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Yet another characterization of the majority rule

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HIGHLIGHTS

- New characterization of the majority rule (unrestricted societies, binary agenda).
- We prove that neither of the axioms in this characterization is superfluous.
- We use neither of the three axioms in the original characterization by May (1952).

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ABSTRACT

We prove an alternative characterization of the majority rule for unrestricted societies and a binary agenda. It uses neither of the three original axioms from the characterization by May (1952) for fixed societies that confront two alternatives.

1. Introduction

Since May (1952) axiomatized two-alternative majority choices in terms of three simple properties for a social welfare function, other axiomatic characterizations have appeared for his or related models. In the original model there are n voters and an issue on which they vote, which can either be accepted, rejected, or left unresolved. Equivalently, there are two alternatives and the society can either choose one of them, or declare them equally desirable. Every member of the society casts an individual vote that has the same characteristics as the social outcome: he or she can either vote for acceptance, rejection, or unresolvedness of the issue. We investigate the model with this voting structure and unrestricted number of voters and provide an alternative characterization of the majority rule that uses neither of the three axioms from May's original characterization.

Section 2 is devoted to basic notation and definitions, and it recaps earlier characterizations of the majority rule. Section 3 presents our main result, and examples that prove the independence of the axioms in this characterization. Two additional characterizations are derived as corollaries.

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2. Notation and preliminaries

Let **A** denote a subset of two alternatives, say x and y. We can also think of an issue that passes (alternative x) or is defeated (alternative y). A *society* is a non-empty set $N = \{1, \ldots, n\}$ of *voters* or agents. Every voter $i \in N$ has a complete preference over **A**, which we represent as $R_i \in \{-1, 0, 1\}$. We let $R_i = 1$, resp., $R_i = -1$, when member i strictly prefers x to y (or i wants that the issue passes), resp., y to x (or i wants that the issue is defeated). In either case we say that voter i is *resolved about* **A**, and also that **A** has a best-liked and a disliked alternative (with obvious meanings).[1] We write $R_i = 0$ when agent i is indifferent between x and y. The society's preferences are collected in a preference *profile* $R = (R_1, \ldots, R_n) \in \{-1, 0, 1\}^n$ whose *length* is n. For profiles of length 1 we just write $R = R_1$ instead of $R = (R_1)$. The collection of preference profiles for arbitrarily large non-empty societies is $\mathbf{P} = \bigcup_{n>0}\{-1, 0, 1\}^n$.

A *social welfare function* is a function $F : \bigcup_{n\geqslant 0}\{-1, 0, 1\}^n \longrightarrow \{-1, 0, 1\}$ with the convention $F(\varnothing) = 0$.[2] For the profile R, the fact $F(R) = 1$ means that the issue passes, $F(R) = -1$ means that it is defeated, and $F(R) = 0$ means that it is unresolved. We are

[1] Woeginger (2005) refers to biased voters.

[2] This is stated as property NSA or Null society assumption in Miroiu (2004)

interested in proving a new characterization of the majority rule, i.e., the social welfare function F_M that assigns to each $R \in \mathbf{P}$ with length n the collective preference $F_M(R) = \mathrm{sgn}(\sum_{i\in N} R_i)$. Here sgn denotes the usual sign function on the real numbers, i.e., sgn(x) equals $1, 0, -1$ when $x > 0, x = 0, x < 0$, respectively.

To complete our vector notation, we denote by $\mathbf{0}_n$ or simply $\mathbf{0}$ the profile of length n where all individual preferences are 0, with the convention $\mathbf{0}_0 = \varnothing$. For each $R, R' \in \mathbf{P}$ with length n, we denote $R \geqslant R'$ when $R_i \geqslant R'_i$ for each $i = 1, \dots, n$, and we write $R > R'$ when $R \geqslant R'$ but $R \neq R'$.

2.1. Our set of necessary and sufficient conditions for the majority rule

In order to prove our characterization we consider the following axioms for social welfare functions. It is routine to check that they are all verified by F_M.

Individual consistency (IC). For any profile $R = R_1$ of length 1, $F(R_1) = R_1$.

Woeginger (2003, p. 91) refers to property IC of social welfare functions. Quesada (2010a) describes it as "the collective preference of a society with only one member is the preference of that member". He also explains that IC is weaker that his Unanimity property, and it is also weaker than his Efficiency property.

Following Woeginger (2003), a *subsociety* of $N = \{1, \dots, n\}$ is $N \setminus \{i\}$ for some $i \in N$. When $R \in \mathbf{P}$ has length n, every $i \in N$ induces the *subprofile* $R^{-i} = (R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n) \in \{-1, 0, 1\}^{n-1}$ associated with the subsociety that results from removing i from N. Similarly we define subsocieties $R^{-i,-j}$ for every $i, j \in N$, and so forth. Inspired by Woeginger we argue that through this notion we can model situations like organizational councils or committees where there is one or more chairpersons, who in order to keep the procedure transparent, abstain from voting themselves.

Let us now put forward a property that establishes that for voters that are resolved about **A**, there are no incentives to manipulate the voting result by not showing up:

Non-swap (NS). For every $R \in \mathbf{P}$ with length $n > 1$, if $R_i \neq 0$ with $i \in N$, then $R_i \neq F(R)$ implies $F(R^{-i}) = -R_i$.

Non-swap bears some similarity to non-reversal (Campbell and Kelly, 2006), which is the key property of majority rule that results in an induced strategy-proof social choice rule (Campbell and Kelly, 2010).

Our last axiom is Social fairness:

Social fairness (SF). For every $R \in \mathbf{P}$ with length $n > 1$, if $F(R^{-k}) = -R_k$ for all $k \in N$ then there are $i < j, i, j \in N$, such that $F(R^{-i,-j}) = F(R)$ and $F(R_i, R_j) = 0$.

Following Woeginger's vindication of subsocieties through chairpersons, our last axiom ensures that for societies where every choice of a resolved chairperson is contrary to his/her interest, a pair of chairpersons can be selected so that the decision when both abstain coincides with the collective decision and they are jointly neutral (i.e., they collectively leave the issue unresolved).

2.2. Further properties of the majority rule

Other authors have found several sets of necessary and sufficient conditions for the majority rule. We now list some properties that have been used to that purpose, starting with the three axioms in the seminal (May, 1952):

Neutrality (N). For any $R \in \mathbf{P}$, $F(-R) = -F(R)$.

Anonymity (A). For any $R \in \mathbf{P}$ with length n and any permutation Π of N, one has $F(R) = F(R_{\Pi(1)}, \dots, R_{\Pi(n)})$.

Positive responsiveness (PR). For any $R, R' \in \mathbf{P}$ with length n, (a) if $R' > R$ then $F(R) \geqslant 0$ implies $F(R') = 1$, and (b) if $R > R'$ then $F(R) \leqslant 0$ implies $F(R') = -1$.

Variations of Pareto optimality are a must in this context. For example:

Pareto optimality (PO). For any $R \in \mathbf{P}$ with length n such that $R \neq \mathbf{0}_n$, (a) if $R \geqslant \mathbf{0}$ then $F(R) = 1$, and (b) if $\mathbf{0} \geqslant R$ then $F(R) = -1$.

Among the three properties in May's characterization, PR has been especially criticized for being too strong. Consequently Aşan and Sanver (2002) show that in May's theorem, PR can be replaced by PO and the following property:

Weak path independence (WPI). For any $R, R' \in \mathbf{P}$ with respective lengths n and n' and such that $|F(R) - F(R')| \neq 2$, we have $F(R_1, \dots, R_n, R'_1, \dots, R'_{n'}) = F(F(R), F(R'))$.

The idea behind WPI goes as follows. Suppose two disjoint societies that are combined into a new society. Then in order to compute the social preference of the enlarged society, *provided that the two initial societies are not in total disagreement* you can instead aggregate their two preference profiles into their respective social representatives and then aggregate them.

Cancellativeness properties are also used e.g., by Llamazares (2006) in his analysis with fixed societies:

Cancellativeness (C). For any $R, R' \in \mathbf{P}$ with length $n > 1$, if there are $i, j \in N$ such that $R_i = 1, R_j = -1, R'_i = R'_j = 0$, and $R'_k = R_k$ when $j \neq k \neq i, k \in N$, then $F(R) = F(R')$.

According to C, the social preference does not change when a pair of resolved agents with opposed preferences about **A** are replaced with two agents that are indifferent about **A**. Llamazares explains that when $n = 2$, C states a property that is already implied by N plus A. And also that cancellative social welfare functions are completely determined by their values on the set of profiles where non-indifferent voters agree that an alternative is better than the other.

Miroiu (2004) introduced the following axiom, which is called *Additive positive responsiveness* in Woeginger (2005):

Additive responsiveness (AR). For any $R \in \mathbf{P}$ with length $n > 1$ and any $i \in N$, (a) if $R_i = 1$ and $F(R^{-i}) \geqslant 0$ then $F(R) = 1$, and (b) if $R_i = -1$ and $F(R^{-i}) \leqslant 0$ then $F(R) = -1$.

In the presence of A, AR is equivalent to the following formulation:

AR′. For any $R \in \mathbf{P}$ with length n and $F(R) \geqslant 0$, resp., $F(R) \leqslant 0$, it must be the case that $F(R_1, \dots, R_n, 1) = 1$, resp., $F(R_1, \dots, R_n, -1) = -1$.

The main contribution in Woeginger (2003) refers to the following axiom:

Reducibility to subsocieties (RS). For any $R \in \mathbf{P}$ with length $n > 1$, $F(R) = F(F(R^{-1}), \dots, F(R^{-n}))$.

In his words, "RS stipulates that the aggregate preference of the total council behaves in the same way as the aggregate preference of all the subsocieties under changing chairmen".

Finally, Fishburn (1973) uses a limited responsiveness notion (Fishburn, 1983, Property A4):

Limited responsiveness (LR). For any $R, R' \in \mathbf{P}$ with length n, if $F(R) = 1$ and there is $j \in N$ such that $R_i - R'_i = 0$ for all $i \in N \setminus \{j\}$ and $R_j - R'_j = 1$, then one has $F(R') \geqslant 0$.

2.3. A digest of earlier characterizations of the majority rule

Table 1 below summarizes some of the better known characterizations of the majority rule under the assumptions of our model (binary agenda, complete preorders for each agent of a finite society, possibly tied alternatives in the social preference). May (1952), Fishburn (1973) and Llamazares (2006) consider a fixed society. Fishburn (1973) uses a variation of PR that is equivalent to a weak specification of PR under N (Fishburn, 1983, Axiom A3).

There are other characterizations that we do not describe in detail here, like Fishburn (1983), who considers a society with a

Table 1
Summary of existing characterizations of the majority rule in the context of this paper. The list is not exhaustive.

	A	N	PR	PO	RS	APR	WPI	LR	C
May (1952)	✓	✓	✓						
Fishburn (1973)		✓	✓					✓	
Aşan and Sanver (2002)	✓	✓		✓			✓		
Woeginger (2003)		✓		✓	✓				
Woeginger (2005)	✓	✓				✓			
Llamazares (2006)		✓		✓					✓

fixed number of agents, or Quesada (2010a,b). We can also cite other contexts where the majority rule has been investigated. For example, Yi (2005) studies majority and weak majority rules for fixed societies and arbitrary agendas. Campbell and Kelly (2013) consider a binary agenda and a social choice function that cannot declare a tie between the two options. Dasgupta and Maskin (2008) assume a continuum of voters who can never be indifferent between two alternatives. Quesada (2013) complements the majority rule with a ranking among individuals such that in case of social indifference, the non-indifferent agent that ranks first determines the social preference. Xu and Zhong (2010) refer to a set of individuals that is variable but whose preferences remain fixed.

3. A new characterization of the majority rule

In this Section we prove our main theorem, namely, Theorem 1 below:

Theorem 1. *A social welfare function* $F : \bigcup_{n\geqslant 0}\{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ *verifies IC, NS, and SF if and only if it is the majority rule.*

Proof. The 'if' part of the proof is straightforward. We prove the 'only if' part of the statement by induction on n. Of course, for $n = 1$ the conclusion is precisely IC.

Suppose $n = 2$. When either $R = (0,0)$ or $R = (1,-1)$ or $R = (-1,1)$, SF applies due to IC, hence the conclusion that there are $i = 1 < 2 = j$ with $F(R_1, R_2) = 0$ is $F(R) = 0$. If $R \in \{(1,1),(1,0),(0,1),(-1,-1),(-1,0),(0,-1)\}$, then NS plus IC give the conclusion.

Suppose that $n > 2$ and the statement is true for all profiles of size $1,\ldots,n-1$. Let $R = (R_1,\ldots,R_n) \in \{-1,0,1\}^n$ be a profile of size n. If $R = (0,\ldots,0)$ then $F(R^{-i}) = F_M(R^{-i}) = 0$ for all i by the induction hypothesis, thus SF assures that for some $i < j$, $F(R^{-i,-j}) = F(R)$ whereas $0 = F_M(R^{-i,-j}) = F_M(\mathbf{0}_{n-2})$. Now suppose $R_i \neq 0$ for some $i \in N$. We distinguish two cases.

Case 1: for all $k \in N$, $F(R^{-k}) = -R_k$. Then SF implies the existence of distinct $i, j \in N$ such that $F(R^{-i,-j}) = F(R)$ and $F(R_i, R_j) = 0$. Now the induction hypothesis assures $F_M(R^{-i,-j}) = F(R^{-i,-j}) = F(R)$ and $F_M(R_i,R_j) = F(R_i,R_j) = 0$. Hence by definition of the majority rule, $F_M(R) = F_M(R^{-i,-j}) = F(R)$.

Case 2: there is $i \in N$ with $F(R^{-i}) \neq -R_i$. Now the induction hypothesis assures $F(R^{-i}) = F_M(R^{-i})$. Because F_M verifies NS, $R_i = F_M(R)$. Because F verifies NS, $R_i = F(R)$. Therefore $F(R) = F_M(R)$. □

Proposition 1 below proves that the axioms in Theorem 1 are tight:

Proposition 1. *There are social welfare functions* F_1, F_2, F_3 *other than* F_M, *such that*

(a) F_1 *verifies IC, SF but not NS.*
(b) F_2 *verifies NS, SF but not IC.*
(c) F_3 *verifies IC, NS but not SF.*

Proof. Function F_1 can be defined as follows: for each $R \in \mathbf{P}$, $F_1(R) = 1$ if $R > \mathbf{0}$, $F_1(R) = -1$ if $\mathbf{0} > R$, and $F_1(R) = 0$ otherwise. Clearly, it verifies IC. To check that F_1 verifies SF select $R \in \mathbf{P}$ with length $n > 1$, and assume $F_1(R^{-k}) = -R_k$ for all $k \in N$. We need to prove that there are $i < j$, $i, j \in N$, such that $F_1(R^{-i,-j}) = F_1(R)$ and $F_1(R_i, R_j) = 0$.

The profile $R = \mathbf{0}$ verifies the hypothesis and the thesis is obvious. Suppose $R \neq \mathbf{0}$, then two cases arise: either there is $j \in N$ with $R_j = 1$, or there is $l \in N$ with $R_l = -1$. We argue with the first situation, the other one being symmetrical.

Because $F_1(R^{-j}) = -R_j = -1$, we deduce $\mathbf{0} > R^{-j}$ and there must be $i \neq j$ with $R_i = -1$. By assumption, $F_1(R^{-i}) = -R_i = 1$ thus $R^{-i} > \mathbf{0}$. The definition of F_1 ensures $F_1(R) = 0$ since $R_j = 1$ and $R_i = -1$. Besides, $F_1(R_i, R_j) = 0$ too, and $F_1(R^{-i,-j}) = 0$ since for every $k \in N$ with $i \neq k \neq j$, one has $R_k = 0$. The latter statement derives from $\mathbf{0} > R^{-j}$ and $R^{-i} > \mathbf{0}$ (observe that $F_1(R^{-i,-j}) = 0$ when $N = 2$ too, because our convention $F_1(\varnothing) = 0$ is implicit to the fact that F_1 is a social welfare function). Finally, we do not lose generality if we assume $i < j$ because F_1 is anonymous.

Since $F_1 \neq F_M$, NS cannot hold true.

To construct F_2 we use the following notation: for each $R \in \mathbf{P}$, $\tilde{R}$ is the vector that results from replacing each 0 by a 1 in R, i.e., R and $\tilde{R}$ have equal length n, and for each $i = 1,\ldots,n$, $\tilde{R}_i = R_i$ if $R_i \neq 0$, $\tilde{R}_i = 1$ if $R_i = 0$. Observe that $\tilde{R}^{-i} = \widetilde{R^{-i}}$, $\tilde{R}^{-i,-j} = \widetilde{R^{-i,-j}}$ throughout. For each $R \in \mathbf{P}$, we let $F_2(R) = F_M(\tilde{R})$. Because $F_2(0) = 1$, F_2 contradicts IC.

To check that F_2 verifies NS select $R \in \mathbf{P}$ with length $n > 1$ and $i \in N$ such that $0 \neq R_i \neq F_2(R) = F_M(\tilde{R})$. Then $\tilde{R}_i \neq F_M(\tilde{R})$ and because F_M verifies NS, $F_2(R^{-i}) = F_M(\tilde{R}^{-i}) = -\tilde{R}_i = -R_i$.

To check that F_2 verifies SF select $R \in \mathbf{P}$ with length $n > 1$, and assume $F_2(R^{-k}) = -R_k$ for all $k \in N$. We need to prove that there are $i < j$, $i, j \in N$, such that $F_2(R^{-i,-j}) = F_2(R)$ and $F_2(R_i, R_j) = 0$. Equivalently: there are $i < j$, $i, j \in N$, such that $F_M(\widetilde{R^{-i,-j}}) = F_M(\tilde{R})$ and $F_M(\tilde{R}_i, \tilde{R}_j) = 0$. The assumption means $F_M(\widetilde{R^{-k}}) = F_M(\tilde{R}^{-k}) = -R_k$ for all $k \in N$. We distinguish two cases. If there is $R_k = 0$ then there must be $i, j \in N$ with $i < j \neq k \neq i$, $\tilde{R}_i = 1$ and $\tilde{R}_j = -1$ (hence $n > 2$). Therefore $F_M(\tilde{R}_i, \tilde{R}_j) = 0$ and by definition of F_M, $F_M(\tilde{R}) = F_M(\tilde{R}^{-i,-j})$ as desired. If $R_k \neq 0$ then for any fixed k, in R^{-k} there must be $\tilde{R}_i = -R_k = -\tilde{R}_k$. Therefore $F_M(\tilde{R}_i, \tilde{R}_k) = 0$ and by definition of F_M, $F_M(\tilde{R}) = F_M(\tilde{R}^{-i,-k})$ as desired.

Function F_3 can be defined as follows: for each $R \in \mathbf{P}$,

$$F_3(R) = \begin{cases} F_M(R), & \text{if either } n = 1 \text{ or } (n > 1 \text{ and } F_M(R) \neq 0), \\ 1, & \text{if } n > 1 \text{ and } F_M(R) = 0 \end{cases}$$

Hence F_3 coincides with the majority rule except for profiles with at least two agents and indifference outcome by the majority rule, a case where option x is selected. Clearly, F_3 verifies IC. To check that F_3 verifies NS select $R \in \mathbf{P}$ with length $n > 1$ and $i \in N$ such that and $0 \neq R_i \neq F_3(R)$. If $R_i = 1$ then $F_3(R) = -1 = F_M(R)$, hence because F_M verifies NS, $F_M(R^{-i}) = -R_i = -1$ and $F(R^{-i}) = -1 = -R_i$. If $R_i = -1$ then $F_3(R) = 1$ and $F_M(R) \geqslant 0$, hence because F_M verifies NS, $F_M(R^{-i}) = -R_i = 1$ and $F(R^{-i}) = 1 = -R_i$. Finally, since $F_3 \neq F_M$, SF cannot hold true. □

It is known that N and PO imply IC (Woeginger, 2003, proof of Theorem 2) and also that N and AR imply IC (Woeginger, 2005, proof of Theorem 3). Therefore one has:

Corollary 1. *Let* $F : \bigcup_{n\geqslant 0}\{-1,0,1\}^n \longrightarrow \{-1,0,1\}$ *be a social welfare function. The following statements are equivalent: (1) F is the majority rule, (2) F verifies N, PO, NS, and SF, and (3) F verifies N, AR, NS, and SF.*

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