

Original articles

On the MS-stability of predictor–corrector schemes for stochastic differential equations

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Abstract

Predictor–corrector schemes are designed to be a compromise to retain the stability properties of the implicit schemes and the computational efficiency of the explicit ones. In this paper a complete analytical study for the linear mean-square stability of the two-parameter family of Euler predictor–corrector schemes for scalar stochastic differential equations is given. The analyzed family is given in terms of two parameters that control the degree of implicitness of the method. For each selection of the parameters the stability region is obtained, letting its comparison. Particular cases of the counter-intuitive fact of losing numerical stability by reducing the step size, is confirmed and proved. Figures of the MS-stability regions and numerical examples that confirm the theoretical results are shown.

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1. Introduction and background

Due to its application for representing the evolution of dynamical systems, frequently subjected to random phenomena, stochastic differential equations (SDEs) are becoming an important tool in many scientific areas like theoretical physics, including dynamics of satellites, mechanical vibrations or linear oscillators, see Langtangen [16], Sagirow [19] or Tocino [23]; investment finance, see e.g. Black and Scholes [4], Jäckel [14] or Platen and Shi [18]; population dynamics, like in Allen [1], Carletti [7], or Gard and Kannan [9]; and many others, see [2] or [15] for a wide number of applications.

Given a filtered probability space $(\Omega, \mathcal{F}_t, P)$ and an SDE of Itô type

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_{t_0} = x_0, \quad t_0 \leq t \leq T, \quad (1)$$

where $x_0 \in \mathbb{R}$, $\{W_t\}$ is the standard scalar Wiener process and the coefficients $a = a(x)$ and $b = b(x)$ satisfy the assumptions of the existence and uniqueness theorem, see [3], in general, its analytic solution is not available; to approximate it, numerical methods represent an indispensable implement. A complete survey on numerical methods for the solution of SDEs can be found in [15].

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The selection of a numerical method to solve the SDE is an important decision. Commonly, convergence is considered the main criterion: strong convergence is used if the sample paths of the exact solution are needed and weak convergence when their moments are required. But some authors, see [18], support that stability should be taking into account firstly. Numerical stability relates with the capacity of the method to control the propagation of errors. As in the deterministic case, the essential components to study the stability of stochastic numerical schemes are a stochastic test equation, a definition of asymptotic stability for the stochastic equation and a characterization of it in terms of equation parameters, a corresponding definition and characterization for numerical schemes and a comparison between the domains of stability of the equation and the scheme. In this paper we consider the multiplicative noise scalar linear test equation

$$dX_t = \lambda X_t dt + \mu X_t dW_t, \tag{2}$$

with $t > 0$, $\lambda, \mu \in \mathbb{R}$ and initial condition $X_0 = x_0 \neq 0$. To study stochastic stability, this test equation has been widely used, see [11–13,20]. Several concepts of stability have been proposed, see e.g. [3] or [10]. Here we shall focus on stability in the mean-square sense or linear MS-stability, characterized by the condition

$$\lim_{t \rightarrow \infty} E|X_t|^2 = 0. \tag{3}$$

Section 1.2 is devoted to recall this concept and its characterization.

As in the deterministic case, predictor–corrector (PC) schemes for SDEs have been developed to take advantage of the good properties, and at the same time to avoid the drawbacks, of explicit and implicit schemes. Implicit methods have been used due to their good stability properties; in return, the computational cost grows because they usually require the solution of an algebraic equation at each time step. Starting from Y_n , the idea behind PC methods is to use first an explicit scheme (called the predictor) to obtain an approximated solution of the next step \bar{Y}_{n+1} ; then, an implicit scheme (the corrector) is used as an explicit one replacing the unknown value of the following step Y_{n+1} by the prediction \bar{Y}_{n+1} . Since both schemes are explicit, PC methods share the computational efficiency of explicit methods. The numerical methods considered in this work are the so-called predictor–corrector Euler methods introduced in [15] as a bi-parametric family of weak order 1.0 schemes. In [5] it was proved that these schemes have strong order 0.5 and their asymptotical stability was studied; later, see [18], asymptotical p -stability was considered for the same family. Varying the parameters of a PC Euler scheme determines the involvement of the predictor, i.e., the degree of implicitness; so, in this work the way how the variation of implicitness determines the stability behavior will be studied. Up to our knowledge the stability analysis of PC methods has been carried on the multiplicative noise linear test used in [5,15] and other works of the same authors; it differs from (2) and is given by

$$dX_t = \left(1 - \frac{3}{2}\alpha\right)\lambda X_t dt + \sqrt{\alpha|\lambda|}X_t dW_t.$$

Similarities and differences of this approach with our proposal will be shown.

The rest of this section is devoted to introduce PC Euler methods and the theoretical tools for mean square stability analysis. In Section 2 this analysis is developed for each scheme in terms of its parameters. Their stability regions will be calculated and pictures that make easier the comparison between different regions will be given. So, it will be shown how the growth in implicitness affects stability. In particular, we are interested in the advised phenomenon, shown graphically in [18], of examples for which there is a loss of stability by decreasing the step size of the method. The theoretical results will be confirmed with the numerical experiments of Section 3. Finally, Section 4 is devoted to expound the conclusions.

1.1. Predictor–corrector Euler schemes

Consider an equidistant discretization $t_0 < t_1 < \dots < t_N = T$ with step-size $\Delta = (T - t_0)/N$. Assuming the differentiability of the diffusion coefficient b , the family of predictor–corrector Euler schemes for computing approximations Y_n to the exact values X_{t_n} of the solution to (1) at t_n , $n = 1, \dots, N$, is given by

$$Y_{n+1} = Y_n + (\theta \bar{a}_\beta(\bar{Y}_{n+1}) + (1 - \theta)\bar{a}_\beta(Y_n))\Delta + (\beta b(\bar{Y}_{n+1}) + (1 - \beta)b(Y_n))\Delta W_n \tag{4}$$

with

$$\bar{a}_\beta = a - \beta b b' \tag{5}$$

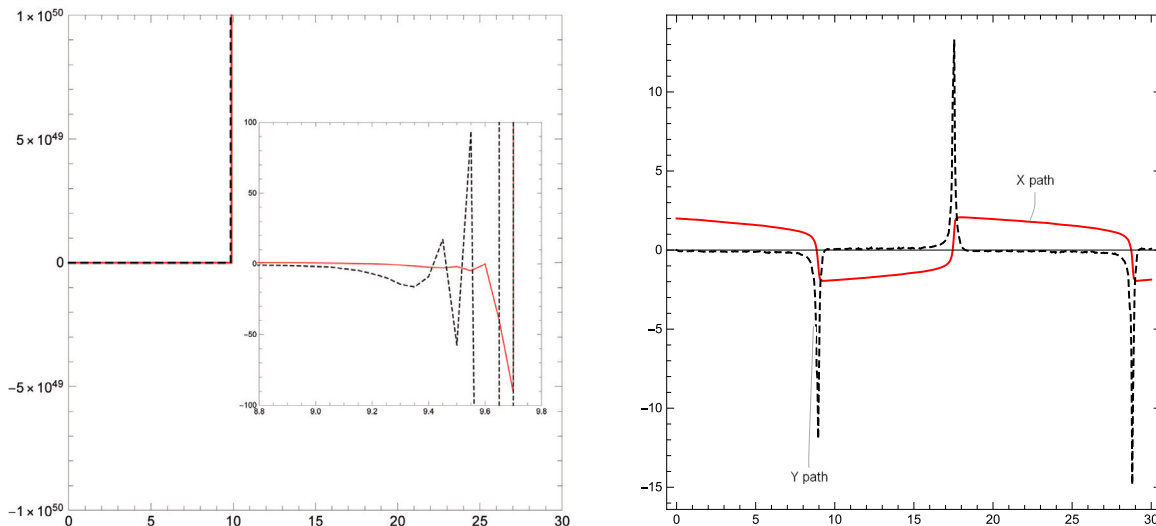


Fig. 1. Approximation of van der Pol equation (7) with initial values $X_0 = 2, Y_0 = 0$ and step size $\Delta = 0.05$ by the Euler scheme (left) and by the predictor–corrector Euler scheme (4) with $\beta = 0$ and $\theta = 1/2$ (right).

and

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n \tag{6}$$

where $\theta, \beta \in [0, 1]$, and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ are Gaussian random variables $\mathcal{N}(0, \Delta)$. The parameters $\theta, \beta \in [0, 1]$ are called the degree of implicitness in the drift and the diffusion coefficients, respectively. Notice also that the drift a is corrected by \bar{a}_β . In this work, for obvious reasons, we shall refer to PC Euler methods as (θ, β) -methods. These methods were introduced in [15]. Also they have been considered in [5] and [18]. For $\beta = 0$ they have been studied in [24]. Notice that Euler method is a particular member of this family; on the other hand if $\theta = \frac{1}{2}, \beta = 0$ Eqs. (4)–(6) become the modified trapezoidal method of weak order 1.0, see [15].

Notice that the expression in (6), the *predictor*, gives the approximation \bar{Y}_{n+1} of the value Y_{n+1} using Euler method; on the other hand, the *corrector* in (4) can be seen as a modified implicit scheme where the value Y_{n+1} has been replaced by its approximation \bar{Y}_{n+1} . This replacement converts an implicit scheme into an explicit one and, in spite of the fact that (4) is an explicit expression, due to $\bar{Y}_{n+1} \simeq Y_{n+1}$, one can expect that the corrector retains part of the features of the original implicit method. This argument is supported by the following example (see also the numerical experiments in Section 3):

Example 1. Consider the stochastic van der Pol system, see [8],

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} Y_t \\ 10(1 - X_t^2)Y_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ (1 - X_t^2)Y_t \end{pmatrix} dW_t, \tag{7}$$

where $\{W_t\}$ stands for the standard scalar Wiener process, with initial value $(X_0, Y_0) = (2, 0)$.

Application of the Euler scheme with step-size $\Delta = 0.05$ to approximate a solution path leads to an explosion, see Fig. 1, left; due to stability problems the program aborts computation around $t = 10$, see the inner window on the left picture of Fig. 1; a similar result was obtained in [8] applying Milstein scheme. On the contrary, if we apply the predictor–corrector Euler scheme (4)–(6) with $\beta = 0$ and $\theta = 1/2$ and the same Brownian path we obtain the result shown in Fig. 1, right. Notice that here the explosion has been avoided by using an explicit scheme, unlike the approach in [8], where an implicit scheme was used.

1.2. Mean square stability analysis tools

Since the exact solution of (2) is given by

$$X_t = x_0 \exp \left\{ \left(\lambda - \frac{1}{2} \mu^2 \right) t + \mu W_t \right\}, \tag{8}$$

one has $E|X_t|^2 = |x_0|^2 \exp \{(2\lambda + \mu^2) t\}$ and MS-stability condition (3) is equivalent to

$$2\lambda + \mu^2 < 0; \tag{9}$$

the set

$$S_{SDE} = \{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} : 2\lambda + \mu^2 < 0\} \tag{10}$$

is called the MS-stability domain of the linear test equation (2).

By analogy with (3), for a numerical method we have

Definition 2. A stochastic method is said to be asymptotically mean-square stable if $\lim_{n \rightarrow \infty} E|X_n|^2 = 0$, where $\{X_n\}$ is the sequence obtained when the method is applied to a test equation with equidistant step-size.

Explicitly, to study the linear MS-stability of a numerical method, we apply the scheme to test problem (2) giving an expression of X_{n+1} in terms of the step-size Δ , the parameters of the problem λ, μ , the parameters and random variables of the method, and X_n ; then, taking mean-square norm in the obtained expression we get a recurrence of the form

$$E[X_{n+1}^2] = R(\lambda, \mu, \Delta)E[X_n^2]. \tag{11}$$

From here, $E[X_n^2] \rightarrow 0$ if and only if $R(\lambda, \mu, \Delta) < 1$; then $R(\lambda, \mu, \Delta)$ is called the MS-stability function of the method and the set

$$S_{SM}(\Delta) = \{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} : R(\lambda, \mu, \Delta) < 1\} \tag{12}$$

is called the stability domain of the stochastic method applied with step Δ . This concept was established in [20] and has been considered in [12,13,21,22]. Following Higham, see [12,13], our approach to analyze numerical MS-stability studies for what values of the step-size the numerical method share the stability of the test problem. In this sense, MS-stability domains $S_{SM}(\Delta)$ and S_{SDE} can be compared, and the inclusion $S_{SDE} \subset S_{SM}(\Delta)$ for all $\Delta > 0$ means that whenever the SDE is stable, then so is the stochastic method for any stepsize. This is a generalization of deterministic A-stability property of numerical methods for deterministic equations.

2. Stability analysis of PC Euler methods

Our aim here is to carry through a linear mean-square stability analysis for the methods of the biparametric family (4)–(6), where $\theta, \beta \in [0, 1]$. On the one hand, varying the parameters θ, β allows to compare the stability regions for different schemes of the family; on the other hand, for a given linear problem, a step-size bound will be given for each (θ, β) -method to ensure MS-stable numerical solution. The work is inspired by [13] and [12], where the author studies mean-square stability of semi-implicit Euler and Milstein θ -methods. Notice in particular that when $\theta = 0, \beta = 0$ the PC method (4)–(6) becomes the Euler method; then it coincides with the stochastic θ -method with $\theta = 0$ in [13]. In that work it was proved that (a) if the test problem is unstable then so is the Euler method for all $\Delta > 0$ and (b) if the problem is stable then so is the Euler method for $\Delta < -(2\lambda + \mu^2)/\lambda^2$. In this section we find a similar step-size bound for each predictor–corrector (θ, β) -method, i.e., a constant $\Delta_{(\theta, \beta)} > 0$ such that if the problem is stable then so is the predictor–corrector method for $0 < \Delta < \Delta_{(\theta, \beta)}$.

Applying the method (4)–(6) to the test problem (2) produces the recurrence

$$Y_{n+1} = (1 + (\lambda - \beta\mu^2)\Delta(1 + \theta\lambda\Delta) + \mu\Delta W_n + ((\theta + \beta)\lambda\mu - \theta\beta\mu^3)\Delta\Delta W_n + \beta\mu^2\Delta W_n^2)Y_n \tag{13}$$

Taking mean-square norm in (13) yields the difference equation

$$E[Y_{n+1}^2] = R_{(\theta, \beta)}(\lambda, \mu, \Delta)E[Y_n^2] \tag{14}$$

with

$$\begin{aligned} R_{(\theta, \beta)}(\lambda, \mu, \Delta) = & (2\lambda + \mu^2)\Delta(1 + \theta\lambda\Delta)^2 + \theta^2\lambda^4\Delta^4 + (2\theta - 2\theta^2)\lambda^3\Delta^3 + (1 - 2\theta)\lambda^2\Delta^2 \\ & + \mu^4\Delta^2\beta(\beta\theta^2\lambda^2\Delta^2 + 2\beta - 2\lambda\Delta\beta\theta - 2\lambda\Delta\theta^2 - 2\theta) \\ & + \mu^2\Delta^2\beta\lambda(-2\theta^2\lambda^2\Delta^2 + \beta\lambda\Delta + 2(1 - \theta)) + \theta^2\beta^2\mu^6\Delta^3 + 1. \end{aligned}$$

A straightforward computation gives that the stability condition $R_{(\theta,\beta)}(\lambda, \mu, \Delta) < 1$ is equivalent to the inequality

$$\begin{aligned} &(2\lambda + \mu^2)\theta^2\beta^2\mu^4\Delta^3 \\ &+ (1 + \theta\lambda\Delta)\Delta(\theta\lambda^3\Delta^2 + \lambda(\theta\mu^2 + \lambda)\Delta + 2\lambda + \mu^2) \\ &+ \mu^4\Delta^2\beta(\beta\theta^2\lambda^2\Delta^2 - 2\theta\lambda(\theta + \beta + \theta\beta)\Delta + 2(\beta - \theta)) \\ &+ \beta\lambda\mu^2\Delta^2(-2\theta^2\lambda^2\Delta^2 + \beta\lambda\Delta + 2(1 - \theta)) < 0. \end{aligned} \tag{15}$$

For each (θ, β) predictor–corrector method the stability domain defined in (12) will be denoted $S_{(\theta,\beta)}(\Delta)$. A geometrical representation of MS-stability regions S_{SDE} and $S_{(\theta,\beta)}(\Delta)$ helps to interpret the bounds found in the results of the next subsections. Following [13] we shall take

$$x = \Delta\lambda, \quad y = \Delta\mu^2$$

to obtain such representation in the real half-plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Notice that, given problem parameters $\lambda, \mu \in \mathbb{R}$, varying Δ correspond to moving along a ray that passes through the origin and (λ, μ^2) . The pairs (x, y) , $y \geq 0$ such that $2x + y < 0$ constitute the MS-stability region of the test equation S_{SDE} and correspond to the area between the ray $y = -2x$ and the negative half-axis x , see, e.g., the squared areas shown in the pictures of Fig. 11. For each (θ, β) -predictor corrector method, with the new coordinates the stability condition (15) becomes

$$\begin{aligned} &y(y + 2) + \theta^2(x + y^2)(y - \beta x)^2 \\ &+ 2\theta(\beta xy + x + y^2 + y)(y - \beta x) + x(2\beta^2x + (\beta y + 1)^2) < 0, \end{aligned} \tag{16}$$

which gives as representation of $S_{(\theta,\beta)}(\Delta)$ the set

$$S_{(\theta,\beta)} = \{(x, y) \in \mathbb{R}^2 : y \geq 0; (16) \text{ holds} \},$$

see, e.g., the shaded areas shown in the pictures of Fig. 11.

2.1. Methods with $\theta = 0$

Euler method, as it was said, corresponds to the case $\theta = \beta = 0$ and it was studied in [13]. Suppose now that $\theta = 0, \beta \in (0, 1]$ and that the stability condition (9) of the test equation holds. Since $\theta = 0$, condition (15) becomes

$$\Delta(\beta^2\lambda^2\mu^2\Delta^2 + (\lambda^2 + 2\beta^2\mu^4 + 2\beta\lambda\mu^2)\Delta + 2\lambda + \mu^2) < 0,$$

which can be written, since $\beta > 0$, as

$$\Delta\beta^2\lambda^2\mu^2(\Delta - \Delta^-)(\Delta - \Delta^+) < 0, \tag{17}$$

where

$$\Delta^\pm = \frac{-(\lambda^2 + 2\beta^2\mu^4 + 2\beta\lambda\mu^2) \pm \sqrt{(\lambda^2 + 2\beta^2\mu^4 + 2\beta\lambda\mu^2)^2 - 4\mu^2\beta^2\lambda^2(2\lambda + \mu^2)}}{2\mu^2\beta^2\lambda^2}.$$

Since $\Delta^- < 0 < \Delta^+$ we conclude that for $0 < \Delta < \Delta^+$ condition (17) holds. Then we have proved:

Theorem 3. *If $\theta = 0$ and $\beta \in [0, 1]$, the (θ, β) -predictor–corrector method (4)–(6) applied with constant step-size $\Delta > 0$ to a stable test problem (2) is stable if $\Delta < \Delta_{(0,\beta)}$ where*

$$\Delta_{(0,\beta)} = \begin{cases} -\frac{2\lambda + \mu^2}{\lambda^2} & \text{if } \beta = 0 \\ \frac{-(\lambda^2 + 2\beta^2\mu^4 + 2\beta\lambda\mu^2) + \sqrt{(\lambda^2 + 2\beta^2\mu^4 + 2\beta\lambda\mu^2)^2 - 4\mu^2\beta^2\lambda^2(2\lambda + \mu^2)}}{2\mu^2\beta^2\lambda^2} & \text{if } 0 < \beta \leq 1 \end{cases}$$

We consider now the geometrical representation of the MS-stability region of the $(0, \beta)$ -methods analyzed in the above Theorem. With $\theta = 0$ condition (16) becomes

$$y(y + 2) + x(2\beta^2x + (\beta y + 1)^2) < 0$$

and then

$$S_{(0,\beta)} = \{(x, y) \in \mathbb{R}^2 : y \geq 0; y(y + 2) + 2x^2\beta^2 + x(\beta y + 1)^2 < 0\}.$$

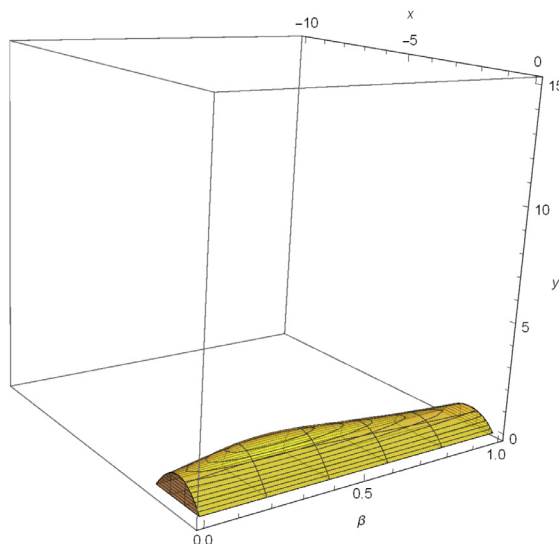


Fig. 2. MS-stability regions of $(0, \beta)$ -predictor corrector schemes, $\beta \in [0, 1]$.

Fig. 2 contains a graphical representation summarizing all these regions: For each $\beta_0 \in [0, 1]$ the $S_{(0,\beta_0)}$ region is obtained by the section of the solid with the plane $\beta = \beta_0$. Notice that all sections are very similar; starting at $\beta = 0$ (Euler method) there is a light size increasing of the regions up to values near to $\beta = 1/4$, followed by a light decrease as the index β grows. This result can be interpreted as the effect of increasing implicitness on the diffusion term (β grows from 0 to 1) of the PC method with explicit drift term ($\theta = 0$): the stability behavior improves slightly with values of β around $1/4$.

In the first row of Fig. 11 the MS-stability regions $S_{(0,0)}$, $S_{(0,1/9)}$, $S_{(0,1/8)}$, $S_{(0,1/2)}$ and $S_{(0,1)}$ have been represented. For each pair $(0, \beta)$ the bound $\Delta_{(0,\beta)}$ found in Theorem 3 can be interpreted in the following sense: Giving a stable problem, i.e. a pair (λ, μ) fulfilling (9) determines the ray $y = (\mu^2/\lambda)x$ (contained in the squared area); there exists a value $\Delta_{(0,\beta)}$ such that the segment $\{(x, y) = (\lambda\Delta, \mu^2\Delta) : 0 < \Delta < \Delta_{(0,\beta)}\}$ is entirely contained in the shaded region.

2.2. Methods with $\beta = 0$

Consider now the case $\beta = 0, \theta \in (0, 1]$. With these values, condition (15) becomes

$$(1 + \theta\lambda\Delta)\Delta (\theta\lambda^3\Delta^2 + \lambda(\theta\mu^2 + \lambda)\Delta + 2\lambda + \mu^2) < 0. \tag{18}$$

Notice that the discriminant of the second order polynomial in Δ which appears in the last factor of (18) is

$$\begin{aligned} D_1 &= \lambda^2(\theta\mu^2 + \lambda)^2 - 4(2\lambda + \mu^2)\theta\lambda^3 \\ &= \lambda^2((1 - 8\theta)\lambda^2 + \mu^2\theta(\mu^2\theta - 2\lambda)) \\ &= \lambda^2((\lambda - \mu^2\theta)^2 - 8\theta\lambda^2); \end{aligned} \tag{19}$$

then if $D_1 \geq 0$, the stability condition (18) becomes

$$\theta\lambda^3(\Delta - \Delta_1^+)(\Delta - \Delta_1^-)(1 + \theta\lambda\Delta)\Delta < 0, \tag{20}$$

where

$$\Delta_1^\pm = \frac{-(\theta\mu^2 + \lambda) \pm \sqrt{(\lambda - \mu^2\theta)^2 - 8\theta\lambda^2}}{2\theta\lambda^2}. \tag{21}$$

On the other hand, note that (18) can also be written

$$(1 + \theta\lambda\Delta)\Delta ((2\lambda + \mu^2)(1 + \theta\lambda\Delta) + \lambda^2\Delta(1 + \theta\lambda\Delta - 2\theta)) < 0. \tag{22}$$

Remark 4. If $D_1 \geq 0$, $0 \leq \theta \leq 1/2$ and (9) holds then $\lambda + \theta\mu^2 < 0$ and $0 < \Delta_1^- < \Delta_1^+$ since the roots can be written

$$\Delta_1^\pm = \frac{-(\theta\mu^2 + \lambda) \pm \sqrt{(\theta\mu^2 + \lambda)^2 - 4(2\lambda + \mu^2)\theta\lambda}}{2\theta\lambda^2}.$$

Theorem 5. If $\beta = 0$ and $0 < \theta \leq 1$, the $(\theta, 0)$ -predictor–corrector method (4)–(6) applied with constant step-size $\Delta > 0$ to a stable test problem (2) is stable if $\Delta < \Delta_{(\theta,0)}$ where

$$\Delta_{(\theta,0)} = \begin{cases} \Delta_1 & \text{for } 0 < \theta \leq \frac{1}{8} \\ \Delta_0 & \text{for } \frac{1}{8} < \theta < \frac{1}{2} \text{ and } \frac{\mu^2}{-\lambda} < \frac{\sqrt{8\theta}-1}{\theta} \\ \Delta_1 & \text{for } \frac{1}{8} < \theta < \frac{1}{2} \text{ and } \frac{\mu^2}{-\lambda} \geq \frac{\sqrt{8\theta}-1}{\theta} \\ \Delta_0 & \text{for } \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

with

$$\Delta_0 = \frac{-1}{\theta\lambda}, \quad \Delta_1 = \frac{-(\theta\mu^2 + \lambda) - \sqrt{(\theta\mu^2 + \lambda)^2 - 4(2\lambda + \mu^2)\theta\lambda}}{2\theta\lambda^2} = \Delta_1^- \tag{23}$$

Proof. Suppose that $\beta = 0$, $0 < \theta \leq 1$ and the test problem is stable, i.e. that condition (9) holds (in particular $\lambda < 0$). We shall prove that one of the inequalities (18), (20) or (22) fulfills for $0 < \Delta < \Delta_{(\theta,0)}$.

Suppose that $0 < \theta \leq \frac{1}{8}$. In this case the second expression in (19) shows that $D_1 \geq 0$; we have, by Remark 4, that $\lambda + \theta\mu^2 < 0$ and $0 < \Delta_1^- < \Delta_1^+$. If $\Delta < \Delta_1^-$ then $\Delta < \frac{-(\theta\mu^2 + \lambda)}{2\theta\lambda^2}$ and

$$\lambda(\theta\lambda\Delta + 1) = \theta\lambda^2\Delta + \lambda < \frac{\lambda - \theta\mu^2}{2} < 0;$$

from here $\theta\lambda\Delta + 1 > 0$ and the stability condition (20) holds if $0 < \Delta < \Delta_1^-$.

Suppose now that $1/8 < \theta < 1/2$ and denote $k_0 = (\sqrt{8\theta} - 1)/\theta$; notice that $0 < k_0 < 2$. We distinguish two cases:

(a) If $-\mu^2/\lambda < k_0$ then $k_0\lambda + \mu^2 < 0$ and

$$\begin{aligned} & (2\lambda + \mu^2)(1 + \theta\lambda\Delta) + \lambda^2\Delta(1 + \theta\lambda\Delta - 2\theta) \\ &= (k_0\lambda + \mu^2)(1 + \theta\lambda\Delta) + (2 - k_0)\lambda(1 + \theta\lambda\Delta) + \lambda^2\Delta(1 + \theta\lambda\Delta - 2\theta) \\ &= (k_0\lambda + \mu^2)(1 + \theta\lambda\Delta) + \lambda^3\theta \left(\Delta - \frac{k_0\theta - 1}{2\lambda\theta} \right)^2 < 0 \end{aligned}$$

if $1 + \theta\lambda\Delta > 0$; from here, (22) holds for any $0 < \Delta < -1/\lambda\theta$.

(b) If $k_0 = (\sqrt{8\theta} - 1)/\theta \leq -\mu^2/\lambda$, then $0 < 8\theta\lambda^2 \leq (\lambda - \mu^2\theta)^2$ and, using the last expression in (19), $D_1 \geq 0$, and, from Remark 4, $0 < \Delta_1^- < \Delta_1^+$. Since $\Delta_1^- < -1/2\lambda\theta < -1/\lambda\theta$, we conclude that if $0 < \Delta < \Delta_1^-$ then $1 + \lambda\theta\Delta > 0$ and (20) holds.

Finally, suppose that $1/2 \leq \theta < 1$ and $\Delta < -1/\theta\lambda$. Since $1 + \theta\lambda\Delta > 0$, the stability condition (22) obviously holds under the assumption (9) because $1 - 2\theta \leq 0$. \square

For the geometrical representation of $(\theta, 0)$ -methods, we have

$$S_{(\theta,0)} = \{(x, y) \in \mathbb{R}^2 : y \geq 0; (\theta y + 1)(\theta y x + x + \theta y^3 + y^2 + 2y) < 0\}.$$

In Fig. 3, varying $\theta \in [0, 1]$ all MS-stability regions with $\beta = 0$ had been represented simultaneously in a 3-dimensional space with coordinates $x = \lambda\Delta$, $y = \mu^2\Delta$ and θ . This picture allows the direct comparison among all $(\theta, 0)$ -methods, $\theta \in [0, 1]$.

For $\theta_0 \in [0, 1]$, the section of the shaded solid with the plane $\theta = \theta_0$ corresponds to the MS-stability region $S_{(\theta_0,0)}$; the regions $S_{(0,0)}$, $S_{(1/9,0)}$, $S_{(1/8,0)}$, $S_{(1/6,0)}$, $S_{(1/4,0)}$, $S_{(1/2,0)}$ and $S_{(1,0)}$ are represented in the first column of Fig. 11. Theorem 5 ensures that for each $(\theta, 0)$ there exists a value $\Delta_{(\theta,\beta)}$ such that the segment $\{(x, y) = (\lambda\Delta, \mu^2\Delta) : 0 < \Delta < \Delta_{(\theta,\beta)}\}$ is entirely contained in the shaded region.

As it was pointed out in [18], we can observe in several of these pictures that for some values of (λ, μ) one can lose numerical stability by reducing the step-size Δ . For example, $S_{(1/9,0)}$ is made up of two disconnected regions;

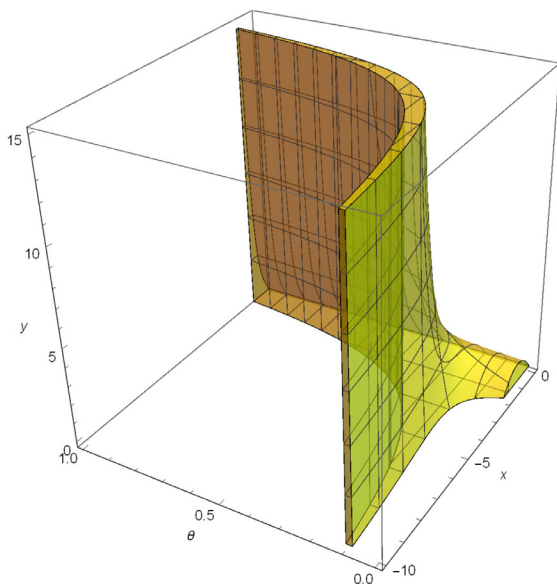


Fig. 3. MS-stability regions of $(\theta, 0)$ -predictor corrector schemes, $\theta \in [0, 1]$.

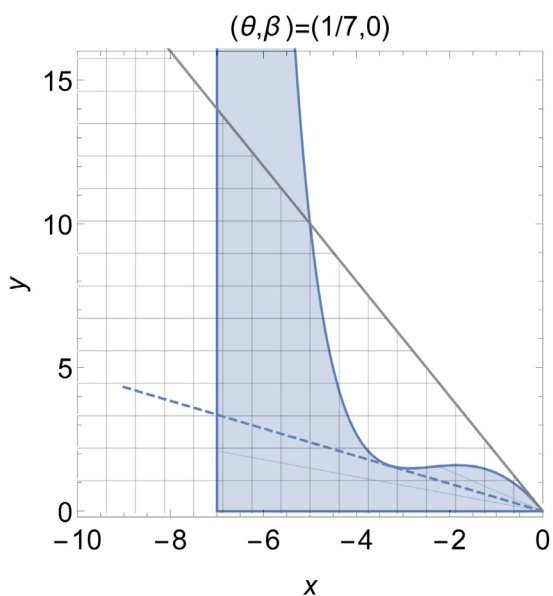


Fig. 4. MS-stability region of $(1/7, 0)$ -predictor corrector scheme.

any ray $y = (\lambda/\mu^2)x$ with $-2 < \lambda/\mu^2 < 0$ directed to the origin, i.e., with decreasing values of Δ , intersects a part of the region for a while, then leave it, and finally enter and remains contained in the second part of the region. This fact is common to $(\theta, 0)$ methods with $0 < \theta \leq 1/8$; and the bound Δ_1 of Theorem 5 determines the value of Δ for which the ray enters in the second connected part of $S_{(\theta,0)}$.

Consider now the picture of $S_{(1/7,0)}$, shown in Fig. 4, as an illustration of $(\theta, 0)$ -regions with $1/8 < \theta < 1/2$; Theorem 5 states that there is a value, in this case equal to $\frac{\sqrt{8\theta}-1}{\theta} = \sqrt{56}-7$, represented by the dashed line in the plot of Fig. 4, such that the extreme of the entirely contained ray belongs to the vertical line $x = -1/\theta$ (when the slope satisfies $0 > \mu^2/\lambda > 7 - \sqrt{56}$) or to the curve $y = -x^2 - 2x/(1+\theta x)$ (when $7 - \sqrt{56} \leq \mu^2/\lambda < -2$). Notice

that in this last case we find again the mentioned counter-intuitive fact of losing numerical stability by reducing the step-size.

2.3. Methods with $\theta > 0$ and $\beta > 0$

For the sake of clarity we separate the rest of the analysis into two cases: firstly we consider $0 < \beta < \theta \leq 1$; later $0 < \theta \leq \beta \leq 1$.

Theorem 6. *If $0 < \beta < \theta \leq 1$ the (θ, β) predictor–corrector method (4)–(6) applied with step-size $\Delta > 0$ to a stable problem is stable if $\Delta < \Delta_{(\theta, \beta)}$ where*

$$\Delta_{(\theta, \beta)} = \begin{cases} \min\{\Delta_1, \Delta_2, \Delta_3\} & \text{for } 0 < \theta \leq \frac{1}{8} \\ \min\{\Delta_0, \Delta_2, \Delta_3\} & \text{for } \frac{1}{8} < \theta < \frac{1}{2} \text{ and } \frac{\mu^2}{-\lambda} < \frac{\sqrt{8\theta}-1}{\theta} \\ \min\{\Delta_1, \Delta_2, \Delta_3\} & \text{for } \frac{1}{8} < \theta < \frac{1}{2} \text{ and } \frac{\mu^2}{-\lambda} \geq \frac{\sqrt{8\theta}-1}{\theta} \\ \min\{\Delta_0, \Delta_2, \Delta_3\} & \text{for } \frac{1}{2} \leq \theta < 1 \\ \min\{\Delta_0, \Delta_4\} & \text{for } \theta = 1 \end{cases}$$

with Δ_0, Δ_1 given in (23) and

$$\Delta_2 = \frac{\theta + \beta + \theta\beta - \sqrt{(\theta + \beta + \theta\beta)^2 - 2\beta(\beta - \theta)}}{\lambda\theta\beta}, \quad \Delta_3 = \frac{\beta - \sqrt{\beta^2 + 16(1 - \theta)\theta^2}}{4\theta^2\lambda},$$

$$\Delta_4 = \frac{-\beta\lambda^2 + 2\lambda\mu^2(1 + 2\beta) + \sqrt{(\beta\lambda^2 - 2\lambda\mu^2(1 + 2\beta))^2 - 8\mu^2(\beta - 1)\lambda^2(\mu^2\beta - 2\lambda)}}{2\lambda^2(\mu^2\beta - 2\lambda)}.$$

Proof. Suppose that $2\lambda + \mu^2 < 0$. For the sake of simplicity we write the stability condition (15) as

$$A + B + C + D < 0 \tag{24}$$

with

$$\begin{aligned} A &= (2\lambda + \mu^2)\theta^2\beta^2\mu^4\Delta^3 \\ B &= (1 + \theta\lambda\Delta)\Delta(\theta\lambda^3\Delta^2 + \lambda(\theta\mu^2 + \lambda)\Delta + 2\lambda + \mu^2) \\ C &= \mu^4\Delta^2\beta(\beta\theta^2\lambda^2\Delta^2 - 2\theta\lambda(\theta + \beta + \theta\beta)\Delta + 2(\beta - \theta)) \\ D &= -\beta\lambda\mu^2\Delta^2(2\theta^2\lambda^2\Delta^2 - \beta\lambda\Delta - 2(1 - \theta)) \end{aligned} \tag{25}$$

Notice that $A < 0$ and Theorem 5 gives conditions ensuring $B < 0$. Since $\beta < \theta$ we can write

$$C = \mu^4\Delta^2\beta(\Delta - \Delta_2^-)(\Delta - \Delta_2^+)$$

with

$$\Delta_2^\pm = \frac{\theta + \beta + \theta\beta \pm \sqrt{(\theta + \beta + \theta\beta)^2 - 2\beta(\beta - \theta)}}{\lambda\theta\beta};$$

since $\Delta_2^+ < 0 < \Delta_2^-$, we conclude that $C < 0$ if $0 < \Delta < \Delta_2$. On the other hand, we can write

$$D = -\beta\lambda\mu^2\Delta^2(\Delta - \Delta_3^-)(\Delta - \Delta_3^+)$$

with

$$\Delta_3^\pm = \frac{\beta \pm \sqrt{\beta^2 + 16(1 - \theta)\theta^2}}{4\theta^2\lambda}.$$

If $\theta < 1$ then $\Delta_3^+ < 0 < \Delta_3^-$; we conclude that $D < 0$ if $0 < \Delta < \Delta_3 = \Delta_3^-$.

Finally, to study the problem $0 < \beta < \theta = 1$, we shall prove that in this case $C + D < 0$ if $0 < \Delta < \Delta_4$. Since $\theta = 1$,

$$\begin{aligned} C + D &= \beta\mu^2\Delta^2((\mu^2\beta\lambda^2 - 2\lambda^3)\Delta^2 + (\beta\lambda^2 - 2\lambda\mu^2(1 + 2\beta))\Delta + 2(\beta - 1)\mu^2) \\ &= \beta\mu^2\Delta^2\lambda^2(\mu^2\beta - 2\lambda)(\Delta - \Delta_4^-)(\Delta - \Delta_4^+) \end{aligned}$$

with

$$\Delta_4^\pm = \frac{-\beta\lambda^2 + 2\lambda\mu^2(1 + 2\beta) \pm \sqrt{(\beta\lambda^2 - 2\lambda\mu^2(1 + 2\beta))^2 - 8\mu^2(\beta - 1)\lambda^2(\mu^2\beta - 2\lambda)}}{2\lambda^2(\mu^2\beta - 2\lambda)}.$$

Since $\Delta_4^- < 0 < \Delta_4^+$, we conclude that $C + D < 0$ if $0 < \Delta < \Delta_4^+ = \Delta_4$. \square

The following result completes the analysis:

Theorem 7. *If $0 < \theta \leq \beta \leq 1$ the (θ, β) predictor–corrector method (4)–(6) applied with step-size $\Delta > 0$ to a stable problem is stable if $\Delta < \min\{\Delta_5, \Delta_6\}$ with*

$$\Delta_5 = \frac{-(2\lambda + \mu^2)}{\lambda^2(1 + 2\theta) + 2\beta\mu^4(\beta - \theta)}$$

and

$$\Delta_6 = - \frac{\lambda\mu^2(\beta^2 + \theta^2) - 2\beta\theta\mu^4(\beta\theta + \beta + \theta) + 2\theta\lambda^2}{2\theta^2\lambda(\lambda - \beta\mu^2)^2} - \frac{\sqrt{(\lambda\mu^2(\beta^2 + \theta^2) - 2\beta\theta\mu^4(\beta\theta + \beta + \theta) + 2\theta\lambda^2)^2 - 8\lambda\theta^2\mu^2(\lambda - \beta\mu^2)^2(\beta + \theta(1 - \beta))}}{2\theta^2\lambda(\lambda - \beta\mu^2)^2}$$

Proof. Suppose that $2\lambda + \mu^2 < 0$. For the sake of simplicity now we write the stability condition (15) as

$$A + E + F < 0$$

where $A < 0$ is given in (25) and

$$E = \Delta(2\lambda + \mu^2 + \Delta(\lambda^2(1 + 2\theta) + 2\beta\mu^4(\beta - \theta))),$$

$$F = \lambda\Delta^2 \left(\theta^2\lambda(\lambda - \beta\mu^2)^2\Delta^2 + (\lambda\mu^2(\beta^2 + \theta^2) - 2\beta\theta\mu^4(\beta\theta + \beta + \theta) + 2\theta\lambda^2)\Delta + 2\mu^2(\beta + \theta(1 - \beta)) \right).$$

Then it is clear that $E < 0$ if $\Delta < \Delta_5$. On the other hand, F can be written

$$F = \lambda^2\Delta^2\theta^2(\lambda - \beta\mu^2)^2(\Delta - \Delta_6^+)(\Delta - \Delta_6^-)$$

where $\Delta_6^+ < 0 < \Delta_6^-$,

$$\Delta_6^\pm = \frac{-b \pm \sqrt{b^2 - 8\lambda\theta^2\mu^2(\lambda - \beta\mu^2)^2(\beta + \theta(1 - \beta))}}{2\theta^2\lambda(\lambda - \beta\mu^2)^2},$$

with $b = \lambda\mu^2(\beta^2 + \theta^2) - 2\beta\theta\mu^4(\beta\theta + \beta + \theta) + 2\theta\lambda^2$, which proves that $F < 0$ if $0 < \Delta < \Delta_6$. \square

Similar to preceding geometrical interpretation, when we fix a value of $\theta > 0$ a solid whose sections with $\beta = \beta_0$ are the MS stability regions for (θ, β_0) -method is obtained, see Fig. 5 for $\theta = 1/9, 1/8, 1/6, 1/2, 1$. The top-center picture is a joint representation of $(1/9, \beta)$ predictor–corrector methods with $\beta \in [0, 1]$; see also the second row of Fig. 11 for $\theta = 1/9$ and some particular values of β . In this case, for every β_0 the stability region $S_{(1/9, \beta_0)}$ is made of two separated components; notice that the component containing the origin is greater than $S_{(0, \beta_0)}$. For values of β near to $1/5$ the stability region increases. When $\theta = 1/8$, see top-right picture in Fig. 5 and the third row of Fig. 11, the MS-stability regions are connected for every $\beta \in [0, 1]$. Larger regions are obtained with β around $1/7$. For $\theta > 1/8$ the MS-stability regions are connected for every $\beta \in [0, 1]$.

Notice that for $1/2 \leq \theta < 1$ the regions decrease when θ increases. If in addition $\beta < \theta$ then the threshold $\Delta_{(\theta, \beta)}$ in Theorem 6 does not depend on μ . This fact geometrically means, see Fig. 11, that these stability regions contain a portion of the line $y = -2x$, or that the line is an inner tangent to the border, which numerically results in a better stability behavior. The intersection of the last two rows with the first three columns in Fig. 11 corresponds to PC methods with this property. Notice finally that for β fixed, Δ_0, Δ_2 and Δ_3 are decreasing functions of θ in

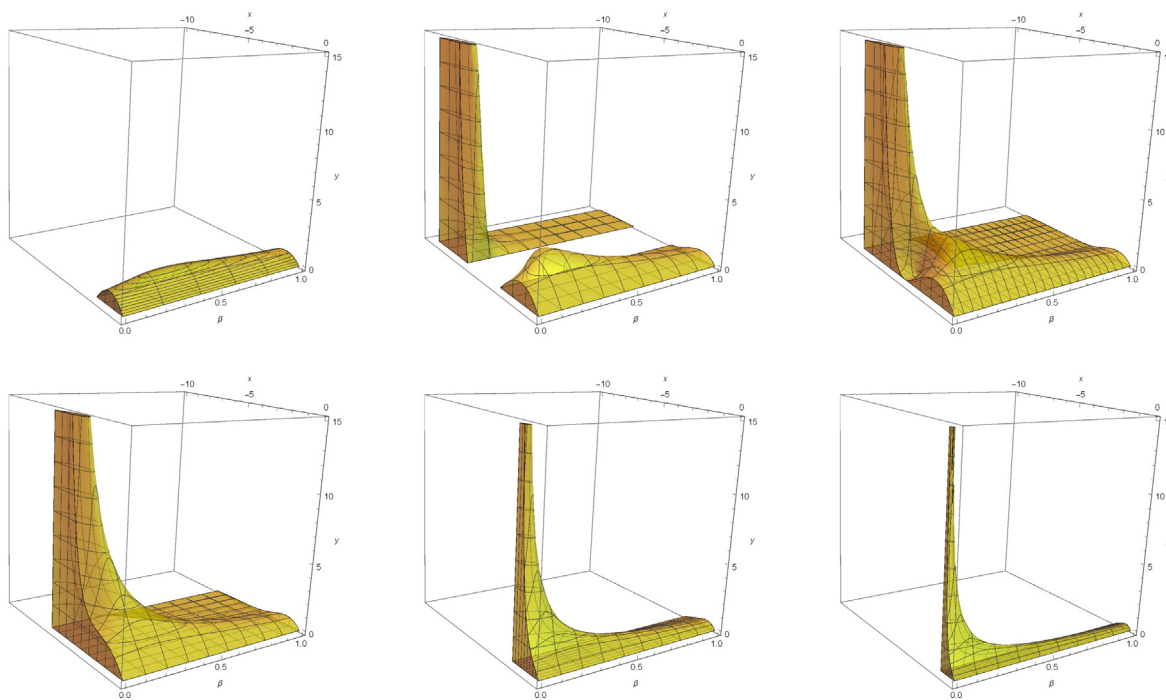


Fig. 5. Comparison between MS-stability region of (θ, β) -predictor corrector schemes for $\theta = 0$ (top left), $\theta = \frac{1}{9}$ (top center), $\theta = \frac{1}{8}$ (top right), $\theta = \frac{1}{6}$ (bottom left), $\theta = \frac{1}{2}$ (bottom center) and $\theta = 1$ (bottom right).

the interval $[1/2, 1]$; then for $\beta < \theta$ and $1/2 \leq \theta < 1$ the threshold $\Delta_{(\theta, \beta)}$ is a decreasing function of $\theta \in [1/2, 1]$. This means that PC method with $(1/2, \beta_0)$ has the greatest stability region within PC methods with (θ, β_0) when $\beta_0 < \theta$ and $1/2 \leq \theta < 1$. We conclude that methods with $\theta = 1/2$ and $\beta < \theta$ have better stability behavior with greater stability regions. **Example 1**, where PC method with $(\theta, \beta) = (1/2, 0)$ was used, illustrates this conclusion.

A similar analysis can be made fixing $\beta > 0$; see **Fig. 6** for $\beta = 1/20, 1/10, 1/4, 1$.

3. Numerical experiments

Two types of numerical experiments were carried out. Experiments (i) and (ii) are devoted to confirm the above stability analysis; experiment (iii) is intended to compare the accuracy and efficiency of PC methods with standard methods in the literature.

In the first two experiments Eq. (2) with initial condition $X_0 = 1$ is solved with different PC methods. For each method we have used 10 000 simulations to approximate $E[|X_t|^2]$. For each experiment, these values have been plotted against t in logarithm scale with basis 10; additionally, to compare the accuracy of each numerical approximation, we have calculated the mean square error between the exact and the numerical solutions, see e.g. [6]. The computations were made with 16 significant digits.

(i) For the first experiment, the parameters $\lambda = -100, \mu = 10$ in (2) were taken and the equation was solved using firstly predictor–corrector (θ, β) -schemes with $(\theta, \beta) = (0, 0), (0, \frac{1}{4}), (0, \frac{1}{2}), (0, \frac{3}{5}), (0, 1)$; then with $(\theta, \beta) = (\frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2}), (\frac{3}{5}, \frac{3}{5}), (\frac{2}{3}, \frac{2}{3})$ and $(1, 1)$; in both cases a constant step size $\Delta = 1/100$ has been used. The results are shown in **Figs. 7** and **8** respectively. They confirm the theoretical results: for the first batch all methods except $(\theta, \beta) = (0, 0)$ and $(\theta, \beta) = (0, 1)$ show stability, which agrees with the values $R_{(0,0)}(-100, 10, \frac{1}{100}) = 1, R_{(0,1/4)}(-100, 10, \frac{1}{100}) = 11/16 < 1, R_{(0,1/2)}(-100, 10, \frac{1}{100}) = 3/4 < 1, R_{(0,3/5)}(-100, 10, \frac{1}{100}) = 22/25 < 1$ and $R_{(0,1)}(-100, 10, \frac{1}{100}) = 2$ in (14). For the second batch all methods except $(\theta, \beta) = (1/4, 1/4)$ show instability, which agrees with the values $R_{(1/4,1/4)}(-100, 10, \frac{1}{100}) = 53/128, R_{(1/2,1/2)}(-100, 10, \frac{1}{100}) = 9/8 > 1, R_{(3/5,3/5)}(-100, 10, \frac{1}{100}) = 1222/625 > 1, R_{(2/3,2/3)}(-100, 10, \frac{1}{100}) = 221/81 > 1$ and $R_{(1,1)}(-100, 10, \frac{1}{100}) = 10 > 1$ in (14).

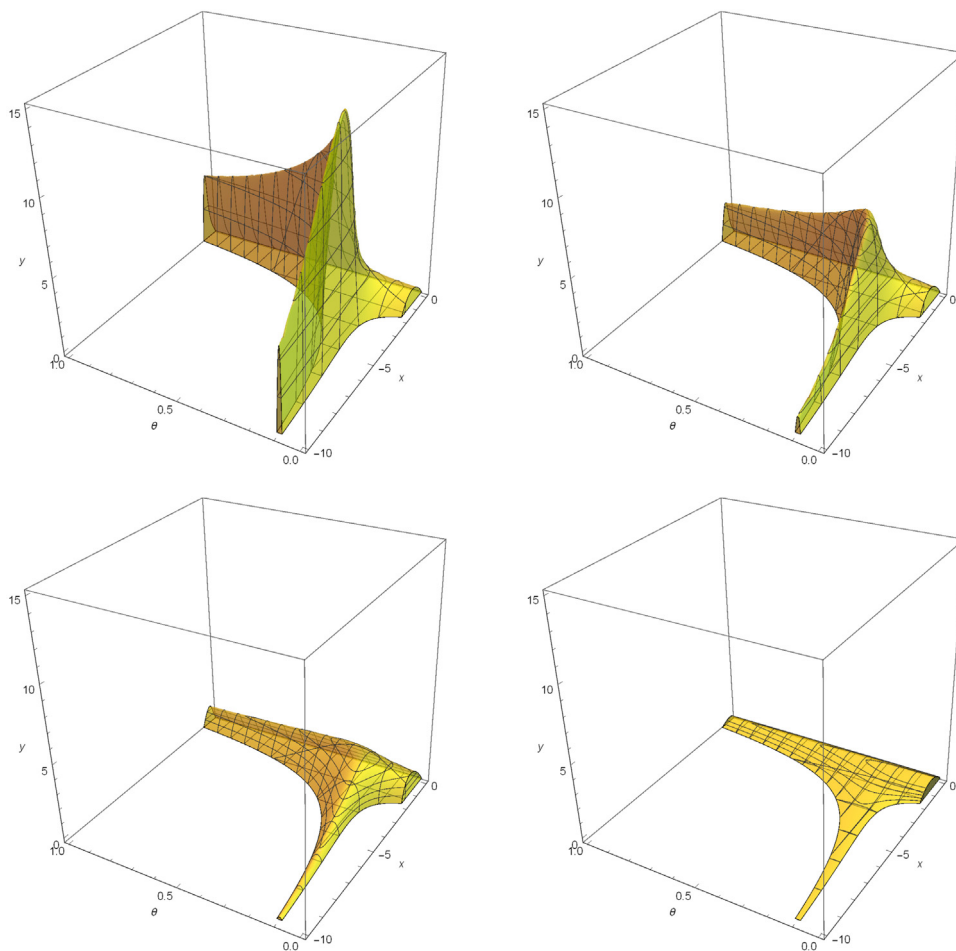


Fig. 6. Comparison between MS-stability region of (θ, β) -predictor corrector schemes for $\beta = \frac{1}{20}$ (top left), $\beta = \frac{1}{10}$ (top right), $\beta = \frac{1}{4}$ (bottom left) and $\beta = 1$ (bottom right).

(ii) The second experiment is devoted to illustrate the observed fact that for some values of (λ, μ) one can lose numerical stability by reducing the step-size Δ . We have integrated the equation

$$dX_t = -60X_t dt + 8X_t dW_t \tag{26}$$

in the interval $[0, 10]$ with the predictor–corrector $(\frac{1}{7}, 0)$ method and step sizes $\Delta = 0.2, 0.1, 0.05, 0.02, 0.01$. The results can be seen in Fig. 9. Notice that $R_{(1/7,0)}(-60, 8, 1/5) = 687/7 > 1$, $R_{(1/7,0)}(-60, 8, 1/10) = 37/245 < 1$, $R_{(1/7,0)}(-60, 8, 1/20) = 381/245 > 1$, $R_{(1/7,0)}(-60, 8, 1/50) = 26913/30625 < 1$ and $R_{(1/7,0)}(-60, 8, 1/100) = 181/245 < 1$. Then, $\Delta = 0.2$ gives instability and the first time we halve the step size ($\Delta = 0.1$) the scheme becomes stable; but when we halve it again ($\Delta = 0.05$), the method becomes unstable; finally for $\Delta = 0.02$ and $\Delta = 0.01$ the scheme is again stable (recall the geometrical interpretation of the ray directed to the origin with decreasing values of Δ that intersects the stability region for a while, then leaves it, and finally enters and remains contained in the region).

(iii) In this experiment we compare some PC methods with well-known methods in the literature, including implicit and higher order ones. The results show mean error values and time of computation in order to compare accuracy and computational effort. For the comparison, the nonlinear stochastic differential equation

$$dX_t = \left(\frac{1}{3}X_t^{1/3} + 6X_t^{2/3} \right) dt + X_t^{2/3} dW_t \tag{27}$$

$$X_0 = 1$$

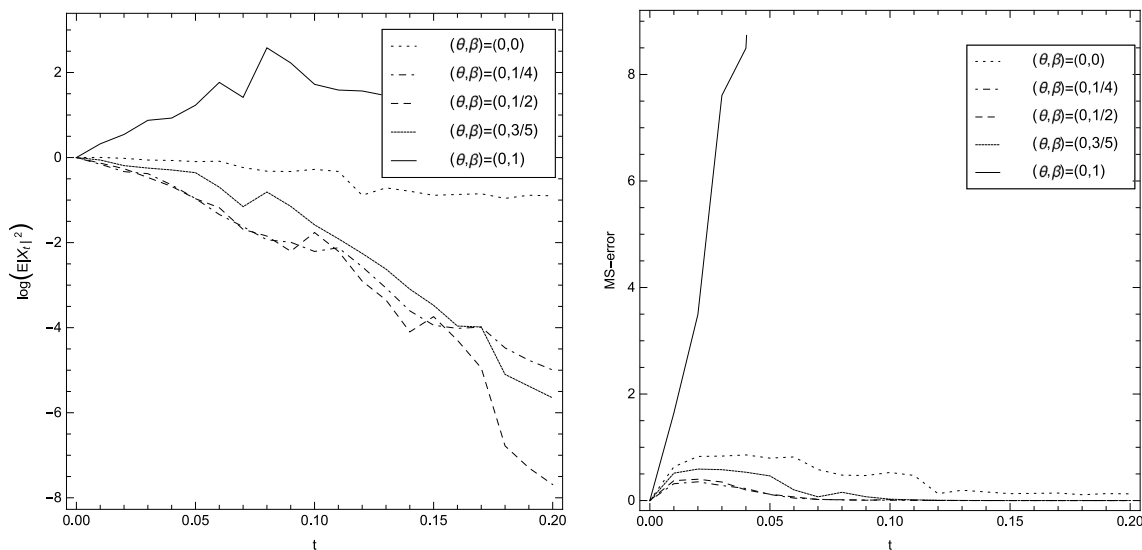


Fig. 7. Values of $\log E|X_t|^2$ (left) and MS-errors (right) against t using different (θ, β) -schemes with $\Delta = 1/100$ to solve $dX_t = -100X_t dt + 10X_t dW_t$.

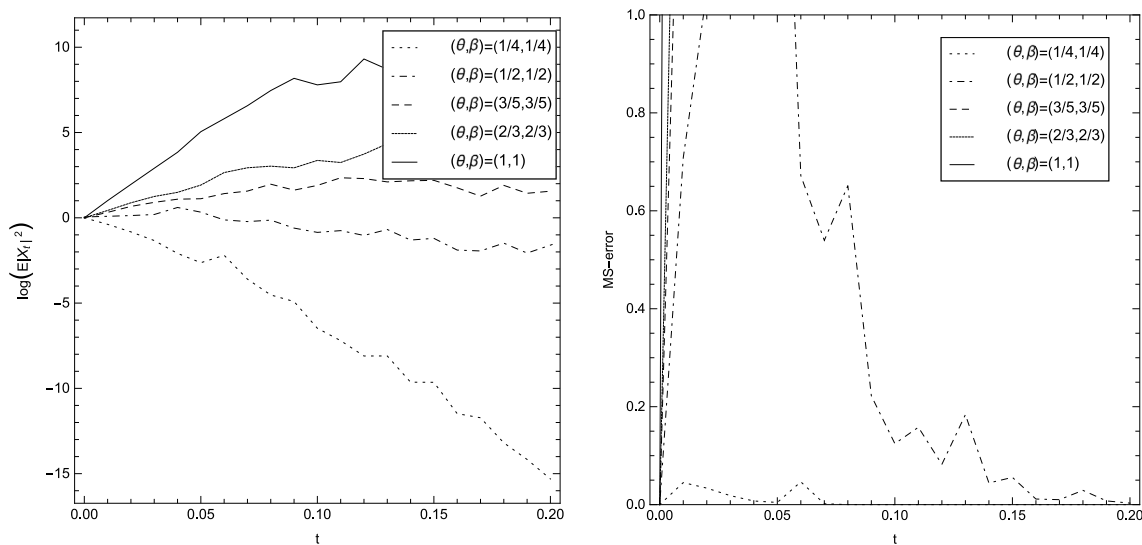


Fig. 8. Values of $\log E|X_t|^2$ (left) and MS-errors (right) against t using different (θ, β) -schemes with $\Delta = 1/100$ to solve $dX_t = -100X_t dt + 10X_t dW_t$.

has been used. Since its exact solution $X_t = (2t + 1 + \frac{W_t}{3})^3$ is known, we can compare the exact values $E[X_1] = 28$ and $E[X_1^2] = 869 + \frac{5}{33}$ with the approximations given by the different schemes. The PC-methods used were Euler (EU) method ($\theta = \beta = 0$), the modified trapezoidal (MT) method ($\theta = 1/2, \beta = 0$) and, due to its relevant stability region, the predictor–corrector method with parameters $\theta = \beta = 1/8$, denoted as PC18. Recall that these methods, as members of the PC family have strong order 0.5 and weak order 1.0. For the comparison, from the so called family of stochastic theta methods (STM), see [13], we have selected, due to its special stability behavior, the semi-implicit Euler method:

$$X_{n+1} = X_n + \frac{1}{2} (a(X_n) + a(X_{n+1})) \Delta + b(X_n) \Delta W_n,$$

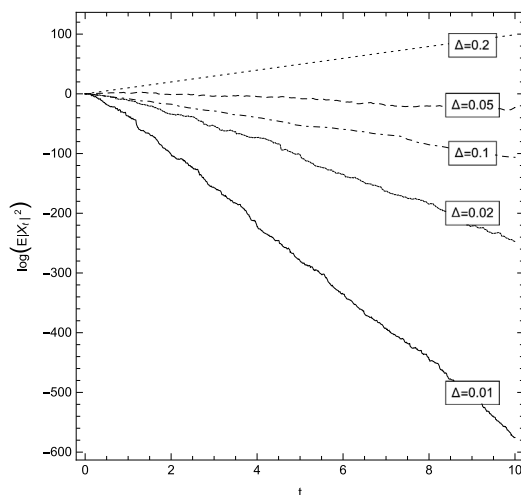


Fig. 9. Values of $\log E|X_t|^2$ against t using PC $(\frac{1}{7}, 0)$ -scheme with different values of Δ to solve (26).

Table 1

Mean errors in the computation of $E[X_1]$ for test equation (27) by Euler (EU), Milstein (MIL), Runge–Kutta (RK), modified trapezoidal (MT), PC with $\theta = \beta = 1/8$ (PC18) and semi-implicit stochastic theta (STM) methods with step sizes $\Delta = 2^{-1}, \dots, 2^{-8}$.

Δ	EU	MIL	RK	MT	PC18	STM
2^{-1}	15.834163	15.838953	15.890211	5.779845	5.859981	3.450810
2^{-2}	10.421102	10.41590	10.454726	1.959769	2.201637	1.146166
2^{-3}	6.145018	6.13697	6.298351	0.557268	0.674469	0.361481
2^{-4}	3.386426	3.37917	3.296597	0.161179	0.200451	0.091673
2^{-5}	1.840789	1.83741	1.318922	0.131892	0.161089	0.041064
2^{-6}	0.914740	0.91158	0.865913	0.042104	0.079803	0.024608
2^{-7}	0.410595	0.40959	0.299432	0.019853	0.028655	0.011547
2^{-8}	0.119174	0.11920	0.092857	0.014884	0.016926	0.009501

i.e., the STM with $\theta = 1/2$. It shares strong and weak orders with the components of PC family. In this experiment, the implicit equation that arises at each step was solved with three iterations of the Newton–Raphson method. In addition two higher (strong) order schemes were employed in the comparison: the explicit strong order 1.0 Milstein scheme (MIL), see [17], which, as PC-methods with $\beta \neq 0$, contains the derivative of the drift coefficient b ; and the explicit strong order 1.0 Runge–Kutta (RK) scheme, proposed by Platen, that appears in [15], pages 374–375.

For each step size $\Delta = 2^{-1}, \dots, 2^{-8}$, $N = 10000$ trajectories were simulated using the same Brownian paths with each scheme. Then the mean values at the endpoint were used to approximate the known expectations $E[X_1]$ and $E[X_1^2]$. The mean errors are shown in Table 1 for $E[X_1]$ and in Table 2 for $E[X_1^2]$. It can be seen that explicit methods (EU, MIL, RK) present similar results with greater mean errors. As it was expected, the implicit method STM shows the best results in both experiments. Predictor–corrector methods MT and PC18 form an intermediate group showing better results than MIL and RK, which are methods with greater (strong) order.

To compare the efficiency, the experiments were repeated for each method separately and the required time employed in the computations was calculated for step sizes $\Delta = 2^{-1}, \dots, 2^{-9}$. The results for both calculations $E[X_1]$ and $E[X_1^2]$ were alike and are shown graphically in Fig. 10, where the time (in seconds) is represented against the logarithm with basis 2 of the step size. The picture shows that explicit and PC methods form a group with a very similar time consumption whereas the implicit method STM doubled the computational cost of the rest of schemes. The results for explicit and PC methods are, in fact, indistinguishable in the graph; for example, with $\Delta = 2^{-9}$ the difference between the fastest (Euler, 252 s) and the slowest (Runge–Kutta, 255 s) methods was about 1.1%.

Table 2

Mean error in the computation of $E[X_1^2]$ for test Eq. (27) by Euler (EU), Milstein (MIL), Runge–Kutta (RK), modified trapezoidal (MT), PC with $\theta = \beta = 1/8$ (PC18) and semi-implicit stochastic theta (STM) methods with step sizes $\Delta = 2^{-1}, \dots, 2^{-8}$.

	EU	MIL	RK	MT	PC18	STM
$\Delta = 2^{-1}$	715.412	715.524	711.735	339.451	358.571	76.905
$\Delta = 2^{-2}$	544.486	544.711	542.843	128.689	159.276	36.764
$\Delta = 2^{-3}$	353.533	353.374	352.126	56.468	58.257	16.837
$\Delta = 2^{-4}$	205.119	204.571	203.436	23.015	26.971	6.005
$\Delta = 2^{-5}$	102.647	102.322	101.857	17.423	22.089	4.651
$\Delta = 2^{-6}$	79.076	79.109	79.107	8.244	12.097	2.775
$\Delta = 2^{-7}$	38.336	37.723	37.464	5.752	6.084	1.840
$\Delta = 2^{-8}$	18.537	17.307	17.223	2.271	3.630	1.648

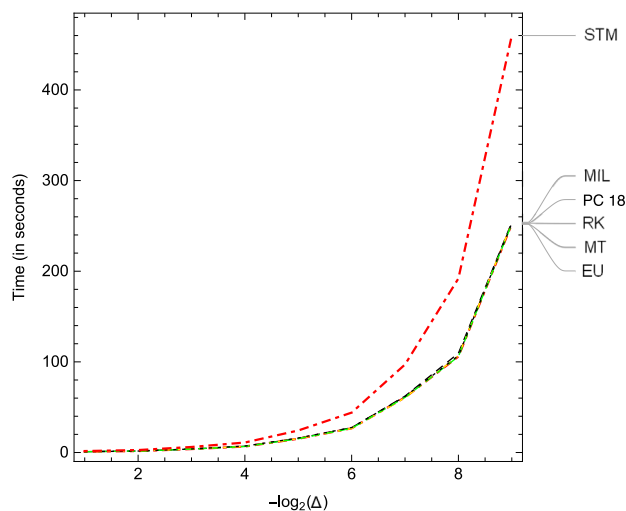


Fig. 10. Time (in seconds) consumed for the calculation of $E[X_1]$ or $E[X_1^2]$ with step sizes $\Delta = 2^{-1}, \dots, 2^{-9}$ for test equation (27) by Euler (EU), Milstein (MIL), Runge–Kutta (RK), modified trapezoidal (MT), PC with $\theta = 1/2, \beta = 1/8$ (PC18) and semi-implicit stochastic theta (STM) methods.

4. Conclusions

A complete analytical study for the linear mean-square stability of the two-parameter family of Euler predictor–corrector schemes for scalar stochastic differential equations has been carried out. The analyzed family is given in terms of the parameters θ, β that control the degree of implicitness of the drift and diffusion terms respectively of the method. Since $S_{SDE} \not\subseteq S_{(\theta, \beta)}(\Delta)$ for all $(\theta, \beta) \in [0, 1] \times [0, 1]$ and any $\Delta > 0$, PC methods are not A-stable; and for each (θ, β) a constant $\Delta_{(\theta, \beta)}$ such that the (θ, β) -PC method applied with step-size $\Delta > 0$ to a stable problem is stable if $\Delta < \Delta_{(\theta, \beta)}$ has been found. In addition we have calculated and represented for each (θ, β) -method its MS-stability region. We have confirmed the unexpected fact of losing numerical stability by reducing the step-size Δ . In some cases this fact is due to the composition in disconnected parts of the stability region, e.g. if $(\theta, \beta) = (1/9, 0)$; in others, to the non convexity of the stability region. We have obtained that the greatest stability regions correspond to (θ, β) -methods with θ around $1/7$ and β between $1/8$ and $1/9$. Increasing the values of the parameters from these values to 1, it can be observed that the stability region declines, corresponding to $\theta = \beta = 1$ the smallest stability region. Finally, we can affirm that the best stability behavior corresponds to (θ, β) -methods with $\theta = 1/2$ and $\beta < \theta$.

Different examples proposed along the text show that PC methods overcome explicit ones with the same (in some cases greater) strong order in accuracy and stability, but with similar computational cost. On the other hand, the better accuracy of implicit stochastic theta methods requires double computational effort than for PC methods.

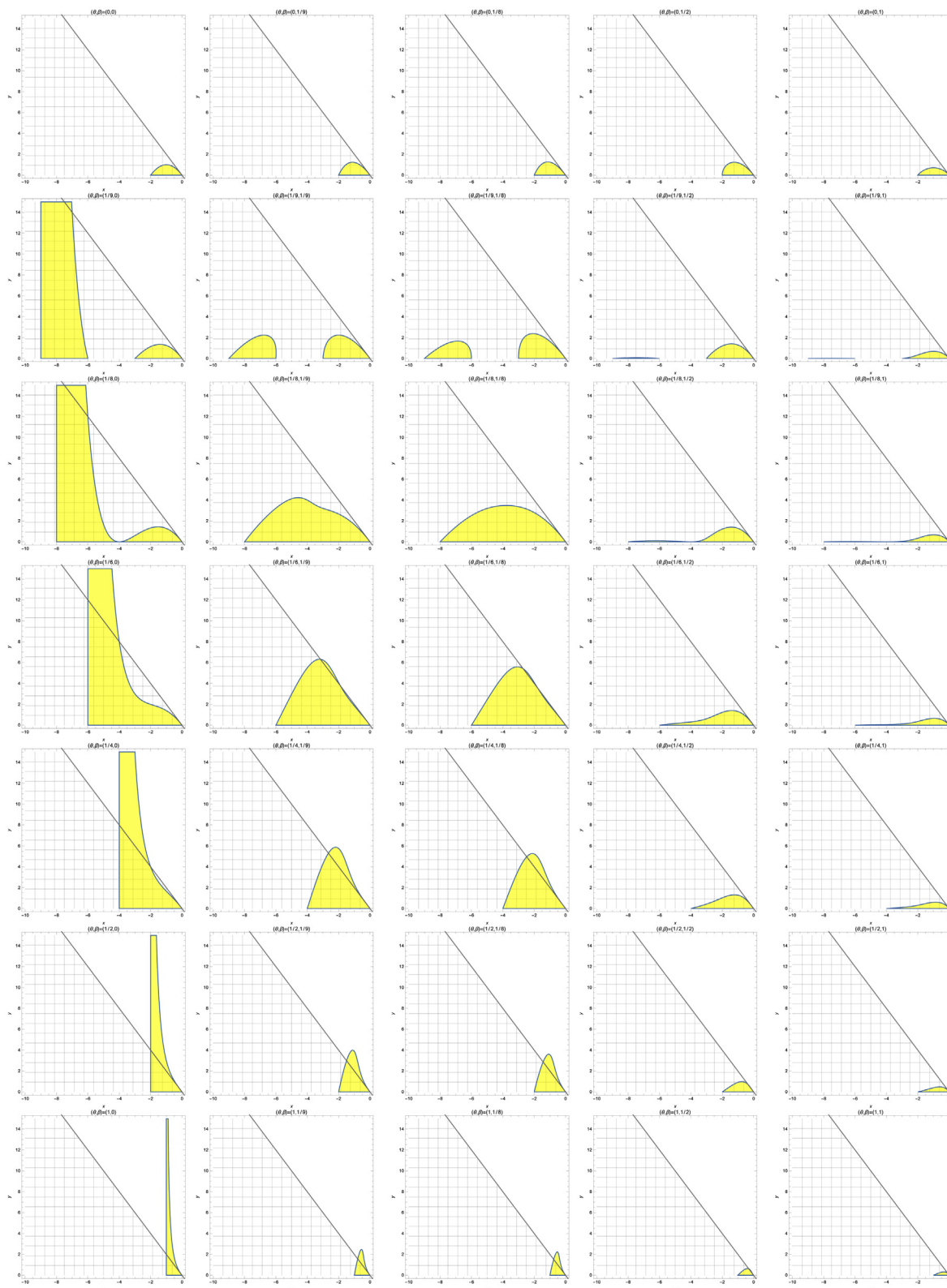


Fig. 11. Comparison between MS-stability region (shaded area) of (θ, β) -predictor corrector schemes for $\theta = 0, \frac{1}{9}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1, \beta = 0, \frac{1}{9}, \frac{1}{8}, \frac{1}{6}, 1$, and the stability region of test equation (squared area).

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