$, thus F has a second spacelike eigenplane with eigenvalue μ in $\Pi^\perp$. Repeating this argument, F^2 has a negative eigenvalue $-\mu^2$ with multilplicity m_a if and only if F has $p_a = m_a/2$ spacelike eigenplanes with eigenvalue μ.

Finally, by Lemma 3.4 and equation (4.11), F has a timelike eigenplane Π with non-zero eigenvalue μ if and only if F^2 has a positive double eigenvalue μ^2. Obviously, the maximum number of timelike eigenplanes that F can have is one. Thus, $F|_{\Pi^\perp}$ cannot have timelike eigenplanes and hence $(F|_{\Pi^\perp})^2$ has no additional positive eigenvalues. Consequently, the multiplicity of μ^2 is exactly two.

[1]We adopt the convention that a root with multiplicity $m \geq 2$ is also double

□

Taking into account Lemma 4.9, we will employ the eigenvalues of $-F^2$ rather than those of F^2, so we assign positive eigenvalues of F^2 with spacelike eigenplanes and negative eigenvalues to timelike eigenplanes. This amounts to employ the roots of the characteristic polynomial $\mathcal{P}_{F^2}(-x)$.

We now discuss how to invariantly define the parameters σ, τ, μ_i for d even and σ, μ_i for d odd. The result of the argument is formalized below in Definition 4.10. Recall that the characteristic polynomial of a direct sum of two or more endomorphisms is the product of their individual characteristic polynomials, in particular, the characteristic polynomial of $-F^2$ equals to the product of the characteristic polynomials of $-F^2_{\sigma\tau}$ or $-F^2_{\sigma}$ times those of each $-F^2_{\mu_i}$ (cf. equation (4.9)). Let us define:

$$\mathcal{Q}_{F^2}(x) := \left(\mathcal{P}_{F^2}(-x)\right)^{1/2} \quad (\text{d even}), \qquad \mathcal{Q}_{F^2}(x) := \left(\frac{\mathcal{P}_{F^2}(-x)}{x}\right)^{1/2} \quad (\text{d odd}), \quad (4.12)$$

Starting with d even, from formula (4.9) it is immediate that μ_i^2 are double roots of $\mathcal{P}_{F^2}(-x^2)$, which by Lemma 4.8 satisfy $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0$. On the other hand, let $\mu_t := \sqrt{(-\sigma+\rho)/2}$ and $i\mu_s := i\sqrt{(\sigma+\rho)/2}$ with $\rho := \sqrt{\sigma^2+\tau^2} \geq 0$, that by Remark 3.11, are roots of $\mathcal{P}_{F_{\sigma\tau}}(x)$, thus roots of $\mathcal{P}_F(x)$. By equation (4.11), $-\mu_t^2, \mu_s^2$ are double roots of $\mathcal{P}_{F^2}(-x)$. The set $\{-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2\}$ are in total $p+2 = [(d-1)/2]+1 = d/2$ elements, each of which is a double root of $\mathcal{P}_{F^2}(-x)$. In other words, $\{-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2\}$ is the set of all roots of the polynomial[2] $\mathcal{Q}_{F^2}(x)$. If $\ker F$ is degenerate, then $\ker F_{\sigma\tau}$ is degenerate and by Remark 3.11 it must happen $\mu_t = \mu_s = 0$. Hence $\mu_1^2 \geq \mu_2^2 \geq \cdots \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$. Otherwise, also by Remark 3.11, $F_{\sigma\tau}$ contains a spacelike eigenplane with eigenvalue μ_s (which by Lemma 4.8 is the largest) as well as a timelike eigenplane with eigenvalue μ_t. In this case $\mu_s^2 \geq \mu_1^2 \geq \cdots \mu_p^2 \geq 0 \geq -\mu_t^2$.

We next discuss σ, μ_i for d odd. Again, from (4.9) we have that μ_i^2 are double roots of $\mathcal{P}_{F^2}(-x^2)$, which by Lemma 4.8 also satisfy $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0$. By Remark 3.12, $\sqrt{\sigma}$ is a root of $P_{F_\sigma}(x)$, thus a root of $P_F(x)$, so by formula (4.11), σ is a double root of $\mathcal{P}_{F^2}(-x)$. Also, $\mathcal{P}_{F^2}(-x)$ has at least one zero root and hence, $\mathcal{P}_{F^2}(-x)/x$ is a polynomial with $d-1$ roots (counting multiplicity). Then, the set $\{\sigma, \mu_1^2, \cdots, \mu_p^2\}$ are all double roots of $\mathcal{P}_{F^2}(-x)/x$, which are $p+1 = [(d-1)/2] = (d-1)/2$ elements. Therefore Q_{F^2} as defined in (4.12) is also a polynomial and $\{\sigma, \mu_1^2, \cdots, \mu_p^2\}$ is the set of all its roots. If $\ker F$ is timelike, then $\ker F_\sigma$ is timelike, which happens if and only if $\sigma > 0$ (cf. Remark 3.12) and also F_σ has a spacelike eigenplane with eigenvalue $\sqrt{|\sigma|}$, that by Lemma 4.8 is the largest eigenvalue among spacelike eigenplanes. Thus $\sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2$. In the case $\ker F$ not timelike, the inequalities become $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq \sigma$.

[2] $\mathcal{Q}_{F^2}(x)$ is a polynomial because all the roots of $\mathcal{P}_{F^2}(-x)$ are double.

Summarizing, the paramaters σ, τ, μ_i correspond to the set of all roots of $\mathcal{Q}_{F^2}$ sorted in a certain order fully determined by the causal character of $\ker F$. This allows us to put forward the following definition:

Definicin 4.10. Let $\text{Roots}\left(\mathcal{Q}_{F^2}\right)$ denote the set of roots of $\mathcal{Q}_{F^2}(x)$ repeated as many times as their multiplicity. Then

a) If d odd, $\left\{\sigma; \mu_1^2, \cdots, \mu_p^2\right\} := \text{Roots}\left(\mathcal{Q}_{F^2}\right)$ sorted by $\sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2$ if $\ker F$ is timelike, where in this case necessarily $\sigma > 0$, and $\mu_p^2 \geq 0 \geq \sigma$ otherwise.

b) If d even, $\left\{-\mu_t^2, \mu_s^2; \mu_1^2, \cdots, \mu_p^2\right\} := \text{Roots}\left(\mathcal{Q}_{F^2}\right)$ sorted by $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$ if $\ker F$ is degenerate and $\mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq -\mu_t^2$ otherwise, where either μ_s^2 or μ_t^2 are non-zero. In addition, in any case, we also define $\sigma := \mu_s^2 - \mu_t^2$, $\tau := 2|\mu_t \mu_s|$.

Observacin 4.11. *In the d even case the parameters σ, τ are useful because they allow one to give a unique unambiguous canonical form for every element in* $\text{SkewEnd}(V)$*, which naturally recovers the canonical form for d odd when $\tau = 0$. However, observe that the sets $\{\sigma, \tau; \mu_1^2, \cdots, \mu_p^2\}$ and $\{-\mu_t^2, \mu_s^2; \mu_1^2, \cdots, \mu_p^2\}$ are equivalent. It is useful to keep both definitions in mind because, depending on the application, we may use one or another.*

In addition, we also summarize the results concerning the canonical form in the following Theorem:

Teorema 4.12. *Let $F \in \text{SkewEnd}(V)$ non-zero, with V Lorentzian of dimension $d \geq 3$ and $p := [(d-1)/2]-1$. Then there exists an orthonormal, future oriented basis such that F is given (4.9) where $F_{\sigma\tau} := F\,|_{\mathbb{M}^{1,3}}$, $F_\sigma := F\,|_{\mathbb{M}^{1,2}}$, $F_{\mu_i} := F\,|_{\Pi_i}$ are given by (3.6), (3.8), (4.7) respectively and σ, τ, μ_i are given in Definition 4.10. In particular, $F_{\sigma\tau}$, F_σ are non-zero and they either do not contain a spacelike eigenplane or they contain one with maximal eigenvalue (among all spacelike eigenplanes of F) and the eigenvalues μ_i are sorted by $\mu_1^2 \geq \mu_2^2 \geq \cdots \mu_p^2$.*

Definicin 4.13. For any $F \in \text{SkewEnd}(V)$, for V Lorentzian d-dimensional, the form of F given in Theorem 4.12 is called **canonical form** and the basis realizing it is called **canonical basis**.

The first and obvious reason why the canonical form is useful is that it allows one to work with all elements $F \in \text{SkewEnd}(V)$ at once. The fact that we can give a canonical form for every element without splitting into cases is a great strenght, since we can perform a general analysis just in terms of the parameters that define the canonical form. Moreover, as we will show in Section 4.4, this form is the same for all the elements in the orbit generated by the adjoint action of the orthochronous Lorentz group $O^+(1, d-1)$. Thus, the canonical form is specially suited for problems with $O^+(1, d-1)$ invariance (or covariance) which, as we have discussed in Chapter 2, is directly related to certain conformally covariant problems in general relativity.

We finish this section with two corollaries that will be useful later. The first one is trivial from the canonical form (4.9)

Corolario 4.14. *The characteristic polynomial of* $F \in \mathrm{SkewEnd}(V)$ *is*

$$\begin{aligned} \mathcal{P}_F(x) &= (x^2 - \mu_t^2)(x^2 + \mu_s^2)\prod_{i=1}^{p}(x^2 + \mu_i^2) \quad (d \text{ even}), \\ \mathcal{P}_F(x) &= x(x^2 + \sigma)\prod_{i=1}^{p}(x^2 + \mu_i^2) \quad (d \text{ odd}), \end{aligned} \tag{4.13}$$

where $-2\mu_t^2 := \sigma - \sqrt{\sigma^2 + \tau^2}$, $2\mu_s^2 := \sigma + \sqrt{\sigma^2 + \tau^2}$.

The second gives a formula for the rank of F. We base our proof in the canonical form (4.9) because it is straightforward. However, we remark that this corollary can also be regarded as a consequence of Theorem 4.6.

Corolario 4.15. *Let* $F \in \mathrm{SkewEnd}(V)$, *with* V *Lorentzian of dimension* d *and* m_0 *the multiplicity of the zero eigenvalue. Then, only of the following exclusive cases hold:*

a) $\ker F$ *is non-degenerate or zero if and only if* $\mathrm{rank}\ F = d - m_0$.

b) $\ker F$ *is degenerate if and only if* $m_0 > 2$ *and* $\mathrm{rank}\ F = d - m_0 + 2$.

Proof. Consider F in canonical form (4.9) and let $k \in \mathbb{N}$ be the number of parameters μ_i that vanish. For d even we have $\dim\ker F = 2k + \dim\ker F_{\sigma\tau}$. On the one hand, $\ker F$ degenerate implies $\ker F_{\sigma\tau}$ degenerate, which by Remark 3.11 happens if and only if $\sigma = \tau = 0$ and in addition $\dim\ker F_{\sigma\tau} = 2$. Therefore $\dim\ker F = 2k+2$ and by (4.13), $m_0 = 2k + 4$ (> 2). Thus $\mathrm{rank}\ F = d - \dim\ker F = d - m_0 + 2$. On the other hand, $\ker F$ non-degenerate if at most one of σ or τ vanish. If $\tau \neq 0$ (so that $\mu_s \neq 0$ and $\mu_t \neq 0$), $\dim\ker F_{\sigma\tau} = 0$ and $m_0 = 2k = \dim\ker F$. Consequently $\mathrm{rank}\ F = d - m_0$. If $\tau = 0$ (and $\sigma \neq 0$, so that exactly one of μ_s, μ_t vanish), by Remark 3.11 $\dim\ker F_{\sigma\tau} = 2$ and by (4.13) $m_0 = 2k+2$. Hence $\dim\ker F = 2k+2$ and $\mathrm{rank}\ F = d - m_0$.

For d odd, we have $\dim\ker F = 2k + \dim\ker F_\sigma = 2k+1$, because $\dim\ker F_\sigma = 1$ (cf. Remark 3.12). $\ker F$ is degenerate if and only if $\ker F_\sigma$ is degenerate, which by Remark 3.12 occurs if and only if $\sigma = 0$. Hence, by equation (4.13), $m_0 = 2k+3$ (> 2) and $\mathrm{rank}\ F = d - \dim\ker F = d - m_0 + 2$. For the $\ker F$ non-degenerate case, $\sigma \neq 0$ and also by (4.13) $m_0 = 2k+1 = \dim\ker F$. Therefore $\mathrm{rank}\ F = d - m_0$.

□

4.3 Simple endomorphisms

In this Section we derive some results which will be useful for the analysis of CKVFs carried out in Section 4.5.

By *simple skew-symmetric endomorphism* we mean a $G \in \mathrm{SkewEnd}(V)$ satisfying rank $G = 2$. As usual $\boldsymbol{e} \equiv \langle e, \cdot \rangle$ is the one-form obtained by lowering index to a vector $e \in V$. Then, a simple skew-symmetric endomorphism can be always written as

$$G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$$

for two linearly independent vectors $e, v \in V$ and its action on any vector $w \in V$ is

$$G(w) = \langle v, w \rangle e - \langle e, w \rangle v.$$

Since the two-fom associated to a simple endomorphism is $\mathbf{G} = \boldsymbol{e} \wedge \boldsymbol{v}$, it follows from elementary algebra that two simple skew-symmetric endomorphisms $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ and $G' = e' \otimes \boldsymbol{v}' - v' \otimes \boldsymbol{e}'$ are proportional if and only if $\mathrm{span}\{e, v\} = \mathrm{span}\{e', v'\}$. This freedom in the pair $\{e, v\}$ defining G can be used to choose them orthogonal.

Lemma 4.16. *Let $G \in \mathrm{SkewEnd}(V)$ be simple. Then there exist two non-zero orthogonal vectors $e, v \in V$ such that $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ with v spacelike.*

Proof. By definition $G = \tilde{e} \otimes \tilde{\boldsymbol{v}} - \tilde{v} \otimes \tilde{\boldsymbol{e}}$ for two linearly indepedent vectors $\tilde{e}, \tilde{v} \in V$. If one of them is non-null, we set $\tilde{v} := v$ and decompose $V = \mathrm{span}\{v\} \oplus v^{\perp}$. Thus $\tilde{e} = av + e$ with $a \in \mathbb{R}$ and $e \in v^{\perp}$ and G takes the form $G = (av + e) \otimes \boldsymbol{v} - v \otimes (a\boldsymbol{v} + \boldsymbol{e}) = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$, as claimed. If $\tilde{e}$ and $\tilde{v}$ are both null, consider $V = \mathrm{span}\{\tilde{e}\} \widetilde{\oplus} (\tilde{e})^c$ (we use $\widetilde{\oplus}$ because this direct sum is not by orthogonal spaces) where $(\tilde{e})^c$ is a spacelike complement of $\mathrm{span}\{\tilde{e}\}$. Then we can write $\tilde{v} = a\tilde{e} + v'$, with $a \in \mathbb{R}$ and $v' \in \tilde{e}^c$ non-null. Thus $G = \tilde{e} \otimes \boldsymbol{v}' - v' \otimes \tilde{\boldsymbol{e}}$, with v' non-null and we fall into the previous case. All in all, $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ with e, v orthogonal. Consequently, either one of the vectors is spacelike or both are null and proportional which would imply $G = 0$, against our hypothesis rank $G = 2$. □

The decomposition $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ is not unique even with the restriction of v being spacelike unit and orthogonal to e. One can easily show that the remaining freedom is given by the transformation $e' = ae - b \langle e, e \rangle v$, $v' = be + av$ with $a, b \in \mathbb{R}$ restricted to $a^2 + b^2 \langle e, e \rangle = 1$. Nevertheless, the square norm $\langle e', e' \rangle$ is invariant under this change, so the following definition makes sense:

Definicin 4.17. Let $G \in \mathrm{SkewEnd}(V)$ be simple, with $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$, $e, v \in V$ orthogonal with v spacelike unit. Then G is said to be *spacelike, timelike* or *null* if the vector e is *spacelike, timelike* or *null* respectively. In the non-null case, G is called spacelike (resp. timelike) unit whenever $\langle e, e \rangle = +1$ (resp. $\langle e, e \rangle = -1$).

By Lemma 4.16, it is immediate that Definition 4.17 comprises any possible simple endomorphism (up to a multiplicative factor).

We next obtain the necessary and sufficient conditions for a simple endomorphism G to commute with a given $F \in \mathrm{SkewEnd}(V)$. We first make the simple observation that the composition of a one-form $\boldsymbol{e}$ and a skew-symmetric endomorphism F satisfies (simply apply for sides to any $w \in V$)

$$\boldsymbol{e} \circ F = -\boldsymbol{F}(\boldsymbol{e}),$$

where we denote $\boldsymbol{F}(\boldsymbol{e}) := \langle F(e), \cdot \rangle$. An immediate consequence is that for any pair of vectors $e, v \in V$ and $F \in \mathrm{SkewEnd}(V)$ it holds

$$F \circ (e \otimes \boldsymbol{v}) = F(e) \otimes \boldsymbol{v}, \qquad (e \otimes \boldsymbol{v}) \circ F = -e \otimes \boldsymbol{F}(\boldsymbol{v}). \tag{4.14}$$

The following commutation result will be used later.

Lemma 4.18. *Let $F, G \in \mathrm{SkewEnd}(V)$ with $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ simple and $e, v \in V$ as in Definition 4.17. Then $[F, G] = 0$ if and only if there exist $\mu \in \mathbb{R}$ such that:*

$$F(e) = \langle e, e \rangle \mu v, \qquad F(v) = -\mu e. \tag{4.15}$$

Proof. The commutator is

$$\begin{aligned}[F, G] = F \circ G - G \circ F &= F \circ (e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}) - (e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}) \circ F \\ &= F(e) \otimes \boldsymbol{v} - F(v) \otimes \boldsymbol{e} + e \otimes \boldsymbol{F}(\boldsymbol{v}) - v \otimes \boldsymbol{F}(\boldsymbol{e}),\end{aligned} \tag{4.16}$$

where we have used (4.14). The "if" part is obtained by direct calculation inserting (4.15) in (4.16). To prove the "only if" part, the condition $[F, G] = 0$ requires the two endomorphisms $F(e) \otimes \boldsymbol{v} - v \otimes \boldsymbol{F}(\boldsymbol{e})$ and $F(v) \otimes \boldsymbol{e} - e \otimes \boldsymbol{F}(\boldsymbol{v})$ to be equal. One such endomorphism is either identically zero or simple. This implies that $\mathrm{span}\{F(e), v\}$ and $\mathrm{span}\{e, F(v)\}$ are either both one dimensional or both two-dimensional and equal. In the first case, $F(v) = -\mu e$ and $F(e) = \alpha v$ for $\mu, \alpha \in \mathbb{R}$, which are determined by skew-symmetry to satisfy $\alpha = \mu \langle e, e \rangle$, so the lemma follows. The second case is empty, for it is necessary that $v = ae + bF(v)$ with $a, b \in \mathbb{R}$, which implies $\langle v, v \rangle = \langle ae + bF(v), v \rangle = b \langle F(v), v \rangle = 0$, against the hypothesis of v being spacelike. □

Corolario 4.19. *Let $G, G' \in \mathrm{SkewEnd}(V)$ be simple, spacelike and linearly independent. Let $\{e, v\}$, $\{e', v'\}$ be orthogonal spacelike vectors such that $G = e \otimes \boldsymbol{v} - v \otimes \boldsymbol{e}$ and $G' = e' \otimes \boldsymbol{v}' - v' \otimes \boldsymbol{e}'$. Then $[G, G'] = 0$ if and only if $\{e, v, e', v'\}$ are mutually orthogonal.*

Proof. By the previous lemma $[G, G'] = 0$ if and only if there exist $\mu \in \mathbb{R}$ such that

$$G(e') = \langle e', v \rangle e - \langle e', e \rangle v = \mu v', \qquad G(v') = \langle v', v \rangle e - \langle v', e \rangle v = -\mu e'. \tag{4.17}$$

If $\mu \neq 0$, then $\text{span}\{e, v\} = \text{span}\{e', v'\}$ and G and G' are proportional, against hypothesis. Thus, $\mu = 0$ and by (4.17) the set $\{e, v, e', v'\}$ is mutually orthogonal. □

4.4 $O^+(1, d-1)$-classes

In this section we use the canonical form of Section 4.2 to characterize skew-symmetric endomorphisms of V under the adjoint action of the orthochronous Lorentz group $O^+(1, d-1)$. Recall that this is the subgroup of $O(1, d-1)$ preserving time orientation. The corresponding classes of skew-symmetric endomorphisms are also known as the adjoint orbits or conjugacy classes and we denote them by $[F]_{O^+}$ for a given element $F \in \text{SkewEnd}(V)$. The characterization of these orbits by a set of independent invariants is known and it can be found in [100] in terms of two-forms, or in [26] where a decomposition into so-called indecomposable types is shown to characterize the conjugacy classes. What we do here is, first, to give an alternative way to characterize the orbits $[F]_{O^+}$ by a convenient set of invariants and second, to show that the canonical form is the same for every element in a given orbit. This makes the canonical form specially useful as a tool for problems with $O(1, d-1)$ invariance.

Observacin 4.20. *We formulate this section in terms of the orthochronous component $O^+(1, d-1)$ because of its relation with conformal transformations of the sphere $\mathbb{S}^{d-2}$ (see Section 2.2.1), but note that the orbits of the full group $O(1, d-1)$ are exactly the same as those of $O^+(1, d-1)$. Recall that the time-reversing component $O^-(1, d-1)$ is one-to-one with $O^+(1, d-1)$. We can map elements $\Lambda^- \in O^-(1, d-1)$ to elements in $\Lambda^+ \in O^+(1, d-1)$ by e.g. $\Lambda^+ := \Lambda^- \Lambda_0$, where $\Lambda_0 = -\text{Id}_d$. Then*

$$\Lambda^+ F (\Lambda^+)^{-1} = \Lambda^- \Lambda_0 F \Lambda_0 (\Lambda^-)^{-1} = \Lambda^- F (\Lambda^-)^{-1},$$

which clearly imples that the orbits generated by the full group $O(1, d-1)$ coincide with the orbits generated by the subgroup $O^+(1, d-1)$.

A consequence of equation (4.10) is that the characteristic polynomial of $F \in \text{SkewEnd}(V)$ must have the form

$$\mathcal{P}_F(x) = x^d + \sum_{b=1}^{q} c_b x^{d-2b}, \tag{4.18}$$

where we have introduced $q := [\frac{d}{2}]$. The coefficients c_b can be obtained using the Fadeev-LeVerrier algorithm, summarized by the following matrix determinant [65]:

$$c_b = \frac{1}{(2b)!} \begin{vmatrix} \text{Tr } F & 2b-1 & 0 & \cdots & 0 \\ \text{Tr } F^2 & \text{Tr } F & 2b-2 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \text{Tr } F^{2b-1} & \text{Tr } F^{2b-2} & \cdots & \cdots & 1 \\ \text{Tr } F^{2b} & \text{Tr } F^{2b-1} & \cdots & \cdots & \text{Tr } F \end{vmatrix}.$$

Since the traces of odd powers vanish by skew-symmetry, the coefficients c_b depend on the entries of F only through the traces of the squared powers of F:

$$I_b := \frac{1}{2}\text{Tr }(F^{2b}), \quad b = 1, \cdots, q.$$

The traces I_b are obviously invariant under the adjoint action (cf. Section 3.5) of $O^+(1, d-1)$ and so are the coefficients c_b. Another invariant that plays an important role in the classification of conjugacy classes is the rank of F. Since this is always even, we denote it by

$$\text{rank } F = 2r,$$

and clearly $r \leq q$. From now we say **rank parameter** to refer to r. In the following proposition we show that this set of invariants actually identifies the canonical form.

Proposicin 4.21. *Let $F, \widetilde{F} \in \text{SkewEnd}(V)$, for V Lorentzian of dimension d. Then the invariants $\{c_b, r\}$ and $\{\widetilde{c}_b, \widetilde{r}\}$ of F and $\widetilde{F}$ respectively are equal if and only if their canonical forms given by Theorem 4.12 are the same.*

Proof. The "if" part ($\Leftarrow$) is trivial, because the invariants c_b, r are independent on the basis, so they can be calculated in a canonical basis. Hence, same canonical form implies same invariants. For the "only if" part ($\Rightarrow$), we notice that if the coefficients c_b and $\widetilde{c}_b$ of $\mathcal{P}_F$ and $\mathcal{P}_{\widetilde{F}}$ are equal, so are their characterisic polynomials, the multiplicities of their zero eigenvalue and the polynomials $\mathcal{Q}_{F^2}$ and $\mathcal{Q}_{\widetilde{F}^2}$ (equation (4.12)). Since rank $F = \text{rank}\widetilde{F}$, Corollary 4.15 implies that ker F and ker $\widetilde{F}$ must have the same causal character. The canonical form only depends on the roots $\mathcal{Q}_{F^2}$ and the causal character of ker F through Definition 4.10. Thus, F and $\widetilde{F}$ must have the same canonical form. □

We now characterize the classes $[F]_{O^+}$ in terms of the same invariants given in Proposition 4.21. As mentioned above, this result is known [100], but we give here an alternative and very simple proof based on our canonical form:

Teorema 4.22. *[100] Let $F, \widetilde{F} \in \text{SkewEnd}(V)$, for V Lorentzian of dimension d. Then their invariants $\{c_b, r\}$ and $\{\widetilde{c}_b, \widetilde{r}\}$ are the same if and only if F and $\widetilde{F}$ are $O^+(1, d-1)$-related.*

Proof. The if ($\Leftarrow$) part is immediate, since it is trivial from their definitions that the quantities $\{c_b, r\}$ are Lorentz invariant. To prove the "only if" ($\Rightarrow$), by Proposition 4.21, F and $\widetilde{F}$ have the same canonical form in canonical bases B and $\widetilde{B}$ respectively. By definition (cf. Theorem 4.12), these bases are unit, future oriented and orthonormal. Thus, the transformation taking B to $\widetilde{B}$ transforms F into $\widetilde{F}$ and both must be $O^+(1, d-1)$-related.

□

Theorem 4.22 establishes the necessary and sufficient conditions for two endomorphisms to be $O^+(1, d-1)$-related. Combining this result with Proposition 4.21, we find that the canonical form (hence the parameters σ, μ_i^2 or σ, τ, μ_i^2) totally define the equivalence class of skew-symmetric endomorphisms up to $O^+(1, d-1)$ transformations. Moreover, we emphasize that this form is the same for every equivalence class, unlike other canonical (or normal) forms based on the classification of $\mathrm{SkewEnd}(V)$, such as the one in [39], where they seek irreducibility of the blocks, so they must give two different forms to cover every case.

Next, we discuss some facts about the coefficients of the characteristic polynomial, also stated in [100], where the proof is only indicated, and which can now be easily proven using the canonical form.

Lemma 4.23. *Let $F \in \mathrm{SkewEnd}(V)$ be non-zero and let $2r = \mathrm{rank}\, F$. Then $c_r > 0$, $c_r = 0$, $c_r < 0$ if and only if $\ker F$ is timelike, null or spacelike (or zero) respectively. Moreover, if $r < q$, $c_q = c_{q-1} = \cdots = c_{r+1} = 0$.*

Proof. Taking into account that the parities of d and m_0 are equal (Lemma 4.9), $q - [\frac{m_0}{2}] = [\frac{d}{2}] - [\frac{m_0}{2}] = \frac{d-m_0}{2}$, so equation (4.18) can be rewritten

$$\mathcal{P}_F(x) = x^{m_0}\left(x^{d-m_0} + \sum_{b=1}^{q-[m_0/2]} c_b x^{d-m_0-2b}\right) = x^{m_0}\left(x^{d-m_0} + \sum_{b=1}^{\frac{d-m_0}{2}} c_b x^{d-m_0-2b}\right), \quad (4.19)$$

where we have explicitly substituted all zero coefficients by extracting the common factor x^{m_0}, thus the remaining coefficients $c_b \neq 0$ for $b = 1, \cdots, (d-m_0)/2$. By Corollary 4.15, $\ker F$ is degenerate if and only if $2r = d - m_0 + 2$ and $m_0 > 2$, so the sum in (4.19) runs up to $(d - m_0)/2 = r - 1$, which means $c_r = c_{r+1} = \cdots = c_q = 0$, as stated in the lemma. Also by Corollary 4.15, $\ker F$ non-degenerate if and only if $2r = d - m_0$. In this case, the sum in (4.19) runs up to $(d - m_0)/2 = r$, hence $c_r \neq 0$ and if $r < q$, the next coefficients vanish $c_{r+1} = c_{r+2} = \cdots = c_q = 0$. In addition c_r is the independent term in the polynomial in parentheses. Let $\mu_1, \cdots, \mu_\lambda$ be all the non-zero parameters among the $\{\mu_i\}$ of the canonical form of F given in (4.9). By equation (4.13), c_r can be written for d odd:

$$c_r = \sigma \mu_1^2 \cdots \mu_\lambda^2.$$

Then, the sign of σ determines the sign of c_r and, by Remark 3.12, also the causal character of $\ker F_\sigma$, hence, the causal character of $\ker F$ in accordance with the stament of the lemma. For d even, also from (4.13) we have

$$c_r = -\frac{\tau^2}{4}\mu_1^2 \cdots \mu_\lambda^2 < 0 \quad (\tau \neq 0), \qquad c_r = \sigma \mu_1^2 \cdots \mu_\lambda^2 \quad (\tau = 0),$$

where the expression for $\tau = 0$ follows because in this case either μ_t or μ_s (or both) vanish, hence either $c_r = \mu_s^2 \mu_1^2 \cdots \mu_\lambda^2$ or $c_r = -\mu_t^2 \mu_1^2 \cdots \mu_\lambda^2$ and σ equals μ_s^2 in the first situation and $-\mu_t^2$ in the second. By Remark 3.11, when $\tau \neq 0$ we have $\ker F_{\sigma\tau} = \{0\}$ and hence ker F is always spacelike or zero and when $\tau = 0$, the causal character of $\ker F_{\sigma\tau}$ (and that of $\ker F$) is determined by the sign of σ in accordance with the statement of the lemma.

□

Observacin 4.24. *A converse version of Lemma 4.23 also holds, in the sense that the number ν of last vanishing coefficients restricts the allowed rank parameters r. Let ν be defined by $\nu = 0$ if $c_q \neq 0$ and, otherwise, by the largest natural number satisfying $c_q = c_{q-1} = \cdots c_{q-\nu+1} = 0$. By equation (4.19) it follows $\nu = [m_0/2]$, and since the dimension d and m_0 have the same parity (cf. Lemma 4.9), $d - m_0 = 2[d/2] - 2[m_0/2] = 2(q-\nu)$ which in particular shows that ν determines m_0 uniquely. If $m_0 > 2$, by Corollary 4.15 the rank parameter admits two possibilities $r = \{q-\nu, q-\nu+1\}$, each of which determined by the causal character of $\ker F$. If $m_0 \leq 2$, also by Corollary 4.15 the $\ker F$ degenerate case cannot occur and $r = (q-\nu)$ is uniquely determined. In particular, if $d = 4$, r is always determined by c_1, c_2, because $r = 2$ happens if and only if $\nu = 0$ and otherwise $r = 1$ (unless F is identically zero, in which case $r = 0$).*

4.4.1 Structure of $\mathrm{SkewEnd}(V)/O^+(1, d-1)$

By Theorem 4.22, the q-tuple $(c_1, \cdots, c_q)$ corresponding to the coefficients of the characteristic polynomial of a skew-symmetric endomorphism, does not suffice to determine a point in the quotient space $\mathrm{SkewEnd}(V)/O^+(1, d-1)$, since generically two ranks are possible (dimensions three and four are an exception). As dicussed in Remark 4.24, for a number ν of last vanishing coefficients c_b, the allowed rank parameters are $r \in \{q-\nu, q-\nu+1\}$, and $r = q-\nu+1$ is only possible provided $m_0 > 2$ (in particular, when $c_q \neq 0$ then necessarily $r = q$). One says that there is a *degeneracy* for the value of the rank at certain points in the space of coefficients c_b. In the submanifold $\{c_q = \cdots = c_{q-\nu+1} = 0, c_{q-\nu} \neq 0\}$, the possible rank parameters are $r \in \{q-\nu, q-\nu+1\}$. When a boundary point where the number of last vanishing coefficients increases by exactly one is approached, the rank parameter may remain equal to $q-\nu$ or jump to $q-\nu-1$ (note that while the coefficients c_i are continuous functions of F, the rank is only lower semicontinuous, e.g. [91]). As we shall see in this section, this behaviour gives rise to special limit points in the space of parameters defining the canonical form (i.e. the space of conjugacy classes).

Recall that the space of skew-symmetric endomorphism $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})$ (being a finite dimensional vector space) carries a canonical topology (see e.g. [35]). The quotient space inherits a natural topology, called "quotient topology" which is the finest one that makes the projection a continuous map. In this topology it is sufficient for a sequence of points s_i to have a limit s that there is a sequence of endomorphisms F_i converging to F with F_i belonging to the class s_i and F belonging to the class s. Therefore, the limits below constructed with explicit endomorphisms $F \in \mathrm{SkewEnd}(V)$, also provide limits of $\mathrm{SkewEnd}(V)/O^+(1,d-1)$ in the quotient topology. This allows to single out some special limits (cf. Remark 4.25 below) which will be useful for the analysis of initial data in Chapter 6.

Let us start by locating these special limit points using the canonical form. Degeneracies can only occur in dimensions $d = 5$ or larger because in dimension three the rank is two for any non-trivial F and in dimension four the rank is uniquely determined by the invariants (cf. Remark 4.24). We thus consider first the case $d = 5$ and then extend to all values $d \geq 5$. In $d = 5$ the space of parameters $\mathcal{A}$ defining the $[F]_{O^+}$ classes is (Figure 4.1)

$$\mathcal{A} := \left\{(\sigma,\mu^2) \in \mathbb{R} \times \mathbb{R}^+ \mid \sigma \geq \mu^2 \text{ if } \sigma > 0\right\}.$$

Consider a $[F]_{O^+}$ in the region

$$\mathcal{R}_+ := \left\{\sigma \geq \mu^2 > 0\right\}$$

and let F be a representative of $[F]_{O^+}$ in a canonical basis $B = \{e_I\}_{I=0,\cdots,4}$, that is

$$F = \begin{pmatrix} 0 & 0 & -1+\frac{\sigma}{4} \\ 0 & 0 & -1-\frac{\sigma}{4} \\ -1+\frac{\sigma}{4} & 1+\frac{\sigma}{4} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}. \tag{4.20}$$

Let us define the functions $C_{\pm}(x) := \frac{1}{x} \pm \frac{x}{4}$. Then, the following change of basis to $B' = \{e'_I\}$ is well defined in $\mathcal{R}_+$:

$$\begin{aligned} e'_0 &= C_+(\mu)\left(C_+(\sqrt{\sigma})e_0 + C_-(\sqrt{\sigma})e_1\right) - C_-(\mu)e_4, \qquad & e'_2 &= -e_3, \\ e'_1 &= -C_-(\mu)\left(C_+(\sqrt{\sigma})e_0 + C_-(\sqrt{\sigma})e_1\right) + C_+(\mu)e_4, \qquad & e'_3 &= -e_2. \\ e'_4 &= C_-(\sqrt{\sigma})e_0 + C_+(\sqrt{\sigma})e_1 \end{aligned} \tag{4.21}$$

By direct calculation, F is written in basis B' as

$$F = \begin{pmatrix} 0 & 0 & -1+\frac{\mu^2}{4} \\ 0 & 0 & -1-\frac{\mu^2}{4} \\ -1+\frac{\mu^2}{4} & 1+\frac{\mu^2}{4} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\sqrt{\sigma} \\ \sqrt{\sigma} & 0 \end{pmatrix}. \tag{4.22}$$

The basis B' is non-canonical because $\mu^2 < \sigma$. However, if we vary the parameters so that $\mu \to 0$ (keeping σ unchanged), the matrix (4.22) becomes canonical (i.e. of the form (4.9)) in the limit and the class $[\lim_{\mu\to 0} F]_{O^+}$ is given by $l_1 = (0,\sigma)$. On the other hand, F in canonical form (4.20) also admits a limit $\mu \to 0$, which is also canonical and whose representative $[\lim_{\mu\to 0} F]_{O^+}$ is given by $l_2 = (\sigma, 0)$. Both limits are defined by the same sequence of points, because the transformation (4.21) is invertible in $\mathcal{R}_+$. However this sequence has two different limit points. As a consequence, the space of canonical matrices, and therefore the quotient space $\mathrm{SkewEnd}(V)/O^+(1,d-1)$, inherits a non-Hausdorff topology.

Something similar happens in the region

$$\mathcal{R}_- := \{\sigma < 0, \mu > 0\}\,.$$

Let F be a representative in canonical form of a point $[F]_{O^+}$ in this region. Then, F has a timelike eigenplane Π_t with eigenvalue $\sqrt{|\sigma|}$ (cf. Remark 3.12), a spacelike eigenvector e as well as a spacelike eigenplane Π_s with eigenvalue μ. Thus $V = \Pi_t \oplus \mathrm{span}\{e\} \oplus \Pi_s$ and there exist a (non-canonical) basis B' adapted to this decomposition, into which F takes the form

$$F = \begin{pmatrix} 0 & \sqrt{|\sigma|} & 0 \\ \sqrt{|\sigma|} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}. \tag{4.23}$$

Keeping μ unchanged, expression (4.23) has a limit $\sigma \to 0$, which has a spacelike eigenplane Π_s of eigenvalue μ and it is identically zero on $\Pi^\perp$. Hence, $\ker F$ is timelike and using Definition 4.10, the canonical form of this limit $\lim_{\sigma\to 0} F$ is given by $\sigma' = \mu^2$ and $\mu' = 0$. Thus $[\lim_{\sigma\to 0} F]_{O^+}$ is represented by the point $l_2 = (\mu^2, 0)$. On the other hand, in a canonical basis (4.20), F also admits a limit $\sigma \to 0$, whose class $[\lim_{\sigma\to 0} F]_{O^+}$ is obviously represented by the point $l_1 = (0, \mu^2)$.

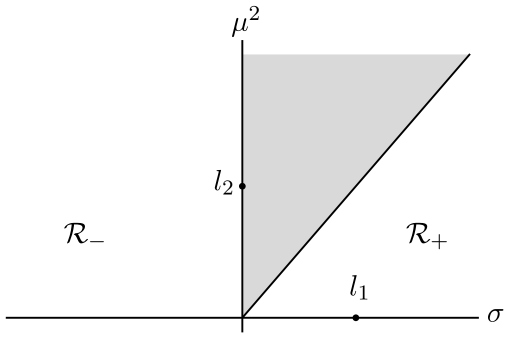

FIGURE 4.1: *Representation of* $\mathrm{SkewEnd}(V)/O^+(1,4)$ *in the subspace* $\mathcal{A} \subset \mathbb{R}^2$*. The shadowed region is not included.*

The same reasoning can be carried out to arbitrary odd dimension. First, define the regions

$$\mathcal{R}_+^{(d,0)} := \left\{\sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\right\} \quad \text{and} \quad \mathcal{R}_-^{(d,0)} := \left\{\sigma < 0, \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\right\}$$

and also the limit regions

$$\mathcal{R}_0^{(d,0)} := \left\{\sigma = 0, \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\right\} \quad \text{and} \quad \mathcal{R}_+^{(d,1)} := \left\{\sigma \geq \mu_1^2 \geq \cdots \geq \mu_{p-1}^2 > \mu_p^2 = 0\right\}.$$

Consider representatives F_+ and F_- (in canonical form) of points $(\sigma^+, (\mu_1^+)^2, \cdots, (\mu_p^+)^2)$ and $(\sigma^-, (\mu_1^-)^2, \cdots, (\mu_p^-)^2)$ in the regions $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_-^{(d,0)}$ respectively. Then F_+ has a spacelike eigenplane Π_s^+ with eigenvalue μ_p^+ as well as a timelike eigenvector e^+ and spacelike eigenplane Π_t^+ with eigenvalue $\sqrt{\sigma^+}$. Restricting to the subspace

$W^+ = \operatorname{span}\{e^+\} \oplus \Pi_t^+ \oplus \Pi_s^+$ we can repeat the procedure followed for the five dimensional case and conclude that $[\lim_{\mu_p^+\to 0} F_+]$ has simultaneously limits at the points $(\sigma^+, (\mu_1^+)^2, \cdots, (\mu_{p-1}^+)^2, 0) \in \mathcal{R}_+^{(d,1)}$ and $(0, (\mu_1^+)^2, \cdots, (\mu_p^+)^2) \in \mathcal{R}_0^{(d,0)}$. Analogously F_- has a spacelike eigenplane Π_s^- with eigenvalue μ_p^- as well as spacelike eigenvector e^- and timelike eigenplane Π'^-_s with eigenvalue $\sqrt{|\sigma^-|}$. Restricting to the subspace $W^- = \Pi_s^- \oplus \operatorname{span}\{e^-\} \oplus \Pi'^-_s$, the above arguments for the five dimensional case show that $[\lim_{\sigma^-\to 0} F_-]$ has simultaneous limits on the points $((\mu_p^-)^2, (\mu_1^-)^2, \cdots, (\mu_{p-1}^-)^2, 0) \in \mathcal{R}_+^{(d,1)}$ and $(0, (\mu_1^-)^2, \cdots, (\mu_p^-)^2) \in \mathcal{R}_0^{(d,0)}$. Thus the regions $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_-^{(d,0)}$ limit simultaneously with $\mathcal{R}_+^{(d,1)}$ and $\mathcal{R}_0^{(d,0)}$ as μ_p and σ tend to zero respectively. Indeed, the same ideas can be applied again to $\mathcal{R}_+^{(d,1)}$ and

$$\mathcal{R}_-^{(d,1)} := \left\{\sigma < 0, \mu_1^2 \geq \cdots \geq \mu_{p-1}^2 > \mu_p^2 = 0\right\},$$

so that they also limit simultaneously, as μ_{p-1} and σ go to zero respectively, with

$$\mathcal{R}_0^{(d,1)} := \left\{\sigma = 0, \mu_1^2 \geq \cdots \geq \mu_{p-1}^2 > \mu_p^2 = 0\right\}$$

and

$$\mathcal{R}_+^{(d,2)} := \left\{\sigma > 0, \mu_1^2 \geq \cdots \geq \mu_{p-1}^2 > \mu_{p-1}^2 = \mu_p^2 = 0\right\}.$$

This same stucture generalizes to any number of last-vanishing μ_i^2 parameters. Namely, the regions with m last-vanishing parameters $\mu_p = \cdots = \mu_{p-m+1} = 0$ and non-zero σ, limit simultaneous the region with m last-vanishing parameters and $\sigma = 0$ and the region m last-vanishing parameters and $\sigma > 0$ (cf. Remark 4.25 below).

For the even dimensional case (with $d \geq 6$), notice that as long as $\dim\ker F \geq 2$, which happens if $\tau = 0$ or $\mu_p^2 = 0$, the restriction $F|_{e^\perp}$, where e is any spacelike vector $e \in \ker\ F$, is equivalent to the odd dimensional case. Hence, the previous reasoning for odd dimensions also applies for even dimensions if $\tau = 0$ or $\mu_p^2 = 0$. For later use, it is convenient to discuss the d even case using parameters $\{-\mu_t^2, \mu_s^2; \mu_1^2, \cdots, \mu_p^2\}$. To start with, assume $d = 6$, where there are only three parameters $\{-\mu_t^2, \mu_s^2, \mu^2\}$. The region

$$\mathcal{R}_+^{(d,0)} := \left\{-\mu_t^2 = 0, \mu_s^2 \geq \mu^2 > 0\right\}$$

contains sequences assuming simultaneously limits in

$$\mathcal{R}_0^{(d,0)} := \left\{-\mu_t^2 = \mu_s^2 = 0, \mu^2 > 0\right\} \quad \text{and} \quad \mathcal{R}_+^{(d,1)} := \left\{-\mu_t^2 = 0, \mu_s^2 > \mu^2 = 0\right\}.$$

These sequences can be constructed as limits $\mu^2 \to 0$, analogous to the n odd case above. In a similar way

$$\mathcal{R}_-^{(d,1)} := \left\{-\mu_t^2 < 0, \mu_s^2 > \mu^2 = 0\right\}$$

contains sequences with limits in $\mathcal{R}_0^{(d,0)}$ and $\mathcal{R}_+^{(d,1)}$ simultaneously. These sequences can be constructed as limits $-\mu_t^2 \to 0$, analogous to the limits $\sigma \to 0$ for the n odd case above.

On the other hand in the region

$$\mathcal{R}_{-}^{(d,0)} := \left\{-\mu_t^2 < 0, \mu_s^2 \geq \mu^2 > 0\right\},$$

one can trivially construct a sequence limiting $\mathcal{R}_{+}^{(d,0)}$ (as $-\mu_t^2 \to 0$) as well as a sequence limiting $\mathcal{R}_{-}^{(d,1)}$ (as $\mu^2 \to 0$). Therefore all the above are also limits[3] of $\mathcal{R}_{-}^{(d,0)}$. Moreover, $\mathcal{R}_{-}^{(d,0)}$ does not assume degenerate limits beyond those above described. This is because it is fact that $\det F = -\mu_t^2\mu_s^2\mu^2$, so if $-\mu_t^2, \mu_s^2, \mu \neq 0$, then $\operatorname{rank} F = 6$. Thus, taking only one of these parameters to zero must lead necessarily to a region in which $\operatorname{rank} F = 4$, which can only be either $\mathcal{R}_{+}^{(d,1)}$ or $\mathcal{R}_{-}^{(d,1)}$, whose limits have already been discussed. Also observe that there cannot be a degenerate limit within two regions with same rank, as the coefficients and the rank determine uniquely the equivalence class.

The generalization to higher even dimensions is straightforward from the $d = 6$ case by an argument similar to the d odd case. Let $d > 6$ be even and define

$$\mathcal{R}_{-}^{(d,0)} := \{-\mu_t^2 < 0, \mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\}.$$

An endomorphism $F \in \operatorname{SkewEnd}(V)$ such that $[F]_{O^+} \in \mathcal{R}_{-}^{(d,0)}$ admits one timelike eigenplane Π_t and two spacelike eigenplanes Π_s, Π_p of eigenvalues μ_t and μ_s and μ_p respectively. Then, $W = \Pi_t \oplus \Pi_s \oplus \Pi_p$ is a Lorentzian vector space of dimension 6, so the restriction $F|_W$ admits the same structure of limits than in the $d = 6$ case above. Namely, in the total space V, the region $\mathcal{R}_{-}^{(d,0)}$ limits trivially with

$$\mathcal{R}_{+}^{(d,0)} := \{-\mu_t^2 = 0, \mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\}$$

and this assumes simultaneous limits at

$$\begin{aligned}
\mathcal{R}_{+}^{(d,1)} &:= \{-\mu_t^2 = 0, \mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_{p-1} > \mu_p^2 = 0\}, \\
\mathcal{R}_{0}^{(d,0)} &:= \{-\mu_t^2 = \mu_s^2 = 0, \mu_1^2 \geq \cdots \geq \mu_p^2 > 0\}.
\end{aligned}$$

Combining all the above arguments, a similar structure of limits extends to the regions with any number m of last-vanishing parameters $\mu_p^2 = \cdots = \mu_{p-m+1}^2 = 0$ and $\{-\mu_t^2, \mu_s^2\}$ (cf. Remark 4.25 below).

The following remark summarizes the above discussion.

Observacin 4.25. *For d odd, consider the space*

$$\begin{aligned}
\mathcal{A}^{(odd)} :=& \{\left(\sigma, \mu_1^2, \cdots, \mu_p^2\right) \in \mathbb{R}^{p+1} \mid \sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2, \quad \textit{with } \sigma > 0\} \\
&\bigcup \{\left(\sigma, \mu_1^2, \cdots, \mu_p^2\right) \in \mathbb{R}^{p+1} \mid \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq \sigma\}
\end{aligned}$$

[3]The regions $\mathcal{R}_{+}^{(d,0)}$ and $\mathcal{R}_{-}^{(d,1)}$ are clearly in the closure of $\mathcal{R}_{-}^{(d,0)}$, as both can be attained from sequences in $\mathcal{R}_{-}^{(d,0)}$. Thus the sequences in $\mathcal{R}_{+}^{(d,0)}$ and $\mathcal{R}_{-}^{(d,1)}$ have limits in the closure of $\mathcal{R}_{-}^{(d,0)}$.

and for d even

$$\begin{aligned}\mathcal{A}^{(even)} := \{ & \big(-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2\big) \in \mathbb{R}^{p+2} \mid \\ & \mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq -\mu_t^2, \quad \textit{with } \mu_s^2 \textit{ or } \mu_t^2 \neq 0\} \\ \bigcup & \{\big(-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2\big) \in \mathbb{R}^{p+2} \mid \ \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 = \mu_s^2 = -\mu_t^2\},\end{aligned}$$

where $\{\sigma, \mu_1^2, \cdots, \mu_p^2\}$ *and* $\{-\mu_s^2, \mu_s^2, \mu_1^2, \cdots \mu_p^2\}$ *are the parameters in Definition 4.10. As a consequence of Proposition 4.21, these parameters are unique for every orbit* $[F]_{O^+} \in \mathrm{SkewEnd}(V)/O^+(1, d-1)$. *Thus* $\mathcal{A}^{(odd)}$ *and* $\mathcal{A}^{(even)}$ *give a good parametrization of* $\mathrm{SkewEnd}(V)/O^+(1, d-1)$.

Define the subsets of $\mathcal{A}^{(odd)}$

$$\begin{aligned}\mathcal{R}_+^{(d,m)} &:= \left\{(\sigma, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(odd)} \mid \sigma \geq \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\right\}, \\ \mathcal{R}_-^{(d,m)} &:= \left\{(\sigma, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(odd)} \mid \sigma < 0, \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\right\}, \\ \mathcal{R}_0^{(d,m)} &:= \left\{(\sigma, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(odd)} \mid \sigma = 0, \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\right\},\end{aligned}$$

and of $\mathcal{A}^{(even)}$

$$\begin{aligned}\mathcal{R}_+^{(d,m)} := \Big\{ & -\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(even)} \mid \\ & -\mu_t^2 = 0, \mu_s^2 \geq \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\}, \\ \mathcal{R}_-^{(d,m)} := \Big\{ & (-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(even)} \mid \\ & -\mu_t^2 < 0, \mu_s^2 \geq \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\}, \\ \mathcal{R}_0^{(d,m)} := \Big\{ & (-\mu_t^2, \mu_s^2, \mu_1^2, \cdots, \mu_p^2) \in \mathcal{A}^{(even)} \mid \\ & -\mu_t^2 = \mu_s^2 = 0, \mu_1^2 \geq \cdots > \mu_{p-m+1}^2 = \cdots = \mu_p^2 = 0\}.\end{aligned}$$

The notation $\mathcal{R}_\epsilon^{(d,m)}$ *generalizes to any dimension as follows: d is the dimension of* V, *m is the number of last-vanishing parameters* $\{\mu_i^2\}$ *and* $\epsilon \in \{0, \pm\}$ *gives the causal character of* $\ker F$*: 0 if degenerate, + if timelike and − if spacelike or zero. We note that* ϵ *is also given by the sign of* σ *in the odd case and closely related to the sign structure of the first two entries* $\{-\mu_t^2, \mu_s^2\}$ *of the point* $s \in \mathcal{A}^{(even)}$ *when d is even.*

In $\mathcal{A}^{(odd)}$, *every sequence in* $\mathcal{R}_+^{(d,m)}$ *and every sequence in* $\mathcal{R}_-^{(d,m)}$ *which has limit at* $\mathcal{R}_+^{(d,m+1)}$ *it also has a limit at* $\mathcal{R}_0^{(d,m)}$ *and viceversa. Similarly, in* $\mathcal{A}^{(even)}$, *every sequence in* $\mathcal{R}_+^{(d,m)}$ *and every sequence in* $\mathcal{R}_-^{(d,m)}$ *which has limit at* $\mathcal{R}_+^{(d,m+1)}$ *it also has a limit at* $\mathcal{R}_0^{(d,m)}$ *and viceversa.*

We conclude this subsection with the following result, stated it in a separate proposition because it will be explicitly required for the analysis in Chapter 6.

Proposicin 4.26. *For d odd,* $\mathcal{R}_+^{(d,0)}$ *and* $\mathcal{R}_-^{(d,0)}$ *are open in the quotient topology. Moreover there exists sequences in* $\mathcal{R}_-^{(d,0)}$ *taking limit at every point* $\mathcal{A}^{(odd)} \backslash \mathcal{R}_+^{(d,0)}$.

For d even, $\mathcal{R}_-^{(d,0)}$ *is open in the quotient topology. Moreover there exists sequences in* $\mathcal{R}_-^{(d,0)}$ *taking limit at every point* $\mathcal{A}^{(even)}$ *(i.e.* $\mathcal{R}_-^{(d,0)}$ *is dense in the quotient topology).*

Proof. We first prove openness of $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_-^{(d,0)}$ for d odd and $\mathcal{R}_-^{(d,0)}$ for d even. Let $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,d-1})$, $[F]_{O^+} \in \mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$ its class in the quotient and π the canonical projection map $\pi : F \mapsto [F]_{O^+}$. The independent term of the characteristic polynomial is an invariant of the class $[F]_{O^+}$. Let $c_{q-\nu}$ be the function that maps F into the independent term of its characteristic polynomial $c_{q-\nu}(F)$. This map is clearly continuous. Let also $[c_{q-\nu}]$ be the induced map in the quotient, i.e. the map satisfying $c_{q-\nu} = [c_{q-\nu}] \circ \pi$. Then $[c_{q-\nu}]$ is also continuous (e.g. [147]). Moreover from (4.13), if d even, $c_{q-\nu}(F) = -\mu_t^2\mu_s^2\mu_1^2 \cdots \mu_p^2$ and, if d odd, $c_{q-\nu}(F) = \sigma\mu_1^2 \cdots \mu_p^2$. Thus when n is odd $\mathcal{R}_+^{(n,0)}$ and $\mathcal{R}_-^{(n,0)}$ are open in $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$ as they are the preimage by $[c_{q-\nu}]$ of the open intervals $(0,\infty)$ and $(-\infty,0)$ respectively. When n is even $\mathcal{R}_+^{(n,0)}$ is also open because it is the preimage of the open interval $(0,\infty)$.

On the other hand, for d odd, by the discussion above, one can construct sequences in $\mathcal{R}_-^{(d,0)}$ assuming limit any point in $\mathcal{R}_+^{(d,1)}$ and $\mathcal{R}_0^{(d,0)}$. Moreover, it is immediate to construct sequences in $\mathcal{R}_+^{(d,1)}$ and $\mathcal{R}_0^{(d,0)}$ with limits into any point in any of the regions $\mathcal{R}_+^{(d,m+1)}$ and $\mathcal{R}_0^{(d,m)}$ respectively. Similarly, there is a trivial sequence in $\mathcal{R}_-^{(d,0)}$ with limit into any point in any of the regions $\mathcal{R}_-^{(d,m)}$. In addition, $\mathcal{R}_+^{(d,0)}$ is open and has empty intersection with $\mathcal{R}_-^{(d,0)}$. Thus, all regions except $\mathcal{R}_+^{(d,0)}$ are accessible as limits of $\mathcal{R}_-^{(d,0)}$.

Similarly, for d even, by the discussion above, one can construct sequences in $\mathcal{R}_-^{(d,0)}$ with limit at $\mathcal{R}_+^{(d,0)}$ and $\mathcal{R}_0^{(d,0)}$. The rest of the argument is analogous to the d odd case. □

4.5 Conformal Killing vector fields

One interesting application of our previous results is based on the relation between skew-symmetric endomorphisms and the set of conformal CKVFs of the n-sphere, $\mathrm{CKill}(\mathbb{S}^n)$, and its local representation in $\mathbb{E}^n$, $\mathrm{CKill}(\mathbb{E}^n)$, discussed in subsection 2.2.1. Our aim in this section is to provide a canonical form for all elements in $\mathrm{CKill}(\mathbb{E}^n)$. Therefore, all the previous results will be applied for dimension $d = n + 2$ with $n > 2$. Restricting to the set of global CKVFs, some of the following results also apply for $n = 2$ (cf. Remarks 2.16 and 3.14). However, this case has been already addressed in detail in Chapter 3, so we shall restrict here to $n > 2$, and only make some remarks on the $n = 2$ case.

We start by making an observation on the construction in subsection 2.2.1, which will allow us to choose suitable Minkowskian coordinates in $\mathbb{M}^{1,n+1}$ in exchange of keeping conformal freedom in the metric of $\mathbb{E}^n$.

Observacin 4.27. *The freedom of choosing a representative for* $\mathbb{S}^n$ *as well as the point* N *and the projection stereographic plane (discussed in subsection 2.2.1), can be also seen in* *a more "passive" picture. Consider two different sets of Minkowskian coordinates* $\{x^I\}$ *and* $\{x'^I\}$ *related by a* $O^+(1,n+1)$ *transformation* Λ, $x'^I = \Lambda^I{}_J x^J$. *Using Theorem 2.11, we obtain two different embeddings* $i, i' : \mathbb{E}^n \hookrightarrow \mathbb{M}^{1,n+1}$ *associated to* $\{x^I\}$ *and* $\{x'^I\}$ *respectively, for which* $i(\mathbb{E}^n) = \{x^0 = x^1 = 1, x^{A+1} =: y^A\}$ *and* $i'(\mathbb{E}^n) = \{x'^0 = x'^1 = 1, x'^{A+1} =: y'^A\}$, *as well as two associated maps* ξ, ξ'. *Let* $F \in \mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$, *defined by (2.26) with parameters* $\{\nu, \mathrm{a}^A, \mathrm{b}^A, \omega^A{}_B\}$ *and* $\{\nu', \mathrm{a}'^A, \mathrm{b}'^A, \omega'^A{}_B\}$ *in the bases* $\{\partial_{x^I}\}$ *and* $\{\partial_{x'^I}\}$ *respectively. Then,* F *can be associated to two vector fields*

$$\xi_F = \left(\mathrm{b}^A + \nu y^A + (\mathrm{a}_B y^B) y^A - \frac{1}{2}(y_B y^B)\mathrm{a}^A - \omega^A{}_B y^B\right)\partial_{y^A},$$
$$\xi'_F = \left(\mathrm{b}'^A + \nu' y'^A + (\mathrm{a}'_B y'^B) y'^A - \frac{1}{2}(y'_B y'^B)\mathrm{a}'^A - \omega'^A{}_B y'^B\right)\partial_{y'^A},$$

which are equal in the following sense. If we transform the representative $\mathbb{S}'^n = \{x'^0 = 1\} \cap \{x'_I x'^I = 0\}$ *with* Λ, *we obtain a new representative of the projective cone which in coordinates* x^I *is precisely* $\mathbb{S}^n = \{x^0 = 1\} \cap \{x_I x^I = 0\}$. *Abusing the notation, the map* $\phi_\Lambda := St_N \circ \Lambda \circ St_{N'}^{-1}$ *is such that* $\phi_{\Lambda\star}(\xi'_F) = \xi_F$. *Then, considering* $i(\mathbb{E}^n)$ *and* $i'(\mathbb{E}^n)$ *as respresentations of the same space in two different global charts* $(y^A, \mathbb{R}^n)$ *and* $(y'^A, \mathbb{R}^n)$, ϕ_Λ *can be seen as a change of coordinates* $y^A = (\phi_\Lambda(y'))^A$, *with the property that the Euclidean metric in coordinates* $\{y'^A\}$ *transforms as*

$$g_E = \delta_{AB} \mathrm{d}y'^A \mathrm{d}y'^B = \Omega^2(y)\delta_{AB}\mathrm{d}y^A\mathrm{d}y^B$$

for a locally smooth (recall that the conformal transformations have generalically two singularities, cf. subsection 2.2.1) positive function Ω. *In other words, changing to different Minkowskian coordinates in* $\mathbb{M}^{1,n+1}$ *induces a change of coordinates in* $\mathbb{E}^n$ *in such a way that the form (2.27) of the map* ξ *is preserved. Notice that a similar result holds if we change the point w.r.t. which we take the stereographic projection, because any two* $N, N' \in \mathbb{S}^n$ *must be related by a* $SO(n) \subset O^+(1,n+1)$ *transformation.*

Therefore, for the rest of this section, we will often adapt our choice of Minkowskian coordinates $\{x^I\}$ of $\mathbb{M}^{1,n+1}$ to simplify the problem at hand. With this choice, it comes a corresponding set of Cartesian coordinates $\{y^A\}$ of $\mathbb{E}^n$ such that ξ_F is given by equation (2.27) and the Euclidean metric is $g_E = \Omega(y)^2\delta_{AB}\mathrm{d}y^A\mathrm{d}y^B$. Which coordinates are adequate obviously depends on the problem. For example, from the block form (4.5) and (4.6) of skew-symmetric endomorphisms, consider each of the blocks $F|_{\mathbb{M}^{1,3}}$ $F|_{\mathbb{M}^{1,2}}$ as endomorphisms of $\mathbb{M}^{1,n+1}$, extended as the zero map in $(\mathbb{M}^{1,3})^\perp$ and $(\mathbb{M}^{1,2})^\perp$ respectively, and similarly for each $F|_{\Pi_i}$. If we denote by $\xi_{F|_{\mathbb{M}^{1,3}}}$, $\xi_{F|_{\mathbb{M}^{1,2}}}$ and $\xi_{F|_{\Pi_i}}$ the corresponding images by ξ, one readily gets following decomposition:

$$\xi_F = \xi_{F|_{\mathbb{M}^{1,3}}} + \sum_{i=1}^{p} \xi_{F|_{\Pi_i}} \quad (n \text{ even}), \qquad \xi_F = \xi_{F|_{\mathbb{M}^{1,2}}} + \sum_{i=1}^{p} \xi_{F|_{\Pi_i}} \quad (n \text{ odd}), \qquad (4.24)$$

where in terms of n, p is given by

$$p = \left[\frac{n+1}{2}\right] - 1 \tag{4.25}$$

(recall that the dimension of the Minkowski space where F is defined is $d = n+2$, cf. Theorem 2.11). The explicit form of each of the terms in (4.24) is direct from (2.28). Namely, the terms $\xi_{F|_{\mathbb{M}^{1,3}}}$ and $\xi_{F|_{\mathbb{M}^{1,2}}}$ are given by (2.27) with vanishing parameters $\mathrm{a}^A, \mathrm{b}^A, \omega^A{}_B$ for $A, B \geq 3$ and $A, B \geq 2$ respectively, and each $\xi_{F|_{\Pi_i}}$ is proportional to a vector field of the form

$$\eta := y^{A_0}\partial_{y^{B_0}} - y^{B_0}\partial_{y^{A_0}} \tag{4.26}$$

with $A_0, B_0 \in \{1, \cdots, n\}$ such that $A_0 \neq B_0$. More specifically, $\xi_{F|_{\Pi_i}} = \mu_i\eta_i$, where η_i is given by equation (4.26) with $B_0 = A_0 + 1$ and $A_0 = 2i$ if n even while $A_0 = 2i + 1$ if n odd. Vector fields of the form (4.26) will play an important role in the following analysis. They have the form of axial Killing vector fields, although in general they are CKVFs because of the conformal factor in $g_E = \Omega(y)^2\delta_{AB}\mathrm{d}y^A\mathrm{d}y^B$. From the discussion in Remark 4.27, it follows that there exists a conformal transformation $\phi_\Lambda \in \mathrm{ConfLoc}(\mathbb{E}^n)$ such that $g'_E := \phi^\star_\Lambda(g_E) = \delta_{AB}\mathrm{d}y^A\mathrm{d}y^B$. Then by the properties of the Lie derivative it is immediate

$$0 = \mathcal{L}_\eta\phi^\star_\Lambda(g_E) = \mathcal{L}_{\phi_{\Lambda\star}(\eta)}g_E.$$

In other words, η is an axial Killing vector of g'_E and $\phi_{\Lambda\star}(\eta)$ is an axial Killing vector of g_E. Thus, we define:

Definicin 4.28. A CKVF of an Euclidean metric g_E, η, is said to be a **conformally axial** Killing vector field (CAKVF) if and only if the exist a $\phi_\Lambda \in \mathrm{ConfLoc}(\mathbb{E}^n)$ such that $\phi_{\Lambda\star}(\eta)$ is an axial Killing vector field of g_E. Equivalently, η is a CAKVF if and only if it is an axial Killing vector field of $\phi^\star_\Lambda(g_E)$.

Observacin 4.29. *Using Theorem 2.11, it is immediate to verify that a CKVF is a CAKVF if and only if it is the image under ξ of a simple unit spacelike endomorphism G.*

Notice that the terms in (4.24) form a commutative subset of $\mathrm{CKill}(\mathbb{E}^n)$. This is an immediate consequence of the fact that ξ is a Lie algebra antihomomorphism (cf. Theorem 2.11) and the blocks $F|_{\mathbb{M}^{1,2}}$ (resp. $F|_{\mathbb{M}^{1,3}}$) and $F|_{\Pi_i}$ are pairwise commuting. In addition, a straightforward calculation shows that they form an orthogonal set

$$g_E(\widetilde{\xi}, \eta_i) = 0, \qquad g_E(\eta_i, \eta_j) = 0 \qquad (i \neq j)$$

where $\widetilde{\xi} := \xi_{F|_{\mathbb{M}^{1,3}}}$ for n even and $\widetilde{\xi} := \xi_{F|_{\mathbb{M}^{1,2}}}$ for n odd. In fact, as we show next, orthogonality of two CKVFs implies commutativity provided one of them is a CAKVF. If both are CAKVF, then orthogonality turns out to be equivalent to commutativity.

Lemma 4.30. *Let η, η' be non-proportional CAKVFs and ξ_F a CKVF. Then $[\eta, \eta'] = 0$ if and only if there exist Cartesian coordinates such that $\eta = y^{n-2}\partial_{y^{n-3}} - y^{n-3}\partial_{y^{n-2}}$ and*

$\eta' = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. *Equivalently* $[\eta,\eta'] = 0$ *if and only if* $g_E(\eta,\eta') = 0$. *In addition,* $[\xi_F,\eta] = 0$ *if* $g_E(\xi_F,\eta) = 0$.

Proof. Let $G, G' \in \mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$ be such that $\xi(G) = \eta$, $\xi(G') = \eta'$. Since G and G' are simple, spacelike and unit (cf. Remark 4.29), we can write $G = e\otimes \boldsymbol{v} - v\otimes \boldsymbol{e}$ and $G' = e'\otimes \boldsymbol{v}' - v'\otimes \boldsymbol{e}'$ for spacelike, unit vectors $\{e, e', v, v'\}$, such that $0 = \langle e, v\rangle = \langle e', v'\rangle$. By Corollary 4.19, it follows that $[G, G'] = 0$ if and only if $\{e, e', v, v'\}$ are mutually orthogonal. Let us take Cartesian coordinates of $\mathbb{M}^{1,n+1}$ such that $e = \partial_{x^{n-2}}, v = \partial_{x^{n-1}}, e' = \partial_{x^n}, v' = \partial_{x^{n+1}}$. Then, in the associated coordinates $\{y^A\}$ of $\mathbb{E}^n$ it follows $\eta = y^{n-2}\partial_{y^{n-3}} - y^{n-3}\partial_{y^{n-2}}$ and $\eta' = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. This proves the first part of the lemma. From this result, it is trivial that $[\eta,\eta'] = 0$ implies $g_E(\eta,\eta') = 0$.

To prove that $g_E(\eta,\xi_F) = 0$ implies $[\eta,\xi_F] = 0$ (which in particular establishes the converse $g_E(\eta,\eta') = 0 \Longrightarrow [\eta,\eta'] = 0$ for CAKVFs), let us take coordinates $\{y^A\}$ such that $\eta = y^{n-1}\partial_{y^n} - y^n\partial_{y^{n-1}}$. Then, writing ξ_F as a general CKVF (2.27), we obtain by direct calculation:

$$g_E(\eta,\xi_F) = \Omega^2\big(y^n\mathrm{b}^{n-1} - y^{n-1}\mathrm{b}^n - \frac{y_By^B}{2}(\mathrm{a}^n y^{n-1} - \mathrm{a}^{n-1}y^n) \\ + \omega^{n-1}{}_B y^B y^n - \omega^n{}_B y^B y^{n-1}\big) = 0.$$

Therefore $\mathrm{a}^n, \mathrm{a}^{n-1}, \mathrm{b}^n, \mathrm{b}^{n-1}, \omega^n{}_B, \omega^{n-1}{}_B$ must vanish. This implies that the associated endomorphisms G and F to η and ξ_F adopt a block structure from which it easily follows that $[G, F] = 0$ and hence $[\eta,\xi_F] = 0$. □

Definicin 4.31. Let $\xi_F \in \mathrm{CKill}(\mathbb{E}^n)$. Then a **decomposed form** of ξ_F is $\xi_F = \widetilde{\xi} + \sum_{i=1}^p \mu_i\eta_i$ for an orthogonal subset $\{\widetilde{\xi}, \eta_i\}$, where η_i are CAKVFs, $\mu_i \in \mathbb{R}$ for $i = 1, \cdots, p$. A set of Cartesian coordinates $\{y^A\}$ such that $\eta_i = y^{A_i}\partial_{y^{A_i+1}} - y^{A_i+1}\partial_{y^{A_i}}$, for $A_i = 2i$ for n odd and $A_i = 2i+1$ for n even, is called a set of **decomposed** coordinates.

Observacin 4.32. *Observe that the* $\widetilde{\xi}$ *is a CKVF. By Lemma 4.30 and its proof, the parameters* $\{\nu, \mathrm{a}, \mathrm{b}, \boldsymbol{\omega}\}$ *defining* $\widetilde{\xi}$ *in a set of decomposed coordinates must all vanish except possibly* $\{\nu, \mathrm{a}^1, \mathrm{a}^2, \mathrm{b}^1, \mathrm{b}^2, \omega^1{}_2 = -\omega^2_1\}$ *when* n *is even or* $\{\nu, \mathrm{a}^1, \mathrm{b}^1\}$ *when* n *is odd. This means that there is a skew-symmetric endomorphism* $\widetilde{F}$ *which restricts to* $\mathbb{M}^{1,3} \subset \mathbb{M}^{1,n}$ *(n even) or* $\mathbb{M}^{1,2} \subset \mathbb{M}^{1,n}$ *(n odd) and vanishes identically on their respective orthogonal complements such that* $\widetilde{\xi} = \xi_{\widetilde{F}}$. *We will exploit this fact in an essential way below.*

With the definition of decomposed form of CKVFs, we can reformulate Theorem 4.6 in terms of CKVFs.

Proposicin 4.33. *Let* $\xi_F \in \mathrm{CKill}(\mathbb{E}^n)$. *Then there exist an orthogonal set* $\{\eta_i\}_{i=1}^p$ *of CAKVFs such that* $[\xi_F, \eta_i] = 0$. *For every such a set* $\{\eta_j\}_{j=1}^p$ *and* $i \in \{1, \cdots, p\}$ *there exist* $\mu_i \in \mathbb{R}$ *such that* $g_E(\eta_i,\eta_i)\mu_i = g_E(\xi_F,\eta_i)$. *In addition, with the definition* $\widetilde{\xi} := \xi_F - \sum \mu_i\eta_i$ *the expression* $\xi_F = \widetilde{\xi} + \sum \mu_i\eta_i$ *provides a decomposed form of* ξ_F.

Proof. The existence of p commuting CAKVFs is a direct consequence of decompositions (4.5) and (4.6) of the associated skew-symmetric endomorphism F, for n even and odd respectively. Indeed, for each such decomposition of F, it follows a set of p CAKVFs commuting with ξ_F. Let us denote $\{\eta_i\}$ any such set. Each η_i is associated to a simple, spacelike unit endomorphism G_i that commutes with F. By Lemma 4.18, G_i defines a spacelike eigenplane Π_i of F. The orthogonality of any two such eigenplanes Π_i, Π_j, $i \neq j$ is a consequence of Corollary 4.19 because $[G_i, G_j] = 0$. In other words, given a set of p CAKVFs commuting with ξ_F, we have a block form of F, thus, defining $\widetilde{\xi} := \xi_F - \sum \mu_i \eta_i$, it is immediate that $\xi_F = \widetilde{\xi} + \sum \mu_i \eta_i$ is a decomposed form with $g_E(\eta_i, \eta_i)\mu_i = g_E(\xi_F, \eta_i)$. □

The next step now is to give a definition of canonical form for CKVFs, which we induce from the canonical form of the associated skew-symmetric endomorphism.

Definicin 4.34. A CKVF ξ_F is in **canonical form** if it is the image of a skew-symmetric endomorphism F in canonical form, i.e. $\xi_F = \widetilde{\xi} + \sum \mu_i \eta_i$ such that $\widetilde{\xi}$ is given, in a Cartesian set of coordinates $\{y^A\}$ denoted **canonical coordinates**, by the parameters $\mathrm{a}^1 = 1$, $\mathrm{b}^1 = \sigma/2$, $\mathrm{a}^2 = 0$, $\mathrm{b}^2 = \tau/2$ if n even and $\mathrm{a}^1 = 1$, $\mathrm{b}^1 = \sigma/2$ if n odd (the non-specified parameters all vanish) and η_i are CAKVFs $\eta_i = y^{A_i}\partial_{y^{A_i+1}} - y^{A_i+1}\partial_{y^{A_i}}$, for $A_i = 2i$ for n odd and $A_i = 2i+1$ for n even, and where σ, τ, μ_i are given by Definition 4.10.

Given a CKVF ξ_F, the existence of a canonical form and canonical coordinates is guaranteed by Theorem 4.12. By Theorem 2.11, the conformal class $[\xi_F]$ of a CKVF ξ_F is equivalent to the equivalence class $[F]_{O^+}$ of F under the adjoint action of $O^+(1, n+1)$, and this is determined by the canonical form of F (cf. Theorem 4.22). Therefore the parameters $\{\sigma, \tau, \mu_i^2\}$ (equivalently $\{-\mu_t^2, \mu_s^2, \mu_i^2\}$) for n even and $\{\sigma, \mu_i^2\}$ for n odd determine a unique conformal class of CKVFs of $\mathbb{E}^n$.

In the following Theorem, we summarize the algorithm to determine the conformal class of CKVFs in locally conformally flat manifolds. This will be applied in the forthcoming Chapters 5 and 6

Teorema 4.35. *Let $\xi_F \in \mathrm{CKill}(\mathbb{E}^n)$, with $\mathbb{E}^n$ endowed with a flat metric γ_E and Cartesian coordinates $\{y^A\}_{A=1}^n$. Contruct the skew-symmetric endomorphism F corresponding to ξ_F according to Theorem 2.11 and consider the parameters $\{\sigma, \tau, \mu_i^2\}$ (equivalently $\{-\mu_t^2, \mu_s^2, \mu_i^2\}$) if n even and $\{\sigma, \mu_i^2\}$ if n odd in Definition 4.10. Then the conformal class $[\xi_F]$ is uniquely determined by these parameters. Moreover, the structure of limits in Remark 4.25 applies for $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$.*

Observacin 4.36. *Obviously, although this quotient is naturally constructed for conformal classes of CKVFs of $\mathbb{E}^n$, i.e. $\mathrm{CKill}(\mathbb{E})^n/\mathrm{ConfLoc}(\mathbb{E}^n)$, by Proposition 2.18, this has a one-to-one correspondence with the global conformal classes in the sphere, namely $\mathrm{CKill}(\mathbb{S}^n)/\mathrm{Conf}(\mathbb{S}^n)$.*

Observacin 4.37. *Theorem 4.35 also applies to the $n = 2$ case for equivalence classes generated by global CKVFs up to conformal transformations of $\mathbb{E}^2$ globally extendable in the sphere (see Remark 2.16), namely, the Möbius and affine transformations. It is interesting to stress this because no analogous result has been given in Chapter 3.*

Given a canonical form $\xi_F = \widetilde{\xi} + \sum \mu_i \eta_i$ the set of vectors $\{\widetilde{\xi}, \eta_i\}$ are pairwise commuting and linearly independent. As we will next prove, in the case of odd dimension this set is a maximal (linearly independent) pairwise commuting set of CKVFs commuting with ξ (i.e. it is not contained in a larger set of linearly independent vectors commuting one to another and with ξ). In the case of even dimension it is not maximal. By Remark 4.32, $\widetilde{\xi}$ equals $\widetilde{\xi}(\nu, \mathrm{a}^1, \mathrm{a}^2, \mathrm{b}^1, \mathrm{b}^2, \omega)$, where the right-hand side denotes a CKVF of the form (4.26) whose parameters vanish, except possibly $\{\nu, \mathrm{a}^1, \mathrm{a}^2, \mathrm{b}^1, \mathrm{b}^2, \omega := {\omega^1}_2\}$. As also mentioned in the Remark, the corresponding skew-symmetric endomorphism $\widetilde{F}$ satisfying $\xi_{\widetilde{F}} = \widetilde{\xi}$ can be understood as an element $\widetilde{F} \in \mathrm{SkewEnd}(\mathbb{M}^{1,3})$, with $\mathbb{M}^{1,3} = \mathrm{span}\{e_0, e_1, e_2, e_3\}$, that is identically zero in $\left(\mathbb{M}^{1,3}\right)^{\perp}$. Then, we may apply the results in Chapter 3 to this block. Namely, fix the orientation in $\mathbb{M}^{1,3}$ so that the basis $\{e_0, e_1, e_2, e_3\}$ is positively oriented. The Hodge star maps two-forms into two-forms. This defines a natural map

$$\begin{aligned} \star : \mathrm{SkewEnd}(\mathbb{M}^{1,3}) &\longrightarrow \mathrm{SkewEnd}(\mathbb{M}^{1,3}), \\ \widetilde{F} &\longmapsto \widetilde{F}^{\star}. \end{aligned}$$

From standard properties of two-forms, (see Section 3.4) it follows that $\widetilde{F}^{\star}$ commutes with $\widetilde{F}$. We may extend $\widetilde{F}^{\star}$ to an endomorphism on $\mathbb{M}^{1,n+1}$ that vanishes identically on $(\mathbb{M}^{1,3})^{\perp}$, just as $\widetilde{F}$. It is clear that the commutation property is preserved by this extension. The image of $\widetilde{F}^{\star}$ under ξ is the vector field

$$\widetilde{\xi}^{\star} := \left(\widetilde{\xi}(\nu, \mathrm{a}^1, \mathrm{a}^2, \mathrm{b}^1, \mathrm{b}^2, \omega)\right)^{\star} = \widetilde{\xi}(-\omega, \mathrm{a}^2, -\mathrm{a}^1, -\mathrm{b}^2, \mathrm{b}^1, \nu),$$

which by construction commutes with $\widetilde{\xi}$. In the case that $\widetilde{\xi}$ is the first element in a decomposed form $\xi_F = \widetilde{\xi} + \sum \mu_i \eta_i$, it is immediately true that $\widetilde{\xi}^{\star}$ also commutes with all of the CAKVFs η_i. Hence, $\{\widetilde{\xi}, \widetilde{\xi}^{\star}, \eta_i\}$ is a pairwise commuting set, all of them commuting with ξ. This set can be proven to be maximal:

Proposicin 4.38. *Let $\xi_F = \widetilde{\xi} + \sum \mu_i \eta_i$ be a CKVF in canonical form. If n is odd, $\{\widetilde{\xi}, \eta_i\}$ is a maximal linearly independent pairwise commuting set of elements that commute with ξ_F. If n is even, $\{\widetilde{\xi}, \widetilde{\xi}^{\star}, \eta_i\}$ is a maximal linearly independent pairwise commuting set of elements that commute with ξ_F.*

Proof. Suppose that there is an additional CKVF ξ' commuting with each element in $\{\widetilde{\xi}, \eta_i\}$ if n odd or $\{\widetilde{\xi}, \widetilde{\xi}^{\star}, \eta_i\}$ if n even (in either case ξ' clearly commutes with ξ_F also). Since it commutes with each η_i, by Proposition 4.33, it admits a decomposed form $\xi' = \widetilde{\xi}' + \sum_{i=1}^{p} \mu_i' \eta_i$, where $\widetilde{\xi}'$ is a CKVF orthogonal to each η_i and which must verify $[\widetilde{\xi}', \widetilde{\xi}] = 0$. Equivalently, their associated endomorphisms satisfy $\widetilde{F}' \in \mathcal{C}(\widetilde{F})$, where $\mathcal{C}(\widetilde{F})$

denotes the centralizer of F, i.e. the set of all skew-symmetric endomorphisms that commute with F. From the results in Section 3.4, $\mathcal{C}(\widetilde{F}\mid_{\mathbb{M}^{1,2}}) = \text{span}\{\widetilde{F}\mid_{\mathbb{M}^{1,2}}\}$ when n is odd and $\mathcal{C}(\widetilde{F}\mid_{\mathbb{M}^{1,3}}) = \text{span}\{\widetilde{F}\mid_{\mathbb{M}^{1,3}}, \widetilde{F}^{\star}\mid_{\mathbb{M}^{1,3}}\}$ when n is even. Here, $\widetilde{F}^{\star}$ is the skew-symmetric endomorphim associated with $\widetilde{\xi}^{\star}$ and we restrict to $\mathbb{M}^{1,3}$ because the action of the endomorphisms is identically zero in $(\mathbb{M}^{1,3})^{\perp}$. Thus $\widetilde{\xi}' = a\widetilde{\xi},\ a \in \mathbb{R}$, if n odd and $\widetilde{\xi}' = b\widetilde{\xi} + c\widetilde{\xi}^{\star},\ b, c \in \mathbb{R}$ if n even.

□

4.6 Adapted coordinates

In the previous Section we obtained a canonical form for each CKVF of the Euclidean space based on the canonical form of skew-symmetric endomorphisms in Section 4.2. As an application, we consider in this section the problem of adapting coordinates in $\mathbb{E}^n$ to a given CKVF ξ_F. The use of the canonical form will allow us to solve the problem for every possible ξ_F essentially in one go. Actually it will suffice to consider the case of even dimension n and assume that at least one of the parameters σ, τ in the canonical form of ξ_F is non-zero. The case where both σ and τ vanish will be obtained as a limit (and we will check that this limit does solve the required equations). The case of odd dimension n wil be obtained from the even dimensional one by exploiting the property that $\mathbb{E}^{2m+1}$ can be viewed as a hyperplane of $\mathbb{E}^{2m+2}$ in such a way that the given CKVF ξ_F in $\mathbb{E}^{2m+1}$ extends conveniently to $\mathbb{E}^{2m+2}$. Restricting the adapted coordinates already obtained in the even dimensional case to the appropriate hyperplane we will be able to infer the odd dimensional case. Recall that we are restricting to $n > 2$, so here we shall assume $n \geq 4$.

4.6.1 Calculation of the adapted coordinates

We start by integrating the PDEs which yield adapted coordinates to an arbitrary CKVF in the case of even n. Consider $\mathbb{E}^n$ endowed with a CKVF ξ_F. First of all, we adapt the Cartesian coordinates of $\mathbb{E}^n$ so that ξ_F takes its canonical form and we fix the metric of $\mathbb{E}^n$ to take the explictly flat form in these coordinates. We further assume (for the moment) that n is even. For notational reasons it is convenient to rename the canonical coordinates[4] as $z_1 := y^1,\ z_2 := y^2$ and $x_i := y^{2i+1},\ y_i := y^{2i+2}$ for $i = 1, \cdots, p$, where in the even case case $p = n/2 - 1$ (see (4.25)). By Proposition 4.33, ξ_F can be decomposed as a sum of CKVFs $\widetilde{\xi}$ and η_i and, additionally one can construct canonically yet another CKVF $\widetilde{\xi}^{\star}$. This collection of CKVFs defines a maximal commutative set. Moreover, $\{\eta_i\}$ are all mutually orthogonal and perpendicular to $\widetilde{\xi}$ and $\widetilde{\xi}^{\star}$. It is therefore most natural to try and find coordinates adapted simultaneously to the whole family $\{\widetilde{\xi}, \widetilde{\xi}^{\star}, \eta_i\}$. This

[4] The fact that we tag the coordinates $\{z_1, z_2, x_i, y_i\}$ with lower indices has no particular meaning. It is simply to avoid a notational clash of upper indices and powers that will appear later

will lead a (collection of) coordinate systems where the components of ξ_F are simply constants. From here one can immediately find coordinates that rectify ξ_F, if necessary. It is important to emphasize that selecting the whole set $\{\widetilde{\xi}, \widetilde{\xi}^\star, \eta_i\}$ to adapt coordinates provides enough restrictions so that the coordinate change(s) can be fully determined. Imposing the much weaker condition that the system of coordinates rectifies only ξ_F is just a too poor condition to solve the problem. This is an interesting example where the structure of the canonical decomposition of ξ_F (or of F) is exploited in full.

By Theorem 2.11, the explicit form of $\{\widetilde{\xi}, \widetilde{\xi}^\star, \eta_i\}$ in the canonical coordinates is

$$\widetilde{\xi} = \left(\frac{\sigma}{2} + \frac{1}{2}\left(z_1^2 - z_2^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right)\right)\partial_{z_1} + \left(\frac{\tau}{2} + z_1 z_2\right)\partial_{z_2} + z_1 \sum_{i=1}^{p}(x_i\partial_{x_i} + y_i\partial_{y_i}) \tag{4.27}$$

$$\widetilde{\xi}^\star = -\left(\frac{\tau}{2} + z_1 z_2\right)\partial_{z_1} + \left(\frac{\sigma}{2} - \frac{1}{2}\left(z_2^2 - z_1^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right)\right)\partial_{z_2} - z_2 \sum_{i=1}^{p}(x_i\partial_{x_i} + y_i\partial_{y_i})$$

$$\eta_i = x_i\partial_{y_i} - y_i\partial_{x_i}.$$

We are seeking coordinates $\{t_1, t_2, \phi_i, v_i\}$ adapted to these vector fields, i.e. such that $\partial_{t_1} = \widetilde{\xi}$, $\partial_{t_2} = \widetilde{\xi}^\star$, $\partial_{\phi_i} = \eta_i$. It is clear that if $\{t_1, t_2, \phi_i, v_i\}$ is an adapted coordinate system, so it is $\{t_1 - t_{0,1}(v), t_2 - t_{0,2}(v), \phi_i - \phi_{0,i}(v), v_i\}$ for arbitrary functions $t_{0,1}(v)$, $t_{0,2}(v)$ and $\phi_{0,i}(v)$, where $v = (v_1, \cdots, v_p)$. This will be used to simplify the process of integration. This freedom, may be restored at the end if so desired. Hence from $\tilde{\xi} = \partial_{t_1}$

$$\frac{\partial z_1}{\partial t_1} = \frac{\sigma}{2} + \frac{1}{2}\left(z_1^2 - z_2^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right), \qquad \frac{\partial z_2}{\partial t_1} = \frac{\tau}{2} + z_1 z_2, \tag{4.28}$$

$$\frac{\partial x_i}{\partial t_1} = z_1 x_i, \qquad \frac{\partial y_i}{\partial t_1} = z_1 y_i, \tag{4.29}$$

from $\tilde{\xi}^\star = \partial_{t_2}$,

$$\frac{\partial z_2}{\partial t_2} = \frac{\sigma}{2} - \frac{1}{2}\left(z_2^2 - z_1^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right), \qquad \frac{\partial z_1}{\partial t_2} = -\frac{\tau}{2} - z_1 z_2, \tag{4.30}$$

$$\frac{\partial x_i}{\partial t_2} = -z_2 x_i, \qquad \frac{\partial y_i}{\partial t_2} = -z_2 y_i, \tag{4.31}$$

and from $\eta_i = \partial_{\phi^i}$

$$\frac{\partial z_1}{\partial \phi_i} = 0 \qquad \frac{\partial z_2}{\partial \phi_i} = 0 \qquad \frac{\partial x_i}{\partial \phi_i} = -y_i \qquad \frac{\partial y_i}{\partial \phi_i} = x_i. \tag{4.32}$$

The additional p coordinates v_i, will appear through functions of integration. It is interesting to observe that, had we allowed n to be $n = 2$, and restricting oneself to global CKVFs, it is clear that the structure of the equations would have been different. This is because there are no $\{x_i, y_i\}$, which implies that the process of integration in this case would require a different route. In any case, as we have already seen in Chapter 3, for the case $n = 2$ the complex structure of $\mathbb{S}^2$ can be exploited to simplify the problem.

We may start by integrating (4.32). The first pair gives $z_1 = z_1(t_1, t_2, v)$, $z_2 = z_2(t_1, t_2, v)$, so that the second pair becomes a harmonic oscillator in x_i, y_i, whose solution is

$$x_i = \rho_i(t_1, t_2, v)\cos(\phi_i - \phi_{0,i}(t_1, t_2, v)), \qquad y_i = \rho_i(t_1, t_2, v)\sin(\phi_i - \phi_{0,i}(t_1, t_2, v)), \tag{4.33}$$

where ρ_i and $\phi_{0,i}$ are arbitrary functions (depending only on the variables indicated) and ρ_i is not identically zero.

Inserting (4.33) in any of equations (4.29) and (4.31) and equating terms multiplying $\sin(\phi_i + \phi_{0,i})$ and $\cos(\phi_i + \phi_{0,i})$ yields:

$$z_1 = \frac{1}{\rho_i}\frac{\partial \rho_i}{\partial t_1}, \qquad z_2 = -\frac{1}{\rho_i}\frac{\partial \rho_i}{\partial t_2}, \qquad \frac{\partial \phi_{0,i}}{\partial t_1} = 0, \qquad \frac{\partial \phi_{0,i}}{\partial t_2} = 0.$$

Thus, $\phi_{0,i}$ is a function only of v, which may be absorbed on the coordinate ϕ_i as discussed above. The two first equations imply

$$\frac{1}{\rho_i}\frac{\partial \rho_i}{\partial t_1} = \frac{1}{\rho_j}\frac{\partial \rho_j}{\partial t_1}, \quad \frac{1}{\rho_i}\frac{\partial \rho_i}{\partial t_2} = \frac{1}{\rho_j}\frac{\partial \rho_j}{\partial t_2} \qquad \Longleftrightarrow \qquad \rho_i = \hat{\alpha}_i(v)\hat{\rho}(t_1, t_2, v),$$

for arbitrary (non-zero) functions $\hat{\alpha}_i$ and $\hat{\rho}$. Defining $\rho^2 := \sum_{i=1}^{p} \rho_i^2 = \left(\sum_{i=1}^{p} \hat{\alpha}_i^2\right)\hat{\rho}^2$ we can write

$$\rho_i = \hat{\alpha}_i\hat{\rho} = \frac{\hat{\alpha}_i\epsilon}{\sqrt{\sum_{j=1}^{p}\hat{\alpha}_j^2}}\rho = \alpha_i\rho,$$

where $\alpha_i := \hat{\alpha}_i\epsilon/\sqrt{\sum_{j=1}^{p}\hat{\alpha}_j^2}$, with $\epsilon^2 = 1$, form a set of arbitrary (non-zero) functions of v such that $\sum_{i=1}^{p} \alpha_i^2 = 1$. The function ρ satisfies

$$z_1 = \frac{1}{\rho}\frac{\partial \rho}{\partial t_1}, \qquad z_2 = -\frac{1}{\rho}\frac{\partial \rho}{\partial t_2}. \tag{4.34}$$

Inserting (4.34) in equations (4.28) and (4.30), with the change of variable $U = \rho^{-1}$, we obtain after some algebra the following covariant system of PDEs (indices $a, b = 1, 2$ refer to $\{t_1, t_2\}$)

$$\nabla_a\nabla_b U = U A_{ab} + \frac{1}{2U}(1 + \nabla_c U \nabla^c U) g_{ab} \tag{4.35}$$

with

$$A := \frac{1}{2}(-\sigma \mathrm{d}t_1^2 + \sigma \mathrm{d}t_2^2 + 2\tau \mathrm{d}t_1 \mathrm{d}t_2), \quad g := \mathrm{d}t_1^2 + \mathrm{d}t_2^2,$$

and where ∇ is the Levi-Civita covariant derivative of g.

Lemma 4.39. *Up to shifts $t_1 \to t_1 - t_{0,1}(v)$ and $t_1 \to t_1 - t_{0,1}(v)$, the general solution of (4.35) with either σ or τ non-zero is given by*

$$U = \frac{\epsilon}{\mu_t^2 + \mu_s^2}\left(\beta\cosh(t_+) - \alpha\cos(t_-)\right) \qquad \textit{with} \qquad \beta = \sqrt{\alpha^2 + \mu_t^2 + \mu_s^2} \tag{4.36}$$

where α is a function of integration (depending on v), $\epsilon^2 = 1$ and $t_+ := \mu_t t_1 + \mu_s t_2$, $t_- := \mu_t t_2 - \mu_s t_1$, with μ_s, μ_t given by (3.9). The solution (4.36) admits a limit $\sigma = \tau = 0$ (i.e. $\mu_t = \mu_s = 0$) provided $\alpha > 0$, which is

$$\lim_{\mu_s\mu_t \to 0} U = \epsilon\frac{\alpha}{2}(t_1^2 + t_2^2) + \frac{\epsilon}{2\alpha}. \tag{4.37}$$

Up to shifts $t_1 \to t_1 - t_{0,1}(v)$ and $t_2 \to t_2 - t_{0,2}(v)$, this function is the general solution of (4.35) for $\sigma = \tau = 0$.

Proof. The coordinates t_+, t_- defined in the lemma diagonalize A and g simultaneously and yield

$$A = \frac{1}{2}(\mathrm{d}t_+^2 - \mathrm{d}t_-^2), \qquad g = \frac{1}{\mu_s^2 + \mu_t^2}(\mathrm{d}t_+^2 + \mathrm{d}t_-^2).$$

From this and equation (4.35) it follows that $\partial^2 U/\partial t_+ \partial t_- = 0$ or, equivalently, $U(t_+, t_-) = U_+(t_+) + U_-(t_-)$. Substracting the $\{t_+, t_+\}$ and $\{t_-, t_-\}$ components of (4.35) one obtains

$$\frac{\mathrm{d}^2 U_+}{\mathrm{d}t_+^2} - \frac{\mathrm{d}^2 U_-}{\mathrm{d}t_-^2} = U = U_+ + U_- \quad \Longrightarrow \quad \frac{\mathrm{d}^2 U_+}{\mathrm{d}t_+^2} - U_+ = \frac{\mathrm{d}^2 U_-}{\mathrm{d}t_-^2} + U_- = \hat{a}$$

for an arbitrary separation function $\hat{a}(v)$. The general solution is clearly

$$U_+ = -\hat{a} + a\cosh(t_+) + b\sinh(t_-) \qquad U_- = \hat{a} + c\cos(t_- - \delta), \tag{4.38}$$

where a, b, c, δ are also functions of v. Since $\hat{a}$ drops out in $U = U_+ + U_-$ we may set $\hat{a} = 0$ w.l.o.g. Inserting (4.38) in (any of) the diagonal terms of (4.35) and one simply gets

$$a^2 - b^2 = \frac{1}{\mu_s^2 + \mu_t^2} + c^2.$$

Hence $|a| > |b|$ and we may use the freedom of translating t_+ by a function of v to write $U_+ = a\cosh(t_+)$ (i.e. $b = 0$). A similar translation in t_- sets $\delta = 0$. Rescaling the functions a, c as $a = (\mu_s^2 + \mu_t^2)^{-1}\beta$ and $c = -(\mu_s^2 + \mu_t^2)^{-1}\alpha$ we get

$$U = U_- + U_- = \frac{\beta}{\mu_s^2 + \mu_t^2}\cosh(t_+) - \frac{\alpha}{\mu_s^2 + \mu_t^2}\cos(t_-), \qquad \beta^2 = \mu_s^2 + \mu_t^2 + \alpha^2. \tag{4.39}$$

It is obvious that $\mathrm{sign}(U) = \mathrm{sign}(\beta)$. Thus taking β as the positive root $\beta = \sqrt{\alpha^2 + \mu_s^2 + \mu_s^2}$

and adding a multiplicative sign ϵ in (4.39), we obtain (4.36). To evaluate the convergence as both σ, τ tend to zero, or equivalently $\mu_s, \mu_t \to 0$, consider the series expansion

$$\beta\cosh(t_+) = \left(|\alpha| + \frac{\mu_s^2+\mu_t^2}{2|\alpha|} + o^{(4)}_{\mu_t,\mu_s}\right)\left(1 + \frac{(\mu_s t_2 + \mu_t t_1)^2}{2} + o^{(4)}_{\mu_t,\mu_s}\right),$$
$$\alpha\cos(t_-) = \alpha - \alpha\frac{(\mu_t t_2 - \mu_s t_1)^2}{2} + o^{(4)}_{\mu_t,\mu_s},$$

where $o^{(4)}_{\mu_t,\mu_s}$ denotes a sum of homogeneous polynomials in μ_t, μ_s starting at order four, whose coeficients may depend on t_1, t_2 and α. Then, the expansion of U is

$$U = \frac{\epsilon}{\mu_s^2+\mu_t^2}\Big((|\alpha|-\alpha)(1+\mu_s\mu_t t_1 t_2) + \frac{|\alpha|\mu_s^2+\alpha\mu_t^2}{2}t_2^2 + \frac{|\alpha|\mu_t^2+\alpha\mu_s^2}{2}t_1^2 + \frac{\mu_s^2+\mu_t^2}{2|\alpha|} + o^{(4)}_{\mu_t,\mu_s}\Big).$$

It is clear that $\lim_{\mu_s,\mu_t\to 0} o^{(4)}_{\mu_t,\mu_s}/(\mu_s^2+\mu_t^2) = 0$ and the rest of the equation converges if and only if $\alpha > 0$ in which case the limit is (4.37). An easy calculation shows that this limit is (up to shifts in t_1, t_2) is the general solution of (4.35) when $\sigma, \tau = 0$. □

Having the general general solution (4.36) of (4.35) we can give the expression of the adapted coordinates

$$z_1 = -\frac{1}{U}\frac{\partial U}{\partial t_1} = \left|\frac{1}{U}\right|\frac{\alpha\mu_s\sin(t_-) - \beta\mu_t\sinh(t_+)}{\mu_s^2+\mu_t^2}, \tag{4.40}$$
$$z_2 = \frac{1}{U}\frac{\partial U}{\partial t_2} = \left|\frac{1}{U}\right|\frac{\alpha\mu_t\sin(t_-) + \beta\mu_s\sinh(t_+)}{\mu_s^2+\mu_t^2}, \tag{4.41}$$
$$x_i = \frac{\alpha_i}{U}\cos(\phi_i), \qquad y_i = \frac{\alpha_i}{U}\sin(\phi_i), \tag{4.42}$$

where no sign of α is in principle assumed[5], except for the case $\mu_s = \mu_t = 0$, where U must be understood as the limit (with $\alpha > 0$) (4.37) and $z_1 = -U^{-1}\partial U/\partial t_1$, $z_2 = U^{-1}\partial U/\partial t_2$. This coincides with the limit of the RHS expressions (4.40), (4.41), which is

$$z_1 = \frac{-2\alpha^2 t_1}{1+\alpha^2(t_1^2+t_2^2)}, \qquad z_2 = \frac{2\alpha^2 t_2}{1+\alpha^2(t_1^2+t_2^2)}. \tag{4.43}$$

From equations (4.40), (4.41) and (4.42) it is obvious that the sign ϵ is not relevant in the definition of the adapted coordinates. This is because the two branches $\epsilon = 1$ and $\epsilon = -1$ correspond to $U > 0$ and $U < 0$ respectively, which in terms of the adapted coordinates, is equivalent to a rotation of π in the ϕ_i angles. Hence, w.l.o.g. we consider $\epsilon = 1$, i.e. $U > 0$. Also notice that the dependence on the variables v_i appears through the functions α_i and α, with $\sum_{i=1}^{p}\alpha_i^2 = 1$. The set $\{\alpha_i, \alpha\}$ define p independent arbitrary

[5]The domain of definition of α will be later restricted under the condition that the adapted coordinates define a one to one map.

functions of the variables v_i, so it is natural to use as coordinates $\{\alpha_i, \alpha\}$ themselves, provided they are restricted to satisfy $\sum_{i=1}^{p} \alpha_i^2 = 1$.

4.6.2 Region covered by the adapted coordinates

We now calculate the region of $\mathbb{E}^n$ covered by the adapted coordinates. It is clear that in no case this region can include neither the zeros of the vector fields $\widetilde{\xi}$ and $\widetilde{\xi}^\star$ and η_i nor the points where these $p+2$ vectors are linearly dependent. We therefore start by locating those points. Denoting the loci of the zeros of $\widetilde{\xi}$ and $\widetilde{\xi}^\star$ and η_i by $\mathcal{Z}(\widetilde{\xi})$, $\mathcal{Z}(\widetilde{\xi})^\star$ and $\mathcal{Z}(\eta_i)$ respectively, a simple calculation gives

$$\begin{aligned}
\mathcal{Z}(\widetilde{\xi}) &= \Big(\{\bigcap_{j=1}^{p}\{x_j = y_j = 0\}\} \cap \{z_1 = \pm\mu_t, z_2 = \mp\mu_s\}\Big) \\
&\qquad \cup \Big(\{z_1 = 0\} \cap \{z_2^2 + \sum_{j=1}^{p}(x_j^2 + y_j^2) = \mu_s^2 - \mu_t^2\} \text{ if } \mu_s\mu_t = 0\Big), \qquad (4.44) \\
\mathcal{Z}(\widetilde{\xi}^\star) &= \Big(\{\bigcap_{j=1}^{p}\{x_j = y_j = 0\}\} \cap \{z_1 = \pm\mu_t, z_2 = \mp\mu_s\}\Big) \\
&\qquad \cup \Big(\{z_2 = 0\} \cap \{z_1^2 + \sum_{j=1}^{p}(x_j^2 + y_j^2) = \mu_t^2 - \mu_s^2\} \text{ if } \mu_s\mu_t = 0\Big), \qquad (4.45) \\
\mathcal{Z}(\eta_i) &= \{x_i = y_i = 0\}\,.
\end{aligned}$$

These expressions are valid for every value of μ_s, μ_t and imply that in the case $\mu_s = \mu_t = 0$, $\mathcal{Z}(\widetilde{\xi}) = \mathcal{Z}(\widetilde{\xi}^\star) = \{\bigcap_{j=1}^{p}\{x_j = y_j = 0\}\} \cap \{z_1 = z_2 = 0\}$, which is contained in each $\mathcal{Z}(\eta_i) = \{x_i = y_i = 0\}$.

On the other hand, since $\{\widetilde{\xi}, \eta_i\}$ is an orthogonal set of CKVFs (cf. Lemma 4.30), they are pointwise linearly independent at all points where they do not vanish. Similarly, $\{\widetilde{\xi}^\star, \eta_i\}$ is also an orthogonal set, so linear independence is guaranteed away from the zero set. Away from this set, the set of vectors $\{\widetilde{\xi}, \widetilde{\xi}^\star, \eta_i\}$ is linearly dependent only at points where $\widetilde{\xi}$ and $\widetilde{\xi}^\star$ are proportional to each other with a non-zero proportionality factor, $\widetilde{\xi} = a\widetilde{\xi}^\star$, $a \neq 0$. One easily checks that, away from $\mathcal{Z}(\widetilde{\xi})$ and $\mathcal{Z}(\widetilde{\xi}^\star)$, the set of point where $\widetilde{\xi} - a\widetilde{\xi}^\star$ vanishes is empty except when $\mu_s \neq 0, \mu_t \neq 0$ and $a = \frac{\mu_t}{\mu_s}$. It turns out to be useful to determine the set of points where $\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star = 0$ when at least one of $\{\mu_s, \mu_t\}$ is non-zero. We call this set $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$, and a straightforward analysis gives

$$\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star) = \begin{cases} \{\mu_s z_1 = -\mu_t z_2\} \\ \cap\{(\mu_s^2 + \mu_t^2)z_2^2 + \mu_s^2 \sum_{i=1}^{p}(x_i^2 + y_i^2) = (\mu_s^2 + \mu_t^2)\mu_s^2\} & \text{if } \mu_s \neq 0, \\ \{\mu_s z_1 = -\mu_t z_2\} \\ \cap\{(\mu_s^2 + \mu_t^2)z_1^2 + \mu_t^2 \sum_{i=1}^{p}(x_i^2 + y_i^2) = (\mu_s^2 + \mu_t^2)\mu_t^2\} & \text{if } \mu_t \neq 0. \end{cases} \qquad (4.46)$$

Obviously, the two expressions are equivalent when both μ_s and μ_t are non-zero. The interest of this set is that it happens to always contain $\mathcal{Z}(\widetilde{\xi})$ and $\mathcal{Z}(\widetilde{\xi}^\star)$. This, together with the fact that when $\mu_s = \mu_t = 0$ these sets are contained in the axes $\mathcal{Z}(\eta_i)$ will allow us to ignore them altogether.

Lemma 4.40. *Assume that at least one of $\{\mu_s, \mu_t\}$ is non-zero. Then $\mathcal{Z}(\widetilde{\xi}), \mathcal{Z}(\widetilde{\xi}^\star) \subset \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$.*

Proof. Consider first $\mu_s, \mu_t \neq 0$. Then at $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star) \cap \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\}$ we have that $z_1 = \pm\mu_t$ and $z_2 = \mp\mu_s$ which establishes $\mathcal{Z}(\widetilde{\xi}), \mathcal{Z}(\widetilde{\xi}^\star) \subset \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$ in this case. When $\mu_t = 0,\ \mu_s \neq 0$, by definition of the respective sets we have $\mathcal{Z}(\widetilde{\xi}) = \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$. Moreover, directly from (4.45) one finds

$$\mathcal{Z}(\widetilde{\xi}^\star) = \bigcap_{j=1}^{p} \{x_j = y_j = 0\} \cap \{z_1 = 0, z_2 = \pm\mu_s\},$$

which (cf. the first expression in (4.46)) is clearly contained in $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$. An analogous argument applies in the case $\mu_t \neq 0,\ \mu_s = 0$. □

Let us define the following auxiliary coordinates

$$\hat{z}_+ := \frac{\mu_s z_1 + \mu_t z_2}{\sqrt{\sum_{i=1}^p (x_i^2 + y_i^2)}}, \qquad \hat{z}_- := \frac{\mu_s z_2 - \mu_t z_1}{\sqrt{\sum_{i=1}^p (x_i^2 + y_i^2)}}, \qquad \hat{x}_i := x_i, \qquad \hat{y}_i := y_i.$$

Except for the case $\mu_s = \mu_t = 0$ (which will be analyzed later) the coordinates $\{\hat{z}_+, \hat{z}_-, \hat{x}_i, \hat{y}_i\}$ obviously cover $\mathbb{R}^n \backslash \left\{ \bigcap_{j=1}^p \{x_j = y_j = 0\} \right\}$. In terms of the adapted coordinates, they read

$$\hat{z}_+ = \alpha \sin(t_-), \qquad \hat{z}_- = \beta \sinh(t_+) \qquad \hat{x}_i = \frac{\alpha_i}{U}\cos(\phi_i), \qquad \hat{y}_i = \frac{\alpha_i}{U}\sin(\phi_i). \tag{4.47}$$

Let us analyze the points where (4.47) fails to be a change of coordinates and hence restrict the domain of definition of $\{\alpha, t_-, t_+, \alpha_i, \phi_i\}$. The first thing to notice is that a change of sign in the coordinate α_i is equivalent to a rotation of angle π in the coordinate ϕ_i. Moreover, at points where $\alpha_i = 0$, i.e. the axis of η_i, the coordinate ϕ_i is completely degenerate, which obviously excludes $\bigcup_{j=1}^p \{x_j = y_j = 0\}$ from the region covered by the adapted coordinates. To avoid duplications, we must restrict $\alpha_i \in (0, 1)$ and $\phi \in [-\pi, \pi)$ or alternatively $\alpha_i \in (-1, 1)\backslash\{0\}$ and $\phi_i \in [0, \pi)$. We choose the former for definiteness.

The hypersurface $\{\alpha = \text{const}, t_- = \text{const}, t_+ = \text{const}\}$ is an $n-3$ dimensional sphere of radius U^{-1}, namely $\{\hat{z}_- = \text{const},\ \hat{z}_+ = \text{const}\} \cap \{\sum_{i=1}^p (x_i^2 + y_i^2) = U^{-2} = \text{const}\}$. This gives a straightforward splitting of $\mathbb{R}^n \backslash \{0_{n-2}\}$, with $0_{n-2} := \{\bigcap_{j=1}^p \{x_j = y_j = 0\}\}$, into $\mathbb{R}^2 \times (\mathbb{R}^{n-2} \backslash \{0_{n-2}\})$, where $\mathbb{R}^{n-2} \backslash \{0_{n-2}\}$ is foliated by $n-3$ dimensional spheres. The set $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$ respects this foliation, so it descends to $\mathbb{R}^2 \times \mathbb{R}^+$ (the last factor is the radius of the $n-3$ sphere). To avoid extra notation we also use $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$ to denote this quotient set. We next show that the adapted coordinates actually cover the largest possible domain, namely $\mathbb{R}^n \setminus \{\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star) \cup \bigcup_{j=1}^p \{x_j = y_j = 0\}\}$. From the previous discussion, this is a consequence of the following result.

Lemma 4.41. *Assume that at least one of $\{\mu_s, \mu_t\}$ is not zero. Then. the transformation*

$$\begin{array}{rcl} (\hat{z}_+, \hat{z}_-, U) : \mathbb{R} \times [-\pi, \pi) \times \mathbb{R}^+ & \longrightarrow & \left(\mathbb{R}^2 \times \mathbb{R}^+\right) \setminus \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star) \\ (t_+, t_-, \alpha) & \longmapsto & (\hat{z}_+, \hat{z}_-, U). \end{array} \tag{4.48}$$

is a diffeomorphism.

Proof. The determinant of the jacobian of (4.48) reads

$$\left| \frac{\partial(\hat{z}_+, \hat{z}_-, U)}{\partial(t_+, t_-, \alpha)} \right| = \alpha U.$$

Since U is strictly positive (cf. (4.36) and recall that we chose $\epsilon = 1$ w.l.o.g.), the conflictive points are $\alpha = 0$. To calculate the locus $\{\alpha = 0\}$ we obtain the inverse transformation of α in terms of $U, \hat{z}_+, \hat{z}_-$ by solving (4.36) and the first two in (4.47). The result is, after a straightforward computation,

$$\alpha = \pm\left(\hat{z}_+^2 + \frac{1}{4U^2(\mu_s^2 + \mu_t^2)^2}(\hat{z}_+^2 + \hat{z}_-^2 - U^2(\mu_s^2 + \mu_t^2)^2 + (\mu_s^2 + \mu_t^2))^2\right)^{1/2}. \tag{4.49}$$

It follows that $\alpha = 0$ is equivalent to $\hat{z}_+ = 0$ and $\hat{z}_-^2 + \mu_s^2 + \mu_t^2 = U^2(\mu_s^2 + \mu_t^2)^2$. When translated into the original coordinates$\{z_1, z_2, x_i, y_i\}$ this set is precisely $\mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$. Also, from (4.49) it is obvious that α is multivalued, which also implies that t_- is multivalued after substituting α as a function of $\hat{z}_+, \hat{z}_-, U$ in the first equation in (4.47)[6]. We solve this issue by restricting α to be strictly positive and let t_- take values in $[-\pi, \pi)$. □

We have shown that the adapted coordinates cover all $\mathbb{R}^n$ except $\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star)$. The domain of definition of the coordinates t_1, t_2 depends on μ_t and μ_s, because $-\pi \leq t_- = \mu_t t_2 - \mu_s t_1 < \pi$. This defines a band $B(\mu_s, \mu_t) := \{-\pi \leq t_- = \mu_t t_2 - \mu_s t_1 < \pi\}$, whose width and tilt is determined by σ, τ through μ_s, μ_t (see Figure 4.2). Nevertheless, the coordinate change is well defined for all values of t_1 and t_2 and involves only periodic functions of t_-. Thus, we can extend the domain of definition of t_1, t_2 to all of $\mathbb{R}^2$. This defines a covering of the original space $\mathbb{R}^n \setminus (\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s\widetilde{\xi} - \mu_t\widetilde{\xi}^\star))$ which unwraps completely the orbits of $\widetilde{\xi}$ and $\widetilde{\xi}^\star$. It is not the universal covering because it does not unwrap the orbits of the axial vectors. This result is a generalization to higher dimensions of the covering dicussed in detail in Chapter 3.

[6] This was already evident by observing that a change of sing in α is cancelled by a rotation of π in t_-

The limit case $\mu_s = \mu_t = 0$ (that is $\sigma = \tau = 0$) corresponds with a band of infinite width, i.e. $B(\mu_s, \mu_t) = \mathbb{R}^2$. In this case, the adapted coordinates also cover the largest possible set $\mathbb{R}^n \backslash (\bigcup_{j=1}^p \mathcal{Z}(\eta_j))$. Recall that in this case the only points where $\{\widetilde{\xi}, \widetilde{\xi}^\star, \eta_i\}$ is not a linearly independent set is the union of $\mathcal{Z}(\widetilde{\xi}), \mathcal{Z}(\widetilde{\xi}^\star)$, and $\mathcal{Z}(\eta_i)$ and we have already seen that in this case $\mathcal{Z}(\widetilde{\xi}) = \mathcal{Z}(\widetilde{\xi}^\star) \subset \mathcal{Z}(\eta_i)$, for $i = 1, \cdots, p$. This limit case is the same result that we would have obtained, had we performed a direct analysis using U as given by (4.37).

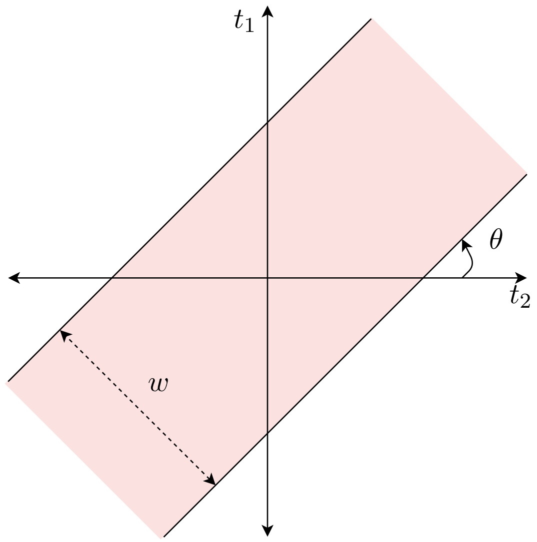

FIGURE 4.2: *Band $B(\mu_s, \mu_t)$ where the coordinates t_1, t_2 are defined. The tilt is given by $\theta = \arctan\left(\frac{\mu_s}{\mu_t}\right)$ and the width w is $2\pi/\mu_t$ if $\mu_t \neq 0$, $2\pi/\mu_s$ if $\mu_t = 0$, $\mu_s \neq 0$ and $w \to \infty$ if $\mu_s = \mu_t = 0$.*

4.6.3 Conformally flat metrics in adapted coordinates

Once we have determined the adapted coordinates and the region they cover, we may proceed to calculate the expression of the Euclidean metric

$$g_E = \mathrm{d}z_1^2 + \mathrm{d}z_2^2 + \sum_{i=1}^p \left(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\right). \tag{4.50}$$

in adapted coordinates. We start with the term $\sum_{i=1}^p \left(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\right)$, which is straightforward

$$\begin{aligned}\sum_{i=1}^p \left(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\right) &= \frac{\mathrm{d}U^2}{U^4} + \frac{1}{U^2}\sum_{i=1}^p \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right)\big|_{\sum_{i=1}^p \alpha_i^2 = 1} - \frac{2\mathrm{d}U}{U^2}\left(\sum_{i=1}^p \alpha_i \mathrm{d}\alpha_i\right) \\ &= \frac{\mathrm{d}U}{U^4} + \frac{1}{U^2}\gamma_{\mathbb{S}^{n-3}},\end{aligned} \tag{4.51}$$

where in the last equality we used $\sum_{i=1}^{p} \alpha_i \mathrm{d}\alpha_i = 0$, which follows from $\sum_{i=1}^{p} \alpha_i^2 = 1$ and we have defined

$$\gamma_{\mathbb{S}^{n-3}} := \sum_{i=1}^{p} \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right)\big|_{\sum_{i=1}^{p}\alpha_i^2=1} . \tag{4.52}$$

The notation is justified because the right-hand side corresponds to the standard unit metric on $\mathbb{S}^{n-3}$. This follows because $\sum_{i=1}^{p}\left(\mathrm{d}\alpha_i^2 + \alpha_i^2\mathrm{d}\phi_i^2\right)$ is obviously flat and the restriction $\sum_{i=1}^{p}\alpha_i^2 = 1$ defines a unit sphere. We emphasize, however that the notation $\gamma_{\mathbb{S}^{n-3}}$ refers to the quadratic form above, not to the spherical metric in any other coordinate system. Observe also that $\mathrm{d}U$ in (4.51) should be understood as a short name for the explicit differential of U in terms of $\mathrm{d}t_1, \mathrm{d}t_2$, $\mathrm{d}\alpha$. Using (4.50) and (4.51), we have

$$g_{t_1t_1} = \left(\frac{\partial z_1}{\partial t_1}\right)^2 + \left(\frac{\partial z_2}{\partial t_1}\right)^2 + \frac{1}{U^4}\left(\frac{\partial U}{\partial t_1}\right)^2,$$

which after an explicit calculation reduces to

$$g_{t_1t_1} = \frac{\alpha^2 + \mu_t^2}{U^2}.$$

Notice that $g_{t_1t_1} = g_E(\widetilde{\xi}, \widetilde{\xi})$, $g_{t_2t_2} = g_E(\widetilde{\xi^\star}, \widetilde{\xi^\star})$ and $g_{t_1,t_2} = g_E(\widetilde{\xi}, \widetilde{\xi^\star})$. From the expressions in Cartesian coordinates it is straightforward to show

$$g_E(\widetilde{\xi}, \widetilde{\xi}) = g_E(\widetilde{\xi^\star}, \widetilde{\xi^\star}) - \sigma\sum_{i=1}^{p}(x_i^2 + y_i^2) = g_E(\widetilde{\xi^\star}, \widetilde{\xi^\star}) - \frac{\sigma}{U^2},$$
$$g_E(\widetilde{\xi}, \widetilde{\xi^\star}) = \frac{\tau}{2}\sum_{i=1}^{p}(x_i^2 + y_i^2) = \frac{\tau}{2U^2}$$

where we have used $U^{-2} = \sum\limits_{i=1}^{p}(x_i^2 + y_i^2)$ (see (4.42)). Thus

$$g_{t_2t_2} = g_{t_1t_1} + \frac{\sigma}{U^2} = \frac{\alpha^2 + \mu_s^2}{U^2}, \qquad g_{t_1t_2} = \frac{\tau}{2U^2} = \frac{\mu_s\mu_t}{U^2}.$$

The remaining terms are rather long to calculate. With the aid of a computer algebra system one gets

$$g_{\alpha\alpha} = \left(\frac{\partial z_1}{\partial \alpha}\right)^2 + \left(\frac{\partial z_2}{\partial \alpha}\right)^2 + \frac{1}{U^4}\left(\frac{\partial U}{\partial \alpha}\right)^2 = \frac{1}{\beta^2 U^2}$$
$$g_{\alpha t_1} = \frac{\partial z_1}{\partial \alpha}\frac{\partial z_1}{\partial t_1} + \frac{\partial z_2}{\partial \alpha}\frac{\partial z_2}{\partial t_1} + \frac{1}{U^4}\frac{\partial U}{\partial \alpha}\frac{\partial U}{\partial t_1} = 0,$$
$$g_{\alpha t_2} = \frac{\partial z_1}{\partial \alpha}\frac{\partial z_1}{\partial t_2} + \frac{\partial z_2}{\partial \alpha}\frac{\partial z_2}{\partial t_2} + \frac{1}{U^4}\frac{\partial U}{\partial \alpha}\frac{\partial U}{\partial t_2} = 0.$$

Notice that no terms in $\mathrm{d}\alpha_i$, $\mathrm{d}\phi_i$ appear but those in $\gamma_{\mathbb{S}^{n-3}}$, since neither U nor z_1, z_2 depend on α_i, ϕ_i. Putting all these results together we obtain the following expression:

Lemma 4.42. *In adapted coordinates $\{t_1, t_2, \alpha, \alpha_i, \phi_i\}$, the Euclidean metric g_E takes the form*

$$g_E = \frac{1}{U^2}\left((\alpha^2+\mu_t^2)\mathrm{d}t_1^2 + (\alpha^2+\mu_s^2)\mathrm{d}t_2^2 + 2\mu_s\mu_t \mathrm{d}t_1\mathrm{d}t_2 + \frac{\mathrm{d}\alpha^2}{\alpha^2+\mu_s^2+\mu_t^2} + \gamma_{\mathbb{S}^{n-3}}\right). \tag{4.53}$$

We would like to stress the simplicity of this result. Except in the conformal factor, the metric does not depend in t_1 and t_2 (so, both $\widetilde{\xi}$ and $\widetilde{\xi}^\star$ are Killing vectors of $U^2 g_E$). The dependence in the coordinate α and the conformal class constants $\{\mu_s, \mu_t\}$ is also extremely simple. Even more, the fact that all dependence in $\{\alpha_i, \phi_i\}$ arises only in $\gamma_{\mathbb{S}^{n-3}}$ allows us to use any other coordinate system on the unit sphere $\mathbb{S}^{n-3}$. Any such coordinate system is still adapted to $\widetilde{\xi}$ and $\widetilde{\xi}^\star$ but (in general) no longer to $\{\eta_i\}$. This enlargement to partially adapted coordinates is an interesting consequence of the foliation of $\mathbb{R}^n$ by $(n-3)$-spheres described above.

4.6.4 Odd dimensional case and Adapted Coordinates Theorem

We now work out the odd n case. As already discussed, we will base the analysis on the even dimensional case by restricting to a suitable a hyperplane. The underlying reason why this is possible is given in the following lemma.

Lemma 4.43. *Fix $n \geq 3$ odd. Let ξ_F be a CKVF of $\mathbb{E}^n$ in canonical form and let $\{z_1, x_i, y_i\}$ be canonical coordinates. Consider the embedding $\mathbb{E}^n \hookrightarrow \mathbb{E}^{n+1}$ where $\mathbb{E}^n$ is identified with the hyperplane $\{z_2 = 0\}$, for a Cartesian coordinate z_2 of $\mathbb{E}^{n+1}$. Then ξ_F extends to a CKVF of $\mathbb{E}^{n+1}$ with the same values of σ, μ_i and $\tau = 0$.*

Proof. By Remark 4.32 and Theorem 2.11, the expression of ξ_F in the canonical coordinates $\{z_1, x_i, y_i\}$ is

$$\begin{aligned}\xi_F &= \left(\frac{\sigma}{2} + \frac{1}{2}\left(z_1^2 - \sum_{i=1}^p (x_i^2+y_i^2)\right)\right)\partial_{z_1} + z_1 \sum_{i=1}^p \left(x_i\partial_{x_i} + y_i\partial_{y_i}\right) + \sum_{i=1}^p \mu_i\left(x_i\partial_{y_i} - y_i\partial_{x_i}\right)\\ &= \widetilde{\xi} + \sum_{i=1}^p \mu_i\eta_i.\end{aligned}$$

Define ξ_F' on $\mathbb{E}^{n+1}$ in Cartesian coordinates $\{z_1, z_2, x_i, y_i\}$ by $\xi_F' = \widetilde{\xi}' + \sum_{i=1}^p \mu_i\left(x_i\partial_{y_i} - y_i\partial_{x_i}\right)$ where $\widetilde{\xi}'$ is given by (4.27) with $\tau = 0$. It is clear that this vector is a CKVF of $\mathbb{E}^{n+1}$ written in canonical form, that it is tangent to the hyperplane $z_2 = 0$ and that it agrees with ξ_F on this submanifold. □

Consequently, introducing adapted coordinates for the extended CKVF and restricting to $\{z_2 = 0\}$ will provide adapted coordinates for ξ_F. The restriction will obviously reduce the domain of definition of the adapted coordinates $\{t_1, t_2, \alpha, \alpha_i, \phi_i\}$ to a hypersurface. It is straightforward from equation (4.41) and the second equation in (4.43) that for the three cases $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$, the hyperplane $\{z_2 = 0\}$ corresponds to $\{t_2 = 0\}$. It follows that the remaining coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ are adapted to $\widetilde{\xi}$ and all η_i. Their domain of definition is $t_1 \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$, $\alpha_i \in (0,1)$, $\phi_i \in [-\pi, \pi)$ and the coordinate change is given by (4.40) (or the first in (4.43)) together with (4.42) after setting $\tau = 0$ and $t_2 = 0$. Depending on the sign of σ one gets for z_1

$$z_1 = \begin{cases} \frac{-1}{|U^+|}\frac{\alpha\sin(\sqrt{\sigma}t_1)}{\sqrt{\sigma}}, & \sigma > 0 \\ \frac{-1}{|U^-|}\frac{\sqrt{\alpha^2+|\sigma|}\sinh(\sqrt{|\sigma|}t_1)}{\sqrt{|\sigma|}}, & \sigma < 0 \\ \frac{-1}{|U^0|}\alpha t_1, & \sigma = 0 \end{cases}, \tag{4.54}$$

where

$$U^+ := \frac{1}{\sigma}(\sqrt{\alpha^2+\sigma} - \alpha\cos(\sqrt{\sigma}t_1)), \qquad U^- := \frac{1}{-\sigma}(\sqrt{\alpha^2-\sigma}\cosh(\sqrt{-\sigma}t_1) - \alpha),$$
$$U^0 := \frac{1}{2}(\alpha t_1^2 + \frac{1}{\alpha}),$$

and for all three cases

$$x_i = \frac{\alpha_i}{U^\epsilon}\cos(\phi_i), \qquad y_i = \frac{\alpha_i}{U^\epsilon}\sin(\phi_i), \tag{4.55}$$

where we write U^ϵ for the function U^+, U^- or U^0 according with sign of σ.

The range of variation of $\{t_1, \alpha, \alpha_i, \phi_i\}$ was inferred before from the corresponding range of variation of $\{t_1, t_2, \alpha, \alpha_i, \phi_i\}$ in $\mathbb{E}^{n+1}$. It may happen, however, that when we restrict to the hyperplane $\{z_2 = 0\}$, the range gets enlarged and additional points get covered by the adapted coordinate system. The underlying reason is that, in effect, we are no longer adapting coordinates to $\widetilde{\xi}'^\star$, so the points on $z_2 = 0$ where this vector is linearly dependent to $\widetilde{\xi}'$ (or zero) are no longer problematic. When $\tau = 0$, one has

$$(\mu_s = \sqrt{\sigma}, \quad \mu_t = 0) \quad \text{if } \sigma \geq 0, \qquad\qquad (\mu_s = 0, \quad \mu_t = \sqrt{|\sigma|}) \quad \text{if } \sigma \leq 0.$$

We may ignore the case $\sigma = 0$ because $\mathcal{Z}(\widetilde{\xi}') = \mathcal{Z}(\widetilde{\xi}'^\star)$. It follows from (4.44) and (4.46) that

$$\mathcal{Z}(\widetilde{\xi}')\Big|_{z_2=0} = \begin{cases} \{z_1 = 0\} \cap \left\{\sum_{i=1}^p (x_i^2 + y_i^2) = \sigma\right\} & \text{if } \sigma > 0 \\ \bigcap_{j=1}^p \{x_j = y_j = 0\} \cap \left\{z_1 = \pm\sqrt{|\sigma|}\right\} & \text{if } \sigma < 0 \end{cases}$$
$$\mathcal{Z}(\mu_s\widetilde{\xi}' - \mu_t\widetilde{\xi}'^\star)\Big|_{z_2=0} = \begin{cases} \{z_1 = 0\} \cap \left\{\sum_{i=1}^p (x_i^2 + y_i^2) = \sigma\right\} & \text{if } \sigma > 0 \\ \{z_1^2 + \sum_{i=1}^p (x_i^2 + y_i^2) = |\sigma|\} & \text{if } \sigma < 0. \end{cases}$$

When $\sigma > 0$, the two sets are the same and no extension of the coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ is possible. However, when $\sigma < 0$, the set $\mathcal{Z}(\mu_s \widetilde{\xi}' - \mu_t \widetilde{\xi}'^{\star})|_{z_2=0}$ is strictly larger than $\mathcal{Z}(\widetilde{\xi}')|_{z_2=0}$. From expressions (4.54) and (4.55) one checks that $\mathcal{Z}(\mu_s \widetilde{\xi}' - \mu_t \widetilde{\xi}'^{\star})|_{z_2=0} \setminus \mathcal{Z}(\widetilde{\xi}')|_{z_2=0}$ corresponds exactly to the value $\alpha = 0$ and that $\mathcal{Z}(\widetilde{\xi}) = \mathcal{Z}(\widetilde{\xi}')|_{z_2=0}$ is at the limit $t_1 \to \pm\infty$. Thus, a priori there is the possibility that the adapted coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ can be extended regularly to $\alpha = 0$ when $\sigma < 0$. It follows directly from (4.54) that this is indeed the case (observe that, to the contrary, the limit $\alpha \to 0$ in (4.54) is singular when $\sigma \geq 0$, in agreement with the previous discussion). Thus, the range of definition of α is $[0, \infty)$ when $\sigma < 0$. The conclusion is that, irrespectively of the value of σ, the adapted coordinates $\{t_1, \alpha, \alpha_i, \phi_i\}$ cover the largest possible domain of $\mathbb{E}^n$, namely all points where $\widetilde{\xi}$ is non-zero away from the axes of $\{\eta_i\}$.

To obtain the Euclidean metric in $\mathbb{E}^n$ for n odd in adapted coordinates we simply restrict (4.53) (with $n \to n+1$) to the hypersurface $t_2 = 0$, and get

$$g_E^{\epsilon} = \frac{1}{(U^{\epsilon})^2}\left(\left(\alpha^2 + \frac{(1-\epsilon)|\sigma|}{2}\right) dt_1^2 + \frac{d\alpha^2}{\alpha^2 + |\sigma|} + \gamma_{\mathbb{S}^{n-2}}\right), \tag{4.56}$$

where $\epsilon = -1, 0, 1$ respectively if $\sigma < 0, \sigma = 0, \sigma > 0$.

Observacin 4.44. *The three odd dimensional cases can be unified into one. The function U^0 coincides with the limits of U^+ and U^- when $\sigma \to 0$. However, the analytical continuation of U^+ to negative values of σ does not directly yield U^-. To solve this we introduce the function*

$$W_1(y) = \frac{1}{\sigma}\left(\sqrt{y^2 + \sigma} - y \cos\left(\sqrt{\sigma} t_1\right)\right),$$

which is analytic in σ and takes real values for real σ. We observe that $U^+(\alpha = y) = W_1(y)$ for $\sigma > 0$, $U^0(\alpha = y) = W_1(y)$ $(\sigma = 0)$ and $U^-(\alpha = +\sqrt{y^2 + \sigma}) = W_1(y)$ $(\sigma < 0)$. This suggests introducing the coordinate change $\alpha = y$ for $\sigma \geq 0$ and $\alpha = +\sqrt{y^2 + \sigma}$ for $\sigma < 0$. From the domain of α, it follows that y takes values in $y > 0$ when $\sigma \geq 0$ and $y \geq \sqrt{-\sigma}$ when $\sigma < 0$. In terms of y, the three metrics metric g^{ϵ} take the unified form

$$g_E^{\epsilon} = \frac{1}{W_1(y)^2}\left(y^2 \mathrm{d}t_1^2 + \frac{\mathrm{d}y^2}{y^2 + \sigma} + \gamma_{\mathbb{S}^{n-2}}\right).$$

The function W_1 is the analytic continuation of U^+ to negative values of σ. We could have started with U^- and continued analytically to positive values of σ. Instead of repeating the argument, we simply introduce a new variable z defined by $y = \sqrt{z^2 - \sigma}$ with range of variation $z > \sqrt{\sigma}$ for $\sigma \geq 0$ and $z \geq 0$ for $\sigma < 0$. The metric takes the (also unified and even more symmetric) form

$$g_E^{\epsilon} = \frac{1}{W_2(z)^2}\left((z^2 - \sigma)\mathrm{d}t_1^2 + \frac{\mathrm{d}z^2}{z^2 - \sigma} + \gamma_{\mathbb{S}^{n-2}}\right), \quad W_2(z) := \frac{1}{\sigma}\left(z - \sqrt{z^2 - \sigma} \cos\left(\sqrt{\sigma} t_1\right)\right).$$

The function $W_2(z)$ is again analytic in σ, takes real values on the real line, and now it extends U^-. More specifically, $U^-(\alpha = z) = W_2(z)$ $(\sigma < 0)$, $U^0(\alpha = z) = W_2(z)$ $(\sigma = 0)$ and $U^+(\alpha = \sqrt{z^2 - \sigma}) = W_2(z)$ $(\sigma > 0)$.

Remark 4.44 allows us to work with all the odd dimensional cases at once, which will be useful for Section 4.7. However, this unified form does not arise naturally when the odd dimensional case is viewed as a consequence of the $n+1$ even dimensional case. So, leaving aside this remark for Section 4.7, we summarize the results of this section in the following Theorem.

Teorema 4.45. *Given a CKVF ξ_F of $\mathbb{E}^n$, with $n \geq 4$ even, in canonical form $\xi_F = \widetilde{\xi} + \sum_{i=1}^p \mu_i \eta_i$, the coordinates $t_1, t_2, \phi_i, \alpha, \alpha_i$, for $i = 1, \cdots p$ and $\sum_{i=1}^p \alpha_i^2 = 1$, defined by*

$$z_1 = -\frac{1}{U}\frac{\partial U}{\partial t_1}, \qquad z_2 = \frac{1}{U}\frac{\partial U}{\partial t_2} \qquad x_i = \frac{\alpha_i}{U}\cos(\phi_i), \qquad y_i = \frac{\alpha_i}{U}\sin(\phi_i)$$

with

$$U = \frac{\sqrt{\alpha^2 + \mu_t^2 + \mu_s^2}\cosh(\mu_t t_1 + \mu_s t_2) - \alpha\cos(\mu_t t_2 - \mu_s t_1)}{\mu_t^2 + \mu_s^2},$$

which admits a limit $\lim_{\mu_s \mu_t \to 0} U = \frac{\alpha}{2}(t_1^2 + t_2^2) + \frac{1}{2\alpha}$, furnish adapted coordinates to $\widetilde{\xi} = \partial_{t_1}$ $\widetilde{\xi}^\star = \partial_{t_2}$ $\eta_i = \partial_{\phi_i}$, which cover the maximal possible domain, namely $\mathbb{E}^n \setminus \Big(\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\mu_s \widetilde{\xi} - \mu_t \widetilde{\xi}^\star) \Big)$ for $t_1, t_2 \in B(\mu_s, \mu_t)$, $\phi_i \in [-\pi, \pi)$, $\alpha_i \in (0,1)$ and $\alpha \in \mathbb{R}^+$. Moreover, the metric g_E, which is flat in canonical Cartesian coordinates, is given by

$$g_E = \frac{1}{U^2}\Big((\alpha^2 + \mu_t^2)\mathrm{d}t_1^2 + (\alpha^2 + \mu_s^2)\mathrm{d}t_2^2 + 2\mu_s\mu_t \mathrm{d}t_1\mathrm{d}t_2 + \frac{\mathrm{d}\alpha^2}{\alpha^2 + \mu_s^2 + \mu_t^2} + \sum_{i=1}^p \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right)\big|_{\sum_{i=1}^p \alpha_i^2 = 1} \Big). \tag{4.57}$$

If $n \geq 3$ is odd and ξ_F is in canonical form, $\xi_F = \widetilde{\xi} + \sum_{i=1}^p \mu_i \eta_i$, the coordinates $\{t_1, \phi_i, \alpha, \alpha_i\}$ adapted to $\widetilde{\xi} = \partial_{t_1}$ $\eta_i = \partial_{\phi_i}$ are given by the case of $n+1$ (even) dimensions, for $\tau = 0$ restricted to $t_2 = 0$ (which defines the embedding $\mathbb{E}^n = \{z_2 = 0\} \subset \mathbb{E}^{n+1}$) and cover again the maximal possible domain, given by $\mathbb{E}^n \setminus \Big(\bigcup_{j=1}^p \mathcal{Z}(\eta_j) \cup \mathcal{Z}(\widetilde{\xi}) \Big)$ for $t_1 \in \mathbb{R}$, $\phi_i \in [-\pi, \pi)$, $\alpha_i \in (0,1)$ and $\alpha \in \mathbb{R}^+$ when $\sigma \geq 0$ and $\alpha \in \mathbb{R}^+ \cup \{0\}$ when $\sigma < 0$. Moreover, the metric g_E, which is flat in canonical Cartesian coordinates, is given by the pull-back of (4.57) at $t_2 = 0$ after setting $\tau = 0$. Explicitly g_E is, depending on the sign of σ, given by (4.56) with $\gamma_{\mathbb{S}^{n-2}}$ as in (4.52).

4.7 TT-Tensors

The adapted coordinates derived in Section 4.6 provide a useful tool to solve geometric equations involving CKVFs. In this section we give an example of this in the context of Λ-vacuum spacetimes admitting a smooth null conformal infinity in the $n = 3$ case.

Our aim is to give a simple yet interesting application of the formalism developed in the previous sections. We stress that the methods that we employ here can also be used in the higher dimensional case (with a considerable amount of extra work).

Consider a Riemannian 3-manifold endowed with a conformally flat metric g and let ξ be an arbitrary CKVF of g with its canonical form $\xi = \tilde{\xi} + \eta$. We shall use the KID equation (cf. Theorem 2.35) to obtain the most general TT tensor D satisfying (2.55) for both $\tilde{\xi}$ and η, so we obtain the asymptotic data which generates a spacetime with two commuting symmetries, one of which is axial. As we shall also justify (cf. Remark 4.50) the requirement of one of these symmetries being axial is not very restrictive and the data corresponding of all spacetimes with two commuting symmetries (with none of them necessarily axial) can be obtained straightforwardly.

A CKVF satisfying (2.55) will be called KID vector for short. An important property of KID vectors is that they form a Lie subalgebra of CKVFs, i.e. if ξ, ξ' are KIDs for a given TT tensor D, then $[\xi, \xi']$ is also a KID for D. The problem of obtaining all TT-tensors with generality for a given conformal structure is hard, even in the conformally flat case (see e.g. [16], [145]). Our approach is not completely general as we impose additional equations but is relevant to study spacetimes with symmetries. Also, $n = 3$ corresponds to the physical case of four spacetime dimensions and the class of solutions we obtain necessarily contains the Kerr-de Sitter family of spacetimes, which is one of our main interests in this thesis. Our strategy is to take an arbitrary CKVF ξ, derive its canonical form $\xi_F = \widetilde{\xi} + \mu\eta$, adapt coordinates to $\widetilde{\xi}$ and η and impose the KID equations[7] to $\widetilde{\xi}$ and η.

The problem simplifies notably in the conformal gauge to $g := (U^\epsilon)^2 g_E^\epsilon$ because both $\widetilde{\xi}$ and η become Killing vector fields. From Remark 4.44, we may treat all cases $\sigma < 0$, $\sigma = 0$, $\sigma > 0$ at the same time by using the form of the metric

$$g = \frac{\mathrm{d}z^2}{z^2 - \sigma} + (z^2 - \sigma)\mathrm{d}t^2 + \mathrm{d}\phi^2, \qquad \widetilde{\xi} = \partial_t, \qquad \eta = \partial_\phi. \tag{4.58}$$

We remark that even though we solve the problem by fixing the coordinates and conformal gauge, we shall write the final result in fully covariant form (cf. Theorem 4.47 below). Also notice that, assuming that we have coordinates adapted to two orthogonal CKVFs $\partial_t, \partial_\phi$, and knowing that these vectors are orthogonal, the vanishing of the Cotton tensor reduces to an ODE in z (in the conformal gauge where $\partial_t, \partial_\phi$ are Killing vectors and $g_{\phi\phi} = 1$) which yields a metric of the form of (4.58).

In the conformal gauge of g, the condition that a TT-tensor D satisfies KID equations for both $\widetilde{\xi}$ and η (which is equivalent to imposing that ξ and η are KID vectors) is trivial

[7] In higher dimensions one could impose the KID equations, for $\widetilde{\xi}$ and each η_i still yielding a tractable problem. One can also enlarge the class by supressing some of the KIDs. Obviously, the less KID equations one imposes the more difficult the problem becomes.

in the adapted coordinates obtained in the previous section:

$$\mathcal{L}_{\widetilde{\xi}} D^{AB} = \partial_t D^{AB} = 0, \qquad \mathcal{L}_\eta D^{AB} = \partial_\phi D^{AB} = 0.$$

Thus, D^{AB} are only functions of z. The transversality condition is also quite simple in adapted coordinates:

$$\frac{\mathrm{d}D^{zz}}{\mathrm{d}z} - z\left(\frac{D^{zz}}{z^2-\sigma} + (z^2-\sigma)D^{tt}\right) = 0, \tag{4.59}$$

$$\frac{\mathrm{d}D^{zt}}{\mathrm{d}z} + \frac{2z}{z^2-\sigma}D^{zt} = 0 \tag{4.60}$$

$$\frac{\mathrm{d}D^{z\phi}}{\mathrm{d}z} = 0, \tag{4.61}$$

while the traceless condition imposes

$$g_{AB}D^{AB} = \frac{D^{zz}}{z^2-\sigma} + (z^2-\sigma)D^{tt} + D^{\phi\phi} = 0. \tag{4.62}$$

There are no equations for $D^{t\phi}$ so $D^{t\phi} = h(z)$ with $h(z)$ an arbitrary function. The general solution of equations (4.60) and (4.61) is obtained at once and reads

$$D^{zt} = \frac{K_1}{z^2-\sigma}, \qquad D^{z\phi} = K_2, \qquad K_1, K_2 \in \mathbb{R}.$$

For equations (4.59) and (4.62), we let $D^{zz} =: f(z)$ be an arbitrary function and obtain the remaining components

$$D^{\phi\phi} = -\frac{1}{z}\frac{\mathrm{d}f}{\mathrm{d}y}, \qquad D^{tt} = \frac{1}{z(z^2-\sigma)}\frac{\mathrm{d}f}{\mathrm{d}z} - \frac{f}{(z^2-\sigma)^2}.$$

Summarizing

Lemma 4.46. *In the three-dimensional conformally flat class $[g]$, let ξ_F be a CKVF. Decompose ξ in canonical form $\xi = \widetilde{\xi} + \mu\eta$ and fix the conformal gauge so that g is given by (4.58). Then the most general symmetric TT-tensor D satifying the KID equations for ξ and η simultaneously is, in adapted coordinates $\{z, t, \phi\}$, a combination (with constants) of the following tensors*

$$\begin{aligned}
D_f &:= f\partial_z \otimes \partial_z + \left(\frac{1}{z(z^2-\sigma)}\frac{\mathrm{d}f}{\mathrm{d}z} - \frac{f}{(z^2-\sigma)^2}\right)\partial_t \otimes \partial_t - \frac{1}{z}\frac{\mathrm{d}f}{\mathrm{d}z}\partial_\phi \otimes \partial_\phi,\\
D_h &:= h(\partial_t \otimes \partial_\phi + \partial_\phi \otimes \partial_t),\\
D_{\widetilde{\xi},\chi} &:= \frac{1}{z^2-\sigma}(\partial_z \otimes \partial_t + \partial_t \otimes \partial_z),\\
D_{\eta,\chi} &:= \partial_z \otimes \partial_\phi + \partial_\phi \otimes \partial_z,
\end{aligned}$$

where f and h are arbitrary functions of z.

Having obtained the general solution in a particular gauge, our next aim is to give a (diffeomorphism and conformal) covariant form of the generators in Lemma 4.46. From [99, 100], we know that, for any CKV ξ of any n-dimensional metric g (not necessarily conformally flat) the following tensors are TT w.r.t. to g and satisfy the KID equation with respect to ξ.

$$D_\xi = \frac{1}{|\xi|_g^{n+2}}\left(\xi\otimes\xi - \frac{|\xi|_g^2}{n}g^\sharp\right),$$

where $|\cdot|_g$ denotes the norm w.r.t. g. Thus, we can rewrite D_f as

$$D_f = \left(-2(z^2-\sigma)^{1/2}f + \frac{(z^2-\sigma)^{3/2}}{z}\frac{\mathrm{d}f}{\mathrm{d}z}\right)D_{\widetilde{\xi}} - \left(\frac{f}{z^2-\sigma} + \frac{1}{z}\frac{\mathrm{d}f}{\mathrm{d}z}\right)D_\eta.$$

We now restore the conformal gauge freedom by considering the metric $\widehat{g} = \Omega^2 g$ and $\widehat{D}_f = D_f/\Omega^5$ (cf. Lemma 2.1), for any (positive) conformal factor Ω. Since the tensors $D_{\widetilde{\xi}}, D_\eta$ are already conformal and diffeomorphism covariant, we must impose their multiplicative factors in $\widehat{D}_f$ to be conformal and diffeomorphism invariant. With the gauge freedom restored, the norms of the CKVFs now are

$$|\widetilde{\xi}|_{\widehat{g}} = \Omega\sqrt{z^2-\sigma}, \qquad |\eta|_{\widehat{g}} = \Omega.$$

Then, considering $f =: \sqrt{\mathrm{X}}\widehat{f}(\mathrm{X})$ as function of the conformal invariant quantity $\mathrm{X} = |\widetilde{\xi}|_{\widehat{g}}/|\eta|_{\widehat{g}} = \sqrt{z^2-\sigma}$, one can directly cast $\widehat{D}_f$ in the following form:

$$\widehat{D}_f = \mathrm{X}^4\frac{\mathrm{d}}{\mathrm{dX}}\left(\frac{\widehat{f}(\mathrm{X})}{\mathrm{X}^{3/2}}\right)D_{\widetilde{\xi}} - \frac{1}{\mathrm{X}^2}\frac{\mathrm{d}}{\mathrm{dX}}\left(\mathrm{X}^{3/2}\widehat{f}(\mathrm{X})\right)D_\eta,$$

which is a conformal and diffeomorphism covariant expression. Notice that the expression is symmetric under the interchange $\widetilde{\xi}\leftrightarrow\eta$ because the coefficient of D_η expressed in the variable $\mathrm{Y} = \mathrm{X}^{-1}$ is identical in form to the coefficient of $D_{\widetilde{\xi}}$.

For the tensor $\widehat{D}_h := D_h/\Omega^5$, redifining $h =: \widehat{h}|\widetilde{\xi}|^{-5/2}$, it is immediate to write

$$\widehat{D}_h = \widehat{D}_{\widehat{h}} := \frac{\widehat{h}}{|\eta|_{\widehat{g}}^{5/2}|\widetilde{\xi}|_{\widehat{g}}^{5/2}}(\widetilde{\xi}\otimes\eta + \eta\otimes\widetilde{\xi}), \tag{4.63}$$

which is obviously conformal and diffeomorphism covariant if and only if $\widehat{h}$ is conformal invariant, e.g. considering $\widehat{h}\equiv\widehat{h}(\mathrm{X})$. Observe that the form (4.63) already appeared in Theorem 3.25 for TT tensors in dimension two satisfying the KID equation.

For the remaining tensors $\widehat{D}_{\widetilde{\xi},\chi} := D_{\widetilde{\xi},\chi}/\Omega^5$ and $\widehat{D}_{\eta,\chi} := D_{\eta,\chi}/\Omega^5$, we define a conformal class of vector fields χ, which in the original gauge coincides with $\chi := \partial_z$. This vector is divergence-free $\nabla_A\chi^A = 0$, and this equation is conformally invariant provided the conformal weight of χ is -3 (i.e. for $\widehat{g} = \Omega^2 g$, the corresponding vector is $\widehat{\chi} = \Omega^{-3}\chi$).

We therefore impose this conformal behaviour[8] of χ. The direction of χ is fixed by orthogonality to $\widetilde{\xi}$ and η. The combination of norms that has this conformal weight and recovers the appropriate expression in the gauge of Lemma 4.46 is $|\chi|_{\widehat{g}} := |\widetilde{\xi}|_{\widehat{g}}^{-1}|\eta|_{\widehat{g}}^{-2}$ (note that the orthogonality and norm conditions fix χ uniquely up to an irrelevant sign in any gauge). Thus, we may write

$$D_{\widetilde{\xi},\chi} = \frac{1}{|\widetilde{\xi}|^2_{\widehat{g}}}(\chi \otimes \widetilde{\xi} + \widetilde{\xi} \otimes \chi), \qquad D_{\eta,\chi} = \frac{1}{|\eta|^2_{\widehat{g}}}(\chi \otimes \eta + \eta \otimes \chi),$$

which are conformally covariant expressions (this explains the notation we have used for $D_{\widetilde{\xi},\chi}$ and $D_{\eta,\chi}$, which up to now may have seemed awkward). Therefore, we get to the final result:

Teorema 4.47. *Let ξ be a CKVF of the class of three dimensional conformally flat metrics and let $\xi = \widetilde{\xi} + \mu\eta$ a canonical form. For each conformal gauge, let us define a vector field χ with norm $|\chi|_{\widehat{g}} := |\widetilde{\xi}|_{\widehat{g}}^{-1}|\eta|_{\widehat{g}}^{-2}$, orthogonal to $\widetilde{\xi}$ and η. Then, any TT-tensor satisfying the KID equations (2.55) for $\widetilde{\xi}$ and η is a combination (with constants) of the following tensors:*

$$\widehat{D}_{\widehat{f}} = \mathrm{X}^4 \frac{\mathrm{d}}{\mathrm{dX}}\left(\frac{\widehat{f}(\mathrm{X})}{\mathrm{X}^{3/2}}\right) D_{\widetilde{\xi}} - \frac{1}{\mathrm{X}^2}\frac{\mathrm{d}}{\mathrm{dX}}\left(\mathrm{X}^{3/2}\widehat{f}(\mathrm{X})\right) D_{\eta}, \quad \widehat{D}_{\widehat{h}} = \frac{\widehat{h}}{|\eta|^{5/2}_{\widehat{g}}|\widetilde{\xi}|^{5/2}_{\widehat{g}}}(\widetilde{\xi}\otimes\eta + \eta\otimes\widetilde{\xi}),$$

$$D_{\widetilde{\xi},\chi} = \frac{1}{|\widetilde{\xi}|^2_{\widehat{g}}}(\chi \otimes \widetilde{\xi} + \widetilde{\xi} \otimes \chi), \qquad D_{\eta,\chi} = \frac{1}{|\eta|^2_{\widehat{g}}}(\chi \otimes \eta + \eta \otimes \chi),$$

for arbitrary functions $\widehat{f}$ and $\widehat{h}$ of $\mathrm{X} = |\widetilde{\xi}|_{\widehat{g}}/|\eta|_{\widehat{g}}$.

Observacin 4.48. *The vector field χ defined in this Theorem is divergence-free. This property would have been difficult to guess (and even to prove) in the original Cartesian coordinate system.*

Observacin 4.49. *A corollary of this theorem is that the general solution of the Λ-vacuum Einstein field equation in four dimensions with a smooth conformally flat null infinity and admitting an axial symmetric and a second commuting Killing vector can be parametrized by two functions of one variable and two constants. Recall that in the $\Lambda = 0$ case, the general asympotically flat stationary and axially symmetric solution of the Einstein field equations can be parametrized (in a neighbourhood of spacelike infinity, by two numerable sets of mass and angular multipole moments (satisfying appropriate convergence properties), see [2], [18], [27] for details. There is an intriguing parallelism between the two situations, at least at the level of crude counting of degrees of freedom. This suggests that maybe in the $\Lambda > 0$ case it is possible to define a set of multipole-type moments that characterizes de data at null infinity (and hence the spacetime), at*

[8]This choice may appear somewhat ad hoc at this point. However, the condition of vanishing divergence appears naturaly when studying (for more general metrics) under which conditions a tensor $\xi \otimes W + W \otimes \xi$ is a TT tensor satisfying the KID equation for ξ. We leave this general analysis for a future work.

least in the case of a conformally flat null infinity. For example, the contraction of an arbitrary TT tensor D with of any CKVF gives a conserved current [9, 10]. In particular $D^{\alpha}{}_{\beta}\widetilde{\xi}^{\beta}$ and $D^{\alpha}{}_{\beta}\eta^{\beta}$ integrated over the surface $\mathcal{S}_t = \{t = const.\}$ give finite conserved charges (under suitable assumptions on $\hat{f}$ and $\hat{h}$), which one could attempt relate to energy and/or angular momenta. We shall comment on this again in Chapter 5. This is an interesting problem, but beyond the scope of this thesis.

Observacin 4.50. *It is natural to ask whether Theorem 4.47 is general for TT-tensors admitting two commuting KIDs, $\widetilde{\xi}, \eta$, without the condition of η being conformally axial. In Appendix C of [100] one can explicitly find, for an arbitrary CKVF ξ, the set $\mathcal{C}(\xi)$ of elements that commute with ξ. Then, from a case by case analysis, one concludes that except in one special situation, for any linearly independent pair ξ, ξ', with $\xi' \in \mathcal{C}(\xi)$ it is the case that there is a CAKVF $\eta \in \mathcal{C}(\xi)$ such that $span\{\xi,\eta\} = span\{\xi,\xi'\}$. Thus, all these cases are covered by Theorem 4.47. The exceptional case is when ξ, ξ' are conformal to translations. It is immediate to solve the TT and KID equations for such a case directly in Cartesian coordinates.*

The solution given in Theorem 4.47 provides a large class of initial data, which we know must contain the so-called Kerr-de Sitter-like *class* with conformally flat $\mathscr{I}$ (cf. [100] and Chapter 6 for precise definition and properties of this class), which in turn contains the Kerr-de Sitter *family* of spacetimes. It is interesting to identify this class within the general solution given in Theorem 4.47. The characterizing property of the Kerr-de Sitter-like class in the conformally flat case is $D = D(\xi)$ for some CKVF ξ, where moreover, only the conformal class of ξ matters to determine the family associated to the data. Decomposing canonically $\xi = \widetilde{\xi} + \mu\eta$, a straightforward computation yields

$$D_{\xi} = \frac{\mathrm{X}^5}{(\mathrm{X}^2+\mu^2)^{5/2}} D_{\widetilde{\xi}} + \frac{\mu^2}{(\mathrm{X}^2+\mu^2)^{5/2}} D_{\eta} + \frac{\mu \mathrm{X}^{5/2}}{(\mathrm{X}^2+\mu^2)^{5/2}} \widehat{D}_{\widehat{h}=1},$$

which comparing with Theorem 4.47 yields the following corollary:

Corolario 4.51. *The Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ is determined by the TT-tensor $D_{KdS} = \widehat{D}_{f} + \widehat{D}_{\widehat{h}}$ with*

$$\widehat{f} = -\frac{1}{3}\frac{\mathrm{X}^{3/2}}{(\mathrm{X}^2+\mu^2)^{3/2}}, \qquad \widehat{h} = \mu \frac{\mathrm{X}^{5/2}}{(\mathrm{X}^2+\mu^2)^{5/2}}.$$

It is also of interest to identify the the Kerr-de Sitter family. To that aim we combine the results in [100] to those in the present chapter to show that this family corresponds to $\sigma < 0$. The classification of conformal classes of ξ in [100] is done in terms of the invariants $\widehat{c} = -c_1$ and $\widehat{k} = -c_2$ together with the rank parameter r, where c_1 and c_2 are the coefficients of the characteristic polynomial of the skew-symmetric endomorphism F associated to ξ. In terms of these objects, it is shown in [100] that the Kerr-de Sitter family corresponds to either $\mathcal{S}_1 = \{\widehat{k} > 0,\ \widehat{c} \in \mathbb{R}$ and $r = 2\}$, or $\mathcal{S}_2 = \{\widehat{k} = 0,\ \widehat{c} > 0$ and

$r = 1\}$, the latter defining the Schwarzschild-de Sitter family. It is immediate to verify that, since (cf. Corollary 4.14) $\widehat{k} = -\sigma\mu^2 < 0$ and $\widehat{c} = -\sigma - \mu^2$, then $\mathcal{S}_1 = \{\sigma < 0, \mu \neq 0\}$ and $\mathcal{S}_2 = \{\sigma < 0, \mu = 0\}$ (the condition $\mu \neq 0$ implies $r = 2$ and $\mu = 0$ implies $r = 1$). Thus, in terms of the classification developed in this chapter, the Kerr-de Sitter family corresponds to $\sigma < 0$. It is interesting that in the present scheme we no longer need to specify the rank parameter to identify the Kerr-de Sitter family (unlike in [100]) and that the whole family is represented by an open domain. These results will be recovered and extended to arbitrary dimensions in Chapter 5. We emphazise that the dependence in σ in the solutions given in Theorem 4.47 and Corollary 4.51 is implicit through the norm of $\widetilde{\xi}$.

Chapter 5

Free data at $\mathscr{I}$ and characterization of Kerr-de Sitter in all dimensions

In this chapter we deal with higher dimensional asymptotic initial value problems of general relativity with non-zero cosmological constant. The contents of this chapter are in the preprint [96] which has been submitted for publication and is currently under referee assessment.

In Section 5.1 we study the relation between the Weyl tensor and the n-th order coefficient of the FG expansion in the conformally flat $\mathscr{I}$ case. In order to remain as general as possible, some of our results are derived for Poincaré and FPG metrics (cf. subsection 2.3.2). We start, by giving two identities for the Weyl tensor, which are specially useful here and in Chapter 6. We believe that they may be of independent interest in general relativity. Then in subsection 5.1.1, in the conformally flat $\mathscr{I}$ case, from the n-th order coefficient, we extract a TT term $\mathring{g}_{(n)}$ which coincides, up to a constant, with the electric part of the rescaled Weyl tensor at $\mathscr{I}$, D. We do this in such a way that a boundary metric γ and $\mathring{g}_{(n)}$ are equivalent to γ and the full coefficient $g_{(n)}$, thus providing a geometric characterization of the initial data. In the case $\Lambda < 0$ and Lorentzian signature, it was known [82] that conformal flatness at $\mathscr{I}$ is sufficient for D and $\mathring{g}_{(n)}$ to agree up to a universal constant. We recover and extend this result to general signature and any sign of non-zero Λ. Moreover, we explore whether conformal flatness of $\mathscr{I}$ is also necessary and link this to the validity of long-standing open conjecture that no non-trivial purely magnetic Λ-vacuum spacetimes exist. In addition we study the non-conformally flat $\mathscr{I}$ case. In this situation, the electric part of rescaled Weyl tensor is in general divergent at $\mathscr{I}$, so we determine a quantity constructed from an auxiliary metric which can be used to retrieve $\mathring{g}_{(n)}$ from the electric part of the rescaled Weyl tensor.

In Sections 5.2 and 5.3 we concentrate in the $\Lambda > 0$ case and Lorentzian signature. In Section 5.2 we obtain a KID equation, analogous to the one in Theorem 2.35, which is a necessary and sufficient condition for analytic data at $\mathscr{I}$ to generate spacetimes with symmetries in all dimensions. In addition, the analysis on the data of the FG expansion is used in Section 5.3 to find a geometric characterization of the Kerr-de Sitter metrics in all dimensions in terms of its geometric data at null infinity. The validity of this characterization in even dimension relies on the fact that the data obtained are analytic, so that existence and uniqueness is guaranteed (cf. Theorem 2.39).

5.1 Formulae for the Weyl tensor

Before starting our analysis on the initial data in the FG expansion, we begin by stating and proving some useful results which help to calculating the electric part of the rescaled Weyl tensor. Recall that for a conformal extension $g = \Omega^2 \widetilde{g}$, we denote $\nabla_\mu \Omega = T_\mu$ and $T^\mu = g^{\mu\nu} T_\mu$. In the first part of this chapter, we shall work with the following components of the Weyl tensor, for which calculations are more natural.

Definicin 5.1. For every metric $\widetilde{g}$ and conformal extension $g = \Omega^2 \widetilde{g}$, the ***T*-electric part** of the Weyl tensor is given by the following contraction of the Weyl tensor

$$(C_T)_{ij} := C^\mu{}_{\alpha\nu\beta} T_\mu T^\nu.$$

Note that this definition is only slighlty different from the standard definition of the electric part of the Weyl tensor (cf. Definition 2.32). For our purposes, it is more convenient to use the definition above, which of course only differs from the standard one by a factor. Moreover, for geodesic conformal extensions, the proportionality is just a constant, namely $C_\perp = \lambda^{-1} C_T$ provided T and u point into the same direction. If, in addition, the metric is ACC (cf. Section 2.2) the rescaled Weyl tensors always satisfy

$$(\Omega^{2-n} C_\perp) \mid_{\mathscr{I}} = \lambda^{-1} (\Omega^{2-n} C_T) \mid_{\mathscr{I}}, \tag{5.1}$$

whenever these quantities are finite. Hence, by adding the constant factor λ we can use interchangeably the electric and T-electric parts of the Weyl tensors at $\mathscr{I}$.

Lemma 5.2. *Let $\widetilde{g}$ be a conformally extendable Einstein metric with $\Lambda \neq 0$ and $g = \Omega^2 \widetilde{g}$ a geodesic conformal extension. Then, in Gaussian coordinates $\{\Omega, x^i\}$, the T-electric part of the Weyl tensor reads*

$$(C_T)_{ij} = \frac{\lambda^2}{2} \left(\frac{1}{2} \partial_\Omega g_{ik} g^{kl} \partial_\Omega g_{lj} + \frac{1}{\Omega} \partial_\Omega g_{ij} - \partial^2_\Omega g_{ij} \right), \tag{5.2}$$

where g_Ω is the metric induced by g on the leaves $\{\Omega = const.\}$.

Proof. Inserting (A.2) and (A.5) in equation (A.3) of Appendix A yields

$$\Omega^2(\widetilde{R}_T)_{\alpha\beta} = \lambda\partial_\Omega A_{\alpha\beta} + A^2_{\alpha\beta} - \frac{\lambda}{\Omega}A_{\alpha\beta} - \frac{\lambda}{\Omega^2}\left(T_\alpha T_\beta + \lambda g_{\alpha\beta}\right). \qquad (5.3)$$

Since $\widetilde{g}$ is Einstein with cosmological constant $\Lambda \neq 0$

$$\widetilde{R}_{\mu\alpha\nu\beta} = \widetilde{C}_{\mu\alpha\nu\beta} + 2\lambda\widetilde{g}_{\mu[\nu}\widetilde{g}_{\beta]\alpha},$$

we can relate $\widetilde{R}_T$ and the T-electric part of the Weyl tensors,

$$(C_T)_{\alpha\beta} := C_{\mu\alpha\nu\beta}T^\mu T^\nu = \Omega^2\widetilde{C}_{\mu\alpha\nu\beta}T^\mu T^\nu,$$

by

$$(\widetilde{R}_T)_{\alpha\beta} = \frac{(C_T)_{\alpha\beta}}{\Omega^2} - \lambda\left(\frac{\lambda g_{\alpha\beta} + T_\alpha T_\beta}{\Omega^4}\right). \qquad (5.4)$$

Combining (5.3) and (5.4) gives

$$(C_T)_{\alpha\beta} = \lambda\partial_\Omega A_{\alpha\beta} + A^2_{\alpha\beta} - \frac{\lambda}{\Omega}A_{\alpha\beta},$$

which yields (5.2) after writting $A_{\alpha\beta}$ in terms of the metric by means of expression (A.4). □

Observacin 5.3. *Note that equation (5.2) implies that C_T is always $O(\Omega)$. In particular, in dimension $n = 3$ it is always the case that*

$$(\Omega^{-1}C_T)\mid_{\mathscr{I}} = -\frac{3\lambda^2}{2}g_{(3)}$$

which recovers the well-known result by Friedrich [58] that for positive Λ the electric part of the rescaled Weyl tensor corresponds to the free data specifiable at $\mathscr{I}$.

Assume that $\hat{g}$ satisfies the hypothesis of Lemma 5.2 and that its FG expansion is of the form

$$\hat{g} \sim \sum_{s=0}^{(n-1)/2} g_{(2s)}\Omega^{2s} + \Omega^{n+1}l$$

with n odd and l at least C^2 up to an including $\{\Omega = 0\}$. Equation (5.2) implies that its T-electric Weyl tensor $\hat{C}_T$ only has even powers of Ω up to and including Ω^{n-1} (higher order terms may be even and odd). As a consequence, the tensor $\Omega^{2-n}\hat{C}_T$ splits as a sum of divergent terms at $\Omega = 0$ plus a regular part which vanishes at $\Omega = 0$.

We now present a general result concerning the Weyl tensors of two general metrics related by

$$g = \hat{g} + \Omega^m q \qquad (5.5)$$

for a natural number $m \geq 2$, where q is a symmetric tensor and all three tensors $g, \hat{g}$ and q are at least C^2 in a neighbourhood including $\{\Omega = 0\}$. No further assumptions besides minimal regularity conditions are imposed on g or $\hat{g}$, such as being Einstein or FPG. The result holds therefore in full generality and has potentially a wide range of applications.

Lemma 5.4. *Let $n \geq 3$ and g, $\hat{g}$ be $(n+1)$-dimensional metrics related by (5.5), for $m \geq 2$, with $g, \hat{g}, q$ and Ω at least C^2 in a neighbourhood of $\{\Omega = 0\}$. Assume that $\nabla\Omega$ is nowhere null at $\Omega = 0$. Then their Weyl tensors satisfy the following equation*

$$C^{\mu}{}_{\nu\alpha\beta} = \hat{C}^{\mu}{}_{\nu\alpha\beta} - K_m(\Omega)\frac{n-2}{n-1}(u^{\mu}u_{[\alpha}\mathring{t}_{\beta]\nu} + \mathring{t}^{\mu}{}_{[\alpha}u_{\beta]}u_{\nu}) + \frac{\epsilon K_m(\Omega)}{n-1}(h^{\mu}{}_{[\alpha}\mathring{t}_{\beta]\nu} + \mathring{t}^{\mu}{}_{[\alpha}h_{\beta]\nu}) + o(\Omega^{m-2}) \tag{5.6}$$

with

$$K_m(\Omega) = m(m-1)\Omega^{m-2}F^2,$$

and where $\nabla\Omega = Fu$, for $g(u,u) = \epsilon = \pm 1$, $h_{\alpha\beta}$ is the projector orthogonal to u, all indices are raised and lowered with g, $t_{\alpha\beta} := q_{\mu\nu}h^{\mu}{}_{\alpha}h^{\nu}{}_{\beta}$ while t and $\mathring{t}_{\alpha\beta}$ are its trace and traceless part respectively.

Proof. First notice that the covariant metrics $g^{\sharp}$ and $\hat{g}^{\sharp}$ (associated to g and $\hat{g}$ respectively) must be related by a similar formula

$$g^{\sharp} = \hat{g}^{\sharp} + \Omega^m l,$$

for a contravariant two-tensor l (also C^2 near $\{\Omega = 0\}$, just as $g^{\sharp}, \hat{g}^{\sharp}$), because the presence of any term of order $\Omega^{m'}$, $m' < m$, would imply terms of order $\Omega^{m'}$ in $g^{\sharp}g$ which could not be cancelled. As mentioned in Chapter 2, when using indices, we will omit the $^{\sharp}$ in the metrics and write upper indices. Also, indices in objects with hats are moved with the metric $\hat{g}$ and its inverse and indices of unhatted tensors are moved with g.

Recall the definition of the Weyl tensor (2.10) and define

$$A^{\mu}{}_{\nu\alpha\beta} := -\frac{2}{n-1}(\delta^{\mu}{}_{[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R^{\mu}{}_{|\beta)}) + \frac{2R}{n(n-1)}\delta^{\mu}{}_{[\alpha}g_{\beta]\nu}.$$

Using the relation of Riemann tensors (2.5) for $g^{(1)} = g$ and $g^{(2)} = \hat{g}$ and (2.10) we find

$$C^{\mu}{}_{\nu\alpha\beta} = \hat{C}^{\mu}{}_{\nu\alpha\beta} + B^{\mu}{}_{\nu\alpha\beta} + A^{\mu}{}_{\nu\alpha\beta} - \hat{A}^{\mu}{}_{\nu\alpha\beta} \qquad \text{with} \qquad B^{\mu}{}_{\nu\alpha\beta} := 2\nabla_{[\alpha}S^{\mu}{}_{\beta]\nu} + S^{\sigma}{}_{[\alpha|\nu|}S^{\mu}{}_{\beta]\sigma}$$

where S is the difference of connections tensor (2.4). We also define $B_{\alpha\beta} = B^{\mu}{}_{\alpha\mu\beta}$ and $B = g^{\alpha\beta}B_{\alpha\beta}$ so that

$$R_{\alpha\beta} - \hat{R}_{\alpha\beta} = B_{\alpha\beta}, \qquad R^{\mu}{}_{\beta} - \hat{R}^{\mu}{}_{\beta} = B^{\mu}{}_{\beta} + \Omega^m l^{\mu\alpha}\hat{R}_{\alpha\beta}, \qquad R - \hat{R} = B + \Omega^m l^{\mu\beta}\hat{R}_{\mu\beta}.$$

With these definitions we expand $A^{\mu}{}_{\nu\alpha\beta}$

$$\begin{aligned}
A^{\mu}{}_{\nu\alpha\beta} = &-\frac{2}{n-1}\left(\delta^{\mu}{}_{[\alpha}\hat{R}_{\beta]\nu} - \hat{g}_{\nu[\alpha}\hat{R}^{\mu}{}_{\beta]}\right) + \frac{2}{n(n-1)}\hat{R}\delta^{\mu}{}_{[\alpha}\hat{g}_{\beta]\nu} - \frac{2}{n-1}\Big(\delta^{\mu}{}_{[\alpha}B_{\beta]\nu} \\
&- \hat{g}_{\nu[\alpha}B^{\mu}{}_{\beta]} - \Omega^{m}\left(\hat{g}_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\mu\sigma} + q_{\nu[\alpha}\hat{R}^{\mu}{}_{\beta]} + q_{\nu[\alpha}B^{\mu}{}_{\beta]}\right) - \Omega^{2m}q_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\sigma\mu}\Big) \\
&+ \frac{2B}{n(n-1)}\delta^{\mu}{}_{[\alpha}\hat{g}_{\beta]\nu} + \frac{2\Omega^{m}}{n(n-1)}\left(l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^{\mu}{}_{[\alpha}\hat{g}_{\beta]\nu} + (\hat{R}+B)\delta^{\mu}{}_{[\alpha}q_{\beta]\nu}\right) \\
&+ \frac{2\Omega^{2m}}{n(n-1)}l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^{\mu}{}_{[\alpha}q_{\beta]\nu},
\end{aligned}$$

so defining

$$\begin{aligned}
G^{\mu}{}_{\nu\alpha\beta} := &-\frac{2}{n-1}\Big(\delta^{\mu}{}_{[\alpha}B_{\beta]\nu} - \hat{g}_{\nu[\alpha}B^{\mu}{}_{\beta]} - \Omega^{m}\left(\hat{g}_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\mu\sigma} + q_{\nu[\alpha}\hat{R}^{\mu}{}_{\beta]} + q_{\nu[\alpha}B^{\mu}{}_{\beta]}\right) \\
&- \Omega^{2m}q_{\nu[\alpha}\hat{R}_{\beta]\sigma}l^{\sigma\mu}\Big) + \frac{2B}{n(n-1)}\delta^{\mu}{}_{[\alpha}\hat{g}_{\beta]\nu} + \frac{2\Omega^{m}}{n(n-1)}\Big(l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^{\mu}{}_{[\alpha}\hat{g}_{\beta]\nu} \\
&+ (\hat{R}+B)\delta^{\mu}{}_{[\alpha}q_{\beta]\nu}\Big) + \frac{2\Omega^{2m}}{n(n-1)}l^{\lambda\sigma}\hat{R}_{\lambda\sigma}\delta^{\mu}{}_{[\alpha}q_{\beta]\nu}.
\end{aligned}$$

gives

$$A^{\mu}{}_{\nu\alpha\beta} = \hat{A}^{\mu}{}_{\nu\alpha\beta} + G^{\mu}{}_{\nu\alpha\beta},$$

from which

$$C^{\mu}{}_{\nu\alpha\beta} = \hat{C}^{\mu}{}_{\nu\alpha\beta} + B^{\mu}{}_{\nu\alpha\beta} + G^{\mu}{}_{\nu\alpha\beta}. \tag{5.7}$$

We now analyze the behaviour near $\{\Omega = 0\}$ of the tensors B and G. Using formula (2.4) (with $g^{(2)} = \hat{g} = g - \Omega^{m}q$) we have

$$\begin{aligned}
\hat{S}_{\nu\alpha\beta} := \hat{g}_{\mu\nu}S^{\mu}{}_{\alpha\beta} &= -F\frac{m}{2}\Omega^{m-1}\left(u_{\nu}q_{\alpha\beta} - u_{\alpha}q_{\beta\nu} - u_{\beta}q_{\alpha\nu}\right) \\
&\quad - \frac{\Omega^{m}}{2}\left(\nabla_{\nu}q_{\alpha\beta} - \nabla_{\alpha}q_{\beta\nu} - \nabla_{\beta}q_{\alpha\nu}\right) \\
&= -F\frac{m}{2}\Omega^{m-1}\left(u_{\nu}q_{\alpha\beta} - u_{\alpha}q_{\beta\nu} - u_{\beta}q_{\alpha\nu}\right) + O(\Omega^{m}) = O(\Omega^{m-1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\nabla_{\mu}\hat{S}_{\nu\alpha\beta} = &-F^{2}\frac{m(m-1)}{2}\Omega^{m-2}u_{\mu}\left(u_{\nu}q_{\alpha\beta} - u_{\alpha}q_{\beta\nu} - u_{\beta}q_{\alpha\nu}\right) - \frac{\Omega^{m}}{2}\nabla_{\mu}(\nabla_{\nu}q_{\alpha\beta} \\
&- \nabla_{\alpha}q_{\beta\nu} - \nabla_{\beta}q_{\alpha\nu}) - m\frac{\Omega^{m-1}}{2}\Big(\nabla_{\mu}\big(F\left(u_{\nu}q_{\alpha\beta} - u_{\alpha}q_{\beta\nu} - u_{\beta}q_{\alpha\nu}\right)\big) \\
&+ Fu_{\mu}\left(\nabla_{\nu}q_{\alpha\beta} - \nabla_{\alpha}q_{\beta\nu} - \nabla_{\beta}q_{\alpha\nu}\right)\Big) \\
= &-F^{2}\frac{m(m-1)}{2}\Omega^{m-2}u_{\mu}\left(u_{\nu}q_{\alpha\beta} - u_{\alpha}q_{\beta\nu} - u_{\beta}q_{\alpha\nu}\right) + O(\Omega^{m-1})
\end{aligned}$$

thus

$$\nabla_\mu S^\nu{}_{\alpha\beta} = \nabla_\mu(\hat{g}^{\sigma\nu}\hat{S}_{\sigma\alpha\beta}) = \nabla_\mu\big((g^{\sigma\nu} - \Omega^m l^{\sigma\nu})\hat{Q}_{\sigma\alpha\beta}\big) = g^{\sigma\nu}\nabla_\mu\hat{Q}_{\sigma\alpha\beta} + O(\Omega^{m-1})$$
$$= -F^2\frac{m(m-1)}{2}\Omega^{m-2}u_\mu\left(u^\nu q_{\alpha\beta} - u_\alpha q^\nu{}_\beta - u_\beta q^\nu{}_\alpha\right) + O(\Omega^{m-1}).$$

Therefore, the leading order terms of B are

$$B^\mu{}_{\nu\alpha\beta} = 2\nabla_{[\alpha}S^\mu{}_{\beta]\nu} + O(\Omega^{2m-2})$$
$$= -m(m-1)F^2\Omega^{m-2}\big(u^\mu u_{[\alpha}q_{\beta]\nu} + q^\mu{}_{[\alpha}u_{\beta]}u_\nu\big) + O(\Omega^{m-1}) = O(\Omega^{m-2}).$$

Next, we calculate the leading order terms of D. Notice that since $\hat{g}$ is C^2 at $\{\Omega = 0\}$, its Ricci tensor is well-defined. Moreover B and all its traces are $O(\Omega^{m-2})$. Thus

$$G^\mu{}_{\nu\alpha\beta} = -\frac{2}{n-1}\big(\delta^\mu{}_{[\alpha}B_{\beta]\nu} - \hat{g}_{\nu[\alpha}B^\mu{}_{\beta]}\big) + \frac{2B}{n(n-1)}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu} + O(\Omega^m).$$

If u is non-null, i.e. $\epsilon \neq 0$, it is useful to decompose q in terms parallel and orthogonal to u, i.e.

$$q_{\alpha\beta} = Uu_\alpha u_\beta + 2u_{(\alpha}V_{\beta)} + t_{\alpha\beta}, \qquad \text{with} \quad u^\mu V_\mu = 0,\ u^\mu t_{\mu\nu} = 0.$$

Similarly, the following decomposition of the metric holds

$$g_{\alpha\beta} = \epsilon u_\alpha u_\beta + h_{\alpha\beta}, \tag{5.8}$$

which defines $h_{\alpha\beta}$ as the projector orthogonal to u. In terms of these quantities

$$B^\mu{}_{\nu\alpha\beta} = -m(m-1)\Omega^{m-2}F^2\big(u^\mu u_{[\alpha}t_{\beta]\nu} + t^\mu{}_{[\alpha}u_{\beta]}u_\nu\big) + O(\Omega^{m-1}) \tag{5.9}$$

and

$$B_{\beta\nu} = B^\mu{}_{\beta\mu\nu} = -\frac{1}{2}m(m-1)\Omega^{m-2}F^2(\epsilon t_{\beta\nu} + tu_\beta u_\nu) + O(\Omega^{m-1})$$
$$B = B^\mu{}_\mu = -\frac{1}{2}m(m-1)\Omega^{m-2}F^2(2\epsilon t) + O(\Omega^{m-1})$$

where $t = g^{\alpha\beta}t_{\alpha\beta} = h^{\alpha\beta}t_{\alpha\beta}$. In consequence,

$$G^\mu{}_{\nu\alpha\beta} = -m(m-1)\Omega^{m-2}F^2 \times \Big(\frac{-1}{n-1}\big(\epsilon\delta^\mu{}_{[\alpha}t_{\beta]\nu} + t\delta^\mu{}_{[\alpha}u_{\beta]}u_\nu - \epsilon\hat{g}_{\nu[\alpha}t^\mu{}_{\beta]} - tu^\mu\hat{g}_{\nu[\alpha}u_{\beta]}\big)$$
$$+ \frac{2\epsilon t}{n(n-1)}\delta^\mu{}_{[\alpha}\hat{g}_{\beta]\nu}\Big) + O(\Omega^{m-1}) \tag{5.10}$$

From (5.8) one has

$$\hat{g}_{\alpha\beta} = \epsilon u_\alpha u_\beta + h_{\alpha\beta} + O(\Omega^m), \qquad \delta^\alpha{}_\beta = \epsilon u^\alpha u_\beta + h^\alpha{}_\beta,$$

so that (5.10) reads

$$\begin{aligned}
G^\mu{}_{\nu\alpha\beta} &= -m(m-1)\Omega^{m-2}F^2 \times \\
&\Big(\frac{-1}{n-1}\big(u^\mu u_{[\alpha} t_{\beta]\nu} + \epsilon h^\mu{}_{[\alpha} t_{\beta]\nu} + t h^\mu{}_{[\alpha} u_{\beta]} u_\nu + t^\mu{}_{[\alpha} u_{\beta]} u_\nu + \epsilon t^\mu{}_{[\alpha} h_{\beta]\nu} \\
&+ t u^\mu u_{[\alpha} h_{\beta]\nu}\big) + \frac{2t}{n(n-1)}\left(u^\mu u_{[\alpha} h_{\beta]\nu} + h^\mu{}_{[\alpha} u_{\beta]} u_\nu + \epsilon h^\mu{}_{[\alpha} h_{\beta]\nu}\right)\Big) + O(\Omega^{m-1}) \\
&= -m(m-1)\Omega^{m-2}F^2 \times \\
&\Big(\frac{-1}{n-1}\big(u^\mu u_{[\alpha} t_{\beta]\nu} + t^\mu{}_{[\alpha} u_{\beta]} u_\nu\big) - t\frac{n-2}{n(n-1)}\left(u^\mu u_{[\alpha} h_{\beta]\nu} + h^\mu{}_{[\alpha} u_{\beta]} u_\nu\right) \\
&- \frac{\epsilon}{n-1}\big(h^\mu{}_{[\alpha} t_{\beta]\nu} - \frac{t}{n} h^\mu{}_{[\alpha} h_{\beta]\nu} + t^\mu{}_{[\alpha} h_{\beta]\nu} - \frac{t}{n} h^\mu{}_{[\alpha} h_{\beta]\nu}\big)\Big) + O(\Omega^{m-1}). \qquad (5.11)
\end{aligned}$$

Denote the traceless part of $t_{\alpha\beta}$ by

$$\mathring{t}_{\alpha\beta} = t_{\alpha\beta} - \frac{t}{n} h_{\alpha\beta}.$$

Also, notice that the lower order terms of all expression are $O(\Omega^{m-1}) = o(\Omega^{m-2})$ for $m \geq 2$. Hence, combining (5.7), (5.9) and (5.11) gives equation (5.6).

□

Lemma 5.4 has an interesting application in the context of data at $\mathscr{I}$. Consider a FGP metric $\widetilde{g}$ and a geodesic conformal extension $g = \Omega^2 \widetilde{g}$ and assume that either n is odd or that the obstruction tensor is identically zero if n is even. The FG expansion of this metric allows one to decompose $g = \hat{g} + \Omega^n q$ where $\hat{g}$ is a metric containing all the terms of the expansion of order strictly lower than n (and possibly also higher order terms, but <u>not</u> the term at order n). The rest of terms are collected in $\Omega^n q$. By construction all these objects are $\mathcal{C}^\infty$ up to and including $\Omega = 0$ (here we use the assumption that the obstruction tensor vanishes in the even case). Hence all the hypothesis of Lemma 5.4 holds with $m = n$. From equation (5.6), the T-electric part of the Weyl tensors of g and of $\hat{g}$ are related by

$$(C_T)_{ij} = (\hat{C}_T)_{ij} - \Omega^{n-2}\lambda^2 n(n-2)\mathring{t}_{ij} + o(\Omega^{n-2}), \qquad (5.12)$$

It follows immediately from the FG expansion and the definition of $\mathring{t}$ in Lemma 5.4 that $\mathring{t}_{ij}|_{\Omega=0} = tf(g_{(n)})$, where tf denotes the trace-free part. Note that taking the trace-free part is unnecessary when n is odd because $g_{(n)}$ is always trace-free in that case. The tensor $(\hat{C}_T)_{ij}$ is in general $O(1)$ in Ω, so $\Omega^{2-n}(\hat{C}_T)_{ij}$ will generically contain $[(n-1)/2]$ divergent terms, and the same divergent terms must appear in $\Omega^{2-n}(C_T)_{ij}$ because of

(5.12). Substracting the divergent terms we get

$$\left(\Omega^{2-n}(C_T)_{ij} - \Omega^{2-n}(\hat{C}_T)_{ij}\right)|_{\mathscr{I}} = -\lambda^2 n(n-2) tf(g_{(n)}), \tag{5.13}$$

which provides a general formula for the free data in terms of the electric parts of the Weyl tensors of g and $\hat{g}$ at $\mathscr{I}$. In the case of n odd more can be said because, as justified below Lemma 5.2, the regular part of $(\hat{C}_T)_{ij}$ vanishes at $\mathscr{I}$ and (5.13) establishes that $g_{(n)}$ arises as the value of $(C_T)_{ij}$ at $\mathscr{I}$ once all its divergent terms have been substracted. This last statement is not true in the n even case with zero obstruction tensor, since $\Omega^{2-n}(\hat{C}_T)_{ij}$ may contain regular non-zero terms.

In the next subsection we will prove that in arbitrary dimension and for conformally flat $\mathscr{I}$, $(\hat{C}_T)_{ij}$ vanishes so the T-electric part of the rescaled Weyl tensor of g actually encodes the trace-free part $tf(g_{(n)})$. In the non-conformally flat case $\Omega^{2-n}C_T$ is generically divergent and (5.13) gives a prescription to remove the divergent terms to retrieve the trace-free part. In the context of AdS/CFT correspondence a useful method to remove divergent terms is by means of the so-called renormalization techniques. One method [117, 118, 139] involves decomposing objects in terms of a basis of eigenfunctions of a dilation operator. It would be interesting to analyze whether this method has any relationship with (5.13), or whether it can be used to be make the removal of divergent quantities more explicit.

5.1.1 Free data and the Weyl tensor

The aim of this subsection is to determine the role that the electric part of the rescaled Weyl tensor plays in the FG expansion coefficients, with particular interest in the conformally flat $\mathscr{I}$ case. We will use formula (5.2) to relate the electric part of the rescaled Weyl tensor to the n-th order coefficient $g_{(n)}$ of the FG expansion. We start with some preliminary results about umbilical submanifolds (also called *totally umbilic*). Recall that a nowhere null submanifold $\Sigma \subset M$ is umbilical if its second fundamental form is

$$K_{ij} = f(x^k)\gamma_{ij}$$

for a smooth function f of Σ and γ the induced metric. This property is well-known to be invariant under conformal scalings of total space metric.

Lemma 5.5. *Let $n \geq 2$. Every nowhere null umbilical hypersurface (Σ, γ) of a conformally flat $(n+1)$-manifold $(M, \hat{g})$, where γ is induced by $\hat{g}$, is conformally flat.*

Proof. For $n = 2$ the result is immediate as every 2-surface is locally conformally flat, so let us assume $n \geq 3$. Since umbilical submanifolds remain umbilical w.r.t. to the whole conformal class of the metrics and $\hat{g}$ is conformally flat, then (Σ, γ) is umbilical w.r.t.

the flat metric $g_E = \omega^2 \hat{g}$. In this gauge, the Gauss equation (2.17) and its trace by γ yield

$$R(\gamma)_{ijkl} = -\epsilon(K_{il}K_{jk} - K_{ik}K_{jl}) = -\epsilon(\gamma_{il}\gamma_{jk} - \gamma_{ik}\gamma_{jl})\kappa^2,$$
$$R(\gamma)_{jl} = -\epsilon(K^2_{jl} - KK_{jl}) = -\epsilon(1-n)\kappa^2\gamma_{jl},$$

where $K_{ij} = \kappa\gamma_{ij}$ is the second fundamental form, for $\kappa \in \mathbb{R}$ constant as a consequence of the Codazzi equation and the fact that the ambient metric g_E is flat, and $K^2_{ij} := \gamma^{kl}K_{ik}K_{jl}$, $K := \gamma^{ij}K_{ij}$, $\epsilon = \hat{g}(u,u)$ with u the unit normal to Σ. The Schouten tensor of γ is

$$P(\gamma)_{ij} = \frac{1}{n-2}\left(R(\gamma)_{ij} - \frac{R(\gamma)}{2(n-1)}\gamma_{ij}\right) = \epsilon\frac{\kappa^2}{2}\gamma_{ij}.$$

Thus for $n = 3$ we can calculate the Cotton tensor

$$Y(\gamma)_{ijk} = \nabla_k P(\gamma)_{ij} - \nabla_j P(\gamma)_{ik} = 0,$$

and for $n \geq 4$ the Weyl tensor (cf. (2.11)) is

$$C(\gamma)_{ijkl} = R(\gamma)_{ijkl} - \gamma_{ik}P(\gamma)_{jl} + \gamma_{jk}P(\gamma)_{il} + \gamma_{il}P(\gamma)_{jk} - \gamma_{jl}P(\gamma)_{ik} = 0.$$

By the standard characterization of locally conformally flat metrics by the vanishing of the Cotton ($n = 3$) or Weyl ($n \geq 4$) tensors, the result follows. □

The following results are stated imposing the minimal conditions of differentiability required near $\mathscr{I}$. We remark that for the cases of our interest, namely FGP metrics, these conditions are always satisfied.

Lemma 5.6. *Let g and $\hat{g}$ be metrics related by $g = \hat{g} + \Omega^m q$, where Ω is a defining function of $\Sigma = \{\Omega = 0\}$ and $g, \hat{g}$ and q are C^1 in a neighbourhood of Σ. Then if $m \geq 2$, Σ is umbilical w.r.t. g if and only if it is umbilical w.r.t. $\hat{g}$.*

Proof. The metrics induced by g and $\hat{g}$ at Σ are the same. Assume that Σ is nowhere null. Thus, the property of being umbilical is preserved if the covariant derivatives ∇u and $\hat{\nabla} u$ w.r.t. the Levi Civita connections of g and $\hat{g}$ respectively of the normal unit (which is the same for g and $\hat{g}$) covector $u \in (T\Sigma)^\perp$ coincide at Σ. The covariant associated metric $g^\sharp$ is $g^\sharp = \hat{g}^\sharp + \Omega^m l$ for l a contravariant tensor $O(1)$ in Ω (cf. proof of Lemma 5.4). Then, the Christoffel symbols are

$$\Gamma^\mu_{\alpha\beta} = (\hat{g}^\sharp + \Omega^m l)^{\mu\nu}(\partial_\alpha(\hat{g} + \Omega^m q)_{\beta\nu} + \partial_\beta(\hat{g} + \Omega^m q)_{\alpha\nu} - \partial_\nu(\hat{g} + \Omega^m q)_{\alpha\beta}) = \hat{\Gamma}^\mu_{\alpha\beta} + O(\Omega^{m-1}),$$

from which it follows $\nabla u \mid_\Sigma = \hat{\nabla} u \mid_\Sigma$ if $m \geq 2$. □

Our interest in umbilical submanifolds is because of the (well-known) fact that $\mathscr{I}$ is umbilical for Poincaré or FGP metrics. This results follows immediately from the Einstein equations at $\mathscr{I}$, and will be the base for an interesting decompostion that we will derive later in this section (cf. Proposition 5.11).

Lemma 5.7. *Let $\widetilde{g}$ be a Poincaré or FGP metric for $\mathscr{I} = (\Sigma, [\gamma])$. Then $\mathscr{I}$ is umbilical.*

Proof. For a geodesic conformal extension $g = \Omega^2 \widetilde{g}$, the relation between the Ricci tensors of g and $\widetilde{g}$ is given by (2.8) with $\nabla_\mu \Omega \nabla^\mu \Omega = -\lambda$ (cf. Lemma 2.9). This expression is not defined at $\Omega = 0$, but it is when multiplied by Ω. Rearranging terms this gives

$$\Omega R_{\alpha\beta} + (n-1)\nabla_\alpha \nabla_\beta \Omega + g_{\alpha\beta} \nabla_\mu \nabla^\mu \Omega = \Omega(\widetilde{R}_{\alpha\beta} - \lambda n \widetilde{g}_{\alpha\beta}), \tag{5.14}$$

where we have used $g = \Omega^2 \widetilde{g}$ in the RHS. Since $\widetilde{g}$ is a Poincaré or FGP metric, the RHS vanishes at $\mathscr{I}$. This also implies that $g_{\alpha\beta}$ is at least C^2 at $\mathscr{I}$, so $R_{\alpha\beta}$ is defined at $\mathscr{I}$. In addition writing $\nabla_\alpha \Omega = |\lambda|^{1/2} u_\alpha$, where u is the unit normal of the hypesurfaces $\Sigma_\Omega = \{\Omega = const.\}$, then $\nabla_i \nabla_j \Omega \mid_{\mathscr{I}} = |\lambda|^{1/2} K_{ij}$, where K_{ij} is the second fundamental form of $\mathscr{I}$. Thus, equation (5.14) gives at $\mathscr{I}$

$$(n-1)|\lambda|^{1/2} K_{ij} + f \gamma_{ij} = 0, \qquad \text{with} \qquad f := \nabla_\mu \nabla^\mu \Omega \mid_{\mathscr{I}} .$$

□

For concreteness, in the remainder of this Section, we state and prove our results in the case of positive cosmological constant and Lorentzian signature. However, they also hold with slight modifications for arbitrary signature and non-vanishing cosmological constant (see Remark 5.9 for the specific correspondence).

We start by giving the general form of the FG expansion of the de Sitter spacetime. We refer the reader to [140] for a similar proof in the case of $\lambda < 0$. Also, see a discussion of general case in [43] (in terms of Fefferman-Graham ambient metrics).

Lemma 5.8. *For every Riemmanian conformally flat boundary metric γ of dimension $n \geq 3$ and positive cosmological constant λ, let g be the spacetime metric defined by*

$$g := -\frac{d\Omega^2}{\lambda} + g_\Omega, \qquad g_\Omega := \gamma + \frac{P}{\lambda}\Omega^2 + \frac{1}{4}\frac{P^2}{\lambda^2}\Omega^4 \tag{5.15}$$

where P is the Schouten tensor of γ and $(P^2)_{ij} := P_{il}\gamma^{kl}P_{lj}$. Then $\widetilde{g}_{dS} := \Omega^{-2} g$ is locally isometric to the de Sitter metric.

Proof. De Sitter spacetime is ACC and its boundary metric γ is (by Lemmas 5.7 and 5.5) necessarily conformally flat. Moreover, given the freedom in scaling any conformal extension by an arbitrary positive function, any conformally flat metric is (locally) a boundary metric for the de Sitter space. In addition, as a consequence of this fact and Lemma 2.10 we have that for any conformally flat metric γ, there exists a local coordinate system of de Sitter near null infinity such that the metric is in normal form with respect to γ. The core of the proof is to verify that this ACC metric in normal form w.r.t any such conformally flat γ takes the explicit form (5.15).

Therefore, consider a conformally flat boundary metric γ for a geodesic conformal extension of de Sitter g. Since de Sitter metric is also conformally flat, it follows that the T-electric part of the Weyl tensor $C_T = 0$. Using formula (5.2) we obtain the coefficients of the FG expansion, which give the normal form of g w.r.t. γ. Let us put (5.2) in matrix notation

$$C_T = \frac{\lambda^2}{2}\left(\frac{1}{2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \frac{1}{\Omega}\dot{g}_\Omega - \ddot{g}_\Omega\right) = 0 \quad \Longrightarrow \quad \ddot{g}_\Omega = \frac{1}{2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \frac{1}{\Omega}\dot{g}_\Omega \tag{5.16}$$

where a dot denotes derivative w.r.t. Ω. First we calculate

$$\partial_\Omega(\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega) = \ddot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega - \dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega + \dot{g}_\Omega g_\Omega^{-1}\ddot{g}_\Omega = \frac{2}{\Omega}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega \tag{5.17}$$

where we have used $\partial_\Omega(g_\Omega^{-1}) = -g_\Omega^{-1}\dot{g}_\Omega g_\Omega^{-1}$ for the first equality and expression of $\ddot{g}_\Omega$ in (5.16) for the second equality. Then, taking two derivatives in Ω of (5.16) gives

$$\partial_\Omega^{(4)} g_\Omega = \frac{3}{2\Omega^2}\dot{g}_\Omega g_\Omega^{-1}\dot{g}_\Omega. \tag{5.18}$$

Thus, taking one more derivative in Ω of (5.18) and combining with (5.17) gives $\partial_\Omega^{(5)} g_\Omega = 0$ and hence all higher derivates also vanish. Expression (5.18) evaluated at $\Omega = 0$ gives the expressions for the coefficients (note $\partial_\Omega^{(k)} g_\Omega \mid_{\Omega=0} = k! g_{(k)}$)

$$g_{(4)} = \frac{1}{4} g_{(2)}\gamma^{-1} g_{(2)}.$$

The coefficient $g_{(2)}$ can be directly calculated from the recursive relations for the FG expansion and it always coincides, up to a constant, with the Schouten tensor of the boundary metric (cf. Corollary A.5)

$$g_{(2)} = \frac{\lambda^{-1}}{n-2}\left(Ric(\gamma) - \frac{R(\gamma)}{2(n-1)}\gamma\right) = \frac{P}{\lambda}.$$

Having calculated the only non-zero coefficients $g_{(2)}$ and $g_{(4)}$, it is straightforward to verify that the FG expansion of de Sitter takes the form (5.15).

We have shown that for any choice of conformally flat γ, there exists a de Sitter metric $\widetilde{g}_{dS}$ and a choice of conformal factor Ω with associated Gaussian coordinates such that, defining g as in (5.15), we have $\Omega^{-2} g = \widetilde{g}_{dS}$. Moreover, the metric (5.15) satisfies all the properties stated in Theorem 2.30 with the choice $h = 0$ if $n \neq 4$ and $h = P^2/(2\lambda)^2$ if $n = 4$ (the latter can be straightforwardly verified from the expressions for $\mathfrak{a}$ and $\mathfrak{b}$ in Appendix A). Recall that we are assuming $n \geq 3$ and that the obstruction tensor vanishes identically when γ is conformally flat. Now the lemma follows as a consequence of the uniqueness part of the FG expansion stated in Theorem 2.30. □

Observacin 5.9. *The result generalizes to arbitrary signature and arbitrary sign of* λ *(see [5, 139] for a discussion on the relation between* λ *positive and negative cases), by changing* γ *to a conformally flat metric of signature* (n_+, n_-) *and* g *to conformal to a metric of constant curvature (instead of conformal to de Sitter) and signature* $(n_+ + 1, n_-)$ *if* $\lambda > 0$ *or* $(n_+, n_- + 1)$ *if* $\lambda < 0$*. Taking this into account, Proposition 5.11 and Theorem 5.14 below easily extend to arbitrary signature and arbitrary sign of* λ*.*

Observacin 5.10. *The proof of Lemma 5.8 shows that the condition* $C_T = 0$ *suffices to obtain a metric of the form (5.15) with* γ *in an arbitrary conformal class. The spacetimes satisfying this condition are the so-called "purely magnetic" and they have a long tradition in general relativity (e.g. [15] and references therein). The purely magnetic condition implies restrictive integrability conditions which lead to a conjecture [103] that no Einstein spacetimes exist in the* $n = 3$ *case, besides the spaces of constant curvature. Although no general proof has been found so far, the conjecture has been established in restricted cases such as Petrov type D, and this not only in dimension four, but in arbitrary dimensions [79]. The explicit form (5.15) that the metric must take whenever* $C_T = 0$ *gives an avenue to analyze the conjecture in the case of metrics admitting a conformal compactification.*

Before proving Theorem 5.14, we state and prove an auxiliary result (Proposition 5.11) which is of independent interest since it provides (when combined with Lemma 5.4) a useful decomposition for calculating leading order terms of the Weyl tensor. This will be exploited in the calculation of initial data of spacetimes which admit a smooth conformally flat $\mathscr{I}$ (cf. Corollary 5.17).

Proposicin 5.11. *Assume* $n \geq 3$*. Let* $\widetilde{g}$ *be a FGP metric with* λ *positive for a Riemannian conformal manifold* $\mathscr{I} = (\Sigma, [\gamma])$*. Then* $\mathscr{I}$ *is locally conformally flat if and only if any geodesic conformal extension* $g = \Omega^2 \widetilde{g}$*, admits the following decomposition*

$$g = \hat{g} + \Omega^n q \tag{5.19}$$

where $\hat{g}$ *is conformally isometric to de Sitter and* $\hat{g}$*,* q *and* Ω *are at least* C^1 *in a neighbourhood of* $\{\Omega = 0\}$*.*

Proof. $\mathscr{I}$ is umbilical w.r.t. g and if g admits the decomposition (5.19), by Lemma 5.6 $\mathscr{I}$ is also umbilical w.r.t. $\hat{g}$. Since $\hat{g}$ is conformally flat, Lemma 5.5 implies that $\mathscr{I}$ is also conformally flat. This proves the proposition in one direction.

The converse follows by considering the FGP metric in normal form constructed from a representative γ in the conformal structure of $\mathscr{I}$. By assumption, γ is conformally flat. The terms up to order n are uniquely generated by γ (cf. 2.29). Thus, by Lemma 5.8

$$g = -\frac{d\Omega^2}{\lambda} + \gamma + \frac{P}{\lambda}\Omega^2 + \frac{1}{4}\frac{P^2}{\lambda^2}\Omega^4 + \Omega^n q := \hat{g} + \Omega^n q,$$

where $\hat{g}$ is locally conformally isometric to de Sitter and $\hat{g}$, q and Ω are smooth at $\Omega = 0$ by construction. □

Now observe that for any set of initial data $(\gamma, g_{(n)})$, one can always add a TT term $\mathring{g}_{(n)}$ to $g_{(n)}$ so that $(\gamma, g_{(n)} + \mathring{g}_{(n)})$ gives a new set of initial data. On the other hand, decomposition (5.19) in the conformally flat $\mathscr{I}$ case reads

$$g = -\frac{d\Omega^2}{\lambda} + \gamma + \frac{P}{\lambda}\Omega^2 + \frac{1}{4}\frac{P^2}{\lambda^2}\Omega^4 + \Omega^n(\mathring{g}_{(n)} + \cdots).$$

Therefore, if $n \neq 4$ and $n \geq 3$, then $g_{(n)} = \mathring{g}_{(n)}$ and if $n = 4$, then $g_{(4)} = \overline{g}_{(4)} + \mathring{g}_{(4)}$, where $\overline{g}_{(4)}$ is the term of order four in (5.15). This forces $\mathring{g}_{(n)}$ to be TT, because it is immediate from Lemma 5.8 that de Sitter is given by data $(\gamma, 0)$ for $n \neq 4$ and $n \geq 3$ and by $(\gamma, \overline{g}_{(4)})$ if $n = 4$. Therefore, in the conformally flat $\mathscr{I}$ case we can always extract the TT term $\mathring{g}_{(n)}$.

Definicin 5.12. For a FGP metric admitting a conformally flat $\mathscr{I}$, the term $\mathring{g}_{(n)}$ is called **free part** of $g_{(n)}$.

Observacin 5.13. *Note that a pair $(\gamma, \mathring{g}_{(n)})$ is equivalent to $(\gamma, g_{(n)})$.*

We stress that it would be interesting to give a definition of free part in the general case. This may facilitate a geometric definition of the n-th order coefficient, but also it would help to clearly establish a conformal equivalence of the asymptotic data in the n even case (see the discussion previous to Theorem 2.39).

We may now extend to the case of arbitrary λ the relation between the electric part of the rescalled Weyl tensor and the coefficient $g_{(n)}$ obtained in [82] for the negative λ case. We observe that this extension could be inferred from the general results in [139]. However, our argument is fully conformally covariant and follows directly from the general identity in Lemma 5.4.

Teorema 5.14. *Assume $n \geq 3$ and let $\widetilde{g}$ be a FGP metric with λ positive for a Riemannian conformal manifold $\mathscr{I} = (\Sigma, [\gamma])$. Then, if $\mathscr{I}$ is conformally flat, $\mathring{g}_{(n)}$, the free part of the n-th order coefficient of the FG expansion, coincides, up to a constant, with the T-electric part of the rescaled Weyl tensor at $\mathscr{I}$*

$$-\frac{\lambda^2}{2}n(n-2)\mathring{g}_{(n)} = \Omega^{2-n}C_T \mid_{\mathscr{I}} .$$

Proof. By Proposition 5.11, admitting a smooth conformally flat $\mathscr{I}$ amounts to admitting a decomposition of the form (5.19). Then, by Lemma 5.8, the associated FG expansion has the form

$$g = -\frac{d\Omega^2}{\lambda} + g_\Omega = -\frac{d\Omega^2}{\lambda} + \gamma + \frac{P}{\lambda}\Omega^2 + \frac{1}{4}\frac{P^2}{\lambda^2}\Omega^4 + \Omega^n\mathring{g}_{(n)} + \cdots = \hat{g} + \Omega^n q,$$

where $q\,|_{\mathscr{I}} = \mathring{g}_{(n)}$ and $\hat{g}$ is conformal to de Sitter. Using the formula (5.6) of Lemma 5.4 with $m = n$ and putting $T = |\lambda|^{1/2}u$, with u unit normal, one obtains

$$(C_T)_{\alpha\beta} = -\frac{\lambda^2}{2}n(n-2)\mathring{t}_{\alpha\beta}\Omega^{n-2} + o(\Omega^{n-2})$$

and the Theorem follows.

□

Observacin 5.15. *Although this theorem concentrates on the electric part of the Weyl tensor, its proof (which is based on Lemma 5.4) actually establishes that the full Weyl tensor decays at $\mathscr{I}$ as Ω^{n-2}. In [112], the authors analyze the asymptotic behaviour along null godesics of vacuum solutions with non-zero cosmological constant. Letting r be an affine parameter along the geodesics and assuming a priori that suitable components of the Weyl tensor decay at infinity faster than r^{-2} the authors prove a certain peeling behaviour of the Weyl tensor, with the fastest components decaying like $r^{-(n+2)}$ and the slowest as r^{2-n}. It is clear that there is a connection between the two results. It would be interesting to establish and analyze this connection, which hopefully would lead to a weakening of the a priori decay rate assumed in [112].*

Observacin 5.16. *It is also interesting to comment on the necessary and sufficient conditions for $\mathring{g}_{(n)}$ and $\Omega^{2-n}C_T\,|_{\mathscr{I}}$ to be the same in the case of Einstein metrics. Just like in the proof of Lemma 5.8, if C_T has a zero of order $m > 3$, we can apply formula (5.2) and find*

$$\partial_\Omega^{(5)} g_\Omega = O(\Omega^{m-3}) \tag{5.20}$$

and all coefficients of the FG expansion vanish up to order $g_{(m+2)}$. If, like in the conformally flat case, C_T has a zero of order $n-2$, its leading order term determines $\mathring{g}_{(n)}$. If n is odd, we can construct (cf. Theorem 2.39) two solutions of the $\Lambda > 0$ Einstein field equations $\hat{g}$ and g in a neighbourhood of $\{\Omega = 0\}$, the first one corresponding to the data $(\Sigma, \gamma, 0)$ and the second to the data $(\Sigma, \gamma, g_{(n)})$ where γ belongs to an arbitrary conformal class. By the FG expansion we also have $g = \hat{g} + \Omega^n q$ with $q = g_{(n)} + O(\Omega)$. As a consequence of (5.20), the metric $\hat{g}$ is of the form (5.15) with γ in the given conformal class. Then, from equation (5.2) it follows that $\hat{g}$ is purely magnetic. The converse is also true, namely, if $g = \hat{g} + \Omega^n q$, with $\hat{g}$ a purely magnetic Einstein spacetime and both $\hat{g}$, q and Ω are C^2 near $\{\Omega = 0\}$, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ and $\mathring{g}_{(n)}$ coincide (up to a constant) provided $n > 2$. The proof involves simply taking the T-electric part in (5.6).

This proves that, for Einstein metrics with positive Λ, of dimension $n + 1 \geq 4$ and admitting a conformal compactification, $\mathring{g}_{(n)}$ and $C_T\,|_{\mathscr{I}}$ coincide up to a constant if and only if $g = \hat{g} + \Omega^n q$, where $\hat{g}$ is a purely magnetic spacetime Einstein with non-zero cosmological constant. However, as mentioned in Remark 5.10, it is not clear (and not an easy question) whether purely magnetic Einstein spacetimes are locally isometric to de Sitter or anti-de Sitter spacetimes.

Note that Theorem 5.14 has been proven for metrics of all dimensions $n \geq 3$ and arbitrary signature. An interesting Corollary arises when applying this to the case of $\Lambda > 0$ Einstein metrics of Lorentzian signature and odd n, because the coefficients of the FG expansion γ and $g_{(n)}$ determine initial data at $\mathscr{I}$ which characterize the spacetime metric (cf. Theorem 2.39). In a similar manner, notice that if n is even and the data $(\gamma, g_{(n)})$ are analytic with γ Riemannian (see also Theorem 2.39), the convergence of the FG expansion holds in general for any sign of Λ. Thus, we obtain a characterization result also for this case.

Corolario 5.17. *Let $n \geq 3$ be odd. Then for every asymptotic data $(\Sigma, \gamma, g_{(n)})$ of Einstein's vacuum equations with $\Lambda > 0$ and γ conformally flat, the free part $\mathring{g}_{(n)}$ is up to a constant, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ of the corresponding spacetime. Similarly, if $n \geq 4$ is even, the same statement holds for every analytic data $(\Sigma, \gamma, g_{(n)})$, with γ Riemannian and for any sign of non-zero Λ.*

5.2 KID for analytic metrics

In this section we prove a result (cf. Theorem 5.18) that determines, in the analytic case, the necessary and sufficient conditions for initial data at $\mathscr{I}$ so that the corresponding spacetime metric it generates admits a local isometry. The proof relies in the FG expansion of FGP metrics. Theorem 5.18 below is a generalization to higher dimensions (but restricted to the analytic case) of a known result [116] (cf. Theorem 2.35) in dimension $n = 3$. We focus in the analytic data case, as we shall require convergence of the FG expansion (cf. subsection 2.4.2) in the proof of the theorem. Also, we impose the obstruction tensor to vanish for simplicity and because all cases we shall later deal with satisfy this condition. However, we discuss at the end of this Section the non-zero obstruction case.

Teorema 5.18. *Let Σ be n dimensional with $n \geq 3$ and let $(\Sigma, \gamma, g_{(n)})$ be asymptotic data in the analytic class, with γ Riemannian and if n even $\mathcal{O} = 0$. Then, the corresponding spacetime admits a Killing vector field if and only if there exist a CKVF ξ of $\mathscr{I}$ satisfying the following Killing initial data (KID) equation*

$$\mathcal{L}_\xi g_{(n)} + \frac{n-2}{n} \mathrm{div}_\gamma(\xi) g_{(n)} = 0. \tag{5.21}$$

Proof. Showing that (5.21) is necessary is proved by direct calculation as follows. Let X be a Killing vector field of $\widetilde{g}$ so that

$$0 = \mathcal{L}_X \widetilde{g} = \mathcal{L}_X(\Omega^{-2} g) = -2\frac{X(\Omega)}{\Omega^3} g + \frac{1}{\Omega^2} \mathcal{L}_X g.$$

It follows that on Int(M), X is a conformal Killing vector of g with a specific right-hand side, namely

$$\mathcal{L}_X g_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 2\frac{\mathrm{div}_g X}{n+1} g_{\alpha\beta}, \qquad X(\Omega) = \frac{\Omega}{n+1}\mathrm{div}_g X. \tag{5.22}$$

The following argument [58] shows that X must be extendable to $\mathscr{I}$. The terms $\mathcal{L}_X g_{0\beta}$ of (5.22) imply a linear, homogeneous symmetric hyperbolic system of propagation equations for X. Thus, putting initial data corresponding to X sufficiently close to $\mathscr{I}$ generates a solution whose domain of dependence must reach $\mathscr{I}$ (and possibly beyond if the manifold is extendable across $\mathscr{I}$). Hence X must admit a smooth extension on $\mathscr{I}$, which vanishes near $\mathscr{I}$ only if $X\mid_{\mathscr{I}} = 0$. The rest of equations $\mathcal{L}_X g_{ij}$ are also satisfied at $\mathscr{I}$ by continuity so the extension is a CKVF.

Then, from the second of equations (5.22), it follows that $X(\Omega) = 0$ when $\Omega = 0$, thus X is tangent to $\mathscr{I}$, so we denote $\xi := X\mid_{\mathscr{I}}$. Putting g in normal form $g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega$ it easily follows that $\Gamma^\alpha_{\alpha j} = \Gamma^i_{ij}$. In consequence, expanding $\mathrm{div}_g X$ and evaluating at $\mathscr{I}$ yields

$$\begin{aligned} \mathrm{div}_g X\mid_{\mathscr{I}} &= \partial_\Omega(X(\Omega))\mid_{\mathscr{I}} + \partial_j \xi^j + \Gamma^i_{ij}\mid_{\mathscr{I}} \xi^j \\ &= \frac{1}{n+1}\mathrm{div}_g X\mid_{\mathscr{I}} + \mathrm{div}_\gamma \xi \quad \Longrightarrow \quad \mathrm{div}_g X\mid_{\mathscr{I}} = \frac{n+1}{n}\mathrm{div}_\gamma \xi \end{aligned} \tag{5.23}$$

where we have used the second equation in (5.22). In addition, the normal form gives the following tangent components of the first equation in (5.22):

$$\mathcal{L}_X g_\Omega = \frac{2}{n+1}\mathrm{div}_g X g_\Omega.$$

Evaluating this expression at $\mathscr{I}$ and taking into account (5.23) shows that ξ is a CKVF of γ. Also, using the FG expansion of g_Ω we have the following expansion of $\mathcal{L}_X g_\Omega$:

$$\begin{aligned} \mathcal{L}_X g_\Omega &= X(\Omega)\partial_\Omega g_\Omega + \mathcal{L}_X \gamma + \Omega^2 \mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots \\ &= \frac{\Omega}{n+1}(\mathrm{div}_g X)\partial_\Omega g_\Omega + \mathcal{L}_X \gamma + \Omega^2 \mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots . \end{aligned} \tag{5.24}$$

Therefore

$$\mathcal{L}_X \gamma + \Omega^2 \mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots = \frac{1}{n+1}(\mathrm{div}_g X)(2g_\Omega - \Omega\partial_\Omega g_\Omega). \tag{5.25}$$

Equating n-th order terms and evaluating at $\mathscr{I}$ yields (5.21) after substituting $\mathrm{div}_g X\mid_{\mathscr{I}}$ as in (5.23).

To prove sufficiency, let us first choose the conformal gauge where ξ is a Killing vector field of $\gamma' = \omega^2\gamma$. Thus, the corresponding KID equation for $g'_{(n)}$ becomes:

$$\mathcal{L}_\xi g'_{(n)} = 0. \tag{5.26}$$

The remainder of the proof in this gauge, so we drop all the primes. By Lemma 2.10 there exist a geodesic extension which recovers the representative γ at $\mathscr{I}$. In addition, there exists a unique vector field X, extended from ξ at $\mathscr{I}$, which satisfies $[T, X] = 0$. This is obvious in geodesic Gaussian coordinates $\{\Omega, x^i\}$, because

$$[T, X]^\alpha = -\lambda \partial_\Omega X^\alpha = 0,$$

with initial conditions $X^\Omega \mid_{\Omega=0} = 0$ and $X^i \mid_{\Omega=0} = \xi^i$ has a unique solution $X^\Omega = 0$ and $X^i = \xi^i$. We now prove that X is a Killing vector field of the physical metric $\widetilde{g}$ provided that (5.26) holds.

Consider the normal form metric $g = -\frac{\mathrm{d}\Omega^2}{\lambda} + g_\Omega$. Since $\mathcal{L}_X \mathrm{d}\Omega = \mathrm{d}(X(\Omega)) = 0$, it follows that $\mathcal{L}_X g = \mathcal{L}_X(g_\Omega)$. Using the FG expansion of g_Ω we have

$$\mathcal{L}_X g_\Omega = \mathcal{L}_X \gamma + \Omega^2 \mathcal{L}_X g_{(2)} + \cdots + \Omega^n \mathcal{L}_X g_{(n)} + \cdots .$$

If g is analytic, the value of the coefficients $\mathcal{L}_X g_{(r)}$ determine $\mathcal{L}_X g$ in a neighbourhood of $\mathscr{I}$. These are

$$\partial_\Omega^{(r)} \left(\mathcal{L}_X g_\Omega\right) \mid_{\Omega=0} = \mathcal{L}_\xi \left(\partial_\Omega^{(r)} g_\Omega \mid_{\Omega=0}\right) = r! \mathcal{L}_\xi g_{(r)}.$$

We want to show that all these quantities are identically zero, for which we exploit the Feffeman-Graham recursive construction (cf. Appendix A). The fundamental equation that determines recursively the coefficients of the FG expansion takes the form (cf. Lemma A.3)

$$(n - r - 1) g_{(r+1)} + \left(\mathrm{Tr}_\gamma g_{(r+1)}\right) \gamma = \mathcal{F}^{(r-1)} \tag{5.27}$$

where we denote

$$\mathcal{F}^{(r-1)} := \frac{r}{(r+1)!} \mathcal{L}^{(r-1)} - \frac{1}{r+1} \mathcal{P}^{(r-1)}$$

which by Lemma A.3 is a sum of terms containing products of coefficients up to order $r - 1$ and tangential derivatives thereof, up to second order. We now prove by induction that the Lie derivative of all cofficients vanish provided equation (5.26) is satisfied.

First, the Lie derivative of (5.27), given that ξ is a Killing of γ, yields

$$(n - r - 1) \mathcal{L}_\xi g_{(r+1)} + \left(\mathrm{Tr}_\gamma \mathcal{L}_\xi g_{(r+1)}\right) \gamma = \mathcal{L}_\xi \mathcal{F}^{(r-1)}.$$

Assume by hypothesis that the Lie derivative $\mathcal{L}_\xi$ of all the coefficients up to a certain order r is zero (for the moment we do not assume neither $r < n$ nor $r > n$). The Lie derivative $\mathcal{L}_\xi \mathcal{F}^{(r-1)}$ is a sum where each terms is multiplied by either $\mathcal{L}_\xi g_{(s)}$, $\mathcal{L}_\xi \partial_i g_{(s)}$ or $\mathcal{L}_\xi \partial_i \partial_j g_{(s)}$, with $s \leq r - 1$. Since ξ commutes with $T = -\lambda \partial_\Omega$, we can locally adapt coordinates to both vector fields, namely $\xi = \partial_j$, so that in these coordinates $\mathcal{L}_\xi \partial_i g_{(s)} = \partial_i \mathcal{L}_\xi g_{(s)}$ and $\mathcal{L}_\xi \partial_i \partial_j g_{(s)} = \partial_i \partial_j \mathcal{L}_\xi g_{(s)}$. Thus each term in $\mathcal{L}_\xi \mathcal{F}^{(r-1)}$ contains a

Lie derivative $\mathcal{L}_\xi g_{(s)}$ with $s < r-1$, or a tangential derivative thereof up to second order. Thus by the induction hypothesis $\mathcal{L}_\xi \mathcal{F}^{(r-1)} = 0$. Therefore, it follows that $\mathcal{L}_\xi g_{(r+1)} = 0$

The induction hypothesis can be assumed for $r < n-1$ because it is true for the first term $\mathcal{L}_\xi \gamma = 0$ and we have equations for the succesive terms. For $r = n-1$ the fundamental equation does not determine the term $g_{(n)}$ any longer (this is the reason why this terms is free-data in the FG expansion), so the induction hypothesis cannot go further in principle. But since we are imposing the condition $\mathcal{L}_\xi g_{(n)} = 0$, the induction hypothesis can be extended to any value of r. Therefore, all the derivatives $\mathcal{L}_\xi g_{(r+1)}$ vanish, so if g is analytic $\mathcal{L}_\xi g = 0$. □

In short, the argument behind the proof of Theorem 5.18 relies on the well-known fact that the recursive relations that determine the coefficients of the FG expansion can be cast in a covariant form, so that ultimately all terms can be expressed in terms of γ, its curvature tensor, $g_{(n)}$ and covariant derivatives thereof. Then the Lie derivative of any coefficient must be zero provided that $\mathcal{L}_\xi \gamma = \mathcal{L}_\xi g_{(n)} = 0$. The case with non-zero obstruction tensor, and hence involving logarithmic terms is likely to admit an analogous proof. However, the recursive equations equivalent to (5.27) are not so explicit, because taking derivatives of order higher than n yields an expression which mixes up coefficients of the regular part $g_{(r)}$ and logarithmic terms $\mathcal{O}_{(r,s)}$ of the expansion. These expressions are notably more involved (see e.g. [129]). If one showed that every coefficient $\mathcal{O}_{(r,s)}$ admits a covariant form which only involves geometric objects constructed from γ, $g_{(n)}$ and its covariant derivatives, a similar argument as in the proof above would establish that equation (5.21) is also sufficient for the spacetime to admit a Killing vector field in the case of analytic data with non-vanishing $\mathcal{O}$. It is hard to imagine that this is not the case, and in fact the result should follow from the expressions in [129], but the details need to be worked out. On the other hand, the necessity of (5.21) is true in general and the argument is totally analogous to the one presented above except that equations (5.24) and (5.25) contain also logarithmic terms. We will not discuss this case any further since for the rest of this thesis we shall focus on conformally flat $\mathscr{I}$ (hence $\mathcal{O} = 0$). We plan to come back to this issue in a future work.

5.3 Characterization of generalized Kerr-de Sitter metrics

In this section, we will apply the results obtained in the previous sections to find a characterization of the higher dimensional Kerr-de Sitter metrics. These were firstly formulated in five dimensions in [77] and latter extended to arbitrary dimensions in [70]. Recall that, as mentioned in the introduction of this thesis, the higher dimensional Kerr-de Sitter metrics in [70] were constructed using heuristical arguments. Our characterization here proves that it is indeed a natural extension of Kerr-de Sitter in four spacetime dimensions. We first prove that these metrics admit a smooth conformally flat

$\mathscr{I}$. Then we combine with Theorem 5.14 to determine their initial data at $\mathscr{I}$, which is straightforwardly computable from equation (5.6). The data corresponding to Kerr-de Sitter in all dimensions are analytic. Therefore, by Theorem 2.39, the identification of their data provide a characterization of the metric also in the case of n even. Hence, we perform the analysis simultaneously for n even and odd.

Like in the four dimensional case, the generalized Kerr-de Sitter metrics are $(n+1)$-dimensional Kerr-Schild type metrics. Namely, they admit the following form

$$\widetilde{g} = \widetilde{g}_{dS} + \widetilde{\mathcal{H}}\, \widetilde{k} \otimes \widetilde{k}$$

with $\widetilde{g}_{dS}$ the de Sitter metric, k is a null (w.r.t. to both $\widetilde{g}$ and $\widetilde{g}_{dS}$) field of 1-forms and $\widetilde{\mathcal{H}}$ is a smooth function. In order to unify the n odd and n even cases in one single expression, we define the following parameters

$$p := \left[\frac{n+1}{2}\right] - 1, \qquad q := \left[\frac{n}{2}\right],$$

where note, $p = q$ if n odd and $p+1 = q$ if n even. The explicit expression of the Kerr-de Sitter metrics will be given using the so-called "spheroidal coordinates" $\{r, \alpha_i\}_{i=1}^{p+1}$ (see [70] for their detailed construction), with the redefinition $\rho := r^{-1}$. Strictly speaking, they do not quite define a coordinate system because the α_i functions are constrainted to satisfy

$$\sum_{i=1}^{p+1} \alpha_i^2 = 1.$$

However, it is safe to abuse the language and still call $\{\alpha_i\}$ coordinates. To complete $\{\rho, \alpha_i\}$ to full spacetime coordinates we include $\{\rho, t, \{\alpha_i\}_{i=1}^{p+1}, \{\phi_i\}_{i=1}^{q}\}$. The α_is and ϕ_is are related to polar and azimuthal angles of the sphere respectively and they take values in $0 \leq \alpha_i \leq 1$ and $0 \leq \phi_i < 2\pi$ for $i = 1, \cdots, q$ and (only when n odd) $-1 \leq \alpha_{p+1} \leq 1$. Associated to each ϕ_i there is one rotation parameter $a_i \in \mathbb{R}$. For notational reasons, it is useful to define a trivial parameter $a_{p+1} = 0$ in the case of n odd. The remaining ρ and t lie in $0 \leq \rho < \lambda^{1/2}$ and $t \in \mathbb{R}$. The domain of definition of ρ can be extended (across the Killing horizon) to $\rho > \lambda^{1/2}$, but this is unnecessary in this work since we are interested in regions near $\rho = 0$.

In addition, as we will work with the conformally extended metric $g = \rho^2 \widetilde{g}$, we directly write down the expresions of the following quantities, which admit a smooth extension to $\rho = 0$,

$$\hat{g} = \rho^2 \widetilde{g}_{dS}, \qquad \mathcal{H} = \rho^2 \widetilde{\mathcal{H}}, \qquad k_\alpha = \widetilde{k}_\alpha \tag{5.28}$$

and

$$g = \hat{g} + \mathcal{H}\; k \otimes k. \tag{5.29}$$

We provide below the expression of k_α (as opposed to k^α) because the metrically associated vector field $k^\alpha = g^{\alpha\beta} k_\beta$ is no longer the same as $\widetilde{k}^\alpha = \widetilde{g}^{\alpha\beta} \widetilde{k}_\beta$. In order for the reader to compare with the original publication [70], we remark that the expressions given there are for the "physical" objects $\widetilde{g}_{dS}$, $\widetilde{\mathcal{H}}, \widetilde{k}$, using the coordinates $r := \rho^{-1}$ and denoting $\mu_i := \alpha_i$ instead.

Let us now introduce the functions

$$W := \sum_{i=1}^{p+1} \frac{\alpha_i^2}{1+\lambda a_i^2} \qquad \Xi := \sum_{i=1}^{p+1} \frac{\alpha_i^2}{1+\rho^2 a_i^2}, \qquad \Pi := \prod_{j=1}^{p+1} (1+\rho^2 a_j^2). \tag{5.30}$$

Note that it is thanks to having introduced the spurious quantity $a_{p+1} \equiv 0$ that these expressions take a unified form in the n odd and n even cases. The explicit form of the objects in (5.28) in the case of generalized Kerr-de Sitter are

$$\hat{g} = -W(\rho^2 - \lambda)\mathrm{d}t^2 + \frac{\Xi}{\rho^2-\lambda}\mathrm{d}\rho^2 + \delta_{p,q}\mathrm{d}\alpha_{p+1}^2 + \sum_{i=1}^{q} \frac{1+\rho^2 a_i^2}{1+\lambda a_i^2}\left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right)$$
$$+ \frac{\lambda}{W(\rho^2-\lambda)} \left(\sum_{i=1}^{p+1} \frac{\left(1+\rho^2 a_i^2\right)\alpha_i \mathrm{d}\alpha_i}{1+\lambda a_i^2} \right)^2, \tag{5.31}$$

$$k = W\mathrm{d}t - \frac{\Xi}{\rho^2-\lambda}\mathrm{d}\rho - \sum_{i=1}^{q} \frac{a_i \alpha_i^2}{1+\lambda a_i^2}\mathrm{d}\phi_i, \tag{5.32}$$

$$\mathcal{H} = \frac{2M\rho^n}{\Pi\Xi}, \quad M \in \mathbb{R}. \tag{5.33}$$

The term $\delta_{p,q}$ only appears when $q = p$, i.e. when n is odd. In the case of even n, all terms multiplying $\delta_{p,q}$ simply go away.

The function $\mathcal{H} = O(\rho^n)$ and $k \otimes k = O(1)$. Therefore g decomposes as

$$g = \hat{g} + \rho^n q, \qquad \text{with} \qquad q = \frac{\mathcal{H}}{\rho^n} k \otimes k = O(1).$$

Let γ be the metric induced at $\Sigma = \{\rho = 0\}$ by g. By Lemma 2.10, we can define a geodesic conformal factor Ω such that $\{\Omega = 0\} = \Sigma$ and which induces the same metric γ at Σ. Hence $\Omega = O(\rho)$ and therefore $\mathcal{H} = O(\Omega^n)$ and $q = O(1)$ (in Ω). So by Proposition 5.11 it follows that the generalized Kerr-de Sitter metrics in all dimensions admit a conformally flat $\mathscr{I}$. This can be also verified by explicit calculation. From (5.31), the induced metric at $\{\rho = 0\}$ has the following expression

$$\gamma = \lambda W \mathrm{d}t^2 + \delta_{p,q}\mathrm{d}\alpha_{p+1}^2 + \sum_{i=1}^{q} \frac{\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2}{1+\lambda a_i^2} - \frac{1}{W}\left(\sum_{i=1}^{p+1} \frac{\alpha_i \mathrm{d}\alpha_i}{1+\lambda a_i^2} \right)^2. \tag{5.34}$$

It is useful to define new coordinates

$$\widetilde{\alpha}_i^2 := \frac{1}{W}\frac{\alpha_i^2}{1+\lambda a_i^2},$$

which from (5.30) are restricted to satisfy $\sum_{i=1}^{p+1}\widetilde{\alpha}_i^2 = 1$. Since also $\sum_{i=1}^{p+1}\alpha_i^2 = 1$, this allows us to express W (given in (5.30)) in terms of the tilde coordinates

$$W = \frac{1}{1+\sum_{i=1}^{p+1}\lambda\widetilde{\alpha}_i^2 a_i^2}. \tag{5.35}$$

A direct calculation shows that the metric (5.34) expressed with $\widetilde{\alpha}_i$s takes the form

$$\gamma = W\Big(\lambda \mathrm{d}t^2 + \delta_{p,q}\mathrm{d}\widetilde{\alpha}_{p+1}^2 + \sum_{i=1}^{q}\big(\mathrm{d}\widetilde{\alpha}_i^2 + \widetilde{\alpha}_i^2\mathrm{d}\phi_i^2\big)\Big)\,|_{\sum_{i=1}^{p+1}\widetilde{\alpha}_i^2=1}\,. \tag{5.36}$$

A explicilty flat representative of the conformal class of γ can be obtained using the coordinates

$$x_i := e^{\sqrt{\lambda}t}\widetilde{\alpha}_i\cos\phi_i \qquad y_i := e^{\sqrt{\lambda}t}\widetilde{\alpha}_i\sin\phi_i, \quad i=1,\cdots,q \tag{5.37}$$

together with $z := e^{\sqrt{\lambda}t}\widetilde{\alpha}_{p+1}$ if n odd, which are Cartesian for the following flat metric

$$\gamma_E := \frac{e^{2\sqrt{\lambda}t}}{W}\gamma = \delta_{p,q}\mathrm{d}z^2 + \sum_{i=1}^{q}\big(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\big). \tag{5.38}$$

This form will be used below to determine the conformal class of a conformal Killing vector ξ which we introduce next. Let us denote the projection of k onto $\mathscr{I}$ by

$$\xi_\alpha = \big(k_\alpha + (k_\beta u^\beta)u_\alpha\big)\,|_{\mathscr{I}}$$

with $u_\alpha = \nabla_\alpha\rho/|\nabla\rho|_g$ the unit timelike normal to $\mathscr{I}$. Explicitly

$$\boldsymbol{\xi} = W\mathrm{d}t - \sum_{i=1}^{q}\frac{a_i\alpha_i^2}{1+\lambda a_i^2}\mathrm{d}\phi_i = W\Big(\mathrm{d}t - \sum_{i=1}^{q}\widetilde{\alpha}_i^2 a_i\mathrm{d}\phi_i\Big), \tag{5.39}$$

where (as already used in Chapter 3) in index-free notation we use boldface to distinguish the metrically associated one-form $\boldsymbol{\xi} = \gamma(\xi,\cdot)$ from the CKVF ξ of $\mathscr{I}$. The latter is, using (5.36),

$$\xi = \frac{1}{\lambda}\partial_t - \sum_{i=1}^{q}a_i\partial_{\phi_i}, \tag{5.40}$$

and in Cartesian coordinates (5.37) of γ takes the form

$$\xi = \frac{1}{\sqrt{\lambda}}\widetilde{\xi} - \sum_{i=1}^{q}a_i\eta_i \tag{5.41}$$

where we have introduced

$$\widetilde{\xi} := \delta_{p,q}\partial_z + \sum_{i=1}^{q}x_i\partial_{x_i} + y_i\partial_{y_i}, \qquad \eta_i := x_i\partial_{y_i} - y_i\partial_{x_i}. \tag{5.42}$$

The vector $\widetilde{\xi}$ is a homothety of γ_E and each η_i is a rotation of this metric. Consequently, ξ is a CKVF of γ.

The T-electric part of the rescaled Weyl tensor can be obtained at once from Lemma 5.4 using $\Omega = \rho$ and $m = n$, because by definition $(t\mid_{\mathscr{I}})_{\alpha\beta} = (\mathcal{H}/\rho^n)\mid_{\mathscr{I}} \xi_\alpha\xi_\beta$ and $\mathring{t}\mid_{\mathscr{I}}$ is its trace-free part. Note also that $H/\rho^n\mid_{\mathscr{I}} = 2M$. Moreover, by formula (5.1), the electric part at $\mathscr{I}$, D, differs by a constant factor λ^{-1} from the T-electric part at $\mathscr{I}$. Thus

$$D_{\alpha\beta} = \lambda^{-1}(\rho^{2-n}C^{\mu}{}_{\alpha\nu\beta}\nabla_\mu\rho\nabla^\nu\rho)\mid_{\mathscr{I}} = -\frac{1}{2}\lambda n(n-2)\mathring{t}_{\alpha\beta}\mid_{\mathscr{I}}$$
$$= -M\lambda n(n-2)\left(\xi_\alpha\xi_\beta - \frac{|\xi|^2_\gamma}{n}\gamma_{\alpha\beta}\right).$$

Since, by equation (5.35) above,

$$|\xi|^2_\gamma = W\left(\frac{1}{\lambda} + \sum_{i=1}^{q} a_i^2\widetilde{\alpha}_i^2\right) = \frac{1}{\lambda},$$

D can be cast as

$$D = \kappa D_\xi, \qquad \text{with} \quad \kappa := -\frac{Mn(n-2)}{\lambda^{\frac{n}{2}}}$$

and

$$D_\xi := \frac{1}{|\xi|^{n+2}_\gamma}\left(\boldsymbol{\xi}\otimes\boldsymbol{\xi} - \frac{|\xi|^2_\gamma}{n}\gamma\right). \tag{5.43}$$

Observacin 5.19. *Following the notation in [100], observe that the primary object defining D_ξ is actually a vector field ξ, while in the RHS of (5.43) there appears the one-form $\boldsymbol{\xi} = \gamma(\xi,\cdot)$, obtained by lowering the index of ξ with the metric γ w.r.t. which D_ξ is TT. This notation generalizes to any CKVF ξ' and metric γ' w.r.t. which $D_{\xi'}$ is TT. This will be useful in order to prove conformal properties of D_ξ which depend only on ξ (cf. Lemma 5.21).*

Summarizing, we have proven the following result.

Proposicin 5.20. *The asymptotic data corresponding to the $(n+1)$-dimensional generalized Kerr-de Sitter metrics is given by the class of conformally flat metrics and the class of TT tensors determined by (5.43), where ξ is the vector field by (5.40) when the metric γ is written in the coordinates where (5.36) holds.*

Now suppose that we let ξ to be any CKVF of γ. By direct calculation one shows that the corresponding D_ξ is still TT w.r.t.γ (see the proof in [99] for $n = 3$, which readily generalizes to arbitrary n). The spacetimes corresponding to the class of data obtained in this way constitute a natural extension to arbitrary dimensions of the so-called *Kerr-de Sitter-like class* with conformally flat $\mathscr{I}$, first defined for $n = 3$ in [100] and [99]. The details of this class of spacetimes is precisely the main subject of Chapter 6. What is remarkable from the class of data of the form $(\Sigma, \gamma, \kappa D_\xi)$ with γ locally conformally flat is that, by conformal invariance of data, suitably restricting to a subset of Σ (cf. Remark 2.37), it turns out that the corresponding spacetime depends only on the conformal class of ξ. Thus, by identifying the conformal class of (5.40) we will obtain a complete geometrical characterization of Kerr-de Sitter in all dimensions.

Lemma 5.21. *For asymptotic data* $(\Sigma, \gamma, \kappa D_\xi)$ *and any trasformation* $\phi \in \mathrm{ConfLoc}(\Sigma, \gamma)$*, the following equivalence of data holds*

$$(\Sigma, \gamma, \kappa D_{\phi_\star \xi}) \simeq (\Sigma, \phi^\star \gamma, \phi^\star(\kappa D_{\phi_\star \xi})) = (\Sigma, \omega^2 \gamma, \omega^{2-n} \kappa D_\xi) \simeq (\Sigma, \gamma, \kappa D_\xi), \qquad (5.44)$$

where the tensor $D_{\phi_\star \xi}$ *is given by* (5.43) *with the notation of Remark 5.19.*

Proof. The first equivalence in (5.44) is a consequence of the diffeomorphism equivalence of data and the last one a consequence of the conformal equivalence of data (cf. [100]), so we must verify the equality in the expression. Denote the one-form $\phi_\star(\boldsymbol{\xi}) := \gamma(\phi_\star \xi, \cdot)$. Then, on the one hand we have for every vector field $X \in T\Sigma$

$$(\phi^\star \phi_\star(\boldsymbol{\xi}))(X) = (\phi_\star(\boldsymbol{\xi}))(\phi_\star X) = \gamma(\phi_\star \xi, \phi_\star X) = \omega^2 \gamma(\xi, X) = \omega^2 \boldsymbol{\xi}(X)$$

that is $\phi^\star(\phi_\star(\boldsymbol{\xi})) = \omega^2 \boldsymbol{\xi}$. Moreover $|\phi_\star(\xi)|_\gamma = \sqrt{\gamma(\phi_\star \xi, \phi_\star \xi)} = \omega |\xi|_\gamma$. Thus

$$\begin{aligned} \phi^\star \left(D_{\phi_\star(\xi)} \right) &= \frac{1}{|\phi_\star(\xi)|_\gamma^{n+2}} \left(\phi^\star(\phi_\star(\boldsymbol{\xi}) \otimes \phi_\star(\boldsymbol{\xi})) - \frac{|\phi_\star(\xi)|_\gamma^2}{n} \phi^\star \gamma \right) \\ &= \omega^{-n+2} \frac{1}{|\xi|_\gamma^{n+2}} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{n} \gamma \right) = \omega^{2-n} D_\xi. \end{aligned}$$

□

We now come back to Kerr-de Sitter and identify the conformal class of (5.40). Following the results in Chapter 4, a direct way to do that is to write ξ in any Cartesian coordinate system for any flat representative γ_E in the conformal class of metrics. One then finds its associated skew-symmetric endomorphism (cf. Theorem 2.11 and Remark 2.12) in $\mathbb{M}^{1,n+1}$. By calculating the parameters $\{-\mu_t^2, \mu_s^2, \mu_i^2\}$ if n even or $\{\sigma, \mu_i^2\}$ if n odd, according to Definition 4.10, the conformal class of ξ is directly obtained (cf. Theorem 4.35).

We have already obtained a flat representative γ_E and have introduced corresponding Cartesian coordinates (5.38) . We have also obtained the explicit form of ξ in these coordinates, namely (5.41) and (5.42). Denote the Cartesian coordinates in (5.37) by $\{X\}_{A=1}^n = \{z, \{x_i, y_i\}_{i=1}^q\}$ if n odd and $\{X\}_{A=1}^n = \{x_i, y_i\}_{i=1}^q$ if n even. From equations (5.41), (5.42) the parameters of ξ written as in (2.27) are $\nu = \lambda^{-1/2}$, $\mathrm{a}^A = \mathrm{b}^A = 0$ and $\omega_{AB} = 2a_i {\delta^{2i}}_{[A} {\delta^{2i+1}}_{B]}$ for n odd and $\omega_{AB} = 2a_i {\delta^{2i-1}}_{[A} {\delta^{2i}}_{B]}$ for n even. Thus, from equation (2.26) it is immediate

$$F(\xi) = \begin{pmatrix} 0 & -\lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{pmatrix} \oplus (0) \bigoplus_{i=1}^{p} \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}, \qquad \text{if } n \text{ is odd}$$

$$F(\xi) = \begin{pmatrix} 0 & -\lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{pmatrix} \bigoplus_{i=1}^{p+1} \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}, \qquad \text{if } n \text{ is even,}$$

where recall, this block form is adapted to the following orthogonal decomposition of $\mathbb{M}^{1,n+1}$ as a sum of F-invariant subspaces

$$\mathbb{M}^{1,n+1} = \Pi_t \oplus \mathrm{span}\{e_2\} \bigoplus_{i=1}^{p} \Pi_i, \quad (n \text{ odd}), \qquad \mathbb{M}^{1,n+1} = \Pi_t \bigoplus_{i=1}^{p+1} \Pi_i, \quad (n \text{ even}),$$

where $\Pi_t = \mathrm{span}\{e_0, e_1\}$ for both cases and $\Pi_i = \mathrm{span}\{e_{2i+1}, e_{2i+2}\}$ for n odd and $\Pi_i = \mathrm{span}\{e_{2i}, e_{2i+1}\}$ for n even. Any timelike or null vector $v \in \mathbb{M}^{1,n+1}$ must have non-zero projection onto Π_t, so it may be written $v = v_t + v_s$, with $0 \neq v_t \in \Pi_t$, $v_s \in (\Pi_t)^\perp$. Hence $F(\xi)(v) = F(\xi)(v_t) + F(\xi)(v_s)$, where from the block form it follows that $0 \neq F(\xi)(v_t) \in \Pi_t$ and $F(\xi)(v_s) \in (\Pi_t)^\perp$, thus $F(\xi)(v) = F(\xi)(v_t) + F(\xi)(v_s) \neq 0$. Therefore, $\ker F(\xi)$ is always spacelike or cero. It is straightforward to compute the polynomial $\mathcal{Q}_{F^2}(x)$ in (4.12)

$$\mathcal{Q}_{F^2}(x) = (x + \lambda) \prod_{i=1}^{q} (x - a_i^2)$$

where we may order the indices i, so that the rotation parameters a_i appear in decreasing order $a_1^2 \geq \cdots \geq a_q^2$. Hence, by Definition 4.10 we identify the parameters $\sigma := -\lambda^{-1}$ and $\mu_i^2 := a_i^2$ for n odd and $-\mu_t^2 := -\lambda^{-1}$, $\mu_s^2 := a_1^2$ and $\mu_i^2 := a_{i+1}^2$ for n even. Therefore:

Teorema 5.22. *Let $\widetilde{g}_{KdS}$ be a metric of the generalized Kerr-de Sitter family of metrics in all dimensions, namely given by (5.29) and (5.31), (5.32), (5.33), with cosmological constant λ and q rotation parameters a_i sorted by $a_1^2 \geq \cdots \geq a_q^2$. Then $\widetilde{g}_{KdS}$ is uniquely characterized by the class of initial data (Σ, γ, D_ξ), where γ is conformally flat and D_ξ is a TT tensor of γ of the form (5.43), where ξ is a CKVF of γ whose conformal class is uniquely determined by the parameters $\{\sigma = -\lambda^{-1}, \mu_1^2 = a_1^2, \cdots, \mu_p^2 = a_p^2\}$ if n odd and $\{-\mu_t^2 = -\lambda^{-1}, \mu_s^2 = a_1^2; \mu_1^2 = a_2^2, \cdots, \mu_p^2 = a_{p+1}^2\}$ if n is even.*

We conclude this chapter by comparing our results with previous literature in the $\lambda < 0$ case. The metrics in [70] admit both signs of λ, so one also has the family of Kerr-anti de Sitter metrics in all dimensions. The boundary metric γ for this case is given by (6.43), which is now Lorentzian. The electric part of the rescaled Weyl tensor is $D = \kappa_{|\lambda|} D_\xi$, where $\kappa_{|\lambda|}$ is obtained from κ above by simply replacing $\lambda \to |\lambda|$, and D_ξ is (5.43), with ξ given by (5.40). These data characterize the spacetime asymptotically.

One of the main focus on the Kerr-anti de Sitter metrics has been to study conserved quantities at infinity. There are various notions of conserved charges (see the references in [82], where the different definitions are compared), but all of them depend on a CKVF ξ of $\mathscr{I}$. Thus, associated to each ξ one defines a conserved charge $Q(\xi)$. This provides a useful method to defined mass in this context. There is no complete agreement as to which CKVF at infinity should be used to define mass. See for instance the $n = 4$ cases in [119] and [69] or higher dimensional cases in [38]. From our analysis, in the Kerr-anti de Sitter case the boundary data itself singles out a privileged CKVF, and it is most natural to use this CKVF to define the mass. It turns out that this CKVF agrees with the choice made in [119] for completely different reasons. It would be worth to investigate whether there is a deeper reason for this, perhaps in the context of holography.

Chapter 6

Classification of Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ in all dimensions

The present is the final chapter of this thesis before the conclusions. Here, we shall employ many of the results derived in the thesis so far. The contents of this chapter have been sent to the ArXiv [97] and will be submitted for publication soon.

Firstly, we shall extend the definition of the Kerr-de Sitter-like class (given in [100] in four spacetime dimensions) to arbitrary $(n+1)$-dimensions. We do this in Section 6.1 through a generalization of the Kerr-de Sitter family data in Section 5.3 (note the difference between class a family, specified in Remark 6.2). In Section 6.1 we also define the Kerr-Schild-de Sitter spacetimes as "almost all" (cf. Remark 6.4 below) Kerr-Schild type spacetimes which solve the $\Lambda > 0$ vacuum field equations and admit a smooth conformally flat $\mathscr{I}$. Our main result proves that the Kerr-de Sitter-like class is the same as the Kerr-Schild-de Sitter spacetimes. For that, in Section 6.2 we prove, via direct calculation of the asymptotic data, the inclusion of the Kerr-Schild-de Sitter spacetimes in the Kerr-de Sitter-like class. Section 6.3 establishes the inverse inclusion by resconstructing all metrics which realize data in the Kerr-de Sitter-like class and explicitly proving that they are Kerr-Schild-de Sitter.

As already mentioned, this chapter employs many of the previous results in this thesis. First, the analysis of data in the FG formalism in Chapter 5 is essential to give a definition of the Kerr-de Sitter-like class, because the geometric definition of the asymptotic data is required. Moreover, the results on conformal classes of CKVFs in Chapter 4, which in turn are an extension of the results in Chapter 3, are of fundamental importance for the characterization of each one of the spacetimes inside the Kerr-de Sitter-like class. Moreover, the structure of limits of the conformal classes of CKVFs in Remark

4.25 of subsection 4.4.1 is the core of the structure of limits of spacetimes within the Kerr-de Sitter-like class.

6.1 Kerr-de Sitter-like class & Kerr-Schild-de Sitter spacetimes in all dimensions

In Chapter 5 we derived a geometric characterization of the initial data at $\mathscr{I}$ of the Kerr-de Sitter family of metrics in all dimensions (see Theorem 5.22). Recall, that all data of the form $(\Sigma, \gamma, \kappa D_\xi)$, with (Σ, γ) conformally flat and D_ξ given by (5.43) with ξ a CKVF of γ and κ a real constant, were proven to be uniquely determined by the conformal class of ξ (cf. Lemma 5.21). As mentioned in Chapter 5, this allows one to define a whole class of spacetimes in all dimensions. Actually, this was first described for the $n = 3$ case in [100] and named *Kerr-de Sitter-like class* of spacetimes with conformally flat $\mathscr{I}$. In [100], the class is defined as the set of spacetimes solving the vacuum Einstein equations with positive cosmological constant, admitting a smooth conformally flat[1] $\mathscr{I}$ as well as a Killing vector field ζ, whose associated Mars-Simon tensor vanishes. This definition implies initial data at $\mathscr{I}$ of the form $(\Sigma, \gamma, \kappa D_\xi)$, whith ξ is the restriction to $\mathscr{I}$ of the Killing vector field ζ. As no analogous to the Mars-Simon tensor is known in higher dimension, the extension of the definition of the Kerr-de Sitter-like class requires a different approach, and by the above discussion (and also mentioned in the introduction of this thesis), an obvious possibility is to give the definition directly in terms of its initial data $(\Sigma, \gamma, \kappa D_\xi)$.

Definicin 6.1. The **Kerr-de Sitter-like class of spacetimes with conformally flat** $\mathscr{I}$ are conformally extendable metrics solving the Einstein vacuum field equations with positive cosmological constant, characterized by data $(\Sigma, \gamma, \kappa D_\xi)$, with γ conformally flat and where D_ξ is given by (5.43) with ξ a CKVF of γ and κ a real constant.

Observacin 6.2. *In order to clarify the terminology, the word* **class** *is used to denote a collection of* **families** *of spacetimes, a family being a set of metrics, depending on a number of parameters and sharing certain properties. For example, the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ and $n = 3$ contains [100]: the Kerr-de Sitter family, the Kottler families, a limit case of Kerr-de Sitter with infinite rotation parameter [101] and the Wick-rotated-Kerr-de Sitter spacetime [88]. In this Chapter we shall extend the definition of these families to higher dimensions.*

The main purpose of this chapter, is to prove that the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ contains exactly all Kerr-Schild type spacetimes, solution of the $\Lambda > 0$ vacuum Einstein equations and sharing $\mathscr{I}$ with its background metric. In particular this requires that, as the background metric is de Sitter, $\mathscr{I}$ is conformally flat. Since we shall

[1] The non-conformally flat $n = 3$ case is defined in [99]

not deal with non-conformally flat cases, we shall simply refer to the "Kerr-de Sitter-like class“. Recall that the Kerr-Schild spacetimes (with positive Λ) are of the form

$$\widetilde{g} = \widetilde{g}_{dS} + \widetilde{\mathcal{H}}\, \widetilde{k} \otimes \widetilde{k} \tag{6.1}$$

where $\widetilde{g}_{dS}$ is de Sitter, k is a field of lightlike one-forms (both w.r.t. $\widetilde{g}_{dS}$ and $\widetilde{g}$) and $\widetilde{\mathcal{H}}$ is a smooth function. It is convenient to give a name to the set of spacetimes we shall be dealing with.

Definicin 6.3. The **Kerr-Schild-de Sitter** spacetimes are of the form (6.1), solve the $\Lambda > 0$ vacuum Einstein equations and admit a smooth conformally flat $\mathscr{I}$ such that for some conformal extension $g = \Omega^2 \widetilde{g}$, the tensor $\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k}$ vanishes at $\mathscr{I}$.

Observacin 6.4. *Notice that asking the metric $\widetilde{g}$ to share $\mathscr{I}$ with $\widetilde{g}_{dS}$, implies more than simply $\widetilde{g}$ to have a conformally flat $\mathscr{I}$. In particular, consider a conformal extension such that $\gamma = \Omega^2 \widetilde{g}|_{\mathscr{I}}$ is conformally flat and assume that $\gamma_{dS} := \Omega^2 \widetilde{g}_{dS}|_{\mathscr{I}}$ and $(\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k})|_{\mathscr{I}}$ are well-defined. Since γ_{dS} is conformally flat, one could naively think that $\gamma = \gamma_{dS} + (\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k})|_{\mathscr{I}}$ implies $(\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k})|_{\mathscr{I}} = 0$, which would then imply the condition on $\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k}$ assumed in Definition 6.3. However, there is still room, in principle, for conformally flat metrics of the form $\gamma_{dS} + \mathcal{H}_0 y \otimes y$ with $\mathcal{H}_0 \neq 0$, $y \neq 0$. A simple example is any conformally flat graph in a flat n-dimensional space endowed with Cartesian coordinates $\{x^i\}$, i.e. a hypersurface defined by $x^n = f(x^i)$, such that the induced metric happens to be conformally flat. The induced metric takes precisely the form $\gamma_S = \gamma_{\mathbb{E}^{n-1}} + y \otimes y$, for a flat $(n-1)$-dimensional metric $\gamma_{\mathbb{E}^{n-1}}$ and $y := \mathrm{d}f$ (as an explicit example one can take a hemisphere).*

Thus, it may be possible that a Kerr-Schild metric, solving the $\Lambda > 0$ vaccum Einstein equations and admitting a smooth conformally flat $\mathscr{I}$ has a term $\Omega^2 \widetilde{\mathcal{H}}\widetilde{k} \otimes \widetilde{k}$ surviving at $\mathscr{I}$. It would be interesting to settle whether any $\Lambda > 0$-vacuum solution of this type can exist.

With the above definitions 6.1 and 6.3 we can now state the main result of this chapter:

Teorema 6.5. *A spacetime belongs to the Kerr-de Sitter-like class if and only if it is Kerr-Schild-de Sitter.*

The proof of Theorem 6.5 involves two steps, which respectively we address in sections 6.2 and 6.3 of this chapter. In Section 6.2 we consider Kerr-Schild-de Sitter metrics and compute their initial data, which by Corollary 5.17, correspond to the conformal geometry of (conformally flat) $\mathscr{I}$ and the electric part of the rescaled Weyl tensor D. The tensor D is easily seen to have the form $D = \kappa D_\xi$, with $\kappa \in \mathbb{R}$ and D_ξ given by (6.3) with ξ the projection of k onto $\mathscr{I}$. The main task of this section is to prove that ξ is a CKVF of $\mathscr{I}$. This is a consequence of the Kerr-Schild-de Sitter spacetimes being algebraically special (cf. Proposition 6.9). This proves that every Kerr-Schild-de Sitter spacetime is contained in the Kerr-de Sitter-like class.

The reverse inclusion is proven in Section 6.3. To do that we generate every spacetime in the Kerr-de Sitter-like class by taking advantage of the topological structure of the space of conformal classes of CKVFs. By Lemma 5.21, one conformal class corresponds exactly to one spacetime in the class. Moreover, from the well-posedness of the Cauchy problem, limiting classes in the quotient space of CKVFs will generate limiting spacetimes. All the metrics one obtains are summarized in the next theorem. In order to simplify the statement, we modify slightly the notation with respect to Section 6.3: all primes and hats are dropped and all rotation parameters are denoted by a_i.

Teorema 6.6. *Let be $(\mathcal{M}, \widetilde{g})$ be an $(n+1)$-dimensional manifold and set $p := \left[\frac{n+1}{2}\right] - 1$, and $q := \left[\frac{n}{2}\right]$. Consider the functions W and Ξ of table 6.1 and α_{p+1} obtained from the implicit equation in table 6.1, for a collection of real parameters $\{a_i\}_{i=1}^{p+1}$ with $a_{p+1} = 0$ if n odd or in case b). Then, in the coordinates $\{\rho, t, \{\alpha_i\}_{i=1}^{p+1}, \{\phi_i\}_{i=1}^{q}\}$ taking values in $\phi_i \in [0, 2\pi)$ and the maximal domain where W and Ξ are positive and α_{p+1} is real, every Kerr-Schild-de Sitter metric*

$$\widetilde{g} = \widetilde{g}_{dS} + \widetilde{\mathcal{H}} \widetilde{k} \otimes \widetilde{k}, \qquad \textit{must have} \quad \widetilde{\mathcal{H}} = \frac{2M\rho^{n-2}}{\Xi \prod_{i=1}^{q} (1 + \rho^2 a_i^2)}, \qquad M \in \mathbb{R},$$

k as given in table 6.1 and the de Sitter metric $\widetilde{g}_{dS}$ in the corresponding following form:

a) Kerr-de Sitter family,

$$\widetilde{g}_{dS} = -W \frac{(\rho^2 - \lambda)}{\rho^2} \mathrm{d}t^2 + \frac{\Xi}{\rho^2 - \lambda} \frac{\mathrm{d}\rho^2}{\rho^2} + \delta_{p,q} \frac{\mathrm{d}\alpha_{p+1}^2}{\rho^2} + \sum_{i=1}^{q} \frac{1 + \rho^2 a_i^2}{\rho^2} \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right) + \frac{(\rho^2 - \lambda)}{\lambda W \rho^2} \frac{\mathrm{d}W^2}{4}.$$

b) $\{a_i \to \infty\}$-limit-Kerr-de Sitter,

$$\widetilde{g}_{dS} = \frac{\lambda \alpha_{p+1}^2}{\rho^2} \mathrm{d}t^2 - \frac{\Xi}{\lambda} \frac{\mathrm{d}\rho^2}{\rho^2} + \delta_{p+1,q} \frac{\alpha_{p+1}^2 \mathrm{d}\phi_q^2}{\rho^2} + \sum_{i=1}^{p} \frac{1 + \rho^2 a_i^2}{\rho^2} \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right) + \left(\frac{1}{\lambda} + \frac{\sum_{i=1}^{p} \beta_i^2}{\rho^2 \hat{\alpha}_{p+1}^2}\right) \mathrm{d}\alpha_{p+1}^2 - \frac{2\mathrm{d}\alpha_{p+1}}{\rho^2 \alpha_{p+1}} \left(\sum_{i=1}^{p} \alpha_i \mathrm{d}\alpha_i\right).$$

c.1) Wick-rotated-Kerr-de Sitter for n even,

$$\widetilde{g}_{dS} = \frac{\lambda W}{\rho^2} \mathrm{d}t^2 - \frac{\Xi}{\lambda} \frac{\mathrm{d}\rho^2}{\rho^2} + \sum_{i=1}^{q} \frac{1 + \rho^2 a_i^2}{\rho^2} \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right) - \frac{1}{W\rho^2} \frac{\mathrm{d}W^2}{4}.$$

c.2) Wick-rotated-Kerr-de Sitter for n odd,

$$\widetilde{g}_{dS} = W \frac{(\rho^2 + \lambda)}{\rho^2} \mathrm{d}t^2 - \frac{\Xi}{\rho^2 + \lambda} \frac{\mathrm{d}\rho^2}{\rho^2} - \frac{\mathrm{d}\alpha_{p+1}^2}{\rho^2} + \sum_{i=1}^{p} \frac{1 + \rho^2 a_i^2}{\rho^2} \left(\mathrm{d}\alpha_i^2 + \alpha_i^2 \mathrm{d}\phi_i^2\right).$$

Case	Constraint on $\{\alpha_i\}$	W	Ξ	$\widetilde{k}$
a)	$\sum_{i=1}^{p+1}(1+\lambda a_i^2)\alpha_i^2 = 1$	$\sum_{i=1}^{p+1}\alpha_i^2$	$\sum_{i=1}^{p+1}\frac{1+\lambda a_i^2}{1+\rho^2 a_i^2}\alpha_i^2$	$W\mathrm{d}t - \frac{\Xi}{\rho^2-\lambda}\mathrm{d}\rho - \sum_{i=1}^{q} a_i\alpha_i^2\mathrm{d}\phi_i$
b)	$\alpha_{p+1}^2 + \sum_{i=1}^{p}\lambda a_i^2\alpha_i^2 = 1$	α_{p+1}^2	$\alpha_{p+1}^2 + \sum_{i=1}^{p}\frac{\lambda a_i^2}{1+\rho^2 a_i^2}\alpha_i^2$	$W\mathrm{d}t + \frac{\Xi}{\lambda}\mathrm{d}\rho - \sum_{i=1}^{p} a_i\alpha_i^2\mathrm{d}\phi_i$
c.1)	$\sum_{i=1}^{p+1}\lambda a_i^2\alpha_i^2 = 1$	$\sum_{i=1}^{p+1}\alpha_i^2$	$\sum_{i=1}^{p+1}\frac{\lambda a_i^2}{1+\rho^2 a_i^2}\alpha_i^2$	$\frac{\Xi}{\lambda}\mathrm{d}\rho - \sum_{i=1}^{q} b_i\alpha_i^2\mathrm{d}\phi_i$
c.2)	$\alpha_{p+1}^2 - \sum_{i=1}^{p}(1-\lambda a_i^2)\alpha_i^2 = 1$	$\alpha_{p+1}^2 - \sum_{i=1}^{p}\alpha_i^2$	$\alpha_{p+1}^2 - \sum_{i=1}^{p}\frac{1-\lambda a_i^2}{1+\rho^2 a_i^2}\alpha_i^2$	$W\mathrm{d}t + \frac{\Xi}{\rho^2+\lambda}\mathrm{d}\rho + \sum_{i=1}^{q} a_i\alpha_i^2\mathrm{d}\phi_i$

TABLE 6.1: *Functions defining the Kerr-Schild-de Sitter families.*

Before starting with the proof of Theorem 6.5 we shall give a refinement of Proposition 5.11 of Chapter 5. This refinement (given below in Proposition 6.7) is relevant here because the Kerr-Schild structure of the metrics entails a decomposition very similar, but not quite the same, as the one given in Proposition 5.11. For the sake of simplicity, we restrict ourselves to conformally extendable Einstein metrics $\widetilde{g}$ for Λ positive and a geodesic conformal extension $g = \Omega^2\widetilde{g}$.

First, we give a refinement of the decomposition in Proposition 5.11 for FGP metrics with conformally flat $\mathscr{I}$, which follows from the next discussion. Lemma 5.8 gives the FG expansion of metrics conformally isometric to de Sitter, but from property 2 of Lemma 2.29, it also determines the terms up to order n of the FG expansion of any metric admitting a smooth conformally flat $\mathscr{I}$. Consequently, for any such metric, the terms generated exclusively by the boundary metric γ stop at fourth order. This implies that for $n = 3$, a conformally flat γ generates a term of order $n+1 = 4$, which is not only independent on the n-th (i.e. third) order one by property 1 of Lemma 2.29, but actually must take the form $g_{(4)} = P^2/(4\lambda^2)$ by Lemma 5.8. On the other hand, for $n > 3$, the $n+1 > 4$ order term only depends on γ by property 2 of Lemma 2.29. Hence, by Lemma 5.8 it must be zero. That is, if $\widetilde{g}$ is an Einstein metric admitting a smooth conformally flat $\mathscr{I}$, then for every geodesic conformal extension $g = \Omega^2\widetilde{g}$, the FG expansion yields the following decomposition

$$g = \overline{g} + Q, \tag{6.2}$$

where $\overline{g}$ is of the form (5.15) (thus conformally isometric to de Sitter) and Q is both $O(\Omega^n)$ and has no term of order Ω^{n+1} (when $n = 3$ this term exists in g but it is included in $\overline{g}$).

On the other hand, by Proposition 5.11 for conformally extendable metrics admitting a decomposition of the form

$$g = \widehat{g} + \widehat{Q}, \tag{6.3}$$

with $\widehat{g}$ conformally isometric to de Sitter and $\widehat{Q} = O(\Omega^n)$, then $\mathscr{I}$ is conformally flat. One must be careful with the fact that $\widehat{g}$ being conformally isometric to de Sitter does not mean that it takes the form (5.15) for the conformal factor Ω which is geodesic for g. Indeed, $\widehat{g}$ does admit an expansion of the form (5.15) for some conformal factor $\widehat{\Omega}$ geodesic w.r.t. $\widehat{g}$, but in general this conformal factor is different to Ω. Thus, decomposition (6.2) is a very particular decomposition for metrics admitting a smooth conformally flat $\mathscr{I}$, while decomposition (6.3) is a sufficient condition for g to admit a conformally flat $\mathscr{I}$. Obviously, a metric which can be decomposed as in (6.3) can also be decomposed as in (6.2), but these decompositions do not in general coincide. Indeed, in general $\widehat{g} \neq \overline{g}$.

Both decompositions (6.2) and (6.3) will be used in this section, so we summarize the above discussion in the following Proposition:

Proposicin 6.7. *Let $\widetilde{g}$ be an $n \geq 3$ dimensional conformally extendable Λ-vacuum Einstein, with $\Lambda > 0$ and let $g = \Omega^2 \widetilde{g}$ be a geodesic conformal extension. Then*

a) If $\mathscr{I}$ is conformally flat, then g admits a decomposition of the form (6.2) with $\overline{g}$ of the form (5.15) and $Q = O(\Omega^n)$ with no terms in Ω^{n+1}.

b) If g admits a decomposition of the form (6.3), with $\widehat{g}$ conformally isometric to de Sitter and $\widehat{Q} = O(\Omega^n)$, then $\mathscr{I}$ is conformally flat.

Observacin 6.8. *As mentioned in subsection 5.1.1, note that by construction, the leading order term of Q in decomposition (6.2) is precisely $\mathring{g}_{(n)}$, the free part of the n-th order coefficient. Recall that this equals $g_{(n)}$ if n odd or if $n > 4$ even. For $n = 4$ $g_{(4)} = \overline{g}_{(4)} + \mathring{g}_{(4)}$, with $\overline{g}_{(4)} = P^2/4$ (cf. equation (5.15)).*

6.2 Kerr-Schild-de Sitter $\subset$ Kerr-de Sitter-like class

In this section we prove the inclusion of the Kerr-Schild-de Sitter spacetimes in the Kerr-de Sitter-like class. This is done by direct calculation of the data at spacelike $\mathscr{I}$ of the Kerr-Schild-de Sitter spacetimes and by showing that the vector field ξ at $\mathscr{I}$ that arises in the expression of D_ξ is in fact a CKVF of γ.

A key ingredient for this result is that all vacuum Kerr-Schild spacetimes are algebraically special in the Petrov classification. This was proven with Minkowski background in [113] and with (Anti-)de Sitter background in [107]. Recall that the Petrov classification is an algebraic classification of the Weyl tensor based on the vanishing of the components with certain boost weight, as we summarize next. In the case of arbitrary dimension this classification was developed in [33, 34, 104, 111] to which we refer for further details (see also the review [114]). Consider a null frame of vectors $\{\widetilde{k}, \widetilde{l}, \widetilde{m}_{(i)}\}$ for $i = 1, \cdots, n-1$ (whose indices are raised/lowered with $\widetilde{g}$), i.e. a frame satisfying

$$\widetilde{k}^\alpha \widetilde{k}_\alpha = \widetilde{l}^\alpha \widetilde{l}_\alpha = \widetilde{k}^\alpha \widetilde{m}_{(i)\alpha} = 0, \qquad \widetilde{k}^\alpha \widetilde{l}_\alpha = -1, \qquad \widetilde{m}^\alpha_{(i)} \widetilde{m}_{(j)\alpha} = \delta_{ij}. \tag{6.4}$$

This frame maintains its properties (6.4) under the following set of boost transformations

$$\widetilde{k}' = b\widetilde{k}, \qquad \widetilde{l}' = b^{-1}\widetilde{l}, \qquad \widetilde{m}'_{(i)} = \widetilde{m}_{(i)},$$

for every real non-zero parameter b. Thus, the components of the Weyl tensor C expressed in this frame have "boost weight" depending on the number of contractions with $\widetilde{k}$, $\widetilde{l}$ and $\widetilde{m}_{(i)}$. Namely, $+1$ for each contraction with $\widetilde{k}$; -1 for each one with $\widetilde{l}$; and 0 for each one with $\widetilde{m}_{(i)}$. From the symmetries of the Weyl tensor, the maximum boost weight of a component is $+2$ and the minimum is -2. The classification proceeds by looking for vectors $\widetilde{k}$ such that the highest boost weight components vanish. One such $\widetilde{k}$ (when it exists) is called a *Weyl aligned null direction* (WAND) and if the components of boost weight 1 or lower also vanish, $\widetilde{k}$ is called a *multiple* WAND. A spacetime which admits a multiple WAND is said to be *algebraically special*.

It turns out [107] that all $\Lambda > 0$-vacuum Kerr-Schild spacetimes are algebraically special. Hence, for this section, the following result will be key:

Proposicin 6.9 ([107])**.** *Kerr-Schild-de Sitter spacetimes* (6.1) *are algebraically special, with $\widetilde{k}$ a multiple WAND satisfying*

$$\widetilde{C}_{\mu\alpha\nu\beta}\widetilde{k}^{\mu}\widetilde{k}^{\nu}\widetilde{m}^{\alpha}_{(i)}\widetilde{m}^{\beta}_{(j)} = \widetilde{C}_{\mu\alpha\nu\beta}\widetilde{k}^{\mu}\widetilde{k}^{\nu}\widetilde{l}^{\alpha}\widetilde{m}^{\beta}_{(i)} = C_{\mu\alpha\nu\beta}\widetilde{k}^{\mu}\widetilde{m}^{\alpha}_{(i)}\widetilde{m}^{\nu}_{(j)}\widetilde{m}^{\beta}_{(k)} = 0,$$

for a suitable null frame $\{\widetilde{k}, \widetilde{l}, \widetilde{m}_{(i)}\}$. Moreover, $\widetilde{k}$ is geodesic, so after rescaling if necessary, it satisfies

$$\widetilde{k}^{\alpha}\widetilde{\nabla}_{\alpha}\widetilde{k}_{\beta} = 0. \tag{6.5}$$

We shall assume for now on that $\widetilde{k}$ has been scaled so that (6.5) holds.

Let $\widetilde{g}$ be a Kerr-Schild-de Sitter spacetime and consider a geodesic conformal extension $g = \Omega^2\widetilde{g}$. Then, the conformal metric and its associated contravariant metric $g^{\sharp}$ are

$$g_{\alpha\beta} = \Omega^2\widetilde{g} = \widehat{g}_{\alpha\beta} + \mathcal{H}k_{\alpha}\, k_{\beta}, \qquad g^{\alpha\beta} = \Omega^{-2}\widetilde{g}^{\alpha\beta} = \widehat{g}^{\alpha\beta} - \mathcal{H}k^{\alpha}k^{\beta}, \tag{6.6}$$

where $\widehat{g} := \Omega^2\widetilde{g}_{dS}$, $\mathcal{H} := \Omega^2\widetilde{\mathcal{H}}$ and $k_{\alpha} = \widetilde{k}_{\alpha}$ is a field of one-forms whose metrically associated vector field k^{α} by g has components $k^{\alpha} = g^{\alpha\beta}k_{\beta} = \Omega^{-2}\widetilde{g}^{\alpha\beta}\widetilde{k}_{\beta} = \Omega^{-2}\widetilde{k}^{\alpha}$, where $\widetilde{k}^{\alpha}$ is the vector field metrically associated to $\widetilde{k}_{\alpha}$ by $\widetilde{g}$. Moreover, remind the notation $T_{\mu} = \nabla_{\mu}\Omega$, $T^{\mu} = g^{\mu\nu}T_{\nu}$ and u denotes the unit normal along T. We also recall the well-known property that k_{α} is geodesic w.r.t. g if and only if $\widetilde{k}_{\alpha}$ is geodesic w.r.t. $\widetilde{g}$. Indeed (see the change of connections tensor (2.6))

$$k^{\alpha}\nabla_{\alpha}k_{\beta} = k^{\alpha}\widetilde{\nabla}_{\alpha}k_{\beta} - Q^{\mu}{}_{\alpha\beta}k^{\alpha}k_{\mu} = k^{\alpha}\widetilde{\nabla}_{\alpha}k_{\beta} = \Omega^{-2}\widetilde{k}^{\alpha}\widetilde{\nabla}_{\alpha}\widetilde{k}_{\beta}. \tag{6.7}$$

Thus combining equation (6.7) with Proposition 6.9, k must be geodesic w.r.t. g. In addition, the conformal invariance of the Weyl tensor implies that k is a multiple WAND for the Weyl tensor of $\widetilde{g}$ if and only if it is a WAND for the Weyl tensor of g. That is, by Proposition 6.9 and the above discussion, k_α is also a geodesic multiple WAND for g. In what follows, it will be useful to decompose k in tangent and normal components to a timelike unit vector u. Specifically, given one such u, we write

$$k_\alpha = s(u_\alpha + y_\alpha), \tag{6.8}$$

which defines both the scalar s and the spacelike unit vector y perpendicular to u. Except in the trivial case that the Kerr-Schild metric is identical to the backgroud metric, it is clear that $\mathcal{H} k \otimes k$ cannot be identically zero. We let U be a domain of the physical spacetime $\widetilde{\mathcal{M}}$ where this quantity is not zero. We are only interested in the case where $\overline{U}$ intersects $\mathscr{I}$ as otherwise the free-data $\mathring{g}_n$ is identically zero, and the Kerr-Schild metric would be identical to the background metric in some neighbourhood of $\mathscr{I}$. Since k is geodesic, affinely parametrized and nowhere zero in (U, g), it must extend smoothly and nowhere zero to $\mathscr{I} \cap \partial U$. This is because g-null geodesics starting sufficiently close to $\mathscr{I}$ with non-zero tangent reach $\mathscr{I}$ (smoothly). Since the tangent vector to the geodesic cannot vanish anywhere along the curve, we conclude that the covector k is nowhere zero in $\mathscr{I} \cap \partial U$. From now on we shall work on the manifold with boundary $\overline{U}$ so that its infinity (still called $\mathscr{I}$) is such that k is nowhere vanishing there.

In the next lemma, we summarize the important properties of k w.r.t. to the conformal metric g

Lemma 6.10. *Let $\widetilde{g}$ be a Kerr-Schild-de Sitter metric and let $g = \Omega^2 \widetilde{g}$ be a conformal extension. Assume that $\widetilde{g}$ is not identically equal to the background metric in some neighbourhood of $\mathscr{I}$. Then, after restricting $\widetilde{\mathcal{M}}$ if necessary, k extends smoothly and nowhere zero to $\mathscr{I}$ and it is both geodesic affinely parametrized w.r.t. to g*

$$k^\alpha \nabla_\alpha k_\beta = 0$$

and a multiple WAND with

$$C_{\mu\alpha\nu\beta} k^\mu k^\nu m^\alpha_{(i)} m^\beta_{(j)} = C_{\mu\alpha\nu\beta} k^\mu k^\nu l^\alpha m^\beta_{(i)} = C_{\mu\alpha\nu\beta} k^\mu m^\alpha_i m^\nu_{(j)} m^\beta_{(k)} = 0,$$

for a suitable null frame $\{k, l, m_{(i)}\}$ for g.

The Kerr-Schild ansatz gives a decomposition for the metrics (6.6) similar to the one in (6.3), where, however, $\widehat{Q} = \mathcal{H} k \otimes k$ is in principle not necessarily $O(\Omega^n)$. We now prove that Definition 6.3 forces that necessarily $\mathcal{H} = O(\Omega^n)$. In the following, we use the same name for a geometric object and its restriction to $\mathscr{I}$ (we let the context clarify the meaning). This applies in particular to the vector y.

Lemma 6.11. *Let $\widetilde{g}$ be a Kerr-Schild de Sitter spacetime and consider a geodesic conformal extension $g = \Omega^2 \widetilde{g}$ as in (6.6), inducing a (conformally flat) metric γ at $\mathscr{I}$.*

Then, $\mathcal{H} = O(\Omega^n)$ *and the electric part of the rescaled Weyl tensor at* $\mathscr{I}$ *is*

$$D_{\alpha\beta} = \mathcal{F}\left(y_\alpha y_\beta - \frac{1}{n}\gamma_{\alpha\beta}\right), \tag{6.9}$$

where the function $\mathcal{F}$ *at* $\mathscr{I}$ *is given by* $(\Omega^{-n}\mathcal{H}s^2)|_{\mathscr{I}} = -\frac{2\mathcal{F}}{\lambda n(n-2)}$.

Proof. By definition 6.3, $\mathcal{H} = \Omega^2\widetilde{\mathcal{H}}$ must be $O(\Omega^m)$ with $m \geq 1$. Assume first that $m = 1$. By property 2 of Lemma 2.29 the FG expansion of $g = -\mathrm{d}\Omega^2/\lambda + g_\Omega$ is even up to order n, where g_Ω is given by (2.34) if n odd or (2.35) if n even (with vanishing logarithmic terms because γ is conformally flat, cf. Theorem 2.22). Then, using the Kerr-Schild form $g = \widehat{g} + \mathcal{H}k \otimes k$ and expanding $\widehat{g}$ and $\mathcal{H}k \otimes k$ in Ω, the non-zero terms of order Ω of the tangent-tangent (i.e. tangent to $\Sigma_\Omega = \{\Omega = const.\}$) components of $\widehat{g}$ must cancel out those of $\mathcal{H}k \otimes k$. To expand $\widehat{g}$ in powers of Ω, consider a geodesic conformal factor[2] $\widehat{\Omega}$ for $\widehat{g}$, which induces the same boundary metric γ at $\mathscr{I} = \{\Omega = 0\} = \{\widehat{\Omega} = 0\}$. The existence of such conformal factor follows by Lemma 2.10 and it must satisfy $\Omega = \widehat{\Omega}\omega$, with $\omega|_{\mathscr{I}} = 1$. By Lemma 5.8, the FG expansion of $\widehat{g}$, in Gaussian coordinates $\{\widehat{\Omega}, \widehat{x}^i\}$ adapted to the foliation $\Sigma_{\widehat{\Omega}} = \{\widehat{\Omega} = const.\}$, is given by (5.15)

$$\widehat{g} = -\frac{\mathrm{d}\widehat{\Omega}^2}{\lambda} + \widehat{g}_{\widehat{\Omega}}, \qquad \widehat{g}_{\widehat{\Omega}} = \gamma + \frac{P}{\lambda}\widehat{\Omega}^2 + \frac{1}{4}\frac{P^2}{\lambda^2}\widehat{\Omega}^4 \tag{6.10}$$

where P is the Schouten tensor of γ. In order to compare with the expansion of g_Ω, one has to relate the conformal factors, but also the tangent directions. First, as $g|_{\mathscr{I}} = \widehat{g}|_{\mathscr{I}} = \gamma$ we can choose tangent coordinates satisfying $\widehat{x}^i = x^i + \Omega z^i$, for a collection of functions $\{z^i\}$ (still depending on Ω). We use now, as shown before, that the vectors ∂_Ω and $\partial_{\widehat{\Omega}}$ are proportional at $\mathscr{I}$

$$\begin{aligned}\partial_\Omega|_{\Omega=0} &= \left(\partial_\Omega \widehat{x}^i \partial_{\widehat{x}^i} + \partial_\Omega \widehat{\Omega}\right)\partial_{\widehat{\Omega}}|_{\Omega=0} \\ &= \left(z^i + \Omega\partial_\Omega z^i\right)\partial_{\widehat{x}^j}|_{\Omega=0} + \left(\omega + \Omega\partial_{\widehat{\Omega}}\omega\right)\partial_{\widehat{\Omega}}|_{\Omega=0} = \partial_{\widehat{\Omega}}|_{\Omega=0}\,.\end{aligned}$$

Thus $z^i|_{\widehat{\Omega}=0} = 0$ so $z^i = O(\Omega)$ and $\widehat{x}^i = x^i + O(\Omega^2)$. This implies that when γ (which recall is extended off $\mathscr{I}$ as independent of $\widehat{\Omega}$ in the Gaussian coordinates $\{\widehat{\Omega}, \widehat{x}^i\}$) is written in coordinates x^i, it does not add tangent-tangent terms $(\mathrm{d}x^i\mathrm{d}x^j)$ of order Ω and obviously neither they do the rest of terms in $\widehat{g}_{\widehat{\Omega}}$ in (6.10), because $\widehat{\Omega} = \Omega\omega$. On the other hand, $\mathrm{d}\widehat{\Omega}^2$ is

$$\mathrm{d}\widehat{\Omega}^2 = (\omega\mathrm{d}\Omega + \Omega\mathrm{d}\omega)^2 = \omega^2\mathrm{d}\Omega + \Omega^2\mathrm{d}\omega^2 + 2\Omega\mathrm{d}\Omega\mathrm{d}\omega$$

and the only tangent-tangent terms can only appear in $\Omega^2\mathrm{d}\omega^2$, thus starting (at least) at order Ω^2. Therefore the expansion of $\widehat{g}$ in the conformal factor Ω does not have first

[2]Notice that $\widehat{g} = \widehat{\Omega}^2\widetilde{g}'_{dS}$, where $\widetilde{g}'_{dS}$ is locally de Sitter, isometric to the original one $\widetilde{g}_{dS}$, but not equal.

order terms, so neither it does $\mathcal{H} k\otimes k$ because the FG expansion of g does not have such a term. This implies that $m \geq 2$.

Let us expand $\mathcal{H}$ as

$$\mathcal{H} = -\frac{2\mathcal{F}}{\lambda n(n-2)}(s^{-2}|_{\mathscr{I}})\Omega^m + o(\Omega^m),$$

and note that s that does not vanish anywhere (because k has this property). By Lemma 5.4, the electric part of the Weyl tensor is straightforwardly calculated

$$C_\perp = \mathcal{F}\left(y\otimes y - \frac{|y|^2}{n} g_\Omega\right)\Omega^{m-2} + o(\Omega^{m-2}) \tag{6.11}$$

where we have used that $\widehat{g}$ is conformally flat, so that $\widehat{C} = 0$, and $\nabla\Omega$ is geodesic, thus $F^2 = \lambda$, and $\epsilon = -1$ (cf. Lemma 2.9). Now applying Theorem 5.14, scaling (6.11) by Ω^{2-n} and evaluating at $\Omega = 0$ must give the free part of the n-th order coefficient of the FG expansion, so $m \geq n$. But $m > n$ gives $\mathring{g}_{(n)} = 0$, which by uniqueness of the FG expansion would imply that $\widetilde{g}$ is equal to its background metric, against hypothesis. Thus $m = n$ and the lemma follows after scaling (6.11) by Ω^{2-n} and evaluating at $\mathscr{I}$. □

In conclusion, the initial data for Kerr-Schild-de Sitter spacetimes are a conformally flat class of metrics $[\gamma]$ and a TT tensor of the form (6.9). The function $\mathcal{F}$ cannot be identically zero at $\mathscr{I}$ (as otherwise $\widetilde{g}$ would equal its background metric in a neighbourhood of $\mathscr{I}$). After restricting $\mathcal{M}$ further we may therefore assume that $\mathcal{F}$ is nowhere zero at $\mathscr{I}$ and we may reparametrize it as $\mathcal{F} =: \kappa/f^n$, with f everywhere positive and $\kappa \in \mathbb{R}$ is a constant that carries the sign of $\mathcal{F}$. For later convenience we do not normalize κ to be ± 1, which means that we keep an arbitrary (positive) scaling freedom in f. Then, the TT tensor D of Lemma 6.11 can be written as

$$D = \kappa D_\xi, \qquad (D_\xi)_{\alpha\beta} := \frac{1}{f^{n+2}}\left(\xi_\alpha\xi_\beta - \frac{f^2}{n}\gamma_{\alpha\beta}\right), \tag{6.12}$$

with $\xi_\alpha := f y_\alpha$. Our next aim is to prove that ξ it must be a CKVF of $\mathscr{I}$. The strategy is to rewrite the conditions of being CKVF in terms of equations for f and y and then show that they are satisfied as a consequence of k being a WAND.

Recall the following standard decomposition of the covariant derivative of a unit vector field y_α in terms parallel and orthogonal to itself

$$\nabla^{(\gamma)}_\alpha y_\beta = y_\alpha a_\beta + \Pi_{\alpha\beta} + \frac{h_{\alpha\beta}}{n-1}L + w_{\alpha\beta}, \qquad L := \nabla^{(\gamma)}_\alpha y^\alpha \tag{6.13}$$

where $\nabla^{(\gamma)}$ the Levi-civita connection of γ, a_β is a covector, $h_{\alpha\beta} = \gamma_{\alpha\beta} - y_\alpha y_\beta$ (the "projector" onto $(\mathrm{span}\{y\})^\perp$) and $\Pi_{\alpha\beta}$ symmetric traceless and $w_{\alpha\beta}$ skew-symmetric, i.e.

$$\Pi_{(\alpha\beta)} = \Pi_{\alpha\beta}, \qquad \Pi^\alpha{}_\alpha = 0, \qquad w_{[\alpha\beta]} = w_{\alpha\beta},$$

satisfying

$$y^\alpha \Pi_{\alpha\beta} = y^\alpha h_{\alpha\beta} = y^\alpha w_{\alpha\beta} = 0, \qquad y^\alpha a_\alpha = 0.$$

In what follows, it will be useful to express the metric γ as

$$\gamma_{\alpha\beta} = y_\alpha y_\beta + h_{\alpha\beta}.$$

Lemma 6.12. *Let $\xi^\alpha = f y^\alpha$, with y^α unit, be a vector field of a Riemannian n-manifold (Σ, γ) and consider the decomposition of $\nabla^{(\gamma)}_\alpha y_\beta$ as in (6.13). Then ξ is a CKVF of γ if and only if the following equations are satisfied*

$$\nabla^{(\gamma)}_\alpha f = \frac{fL}{n-1} y_\alpha - f a_\alpha, \qquad \Pi_{\alpha\beta} = 0. \tag{6.14}$$

Proof. We rewrite the conformal Killing equation

$$\nabla^{(\gamma)}_\alpha \xi_\beta + \nabla^{(\gamma)}_\beta \xi_\alpha = \frac{2}{n} \nabla^{(\gamma)}_\mu \xi^\mu \gamma_{\alpha\beta}$$

in terms of the kinematical quantities above. Since

$$\begin{aligned}\nabla^{(\gamma)}_\alpha \xi_\beta + \nabla^{(\gamma)}_\alpha \xi_\beta &= (\nabla^{(\gamma)}_\alpha f) y_\beta + (\nabla^{(\gamma)}_\beta f) y_\alpha + f(\nabla^{(\gamma)}_\alpha y_\beta + \nabla^{(\gamma)}_\beta y_\alpha) \\ &= (\nabla^{(\gamma)}_\alpha f) y_\beta + (\nabla^{(\gamma)}_\beta f) y_\alpha + f\left(y_\alpha a_\beta + y_\beta a_\alpha + 2\Pi_{\alpha\beta} + \frac{2h_{\alpha\beta}}{n-1} L\right)\end{aligned}$$

and

$$\frac{2}{n} \nabla^{(\gamma)}_\mu \xi^\mu \gamma_{\alpha\beta} = \frac{2}{n}(y^\mu \nabla^{(\gamma)}_\mu f + fL)(y_\alpha y_\beta + h_{\alpha\beta}),$$

ξ is a CKVF if and only if

$$\begin{aligned}&(\nabla^{(\gamma)}_\alpha f) y_\beta + (\nabla^{(\gamma)}_\beta f) y_\alpha + f\left(y_\alpha a_\beta + y_\beta a_\alpha + 2\Pi_{\alpha\beta} + \frac{2h_{\alpha\beta}}{n-1} L\right) \\ =&\frac{2}{n}(y^\mu \nabla^{(\gamma)}_\mu f + fL)(y_\alpha y_\beta + h_{\alpha\beta}).\end{aligned} \tag{6.15}$$

One contraction with y^α gives

$$(y^\alpha \nabla^{(\gamma)}_\alpha f) y_\beta + \nabla^{(\gamma)}_\beta f + f a_\beta = \frac{2}{n}(y^\mu \nabla^{(\gamma)}_\mu f + fL) y_\beta \tag{6.16}$$

and a second contraction with y^β

$$y^\alpha \nabla^{(\gamma)}_\alpha f + y^\beta \nabla^{(\gamma)}_\beta f = \frac{2}{n}(y^\mu \nabla^{(\gamma)}_\mu f + fL) \iff y^\alpha \nabla^{(\gamma)}_\alpha f = \frac{fL}{n-1}. \tag{6.17}$$

Inserting (6.17) in (6.16) gives the first of equation (6.14). Projecting (6.15) with $h^\alpha{}_\mu h^\beta{}_\nu$ gives

$$2f\left(\Pi_{\mu\nu} + \frac{h_{\mu\nu}}{n-1} L\right) = \frac{2}{n}(y^\mu \nabla^{(\gamma)}_\mu f + fL) h_{\mu\nu}$$

which is equivalent to $\Pi = 0$ after using (6.17). This proves the result in one direction. The converse follows immediately because (6.16) is identically satisfied when (6.14) hold. □

Coming back to the data corresponding to Kerr-Schild de Sitter metrics, we prove that the first equation in (6.14) is satisfied just by imposing D to be TT. The argument for the second equation is more subtle and will be addressed right after.

Lemma 6.13. *Let $\widetilde{g}$ be a Kerr-Schild de Sitter metric and $g = \Omega^2 \widetilde{g}$ a geodesic conformal extension. Then*

$$\nabla^{(\gamma)}_\alpha f = \frac{fL}{n-1} y_\alpha - f a_\alpha. \tag{6.18}$$

Proof. Consider $D_\xi = f^{-n}\left(y \otimes y - (1/n)\gamma\right)$, which by Lemma 6.11 is, up to a constant, the electric part of the rescaled Weyl tensor of $\widetilde{g}$. Since $\widetilde{g}$ is Einstein and γ locally conformally flat, then D_ξ must be TT (because it coincides with the n-th order coefficient of the FG expansion, cf. Theorem 5.14) and the vanishing of its divergence gives by (6.13)

$$\nabla^{(\gamma)}_\alpha (D_\xi)^\alpha{}_\beta = -\frac{n}{f^{n+1}} \left(y^\alpha \nabla^{(\gamma)}_\alpha f y_\beta - \frac{\nabla^{(\gamma)}_\beta f}{n} \right) + \frac{1}{f^n}\left(L y_\beta + a_\beta\right) = 0. \tag{6.19}$$

Contracting with y^β one has

$$y^\alpha \nabla^{(\gamma)}_\alpha f = \frac{fL}{n-1}$$

and inserting back into (6.19) we get (6.18). This condition, which is precisely the first in (6.14), is not only necessary for (6.19) but also sufficient. □

We next show that $\Pi_{\alpha\beta} = 0$. First notice that K_Ω, the second fundamental form of the leaves $\Sigma_\Omega = \{\Omega = const.\}$, can be written

$$K_\Omega = \frac{1}{2}(\mathcal{L}_u g_\Omega) = -\frac{\lambda^{1/2}}{2}(2\Omega g_{(2)} + \cdots + n\Omega^{n-1} g_{(n)} + \cdots),$$

where $\mathcal{L}_u$ denotes the Lie derivative w.r.t. the unit vector $u^\alpha \partial_\alpha = \lambda^{-1/2}\nabla^\alpha \Omega \partial_\alpha = -\lambda^{1/2}\partial_\Omega$. This tensor appears in the Codazzi equation (2.18)

$$(\nabla_k (K_\Omega)_{ij} - \nabla_i (K_\Omega)_{kj}) = R^\mu{}_{jik} u_\mu, \tag{6.20}$$

where i, j, k denote tangent directions to Σ_Ω. The strategy consists in analyzing the Ω^{n-1} order terms of the following components of the Codazzi equation

$$(\nabla_\nu (K_\Omega)_{\beta\alpha} - \nabla_\beta (K_\Omega)_{\nu\alpha})\, h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} y^\nu = R^\mu{}_{\alpha\nu\beta} u_\mu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} y^\nu, \tag{6.21}$$

where we extend h away from $\mathscr{I}$ as the projector orthogonal to y and u, i.e. $h := g + u \otimes u - y \otimes y$. The proof that $\Pi_{\alpha\beta} = 0$ consists in two main steps. Firstly, we prove that the Ω^{n-1} order term of the LHS of (6.21) only involves the free part $\mathring{g}_{(n)}$. This, by Theorem 5.14, coincides up to a constant with the electric part of the rescaled by tensor, which in turn, by Lemma 6.11, is given by equation (6.12). From these facts it follows that the LHS of (6.21) is (up to a non-zero factor) $\Pi_{\alpha\beta}$. The second step consist in analyzing the RHS of (6.21). From the algebraically special condition, it follows that the symmetric part of its Ω^{n-1} order term is pure trace. Since $\Pi_{\alpha\beta}$ is traceless,it follows $\Pi_{\alpha\beta} = 0$.

Before carrying out this program, we derive some identities that will be required for the rest of this section. Consider a conformally extendable Einstein metric $\widetilde{g}$ and let $g = \Omega^2\widetilde{g}$ be a geodesic conformal extension. As before let u be unit normal along $\nabla\Omega$ and i, j, k denote orthogonal directions to span$\{u\}$ (in Gaussian coordinates $\{\Omega, x^i\}$). Then, from the definition of (2.10), a straightforward calculation gives

$$R^{\mu}{}_{jik}u_{\mu} = C^{\mu}{}_{jik}u_{\mu} - \frac{2}{n-1}g_{j[k}R_{i]\mu}u^{\mu}.$$

On the other hand, since $\widetilde{g}$ is Einstein and Ω geodesic, from (2.8) follows

$$R_{\alpha\beta} = -\frac{n-1}{\Omega}\nabla_\alpha\nabla_\beta\Omega - g_{\alpha\beta}\frac{\nabla_\mu\nabla^\mu\Omega}{\Omega}. \tag{6.22}$$

Hence

$$R_{i\mu}u^{\mu} = -\lambda^{-1/2}\frac{n-1}{\Omega}(\nabla_i\nabla_\mu\Omega)\nabla^\mu\Omega = -\lambda^{-1/2}\frac{n-1}{2\Omega}\nabla_i(\nabla^\mu\Omega\nabla_\mu\Omega) = 0$$

and

$$R^{\mu}{}_{jik}u_{\mu} = C^{\mu}{}_{jik}u_{\mu}. \tag{6.23}$$

In particular

$$R^{\mu}{}_{\alpha\nu\beta}u_\mu y^\nu h^\alpha{}_\lambda h^\beta{}_\sigma = C^{\mu}{}_{\alpha\nu\beta}u_\mu y^\nu h^\alpha{}_\lambda h^\beta{}_\sigma. \tag{6.24}$$

Lemma 6.14. *Let $\widetilde{g}_{dS}$ be the metric of de Sitter, $\overline{g} = \Omega^2\widetilde{g}_{dS}$ a geodesic conformal extension and $\overline{K}_\Omega$ the second fundamental form on the leaves $\Sigma_\Omega = \{\Omega = const.\}$. Then, the Codazzi equation (6.20) is*

$$\overline{\nabla}_k(\overline{K}_\Omega)_{ij} - \overline{\nabla}_i(\overline{K}_\Omega)_{kj} = 0,$$

Proof. The lemma follows by simply applying the Codazzi equation (6.20) to $\overline{g}$ together with identity (6.23), where the Weyl tensor vanishes because $\overline{g}$ is conformally flat. □

Proposicin 6.15. *Let $\widetilde{g}$ be an Einstein metric admitting a smooth conformally flat $\mathscr{I}$, $g = \Omega^2\widetilde{g}$ a geodesic conformal extension and K_Ω the second fundamental form on the leaves $\Sigma_\Omega = \{\Omega = const.\}$. Then the leading order term of the LHS of the Codazzi equation (6.20) is*

$$-\frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\left(\nabla^{(\gamma)}_k(\mathring{g}_{(n)})_{ij} - \nabla^{(\gamma)}_i(\mathring{g}_{(n)})_{kj}\right),$$

where γ is extended off $\mathscr{I}$ as independent of Ω and $\nabla^{(\gamma)}$ denotes its Levi-Civita connection.

Proof. Consider the decomposition a) of Proposition 6.7, $g = \overline{g} + Q$ with $\overline{g} = -\mathrm{d}\Omega^2/\lambda + \overline{g}_\Omega$ conformal to de Sitter. Since $\overline{g}^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega = -\lambda$, the conformal factor Ω is geodesic for both g and $\overline{g}$. On the other hand, the second fundamental forms K_Ω and $\overline{K}_\Omega$, respectively induced by g and $\overline{g}$ on Σ_Ω, are related by

$$K_\Omega = \frac{-\lambda^{1/2}}{2}\partial_\Omega g_\Omega = \frac{-\lambda^{1/2}}{2}\partial_\Omega(\overline{g}_\Omega + Q) = \overline{K}_\Omega - \frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\mathring{g}_{(n)} + O(\Omega^{n+1}),$$

where we have used that by construction $Q = \Omega^n \mathring{g}_{(n)} + O(\Omega^{n+2})$. For every tensor $\mathcal{T}_{ij}$, tangent to Σ_Ω, it follows that its covariant derivatives w.r.t. ∇ and $\overline{\nabla}$ satisfy (we use that the coordinates are Gaussian with respect to g)

$$\nabla_k \mathcal{T}_{ij} = \overline{\nabla}_k \mathcal{T}_{ij} - S^l{}_{ki}\mathcal{T}_{lj} - S^l{}_{kj}\mathcal{T}_{il}$$

where the tangent components of S, given by (2.1) for $g^{(1)} = g$ and $g^{(2)} = \overline{g}$, satisfy

$$S^l{}_{ki} = \frac{1}{2}g^{lm}\left(\overline{\nabla}_k g_{im} + \overline{\nabla}_i g_{km} - \overline{\nabla}_m g_{ki}\right) = \frac{1}{2}g^{lm}\left(\overline{\nabla}_k Q_{im} + \overline{\nabla}_i Q_{km} - \overline{\nabla}_m Q_{ki}\right) = O(\Omega^n).$$

Thus $\nabla_k \mathcal{T}_{ij} = \overline{\nabla}_k \mathcal{T}_{ij} + O(\Omega^n)$. In particular, for K_Ω

$$\begin{aligned}\nabla_k(K_\Omega)_{ij} &= \nabla_k(\overline{K}_\Omega)_{ij} - \frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\nabla_k(\mathring{g}_{(n)})_{ij} + O(\Omega^{n+1}) \\ &= \overline{\nabla}_k(\overline{K}_\Omega)_{ij} - \frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\nabla_k(\mathring{g}_{(n)})_{ij} + O(\Omega^n),\end{aligned}$$

and the LHS of the Codazzi equation (6.20) for K_Ω is

$$\begin{aligned}\nabla_k(K_\Omega)_{ij} - \nabla_i(K_\Omega)_{kj} &= \overline{\nabla}_k(\overline{K}_\Omega)_{ij} - \overline{\nabla}_i(\overline{K}_\Omega)_{kj} \\ &\quad - \frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\left(\nabla_k(\mathring{g}_{(n)})_{ij} - \nabla_i(\mathring{g}_{(n)})_{kj}\right) + O(\Omega^n) \\ &= -\frac{\lambda^{1/2}}{2}(n-1)\Omega^{n-1}\left(\nabla_k(\mathring{g}_{(n)})_{ij} - \nabla_i(\mathring{g}_{(n)})_{kj}\right) + O(\Omega^n),\end{aligned}$$

where the second equality is a consequence of Lemma 6.14. Now, since $g_\Omega = \gamma + O(\Omega^2)$, the covariant derivatives $\nabla_k(\mathring{g}_{(n)})_{ij}$ and $\nabla_i(\mathring{g}_{(n)})_{kj}$ are, to lowest order in Ω, $\nabla^{(\gamma)}{}_k(\mathring{g}_{(n)})_{ij}$ and $\nabla^{(\gamma)}{}_i(\mathring{g}_{(n)})_{kj}$. □

Therefore, for the particular case of Kerr-Schild-de Sitter metrics and the components of the Codazzi equation in (6.21) we obtain:

Corolario 6.16. *The Ω^{n-1} order term of the LHS of (6.21) is, up to a non-zero constant, equal to the following tensor*

$$(\mathcal{LHS})_{\lambda\sigma} := -\frac{1}{f^n}\Pi_{\lambda\sigma}.$$

Proof. From Proposition 6.15, the term of order Ω^{n-1} of (6.21) only involves derivatives of $\mathring{g}_{(n)}$. By Theorem 5.14, $\mathring{g}_{(n)}$ is up to a constant the electric part of the rescaled Weyl tensor, which by Lemma 6.11, is given by expression (6.12). Hence, substituting $\gamma_{\alpha\beta} = y_\alpha y_\beta + h_{\alpha\beta}$, the $(n-1)$-th order of the LHS of (6.21) is (up to a non-zero constant)

$$y^\nu(\nabla^{(\gamma)}_\nu (D_\xi)_{\beta\alpha} - \nabla^{(\gamma)}_\beta (D_\xi)_{\nu\alpha}) = -\frac{n}{f^{n+1}} y^\nu \nabla^{(\gamma)}_\nu f \left(y_\beta y_\alpha - \frac{y_\beta y_\alpha}{n} - \frac{h_{\beta\alpha}}{n}\right) + \frac{1}{f^n}(a_\beta y_\alpha + a_\alpha y_\beta) + \frac{n}{f^{n+1}}\nabla^{(\gamma)}_\beta f \frac{n-1}{n} y_\alpha - \frac{1}{f^n}\nabla^{(\gamma)}_\beta y_\alpha.$$

Inserting the decomposition (6.13) and using the first equation in (6.14)

$$y^\nu(\nabla^{(\gamma)}_\nu (D_\xi)_{\beta\alpha} - \nabla^{(\gamma)}_\beta (D_\xi)_{\nu\alpha}) = -\frac{n}{f^{n+1}}\frac{fL}{n-1}\left(\frac{n-1}{n} y_\beta y_\alpha - \frac{h_{\beta\alpha}}{n}\right) + \frac{1}{f^n}(a_\beta y_\alpha + a_\alpha y_\beta)$$

$$+ \frac{n}{f^{n+1}}\left(\frac{fL}{n-1} y_\beta - f a_\beta\right)\frac{n-1}{n} y_\alpha - \frac{1}{f^n}\left(y_\beta a_\alpha + \Pi_{\beta\alpha} + \frac{L}{n-1} h_{\beta\alpha} + w_{\beta\alpha}\right)$$
$$= -\frac{1}{f^n}\left((n-2) a_\beta y_\alpha + \Pi_{\beta\alpha} + w_{\beta\alpha}\right).$$

Contracting both indices with h and symmetrizing yields the following tensor

$$(\mathcal{LHS})_{\lambda\sigma} := y^\nu(\nabla^{(\gamma)}_\nu (D_\xi)_{\beta\alpha} - \nabla^{(\gamma)}_\beta (D_\xi)_{\nu\alpha}) h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} = -\frac{1}{f^n}\Pi_{\lambda\sigma}.$$

□

In the remainder of this section, we ellaborate the RHS of (6.21). Applying identity (6.24) it follows

$$(\mathcal{RHS})_{\sigma\lambda} := R^\mu{}_{\alpha\nu\beta} u^\nu y_\mu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} = C^\nu{}_{\beta\mu\alpha} u_\nu y^\mu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)}. \tag{6.25}$$

Now we use the algebraic special condition to prove that the Ω^{n-1} order components of the Weyl tensor in (6.25) are pure trace. Recall the decomposition (6.8) of k. One can then define $l = 2s^{-1}(u-y)$ such that $l_\alpha k^\alpha = -1$ and complete to a null frame $\{k, l, m_{(i)}\}$. Then, h is the projector onto span$\{m_{(i)}\}$. Thus, contracting $C^\mu{}_{\alpha\nu\beta}$ with

$k_\mu k^\nu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)}$ gives by Proposition 6.9

$$\begin{aligned} & 0 = C^\mu{}_{\alpha\nu\beta} k_\mu k^\nu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} \\ \iff \quad & 0 = \left(C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu + C^\mu{}_{\alpha\nu\beta} y_\mu y^\nu + 2C^\mu{}_{(\alpha|\nu|\beta)} u_\mu y^\nu \right) h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} \\ \iff \quad & 2C^\mu{}_{(\alpha|\nu|\beta)} u^\nu y_\mu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} = -C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu h^\alpha{}_\lambda h^\beta{}_\sigma - C^\mu{}_{\alpha\nu\beta} y_\mu y^\nu h^\alpha{}_\lambda h^\beta{}_\sigma . \end{aligned}$$

In addition

$$g_{\alpha\beta} = -u_\alpha u_\beta + y_\alpha y_\beta + h_{\alpha\beta},$$

and the traceless property of the Weyl tensor gives

$$\begin{aligned} & 0 = C^\mu{}_{\alpha\mu\beta} = -C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu + C^\mu{}_{\alpha\nu\beta} y_\mu y^\nu + C^\mu{}_{\alpha\nu\beta} h^\nu{}_\mu \\ \Longrightarrow \quad & C^\mu{}_{\alpha\nu\beta} y_\mu y^\nu = C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu - C^\mu{}_{\alpha\nu\beta} h^\nu{}_\mu . \end{aligned}$$

Therefore

$$2C^\mu{}_{(\alpha|\nu|\beta)} u^\nu y_\mu h^\alpha{}_{(\lambda} h^\beta{}_{\sigma)} = -2C^\mu{}_{\alpha\nu\beta} u_\mu u^\nu h^\alpha{}_\lambda h^\beta{}_\sigma + C^\mu{}_{\alpha\nu\beta} h^\nu{}_\mu h^\alpha{}_\lambda h^\beta{}_\sigma . \tag{6.26}$$

The first term in the RHS of (6.26) only involves the electric part of the Weyl tensor. Using the previous results we next prove that, at order Ω^{n-1}, it can only contain trace terms.

Lemma 6.17. *Let $\widetilde{g}$ be a conformally extendable metric admitting a smooth conformally flat $\mathscr{I}$. Then, for every geodesic conformal extension $g = \Omega^2 \widetilde{g}$, the electric part of Weyl tensor w.r.t. the normal vector $C_\perp$ has no terms in Ω^{n-1}. Moreover, if $\widetilde{g}$ is Kerr-Schild-de Sitter, the possible terms of order Ω^{n-1} added by contracting twice with h, i.e. $(C_\perp)_{\alpha\beta} h^\alpha{}_\lambda h^\sigma{}_\beta$, are pure trace.*

Proof. First consider $g = -\mathrm{d}\Omega^2 + g_\Omega$ in normal form w.r.t. a boundary metric γ. Since γ is conformally flat, we can decompose g_Ω as in statement $a)$ of Proposition 6.7

$$g_\Omega = \overline{g}_\Omega + Q$$

where $\overline{g} = -\mathrm{d}\Omega^2 + \overline{g}_\Omega$ is conformally isometric to de Sitter, $\overline{g}_\Omega$ is given by (5.15) and $Q = O(\Omega^n)$ contains no terms of order Ω^{n+1}. We now insert this decomposition into formula (5.2), which we write in terms of the electric part of the Weyl tensor $C_\perp$ (cf. equation 5.1), which for simplicity we write using matrix notation as

$$(C_\perp) = \frac{\lambda}{2} \left(\frac{1}{2} \dot{g}_\Omega g_\Omega^{-1} \dot{g}_\Omega + \frac{1}{\Omega} \dot{g}_\Omega - \ddot{g}_\Omega \right). \tag{6.27}$$

where $\dot{}$ stands for derivative in Ω and note, g_Ω^{-1} must decompose as

$$g_\Omega^{-1} = \overline{g}_\Omega^{-1} + V$$

with $V = O(\Omega^n)$, because $g_\Omega^{-1} g_\Omega$ equals the identity and terms of order $m < n$ in V could not be cancelled out. We compute the terms in (6.27). Firstly

$$\begin{aligned}\dot{g}_\Omega g^{-1}\dot{g}_\Omega &= \dot{\overline{g}}_\Omega g_\Omega^{-1}\dot{\overline{g}}_\Omega + \dot{\overline{g}}_\Omega g_\Omega^{-1}\dot{Q} + \dot{Q} g_\Omega^{-1}\dot{\overline{g}}_\Omega + \dot{Q} g_\Omega^{-1}\dot{Q} \\ &= \dot{\overline{g}}_\Omega \overline{g}_\Omega^{-1}\dot{\overline{g}}_\Omega + \dot{\overline{g}}_\Omega \overline{g}_\Omega^{-1}\dot{Q} + \dot{Q}\overline{g}_\Omega^{-1}\dot{\overline{g}}_\Omega + \dot{Q}\overline{g}_\Omega^{-1}\dot{Q} \\ &+ \dot{\overline{g}}_\Omega V \dot{\overline{g}}_\Omega + \dot{\overline{g}}_\Omega V \dot{Q} + \dot{Q} V \dot{\overline{g}}_\Omega + \dot{Q} V \dot{Q},\end{aligned}$$

and second

$$\frac{1}{\Omega}\dot{g}_\Omega - \ddot{g}_\Omega = \frac{1}{\Omega}\dot{\overline{g}}_\Omega - \ddot{\overline{g}}_\Omega + \frac{1}{\Omega}\dot{Q} - \ddot{Q}.$$

Adding them and taking into account that

$$\frac{\lambda}{2}\left(\frac{1}{2}\dot{\overline{g}}_\Omega \overline{g}_\Omega^{-1}\dot{\overline{g}}_\Omega + \frac{1}{\Omega}\dot{\overline{g}}_\Omega - \ddot{\overline{g}}_\Omega\right) = \overline{C}_\perp = 0$$

where $(\overline{C}_\perp)$ is the electric part of the Weyl tensor of $\overline{g}$, we are left with

$$\begin{aligned}\frac{2}{\lambda}(C_\perp) &= \frac{1}{2}\left(\dot{\overline{g}}_\Omega \overline{g}_\Omega^{-1}\dot{Q} + \dot{Q}\overline{g}_\Omega^{-1}\dot{\overline{g}}_\Omega + \dot{Q}\overline{g}_\Omega^{-1}\dot{Q} + \dot{\overline{g}}_\Omega V \dot{\overline{g}}_\Omega + \dot{\overline{g}}_\Omega V \dot{Q} + \dot{Q} V \dot{\overline{g}}_\Omega + \dot{Q} V \dot{Q}\right) \\ &+ \frac{1}{\Omega}\dot{Q} - \ddot{Q} = \frac{1}{\Omega}\dot{Q} - \ddot{Q} + O(\Omega^n).\end{aligned} \tag{6.28}$$

Since Q does not contain terms of order Ω^{n+1}, then (6.28) does not contain terms of order Ω^{n-1}. This proves the first part of the lemma.

Combining this fact with equation (6.12), we can write the leading order of $C_\perp$ and its tail order terms as

$$(C_\perp)_{\alpha\beta} = \Omega^{n-2}\frac{\kappa}{f^{n+2}}\left(\xi_\alpha\xi_\beta - \frac{f^2}{n}\gamma_{\alpha\beta}\right) + O(\Omega^n),$$

where γ must be understood as the leading order term of g_Ω, i.e. the extension of $\gamma|_{\mathscr{I}}$ to the spacetime as a tensor independent of Ω and similarly with ξ. Contracting this expression twice with h gives

$$(C_\perp)_{\alpha\beta}h^\alpha{}_\mu h^\beta{}_\nu = -\Omega^{n-2}\frac{\kappa}{n}\frac{1}{f^n}h_{\mu\nu} + O(\Omega^n).$$

We cannot exclude that the presence of $h_{\alpha\beta}$ in this expression introduces terms of order Ω^{n-1}, but if present, they are clearly trace terms, as claimed in the Lemma. □

We next look for a similar result for the components of the Weyl tensor $C^{\mu}{}_{\beta\nu\alpha}h^{\nu}{}_{\mu}h^{\alpha}{}_{(\lambda}h^{\beta}{}_{\sigma)}$ which arise in (6.26). From the definition (2.10) one has

$$\begin{aligned}C^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} &= R^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} \\ &+ \left(-\frac{2}{n-1}(\delta^{\mu}{}_{[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R^{\mu}{}_{|\beta]}) + \frac{2R}{n(n-1)}\delta^{\mu}{}_{[\nu}g_{\beta]\alpha}\right)h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} \\ &= R^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} \\ &- \frac{n-3}{n-1}R_{\alpha\beta}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} + \left(-\frac{R^{\mu}{}_{\nu}h^{\nu}{}_{\mu}}{n-1} + \frac{n-2}{n(n-1)}R\right)h_{\lambda\sigma},\end{aligned}$$

which using (6.22) gives

$$\begin{aligned}C^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} &= R^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} + (n-3)\frac{\nabla_{\alpha}\nabla_{\beta}\Omega}{\Omega}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} \\ &+ \left(\frac{n-3}{n-1}\frac{\nabla_{\mu}\nabla^{\mu}\Omega}{\Omega} - \frac{R^{\mu}{}_{\nu}h^{\nu}{}_{\mu}}{n-1} + \frac{n-2}{n(n-1)}R\right)h_{\lambda\sigma}. \qquad (6.29)\end{aligned}$$

The term containing $h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma}\nabla_{\alpha}\nabla_{\beta}\Omega$ will be left unaltered as it will cancel out after expanding the rest of terms. Our next aim is to analyze the components of the Riemann tensor $R^{\mu}{}_{\alpha\nu\beta}h^{\nu}{}_{\mu}h^{\alpha}{}_{(\lambda}h^{\alpha}{}_{\sigma)}$, and relate them to the same components of the Riemann tensor of $\widehat{g}$:

Lemma 6.18. *The Riemann tensors of g and $\widehat{g}$ satisfy*

$$\begin{aligned}\widehat{R}^{\mu}{}_{\alpha\nu\beta}h^{\delta}{}_{\mu}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} &= R^{\mu}{}_{\alpha\nu\beta}h^{\delta}{}_{\mu}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} \\ &- 2\mathcal{H}h^{\delta\tau}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma}\left(\nabla_{[\nu|}k_{[\tau}\nabla_{\alpha]}k_{|\beta]} + \nabla_{[\nu}k_{\beta]}\nabla_{[\alpha}k_{\tau]}\right).\end{aligned}$$

Proof. We apply the formula for the difference of Riemann tensors (2.2) with $g^{(1)} = \widehat{g}$ and $g^{(2)} = g$. Setting $g = \widehat{g} + \mathcal{H}k \otimes k$, the tensor S reads

$$S^{\mu}{}_{\alpha\beta} = -\frac{1}{2}\widehat{g}^{\mu\nu}\left(\nabla_{\alpha}(\mathcal{H}k_{\beta}k_{\nu}) + \nabla_{\beta}(\mathcal{H}k_{\alpha}k_{\nu}) - \nabla_{\nu}(\mathcal{H}k_{\alpha}k_{\beta})\right). \qquad (6.30)$$

Hence,

$$S^{\kappa}{}_{\nu\alpha}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda} = -\frac{1}{2}\mathcal{H}k^{\kappa}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda}(\nabla_{\nu}k_{\alpha} + \nabla_{\alpha}k_{\nu})$$

and since (recall that k is null geodesic $k^{\kappa}\nabla_{\kappa}k_{\beta} = k^{\kappa}\nabla_{\beta}k_{\kappa} = 0$)

$$k^{\kappa}h^{\beta}{}_{\sigma}S^{\mu}{}_{\beta\kappa} = -\frac{1}{2}k^{\kappa}h^{\beta}{}_{\sigma}\widehat{g}^{\mu\tau}(\nabla_{\beta}(\mathcal{H}k_{\kappa}k_{\tau}) + \nabla_{\kappa}(\mathcal{H}k_{\beta}k_{\tau}) - \nabla_{\tau}(\mathcal{H}k_{\kappa}k_{\beta})) = 0,$$

it follows

$$2S^{\kappa}{}_{[\nu|\alpha|}S^{\mu}{}_{\beta]\kappa}h^{\nu}{}_{\gamma}h^{\alpha}{}_{\lambda}h^{\beta}{}_{\sigma} = 0.$$

On the other hand

$$\nabla_\nu S^\mu{}_{\alpha\beta} = -\frac{1}{2}\nabla_\nu \widehat{g}^{\mu\tau}\left(\nabla_\alpha(\mathcal{H}k_\beta k_\tau) + \nabla_\beta(\mathcal{H}k_\alpha k_\tau) - \nabla_\tau(\mathcal{H}k_\alpha k_\beta)\right), \\ -\frac{1}{2}\widehat{g}^{\mu\tau}\nabla_\nu\left(\nabla_\alpha(\mathcal{H}k_\beta k_\tau) + \nabla_\beta(\mathcal{H}k_\alpha k_\tau) - \nabla_\tau(\mathcal{H}k_\alpha k_\beta)\right). \qquad (6.31)$$

The first three terms in (6.31) vanish when contracted with $h^\alpha{}_\lambda h^\beta{}_\sigma$ because, taking into account (6.6) and that k is null geodesic,

$$\frac{1}{2}\nabla_\nu \widehat{g}^{\mu\tau}\left(\nabla_\alpha(\mathcal{H}k_\beta k_\tau) + \nabla_\beta(\mathcal{H}k_\alpha k_\tau) - \nabla_\tau(\mathcal{H}k_\alpha k_\beta)\right) h^\alpha{}_\lambda h^\beta{}_\sigma \\ = \frac{1}{2}\nabla_\nu(\mathcal{H}k^\mu k^\tau)k_\tau\left(\nabla_\alpha k_\beta + \nabla_\beta k_\alpha\right) h^\alpha{}_\lambda h^\beta{}_\sigma = \frac{1}{2}\mathcal{H}^2 k^\mu(\nabla_\nu k^\tau)k_\tau\left(\nabla_\alpha k_\beta + \nabla_\beta k_\alpha\right) h^\alpha{}_\lambda h^\beta{}_\sigma = 0.$$

We calculate the contraction of the last three terms in (6.31) with h four times. The expansion of each term gives

$$h^\delta{}_\mu h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \widehat{g}^{\mu\tau}\nabla_\nu\nabla_\alpha(\mathcal{H}k_\beta k_\tau) = h^{\delta\tau}h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \mathcal{H}\left(\nabla_\nu k_\tau \nabla_\alpha k_\beta + \nabla_\nu k_\beta \nabla_\alpha k_\tau\right), \\ h^\delta{}_\mu h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \widehat{g}^{\mu\tau}\nabla_\nu\nabla_\beta(\mathcal{H}k_\alpha k_\tau) = h^{\delta\tau}h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \mathcal{H}\left(\nabla_\nu k_\tau \nabla_\beta k_\alpha + \nabla_\nu k_\alpha \nabla_\beta k_\tau\right), \\ h^\delta{}_\mu h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \widehat{g}^{\mu\tau}\nabla_\nu\nabla_\tau(\mathcal{H}k_\alpha k_\beta) = h^{\delta\tau}h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \mathcal{H}\left(\nabla_\nu k_\alpha \nabla_\tau k_\beta + \nabla_\nu k_\beta \nabla_\tau k_\alpha\right).$$

Then, rearraging terms,

$$2h^\delta{}_\mu h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma \nabla_{[\nu}S^\mu{}_{\beta]\alpha} = -2\mathcal{H}h^{\delta\tau}h^\nu{}_\gamma h^\alpha{}_\lambda h^\beta{}_\sigma\left(\nabla_{[\nu|}k_{[\tau}\nabla_{\alpha]}k_{|\beta]} + \nabla_{[\nu}k_{\beta]}\nabla_{[\alpha}k_{\tau]}\right),$$

and the Lemma follows from the identity (2.2). □

Specifically for our purposes, Lemma 6.18 yields

$$R^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_\lambda h^\beta{}_\sigma = \widehat{R}^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_\lambda h^\beta{}_\sigma + O(\Omega^n), \qquad (6.32)$$

so we do not have to take into account the tail order terms. To calculate $\widehat{R}^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_\lambda h^\beta{}_\sigma$, we use the definition of the Weyl tensor (2.10), which for $\widehat{g}$ vanishes, and contractions with h give:

$$\widehat{R}^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_\lambda h^\beta{}_\sigma = \frac{n-3}{n-1}\widehat{R}_{\alpha\beta}h^\alpha{}_\lambda h^\beta{}_\sigma - \left(-\frac{\widehat{R}^\mu{}_\nu h^\nu{}_\mu}{n-1} + \frac{n-2}{n(n-1)}\widehat{R}\right)h_{\lambda\sigma}. \qquad (6.33)$$

We finally relate the term $\widehat{R}_{\alpha\beta}h^\alpha{}_\lambda h^\beta{}_\sigma$ with the same components of the Ricci tensor of de Sitter. To do that, we use equation (2.8), substituting g by $\widehat{g}$ and $\widetilde{g}$ by $\widetilde{g}_{dS}$

$$\widehat{R}_{\alpha\beta} - \widetilde{R}^{dS}_{\alpha\beta} = -\frac{n-1}{\Omega}\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\Omega - \widehat{g}_{\alpha\beta}\frac{\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Omega}{\Omega} + \widehat{g}_{\alpha\beta}\frac{n}{\Omega^2}\widehat{\nabla}_\mu\Omega\widehat{\nabla}^\mu\Omega. \qquad (6.34)$$

We may now use that $\widetilde{g}_{dS}$ is Einstein to cancel out terms, but Ω is geodesic w.r.t. to g, which means

$$\begin{aligned}\widehat{g}_{\alpha\beta}\frac{n}{\Omega^2}\widehat{\nabla}_\mu\Omega\widehat{\nabla}^\mu\Omega &= \widehat{g}_{\alpha\beta}\frac{n}{\Omega^2}\left(g^{\mu\nu}+\mathcal{H}k^\mu k^\nu\right)\widehat{\nabla}_\mu\Omega\widehat{\nabla}_\nu\Omega \\ &= -\lambda n\widehat{g}_{\alpha\beta}\frac{n}{\Omega^2}+\widehat{g}_{\alpha\beta}\frac{n}{\Omega^2}\mathcal{H}k^\mu k^\nu\nabla_\mu\Omega\nabla_\nu\Omega \\ &= -\lambda n\widetilde{g}^{dS}_{\alpha\beta}+\widehat{g}_{\alpha\beta}\frac{ns^2}{\lambda\Omega^2}\mathcal{H} \end{aligned} \tag{6.35}$$

where we have used that $g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega = -\lambda$ and $s = -\lambda^{-1/2}k^\mu\nabla_\mu\Omega$. Now, since the de Sitter metric is Einstein, equation (6.34) with (6.35) gives

$$\widehat{R}_{\alpha\beta}h^\alpha{}_\lambda h^\beta{}_\sigma = -\frac{n-1}{\Omega}(\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\Omega)h^\alpha{}_\lambda h^\beta{}_\sigma + \left(-\frac{\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Omega}{\Omega}+\frac{ns^2}{\lambda\Omega^2}\mathcal{H}\right)h_{\lambda\sigma}.$$

The tensor $\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\Omega$ can be related with $\nabla_\alpha\nabla_\beta\Omega$ using the difference of connections

$$\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\Omega = \nabla_\alpha\nabla_\beta\Omega - S^\mu{}_{\alpha\beta}\nabla_\mu\Omega$$

with the tensor S given in (6.30) and

$$S^\mu{}_{\alpha\beta}h^\alpha{}_\sigma h^\beta{}_\sigma = \frac{1}{2}\mathcal{H}k_\nu\widehat{g}^{\mu\nu}(\nabla_\alpha k_\beta+\nabla_\beta k_\alpha)h^\alpha{}_\lambda h^\beta{}_\sigma = O(\Omega^n).$$

Thus

$$\widehat{R}_{\alpha\beta}h^\alpha{}_\lambda h^\beta{}_\sigma = -\frac{n-1}{\Omega}(\nabla_\alpha\nabla_\beta\Omega)h^\alpha{}_\lambda h^\beta{}_\sigma + \left(-\frac{\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Omega}{\Omega}+\frac{ns^2}{\lambda\Omega^2}\mathcal{H}\right)h_{\lambda\sigma}+O(\Omega^n)$$

so that from equation (6.33) it follows

$$\begin{aligned}\widehat{R}^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_{(\lambda}h^\alpha{}_{\sigma)} &= -\frac{n-3}{\Omega}(\nabla_\alpha\nabla_\beta\Omega)h^\alpha{}_\lambda h^\beta{}_\sigma \\ &+\left(-\frac{n-3}{n-1}\frac{\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Omega}{\Omega}+\frac{n-3}{n-1}\frac{ns^2}{\lambda\Omega^2}\mathcal{H}+\frac{\widehat{R}^\mu{}_\nu h^\nu{}_\mu}{n-1}-\frac{n-2}{n(n-1)}\widehat{R}\right)h_{\lambda\sigma} \\ &+O(\Omega^n).\end{aligned} \tag{6.36}$$

Combining equation (6.36) and (6.32) and putting the result back in (6.29) , we have proven

$$\begin{aligned}C^\mu{}_{\alpha\nu\beta}h^\nu{}_\mu h^\alpha{}_\lambda h^\alpha{}_\sigma = &\left(\frac{n-3}{n-1}\frac{\nabla_\mu\nabla^\mu\Omega}{\Omega}-\frac{R^\mu{}_\nu h^\nu{}_\mu}{n-1}+\frac{n-2}{n(n-1)}R\right. \\ &\left.-\frac{n-3}{n-1}\frac{\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Omega}{\Omega}+\frac{\widehat{R}^\mu{}_\nu h^\nu{}_\mu}{n-1}-\frac{n-2}{n(n-1)}\widehat{R}+\frac{n-3}{n-1}\frac{ns^2}{\lambda\Omega^2}\mathcal{H}\right)h_{\lambda\sigma} \\ &+O(\Omega^n)\end{aligned} \tag{6.37}$$

which is pure trace plus terms of order n. Now the following result is straightforward

Proposicin 6.19. *Let $\widetilde{g}$ be a Kerr-Schild-de Sitter metric and $g = \Omega^2 \widetilde{g}$ a geodesic conformal extension, with $\gamma = g|_{\mathscr{I}}$ conformally flat by definition. Then the electric part of the rescaled Weyl tensor is*

$$D_{\alpha\beta} = \frac{\kappa}{f^{n+2}} \left(\xi_\alpha \xi_\beta - \frac{|\xi|^2_\gamma}{n} \gamma_{\alpha\beta} \right)$$

where f is a function of $\mathscr{I}$ defined by $(\Omega^{-n}\mathcal{H})|_{\mathscr{I}} = \frac{2f^{-n}}{\lambda n(n-2)}$ and $\xi = fy$ is a CKVF of γ. Thus, the Kerr-Schild-de Sitter metrics are in the Kerr-de Sitter-like class.

Proof. By Lemma 6.11, we only have to prove that ξ is a CKVF of γ. The RHS of the Codazzi equation (6.21) is given by (6.25). Combining equation (6.26), Lemma 6.17 and equation (6.37), the non-zero terms of order Ω^{n-1} of $C^{\mu}{}_{(\alpha|\nu|\beta)} u^\nu y_\mu$ are pure trace. Thus, the traceless part of (6.25) is identically zero. By Corollary 6.16 this is precisely $0 = \Pi_{\alpha\beta}$ up to a multiplicative constant. Now the Proposition follows from Lemma 6.12 and Lemma 6.13. □

Observacin 6.20. *Throughout this section we restricted $\mathscr{I}$ to the set of points where $\mathcal{H}$ (and k) are not zero, because we assumed that $\kappa/f^n = \mathcal{F} \neq 0$ to write down (6.12) (i.e. we assume that f does not diverge). Now, we know that the vector ξ is a CKVF of $\mathscr{I}$, hence this vector is smooth everywhere. The set of points where it vanishes (i.e. where $f = 0$) must be removed from $\mathscr{I}$ as soon as the constant κ in the data $D = \kappa D_\xi$ is not zero because the tensor D_ξ is certainly singular at points where ξ vanishes.*

6.3 Kerr-Schild-de Sitter $\supset$ Kerr-de Sitter-like class

In this section we will prove the converse inclusion than in Section 6.2, namely, that every spacetime in the Kerr-de Sitter-like class is Kerr-Schild-de Sitter. Our strategy is to explicitly construct every Kerr-de Sitter-like spacetime in Kerr-Schild form. To do that, we take advantage of the property that the data in the Kerr-de Sitter-like class depends solely on the conformal class of the CKVF ξ (Lemma 5.21) and a mutiplicative constant. Since the initial value problem is well-posed and each spacetime with data $(\Sigma, \gamma, \kappa D_\xi)$ is uniquely determined by κ and the conformal class of ξ, we can infer all limits of spacetimes from the limits of data, which in turn are consequence of limits of conformal classes of CKVFs. The quotient space of conformal classes of CKVFs was studied in detail in Chapter 4.

6.3.1 Kerr-de Sitter and its limits at $\mathscr{I}$

The explicit form of the metrics in the full Kerr-de Sitter-like class will be obtained via either limits or analytic extensions of the Kerr-de Sitter family of metrics in all dimensions in [70]. The conformally extendable version of this family of metrics was given Section 5.3, i.e. the metrics $g = \Omega^2 \widetilde{g}$ were given, where $\widetilde{g}$ solves the Λ-vacuum field equations and g is smoothly extendable to $\{\Omega = 0\}$. Recall that in Section 5.3 some modifications w.r.t. to the original publication [70] were introduced which we keep here because they make our analysis more direct. Namely, as the limits will be inferred from its data at $\mathscr{I}$, it is convenient to give the metrics in coordinates such that, in the conformally extended space, the conformal factor vanishes at a finite value of the coordinates. Here, as we are interested in the calculation of limits of physical metrics, we shall use the physical version of the metrics $\widetilde{g} = \Omega^{-2} g$. We will also absorb some constants depending on the rotation parameters into the coordinates. This will allow us to perform several limits at once. Moreover, we give the metric already in Kerr-Schild form (6.1). This will be useful to show that the limits also belong to the Kerr-Schild-de Sitter class.

We remark, just like in Section 5.3, that in the following, when using index-free notation, the boldface font shall be used to distinguish a the metrically associated one-form $\boldsymbol{\xi} = \gamma(\xi, \cdot)$ to a CKVF ξ of $\mathscr{I}$.

Recall that the conformally extendable Kerr-de Sitter metric in Kerr-Schild form is given by

$$\widetilde{g} = \widetilde{g}_{dS} + \widetilde{\mathcal{H}}\, \widetilde{k} \otimes \widetilde{k}$$

where $\widetilde{g}_{dS}$, $\widetilde{k}$ and $\widetilde{\mathcal{H}}$ are the physical version of (5.31), (5.32) and (5.33) respectively, directly obtainable by (5.28). In the following, it will be convenient to rewrite these terms using the coordinates

$$\hat{\alpha}_i := \frac{\alpha_i}{(1+\lambda a_i^2)^{1/2}} \quad \Longrightarrow \quad \sum_{i=1}^{p+1} \alpha_i^2 = \sum_{i=1}^{p+1} (1+\lambda a_i^2)\hat{\alpha}_i^2 = 1 \tag{6.38}$$

so that the functions W, Ξ and Π in (5.30) are

$$W = \sum_{i=1}^{p+1} \hat{\alpha}_i^2, \qquad \Xi = \sum_{i=1}^{p+1} \frac{1+\lambda a_i^2}{1+\rho^2 a_i^2} \hat{\alpha}_i^2 \qquad \Pi = \prod_{j=1}^{p+1} (1+\rho^2 a_j^2), \tag{6.39}$$

and

$$\begin{aligned}\widetilde{g}_{dS} = &-W \frac{(\rho^2-\lambda)}{\rho^2} \mathrm{d}t^2 + \frac{\Xi}{\rho^2-\lambda} \frac{\mathrm{d}\rho^2}{\rho^2} + \delta_{p,q} \frac{\mathrm{d}\hat{\alpha}_{p+1}^2}{\rho^2} \\ &+ \sum_{i=1}^{q} \frac{1+\rho^2 a_i^2}{\rho^2} \left(\mathrm{d}\hat{\alpha}_i^2 + \hat{\alpha}_i^2 \mathrm{d}\phi_i^2\right) + \frac{(\rho^2-\lambda)}{\lambda W \rho^2} \frac{\mathrm{d}W^2}{4},\end{aligned} \tag{6.40}$$

$$\widetilde{k} = W \mathrm{d}t - \frac{\Xi}{\rho^2-\lambda} \mathrm{d}\rho - \sum_{i=1}^{q} a_i \hat{\alpha}_i^2 \mathrm{d}\phi_i, \tag{6.41}$$

$$\widetilde{\mathcal{H}} = \frac{2M\rho^n}{\Pi\Xi}, \qquad M \in \mathbb{R}. \tag{6.42}$$

where for (6.40) we have used the differential of (6.38)

$$\sum_{i=1}^{p+1}(1+\lambda a_i^2)\hat{\alpha}_i \mathrm{d}\hat{\alpha}_i = 0 \implies \sum_{i=1}^{p+1}\lambda a_i^2 \hat{\alpha}_i \mathrm{d}\hat{\alpha}_i = -\sum_{i=1}^{p+1}\hat{\alpha}_i \mathrm{d}\hat{\alpha}_i = -\frac{\mathrm{d}W}{2}$$
$$\implies \left(\sum_{i=1}^{p+1}\frac{\left(1+\rho^2 a_i^2\right)\alpha_i \mathrm{d}\alpha_i}{1+\lambda a_i^2}\right)^2 = \left(\sum_{i=1}^{p+1}\left(1+\rho^2 a_i^2\right)\hat{\alpha}_i \mathrm{d}\hat{\alpha}_i\right)^2$$
$$= \left(\frac{\rho^2-\lambda}{\lambda}\right)^2 \frac{\mathrm{d}W^2}{4}.$$

Recall the initial data $(\Sigma, \gamma, \kappa D_\xi)$ of the Kerr-de Sitter family calculated in Chapter 5, which in hatted $\{\hat{\alpha}_i\}$ coordinates, γ is

$$\gamma = \widetilde{g}_{dS} \mid_{\mathscr{I}} = \lambda W \mathrm{d}t^2 + \delta_{p,q}\mathrm{d}\hat{\alpha}_{p+1}^2 + \sum_{i=1}^{q}\left(\mathrm{d}\hat{\alpha}_i^2 + \hat{\alpha}_i^2 \mathrm{d}\phi_i^2\right) - \frac{1}{W}\frac{\mathrm{d}W^2}{4},$$

and the conformal Killing vector ξ is exactly the same

$$\xi = \frac{1}{\lambda}\partial_t - \sum_{i=1}^{q} a_i \partial_{\phi_i}. \tag{6.43}$$

Recall also that, after a suitable reordering of the rotational parameters $\{a_i\}$, the conformal class of ξ is determined by the parameters $\{\sigma = -\lambda^{-1}, \mu_i^2 = a_i^2\}$ for n odd and $\{-\mu_t^2 = -\lambda^{-1},\ \mu_s^2 = a_1^2, \mu_i^2 = a_{i+1}^2\}$ for n even (cf. Theorem 4.35). Observe that λ is one of the parameters which determines the conformal class of ξ. This is a priori fixed by the Einstein equations, so it is not a freely specifiable parameter of the metric. However, under scalings of ξ, σ is also scaled with the same factor. From the structure of D_ξ in (6.12), we have the freedom of scaling ξ and leave the data κD_ξ unaltered if we absorb the inverse (squared) scaling factor in κ, which is essentially the mass parameter of Kerr-de Sitter, therefore freely specifiable. In this way, we may cover the full domain defining the family $\mathcal{R}_-^{(n+2,0)}$. Obviously any point in any region $\mathcal{R}_-^{(n+2,m)}$ is also covered by considering the cases with m vanishing rotation parameters.

From Lemma 5.21, each metric in the Kerr-de Sitter-like class is determined by the parameter κ and the conformal class of ξ. Thus, for a fixed value of κ, one can associate exactly one metric in the Kerr-de Sitter-like class to each point in $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$. Moreover, the limits of regions in $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$, must induce limits of data $(\Sigma, \gamma, \kappa D_\xi)$ which in turn, from the well-posedness of the Cauchy problem, also induce limit of spacetimes corresponding to such data. In this way, we can endow the space of metrics in the Kerr-de Sitter-like class with the topology of $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$. Now, from the above discussion and Proposition 4.26, it follows

Proposicin 6.21. *The conformal class of the Kerr-de Sitter family with m vanishing rotation parameters belongs to the region $\mathcal{R}_-^{(n+2,m)}$ with $\sigma := -\lambda^{-1}$ and $\mu_i^2 := a_i^2$ for n odd and $-\mu_t^2 := -\lambda^{-1}$, $\mu_s^2 := a_1^2$ and $\mu_i^2 := a_{i+1}^2$ for n even. Thus, the Kerr-de Sitter* *family of metrics with all non-zero rotation parameters covers the whole* $\mathcal{R}_-^{(n+2,0)}$. *For* n *even, Kerr-de Sitter family data and its limits cover all data in the Kerr-de Sitter-like class.*

In the rest of this section, we will construct all spacetime metrics in the Kerr-de Sitter-like class taking advantage of the topological structure given in subsection 4.4.1, in particular in Proposition 4.26 and Remark 4.25. By these results all points in all regions $\{\mathcal{R}_-^{(n+2,m)}, \mathcal{R}_+^{(n+2,m)}, \mathcal{R}_0^{(n+2,m)}\}$ (recall that $d = n+2$ now) are attainable as limits of sequences in $\mathcal{R}_-^{(n+2,0)}$, except the region $\mathcal{R}_+^{(n+2,0)}$ when n is odd. Thus, the metrics corresponding to such data cannot be obtained as a limit of the Kerr-de Sitter family. This family will be obtained by analytic extension of Kerr-de Sitter.

Observacin 6.22. *For data* $(\Sigma, \gamma, \kappa D_\xi)$ *in the Kerr-de Sitter-like class the conformal class of* ξ *will be obtained always following the procedure of Theorem 4.35, as we did in Section 5.3 for the Kerr-de Sitter family. For the* n *even cases we shall give the conformal class of* ξ *in terms of the parameters* $\{-\mu_t^2, \mu_s^2, \mu_i^2\}$ *because they are directly related with the "rotation parameters" of Kerr-de Sitter and its limit metrics.*

The spacetime limits will be inferred from limits of data as follows. Start with data corresponding to Kerr-de Sitter $(\Sigma, \gamma, \kappa D_\xi)$ in $\mathcal{R}_-^{(n+2,m)}$, and consider the uniparametric set of equivalent data $(\Sigma, \gamma_\zeta := \zeta^{-2}\gamma, \zeta^{n-2}\kappa D_\xi)$ for a constant parameter $\zeta \in \mathbb{R}$. Scaling the following quantities as

$$M_\zeta := M\zeta^n, \qquad \xi_\zeta := \zeta\xi \qquad \boldsymbol{\xi}_\zeta := \gamma_\zeta(\xi_\zeta, \cdot) = \zeta^{-1}\boldsymbol{\xi}$$

we have

$$\begin{aligned}
\zeta^{n-2}\kappa D_\xi &= -\lambda^{\frac{-n}{2}} \frac{n(n-2)}{|\xi|_\gamma^{n+2}} M\zeta^{n-2} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma}{n}\gamma \right) \\
&= -\lambda^{\frac{-n}{2}} \frac{n(n-2)}{|\xi_\zeta|_{\gamma_\zeta}^{n+2}} M_\zeta \left(\boldsymbol{\xi}_\zeta \otimes \boldsymbol{\xi}_\zeta - \frac{|\xi_\zeta|_{\gamma_\zeta}^2}{n}\gamma_\zeta \right).
\end{aligned}$$

Thus, we obtain the uniparametric family of data $(\Sigma, \gamma_\zeta, \kappa_\zeta D_{\xi_\zeta})$, where κ_ζ is given by

$$\kappa_\zeta := -\frac{M_\zeta n(n-2)}{\lambda^{\frac{n}{2}}}.$$

As we shall describe, after a suitable rescaling of the coordinates and the rotation parameters, the data $(\Sigma, \gamma_\zeta, \kappa_\zeta D_{\xi_\zeta})$ admits regular limits as $\zeta \to 0$, which are no longer equivalent to the original family, but are still in the Kerr-de Sitter-like class.

By Lemma 5.21, the limit data are uniquely determined by the limit mass $M' := \lim_{\zeta\to 0} M_\zeta$ and the conformal class of $\xi' := \lim_{\zeta\to 0}\xi_\zeta$. In all cases, the scaling of the rotation parameters will be of the form

$$a_i = \zeta^{-1} b_i,$$

where we still allow b_i to smoothly depend on ζ. For the CKVF itself, in the following subsections we distinguish the limits of the vector field

$$\xi_\zeta = \zeta \left(\frac{1}{\lambda} \partial_t - \sum_{i=1}^{q} \zeta^{-1} b_i \partial_{\phi_i} \right)$$

as $\zeta \to 0$ into two types, depending on whether or not the parameter ζ is absorbed in the t coordinate by means of the change $t = \zeta t'$. The limits performed with the coordinate t' will be proven to correspond to the region $\mathcal{R}_0^{(n+2,m)}$. The limits with the t coordinate unchanged will only be calculated in the n even case and will be proven to lie in the region $\mathcal{R}_+^{(n+2,m)}$, where m is given by the number of vanishing b_i. The reason why we calculate them only for n even is because only in this case we may attain every point in every region $\mathcal{R}_+^{(n+2,m)}$ from $\mathcal{R}_-^{(n+2,0)}$ (cf. Proposition 4.26). For the n odd case we need to perform an analytic extension to obtain the spacetimes with data in $\mathcal{R}_+^{(n+2,m)}$. For any limit data at $\mathscr{I}$, there is one corresponding spacetime, which from the well-possedness of the Cauchy problem, must be a limit of Kerr-de Sitter. In general, these limit spacetimes are obtained with the same changes than those performed at $\mathscr{I}$ plus the redefinition $\rho' = \zeta \rho$, as we shall also explicitly demonstrate.

Observacin 6.23. *In all the situations, the term $\widetilde{g}_{dS}$ takes a well-defined limit independently of the term $\widetilde{\mathcal{H}}\ \widetilde{k} \otimes \widetilde{k}$. Morever, we will show that, in all cases, $\widetilde{g}_{dS}$ and its derivatives up to second order depend continuously on ζ. Consequently, the Riemann tensor of the limit metric $\widetilde{g}'_{dS} = \lim_{\zeta \to 0} \widetilde{g}_{dS}$ is the limit of the Riemann tensor of $\widetilde{g}$, i.e.*

$$\begin{aligned} R'_{\alpha\beta\mu\nu} = \lim_{\zeta \to 0} R_{\alpha\beta\mu\nu} &= \lambda \lim_{\zeta \to 0} \left((\widetilde{g}_{dS})_{\alpha\mu} (\widetilde{g}_{dS})_{\beta\nu} - (\widetilde{g}_{dS})_{\alpha\nu} (\widetilde{g}_{dS})_{\beta\mu} \right) \\ &= \lambda \left((\widetilde{g}'_{dS})_{\alpha\mu} (\widetilde{g}'_{dS})_{\beta\nu} - (\widetilde{g}'_{dS})_{\alpha\nu} (\widetilde{g}'_{dS})_{\beta\mu} \right). \end{aligned} \quad (6.44)$$

Thus background limit metric is still Einstein of constant curvature, therefore locally isometric to de Sitter.

As already mentioned, in the n even case all spacetimes in the Kerr-de Sitter-like class are limits of the Kerr-de Sitter family. In the n odd case, the spacetimes corresponding to the set $\mathcal{R}_+^{(n+2,0)}$ will be constructed by analytic continuation, and the rest of them as limits of Kerr-de Sitter. For given data, the corresponding spacetimes will be assigned to a family depending on the region $\mathcal{R}_\epsilon^{(n+2,m)}$ to which the defining CKVF at $\mathscr{I}$ belongs. In analogy with the $n = 3$ case [100], these families will be called *generalized* $\{a_i \to \infty\}$-*limit Kerr-de Sitter* if ξ lies in $\mathcal{R}_0^{(n+2,m)}$ (extending the definition [101]), or *generalized Wick-rotated Kerr-de Sitter* if ξ lies in $\mathcal{R}_+^{(n+2,m+1)}$ (also by analogy with [100]).

6.3.2 Limits n-even

We start by determining all limits of Kerr-de Sitter family in the n even case. In principle the limits can be performed in multiple ways. However, by the classification of conformal classes of CKVF described above it suffices to exhibit one limit for each case. To obtain the spacetimes whose CKVF class at $\mathscr{I}$ lies in $\mathcal{R}_{+}^{(n+2,m)}$, we will assume that the starting family has all its rotation parameters different from zero, i.e. that it belongs to the region $\mathcal{R}_{-}^{(n+2,0)}$. Similarly, to obtain those whose CKVF class lies in $\mathcal{R}_{0}^{(n+2,m)}$ we shall start from Kerr-de Sitter with exactly one rotation parameter equal to zero, i.e. whose CKVF is in $\mathcal{R}_{-}^{(n+2,1)}$. Obviously, all spacetimes in $\mathcal{R}_{-}^{(n+2,m)}$ are simply obtained by setting m rotation parameters a_i to zero, so there is no need to explicitly calculate any limit.

6.3.2.1 Generalized Wick-rotated

In this subsection we shall not absorb ζ in the coordinate t. As mentioned in subsection 6.3.1, we will obtain in this way all spacetimes whose corresponding CKVF at $\mathscr{I}$ lies in $\mathbb{R}_{+}^{(n+2,m)}$. We will call these *Wick-rotated-Kerr-de Sitter* family of spacetimes because in the n odd case (cf. subsection 6.3.3.1) they will actually be obtained by a Wick-rotation of Kerr-de Sitter.

We start with a metric in the Kerr-de Sitter family, with every rotation parameter being non-zero and apply the redefinitions

$$\rho = \zeta \rho', \qquad \hat{\alpha}_i = \zeta \beta_i \qquad a_i = \zeta^{-1} b_i, \qquad M = M' \zeta^n.$$

Observe that if any of the rotation parameters were zero, say $a_i = 0$, then the scaling of $\hat{\alpha}_i = \zeta \beta_i$ would not be allowed because (6.38) would imply that β_i is divergent in the limit $\zeta \to 0$. The parameters b_i are still allowed to depend smoothly[3] on ζ, so that their limit at ζ may take the value zero. For notational simplicity we shall not include the dependence on ζ. In particular, the limit at $\zeta \to 0$ will still be called b_i. The context will make clear the intended meaning.

In the limit $\zeta \to 0$, by (6.38) the coordinates $\{\beta_i\}$ satisfy

$$\sum_{i=1}^{p+1} \lambda b_i^2 \beta_i^2 = 1,$$

thus, at least one b_i must be non-zero. Note that if all were zero, the limit vector field $\xi' = \lim_{\zeta \to 0} \xi_\zeta$ would be identically zero, and we would at best fall outside the Kerr-de Sitter-like class.

[3]Sufficient differentiability is necessary in order to make sure that the background metric is de Sitter in the limit. W.l.o.g. we can assumme smooth dependence on ζ as we only want to allow vanishing values in the limit.

The function W goes to zero as ζ^2 while Ξ and Π take finite and smooth limits (cf. (6.39)). We therefore introduce the following limit quantities

$$W' := \lim_{\zeta\to 0} \zeta^{-2} W = \sum_{i=1}^{p+1} \beta_i^2, \quad \Xi' := \lim_{\zeta\to 0} \Xi = \sum_{i=1}^{p+1} \frac{\lambda b_i^2 \beta_i^2}{1+\rho'^2 b_i^2}, \quad \Pi' = \lim_{\zeta\to 0} \Pi = \prod_{j=1}^{q} (1+\rho'^2 b_j^2).$$

On the other hand, by (6.41), the terms of $\widetilde{k}$ in $\mathrm{d}\rho$ and $\mathrm{d}\phi_i$ tend to zero with ζ, while the term in $\mathrm{d}t$ goes with ζ^2. Hence we set

$$\widetilde{k}' := \lim_{\zeta\to 0} \zeta^{-1} \widetilde{k} = \frac{\Xi'}{\lambda} \mathrm{d}\rho' - \sum_{i=1}^{q} b_i \beta_i^2 \mathrm{d}\phi_i,$$

and the redefinition of mass $M' = \zeta^n M$ absorbs the zero of $\widetilde{k} \otimes \widetilde{k}$ and that of $\rho^{n-2} = \zeta^{n-2}\rho'$ in $\widetilde{\mathcal{H}}\, \tilde{k} \otimes \tilde{k}$ (cf. (6.42)). Thus, the limit metric has the Kerr-Schild form

$$\widetilde{g}' = \widetilde{g}'_{dS} + \widetilde{\mathcal{H}}' \widetilde{k}' \otimes \widetilde{k}', \qquad \widetilde{\mathcal{H}}' = \frac{2M'\rho'^{n-2}}{\Pi'\Xi'}, \qquad M' \in \mathbb{R} \tag{6.45}$$

with

$$\widetilde{g}'_{dS} = \frac{\lambda W'}{\rho'^2} \mathrm{d}t^2 - \frac{\Xi'}{\lambda} \frac{\mathrm{d}\rho'^2}{\rho'^2} + \sum_{i=1}^{q} \frac{1+\rho'^2 b_i^2}{\rho'^2} \left(\mathrm{d}\beta_i^2 + \beta_i^2 \mathrm{d}\phi_i^2 \right) - \frac{1}{W'\rho'^2} \frac{\mathrm{d}W'^2}{4}. \tag{6.46}$$

One can easily check that the original de Sitter metric $\widetilde{g}_{dS}$ in (6.40), written in primed coordinates is C^2 in ζ. Hence, by the above argument $\widetilde{g}'_{dS}$ (cf. (6.44)), the limit metric (6.46), is (locally) isometric to de Sitter.

Consider the conformal extension $g' = \rho'^2 \widetilde{g}'$. The boundary metric induced by g' coincides with the one induced by $\widetilde{g}'_{dS}$, which is

$$\gamma' = \rho'^2 \widetilde{g}'_{dS}|_{\mathscr{I}} = \lambda W' \mathrm{d}t^2 + \sum_{i=1}^{q} \left(\mathrm{d}\beta_i^2 + \beta_i^2 \mathrm{d}\phi_i^2 \right) - \frac{1}{W'} \frac{\mathrm{d}W'^2}{4}.$$

As $\widetilde{g}_{dS}$ is locally isometric to de Sitter, γ' must be locally conformally flat.

To calculate the electric part of the rescaled Weyl tensor, we use formula (5.6), after which it follows

$$D = \rho'^{n-2} C_{\perp}|_{\mathscr{I}} = -\lambda n(n-2) M' \left(\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' - \frac{|\xi'|^2_{\gamma'}}{n} \gamma' \right),$$

where $\boldsymbol{\xi}'$ is the projection of $\widetilde{k}'$ onto $\mathscr{I}$

$$\boldsymbol{\xi}' = -\sum_{i=1}^{q} b_i \beta_i^2 \mathrm{d}\phi_i \quad \Longrightarrow \quad \xi' = -\sum_{i=1}^{q} b_i \partial_{\phi_i}.$$

This is obviously a (conformal) Killing vector field of γ'. Therefore, the metric (6.45) is in the Kerr-de Sitter-like class. To calculate the conformal class of ξ, we find an explicitly flat representative in $[\gamma']$. It is a matter of direct computation to check that the coordinate change[4]

$$x_i = \frac{e^{\sqrt{\lambda}t}}{\sqrt{W'}}\beta_i \cos\phi_i, \qquad y_i = \frac{e^{\sqrt{\lambda}t}}{\sqrt{W'}}\beta_i \sin\phi_i,$$

brings the metric γ' into the form

$$\gamma' = \frac{W'}{e^{2\sqrt{\lambda}t}} \sum_{i=1}^{q} \left(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\right).$$

Hence $\gamma_E := e^{2\sqrt{\lambda}t} W^{-1}\gamma$ is flat and ξ' is in Cartesian coordinates $\{x_i, y_i\}$:

$$\xi' = -\sum_{i=1}^{q} b_i (x_i \partial_{y_i} - y_i \partial_{x_i}).$$

Thus, ξ is the sum of generators of rotations within q different orthogonal planes. Its corresponding skew-symmetric endomorphism of $\mathbb{M}^{1,n+1}$, with respect to an orthogonal unit basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ with e_0 timelike, can be directly calculated from (2.26):

$$F(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \bigoplus_{i=1}^{q} \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix}. \tag{6.47}$$

The orthogonal sum of two-dimensional blocks is adapted to the decomposition

$$\mathbb{M}^{1,n+1} = \Pi_0 \bigoplus_{i=1}^{q} \Pi_i$$

where $\Pi_0 = \mathrm{span}\{e_0, e_1\}$ and $\Pi_i = \mathrm{span}\{e_{2i}, e_{2i+1}\}$ are F-invariant planes. The causal character of $\ker F(\xi)$ is evidently timelike because $e_0 \in \ker F(\xi)$ and the polynomial $\mathcal{Q}_{F^2}$ in Definition 4.12 is also straightforwardly computable from the block form (6.47)

$$\mathcal{Q}_{F^2}(x) = \prod_{i=1}^{q} (x - b_i^2).$$

Then, permuting the indices i so that the rotation parameters b_i^2 appear in decreasing order $b_1^2 \geq \cdots \geq b_q^2$, and applying Theorem 4.35, the conformal class of ξ is defined by the parameters

$$\{-\mu_t^2 = 0, \mu_s^2 = b_1^2; \mu_1^2 = b_2^2, \cdots, \mu_p^2 = b_q^2\}.$$

In consequence, for b_is taking arbitrary values, this family covers every point in every region $\mathcal{R}_+^{(n+2,m)}$ of the quotient $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$, where m is the number of

[4]This and the following Cartesian coordinates in this section are inspired from the Kerr-de Sitter case in equation (5.37).

vanishing b_is.

6.3.2.2 Generalized $\{a_i \to \infty\}$-limit Kerr-de Sitter.

In this subsection we perform the limits that cover the regions $\mathcal{R}_0^{(n+2,m)}$ of the quotient $\mathrm{CKill}(\mathbb{E}^n)/\mathrm{ConfLoc}(\mathbb{E}^n)$. In this case, the limits are achieved by absorbing ζ in the t coordinate, i.e. defining $t' = \zeta^{-1}t$, so that the limit vector field $\xi' = \lim_{\zeta\to 0} \xi_\zeta$ has a non-zero term in $\partial_{t'}$. It turns out that these limits lie in the Kerr-de Sitter-like class provided that the Kerr-de Sitter metric from which they are calculated have one rotation parameter vanishing. Otherwise the limit of the boundary metric is degenerate. Thus we will assume that $a_q = 0$. We name the limit spacetimes obtained in this way $\{a_i \to \infty\}$-*limit-Kerr-de Sitter* because the conformal class that characterizes them is similar to the $n = 3$ case [101].

Consider the de Sitter metric (6.40) with the change of coordinates

$$\rho = \zeta\rho', \qquad t = \zeta t', \qquad \phi_q = \zeta\Phi, \qquad \hat{\alpha}_i = \zeta\beta_i \qquad (i = 1, \cdots, p),$$

where note that the coordinate $\hat{\alpha}_q$ and the angles ϕ_i $(i = 1, ..., p)$ remain unaltered. In addition, let us redefine the parameters

$$M = M'\zeta^n, \qquad a_i = \zeta^{-1}b_i \qquad (i = 1, \cdots, p).$$

By (6.38), the coordinates $\{\beta_i, \hat{\alpha}_q\}$ satisfy in the limit $\zeta \to 0$:

$$\hat{\alpha}_q^2 + \sum_{i=1}^{p} \lambda b_i^2 \beta_i^2 = 1. \tag{6.48}$$

The limits of W, Ξ and Π are obtained immediately from (6.39) respectively. They are

$$\Pi' = \lim_{\zeta\to 0} \Pi = \prod_{j=1}^{p} (1+\rho'^2 b_j^2), \qquad W' = \lim_{\zeta\to 0} W = \hat{\alpha}_q^2, \qquad \Xi' = \lim_{\zeta\to 0} \Xi = \hat{\alpha}_q^2 + \sum_{i=1}^{p} \frac{\lambda b_i^2}{1+\rho'^2 b_i^2}\beta_i^2.$$

In addition from (6.42) and (6.41) and the redefinitions above it follows

$$\widetilde{\mathcal{H}}'\widetilde{k}' \otimes \widetilde{k}' := \lim_{\zeta\to 0} \mathcal{H}\widetilde{k} \otimes \widetilde{k} = \underbrace{\frac{2M'}{\Pi'\Xi'}\rho'^{n-2}}_{=:\widetilde{\mathcal{H}}'} \underbrace{\left(W'\mathrm{d}t' + \frac{\Xi'}{\lambda}\mathrm{d}\rho' - \sum_{i=1}^{p} b_i \beta_i^2 \mathrm{d}\phi_i \right)}_{=:\widetilde{k}'}{}^2.$$

Before taking the limit, we rewrite the de Sitter metric (6.40) in the new coordinates and separate the terms multiplying $\mathrm{d}\hat{\alpha}_q$

$$\begin{aligned}\widetilde{g}_{dS} = &-W\frac{(\zeta^2\rho'^2-\lambda)}{\rho'^2}\mathrm{d}t'^2 + \frac{\Xi}{\zeta^2\rho'^2-\lambda}\frac{\mathrm{d}\rho'^2}{\rho'^2} + \frac{1}{\zeta^2\rho'^2}\left(\mathrm{d}\hat{\alpha}_q^2 + \hat{\alpha}_q^2\zeta^2\mathrm{d}\Phi^2\right) \\ &+\sum_{i=1}^{p}\frac{1+\rho'^2b_i^2}{\rho'^2}\left(\mathrm{d}\beta_i^2+\beta_i^2\mathrm{d}\phi_i^2\right) + \frac{(\zeta^2\rho'^2-\lambda)}{\lambda W\zeta^2\rho'^2}\left(\hat{\alpha}_q\mathrm{d}\hat{\alpha}_q + \zeta^2\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)^2, \end{aligned} \tag{6.49}$$

with

$$W = \hat{\alpha}_q^2 + \zeta^2\sum_{i=1}^{p}\beta_i^2, \qquad \Xi = \hat{\alpha}_q^2 + \sum_{i=1}^{p}\frac{\zeta^2+\lambda b_i^2}{1+\rho'^2b_i^2}\beta_i^2.$$

Only the terms involving $\mathrm{d}\hat{\alpha}_q$ are troublesome in the limit $\zeta \to 0$. Let us gather them to get

$$\begin{aligned} g_{(\alpha_q)} :=& \frac{1}{\rho'^2\zeta^2}\mathrm{d}\hat{\alpha}_q^2 + \frac{(\zeta^2\rho'^2-\lambda)}{\lambda W\zeta^2\rho'^2}\left(\hat{\alpha}_q\mathrm{d}\hat{\alpha}_q + \zeta^2\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)^2 \\ =& \frac{1}{\zeta^2\rho'^2}\left(1+\frac{(\zeta^2\rho'^2-\lambda)\hat{\alpha}_q^2}{\lambda W}\right)\mathrm{d}\hat{\alpha}_q^2 \\ &+ \frac{(\zeta^2\rho'^2-\lambda)}{\lambda W\zeta^2\rho'^2}\left(\zeta^4\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)^2 + 2\zeta^2\hat{\alpha}_q\mathrm{d}\hat{\alpha}_q\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)\right). \end{aligned}$$

Writting W in coordinates $\{\beta_i, \hat{\alpha}_q\}$, the term in $\mathrm{d}\hat{\alpha}_q^2$ takes the limit

$$\begin{aligned}\lim_{\zeta\to 0}\frac{1}{\zeta^2\rho'^2}\left(1+\frac{(\zeta^2\rho'^2-\lambda)\hat{\alpha}_q^2}{\lambda W}\right)\mathrm{d}\hat{\alpha}_q^2 &= \lim_{\zeta\to 0}\frac{\lambda\zeta^2\left(\sum_{i=1}^p\beta_i^2\right)+\zeta^2\rho'^2\hat{\alpha}_q^2}{\zeta^2\rho'^2\lambda(\hat{\alpha}_q^2+\zeta^2\sum_{i=1}^p\beta_i^2)}\mathrm{d}\hat{\alpha}_q^2 \\ &= \frac{1}{\lambda}+\frac{\sum_{i=1}^p\beta_i^2}{\rho'^2\hat{\alpha}_q^2}, \end{aligned}$$

while the limit of the last two terms is direct

$$\begin{aligned}&\lim_{\zeta\to 0}\frac{(\zeta^2\rho'^2-\lambda)}{\lambda W\zeta^2\rho'^2}\left(\zeta^4\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)^2 + 2\zeta^2\hat{\alpha}_q\mathrm{d}\hat{\alpha}_q\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right)\right) \\ &= -\frac{2\mathrm{d}\hat{\alpha}_q}{\rho'^2\hat{\alpha}_q}\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right). \end{aligned}$$

Thus, the limit $\zeta \to 0$ of (6.49) is

$$\begin{aligned}\widetilde{g}'_{dS} =& \frac{\lambda\hat{\alpha}_q^2}{\rho'^2}\mathrm{d}t'^2 - \frac{\Xi'}{\lambda}\frac{\mathrm{d}\rho'^2}{\rho'^2} + \frac{\hat{\alpha}_q^2\mathrm{d}\Phi^2}{\rho'^2} + \sum_{i=1}^{p}\frac{1+\rho'^2b_i^2}{\rho'^2}\left(\mathrm{d}\beta_i^2+\beta_i^2\mathrm{d}\phi_i^2\right) \\ &+\left(\frac{1}{\lambda}+\frac{\sum_{i=1}^p\beta_i^2}{\rho'^2\hat{\alpha}_q^2}\right)\mathrm{d}\hat{\alpha}_q^2 - \frac{2\mathrm{d}\hat{\alpha}_q}{\rho'^2\hat{\alpha}_q}\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right), \end{aligned}$$

where we have already substituted $W' = \hat{\alpha}_q^2$. From the argument above (cf. (6.44)) $\widetilde{g}'_{dS}$ is locally isometric to de Sitter. Thus, we have all the ingredients to build up the limit Kerr-Schild metric, namely

$$\widetilde{g}' = \widetilde{g}'_{dS} + \widetilde{\mathcal{H}}'\widetilde{k}' \otimes \widetilde{k}', \qquad \widetilde{\mathcal{H}}' = \frac{2M'\rho'^{n-2}}{\Pi'\Xi'}, \quad M' \in \mathbb{R}.$$

We now calculate the asymptotic structure and verify that indeed, these spacetimes correspond to the regions $\mathcal{R}_0^{(n+2,m)}$ in the space of orbits. The boundary metric is

$$\begin{aligned}\gamma' = \rho'^2\widetilde{g}'|_{\mathscr{I}} &= \lambda\hat{\alpha}_q^2\mathrm{d}t'^2 + \hat{\alpha}_q^2\mathrm{d}\Phi^2 + \sum_{i=1}^{p}\left(\mathrm{d}\beta_i^2 + \beta_i^2\mathrm{d}\phi_i^2\right) \\ &+ \left(\sum_{i=1}^{p}\beta_i^2\right)\frac{\mathrm{d}\hat{\alpha}_q^2}{\hat{\alpha}_q^2} - \frac{2\mathrm{d}\hat{\alpha}_q}{\hat{\alpha}_q}\left(\sum_{i=1}^{p}\beta_i\mathrm{d}\beta_i\right). \end{aligned} \tag{6.50}$$

As usual, the TT tensor $D_{\xi'}$ is directly calculated with equation (5.6)

$$D = \rho'^{n-2}C_{\perp}|_{\mathscr{I}} = -\lambda n(n-2)M'\left(\boldsymbol{\xi}'\otimes\boldsymbol{\xi}' - \frac{|\xi'|^2_{\gamma'}}{n}\gamma'\right),$$

where $\boldsymbol{\xi}'$ is the projection of $\widetilde{k}'$ onto $\mathscr{I}$

$$\boldsymbol{\xi}' = \hat{\alpha}_q^2\mathrm{d}t' - \sum_{i=1}^{p}b_i\beta_i^2\mathrm{d}\phi_i \implies \xi' = \frac{1}{\lambda}\partial_{t'} - \sum_{i=1}^{p}b_i\partial_{\phi_i}.$$

To calculate the conformal class of ξ', we look for a flat representative in $[\gamma']$ written in Cartesian cordinates. It turns out to be useful to scale the coordinates $\{\beta_i\}_{i=1}^p$ as

$$\widetilde{\beta}_i = \frac{\beta_i}{\hat{\alpha}_q}.$$

Replacing $\beta_i = \widetilde{\beta}_i\hat{\alpha}_q$ and $\mathrm{d}\beta_i = \hat{\alpha}_q\mathrm{d}\widetilde{\beta}_i + \widetilde{\beta}_i\mathrm{d}\hat{\alpha}_q$ in equation (6.50), all terms in $\mathrm{d}\hat{\alpha}_q$ cancel out and we are left with the expression

$$\gamma' = \hat{\alpha}_q^2\left(\lambda\mathrm{d}t'^2 + \mathrm{d}\Phi^2 + \sum_{i=1}^{p}\left(\mathrm{d}\widetilde{\beta}_i^2 + \widetilde{\beta}_i^2\mathrm{d}\phi_i^2\right)\right).$$

This determines a flat representative $\gamma_E := \hat{\alpha}_q^{-2}\gamma'$, where by (6.48), $\hat{\alpha}_q$ is written explictly in terms of $\{\widetilde{\beta}_i\}$ as $\hat{\alpha}_q^2 = (1+\lambda\sum_{i=1}^p b_i^2\widetilde{\beta}_i^2)^{-1}$. The set $\{\tau := \lambda^{1/2}t', \Phi, x_i := \widetilde{\beta}_i\cos\phi_i, y_i := \widetilde{\beta}_i\sin\phi_i\}$ define Cartesian coordinates for γ', into which vector field ξ' reads

$$\xi' = \frac{1}{\lambda^{1/2}}\partial_\tau - \sum_{i=1}^{p}b_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

i.e. is the sum of translation along the coordinate τ plus the sum of p independent orthogonal rotations. Its correspoding skew-symmetric endomorphism of $\mathbb{M}^{1,n+1}$ is by

(2.26)

$$F(\xi) = \begin{pmatrix} 0 & 0 & \frac{\lambda^{-1/2}}{2} & 0 \\ 0 & 0 & -\frac{\lambda^{-1/2}}{2} & 0 \\ \frac{\lambda^{-1/2}}{2} & \frac{\lambda^{-1/2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \bigoplus_{i=1}^{p} \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix} \tag{6.51}$$

in an orthogonal unit basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ with e_0 timelike. Similar to subsection 6.3.2.1, the direct sum (6.51) is adapted to the decomposition

$$\mathbb{M}^{1,n+1} = \mathbb{M}^{1,3} \bigoplus_{i=1}^{p} \Pi_i,$$

where $\mathbb{M}^{1,3} = \text{span}\{e_0, e_1, e_2, e_3\}$ and $\Pi_i = \text{span}\{e_{2(i+1)}, e_{2(i+1)+1}\}$ are F-invariant subspaces. The causal character of $\ker F(\xi')$ is determined by the causal character of $\ker F(\xi')|_{\mathbb{M}^{1,3}}$, because every non-spacelike vector $v \in \ker F(\xi')$ must have non-zero projection $v_0 \in \mathbb{M}^{1,3}$ with $v_0 \in \ker F(\xi')|_{\mathbb{M}^{1,3}}$. It is immediate to calculate $\ker F(\xi')|_{\mathbb{M}^{1,3}} = \text{span}\{e_0 - e_1, e_3\}$, where $e_0 - e_1$ is a null vector in $\ker F(\xi')$, thus $\ker F(\xi')$ is degenerate. The polynomial $\mathcal{Q}_{F^2}$ in Definition 4.12 is by direct calculation

$$\mathcal{Q}_{F^2} = x^2 \prod_{i=1}^{p} (x - b_i^2).$$

This, by Theorem 4.35, gives the parameters for the conformal class of ξ'

$$\{-\mu_t^2 = 0, \mu_s^2 = 0; \mu_1^2 = b_1^2, \cdots, \mu_p^2 = b_p^2\}.$$

This collection of conformal classes covers every point in every region $\mathcal{R}_0^{(n+2,m)}$, where m is the number of zero b_i parameters.

6.3.3 Limits n-odd

One major difference between the n odd and even cases is that, only when n is even the region $\mathcal{R}_-^{(n+2,0)}$ (namely the portion corresponding to Kerr-de Sitter with none of the rotation parameters vanishing) admits limit in the whole of $\text{CKill}(\mathbb{E}^n)/\text{ConfLoc}(\mathbb{E}^n)$ (cf. Proposition 4.26). This is what allowed us to construct all spacetimes in the Kerr-de Sitter-like class directly as limits of Kerr-de Sitter in subsection 6.3.2. In the n odd case, no sequence in $\mathcal{R}_-^{(n+2,0)}$ takes limit at $\mathcal{R}_+^{(n+2,0)}$ and viceversa, because they are disjoint and open subspaces by Proposition 4.26. In subsection 6.3.3.1 we deal with this issue by constructing, using analytic continuation of Kerr-de Sitter, the set spacetimes whose CKVF class corresponds to $\mathcal{R}_+^{(n+2,0)}$. To do this, we define a Wick rotation in arbitrary $n+1$ even dimensions (generalizing the transformation in [88]). We name the resulting family Wick-rotated-Kerr-de Sitter, in analogy with the $n = 3$ case in [100]. From these, all spacetimes in $\mathcal{R}_{+}^{(n+2,m)}$ can be obtained easily. Subection 6.3.3.2 is devoted to finding the spacetimes whose CKVF class corresponds to $\mathcal{R}_{0}^{(n+2,m)}$. These are obtained by performing limits to Kerr-de Sitter, similar to those in subsection 6.3.2.2.

6.3.3.1 Generalized Wick-rotated

Let now n be odd and let us consider the Kerr-de Sitter metric with none of the rotation parameters a_i equal to zero. The generalization of the Wick rotation is given by the following complex coordinate transformation

$$t = it', \qquad \rho = i\rho', \qquad \hat{\alpha}_i = i\beta_i, \quad i = 1, \cdots, p, \tag{6.52}$$

with $t', \rho', \beta_i \in \mathbb{R}$, and the redefinition of parameters

$$a_i = -ib_i, \qquad M = (-1)^{\frac{n+1}{2}} iM', \qquad M' \in \mathbb{R}.$$

Note that the only the first p $\hat{\alpha}_i$ coordinates have been "rotated". Introducing $\beta_{p+1} := \hat{\alpha}_{p+1}$, (6.38) gives:

$$\beta_{p+1}^2 - \sum_{i=1}^{p} (1 - \lambda b_i^2)\beta_i^2 = 1.$$

By performing the Wick rotation (6.52), the functions W, Ξ in and Π in (6.39) are now redefined

$$W' := \beta_{p+1}^2 - \sum_{i=1}^{p} \beta_i^2, \qquad \Xi' := \beta_{p+1}^2 - \sum_{i=1}^{p} \frac{1 - \lambda b_i^2}{1 + \rho'^2 b_i^2} \beta_i^2, \qquad \Pi' = \prod_{j=1}^{p} (1 + \rho'^2 b_j^2). \tag{6.53}$$

The spacetime metric is given by

$$\widetilde{g}' = \widetilde{g}'_{dS} + \widetilde{\mathcal{H}}' \widetilde{k}' \otimes \widetilde{k}, \qquad \widetilde{\mathcal{H}}' = \frac{2M'\rho'^{n-2}}{\Pi'\Xi'} \quad M' \in \mathbb{R}$$

with

$$\begin{aligned} \widetilde{g}'_{dS} &= W' \frac{(\rho'^2 + \lambda)}{\rho'^2} \mathrm{d}t'^2 - \frac{\Xi'}{\rho'^2 + \lambda} \frac{\mathrm{d}\rho'^2}{\rho'^2} - \frac{\mathrm{d}\beta_{p+1}^2}{\rho'^2} \\ &\quad + \sum_{i=1}^{p} \frac{1 + \rho'^2 b_i^2}{\rho'^2} \left(\mathrm{d}\beta_i^2 + \beta_i^2 \mathrm{d}\phi_i^2 \right) + \frac{(\rho'^2 + \lambda)}{\lambda W' \rho'^2} \frac{\mathrm{d}W'^2}{4}, \\ \widetilde{k} &:= \left(W' \mathrm{d}\widetilde{t}' + \frac{\Xi'}{\rho'^2 + \lambda} \mathrm{d}\rho' - \sum_{i=1}^{p} b_i \beta_i^2 \mathrm{d}\phi_i \right). \end{aligned} \tag{6.54}$$

The domain of definition of the coordinates is $t', \rho' \in \mathbb{R}$, and the $\phi_i \in [0, 2\pi)$ are still angles. Moreover, $\beta_{p+1}^2 > 0$, and $\{\beta_i\}_{i=1}^p$ are restricted to a sufficiently small neighbourhood of $\{\beta_i = 0\}_{i=1}^p$ so that W', Ξ' are positive (see (6.53)). With this restriction of the coordinates, vanishing values of the b_i parameters are allowed.

The signature is not necessarily preserved after a Wick rotation, so we still need to prove that the Wick-rotated Kerr-de Sitter metrics are Lorentzian. They are obviously $\Lambda > 0$-vacuum Einstein because we have only performed a (complex) change of coordinates. From the Einstein equations and positivity of the cosmological constant, it follows that the boundary metric is positive definite if and only if the spacetime metric is Lorentzian in a neighbourhood of $\mathscr{I}$. In addition, note that the boundary metric induced by $\widetilde{g}'$ is the same as the one induced by $\widetilde{g}'_{dS}$. Moreover, $\widetilde{g}'_{dS}$ is clearly Einstein of constant curvature. Thus, proving that γ' is positive definite, in turn, also proves that $\widetilde{g}'_{dS}$ is Lorentzian and therefore locally isometric to de Sitter.

The metric induced at $\mathscr{I}$ is, directly from (6.54),

$$\gamma' = W'\lambda \mathrm{d}t'^2 - \mathrm{d}\beta_{p+1}^2 + \sum_{i=1}^{p}\left(\mathrm{d}\beta_i^2 + \beta_i^2 \mathrm{d}\phi_i^2\right) + \frac{1}{W'}\frac{\mathrm{d}W'^2}{4}. \tag{6.55}$$

The explicitly conformally flat form is obtained under the change of coordinates

$$\widetilde{\beta}_i = \frac{\beta_i}{W'^{1/2}}, \qquad i = 1, \cdots, p+1. \tag{6.56}$$

Observe that by redefining all the $p+1$ coordinates we now have

$$W' = \beta_{p+1}^2 - \sum_{i=1}^{p}\beta_i^2 = W'(\widetilde{\beta}_{p+1}^2 - \sum_{i=1}^{p}\widetilde{\beta}_i^2) \implies \widetilde{\beta}_{p+1}^2 - \sum_{i=1}^{p}\widetilde{\beta}_i^2 = 1 \tag{6.57}$$

and

$$W'\widetilde{\beta}_{p+1}^2 = \beta_{p+1}^2 = 1 + \sum_{i=1}^{p}(1-\lambda b_i^2)\beta_i^2 = 1 + W'\sum_{i=1}^{p}(1-\lambda b_i^2)\widetilde{\beta}_i^2$$
$$\implies W' = \frac{1}{1 + \sum_{i=1}^{p}\lambda b_i^2 \widetilde{\beta}_i^2}.$$

Inserting the coordinate change (6.56) into (6.55) gives

$$\gamma' = W'\left(\lambda \mathrm{d}t'^2 - \mathrm{d}\widetilde{\beta}_{p+1}^2 + \sum_{i=1}^{p}\left(\mathrm{d}\widetilde{\beta}_i^2 + \widetilde{\beta}_i^2 \mathrm{d}\phi_i^2\right)\right).$$

From this expression it already follows that γ' is Riemannian, because the restriction of γ' to the hypersurfaces $\{t' = const.\}$ is clearly the standard metric of the hyperboloid (cf. (6.57)). More specifically, let us introduce the parametrization

$$\widetilde{\beta}_{p+1} = \cosh\chi, \qquad \widetilde{\beta}_i = \nu_i \sinh\chi, \quad i = 1, \cdots, p, \quad \text{with} \quad \sum_{i=1}^{p}\nu_i^2 = 1,$$

so that

$$\gamma' = W'\left(\lambda \mathrm{d}t'^2 + \mathrm{d}\chi^2 + \sinh^2\chi\ \gamma_{\mathbb{S}^{n-2}}\right),$$

where

$$\gamma_{\mathbb{S}^{n-2}} := \sum_{i=1}^{p} \left(\mathrm{d}\nu_i^2 + \nu_i^2 \mathrm{d}\phi_i^2\right)|_{\sum_{i=1}^{p} \nu_i^2 = 1}$$

is an $(n-2)$-dimensional spherical metric. Finally, defining the coordinates

$$z := \frac{\sin\sqrt{\lambda}t'}{\cos\sqrt{\lambda}t' + \cosh\chi}, \qquad x_i := \frac{\nu_i \cos\phi_i \sinh\chi}{\cos\sqrt{\lambda}t' + \cosh\chi}, \qquad y_i := \frac{\nu_i \sin\phi_i \sinh\chi}{\cos\sqrt{\lambda}t' + \cosh\chi}$$

for $i = 1, \cdots, p$, one has

$$\gamma_E := \frac{1}{W'(\cos\sqrt{\lambda}t' + \cosh\chi)^2}\gamma' = \mathrm{d}z^2 + \sum_{i=1}^{p}\left(\mathrm{d}x_i^2 + \mathrm{d}y_i^2\right).$$

Thus γ_E is a flat representative $\gamma_E \in [\gamma']$ and $\{z, x_i, y_i\}$ are Cartesian coordinates of γ_E.

We continue by calculating the electric part of the rescaled Weyl tensor at $\mathscr{I}$. As usual, the expression follows from formula (5.6). We give it first in coordinates $\{t', \rho', \beta_i, \phi_i\}$:

$$D_{\xi'} = \rho'^{n-2} C_{\perp}|_{\mathscr{I}} = -\lambda n(n-2)M'\left(\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' - \frac{|\xi'|^2_{\gamma'}}{n}\gamma'\right),$$

where $\boldsymbol{\xi}'$ is the projection of $\widetilde{k}'$ onto $\mathscr{I}$

$$\boldsymbol{\xi}' = W'\mathrm{d}t' - \sum_{i=1}^{p} b_i\beta_i^2 \mathrm{d}\phi_i \quad\Longrightarrow\quad \xi' = \frac{1}{\lambda}\partial_{t'} - \sum_{i=1}^{q} b_i \partial_{\phi_i}.$$

To express ξ' in Cartesian coordinates $\{z, \{x_i, y_i\}_{i=1}^{p}\}$, firstly observe

$$\frac{\partial z}{\partial t'} = \sqrt{\lambda}\,\frac{\cos\sqrt{\lambda}t'(\cos\sqrt{\lambda}t' + \cosh\chi) + \sin^2\sqrt{\lambda}t'}{(\cos\sqrt{\lambda}t' + \cosh\chi)^2} = \sqrt{\lambda}\left(\frac{1}{2} - \frac{1}{2} + \frac{1 + \cos\sqrt{\lambda}t'\cosh\chi}{(\cos\sqrt{\lambda}t' + \cosh\chi)^2}\right)$$

$$= \frac{\sqrt{\lambda}}{2} + \frac{\sqrt{\lambda}}{2}\left(z^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right),$$

and it is also straightforward that

$$\frac{\partial x_i}{\partial t'} = \sqrt{\lambda} z x_i, \qquad \frac{\partial y_i}{\partial t'} = \sqrt{\lambda} z y_i.$$

Then

$$\begin{aligned}\partial_t &= \frac{\partial z}{\partial t}\partial_z + \sum_{i=1}^{p}\left(\frac{\partial x_i}{\partial t}\partial_{x_i} + \frac{\partial y_i}{\partial t}\partial_{y_i}\right)\\ &= \frac{\sqrt{\lambda}}{2}\left(1 + z^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right)\partial_z + \sqrt{\lambda} z \sum_{i=1}^{p}(x_i\partial_{x_i} + y_i\partial_{y_i})\end{aligned}$$

and on the other hand

$$\partial_{\phi_i} = \frac{\partial x_i}{\partial \phi_i}\partial_{x_i} + \frac{\partial y_i}{\partial \phi_i}\partial_{y_i} = x_i\partial_{y_i} - y_i\partial_{x_i}.$$

Therefore

$$\xi = \frac{1}{2\sqrt{\lambda}}\left(1 + z^2 - \sum_{i=1}^{p}(x_i^2 + y_i^2)\right)\partial_z + \frac{z}{\sqrt{\lambda}}\sum_{i=1}^{p}(x_i\partial_{x_i} + y_i\partial_{y_i}) - \sum_{i=1}^{p} b_i(x_i\partial_{y_i} - y_i\partial_{x_i}). \tag{6.58}$$

Denoting the coordinates as $\{X^A\}_{A=1}^n := \{z, \{x_i, y_i\}_{i=1}^p\}$, ξ' is a CKVF with $\mathrm{a}^A = \delta^A{}_1\lambda^{-1/2}$, $\mathrm{b}^A = \mathrm{a}^A/2$, plus a sum of orthogonal rotations with parameters b_i. The associated skew-symmetric endomorphism of $\mathbb{M}^{1,n+1}$ is directly computable from expression (6.58) and (2.26)

$$F(\xi) = \begin{pmatrix} 0 & 0 & -\frac{3\lambda^{-1/2}}{4} \\ 0 & 0 & -\frac{5\lambda^{-1/2}}{4} \\ -\frac{3\lambda^{-1/2}}{4} & \frac{5\lambda^{-1/2}}{4} & 0 \end{pmatrix} \bigoplus_{i=1}^{p} \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix}. \tag{6.59}$$

$F(\xi)$ is referred to an orthogonal unit basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ with e_0 timelike and as in the previous sections the direct sum (6.59) is adapted to the decomposition

$$\mathbb{M}^{1,n+1} = \mathbb{M}^{1,2} \bigoplus_{i=1}^{p} \Pi_i$$

where $\mathbb{M}^{1,2} = \mathrm{span}\{e_0, e_1, e_2\}$ and $\Pi_i = \mathrm{span}\{e_{2i+1}, e_{2(i+1)}\}$ are F-invariant subspaces. The causal character of $\ker F(\xi)$ is straightforwardly determined by checking that $v := 5e_0 + 3e_1$ is timelike and that it belongs to $\ker F(\xi)$. Thus $\ker F(\xi)$ is timelike.

On the other hand, the polynomial $\mathcal{Q}_{F^2}$ in Definition 4.12 is

$$\mathcal{Q}_{F^2}(x) = (x - \frac{1}{\lambda})\prod_{i=1}^{p}(x - b_i^2) = \prod_{i=0}^{p}(x - b_i^2)$$

where for the last equality we have set $b_0^2 := 1/\lambda$. Now let $\{\widetilde{b}_i\}_{i=0}^p$ the parameters b_i sorted in decreasing order $\widetilde{b}_0^2 \geq \cdots \geq \widetilde{b}_p^2$. Then by Theorem 4.35, the conformal class of ξ is given by $\{\sigma = \widetilde{b}_0^2; \mu_1^2 = \widetilde{b}_1^2, \cdots, \mu_p^2 = \widetilde{b}_p^2\}$. Note that the value of one of the parameters is $1/\sqrt{\lambda}$, so it is a priori fixed. To cover the whole space of parameters $\mathcal{R}_+^{(n+2,m)}$, we must consider the scaling freedom of ξ, just like we explained in the case of Kerr-de Sitter. Taking this into account, this family of metrics covers every point in all the regions $\mathcal{R}_+^{(n+2,m)}$ in the space of conformal classes.

6.3.3.2 Generalized $\{a_i \to \infty\}$-limit Kerr-de Sitter.

In this subsection we calculate the remaining family of metrics which completes the Kerr-de Sitter-like class for n odd, i.e. those corresponding to the regions $\mathcal{R}_0^{(n+2,m)}$ in the space of conformal classes. Analogously to the case of n even (cf. subsection 6.3.2.2), these are called generalized $\{a_i \to \infty\}$-limit Kerr-de Sitter, also extending the definition in [101].

Contrary to the n even case, if n is odd we obtain a good limit from Kerr-de Sitter with none of the rotation parameters initially vanishing. The reason is that having only p non-vanishing rotation parameters $a_i = \zeta^{-1} b_i$ $(i = 1, \cdots, p)$ (recall that $a_{p+1} = 0$ was defined for notational reasons) the function W remains finite in the limit $\lim_{\zeta\to 0} W = \alpha_{p+1}^2$ if we scale the first p coordinates $\hat{\alpha}_i = \zeta \beta_i$. Thus, $\gamma_\zeta = \zeta^{-2}\gamma$ and ξ_ζ both admit a finite limit $\zeta \to 0$, as soon as the coordinate t is rescaled to $t = \zeta t'$ (see subsection 6.3.2.2 for comparison).

Consider the de Sitter metric (6.40) with the change of coordinates

$$\rho = \zeta \rho', \qquad t = \zeta t', \qquad \hat{\alpha}_i = \zeta \beta_i \qquad (i = 1, \cdots, p), \tag{6.60}$$

where notice that $\hat{\alpha}_{p+1}$ has not been scaled. Also consider the redefinition of parameters

$$M = M' \zeta^n, \qquad a_i = \zeta^{-1} b_i \qquad (i = 1, \cdots, p). \tag{6.61}$$

Unlike in the n even case, no ϕ angle is associated to $\hat{\alpha}_{p+1}$, so there is no need the rescale any of the ϕ_i coordinates. All calculations are analogous to those in subsection 6.3.2.2, so we provide here less detail.

First, the scaled coordinates $\{\beta_i\}_{i=1}^p$ and $\hat{\alpha}_{p+1}$ satisfy when $\zeta \to 0$

$$\hat{\alpha}_{p+1}^2 + \sum_{i=1}^p \lambda b_i^2 \beta_i^2 = 1.$$

The functions W, Ξ and Π (cf. (6.39)) take the limit

$$W' := \lim_{\zeta\to 0} W = \hat{\alpha}_{p+1}^2, \qquad \Xi' := \lim_{\zeta\to 0} \Xi := \hat{\alpha}_{p+1}^2 + \sum_{i=1}^p \frac{\lambda b_i^2 \beta_i^2}{1 + \rho'^2 b_i^2}, \qquad \Pi' = \prod_{j=1}^p (1 + \rho'^2 b_j^2).$$

The limit of the term $\widetilde{\mathcal{H}}\, \widetilde{k} \otimes \widetilde{k}$ present no difficulties since the scalings defined in (6.60) and (6.61) compensate each other so that no divergences appear. Then

$$\widetilde{\mathcal{H}}' \widetilde{k}' \otimes \widetilde{k}' := \lim_{\zeta\to 0} \mathcal{H} \widetilde{k} \otimes \widetilde{k} = \underbrace{\frac{2M'}{\Pi' \Xi'} \rho'^{n-2}}_{=:\widetilde{\mathcal{H}}'} \underbrace{\left(W' \mathrm{d}t' + \frac{\Xi'}{\lambda} \mathrm{d}\rho' - \sum_{i=1}^p b_i \beta_i^2 \mathrm{d}\phi_i \right)}_{=:\widetilde{k}'}{}^2.$$

For the de Sitter background (6.40) a computation analogous to the case of n even shows that the terms in $\mathrm{d}\hat{\alpha}_{p+1}$ do not diverge. In fact, the limit of de Sitter as $\zeta \to 0$ is

$$\widetilde{g}'_{dS} = \frac{\lambda \hat{\alpha}^2_{p+1}}{\rho'^2}\mathrm{d}t'^2 - \frac{\Xi'}{\lambda}\frac{\mathrm{d}\rho'^2}{\rho'^2} + \sum_{i=1}^{p} \frac{1+\rho'^2 b_i^2}{\rho'^2}\left(\mathrm{d}\beta_i^2 + \beta_i^2 \mathrm{d}\phi_i^2\right) + \left(\frac{1}{\lambda} + \frac{\sum_{i=1}^p \beta_i^2}{\rho'^2 \hat{\alpha}^2_{p+1}}\right)\mathrm{d}\hat{\alpha}^2_{p+1} - \frac{2}{\rho'^2}\frac{\mathrm{d}\hat{\alpha}_{p+1}}{\hat{\alpha}_{p+1}}\left(\sum_{i=1}^{p}\beta_i \mathrm{d}\beta_i\right).$$

The limit metric is thus

$$\widetilde{g}' = \widetilde{g}'_{dS} + \widetilde{\mathcal{H}}'\widetilde{k}' \otimes \widetilde{k}', \qquad \widetilde{\mathcal{H}}' = \frac{2M'\rho'^{n-2}}{\Pi'\Xi'}, \quad M' \in \mathbb{R}.$$

In addition, $\widetilde{g}'_{dS}$ must be locally isometric to de Sitter, because the metric $\widetilde{g}_{dS}$ is C^2 in ζ (up to and including $\zeta = 0$) when written in the primed coordinates.

We next analyze the asymptotic structure. First, the boundary metric

$$\gamma' = \lambda \hat{\alpha}^2_{p+1}\mathrm{d}t'^2 + \sum_{i=1}^{p}\left(\mathrm{d}\beta_i^2 + \beta_i^2\mathrm{d}\phi_i^2\right) + \left(\sum_{i=1}^p \beta_i^2\right)\frac{\mathrm{d}\hat{\alpha}^2_{p+1}}{\hat{\alpha}^2_{p+1}} - 2\frac{\mathrm{d}\hat{\alpha}_{p+1}}{\hat{\alpha}_{p+1}}\left(\sum_{i=1}^p \beta_i \mathrm{d}\beta_i\right),$$

which is explicitly conformally flat in coordinates

$$\widetilde{\beta}_i = \frac{\beta_i}{\hat{\alpha}_{p+1}}, \qquad i = 1, \cdots, p$$

because

$$\gamma' = \hat{\alpha}^2_{p+1}\left(\lambda \mathrm{d}t'^2 + \sum_{i=1}^p \left(\mathrm{d}\widetilde{\beta}_i^2 + \widetilde{\beta}_i^2 \mathrm{d}\phi_i^2\right)\right).$$

This also determines a flat representative $\gamma_E := \hat{\alpha}^{-2}_{p+1}\gamma'$ with Cartesian coordinates $\{\tau := \sqrt{\lambda}t', x_i := \widetilde{\beta}_i \cos\phi_i, y_i := \widetilde{\beta}_i \sin\phi_i\}$.

The electric part of the rescaled Weyl tensor D follows from equation (5.6)

$$D_{\xi'} = \rho^{n-2} C_{\perp}|_{\mathscr{I}} = -\lambda n(n-2)M'\left(\boldsymbol{\xi}' \otimes \boldsymbol{\xi}' - \frac{|\xi'|^2_{\gamma'}}{n}\gamma'\right),$$

where

$$\boldsymbol{\xi}' = \hat{\alpha}^2_{p+1}\mathrm{d}t' - \sum_{i=1}^p b_i \beta_i^2 \mathrm{d}\phi_i \implies \xi' = \frac{1}{\lambda}\partial_{t'} - \sum_{i=1}^p b_i \partial_{\phi_i},$$

which in Cartesian coordinates is simply

$$\xi' = \frac{1}{\lambda^{1/2}}\partial_\tau - \sum_{i=1}^p b_i (x_i \partial_{y_i} - y_i \partial_{x_i}).$$

Letting $\{X^A\}_{A=1}^n := \{\tau, \{x_i, y_i\}_{i=1}^p\}$, the skew-symmetric endomorphism of $\mathbb{M}^{1,n+1}$ associated to ξ' is by (2.26)

$$F(\xi) = \begin{pmatrix} 0 & 0 & \frac{\lambda^{-1/2}}{2} \\ 0 & 0 & -\frac{\lambda^{-1/2}}{2} \\ \frac{\lambda^{-1/2}}{2} & \frac{\lambda^{-1/2}}{2} & 0 \end{pmatrix} \bigoplus_{i=1}^{p} \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix} \tag{6.62}$$

referred to an orthogonal unit basis $\{e_\alpha\}_{\alpha=0}^{n+1}$ with e_0 timelike. The direct sum (6.62) is adapted to the decomposition

$$\mathbb{M}^{1,n+1} = \mathbb{M}^{1,2} \bigoplus_{i=1}^{p} \Pi_i$$

where $\mathbb{M}^{1,2} = \text{span}\{e_0, e_1, e_2\}$ and $\Pi_i = \text{span}\{e_{2i+1}, e_{2(i+1)}\}$ are F-invariant subspaces. For analogous reasons than in the n even case, $\ker F(\xi')$ is degenerate. The polynomial $\mathcal{Q}_{F^2}$ in Definition 4.12 is

$$\mathcal{Q}_{F^2} = x \prod_{i=1}^{p} (x - b_i^2),$$

and by Theorem 4.35, the parameters determining the conformal class of ξ' are

$$\{\sigma = 0; \mu_1^2 = b_1^2, \cdots, \mu_p^2 = b_p^2\}.$$

Hence, this set of conformal classes covers every point in every region $\mathcal{R}_0^{(n+2,m)}$.

Chapter 7

Conclusions and outlook

In this thesis I have studied the asymptotic initial value problem of general relativity in all dimensions with positive cosmological constant. To do that, several tools related to conformal geometry have been developed. Highlights among them are the study of CKVFs of locally conformally flat metrics and their classes up to conformal transformations, as well as the initial data in the Fefferman-Graham formalism. These tools have been applied to obtain characterizations of Kerr-de Sitter and related spacetimes. We now discuss the main conclusions of the this work and also the points which are left open for a future study.

In Chapter 3 we have studied the skew-symmetric endomorphisms of $\mathbb{M}^{1,3}$ and $\mathbb{M}^{1,2}$ as well as the global CKVFs of $\mathbb{S}^2$. Firstly we have derived a unified canonical form for every skew-symmetric endomorphisms in $\mathbb{M}^{1,3}$ depending on just two parameters σ, τ (cf. Proposition 3.8). As a corollary, simply by setting $\tau = 0$ a canonical form for $\mathrm{SkewEnd}(\mathbb{M}^{1,2})$ is obtained (cf. Corollary 3.9). Both canonical forms posses an invariance group, which has been calculated and analyzed along with its generators. As mentioned in Chapter 3, another notion of "canonical form" in the context of two-forms is also commonly found in the literature. This, however, requires a separation into two different cases, namely

$$\boldsymbol{F}_1 = a\boldsymbol{e} \wedge \boldsymbol{w} + b\boldsymbol{u} \wedge \boldsymbol{v}, \qquad \boldsymbol{F}_2 = \boldsymbol{k} \wedge \boldsymbol{v}, \qquad a, b \in \mathbb{R}, \tag{7.1}$$

were $\{\boldsymbol{e}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}\}$ are orthogonal one-forms unit with $\boldsymbol{e}$ timelike, and $\boldsymbol{k}$ null orthogonal to $\boldsymbol{v}$. A remarkable feature of the unified canonical form that we obtain is that, when translated to two-forms and by taking an adequate limit $\sigma, \tau \to 0$ a two-form of the type $\boldsymbol{F}_1$ (with $a, b \neq 0$) in (7.1) may take as limit a two-form of the type $\boldsymbol{F}_2$. This, which in the canonical form (3.6) is obvious, is not apparent the form (7.1). Besides the applications that we have given and shall be discussed next, this unified form has potential interest in other areas, such as the study of the electromagnetic field tensor in special or general theory of relativity.

The second part of Chapter 3 is devoted to the study of CKVFs of $\mathbb{S}^2$. It is worth highlighting that we distinguish the global CKVFS of the 2-sphere from the rest. Most of our results hold specifically for global CKVFs, which are the generators of the global diffeomorphisms of $\mathbb{S}^2$, although the results in subsection 3.9.2 apply for a general CK-VFs. We have first discussed some generalities on global CKVFs and global conformal transformations. Then, in Section 3.7 we have obtained a canonical form for the global CKVFs of $\mathbb{S}^2$, induced from the one for $\mathrm{SkewEnd}(\mathbb{M}^{1,3})$. In the first place, this allowed us to explicitly obtain adapted coordinates which fit every global CKVF ξ of the sphere. With these coordinates, we have calculated a class of metrics of constant curvature for which ξ is a Killing vector field. In addition, in Theorem 3.25 we have found the class of all Lie-constant TT tensors w.r.t to a general CKVF (i.e. non necessarily global) ξ. The solution has been given in covariant form in using second CKVF $\xi^{\perp}$, canonically obtainable from ξ. This has found interesting applications in radiation at null infinity [51].

In Chapter 4 we have extended the main points analyzed in Chapter 3 to arbitrary d dimension. Firstly, we have given a new and very direct proof of a known classification result for $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})$ in Theorem 4.6, using only elementary algebra methods. Then we have generalized the canonical form of skew-symmetric endomorphisms to arbitrary dimension d (cf. Theorem 4.12). Using this canonical form, we have studied the structure of the quotient space $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$. It is remarkable that the canonical form gives a good representation of this quotient. In $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$, we have constructed sequences which have two simultaneous limit points in the quotient topology. In other words, we have shown that the quotient topology is non-Hausdorff. In addition, we have proven that, for even d dimension the region $\mathcal{R}_-^{(d,0)} \subset \mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$ (see Remark 4.25) is open in the quotient topology and moreover its closure exhausts $\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)$ (as proven in Proposition 4.26). On the contrary, for d odd dimension both $\mathcal{R}_-^{(d,0)}$ and $\mathcal{R}_+^{(d,0)}$ are open, and the closure of $\mathcal{R}_-^{(d,0)}$ exhausts $\left(\mathrm{SkewEnd}(\mathbb{M}^{1,d-1})/O^+(1,d-1)\right)\backslash\mathcal{R}_+^{(d,0)}$. This structure finds an important application in the last Chapter of this thesis. We will come back to this later.

In the second part of Chapter 4, we apply the results obtained for skew-symmetric endomorphisms to the set of CKVFs of $\mathbb{S}^n$. Firstly, we have obtained a classification result in Theorem 4.33, analogous to that for $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})$. We have derived a canonical form for every CKVF ξ (cf. Definition 4.34), which moreover, determines the conformal class $[\xi]$ (cf. Theorem 4.35). An interesting property of the canonical form is that it always gives a maximal set of pairwise commuting linearly independent CKVFs $\{\widetilde{\xi}, \widetilde{\xi}^{\star}, \eta_i\}$ for n even and $\{\widetilde{\xi}, \eta_i\}$ for n odd, where η_i are in the conformal class of the generators of rotations, which we have defined as conformally axial Killing vector fields (CAKVFs).

In order to obtain the canonical form ξ, one calculates first a flat representative γ_E of the class of locally conformally flat metrics and Cartesian coordinates for it. Then, it is easy to associate a skew-symmetric endomorphism in $\mathbb{M}^{1,n+1}$, whose canonical parameters $\{-\mu_t^2, \mu_s^2, \mu_i^2\}$ for n even and $\{-\sigma, \mu_i^2\}$ for n odd in Definition 4.10 are straighforwardly obtainable. However, finding explicitly a flat representative and corresponding Cartesian coordinates may not be an easy problem in many cases. It would therefore be of interest to have a completely covariant method to determine the conformal class of ξ. This turns out to be possible. The underlying idea is, roughly speaking, that the CKVF is determined by its value and certain derivatives at a unique point. This method has not been included in this thesis because it is very recent and not yet written up. We expect to make it available very soon.

In Section 4.6 we obtain and study a set of coordinates adapted to an arbitary CKVF ξ, for which the canonical form of CKVFs is essential. It is remarkable that all calculations in this section are carried for n even in just one go and the n odd case is obtained by a suitable particularization. The results are summarized in Theorem 4.45. As an application of the adapted coordinates, we have calculated all TT tensors in $n = 3$ which solve the KID equation for two commuting CKVFs $\{\widetilde{\xi}, \eta\}$, with η conformally axial. Both vector fields $\{\widetilde{\xi}, \eta\}$ arise from the canonical form $\xi = \widetilde{\xi} + \eta$. We emphasize that the final form of this class of TT tensors is given in diffeomorphism and conformal covariant form in Theorem 4.47.

Several things should be stressed about the class of TT tensors above. Firstly, it is an infinite dimensional class containing the data of Kerr-de Sitter, which have explicitly identified. As we mentioned in the main text, by comparison with results for stationary axi-symmetric spacetimes in the $\Lambda = 0$ case [2, 18, 27], one could conjecture that a set of momenta, related to mass and angular momentum, could be derived from these data. Several proposals of conserved quantities can be found in the high energy physics literature, with focus in $\Lambda < 0$ case (see e.g. [9, 10] and also [82] and references therein), which perhaps could be applicable to this setting. In addition, the two KID vectors $\{\widetilde{\xi}, \eta\}$ generate two commuting symmetries, with η being associated to an axial symmetry. We have justified that the class of TT tensors we get contains all data corresponding to spacetimes with (at least) two commuting symmetries, except for one case. The remaining case is the class of TT tensors with two independent KID vectors $\{\xi_1, \xi_2\}$ that are conformal to translations. This class can be readily calculated in Cartesian coordinates adapted to both translations, so that we may easily obtain the complete set of initial data at spacelike conformal flat infinity, for spacetimes admitting two commuting symmetries. We note that although the integration of the remaining case is direct in Cartesian coordinates, giving a covariant form is not so straighforward and requires further analysis.

In Chapter 5 we have addressed the higher dimensional Cauchy problem of general relativity. We recall that the spacetime dimension is $n + 1$ so the dimension of $\mathscr{I}$ is n.

In this and the next Chapter, the Fefferman-Graham formalism plays an important role.

In Section 5.1, we have derived two useful formulas for the Weyl tensor. The first one gives a formula for the T-electric part of the Weyl tensor of an Einstein metric, with T the gradient of a geodesic conformal factor. The second one relates, to leading order, the Weyl tensors of two conformally extendable metrics $g = \hat{g} + \Omega^m q$, with $m \geq 2$ and Ω positive at least C^2. Both formulas are very useful in the Fefferman-Graham setting, and we have given several applications of them in this thesis. One is the calculation of the FG expansion of the de Sitter metrics for $n \geq 3$, which easily generalizes to all Einstein metrics of constant curvature of any signature and any sign of Λ. This extends previous results [140] in the $\Lambda < 0$ case of Lorentzian signature. Another application, in Proposition 5.11, is a decomposition of FGP metrics admitting a smooth conformally flat $\mathscr{I}$. This decomposition allows us to extract a well-defined free TT part $\mathring{g}_{(n)}$ from the n-th order coefficient of the FG expansion $g_{(n)}$ of every FGP metric admitting a smooth conformally flat $\mathscr{I}$. By an straighforward combination of the above results, we have proven in Theorem 5.14 that $\mathring{g}_{(n)}$ agrees, up to an explicit constant, with the electric part of the rescaled Weyl tensor at $\mathscr{I}$. Our analysis extends previous results [82], restricted to negative Λ and Lorentzian signature, to any non-zero Λ and arbitrary signature.

It is worth at this point to discuss in a general setting the problem of how to extract the free (TT) part $\mathring{g}_{(n)}$ from the n-th order coefficient $g_{(n)}$ and relate this to the conformal equivalence of data (see discussion above Theorem 2.39). For n odd $g_{(n)}$ is always TT so we can set in general $g_{(n)} = \mathring{g}_{(n)}$. In this case, under conformal scalings of the metric $\gamma' = \omega^2\gamma$, the corresponding TT tensor is $g'_{(n)} = \omega^{2-n} g_{(n)}$. For n even one should find a way to extract the trace and divergence terms from $g_{(n)}$. For a fixed, but arbitrary, conformal class of the boundary metric γ, this could be achieved by canonically selecting "background" data $(\gamma, \overline{g}_{(n)})$, in such a way that for any other set of initial data $(\gamma, g_{(n)})$, we define the free part by $\mathring{g}_{(n)} := g_{(n)} - \overline{g}_{(n)}$. Observe that the trace and divergence of $g_{(n)}$ and $\overline{g}_{(n)}$ only depend on γ, thus $\mathring{g}_{(n)}$ is TT. Once the background data are selected, all data are $(\gamma, \overline{g}_{(n)} + \mathring{g}_{(n)})$ so they are equivalent to the pair $(\gamma, \mathring{g}_{(n)})$. For the free part, the expected conformal equivalence of data is given by the class $(\omega^2\gamma, \omega^{2-n}\mathring{g}_{(n)})$ for every smooth positive function ω of Σ. In the conformally flat γ case, the obvious choice for background data are those corresponding to de Sitter spacetime, and this is what we have used throughout. Observe that in this case, the conformal transformation of the free part $\mathring{g}_{(n)}$ follows directly from Theorem 5.14 and confirms the expectation above that $\mathring{g}_{(n)}$ is conformally covariant of weight $2 - n$. However, it is not clear how this same idea could be extended when γ belongs to an arbitrary conformal class. This is an interesting problem that would deserve further investigation.

We have also discussed in the n odd case, where existence and uniqueness is guaranteed, under which conditions the conformal flatness of $\mathscr{I}$ is sufficient for $\mathring{g}_{(n)}$ to coincide, up to a constant, with the electric part of the rescaled Weyl tensor at $\mathscr{I}$. We have linked this to a conjecture [103] (see also [15, 79]) which asserts that purely magnetic spacetimes, i.e. with zero electric part $C_\perp = 0$, do not exist beyond the conformally flat case. We have found that provided that the conjecture is true, the electric part of the rescaled Weyl tensor at $\mathscr{I}$ and $\mathring{g}_{(n)}$ coincide if and only if $\mathscr{I}$ is conformally flat. Another reason to believe that this relation between electric part of the rescaled and $\mathring{g}_{(n)}$ is very exceptional is that the former is generally divergent at $\mathscr{I}$, as we have justified in Chapter 5.

In connection with the behaviour at infinity of the electric part of the Weyl tensor, we have already mentioned the results in [112] where the peeling of the Weyl tensor in arbitrary dimensions is established under the assumption that certain Weyl components, namely those of highest boost weight, decay faster than r^{-2}. It is an interesting problem to establish a connection between the two results. The idea is to determine the minimal decay rate under which one can guaranteee that a smooth conformal compactification with conformally flat $\mathscr{I}$ exists. This may lead to an interesting weakening of the hypothesis in [112] that imply the peeling behaviour.

A core result of Chapter 5 is the KID equation that we have derived in Theorem 5.18. This is a natural generalization to higher dimensions of the KID equation of the four spacetime dimensional case by Paetz [116]. We have proven that this equation gives a sufficient condition for the Cauchy development of analytic asymptotic data with zero obstruction tensor to admit a Killing vector field. Nevertheless, we have argued that the proof extends in the non-zero obstruction tensor and analytic data case, provided that the logarithmic coefficients $\mathcal{O}_{(r,s)}$ in the FG expansion can be generated by a recursive formula dependending only on γ, $g_{(n)}$ and covariant derivatives of them. This should follow from the FG equations, but requires further analysis. In addition, we have also proven that our KID equation is necessary also in the non-analytic case. Sufficiency, is much more difficult to establish in the general case and is left open for a future work. We can, however, conjecture that the KID equation in Theorem 5.18 extends to the general case.

Chapter 5 concludes with Theorem 5.22, which gives a geometric characterization of the Kerr-de Sitter family of spacetimes in all dimension by direct calculation of their asymptotic initial data. These data have been proven to be a conformally flat manifold (Σ, γ) and a free term $\mathring{g}_{(n)}$ of the form κD_ξ, where

$$D_\xi = \frac{1}{|\xi|_\gamma^{n+2}} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{n} \gamma \right) \tag{7.2}$$

is TT tensor with ξ a CKVF of γ. We stress the simplicity of this TT tensor. We have proven that for any data of the form $(\Sigma, \gamma, \kappa D_\xi)$, with (Σ, γ) locally conformally flat and ξ a CKVF of γ, the conformal class of ξ and the constant κ characterize the resulting spacetime. Hence, the results of Chapter 4 on conformal clases of CKVFs are key in this characterization theorem.

The final chapter of this thesis, Chapter 6, is devoted to the definition and characterization of the so-called Kerr-de Sitter-like class with conformally flat $\mathscr{I}$. This class is defined via extension of the asymptotic data obtained in Chapter 5 for Kerr-de Sitter. Namely, fixing the initial manifold (Σ, γ) to be conformally flat, the CKVF ξ tensor κD_ξ is allowed to belong to an arbitrary conformal class. As already mentioned, only the conformal class of ξ matters to determine the evolving spacetime. This is an extension of the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ in [100] for $n = 3$, which in turn is a particular case of the Kerr-de Sitter-like class in [99], where the conformal flatness of $\mathscr{I}$ is not required. Since our extension only applies for the conformally flat $\mathscr{I}$, this will be implicit in the remainder when referring to the Kerr-de Sitter-like class.

We have defined the Kerr-Schild-de Sitter spacetimes as conformally extendable, Λ-positive-vacuum Kerr-Schild type spacetimes, such that for every conformal factor Ω the term $\Omega^2 \widetilde{\mathcal{H}}\ \widetilde{k} \otimes \widetilde{k}$ vanishes at $\mathscr{I}$. Note that in particular they admit a smooth conformally flat $\mathscr{I}$. We have also observed that being Kerr-Schild and admitting a smooth conformally flat $\mathscr{I}$ may not be sufficient for being Kerr-Schild-de Sitter, but we expect few exceptions, if any at all. It would be interesting to answer this question.

We have proven that, in all dimensions, every Kerr-Schild-de Sitter spacetime belongs to the Kerr-de Sitter-like class and viceversa. Moreover, we have explicitly constructed all these metrics (see Theorem 6.6 for the full list). The proof involves two steps. First, in Section 6.2 we have proven that the asymptotic data of the Kerr-Schild-de Sitter spacetimes belongs to the Kerr-de Sitter-like class. By direct calculation of the initial data of a generic Kerr-Schild-de Sitter spacetime in the Fefferman-Graham picture, we obtain data $(\Sigma, \gamma, \kappa D_\xi)$ with (Σ, γ) conformally flat and D_ξ of the form (7.2), and we prove that ξ is a CKVF of γ, for which the fact [107] that all Λ-vacuum Kerr-Schild spacetimes are algebraically special is of great relevance.

For the converse inclusion we exploit the structure of $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})/O^+(1, n+1)$ developed in Chapter 4. Recall that this space is equivalent to the space of conformal classes of CKVFs (for locally conformally flat n-manifolds). From limits of conformal classes of CKVFs, we obtain limits of data of the form $(\Sigma, \gamma, \kappa D_\xi)$, which in turn, must correspond to limiting spacetimes because of the well-posedness of the Cauchy problem. As a consequence, in the n even case, all spacetimes in the Kerr-de Sitter-like class are limits of the Kerr-de Sitter family. This is because the data for the latter families cover the region $\mathcal{R}_-^{(n+2,0)}$ of $\mathrm{SkewEnd}(\mathbb{M}^{1,n+1})/O^+(1, n+1)$, which we have proven in Proposition 4.26 to be dense in the quotient topology (only if n is even). The n odd case is similar, with the exception of the so-called Wick-rotated-Kerr-de Sitter family with none of the rotation parameters vanishing. These correspond to data in the region $\mathcal{R}_+^{(n+2,0)}$ of conformal classes and therefore cannot be obtained as limits of the Kerr-de Sitter family (cf. Proposition 4.26). They have been obtained by analytic extension (i.e. through a Wick rotation) of Kerr-de Sitter.

It is worth here to make a link with the results in [19]. In this paper, the authors characterize all algebraically special spacetimes, with non-degenerate optical matrix, in dimesion five (i.e. $n = 4$) as the Kerr-de Sitter family or a limit of it. We have obtained a proof, to be presented in a future work, that the spacetimes they obtain exhaust the Kerr-de Sitter-like class. To do that, we use the covariant characterization of conformal classes mentioned above in this chapter, because from the expressions in [19] it is hard to obtain an explicitly flat conformal representative at $\mathscr{I}$ written in Cartesian coordinates. Our results endow the Kerr-de Sitter-like class with conformally flat $\mathscr{I}$ with a structure of limits which helps to understand why the limits performed in [19] were of relevance. Moreover, our methods extend to arbitrary dimension, although the n even and n odd cases have remarkable differences. It would be interesting to study whether the characterization in [19] extends to higher even dimensions, namely, if the algebraic type and the non-degeneracy of the optical matrix characterize the Kerr-de Sitter-like class. In addition, one interesting difference between both approaches is that in [19] the conformal extendability is not imposed.

We notice that the Kerr-de Sitter-like class has other interesting properties which, for the sake of brevity, have not been included in this thesis. Let us conclude this chapter with a brief description of them. The spacetimes in the Kerr-de Sitter-like class whose (non-trivial) data lie in the regions with maximal number of vanishing rotation parameters, namely $\mathcal{R}^{n+1,p-1}_{-\epsilon}$ with $\epsilon \in \{\pm, 0\}$, can be shown to correspond to the so-called generalized Kottler spacetimes in all dimensions. This is the class of metrics

$$g = -\left(\epsilon - \lambda r^2 - \frac{2M}{r^{n-2}}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\epsilon - \lambda r^2 - \frac{2M}{r^{n-2}}} + \frac{r^2}{\lambda} g_\epsilon,$$

where g_ϵ is a $n-1$ dimensional metric of constant curvature ϵ. What is remarkable is that they are limits of Kerr-de Sitter in every dimension. In particular, this implies that in the physical $n = 3$ case, Kerr-de Sitter has three limits which, despite its similar form, are qualitatively different. For instance, for positive M only Schwarzschild-de Sitter ($\epsilon = 1$) includes a static region. The $\epsilon = -1$ case corresponds, in the space of conformal classes, to a point with $(\sigma > 0, \mu^2 = 0)$. This is one of the degenerate limits studied in subsection 4.4.1, i.e. the sequences in the region $(\sigma < 0, \mu^2 > 0)$ (corresponding to Kerr-de Sitter) with limit at $(\sigma > 0, \mu^2 = 0)$ have also limit at $(\sigma' = 0, \mu^2 = \sigma)$. The latter corresponds to a metric in the $a \to \infty$-limit-Kerr-de Sitter, given first in [101], and also here in Chapter 6. In other words, there exists a sequence of metrics in the Kerr-de Sitter family which limits simultaneously with a Kottler metric with $\epsilon = -1$ and a metric in the $a \to \infty$-limit-Kerr-de Sitter.

Another interesting property follows by an straighforward application of the KID equation in Theorem 5.18 to the set of data in the Kerr-de Sitter-like class. As proven in [99] for $n = 3$ and data in the Kerr-de Sitter-like class $(\Sigma, \gamma, \kappa D_\xi)$, any other CKVF ξ' satisfies the KID equation if and only if it commutes with ξ. This fact easily extends to arbitrary n with the KID equation in Theorem 5.18. In other words, $\mathcal{C}(\xi)$, the centralizer of the CKVF ξ, gives the number of independent symmetries of the corresponding spacetime in the Kerr-de Sitter-like class. The centralizer can in turn be obtained through the centralizer of the corresponding skew-symmetric endomorphism $\mathcal{C}(F(\xi))$.

All the above properties of the Kerr-de Sitter-like class and their consequences, e.g. concerning the number and properties of Killing vectors for spacetimes in the class depending on the defining conformal class, are under current investigation and will be subject of a future work.

Appendix A

Fefferman-Graham Equations

In order to derive the Fefferman-Graham (FG) recursive equations, we need to set some identities first. Let $\widetilde{g}$ be a FGP metric $\lambda > 0$ and $g = \Omega^2 \widetilde{g}$ a geodesic conformal extension. Recall that we defined $T_\alpha = \nabla_\alpha \Omega$ and its g-metrically associated vector field $T^\alpha = g^{\alpha\beta} T_\beta$. In this Appendix we derive all expressions assuming $\lambda > 0$. The $\lambda < 0$ case is slightly different, but the procedure is analogous. The FG equations in this case can be found in e.g. [5].

Let us introduce the contraction of the Riemann tensors of g and $\widetilde{g}$ with T^α twice

$$(R_T)_{\alpha\beta} := R_{\mu\alpha\nu\beta} T^\mu T^\nu, \qquad \left(\widetilde{R}_T\right)_{\alpha\beta} := \widetilde{R}_{\mu\alpha\nu\beta} T^\mu T^\nu \tag{A.1}$$

and define

$$(A)_{\alpha\beta} := \nabla_\alpha T_\beta, \quad \left(A^2\right)_{\alpha\beta} := \nabla_\alpha T^\mu \nabla_\mu T_\beta.$$

Firstly, observe that A is symmetric. Since T is geodesic,

$$\begin{aligned}(R_T)_{\alpha\beta} &= T^\nu(-\nabla_\nu \nabla_\beta T_\alpha + \nabla_\beta \nabla_\nu T_\alpha) \\ &= -\nabla_T \nabla_\beta T_\alpha + \nabla_\beta \nabla_T T_\alpha - \nabla_\beta T^\nu \nabla_\nu T_\alpha = -\nabla_T A_{\alpha\beta} - A^2_{\alpha\beta}.\end{aligned} \tag{A.2}$$

The difference of tensors R_T and $\widetilde{R}_T$ in (A.1) is straightforward from expression (2.7). Notice that the first index in $\widetilde{R}^\mu{}_{\alpha\nu\beta}$ is lowered with $\widetilde{g}_{\mu\sigma}$ and that $T_\mu T^\mu = -\lambda$ since T is geodesic for g. Hence, formula (2.7) when contracted with $T^\mu T^\nu$ is

$$(R_T)_{\alpha\beta} - \Omega^2 (\widetilde{R}_T)_{\alpha\beta} = \frac{\lambda}{\Omega} A_{\alpha\beta} + \frac{\lambda}{\Omega^2} \left(T_\alpha T_\beta + \lambda g_{\alpha\beta}\right). \tag{A.3}$$

Now, from (2.32), in Gaussian coordinates $\{\Omega, x^i\}$ we have

$$T = -\lambda \partial_\Omega.$$

Also, denoting $\dot{g}_\Omega = \partial_\Omega g_\Omega$, A is in these coordinates,

$$A_{\alpha\beta} = \nabla_\alpha T_\beta = -\Gamma^0_{\alpha\beta} = \frac{g^{00}}{2}\partial_\Omega g_{\alpha\beta} = -\frac{\lambda}{2}\partial_\Omega g_{\alpha\beta} = -\frac{\lambda}{2}\dot{g}_\Omega \tag{A.4}$$

and its covariant derivative w.r.t. T

$$\nabla_T A_{\alpha\beta} = -\lambda\partial_\Omega A_{\alpha\beta} + \lambda\left(\Gamma^\mu_{0\alpha}A_{\mu\beta} + \Gamma^\mu_{0\beta}A_{\alpha\mu}\right),$$

with

$$\Gamma^\mu_{0\alpha} = \frac{1}{2}g^{\mu\nu}\left(\partial_\Omega g_{\alpha\nu} + \partial_\alpha g_{0\nu} - \partial_\nu g_{0\alpha}\right) = \frac{1}{2}g^{\mu\nu}\partial_\Omega g_{\alpha\nu} = -\frac{1}{\lambda}A^\mu{}_\alpha,$$

so in consequence

$$\nabla_T A = -\lambda\partial_\Omega A - 2A^2. \tag{A.5}$$

The tensor A is related to the second fundamental form of the leaves Σ_Ω by a constant factor $A = \lambda^{1/2}K$. Then, the Gauss identity (2.17) gives

$$R_{ijkl} = R^{(\Omega)}_{ijkl} + \frac{1}{\lambda}(A_{ik}A_{jl} - A_{il}A_{jl}) = R^{(\Omega)}_{ijkl} + \frac{\lambda}{4}(\dot{g}_{ik}\dot{g}_{jl} - \dot{g}_{il}\dot{g}_{jk}), \tag{A.6}$$

where R_{ijkl} are the space components of the Riemann tensor of g and $R^{(\Omega)}_{ijkl}$ the Riemann tensor of g_Ω. The Ricci tensor of g is

$$R_{\alpha\beta} = g^{\mu\nu}R_{\alpha\mu\beta\nu} = -\frac{1}{\lambda}(R_T)_{\alpha\beta} + g^{ij}R_{\alpha i\beta j},$$

so that the contraction of (A.6) with g^{ik} reads

$$R_{jl} + \frac{1}{\lambda}(R_T)_{jl} = R^{(\Omega)}_{jl} - \frac{\lambda}{2}H\dot{g}_{jl} - \frac{\lambda}{4}\dot{g}^2_{jl},$$

where $H := g^{ij}A_{ij}/\lambda = -\frac{1}{2}g^{ij}\dot{g}_{ij}$, $\dot{g}^2_{jl} = g^{ik}\dot{g}_{il}\dot{g}_{kj}$ and $R^{(\Omega)}_{ij}$ is the Ricci tensor of $g^{(\Omega)}$. From (A.2), (A.4) and (A.5) one gets

$$(R_T)_{ij} = -(\nabla_T A_{ij} + A^2_{ij}) = \lambda\dot{A}_{ij} + 2A^2_{ij} - A^2_{ij} = -\frac{\lambda^2}{2}\ddot{g}_{ij} + \frac{\lambda^2}{4}\dot{g}^2_{ij}.$$

Therefore

$$R_{jl} - \frac{\lambda}{2}\ddot{g}_{jl} = R^{(\Omega)}_{jl} - \frac{\lambda}{2}H\dot{g}_{jl} - \frac{\lambda}{2}\dot{g}^2_{jl}. \tag{A.7}$$

Finally, we relate the tangent components of the Ricci tensors of g and $\widetilde{g}$ in terms of the above quantities. First

$$\nabla_\alpha\nabla^\alpha\Omega = g^{\alpha\beta}\nabla_\alpha T_\beta = g^{ij}\nabla_i T_j = \lambda H$$

and from (2.8) it follows (recall that $g, \widetilde{g}$ are metrics in an $(n+1)$-dimensional manifold)

$$R_{ij} = \frac{n-1}{2\Omega}\lambda\dot{g}_{ij} - \frac{1}{\Omega}\lambda H g_{ij} + \widetilde{R}_{ij} - \lambda n\widetilde{g}_{ij}.$$

Inserting this into (A.7) and multiplying by $\frac{2\Omega}{\lambda}$ yields

$$-\Omega\ddot{g}_\Omega + (n-1)\dot{g}_\Omega - 2Hg_\Omega = \Omega\left(\frac{2}{\lambda}Ric(g_\Omega) - H\dot{g}_\Omega - \dot{g}_\Omega^2 - \frac{2}{\lambda}\widetilde{G}_\parallel\right),$$

where $\widetilde{G}_\parallel$ denotes the tangent components of

$$\widetilde{G}_{\alpha\beta} := \widetilde{R}_{\alpha\beta} - \lambda n\widetilde{g}_{\alpha\beta}.$$

In addition, we shall need the trace of equation (A.2)

$$g^{\alpha\beta}(R_T)_{\alpha\beta} = -\nabla_T(g^{\alpha\beta}A_{\alpha\beta}) - g^{\alpha\beta}A^2_{\alpha\beta}.$$

Since T is geodesic, $A_{\alpha\beta}$ has only tangent components and

$$\nabla_T(g^{\alpha\beta}A_{\alpha\beta}) = -\lambda\partial_\Omega(g^{ij}A_{ij}) = -\lambda^2\dot{H}.$$

The term $(R_T)_{\alpha\beta}g^{\alpha\beta}$ can be obtained from (A.3)

$$g^{\alpha\beta}(R_T)_{\alpha\beta} = \frac{\lambda}{\Omega}g^{\alpha\beta}A_{\alpha\beta} + \frac{\lambda^2}{\Omega^2}n + \widetilde{R}_{TT} = \frac{\lambda^2}{\Omega}H + \widetilde{G}_T,$$

where $\widetilde{R}_{TT} = \Omega^2 g^{\alpha\beta}(\widetilde{R}_T)_{\alpha\beta} = \widetilde{g}^{\alpha\beta}(\widetilde{R}_T)_{\alpha\beta}$ is the normal-normal component of the Ricci tensor, $\widetilde{G}_T := \widetilde{G}_{\alpha\beta}T^\alpha T^\beta$ is the normal-normal component of the tensor $\widetilde{G}$, and we have used that $-\lambda/\Omega^2 = \widetilde{g}_{\alpha\beta}T^\alpha T^\beta$. Hence, writing $g^{\alpha\beta}A^2_{\alpha\beta} =: |A|^2$, the trace of (A.2) gives the following expression

$$\lambda^2(\Omega\dot{H} - H) = \Omega|A|^2 + \Omega\widetilde{G}_T.$$

The last equation that we require is the trace of the Codazzi identity (2.18), namely

$$T^\mu R_{\mu ijk}g^{ik} = g^{ik}(\nabla^{(\Omega)}_j A_{ik} - \nabla^{(\Omega)}_k A_{ij}) = \lambda\nabla^{(\Omega)}_j H - \nabla^{(\Omega)}_k A^k{}_j.$$

The LHS of this equation is

$$T^\mu R_{\mu ijk}g^{ik} = T^\mu R_{\mu\alpha j\beta}g^{\alpha\beta} + \frac{1}{\lambda}T^\mu R_{\mu\alpha j\beta}T^\alpha T^\beta = R_{\mu j}T^\mu = \widetilde{G}_{\mu j}T^\mu$$

where the last equality follows form (2.8). In index-free notation the term in the RHS will be denoted $\widetilde{G}_{T\parallel}$. Therefore

$$\mathrm{div}_{g_\Omega}A - \lambda \mathrm{d}H = -\widetilde{G}_{T\parallel}$$

Summarizing,

Definicin A.1. The Fefferman-Graham equations are

$$-\Omega\ddot{g}_\Omega + (n-1)\dot{g}_\Omega - 2Hg_\Omega = \Omega L, \tag{A.8}$$

where

$$L := \frac{2}{\lambda} Ric(g_\Omega) - H\dot{g}_\Omega - (\dot{g}_\Omega)^2 - \frac{2}{\lambda}\widetilde{G}_\|,$$

and

$$\lambda^2(\Omega\dot{H} - H) = \Omega|A|^2 + \Omega\widetilde{G}_T, \qquad \mathrm{div}_{g_\Omega} A - \lambda \mathrm{d}H = -\widetilde{G}_{T\|}. \tag{A.9}$$

Next, we use expressions (A.8) and (A.9) to obtain the recursive relations that generate the coefficients in (2.34) and (2.35). For simplicity, we will assume that for the n even case $\mathcal{O} = 0$, although some remarks will be made concerning the $\mathcal{O} \neq 0$ case. By definition of Poincaré metric, the tensor $\widetilde{G}$ vanishes to all orders at $\mathscr{I}$, therefore we simply omit it during the calculations, since in the end everything will be evaluated at $\mathscr{I}$.

To calculate the coefficients of the expansion, we take derivatives in Ω of (A.8) and evaluate at $\{\Omega = 0\}$. First, evaluating (A.9) and (A.8) at $\Omega = 0$ it follows

$$H\mid_{\Omega=0} = 0 \qquad \dot{g}_\Omega = g_{(1)} = 0.$$

For r^{th} order derivatives we use the generalized Leibniz rule, namely, for every two smooth functions f_1, f_2

$$\partial^r_\Omega(f_1 f_2) = \sum_{s=0}^{r} \binom{r}{s} \partial^{r-s}_\Omega f_1 \; \partial^s_\Omega f_2.$$

Recall that $g_{(r)}$ denote the coefficients in the FG expansions (cf. (2.34) and (2.35)). In a similar manner, we denote $g^\sharp{}_{(r)}$ the r^{th} order coefficient of the corresponding expansion for the inverse metric $g^\sharp_\Omega$. Also, observe that the coefficients and the derivatives of g_Ω at $\Omega = 0$ are related by

$$\partial^r_\Omega g_\Omega|_{\Omega=0} = r! g_{(r)}.$$

Lemma A.2. *The coefficients $g^\sharp{}_{(r)}$ can be written in terms of the coefficients $g_{(s)}$ up to order $s = r$ with $s \neq r - 1$. In particular $g^\sharp_{(1)} = 0$.*

Proof. Taking the r^{th} order derivative in Ω of $g^{ij} g_{ik} = \delta^j{}_k$ and evaluating at $\Omega = 0$

$$\partial^r_\Omega \left(g^{ij} g_{ik}\right)\big|_{\Omega=0} = \sum_{s=0}^{r} r! \left(g^\sharp_{(r-s)}\right)^{ij} g_{(s)ik} = 0, \tag{A.10}$$

shows that the r^{th} order coefficient $g^\sharp_{(r)}$ can be obtained as a combination of coefficients $g_{(s)}$ up to order r and $g^\sharp_{(s)}$ up to order $r-2$. The term $g^\sharp_{(r-1)}$ does not appear because it is multiplied by $g_{(1)}$, which is zero. Inductively, this implies that $g^\sharp_{(r)}$ can be written in terms of coefficients $g_{(s)}$ up to order r. In addition, for $r = 1$ it follows that $g^\sharp_{(1)} = 0$, so $g^\sharp_{(r)}$ does not depend on $g_{(r-1)}$. □

It follows easily from the generalized Leibniz rule that

$$\partial_\Omega^r(\Omega f) = \Omega\partial_\Omega^{(r)} f + r\partial_\Omega^{r-1} f.$$

Using this, the r-th order derivative of (A.8) at $\Omega = 0$ is

$$(r+1)!(n-r-1)g_{(r+1)} - 2\sum_{s=0}^{r}\frac{r!}{(r-s)!}\mathcal{H}^{(r-s)}g_{(s)} = r\mathcal{L}^{(r-1)} \tag{A.11}$$

with $\mathcal{H}^{(s)} := \partial_\Omega^s H\mid_{\Omega=0}$ and $\mathcal{L}^{(s)} := \partial_\Omega^s L\mid_{\Omega=0}$. Since $H = -\frac{1}{2}g^{ij}\dot{g}_{ij}$, we may apply the Leibniz rule again to compute $\mathcal{H}^{(r-s)}$

$$\mathcal{H}^{(r-s)} = -\frac{1}{2}\sum_{t=0}^{r-s}(r-s)!(t+1)g^{\sharp ij}_{(r-s-t)}g_{(t+1)_{ij}}. \tag{A.12}$$

Isolating the term involving the highest order coefficient $g_{(r+1)}$ we have

$$\sum_{s=0}^{r}\frac{r!}{(r-s)!}\mathcal{H}^{(r-s)}g_{(s)} = -\frac{(r+1)!}{2}g^{\sharp ij}_{(0)}g_{(r+1)ij}g_{(0)} - \frac{r!}{2}\mathcal{P}^{(r-1)},$$

where

$$\mathcal{P}^{(r-1)} := \sum_{t=0}^{r-1}(t+1)g^{\sharp ij}_{(r-t)}g_{(t+1)_{ij}}g_{(0)} + \sum_{s=1}^{r}\sum_{t=0}^{r-s}(t+1)g^{\sharp ij}_{(r-s-t)}g_{(t+1)_{ij}}g_{(s)}. \tag{A.13}$$

Then, writing $g_{(0)} = \gamma$ and $g^\sharp_{(0)} = \gamma^\sharp$, expression (A.11) is easily arranged to

$$(n-r-1)g_{(r+1)} + \left(\mathrm{Tr}_\gamma g_{(r+1)}\right)\gamma + \frac{1}{r+1}\mathcal{P}^{(r-1)} = \frac{r}{(r+1)!}\mathcal{L}^{(r-1)}.$$

Since every term in (A.13) containing either $g_{(r)}$ or $g^\sharp_{(r)}$ is multiplied by either $g_{(1)}$ or $g^\sharp_{(1)}$, it follows by Lemma A.2 that $\mathcal{P}^{(r-1)}$ can be written in terms of coefficients $g_{(s)}$ up to order $s \leq r-1$.

On the other hand, consider

$$\mathcal{L}^{(r-1)} = \partial_\Omega^{r-1}\left(\frac{2}{\lambda}Ric(g_\Omega) - H\dot{g}_\Omega - (\dot{g}_\Omega)^2\right)\Big|_{\Omega=0}.$$

The term $\partial_\Omega^{r-1}Ric(g_\Omega)\big|_{\Omega=0}$ obviously contains only coefficients $g_{(s)}$ up to order $s \leq r-1$ and tangential derivatives of them. The term $\partial_\Omega^{r-1}\left(H\dot{g}_\Omega\right)|_{\Omega=0}$ can be cast as

$$\begin{aligned}\partial_\Omega^{r-1}\left(H\dot{g}_\Omega\right)|_{\Omega=0} &= \sum_{s=0}^{r-1}\frac{(r-1)!}{(r-1-s)!}(s+1)\mathcal{H}^{(r-1-s)}g_{(s+1)} \\ &= -\frac{(r-1)!}{2}\sum_{s=0}^{r-1}\sum_{t=0}^{r-1-s}(t+1)(s+1)g^{\sharp ij}_{(r-1-s-t)}g_{(t+1)_{ij}}g_{(s+1)}.\end{aligned} \tag{A.14}$$

Terms involving $g_{(r)}$ or $g^{\sharp}_{(r)}$ arise only for the values $s = 0,\ t = r-1$ and $s = r-1,\ t = 0$. In each case, the product also involves $g_{(1)}$ or $g^{\sharp}_{(1)}$. So, no such terms survive and by Lemma A.2, (A.14) only depends on coefficients $g_{(s)}$ up to order $s \leq r-1$. The same holds for $\partial^{r-1}_{\Omega}\, \dot{g}^2_{\Omega}\big|_{\Omega=0}$, because

$$\partial^{r-1}_{\Omega}\, \dot{g}^2_{ij}\big|_{\Omega=0} = \partial^{r-1}_{\Omega}\left(\dot{g}_{ik} g^{kl} \dot{g}_{lj}\right)\Big|_{\Omega=0}$$

and applying the Leibniz rule yields a similar expression to (A.14) with the indices contracted in a different way. Thus:

Lemma A.3. *For n odd and n even with zero obstruction tensor, the r-th order derivative of equation (A.8) at $\Omega = 0$ has the form*

$$(n-r-1)g_{(r+1)} + \left(\mathrm{Tr}_{\gamma} g_{(r+1)}\right)\gamma + \frac{1}{r+1}\mathcal{P}^{(r-1)} = \frac{r}{(r+1)!}\mathcal{L}^{(r-1)} \tag{A.15}$$

where $\mathcal{P}^{(r-1)}$ and $\mathcal{L}^{(r-1)}$ depend on previous coefficients $g_{(s)}$ and their tangential derivatives up to second order with $s \leq r-1$.

Observacin A.4. *In the n even case with non-zero obstruction tensor, the same analysis shows that Lemma A.3 holds for $r+1 < n$, because all logarithmic terms are multiplied by a factor Ω^{n+s}. Thus, the presence of logarithmic terms do not affect the derivatives of (A.8) of order $r \leq n-1$.*

In the main text we shall need the explicit form of (A.15) when $r = 1$. We write the result as a Corollary.

Corolario A.5. *For any boundary metric γ of dimension $n > 2$, the second order coefficient of the FG expansion is, up to a constant, the Schouten tensor of γ:*

$$g_{(2)} = \frac{\lambda^{-1}}{n-2}\left(Ric(\gamma) - \frac{Scal(\gamma)}{2(n-1)}\gamma\right) = \lambda^{-1} Sch(\gamma). \tag{A.16}$$

Proof. We set $r = 1$ in (A.15) and use that $\mathcal{L}^{(0)} = (2/\lambda)Ric(\gamma)$ and $\mathcal{P}^{(0)} = 0$. Thus

$$(n-2)g_{(2)} + \left(\mathrm{Tr}_{\gamma} g_{(2)}\right)\gamma = \frac{1}{\lambda}Ric(\gamma).$$

Taking trace with γ, expression (A.16) follows at once. □

We now use the formula derived in Lemma A.3 to show that the expansion is even up to order n.

Proposicin A.6. *To all orders strictly smaller than n, the expansions (2.34) and (2.35) are even. For n odd, there may be odd and even terms of order $r \geq n$, while for n even, if $\mathcal{O} = 0$, the expansion remains even to infinite order. Moreover, all terms of even order $r < n$ are solely generated from γ.*

Proof. We proceed by induction. Assume that up to an odd order $r-1$, all previous odd order terms vanish, including $r-1$. We show that this induction hypothesis and (A.15) implies $g_{(r+1)} = 0$.

First note that r is even. The first sum in (A.13) vanishes because $g_{(t+1)} = 0$ unless t odd, but then $g_{(r-t)} = 0$. For a similar reason, the second sum in (A.13) also vanishes and therefore $\mathcal{P}^{(r-1)} = 0$. The same kind of argument applies to $\partial_\Omega^{r-1}(H\dot{g}_\Omega) \mid_{\Omega=0} = 0$ (cf. (A.14)). Also, $\partial_\Omega^{r-1}(\dot{g}_\Omega^2) \mid_{\Omega=0} = 0$, because as already noted above, its expression is just (A.14) with the indices contracted in a different way.

For the derivative of the Ricci tensor, first note that this tensor involves quadratic terms in the Christoffel symbols $\Gamma^l_{mq}\Gamma^i_{jk}$ and tangential derivatives of them $\partial_l\Gamma^i_{jk}$. The latter yield, when taking the derivative $\partial_\Omega^{r-1}Ric(g_\Omega) \mid_{\Omega=0}$,

$$\partial_\Omega^{r-1}\left(\partial_l\Gamma^i_{jk}\right)\Big|_{\Omega=0} = \partial_l\left(\partial_\Omega^{r-1}\Gamma^i_{jk}\right)\Big|_{\Omega=0}$$

and the former

$$\partial_\Omega^{r-1}\left(\Gamma^l_{mq}\Gamma^i_{jk}\right)\Big|_{\Omega=0} = \sum_{s=0}^{r-1}\binom{r-s-1}{s}\left(\partial_\Omega^{r-s-1}\Gamma^l_{mq}\partial_\Omega^s\Gamma^i_{jk}\right)\Big|_{\Omega=0}.$$

Observe that both expressions have a derivative of odd order (lower or equal to $r-1$) in Ω of a Christoffel symbol. Thus, we evaluate $\left(\partial_\Omega^{r'}\Gamma^i_{jk}\right)\Big|_{\Omega=0}$ for $r' \leq r-1$ odd. The Christoffel symbols are a combination of contractions of $g^{ij}\partial_k g_{lm}$. Hence

$$\partial_\Omega^{r'}(g^{ij}\partial_k g_{lm}) \mid_{\Omega=0} = \sum_{s=0}^{r'} r'! g^{\sharp ij}_{(r'-s)}\partial_k g_{(s)lm}, \tag{A.17}$$

which vanishes because $g^{\sharp ij}_{(r'-s)} = 0$ unless s is odd, but then $\partial_k g_{(s)lm} = 0$. Thus $\partial_\Omega^{r-1}Ric(g_\Omega) \mid_{\Omega=0} = 0$ and the induction hypothesis implies $g_{(r+1)} = 0$. Since $g_{(1)} = 0$, the induction hypothesis holds as long as (A.15) provides an equation for the term $g_{(r+1)}$. Namely, if n is odd, they hold to any order strictly smaller than n. If n is even and $\mathcal{O} = 0$, it goes on for all values of r.

By Lemma A.3, the coefficient $g_{(r)}$ is generated by previous ones up to order $r-2$. Since all orders strictly smaller than n are even, it follows that all coefficients $g_{(r)}$ with $r < n$ are exclusively generated by $g_{(0)}$. i.e. γ. □

Summarizing, the main argument in the proof of Proposition A.6 is to inductively apply equation (A.15) to, first, establish the vanishing of odd order coefficients up to a certain order, and then, establish the dependence only on γ of the even (lower than n) order coefficients. The inductive argument applies to a coefficient as long as (A.15) provides a recursive expression for it, and this fails at the level $r = n$ irrespectively of the parity of n. This is the reason why evenness does not extend beyond $g_{(n)}$ for n odd. For the same reason, the dependence only on on γ of the non-zero even coefficients $g_{(r)}$ does no longer apply for $r \geq n$ when n is even. The absence of an equation (A.15) for $g_{(n)}$ implies an indeterminacy for this term. However, this term is not totally independent of γ. As we show in the next lemma, its trace and divergence are constrainted by γ as a consequence of equations (A.9).

Lemma A.7. *The trace and divergence of $g_{(n)}$ satisfy*

$$\mathrm{Tr}_\gamma g_{(n)} = \mathfrak{a}, \qquad \mathrm{div}_\gamma g_{(n)} = \mathfrak{b},$$

where $\mathfrak{a} = 0$, $\mathfrak{b} = 0$ for n odd and $\mathfrak{a}$ is a scalar and $\mathfrak{b}$ a one-form determined by γ for n even.

Proof.

Trace of $g_{(n)}$

Taking the r^{th} order derivative in the first of equations (A.9) and evaluating at $\Omega = 0$ yields

$$\lambda^2(r-1)\mathcal{H}^{(r)} - r\left(|\mathcal{A}|^2\right)^{(r-1)} = 0$$

where $(|\mathcal{A}|^2)^{(r-1)} = \partial_\Omega^{r-1}\left.|A|^2\right|_{\Omega=0}$. Expanding the terms, we have on the one hand that by (A.12)

$$\mathcal{H}^{(r)} = -\frac{(r+1)!}{2}\mathrm{Tr}_\gamma\left(g_{(r+1)}\right) - \frac{r!}{2}\sum_{s=0}^{r-1}(s+1)g^{\sharp ij}_{(r-s)}g_{(s+1)ij}. \tag{A.18}$$

In order to calculate $|\mathcal{A}|^2$ note

$$A^{ij} = g^{ik}g^{jl}A_{kl} = -\frac{\lambda}{2}g^{ik}g^{jl}\partial_\Omega g_{jk} = \frac{\lambda}{2}\partial_\Omega g^{ij} = \frac{\lambda}{2}\dot{g}^{ij}$$

hence $|A|^2 = -(\lambda^2/4)\dot{g}^{ij}\dot{g}_{ij}$ and

$$(|\mathcal{A}|^2)^{(r-1)} = -\frac{\lambda^2}{4}(r-1)!\sum_{s=0}^{r-1}(r-s)(s+1)g^{\sharp ij}_{(r-s)}g_{(s+1)ij}.$$

Then

$$\begin{aligned}\lambda^2(r-1)\mathcal{H}^{(r)} - r\left(|\mathcal{A}|^2\right)^{(r-1)} &= -\lambda^2\frac{r-1}{2}(r+1)!\mathrm{Tr}_\gamma\left(g_{(r+1)}\right)\\ &- \lambda^2\frac{r-1}{2}r!\sum_{s=0}^{r-1}(s+1)g^{\sharp ij}_{(r-s)}g_{(s+1)ij}\\ &+ \lambda^2\frac{r!}{4}\sum_{s=0}^{r-1}(r-s)(s+1)g^{\sharp ij}_{(r-s)}g_{(s+1)ij}\\ &= -\lambda^2\frac{r-1}{2}(r+1)!\mathrm{Tr}_\gamma\left(g_{(r+1)}\right) + \lambda^2\frac{r!}{2}\sum_{s=0}^{r-1}K_{rs}g^{\sharp ij}_{(r-s)}g_{(s+1)ij} = 0\end{aligned}$$

where

$$K_{rs} := \frac{(r-s)(s+1)}{2} - (r-1)(s+1) = \frac{(s+1)(2-r-s)}{2}.$$

Rearranging terms, we obtain

$$(r^2-1)\mathrm{Tr}_\gamma\left(g_{(r+1)}\right) = \sum_{s=0}^{r-1} K_{rs} g^{\sharp ij}_{(r-s)} g_{(s+1)ij}. \tag{A.19}$$

Equation (A.19) for $r+1 = n(>2)$ becomes

$$\mathrm{Tr}_\gamma\left(g_{(n)}\right) = \frac{1}{n(n-2)} \sum_{s=0}^{n-2} K_{(n-1)s} g^{\sharp ij}_{(n-1-s)} g_{(s+1)ij} =: \mathfrak{a}. \tag{A.20}$$

For n odd, all coefficients of odd order lower than n vanish. Since $r = n-1$ is even, $g^{\sharp ij}_{(r-s)}$ vanishes unless s even, but this makes $s+1$ odd and $g_{(s+1)ij} = 0$. Thus for n odd we conclude from (A.20) that $\mathrm{Tr}_\gamma g_{(n)} = 0$. For n even this does no longer holds, because the non-zero terms $g^{\sharp ij}_{(r-s)}$ ($r = n-1$ odd and s odd) multiply terms $g_{(s+1)ij}$ that do not necessarily vanish. In this case $\mathfrak{a}$ is generated by γ, because (A.19) contains terms up to order $n-2$ (those with of order $n-1$ are zero), which only depend on γ (cf. Proposition A.6).

Divergence of $g_{(n)}$

For this proof we use the second of equations (A.9)

$$\mathrm{div}_{g_\Omega} A - \lambda \mathrm{d}H = 0.$$

We write the divergence of A with indices (recall that, in the Gaussian coordinates we are using, $\nabla^{(\Omega)}_i A^{ij} = \nabla_i A^{ij}$)

$$\mathrm{div}_{g_\Omega} A^j = \nabla_i A^{ij} = \partial_i A^{ij} + \Gamma^i_{ik} A^{kj} + \Gamma^j_{ik} A^{ik}.$$

Taking the r^{th} order derivative at $\Omega = 0$ we obtain, for the first term

$$\partial^{(r)}_\Omega \, \partial_i A^{ij}\big|_{\Omega=0} = \frac{\lambda}{2}(r+1)! \partial_i g^{\sharp ij}_{(r+1)},$$

and for the second and the third

$$\begin{aligned}
\partial^r_\Omega \left(\Gamma^i_{ik} A^{kj}\right)\Big|_{\Omega=0} &= \frac{\lambda}{2} \sum_{s=0}^{r} \binom{r}{s} (s+1)! \, \partial^{r-s}_\Omega \left(\Gamma^i_{ik}\right)\big|_{\Omega=0} g^{\sharp kj}_{(s+1)} \\
&= \frac{\lambda}{2}(r+1)! \, \Gamma^i_{ik}\big|_{\Omega=0} g^{\sharp kj}_{(r+1)} + \lambda S_1, \\
\partial^r_\Omega \left(\Gamma^j_{ik} A^{ik}\right)\Big|_{\Omega=0} &= \frac{\lambda}{2} \sum_{s=0}^{r} \binom{r}{s} (s+1)! \left(\partial^{r-s}_\Omega \Gamma^j_{ik}\right)\Big|_{\Omega=0} g^{\sharp ik}_{(s+1)} \\
&= \frac{\lambda}{2}(r+1)! \, \Gamma^i_{ik}\big|_{\Omega=0} g^{\sharp kj}_{(r+1)} + \lambda S_2,
\end{aligned}$$

where

$$S_1 := \frac{1}{2}\sum_{s=0}^{r-1}\binom{r}{s}(s+1)!\,\partial_\Omega^{r-s}\left(\Gamma^i_{ik}\right)\big|_{\Omega=0}\,g^{\sharp kj}_{(s+1)}, \tag{A.21}$$

$$S_2 := \frac{1}{2}\sum_{s=0}^{r-1}\binom{r}{s}(s+1)!\,\partial_\Omega^{r-s}\left(\Gamma^j_{ik}\right)\Big|_{\Omega=0}\,g^{\sharp ik}_{(s+1)}. \tag{A.22}$$

Hence

$$\partial^r_\Omega\left(\mathrm{div}_{g_\Omega}A\right)|_{\Omega=0} = \frac{\lambda}{2}(r+1)!\mathrm{div}_\gamma g^\sharp_{(r+1)} + \lambda S_1 + \lambda S_2.$$

On the other hand, the r^{th} order derivative of $\mathrm{d}H$ at $\Omega=0$ is simply $\mathrm{d}\mathcal{H}^{(r)}$ because the diferential is taken in the submanifold Σ_Ω. Therefore

$$\frac{(r+1)!}{2}\mathrm{div}_\gamma g^\sharp_{(r+1)} = -S_1 - S_2 + \mathrm{d}\mathcal{H}^{(r)}. \tag{A.23}$$

For n odd, set $r=n-1$ (thus r even). The terms $g^{\sharp ik}_{(s+1)}$ in (A.21) and (A.22) are zero unless s is odd, hence $r-s$ is odd and lower or equal than $n-2$. Thus, all the derivatives $\partial^{r'}_\Omega\left(\Gamma^j_{ik}\right)\Big|_{\Omega=0}$ in (A.21) and (A.22) have r' odd. As noted above, the Christoffel symbols are a combination of contractions of $g^{ij}\partial_k g_{lm}$, hence formula (A.17) gives $\partial^{r'}_\Omega\left(\Gamma^j_{ik}\right)\Big|_{\Omega=0} = 0$ and therefore $S_1=S_2=0$. For $\mathrm{d}\mathcal{H}^{(n-1)}$, we look at equation (A.18). We have already proven that the trace of $g_{(n)}$ vanishes. The remaining terms in (A.18) also vanish: the terms with s even because $g_{(s+1)ij}$ is zero, the terms with s odd, because $g^{\sharp ij}_{(r-s)}=0$ as $r-s$ is odd. Therefore

$$\mathrm{div}_\gamma g^\sharp_{(n)} = 0.$$

Now, to relate $g^\sharp_{(n)}$ and $g_{(n)}$ we use formula (A.10) with $r=n$. Since n is odd, the only term survivings in the sum are the first and the last ones,

$$g^{\sharp ij}_{(n)}\gamma_{ik} + \gamma^{ij}g_{(n)ik} = 0 \quad\Longleftrightarrow\quad g^{\sharp ij}_{(n)} = -\gamma^{ik}g_{(n)kl}\gamma^{lk}$$

and therefore $\mathrm{div}_\gamma g^\sharp_{(n)}=0$ if and only if $\mathrm{div}_\gamma g_{(n)}=0$.

For n even, the above argument does not apply because $r=n-1$ is odd. Thus the RHS of (A.23) does not vanish in general. Looking at (A.21), (A.22) and (A.18) it follows that $\mathfrak{b} := \mathrm{div}_\gamma g^\sharp_{(n)}$ depends on coefficients of order up to $n-2$ and tangent derivatives thereof. Hence, $\mathrm{div}_\gamma g^\sharp_{(n)}$ is generated by γ. The same conclusion follows for $\mathrm{div}_\gamma g_{(n)}$ by an immediate application of Lemma A.2. □

Combining the results in this appendix, we may now show how the FG expansions are generated, under the assumption that $\mathcal{O}=0$ if n even. The zero-th order coefficient γ must be prescribed. This generates all coefficients $g_{(r)}$ with $r<n$. If n is even, γ also generates restrictions to the order n, so that given any $\overline{g}_{(n)}$ satisfying them, one

can always add freely a TT term $\mathring{g}_{(n)}$, so that $g_{(n)} = \overline{g}_{(n)} + \mathring{g}_{(n)}$ keeps satisfying all the equations. The generation of coefficients keeps going on recursively from $g_{(n)}$, so only even terms arise. For n odd, the recursive relations only restrict the term of order n to be TT. Thus the n-th order term is a freely prescribable TT tensor $g_{(n)} = \mathring{g}_{(n)}$. Similarly, the generation of coefficients keeps going on recursively from $g_{(n)}$, but it is no longer even. This is obvious at order n because of the presence of $g_{(n)}$ itself, and also true at higher order where further odd terms will generically appear. This is because the argument above proving evenness to a certain order relies on the vanishing of all previous odd order coefficients. In the case $g_{(n)} = 0$, the expansion is even to infinite order also for n odd.

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