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Étale Covers and Fundamental Groups of Schematic Finite Spaces

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Abstract. We introduce the category of finite étale covers of an arbitrary schematic space X and show that, equipped with an appropriate natural fiber functor, it is a Galois Category. This allows us to define the étale fundamental group of schematic spaces. If X is a finite model of a scheme S, we show that the resulting Galois theory on X coincides with the classical theory of finite étale covers on S, and therefore, we recover the classical étale fundamental group introduced by Grothendieck. To prove these results, it is crucial to find a suitable geometric notion of connectedness for schematic spaces and also to study their geometric points. We achieve these goals by means of the strong cohomological constraints enjoyed by schematic spaces.

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1. Introduction

Schematic (finite) spaces are finite ringed spaces (*not* locally ringed in most cases of interest) admitting a «good »theory of quasi-coherent sheaves with minimal natural conditions. They were first constructed in [7]. These spaces can be used to study the category of quasi-compact and quasi-separated (qcqs) schemes via the construction of «finite models»: that is, a projection $\pi\colon S \to X$ from a scheme S to a schematic space X inducing an equivalence of quasi-coherent sheaves. The scheme S is reconstructed from X via a colimit that we denote $\mathrm{Spec}(X)$ and, actually, qcqs schemes are embedded into a suitable localization—by *qc-isomorphisms* (defined before Proposition 2.4)—of the category of schematic spaces **SchFin**. However, **SchFin** contains

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objects X, such that $\mathrm{Spec}(X)$ is not a scheme (in particular, X does not arise as a finite model of any scheme), but a locally ringed space obtained by gluing affine schemes along flat monomorphisms. These generalized spaces, which we shall study via their finite incarnations, are potentially useful in the study of singularities and Prüfer spaces (see [3,11]). Besides that, while sheaf theory on finite spaces admits a very simple description, schematic spaces still manage to retain a good deal of the «geometry»that other related constructions (such as simplicial schemes) do not. This compromise makes them useful to give an intuitive approach to Grothendieck's Algebraic Geometry (see [9]).

Motivated by this setting and the fact that strictly topological finite models play the role of the Čech nerve of a covering (inducing weak homotopy equivalences, see [6]), we construct the analogue of Grothendieck's étale fundamental group for schematic spaces via Galois Categories. There are two preliminary steps: defining a notion of connectedness for a schematic space X that is «geometric», not only in the sense of reflecting connectedness of the associated locally ringed space $\mathrm{Spec}(X)$, but also in admitting constructions such as decomposition into «connected»components; and describing geometric points of schematic spaces (in this paper, analogue to the scheme-theoretic version, rather than from a topos-theoretic point of view), which are notably well-behaved thanks to the schematic condition. For the first part, we define a subcategory of *pw-connected spaces*, $\mathbf{SchFin}^{\mathrm{pw}}$, show that its objects are well-behaved (for them, combinatorial and «algebraic»connectedness coincide) and prove that $\mathbf{SchFin}^{\mathrm{pw}} \subseteq \mathbf{SchFin}$ admits a right adjoint $\mathbf{pw}$, such that $\mathbf{pw}(X) \to X$ is a qc-isomorphism for all X schematic. This is covered in Sects. 3 and 4.

In Sect. 5, we define the category of finite étale covers of a schematic space X, denoted $\mathbf{Qcoh}^{\mathrm{fet}}(X)$, as quasi-coherent sheaves of algebras that are finite étale at stalks. If $\pi\colon S \to X$ is a finite model, $\mathbf{Fet}_S \simeq \mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}$ (where $\mathbf{Fet}_S$ is the category of finite étale covers of S, which are affine morphisms, hence defined by quasi-coherent sheaves of $\mathcal{O}_S$-algebras). We also prove that it is the subcategory of locally constant objects of a *cosite* $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ and—in Sect. 6—that it is stable under qc-isomorphisms (hence defined in the localization of $\mathbf{SchFin}$). Finally, in Sect. 7, for any given geometric point $\overline{x}$, we construct a fiber functor $\mathrm{Fib}_{\overline{x}}$ (via a fibered product in $\mathbf{SchFin}^{\mathrm{pw}}$) and prove the main result:

Theorem 7.3 If X is schematic and $\mathcal{O}_X(X)$ has connected spectrum, for any geometric point $\overline{x}$ of X, the pair $(\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}, \mathrm{Fib}_{\overline{x}})$ is a Galois Category. Furthermore, if $\pi\colon S \to X$ is a finite model of a scheme and $\overline{s}$ the corresponding geometric point (4.8), there is an isomorphism of profinite groups $\pi_1^{\mathrm{et}}(S, \overline{s}) \simeq \pi_1^{\mathrm{et}}(X, \overline{x})$, where $\pi_1^{\mathrm{et}}(X, \overline{x}) {:=} \mathrm{Aut}_{[\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}, \mathbf{Set}_f]}(\mathrm{Fib}_{\overline{x}})$.

Due to space constraints, examples and further applications of the combinatorial structure of schematic spaces (in this context) will be provided in future work. For instance, we will prove a very general version of Seifert–Van Kampen's Theorem with rather elementary technology, which will recover the aforementioned theorem for the topology of flat monomorphisms of schemes; introduce the pro-étale fundamental group [1] in this context or tackle Étale

Homotopy (remember that topological finite models play the role of Čech nerves). *Unless stated otherwise, all ringed spaces are Noetherian. Given a ring homomorphism* $\varphi\colon R \to R'$ *to ease the notation, we just denote by* $\varphi^\sharp$ *the induced map* $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(R') \to \operatorname{Spec}(R)$ *between the corresponding spectra.*

2. Quick Overview of Schematic Finite Spaces

The content of this section is treated in [7] or is straightforward to prove. A finite ringed space is a ringed space $(X, \mathcal{O}_X)$ where X is a finite T_0 topological space or, equivalently, a finite poset; thus, X is an Alexandrov space: in terms of the partial order, the minimal open neighborhood of $x \in X$ is $U_x = \{y \in X : x \leq y\}$. We denote the closure of x by $C_x = \{y \leq x\}$. In particular, this implies that the stalks of any sheaf $\mathcal{F}$ are $\mathcal{F}_x = \mathcal{F}(U_x)$. If one sees X as a category (with arrows given by the order), a sheaf with values in a category $\mathfrak{C}$ is equivalent to a functor $\mathcal{F}\colon X \to \mathfrak{C}$. For $(X, \mathcal{O}_X)$ and each ordered pair $x \leq y$, we denote the restriction map by $r_{xy}\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,y}$.

Proposition 2.1. *A sheaf of* $\mathcal{O}_X$*-modules* $\mathcal{M}$ *is quasi-coherent iff for every* $x \leq y$*, the natural map* $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y} \xrightarrow{\sim} \mathcal{M}_y$ *is an isomorphism.*

Now, given a qcqs scheme S and a finite covering $\mathcal{U} = \{U_i\}$, such that the intersections $U^s = \bigcap_{s \in U_i} U_i$ are affine, we introduce the equivalence relation on S that establishes $s \sim s'$ iff $U^s = U^{s'}$. This relation defines the *finite model* of S relative to $\mathcal{U}$ as the quotient $\pi\colon S \to X := (S/\sim, \pi_*\mathcal{O}_S)$. There is an adjoint equivalence between the categories of quasi-coherent sheaves $(\pi^*, \pi_*)\colon \mathbf{Qcoh}(S) \xrightarrow{\sim} \mathbf{Qcoh}(X)$.

Remark 2.1. Every equivalence between abelian categories is automatically exact (and additive). In particular, all equivalences between categories of quasi-coherent sheaves described throughout the paper preserve cohomology.

Conversely, we construct a functor $\operatorname{Spec}\colon \mathbf{FRS} \to \mathbf{LRS}$ from finite ringed spaces to locally ringed spaces, such that, if X is a finite model of S, then $\operatorname{Spec}(X) \simeq S$. Explicitly, define the functor $\operatorname{Spec}(\mathcal{O}_X)\colon X \to \mathbf{LRS}^{\mathrm{op}}$, such that $x \mapsto \operatorname{Spec}(\mathcal{O}_{X,x})$ and define $\operatorname{Spec}(X)$ to be the limit of this functor. If the r_{xy} induce open immersions of schemes, we get a scheme by descent.

We define the subcategory of (finite) *schematic spaces* $\mathbf{SchFin} \subset \mathbf{FRS}$ as the category of spaces that admit a «good» theory of quasi-coherent sheaves. In the process, we also define *affine* finite spaces:

Definition 2.2. A *finite space* (or Fr-space) is a finite ringed space with flat restriction maps. It is said to be *affine* if $\pi\colon X \to (\star, \mathcal{O}_X)$ induces an equivalence $(\pi^*, \pi_*)\colon \mathbf{Qcoh}(X) \xrightarrow{\sim} \mathcal{O}_X(X)$-$\mathbf{mod}$.

If X is affine, exactness of π_* means that $H^i(X, \mathcal{M}) = 0$ for all $i > 0$ and $\mathcal{M} \in \mathbf{Qcoh}(X)$. Furthermore, $\mathcal{O}_X(X) \to \prod_{x \in X} \mathcal{O}_{X,x}$ is faithfully flat, which can be shown to characterize affine subspaces of an affine space (see [8]). Additionally, for $\mathcal{M} \in \mathbf{Qcoh}(X)$, we have $\mathcal{M}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{M}_x$ for all x. Combined with the previous faithful flatness condition, this also proves that, for all affine open subspaces $U \subseteq X$, $\mathcal{M}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{M}(U)$.

For any (finite) ringed space X, let $\mathrm{D}_{\mathrm{qc}}(X)$ denote the derived category of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology.

Definition 2.3. A morphism of finite spaces $f\colon X \to Y$ is *affine* if $f^{-1}(U_y)$ is affine for all $y \in Y$ and it preserves quasi-coherence: $\mathbb{R}f_*\mathcal{M} \in \mathrm{D}_{\mathrm{qc}}(Y)$ for $\mathcal{M} \in \mathbf{Qcoh}(X)$. It is *schematic* if its graph $\Gamma_f\colon X \to X \times Y$ preserves quasi-coherence for the structure sheaf: $\mathbb{R}\Gamma_{f*}\mathcal{O}_X \in \mathrm{D}_{\mathrm{qc}}(X \times Y)$, that is

$$H^i(U_x \cap f^{-1}(U_y), \mathcal{O}_X) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} \xrightarrow{\sim} H^i(U_{x'} \cap f^{-1}(U_y), \mathcal{O}_X)$$
$$H^i(U_x \cap f^{-1}(U_y), \mathcal{O}_X) \otimes_{\mathcal{O}_{X,y}} \mathcal{O}_{X,y'} \xrightarrow{\sim} H^i(U_x \cap f^{-1}(U_{y'}), \mathcal{O}_X)$$

for all $(x, y) \leq (x', y')$. By [7, Theorem 5.5], schematic morphisms preserve quasi-coherence. A finite space X is *schematic* if $\mathrm{Id}\colon X \to X$ is schematic.

Remark 2.2. It follows that schematic spaces have flat epimorphisms of rings as restrictions maps, which are local isomorphisms [4, Prop 2.4].

SchFin has finite fibered products and the forgetful **SchFin** $\to$ **FRS** preserves them. A schematic morphism $f\colon X \to Y$ is a *qc-isomorphism* if it is affine and $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$. The qc-isomorphisms define a multiplicative class of arrows of **SchFin** and we denote the corresponding localization by $\mathbf{SchFin}_{\mathrm{qc}}$. The Spec functor factors through $\mathbf{SchFin}_{\mathrm{qc}}$. Note that all finite models of the same scheme are qc-isomorphic, so their construction is functorial into this localization and qcqs schemes are embedded fully faithfully in $\mathbf{SchFin}_{\mathrm{qc}}$.

Proposition 2.4. *A schematic morphism $f : X \to Y$ is a qc-isomorphism iff it induces an equivalence $(f_*, f^*) : \mathbf{Qcoh}(X) \simeq \mathbf{Qcoh}(Y)$.*

Proof. The «only if»part is proven in [7, Theorem 5.26]. For the converse: from the equivalence of categories, we have $f_*f^*\mathcal{O}_Y \simeq \mathcal{O}_Y$, so $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. To prove that $f^{-1}(U_y)$ is affine for every $y \in Y$, we have to check that taking global sections induces an equivalence of categories. This follows from the fact that U_y is affine, that $\mathcal{O}_{Y,y} \simeq \mathcal{O}_X(f^{-1}(U_y))$ by the previous argument, and that the equivalence of the hypothesis restricts to $\mathbf{Qcoh}(f^{-1}(U_y)) \simeq \mathbf{Qcoh}(U_y)$ by the extension theorem for quasi-coherent sheaves [7, Theorem 4.4]. □

3. Connectedness

Topological—combinatorial—connectedness of a schematic space X does not reflect the geometry of the locally ringed space $\mathrm{Spec}(X)$ that it represents. We enhance it in this section.

Definition 3.1. Let X be a schematic space. We say that

- X is *top-connected* if its underlying poset is connected.
- X is *connected* if $\mathrm{Spec}(\mathcal{O}_X(X))$ is connected.
- X is *pw-connected* if $\mathrm{Spec}(\mathcal{O}_{X,x})$ is connected for every $x \in X$.

- X is *well-connected* if it is connected and pw-connected.

Recall that the prime spectrum of a non-zero ring is connected if and only if the ring has no non-trivial idempotent elements. A schematic space X is connected if and only if $\operatorname{Spec}(X)$ is connected in the usual sense, since X and $\operatorname{Spec}(X)$ have the same global sections. The empty set $\emptyset$ is not considered connected. In particular, a pw-connected space has non-zero stalks.

Example 3.2. Given two rings A, B with connected spectrum, the finite space $(\star, A \times B)$ is top-connected, but not pw-connected; while $(\star, A) \amalg (\star, B)$ is pw-connected, but not top-connected. However, $(\star, A) \amalg (\star, B) \to (\star, A \times B)$ is a qc-isomorphism. None of these spaces is connected or well-connected.

Proposition 3.3. *A schematic space X is well-connected iff it is top-connected and pw-connected.*

Proof. The «only if»part is clear. The converse holds, because a connected colimit of connected topological spaces $\{\operatorname{Spec}(\mathcal{O}_{X,x})\}_{x\in X}$ is connected. □

Let $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ denote the connected components functor. For any finite ringed space X, we construct a pw-connected space as follows:

- As a set, $\mathbf{pw}(X) = \coprod_{x\in X} \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x}))$. This set comes with a natural projection $\pi\colon \mathbf{pw}(X) \to X$.
- $\mathbf{pw}(X)$ is endowed with the partial order defined as follows: for every $x \leq y$ in X, let $\varphi_{xy}\colon \pi_0(\operatorname{Spec}(\mathcal{O}_{X,y})) \to \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x}))$ denote the map induced by the restriction morphism. Now, for $\alpha, \beta \in \mathbf{pw}(X)$ with $\pi(\alpha) = x$ and $\pi(\beta) = y$, we define

  $$\alpha \leq \beta \iff x \leq y \text{ and } \varphi_{xy}(\beta) = \alpha.$$

  This partial order makes $\pi\colon \mathbf{pw}(X) \to X$ monotone, hence continuous.

For every $x \in X$, consider $\pi^{-1}(x) = \{\alpha_1, \ldots, \alpha_n\} = \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x}))$. Each connected component is affine (connected components of a locally Noetherian topological space are open), so α_i corresponds to a non-zero ring with connected spectrum: $A_x^{\alpha_i}$. There is a decomposition $\mathcal{O}_{X,x} \simeq A_x^{\alpha_1} \times \cdots \times A_x^{\alpha_n}$.

Furthermore, for each $x \leq y$, since the continuous image of a connected topological space is connected, the morphism φ_{xy} induces a unique ring homomorphism $A_x^{\alpha_i} \to A_y^{\beta_j}$ for every $\beta_j \in \pi^{-1}(y)$ and $\alpha_i = \varphi_{xy}(\beta_j)$.

- We endow $\mathbf{pw}(X)$ with the sheaf of rings that for all α_i with $\pi(\alpha_i) = x$

  $$\mathcal{O}_{\mathbf{pw}(X),\alpha_i} = A_x^{\alpha_i},$$

  and, for every $\alpha_i \leq \beta_j$ with $\pi(\alpha_i) = x$ and $\pi(\beta_j) = y$, we define the restriction morphism as the ring map $A_x^{\alpha_i} \to A_y^{\beta_j}$ described above.
- The projection $\pi\colon \mathbf{pw}(X) \to X$ becomes a morphism of (finite) ringed spaces with $\pi_{\alpha_i}^{\#}\colon \mathcal{O}_{X,x} \to A_x^{\alpha_i}$ being the natural projection for all α_i with $\pi(\alpha_i) = x$. If X is pw-connected, π is clearly the identity.

This construction is functorial: given a morphism $f\colon X \to Y$, the maps $f_x^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induce $\psi_x\colon \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x})) \to \pi_0(\operatorname{Spec}(\mathcal{O}_{Y,f(x)}))$ for every $x \in X$. Their disjoint union defines a continuous map

$$\mathbf{pw}(f)\colon \mathbf{pw}(X) \to \mathbf{pw}(X'),$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{pw}(X) & \xrightarrow{\mathbf{pw}(f)} & \mathbf{pw}(Y) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

We turn this into a commutative diagram of ringed spaces by endowing $\mathbf{pw}(f)$ with the morphism of sheaves of rings, such that, with the previous notation, for each $\gamma_j \in \pi_Y^{-1}(f(x))$ and $\alpha_i = \psi_x(\gamma_j) = \mathbf{pw}(f)(\gamma_j)$

$$\mathbf{pw}(f)^{\#}_{\gamma_j}\colon A^{\alpha_i}_{f(x)} \to A^{\gamma_j}_x$$

is the unique ring homomorphism defined by the same connectedness argument as the restriction morphism.

Remark 3.1. Minimal open neighborhoods are well-behaved with respect to the projection π. Indeed, for each $x \in X$ we have $r_x\colon \mathcal{O}_{X,x} \to \prod_{x' \geq x} \mathcal{O}_{X,x'}$ inducing a continuous map $\pi_0(r_x^{\sharp})\colon \coprod_{x' \geq x} \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x'})) \to \pi_0(\operatorname{Spec}(\mathcal{O}_{X,x}))$, where both spaces are topologized as subspaces of $\mathbf{pw}(X)$; hence, the target is discrete. Now, for all $\alpha \in \pi^{-1}(x)$

$$U_\alpha = \pi_0(r_x^{\sharp})^{-1}(\alpha).$$

In particular, for $\alpha, \beta \in \mathbf{pw}(X)$ with $\pi(\alpha) = \pi(\beta) = x$, $U_\alpha \cap U_\beta = \emptyset$, that is

$$\pi^{-1}(U_x) = U_{\alpha_1} \amalg \cdots \amalg U_{\alpha_n}.$$

Proposition 3.4. *If X is a schematic space, $\mathbf{pw}(X)$ is a schematic space and the projection $\pi\colon \mathbf{pw}(X) \to X$ is schematic and a qc-isomorphism.*

Proof. First, note that π will be affine, because, by Remark 3.1, for every x, such that $\pi^{-1}(x) = \{\alpha_1, \ldots, \alpha_n\}$, we have $\pi^{-1}(U_x) = U_{\alpha_1} \amalg \cdots \amalg U_{\alpha_n}$. Furthermore, the definition of the structure sheaf of $\mathbf{pw}(X)$ yields

$$(\pi_* \mathcal{O}_{\mathbf{pw}(X)})_x = \mathcal{O}_{\mathbf{pw}(X)}(U_{\alpha_1} \amalg \cdots \amalg U_{\alpha_n}) = \mathcal{O}_{\mathbf{pw}(X),\alpha_1} \times \cdots \times \mathcal{O}_{\mathbf{pw}(X),\alpha_n} \simeq \mathcal{O}_{X,x}.$$

We now show that $\mathbf{pw}(X)$ and π are schematic (so π is a qc-isomorphism). We begin with the former. If $\alpha, \alpha' \in \mathbf{pw}(X)$ ($\alpha \neq \alpha'$) verify $\pi(\alpha) = \pi(\alpha')$, then $U_\alpha \cap U_{\alpha'} = \emptyset$ by Remark 3.1, and we also have that $U_\alpha \cap U_{\alpha'} = \emptyset$ whenever $U_{\pi(\alpha)} \cap U_{\pi(\alpha')} = \emptyset$; so we only need to check that, for all α, α' with $\pi(\alpha) \neq \pi(\alpha')$, $U_{\pi(\alpha)} \cap U_{\pi(\alpha')} \neq \emptyset$ and any $\alpha'' \geq \alpha'$, one has

$$H^i(U_\alpha \cap U_{\alpha'}, \mathcal{O}_{\mathbf{pw}(X)}) \otimes_{\mathcal{O}_{\mathbf{pw}(X),\alpha'}} \mathcal{O}_{\mathbf{pw}(X),\alpha''} \xrightarrow{\sim} H^i(U_\alpha \cap U_{\alpha''}, \mathcal{O}_{\mathbf{pw}(X)}).$$

To see this, we compare the standard resolutions of $\mathcal{O}_{\mathbf{pw}(X)}$ and $\mathcal{O}_X$ [7, After Remark 2.14]. Denote all intersections of minimal sets by $U_a \cap U_b = U_{ab}$ (a, b generic notation for points), $x = \pi(\alpha)$ and $x' = \pi(\alpha')$. We have

$$\pi^{-1}(U_{xx'}) = \coprod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha'_k \in \pi^{-1}(x')}} U_{\alpha_j \alpha'_k}.$$

For every i, denoting by $C^i\mathcal{O}_X$ the i-th degree term of the *standard resolution* described in [7, Section 2.3], we have the decomposition (compatible with differentials)

$$\begin{aligned}(C^i\mathcal{O}_X)(U_{xx'}) &= \prod_{t_i > \cdots > t_0 \in U_{xx'}} \mathcal{O}_{X,t_i} \simeq \prod_{\substack{t_i > \cdots > t_0 \in U_{xx'} \\ \beta_i \in \pi^{-1}(t_i)}} \mathcal{O}_{\mathbf{pw}(X),\beta_i} \\ &= \prod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha'_k \in \pi^{-1}(x')}} \prod_{\substack{t_i > \cdots > t_0 \in U_{xx'} \\ \beta_i \in \pi^{-1}(t_i) \\ \beta_i \in U_{\alpha_j \alpha'_k}}} \mathcal{O}_{\mathbf{pw}(X),\beta_i} \\ &= \prod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha'_k \in \pi^{-1}(x')}} \prod_{\beta_i > \cdots > \beta_0 \in U_{\alpha_j \alpha'_k}} \mathcal{O}_{\mathbf{pw}(X),\beta_i} \\ &= \prod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha'_k \in \pi^{-1}(x')}} (C^i\mathcal{O}_{\mathbf{pw}(X)})(U_{\alpha_j \alpha'_k}),\end{aligned}$$

where we have used that, by definition of $\pi\colon \mathbf{pw}(X) \to X$, there is an identification

$$\{\beta_i > \cdots > \beta_0 \in U_{\alpha_j \alpha'_k}\} \equiv \{t_i > \cdots > t_0 \in U_{xx'} : \pi(\beta_i) = t_i \text{ and } \beta_i \in U_{\alpha_j \alpha'_k}\}.$$

Indeed, each $\beta_i > \cdots > \beta_0$ clearly produces a chain $\pi(\beta_i) > \cdots > \pi(\beta_0)$ verifying the conditions. Conversely, given $\pi(\beta_i) > t_{i-1} > \cdots > t_0$ with $\beta_i \in U_{\alpha_j \alpha'_k}$, we define $\beta_j = \pi_0(r^\sharp_{t_j t_i})(\beta_i)$ for all $j < i$.

Now, since X is schematic, $H^i(U_{xx'}, \mathcal{O}_X) \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x''} \xrightarrow{\sim} H^i(U_{xx''}, \mathcal{O}_X)$ for all $x'' > x'$, and the previous discussion implies that we have

$$\begin{aligned} H^i(U_{xx'}, \mathcal{O}_X) &\simeq \prod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha'_k \in \pi^{-1}(x')}} H^i(U_{\alpha_j \alpha'_k}, \mathcal{O}_{\mathbf{pw}(X)}), \\ H^i(U_{xx''}, \mathcal{O}_X) &\simeq \prod_{\substack{\alpha_j \in \pi^{-1}(x) \\ \alpha''_k \in \pi^{-1}(x'')}} H^i(U_{\alpha_j \alpha''_k}, \mathcal{O}_{\mathbf{pw}(X)}),\end{aligned}$$

and thus, for all $\alpha, \alpha' < \alpha''$ with $\pi(\alpha) = x, \pi(\alpha') = x', \pi(\alpha'') = x''$

$$H^i(U_{\alpha\alpha'}, \mathcal{O}_{\mathbf{pw}(X)}) \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x''} \simeq H^i(U_{\alpha\alpha''}, \mathcal{O}_{\mathbf{pw}(X)}).$$

Since $\mathcal{O}_{X,x'} \simeq \prod_{\alpha' \in \pi^{-1}(x')} \mathcal{O}_{\mathbf{pw}(X),\alpha'}$, $\mathcal{O}_{X,x''} \simeq \prod_{\alpha'' \in \pi^{-1}(x'')} \mathcal{O}_{\mathbf{pw}(X),\alpha''}$, finite direct products are isomorphic to direct sums, so they commute with tensor products; and, by construction of $\pi\colon \mathbf{pw}(X) \to X$, the tensor product $H^i(U_{\alpha\alpha'}, \mathcal{O}_{\mathbf{pw}(X)}) \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{\mathbf{pw}(X),\alpha''}$ is non-zero—and isomorphic to the desired one—only when $\alpha'' \geq \alpha'$, we conclude.

To prove that π is schematic it suffices to see that $\mathbb{R}\pi_*\mathcal{M}$ is quasi-coherent for any quasi-coherent module $\mathcal{M}$ [7, Theorem 5.6], i.e., that for all $x \leq x'$ in X, the morphism $H^i(\pi^{-1}(U_x), \mathcal{M}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} \xrightarrow{\sim} H^i(\pi^{-1}(U_x), \mathcal{M})$ is an isomorphism for all i. Since quasi-coherent modules on the affine space $\pi^{-1}(U_x) = U_{\alpha_1} \amalg \cdots \amalg U_{\alpha_n}$ are acyclic, the only non-trivial case is $i = 0$

$$\mathcal{M}(U_{\alpha_1} \amalg \cdots \amalg U_{\alpha_n}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} \simeq \left(\prod_{j=1}^{n} \mathcal{M}(U_{\alpha_j}) \right) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'}$$

$$\simeq \prod_{j=1}^{n} \mathcal{M}(U_{\alpha_j}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} \simeq \prod_{j=1}^{n} \prod_{\alpha' \in \pi^{-1}(x') \cap U_{\alpha_j}} \mathcal{M}(U_{\alpha'}) \simeq \mathcal{M}(\amalg_{\alpha' \in \pi^{-1}(x')} U_{\alpha'}),$$

where $\mathcal{M}(U_{\alpha_j}) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} \simeq \mathcal{M}(U_{\alpha_j}) \otimes_{\mathcal{O}_{\mathbf{pw}(X),\alpha_j}} \left(\prod_{\alpha' \in \pi^{-1}(x')} \mathcal{O}_{\mathbf{pw}(X),\alpha'} \right)$ and, as before, the only non-zero components of this tensor product are those with $\alpha' \geq \alpha_j$. □

Proposition 3.5. *If $f\colon X \to Y$ is a schematic morphism between schematic spaces, the induced morphism $\mathbf{pw}(f)\colon \mathbf{pw}(X) \to \mathbf{pw}(Y)$ is schematic.*

Proof. The proof is routinary and similar to the last part of the previous one. If π_X and π_Y are the corresponding natural projections, we have to see that for all $\alpha < \alpha' \in \mathbf{pw}(X)$, $\beta < \beta' \in \mathbf{pw}(Y)$, denoting $g = \mathbf{pw}(f)$

$$H^i(U_\alpha \cap g^{-1}(U_\beta), \mathcal{O}_{\mathbf{pw}(X)}) \otimes_{\mathcal{O}_{\mathbf{pw}(Y),\beta}} \mathcal{O}_{\mathbf{pw}(Y),\beta'} \simeq H^i(U_\alpha \cap g^{-1}(U_{\beta'}), \mathcal{O}_{\mathbf{pw}(X)}),$$
$$H^i(U_\alpha \cap g^{-1}(U_\beta), \mathcal{O}_{\mathbf{pw}(X)}) \otimes_{\mathcal{O}_{\mathbf{pw}(X),\alpha}} \mathcal{O}_{\mathbf{pw}(X),\alpha'} \simeq H^i(U_{\alpha'} \cap g^{-1}(U_\beta), \mathcal{O}_{\mathbf{pw}(X)})$$

for all i. Since $\pi_Y \circ g = f \circ \pi_X$, we have that

$$\pi_X^{-1}(f^{-1}(U_y)) = g^{-1}\left(\coprod_{\beta \in \pi_Y^{-1}(U_y)} U_\beta \right) = \coprod_{\beta \in \pi_Y^{-1}(U_y)} g^{-1}(U_\beta).$$

Now, as in Proposition 3.4, one relates the sections of the standard resolution of $\mathcal{O}_X$ on $U_{\pi_X(\alpha)} \cap f^{-1}(U_{\pi_Y(\beta)})$ with those of $\mathcal{O}_{\mathbf{pw(X)}}$ on $U_\alpha \cap g^{-1}(U_\beta)$. Finally, the proof concludes using the hypothesis that f is schematic and the decomposition of the stalk rings of $\mathcal{O}_X$ and $\mathcal{O}_Y$ at each point. Explicit computations are left to the dedicated reader. □

Let $\mathbf{SchFin}^{\mathrm{pw}}$ denote the subcategory of pw-connected schematic spaces. We have seen that there is a functor $\mathbf{pw}\colon \mathbf{SchFin} \to \mathbf{SchFin}^{\mathrm{pw}}$.

Theorem 3.6. *The functor $\mathbf{pw}\colon \mathbf{SchFin} \to \mathbf{SchFin}^{\mathrm{pw}}$ is right adjoint to $i\colon \mathbf{SchFin}^{\mathrm{pw}} \hookrightarrow \mathbf{SchFin}$. The map $\mathbf{pw}(X) \to X$ is a qc-isomorphism for all X and verifies $\mathbf{pw} \circ i = \mathrm{Id}$. In particular, $\mathbf{SchFin}^{\mathrm{pw}}_{\mathrm{qc}} \simeq \mathbf{SchFin}_{\mathrm{qc}}$.*

Proof. It only remains to check the adjunction. If Y is pw-connected and X arbitrary, there is a bijection $\mathrm{Hom}_{\mathbf{SchFin}}(Y, X) \simeq \mathrm{Hom}_{\mathbf{SchFin}^{\mathrm{pw}}}(Y, \mathbf{pw}(X))$. Indeed, given $f\colon Y \to X$, we apply $\mathbf{pw}$ and obtain $Y = \mathbf{pw}(Y) \to X$. Conversely, any $g\colon Y \to \mathbf{pw}(X)$ induces $\pi_X \circ g\colon Y \to X$, where $\pi_X\colon \mathbf{pw}(X) \to X$ is the natural projection. □

Lemma 3.7. *A schematic space X is connected if and only if* $\mathbf{pw}(X)$ *is top-connected (hence well-connected), i.e., in* $\mathbf{SchFin}^{\mathrm{pw}}$*, connectedness, well-connectedness, and top-connectedness are equivalent.*

Proof. It follows from the definition of **pw**, Proposition 3.3 and the fact that qc-isomorphisms preserve global sections. □

Definition 3.8. (*Well-connected components*) The topological connected components of $\mathbf{pw}(X)$ are called the *well-connected* components of X. We denote by $\pi_0^{wc}(X) := \pi_0(\mathbf{pw}(X))$ the set of well-connected components of X.

If $i_k \colon X_k \hookrightarrow \mathbf{pw}(X) \to X$ is a connected component of $\mathbf{pw}(X)$, it is straightforward to see that $\mathcal{O}_X \simeq \prod_k i_{k*}\mathcal{O}_{X_k}$. We can adapt this construction for sheaves of quasi-coherent algebras: let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra and $(X, \mathcal{A}) \to X$ the corresponding schematic space, which is affine over X. The same argument gives us a decomposition $\mathcal{A} \simeq \prod_k \mathcal{A}_k$.

Definition 3.9. (*Well-connected components of a sheaf of algebras*) The algebras $\mathcal{A}_k$ just introduced are called the *well-connected components of* $\mathcal{A}$. In this sense, $\mathcal{A}$ is *connected* iff for every $\mathcal{A} \simeq \mathcal{A}_1 \times \mathcal{A}_2$, either $\mathcal{A}_1 = 0$ or $\mathcal{A}_2 = 0$.

Theorem 3.10. *A schematic space X is well-connected iff for all decompositions $X = X_1 \amalg X_2$ in* **SchFin***, either X_1 or X_2, are qc-isomorphic to $\emptyset$.*

Proof. If $X = X_1 \amalg X_2$ and X is well-connected, $\mathbf{pw}(X) = \mathbf{pw}(X_1) \amalg \mathbf{pw}(X_2)$ with $\mathbf{pw}(X)$ top-connected by Lemma 3.7. We conclude that $\mathbf{pw}(X_i) = \emptyset$ for $i = 1$ or $i = 2$, i.e., that X_i is qc-isomorphic to $\emptyset$. Conversely, X admits a decomposition $X = (X, \mathcal{O}_1) \amalg \cdots \amalg (X, \mathcal{O}_k)$ with $\mathcal{O}_1, \ldots, \mathcal{O}_k$ the well-connected components of $\mathcal{O}_X$. By the hypothesis, only one of them is non-zero. □

4. Geometric Points

We introduce geometric points of schematic spaces. In this paper, we employ a straightforward analogy with Scheme Theory rather than the topos-theoretic approach. From now on, we consider spaces over $(\star, k)$ with k a field. The functor of points of a schematic space X is its image $X^\bullet$ in $[\mathbf{SchFin}^{\mathrm{op}}, \mathbf{Set}]$ via Yoneda. For any ring A we denote $X^\bullet(A) \equiv X^\bullet((\star, A))$.

Definition 4.1. Given a schematic space X and an algebraically closed field Ω, a *geometric point* (*with values in* Ω) is a point $\overline{x} \in X^\bullet(\Omega)$.

Proposition 4.2. (Characterization of geometric points) *A morphism of ringed spaces $\overline{x} : (\star, \Omega) \to X(\star \mapsto x)$ is a geometric point if and only if the prime ideal $\mathfrak{p} := \ker(\mathcal{O}_{X,x} \to \Omega)$ does not «lift»to any $x' > x$; i.e., for every $x' > x$ there is no prime $\mathfrak{p}' \subseteq \mathcal{O}_{X,x'}$, such that $r_{xx'}^{-1}(\mathfrak{p}') = \mathfrak{p}$.*

Proof. It is a simply consequence of the definition of schematic morphism for this particular case. The only non-trivial condition is that $\Omega \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x'} = 0$ for every $x < x'$. In other words, that the fiber of the morphism of schemes $\operatorname{Spec}(\mathcal{O}_{X,x'}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ at the point $\operatorname{Spec}(\Omega) \hookrightarrow \operatorname{Spec}(\mathcal{O}_{X,x})$ is empty. In terms of rings, this is exactly the «lifting»condition of the proposition. □

Consider pairs $(x,\mathfrak{p})$, where $x\in X$ and $\mathfrak{p}\subseteq\mathcal{O}_{X,x}$ is a prime ideal. For every such pair, we define its *residue field* $\kappa(x,\mathfrak{p})$ as

$$\kappa(x,\mathfrak{p}):=(\mathcal{O}_{X,x})_{\mathfrak{p}}/\mathfrak{p}(\mathcal{O}_{X,x})_{\mathfrak{p}}. \tag{4.1}$$

Proposition 4.3. *If X is schematic and $(x,\mathfrak{p})$, $(x',\mathfrak{p}')$ are pairs, such that $x\leq x'$ and $\mathfrak{p}=r_{xx'}^{-1}(\mathfrak{p}')$, then the natural map $\kappa(x,\mathfrak{p})\to\kappa(x',\mathfrak{p}')$ is an isomorphism.*

Proof. Restriction maps are local isomorphisms (Remark 2.2). □

We define the following binary relation in the set of these pairs:

$$(x,\mathfrak{p})\sim(y,\mathfrak{q})\iff\exists(z,\mathfrak{r})\text{ s.t. }z\geq x,y\text{ and }r_{xz}^{-1}(\mathfrak{r})=\mathfrak{p},\ r_{yz}^{-1}(\mathfrak{r})=\mathfrak{q}. \tag{4.2}$$

We prove that it is the equivalence relation realizing $\operatorname{Spec}(X)$ as a quotient set of $\coprod_{x\in X}\operatorname{Spec}(\mathcal{O}_{X,x})$. This relies on X being schematic.

Lemma 4.4. *For any $(x,\mathfrak{p})$ and $(y,\mathfrak{q})$, such that $x,y\geq s$ for some $s\in X$ and $r_{sx}^{-1}(\mathfrak{p})=r_{sx}^{-1}(\mathfrak{q})$, we have $(x,\mathfrak{p})\sim(y,\mathfrak{q})$.*

Proof. First, we see that $U_x\cap U_y\neq\emptyset$: otherwise, since X is schematic, we would have $\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{X,s}}\mathcal{O}_{X,y}=0$, i.e. $\operatorname{Spec}(\mathcal{O}_{X,x})\times_{\operatorname{Spec}(\mathcal{O}_{X,s})}\operatorname{Spec}(\mathcal{O}_{X,y})\neq\emptyset$; by Proposition 4.3, the underlying set of this scheme-theoretic fiber product is the set-theoretic fiber product, which contradicts $r_{sx}^{-1}(\mathfrak{p})=r_{sx}^{-1}(\mathfrak{q})$. Next, we see that $U_x\cap U_y\subseteq U_s$ is affine: by [7, Corollary 4.11], it suffices to prove that it is acyclic ($H^i(U_x\cap U_y,\mathcal{O}_X)=0$ for $i>0$), which holds, because $H^i(U_x\cap U_y,\mathcal{O}_X)\simeq H^i(U_x,\mathcal{O}_X)\otimes_{\mathcal{O}_{X,z}}\mathcal{O}_{X,y}\simeq 0$ for $i>0$ (see comments below Definition 2.2). This implies that the natural morphism

$$R^\sharp\colon\coprod_{t\geq x,y}\operatorname{Spec}(\mathcal{O}_{X,t})\to\operatorname{Spec}(\mathcal{O}_X(U_x\cap U_y))$$

is surjective, so there is a point $z\geq x,y$ and a prime $\mathfrak{r}\in\operatorname{Spec}(\mathcal{O}_{X,z})$, such that $R^\sharp(\mathfrak{r})=(\mathfrak{p},\mathfrak{q})$. The pair $(z,\mathfrak{r})$ verifies (4.2). □

Lemma 4.5. *The binary relation of Eq.* (4.2) *is an equivalence relation.*

Proof. Transtivity holds by Lemma 4.4. The rest is obvious. □

Corollary 4.5.1 *If $(x,\mathfrak{p})\sim(x,\mathfrak{p}')$, then $\mathfrak{p}=\mathfrak{p}'$. Each class of pairs $(x,\mathfrak{p})$ has a unique maximal representative, i.e., all $(y,\mathfrak{q})$ in its class verify $y\leq x$.*

Proof. The first part follows, since restriction maps are flat epimorphisms. The last statement is proved by contradiction with Lemma 4.4. □

Definition 4.6. A *schematic point* of X is an *equivalence class* of pairs $(x,\mathfrak{p})$. Unless stated otherwise, we identify them with their maximal representatives.

Remark 4.1. By Proposition 4.3, the residue field of a schematic point $(x,\mathfrak{p})$ is well-defined up to isomorphism. We still denote it $\kappa(x,\mathfrak{p})$.

Proposition 4.7. *Let X be schematic and Ω be an algebraically closed field. There is a correspondence*

$$\{\textit{Morphisms of ringed spaces } (\star,\Omega)\to X\} \overset{1:1}{\leftrightarrow} \{\textit{Pairs } (x,\mathfrak{p}) \textit{ and } \kappa(x,\mathfrak{p})\to\Omega\}$$

that restricts to the schematic category as

$$X^\bullet(\Omega) \overset{1:1}{\longleftrightarrow} \{\textit{Schematic points } (x,\mathfrak{p}) \textit{ and } \kappa(x,\mathfrak{p})\to\Omega\}.$$

Proof. For every $\overline{x} : (\star,\Omega)\to X$, we define $x{:=}\overline{x}(\star)$, $\mathfrak{p}{:=}\ker(\mathcal{O}_{X,x}\to\Omega)$ and the extension is obtained by factoring the map $\mathcal{O}_{x,x}\to\Omega$ through $\kappa(x,\mathfrak{p})$. Conversely, for every schematic point $(x,\mathfrak{p})$, we define $(\star,\kappa((s,\mathfrak{p})))\to X$ as $\star\mapsto x$ at the level of sets and as the natural map $\mathcal{O}_{X,x}\to\kappa(x,\mathfrak{p})$ at the level of rings; thus, composing with $(\star,\Omega)\to(\star,\kappa(x,\mathfrak{p}))$, we obtain the desired morphism. The second part follows from the fact that every schematic point $(x,\mathfrak{p})$ has a unique maximal representative, so the morphism of ringed spaces constructed using this representative is a geometric point according to Proposition 4.2. □

We see that schematic points are analogous to points in the sense of schemes. Let S be a qcqs scheme and $\pi : S\to X$ a finite model.

Proposition 4.8. *There is a 1:1 correspondence between topological points of S and schematic points of its finite model X*

$$|S| \overset{1:1}{\longleftrightarrow} \{\textit{Schematic points } (x,\mathfrak{p})\}.$$

Moreover, for any algebraically closed field Ω, the projection π induces a residue field-preserving bijection $Hom_{\mathbf{LRS}}(\mathrm{Spec}(\Omega),S)\overset{\sim}{\to}X^\bullet(\Omega)$.

Proof. Let $s\in S$ be a point and $\mathfrak{p}_s\in\mathrm{Spec}(\mathcal{O}_S(U^s))$ the ideal it defines as an element of the affine open U^s. Then, $(\pi(s),\mathfrak{p}_s)$ is a schematic point of X. Conversely, given a schematic point $(x,\mathfrak{p}_x)$ of X, we have that $\mathfrak{p}_x\subseteq\mathcal{O}_X(U_x)=\mathcal{O}_S(U^s)$ for any $s\in\pi^{-1}(x)$. This ideal determines a point of S independent of the chosen representative of $(x,\mathfrak{p}_x)$. These correspondences are mutually inverse. From the definitions, it also follows that $\kappa(s)=\kappa(\pi(s),\mathfrak{p}_s)$.

Since $\mathrm{Spec}(\Omega)=(\star,\Omega)$ in **LRS**, the second bijection is explicitly given by composition with $S\to X$, with inverse defined by the Spec functor. □

In general, schematic points describe $|\mathrm{Spec}(X)|=|\mathrm{colim}_x\,\mathrm{Spec}(\mathcal{O}_{X,x})|$:

Proposition 4.9. *If X is schematic, $|\mathrm{Spec}(X)| \overset{1:1}{\longleftrightarrow} \{\textit{Schematic points } (x,\mathfrak{p})\}$.*

Theorem 4.10. *Let $f : X\to Y$ be a schematic morphism over k. If f is a qc-isomorphism, then, for every algebraically closed field extension $k\hookrightarrow\Omega$, the natural morphism $X^\bullet(\Omega)\to Y^\bullet(\Omega)$ is an isomorphism.*

Proof. Consider $\Psi : \{\text{Schematic points of } X\}\to\{\text{Schematic points of } Y\}$ with $(x,\mathfrak{p})\mapsto(f(x),(f_x^{\#})^{-1}(\mathfrak{p}))$. By Proposition 4.9, this is just the map $|\mathrm{Spec}(f)|\colon|\mathrm{Spec}(X)|\to|\mathrm{Spec}(Y)|$, which is a bijection when f is a qc-iso ([7, Proposition 6.6], where qc-isomorphisms are called *weak equivalences*). □

5. The Category of Finite Étale Covers

Étale morphisms and ring maps are of finite presentation, and hence not well-suited to work with general schematic spaces if one intends to obtain results that mimic scheme theory (this is because the notion is not stable under qc-isomorphisms, so functors will not factor through $\mathbf{SchFin}_{\mathrm{qc}}$). There are two initial ways of avoiding this problem: working with weakly étale morphisms and defining the pro-étale topology (*à la* Scholze and Bhatt [1]), or exploiting the fact that *finite* étale covers are affine and thus can be interpreted as sheaves of algebras, which will behave well, because the restriction maps of our spaces are local isomorphisms. We proceed with the second approach.

Definition 5.1. We say that a ring homomorphism $A \to B$ is *pointwise-étale* if, for every prime $\mathfrak{p} \subset A$, $B \otimes_A \kappa(\mathfrak{p})$ is an étale $\kappa(\mathfrak{p})$-algebra, i.e., it is a finite direct product of finite separable field extensions of $\kappa(\mathfrak{p})$.

Proposition 5.2. [2, I, 5.9] *A ring homomorphism $A \to B$ is étale if and only if it is flat, of finite presentation and pointwise-étale.*

Remark 5.1. If A is Noetherian, an A-algebra is flat and of finite presentation if and only if it is finite and locally free. This will be assumed to be the case.

Recall the definition of étale morphism in terms of ring homomorphisms:

Definition 5.3. A morphism of schemes $f: T \to S$ is said to be étale if for all affine open subsets $U \subseteq S$ and $V \subseteq T$, such that $U \subseteq f^{-1}(V)$, the natural ring homomorphism $\mathcal{O}_S(U) \to \mathcal{O}_T(V)$ is étale.

It is well known that the fibers of an étale morphism f are disjoint unions of spectra of finite separable field extensions of the residue field (which actually characterizes étale morphisms among all flat morphisms that are of finite presentation). Clearly, if f has compact fibers, these disjoint unions are finite. In particular, finite morphisms of schemes have compact fibers.

Let S be a scheme and denote by $\mathbf{Fet}_S$ the category of finite and étale morphisms to S, also known as the category of *finite étale covers* of S, which is a full and faithful subcategory of $\mathbf{Schemes}_{/S}$; and let $\mathbf{Qcoh}^{\mathrm{alg}}(S)$ denote the category of quasi-coherent $\mathcal{O}_S$-algebras (we shall use this notation for *any* ringed space S with its natural topology). The classical relative spectrum functor $\underline{\mathrm{Spec}}_S: \mathbf{Qcoh}^{\mathrm{alg}}(S) \to (\mathbf{Schemes}^{\mathrm{Affine}}_{/S})^{\mathrm{op}}$ induces an isomorphism of categories between $\mathbf{Qcoh}^{\mathrm{alg}}(S)$ and the opposite category of affine morphisms to S, whose inverse is given by $(f: T \to S) \mapsto f_* \mathcal{O}_T$. Since finite morphisms are affine, this gives us a fully faithful functor

$$\mathbf{Fet}_S^{\mathrm{op}} \to \mathbf{Qcoh}^{\mathrm{alg}}(S). \tag{5.1}$$

Let $\mathbf{Qcoh}^{\mathrm{fet}}(S)$ denote the essential image of this functor. From the definition of étale morphisms of schemes and the local characterization of étale algebras:

Lemma 5.4. *Let S be a scheme. A quasi-coherent algebra $\mathcal{A}$ lies in $\mathbf{Qcoh}^{\mathrm{fet}}(S)$ iff it is finite, flat and $\mathcal{A}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ is a finite étale $\kappa(s)$-algebra for every $s \in S$. If this holds, then for every affine $U \subseteq S$, $\mathcal{O}_S(U) \to \mathcal{A}(U)$ is étale.*

Remark 5.2. It is well known that $\mathbf{Qcoh}^{\mathrm{fet}}(S) \simeq \mathbf{Fet}_S^{\mathrm{op}} \simeq \mathbf{Loc}(S_{et})$, where the latter is the category of finite, locally constant sheaves of sets on the small étale site of S (which may also be considered as sheaves of $\mathbb{Z}$-modules). We will partially bring the last point of view back in Sect. 5.2.

Definition 5.5. Let X be a schematic space. $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ is an *étale cover sheaf* (or simply an *étale cover*) if it is finite, flat, and for each schematic point $(x, \mathfrak{p})$, $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x, \mathfrak{p})$ is a finite étale $\kappa(x, \mathfrak{p})$-algebra. Equivalently (by Proposition 5.2), if $\mathcal{O}_{X,x} \to \mathcal{A}_x$ is finite étale for every $x \in X$.

By Remark 2.2, this notion is well-defined. Since we are assuming quasi-coherence, finiteness is a condition at stalks (c.f. [7, Theorem 2.8]).

Given a schematic space X, let $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ denote the subcategory of $\mathbf{Qcoh}^{\mathrm{alg}}(X)$ whose objects are étale cover sheaves.

Theorem 5.6. *Let S be a qcqs scheme and $\pi : S \to X$ a finite model. The equivalence $(\pi^*, \pi_*)\colon \mathbf{Qcoh}(S) \simeq \mathbf{Qcoh}(X)$ restricts to*

$$\mathbf{Qcoh}^{\mathrm{fet}}(S) \simeq \mathbf{Qcoh}^{\mathrm{fet}}(X).$$

Proof. It follows from Proposition 4.8, Lemma 5.4, the definition of étale cover sheaf and a straightforward computation at stalks. □

5.1. Finite Locally Free Sheaves on Schematic Finite Spaces

Let $A \to B$ be a morphism of rings, such that B is finitely generated and locally free over A. Once again, if A is Noetherian, this is equivalent to B being finite and flat. The degree (or rank) of $A \to B$ is classically defined to be the map $\deg(B)\colon \mathrm{Spec}(\mathcal{A}) \to \mathbb{Z}^+$ with $\mathfrak{p} \mapsto \mathrm{rank}_{A_\mathfrak{p}} B_\mathfrak{p}$. It is continuous and locally constant, and hence can be identified with a non-negative integer if $\mathrm{Spec}(A)$ is connected. We wish to extend this notion to schematic spaces.

Now, let X be a schematic $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$. We know that $\mathcal{A}_x \to \mathcal{A}_y$ $(x \le y)$ are flat epimorphisms of rings, hence local isomorphisms. We leave Lemma 5.7 below as an algebra exercise, which states that they are also local isomorphisms with respect to localization at primes of $\mathcal{O}_{X,y}$.

Lemma 5.7. *Let X be schematic and $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$. For every $x \le y$ and $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_y)$, one has $(\mathcal{A}_x)_{r_{xy}^{-1}(\mathfrak{p})} \simeq (\mathcal{A}_y)_\mathfrak{p}$.*

Proposition 5.8. *If X is pw-connected and $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ has finite locally free stalks, there is a locally constant «degree»function $\deg(\mathcal{A}) : X \to \mathbb{Z}^+$, such that $\deg(\mathcal{A})(x) = \deg(\mathcal{A}_x)$. In particular, if X is connected, the degree of $\mathcal{A}$ is identified with an integer $\deg(\mathcal{A}) \in \mathbb{Z}^+$.*

Proof. We have to prove $\deg(\mathcal{A}_x) = \deg(\mathcal{A}_y)$ for any two points x, y in the same connected component. By transitivity, it is enough to see it for $x \le y$. The result follows from Lemma 5.7 and the hypothesis on X. □

5.2. Étale Cover Sheaves Are Locally Trivial

We use ideas of Lenstra [10, Proposition 5.2.9, p. 155] as a guide, proceeding fiberwise with an eye on connectedness and quasi-coherence.

Definition 5.9. Let X be schematic. We say that $\mathcal{B} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ is a «*covering*»if it is finite and faithfully flat ($\mathcal{O}_{X,x} \to \mathcal{B}_x$ faithfully flat for all $x \in X$). Equipped with this family of coverings, $\mathbf{Qcoh}^{\mathrm{alg}}(X)$ becomes a *cosite* $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$.

Remark 5.3. *Cosite* means that the opposite category is a site and $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ is the analogue of the fppf site of (Noetherian) schemes.

Given $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$, the natural morphism $f\colon (X,\mathcal{A}) \to X$ induces

$$\begin{aligned} f^*\colon X^{\mathrm{fppf}}_{\mathbf{Qcoh}} &\to \mathcal{A}^{\mathrm{fppf}}_{\mathbf{Qcoh}} := \mathbf{pw}(X,\mathcal{A})^{\mathrm{fppf}}_{\mathbf{Qcoh}} \\ \mathcal{B} &\mapsto \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{A}. \end{aligned}$$

Remark 5.4. For simplicity, we omit the pullback via $\mathbf{pw}(X,\mathcal{A}) \to (X,\mathcal{A})$.

Let X be schematic and $\mathcal{O}_1,\dots,\mathcal{O}_k$ the well-connected components of $\mathcal{O}_X$ (Definitions 3.8, 3.9). Given a finite set F, consider the constant functor: $\underline{F}\colon X^{\mathrm{fppf}}_{\mathbf{Qcoh}} \to \mathbf{Set}$ (with $\mathcal{A} \mapsto F$) and its (co)sheafification, denoted $\underline{F}^{\#}$, which sends $\mathcal{A}$ to $\coprod_{\pi_0^{wc}(X,\mathcal{A})} F$, where $\pi_0^{wc}(X,\mathcal{A})$ is the set of well-connected components (Definition 3.8) of $(X,\mathcal{A})$. Just as in the case of constant covers in ordinary theories, it follows that $\underline{F}^{\#}$ is representable (in the sense of the covariant functor of points) by $\prod_{1\le j\le k}\prod_F \mathcal{O}_j = \mathcal{O}_1^{\times n} \times \cdots \times \mathcal{O}_k^{\times n} \simeq \mathcal{O}_X^{\times n}$, which is called a *constant object* of cardinality F (or degree $n = \#F$).

Remark 5.5. If X is pw-connected, $\mathbf{pw}(X,\mathcal{O}_X^{\times n}) \to X$ is just $X \amalg \overset{n)}{\cdots} \amalg X \to X$.

Definition 5.10. An object $\mathcal{A}$ in $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ is *locally constant* if there exists a covering $\mathcal{B}$, such that, if $\mathcal{B} \simeq \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ denotes the decomposition into well-connected components of $\mathcal{B}$, $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}_j$ is constant of degree n_j in $(\mathcal{B}_j)^{\mathrm{fppf}}_{\mathbf{Qcoh}}$.

Lemma 5.11. *If X is well-connected, $\mathcal{A}$ is locally constant if and only if there exists a covering $\mathcal{B}$, such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B} \simeq \mathcal{B}^n$ for some $n \ge 0$ in $\mathcal{B}^{\mathrm{fppf}}_{\mathbf{Qcoh}}$.*

Proof. The «if»part is trivial. For the converse, recall that well-connectedness implies top-connectedness and pw-connectedness (Proposition 3.3). Using the notation of Definition 5.10, assume that $n = 2$, so $\mathbf{pw}(X,\mathcal{B})$ has two connected components (for $n > 2$, the idea is identical, but the argument is longer). We denote by $\mathcal{B}_1$ and $\mathcal{B}_2$ the corresponding connected components of $\mathcal{B}$ (Definition 3.9). We have to prove that $n_1 = n_2$. Notice that, since $\mathcal{B}$ is faithfully flat and X is pw-connected, $(X,\mathcal{B})$ has non-zero stalks.

Claim 1 There exists some $x \in X$, such that both $\mathcal{B}_{1x}$ and $\mathcal{B}_{2x}$ are non-zero.

Proof of Claim 1. Consider $\pi\colon X^1_{\mathcal{B}} \amalg X^2_{\mathcal{B}} = \mathbf{pw}(X,\mathcal{B}) \to (X,\mathcal{B}) \to X$, with $X^i_{\mathcal{B}}$ the connected components of $\mathbf{pw}(X,\mathcal{B})$. Note that $\mathcal{B}_i = \pi_{i*}\mathcal{O}_{X^i_{\mathcal{B}}}$ with $\pi_i\colon X^i_{\mathcal{B}} \to X$ for $i = 1, 2$ and that π is surjective, because $\mathcal{B}$ has non-zero stalks. Let us prove that it is injective. Assume that either $\mathcal{B}_{1x} = 0$ or $\mathcal{B}_{2x} = 0$ for all $x \in X$. Note that both cannot be zero at the same time, because $\mathcal{B}_x \neq 0$. In this case $\pi^{-1}(x)$ has exactly one element for all $x \in X$ by construction, so π is injective and thus a homeomorphism. In particular, X would be homeomorphic to a disjoint union of non-empty topological spaces, contradicting the top-connectedness of X. We conclude that there is some $x \in X$, such that both $\mathcal{B}_{1x}, \mathcal{B}_{2x} \neq 0$. □

Claim 2 $\mathcal{B}_1 \otimes_{\mathcal{O}_X} \mathcal{B}_2 \neq 0$.

Proof of Claim 2. Let $f_x\colon \mathcal{O}_{X,x} \hookrightarrow \mathcal{B}_{1x} \times \mathcal{B}_{2x}$ denote the natural faithfully flat structure morphism and let $f_{ix}\colon \mathcal{O}_{X,x} \to \mathcal{B}_{ix}$ (for $i = 1, 2$) denote its composition with the natural projections. We have a surjective morphism $f^\sharp_x\colon \operatorname{Spec}(\mathcal{B}_{1x}) \amalg \operatorname{Spec}(\mathcal{B}_{2x}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ and $f^\sharp_{ix}\colon \operatorname{Spec}(\mathcal{B}_{ix}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ with closed and open image for $i = 1, 2$. By Claim 1, there exists some $x \in X$, such that both $\mathcal{B}_{1x}, \mathcal{B}_{2x} \neq 0$. If for such an x, $\mathcal{B}_{1x} \otimes_{\mathcal{O}_{X,x}} \mathcal{B}_{2x} = 0$, taking spectra, we would obtain $f^\sharp_{1x}(\mathfrak{p}_1) \neq f^\sharp_{2x}(\mathfrak{p}_2)$ for any $\mathfrak{p}_i \in \operatorname{Spec}(\mathcal{B}_{ix})$ ($i = 1, 2$), which would imply that $\operatorname{Spec}(\mathcal{O}_{X,x}) = \operatorname{Im}(f^\sharp_x) \simeq \operatorname{Im}(f^\sharp_{1x}) \amalg \operatorname{Im}(f^\sharp_{2x})$, contradicting the pw-connectedness of X. □

Finally, since $\mathcal{B}_1 \otimes_{\mathcal{O}_X} \mathcal{B}_2 \neq 0$, we have isomorphisms of $\mathcal{O}_X$-modules

$$(\mathcal{B}_1 \otimes_{\mathcal{O}_X} \mathcal{B}_2)^{\times n_1} \simeq \mathcal{B}_1^{\times n_1} \otimes_{\mathcal{O}_X} \mathcal{B}_2 \simeq \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}_1 \otimes_{\mathcal{O}_X} \mathcal{B}_2 \simeq (\mathcal{B}_1 \otimes_{\mathcal{O}_X} \mathcal{B}_2)^{\times n_2},$$

from which it follows that $n_1 = n_2$, which proves the lemma. □

Lemma 5.12. *Let $f\colon A \to B$ be a morphism of rings with $\Omega_{A|B} = 0$ and $I = \ker(B \otimes_A B \to B)$ finitely generated. Then, there is a decomposition $B \otimes_A B = C \times D$ (of rings) with $I \otimes_{B \otimes_A B} C \simeq 0$, where C is a $B \otimes_A B$-algebra via the natural projection. Furthermore, $C \simeq B$.*

Proof. Since $\Omega_{A|B} = I/I^2 = 0$, by Nakayama, there is a non-zero idempotent $e \in I$, such that $I = (e)$. Then, $C = (B \otimes_A B)/I$ and $D = (B \otimes_A B)/(1 - e)$. □

Proposition 5.13. *(Local triviality of étale cover sheaves) Let X be well-connected and schematic, and then, $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ is a degree n étale cover sheaf iff it is finite locally constant (of cardinality n) as an object of $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$.*

Proof. Assume $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B} \simeq \mathcal{B}^{\times n}$ for some $n > 0$ (otherwise $\mathcal{A} = 0$) and some faithfully flat algebra $\mathcal{B} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ (Lemma 5.11). Note that this also implies that $\mathcal{A}_x \neq 0$ for all $x \in X$.

Since for every $x \in X$, $\mathcal{O}_{X,x} \to \mathcal{B}_x$ is finite locally free and finiteness is a local condition, we can assume that it is finite and free, $\mathcal{B}_x \simeq \mathcal{O}_{X,x}^{\oplus m}$. Since $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{B}_x$ is a finite $\mathcal{O}_{X,x}$ module and it is also a finite $\mathcal{A}_x$-module, $\mathcal{A}_x$ is a finite $\mathcal{O}_{X,x}$-module for all $x \in X$. Finally, for a schematic point $(x, \mathfrak{p})$, $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \overline{\kappa(x,\mathfrak{p})} \simeq (\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{B}_x) \otimes_{\mathcal{B}_x} \overline{\kappa(x,\mathfrak{p})} \simeq \mathcal{B}_x^{\times n} \otimes_{\mathcal{B}_x} \overline{\kappa(x,\mathfrak{p})} \simeq \overline{\kappa(x,\mathfrak{p})}^{\times n}$.

Conversely, if $\mathcal{A} \neq 0$ is finite étale, it is locally trivial of constant positive degree n. By induction over n: if $n = 1$, $\mathcal{A} \simeq \mathcal{O}_X$ and there is nothing to say. In general, if $\mathcal{I} = \ker(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A})$, which is a quasi-coherent sheaf of ideals ($\mathbf{Qcoh}(X)$ is abelian) both as a sheaf of $\mathcal{O}_X$-modules and of $\mathcal{A}$-modules; $\mathcal{I}/\mathcal{I}^2$ is the sheaf of relative differentials of $\mathcal{O}_X \to \mathcal{A}$, which is trivial (at stalks) by étaleness. Since $\mathcal{A}_x$ is finite and finitely presented, $\mathcal{I}_x$ is finitely generated, and hence, Lemma 5.12 applies and $\mathcal{I}_x = (e_x)$ for some non-trivial idempotent $e_x \in \mathcal{O}_{X,x}$ and $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x \simeq \mathcal{A}_x \times C_x$ for some ring C_x. Since both the natural morphism $\mathcal{A}_x \to \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x$ and $\mathcal{A}_x \times C_x \to C_x$ are finite étale, C_x is a finite étale A_x-algebra for all $x \in X$.

We sheafify the situation: define a sheaf of ideals $\mathcal{J}$ by $\mathcal{J}_x{:=}(1-e_x)$ and setting, for every $x \leq y$, $J_x \to J_y$ to be the obvious map induced by $\mathcal{I}_x \to \mathcal{I}_y$ (note that $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ as $\mathcal{O}_X$-modules). Since $\mathcal{I}$ is quasi-coherent, $\mathcal{J}$ is readily seen to be quasi-coherent. Now, we consider the quotient $\mathcal{C}{:=}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A})/\mathcal{J}$, which is quasi-coherent and verifies $\mathcal{C}_x = C_x$, so it is an étale cover sheaf.

By construction, $\mathcal{C}$ (since X is well-connected) has constant degree $n-1$ as a finite locally free $\mathcal{A}$-algebra (and $n^2 - n$ as an $\mathcal{O}_X$-algebra). Consider the qc-isomorphism $\mathbf{pw}(X, \mathcal{A}) \to (X, \mathcal{A})$ and denote by $\mathcal{C}'$ the pullback of $\mathcal{C}$, which also has constant degree $n - 1$. By induction, there is a faithfully flat finite algebra $\mathcal{B}'$ in $\mathbf{Qcoh}(\mathbf{pw}(X, \mathcal{A}))$ with $\mathcal{C}' \otimes_{\mathcal{O}_{\mathbf{pw}(X,\mathcal{A})}} \mathcal{B}' \simeq \mathcal{B}'^{\times n-1}$. Let $\mathcal{B}$ be its push-forward to $(X, \mathcal{A})$, which verifies $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{B}^{\times n-1}$. Finally, $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B} \simeq (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{B} \simeq (\mathcal{A} \times \mathcal{C}) \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{B} \times \mathcal{B}^{\times n-1} \simeq \mathcal{B}^{\times n}$. □

Lemma 5.14. (Triviality of morphisms) *Let X be well-connected and consider $\mathcal{A} \simeq \mathcal{O}_X^{\times F}$, $\mathcal{B} \simeq \mathcal{O}_X^{\times E}$ for some finite sets F, E. Any morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ is induced by composition with some map $\phi : F \to E$.*

Proof. Pick $x \in X$ and consider $f_x : \mathcal{A}_x \to \mathcal{B}_x$. Since $\mathcal{O}_{X,x}$ is connected (it has no non-trivial idempotents), it is an exercise of basic algebra to check that f_x is indeed induced by a morphism $\phi : F \to E$ [5, ex. 5.11(d)]. This pointwise argument is compatible with restrictions maps. □

6. Stability by qc-Isomorphisms

Let X be schematic. This section will reduce our study to $\mathbf{SchFin}^{\mathrm{pw}}$.

Lemma 6.1. *If $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$, $U \subseteq X$ is an affine open and $x \in U$, then $\mathcal{A}(U) \to \mathcal{A}_x$ is a flat ring epimorphism with $\mathcal{A}(U) \to \prod_{x \in U} \mathcal{A}_x$ faithfully flat.*

Proof. It follows, since $(X, \mathcal{A}) \to X$ is schematic and affine [7, Ex. 5.25]. □

Lemma 6.2. *A sheaf $\mathcal{A} \in \mathbf{Qcoh}^{\mathrm{alg}}(X)$ is flat (resp. finite, étale, faithfully flat) if and only if for every affine open $U \subseteq X$, the natural morphism $\mathcal{O}_X(U) \to \mathcal{A}(U)$ is flat (resp. finite, étale, faithfully flat). These properties can be checked on an affine open cover.*

Proof. We prove it for flatness. The rest are carried out in a similar fashion and left to the reader. For every such U, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \longrightarrow & \mathcal{A}(U) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \prod_{x\in U}\mathcal{O}_{X,x} & \longrightarrow & \prod_{x\in U}\mathcal{A}_x \end{array},$$

where φ and ψ are faithfully flat by the previous Lemma, and thus, the result follows, since surjectivity and exactness, hence flatness, are compatible with direct products. For the final statement, note that if $x \in U$ with U affine, we have $\mathcal{A}_x \simeq \mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X,x}$ and we conclude by the stability under base change of flat and faithfully flat morphisms. □

Theorem 6.3. *If* $f\colon X \to Y$ *is a qc-iso,* $(f^*, f_*)\colon \mathbf{Qcoh}^{\mathrm{alg}}(X) \xrightarrow{\sim} \mathbf{Qcoh}^{\mathrm{alg}}(Y)$ *gives an adjoint equivalence of cosites* $f_*\colon X^{\mathrm{fppf}}_{\mathbf{Qcoh}} \xrightarrow{\sim} Y^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ *with adjoint* f^*.

Proof. We know that (f^*, f_*) is an adjoint equivalence of categories. Furthermore, these functors preserve coverings: if $\mathcal{A} \in X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ and $\mathcal{A} \to \mathcal{B}$ is a covering (finite and faithfully flat), since qc-isomorphisms are affine, $f_*\mathcal{A} \to f_*\mathcal{B}$ is a covering of $f_*\mathcal{A}$ on $Y^{\mathrm{fppf}}_{\mathbf{Qcoh}}$ by Lemma 6.2. Conversely, let $\mathcal{A} \to \mathcal{B}$ be a covering in $Y^{\mathrm{fppf}}_{\mathbf{Qcoh}}$. Since $f^*\mathcal{A} \simeq f^{-1}\mathcal{A}$ and $f^*\mathcal{B} \simeq f^{-1}\mathcal{B}$ due to the qc-isomorphism assumption, $f^*\mathcal{A} \to f^*\mathcal{B}$ is a covering in $X^{\mathrm{fppf}}_{\mathbf{Qcoh}}$. □

Corollary 6.4. *If* $f : X \to Y$ *is a qc-iso,* $(f^*, f_*)\colon \mathbf{Qcoh}^{\mathrm{fet}}(X) \xrightarrow{\sim} \mathbf{Qcoh}^{\mathrm{fet}}(Y)$.

Proof. Apply Theorem 6.3 and Proposition 5.13; or directly Lemma 6.2. □

7. Fiber Functors and Main Result

Let X be schematic and $\overline{x} \in X^{\bullet}(\Omega)$ a geometric point. We define the *fiber functor of* X *at* $\overline{x}$ as follows, via a product in $\mathbf{SchFin}^{\mathrm{pw}}$:

$$\begin{aligned} \mathrm{Fib}_{\overline{x}}\colon \mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}} &\to \mathbf{Set}_f \\ \mathcal{A} &\to |\mathbf{pw}((\star, \Omega) \times_X (X, \mathcal{A}))|. \end{aligned}$$

This functor coincides with the one coming from schemes.

Proposition 7.1. *One has the equality* $\mathrm{Fib}_{\overline{x}}(\mathcal{A}) = |\mathrm{Spec}(\Omega \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x)|$.

Proof. By étaleness, $(\star, \Omega) \times_X (X, \mathcal{A}) = (\star, \Omega \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x) = (\star, \prod_I \Omega)$, I finite. Now, $|\mathbf{pw}(\star, \prod_I \Omega)| = |\coprod_I (\star, \Omega)| = |\mathrm{Spec}(\prod_I \Omega)| = |\mathrm{Spec}(\Omega \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x)|$. □

Remark 7.1. Notice that $|-|\colon \mathbf{SchFin}^{\mathrm{pw}} \to \mathbf{Set}$ does not factor through $\mathbf{SchFin}^{\mathrm{pw}}_{\mathrm{qc}}$. A related and interesting open question is the following *rigidification* problem: is there a subcategory $\mathfrak{C} \subseteq \mathbf{SchFin}^{\mathrm{pw}}$ such that the restriction of $|-|$ to $\mathfrak{C}$ factors through the localization by qc-isomorphisms?

Let us see now that $(\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}, \mathrm{Fib}_{\overline{x}})$ is a Galois category. First, we recall the general definition, whose main properties are compiled in [2,5].

Definition 7.2. (*Galois Category*) Let $\mathfrak{C}$ be a category and $F : \mathfrak{C} \to \mathbf{Set}_f$ a covariant functor to the category of finite sets. We say that $(\mathfrak{C}, F)$ is a *Galois category* with fundamental functor F if:

1. $\mathfrak{C}$ has a terminal object and finite fibered products.
2. $\mathfrak{C}$ has finite sums, in particular an initial object, and the quotients by a finite group of automorphisms exist for every object of $\mathfrak{C}$.
3. Any morphism u in $\mathfrak{C}$ can be written as $u = u' \circ u''$ where u'' is an epimorphism and u' a monomorphism. Additionally, any monomorphism $u : X \to Y$ in $\mathfrak{C}$ is an isomorphism of X with a direct summand of Y.
4. F preserves terminal objects and epimorphisms, and commutes with fibered products, finite sums, and quotients by finite groups of automorphisms.
5. F is conservative.

Theorem 7.3. *If X is a connected schematic space and $\overline{x} \in X^\bullet(\Omega)$ is a geometric point, then $(\mathbf{Qcoh}^{\text{fet}}(X)^{\text{op}}, \text{Fib}_{\overline{x}})$ is a Galois Category. Furthermore, if $\pi\colon S \to X$ is a finite model and $\overline{s} \in S^\bullet(\Omega)$ is the corresponding geometric point (Proposition 4.8), then we have an isomorphism of profinite groups*

$$\pi_1^{\text{et}}(S, \overline{s}) \simeq \pi_1^{\text{et}}(X, \overline{x}) \tag{7.1}$$

where $\pi_1^{\text{et}}(X, \overline{x}) := \text{Aut}_{[\mathbf{Qcoh}^{\text{fet}}(X)^{\text{op}}, \mathbf{Set}_f]}(\text{Fib}_{\overline{x}})$.

Proof. We can assume that X is well-connected due to Theorems 3.6, 4.10 and Corollary 6.4. For such an X, we have a well-defined notion of degree of $\mathcal{A} \in \mathbf{Qcoh}^{\text{fet}}(X)$ (identified with an integer). Sects. 7.1 and 7.2 below prove that, indeed, the axioms of Definition 7.2 are satisfied in this case. The good behaviour of quasi-coherent sheaves on schematic spaces allows us to reduce the proof to one at stalks, where we (essentially) invoke a contravariant version of the usual ideas for scheme theory shown in [2,5] or [10]. As such, we just give the general idea and suitable remarks. □

7.1. Verification of the Axioms on the Category

Proposition 7.4. *The category $\mathbf{Qcoh}^{\text{fet}}(X)$ has an initial object, finite direct sums and products, and contravariant quotients by finite subgroups of automorphisms; thus, $\mathbf{Qcoh}^{\text{fet}}(X)^{\text{op}}$ verifies axioms 1 and 2 of Definition 7.2.*

Proof. The initial object is $\mathcal{O}_X$. The direct sum is the tensor product of algebras. Finite direct products are direct products of algebras. To construct quotients, one may assume that X is pw-connected and consider a subgroup $G \subseteq \text{Aut}_{\mathcal{O}_X}(\mathcal{A}) \subseteq \prod_{x \in X} \text{Aut}_{\mathcal{O}_{X,x}} \mathcal{A}_x$. Then, one defines

$$Q(\mathcal{A}, G) : \mathbf{Qcoh}^{\text{fet}}(X)^{\text{op}} \to \mathbf{Set}$$
$$\mathcal{B} \longmapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{A})^G,$$

where $\text{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{A})^G := \{f \in \text{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{A}) : \phi \circ f = f \forall \phi \in G\}$ is the subset of G-invariant morphisms. Up to isomorphism, the categorical quotient is a representing object for this functor. In this case, the sheaf $\mathcal{A}^G$ of invariant elements defined via $\mathcal{A}_x^G := \{a \in \mathcal{A}_x : \phi_x(a) = a \text{ for every } \phi \in G\} \subseteq \mathcal{A}_x$ for every $x \in X$; where $\phi_x\colon \mathcal{A}_x \to \mathcal{A}_x$ is the stalk of ϕ at x. The proof of étaleness is carried out via local triviality (Proposition 5.13), because the action of G translates to a permutation of the fibers. ☐

For the third axiom, recall the following standard results:

Lemma 7.5. *Let A be a ring and let C be a clopen subset of* $\mathrm{Spec}(A)$*, such that $I = I(C)$. Then, for every $\mathfrak{p} \in C$, $(A/I)_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$.*

Lemma 7.6. *If $f\colon A \to B$ is étale,* $\mathrm{Supp}(B)$ *(as an A-module) is clopen.*

We say that a morphism of sheaves of rings is *injective* (resp. *surjective*) if it is injective (resp. surjective) at stalks. It is clear that they are monomorphisms (resp. epimorphisms) in $\mathbf{Qcoh}^{\mathrm{alg}}(X)$, so:

Lemma 7.7. *Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$. If f is injective (resp. surjective), then it is a monomorphism (resp. an epimorphism).*

If $f\colon \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves of rings on X, then for every $x \in X$, one has $\ker(f)_x = \mathrm{Ann}(\mathcal{B}_x)$ (where $\mathrm{Ann}(\mathcal{B}_x)$ is the annihilator ideal of the $\mathcal{A}_x$-module $\mathcal{B}_x$, i.e., $\mathrm{Ann}(\mathcal{B}_x) = I(\mathrm{Supp}(\mathcal{B}_x))$, because $\mathcal{B}_x$ is a finite $\mathcal{A}_x$-module. Finally, if $\mathcal{A}$ and $\mathcal{B}$ are quasi-coherent, $\ker(f)$ is quasi-coherent.

Proposition 7.8. *Any $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ can be written as $f = h \circ g$ with g an epimorphism and h a monomorphism.*

Proof. We have the ordinary factorization $\mathcal{A} \xrightarrow{g} \mathcal{A}/\ker(f) \xrightarrow{h} \mathcal{B}$. Since $\ker(f)$ is quasi-coherent, $\mathcal{A}/\ker(f)$ is a quasi-coherent algebra, g is an epimorphism (because it is a quotient map), and h is injective at stalks by the first isomorphism Theorem, thus a monomorphism by Lemma 7.7. Étaleness is checked at stalks via Lemmas 7.5 and 7.6. ☐

Next, we characterize monomorphisms and epimorphisms in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$.

Lemma 7.9. *A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ is an epimorphism if and only if for every $x \in X$, $f_x : \mathcal{A}_x \to \mathcal{B}_x$ is an epimorphism in the category of (flat, finite) étale algebras over $\mathcal{O}_{X,x}$.*

Proof. If all the local rings verify the condition, it is clear that the global morphism also verifies it. For the converse, it suffices to see that for any $x \in X$ and any morphism $g : \mathcal{B}_x \to R$ (with R finite, flat, and étale over $\mathcal{O}_{X,x}$), there exists a morphism of sheaves $f : \mathcal{B} \to \mathcal{R}$, such that $\mathcal{R}_x = R$, $f_x = g$ and $\mathcal{R} \in \mathbf{Qcoh}^{\mathrm{fet}}(X)$. Every R determines a sheaf of algebras $\mathcal{R}^x$ on U_x defined as $\mathcal{R}^x_y{:=}R \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y}$, with a natural morphism $i^*\mathcal{B} \to \mathcal{R}^x$ (where $i : U_x \hookrightarrow X$ is the inclusion). By the «extension theorem»[7, Theorem 4.4], we obtain $i_*i^*\mathcal{B} \to \mathcal{R}{:=}i_*\mathcal{R}^x$; which composed with $\mathcal{B} \to i_*i^*\mathcal{B}$ gives us the desired map. Moreover, $\mathcal{R} \in \mathbf{Qcoh}^{\mathrm{fet}}(X)$, because, for $y \geq x$, $\mathcal{R}_y = R \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y}$, for any other y, $\mathcal{R}_y = \mathcal{R}^x(U_x \cap U_y) = R \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(U_x \cap U_y)$; all the conditions are stable under base change. ☐

From this Lemma and the classical theory of étale $\mathcal{O}_{X,x}$-algebras, we get the following:

Proposition 7.10. *A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ is an epimorphism iff f_x is surjective for every $x \in X$, i.e., iff f is surjective.*

Proposition 7.11. *A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ is a monomorphism iff f_x is injective for every $x \in X$, i.e., iff f is injective.*

Proof. One direction is Lemma 7.7. Conversely, since $\mathrm{Supp}(\mathcal{B}_x)$ (as an $\mathcal{A}_x$-module) is open and closed, then for every $x \in X$, $C_x = \mathrm{Spec}(\mathcal{A}_x) \setminus \mathrm{Spec}(\mathcal{B}_x)$ defines an ideal $I(C_x) \subseteq \mathcal{A}_x$. These define a quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{A}$, such that $\mathcal{I}_x = I(C_x)$ and $\mathcal{A} \simeq \mathcal{A}/\ker(f) \times \mathcal{A}/\mathcal{I}$. Denoting $\mathcal{A}_0 = \mathcal{A}/\ker(f)$ and $\mathcal{A}_1 = \mathcal{A}/\mathcal{I}$, we use the fact that f is monic in

$$\mathcal{A}_0 \times \mathcal{A}_0 \times \mathcal{A}_1 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathcal{A}_0 \times \mathcal{A}_1 \simeq \mathcal{A} \xrightarrow{f} \mathcal{B}$$

to conclude that $\ker(f) = 0$. □

Proposition 7.12. *Any epimorphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ induces $\mathcal{C} \xrightarrow{\sim} \mathcal{B}$, such that $\mathcal{A} \simeq \mathcal{C} \times \mathcal{D}$.*

Proof. It is the decomposition of Proposition 7.11 with $\mathcal{C} = \mathcal{A}_0$, $\mathcal{D} = \mathcal{A}_1$. □

Corollary 7.13. *The category $\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}$ satisfies axiom 3 of Definition 7.2.*

7.2. Verification of the Axioms on the Category and Functor

Proposition 7.14. *The pair $(\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}, \mathrm{Fib}_{\overline{x}})$ verifies axiom 4 of Definition 7.2.*

Proof. All the conditions are straightforward (note that $\mathrm{Fib}_{\overline{x}}$ has the same expression as the fiber functor for schemes, where all these properties hold). For the quotient, note that $\mathrm{Spec}(\mathcal{A})/G$ is $\mathrm{Spec}(\mathcal{A}^G)$. □

Lemma 7.15. *Let X be well-connected and let $f : \mathcal{A} \to \mathcal{B}$ be injective in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$. If $\deg(\mathcal{A}) = \deg(\mathcal{B})$ as $\mathcal{O}_X$-modules, then f is an isomorphism.*

Proof. Take non-zero finite and locally free $\mathcal{O}_X$-modules $\mathcal{C}$ and $\mathcal{D}$ trivializing $\mathcal{A}$ and $\mathcal{B}$ as in Proposition 5.13. Their tensor product as $\mathcal{O}_X$-modules, denoted $\mathcal{G}$, is a covering (i.e., a faithfully flat $\mathcal{O}_X$-module) trivializing both modules simultaneously. Lemma 5.14 yields that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{G}$, which is injective due to faithful flatness, is an isomorphism. We win by faithful flatness. □

Corollary 7.16. *If X is well-connected, $(\mathbf{Qcoh}^{\mathrm{fet}}(X)^{\mathrm{op}}, \mathrm{Fib}_{\overline{x}})$ verifies axiom 5 of Definition 7.2.*

Proof. Consider $u : \mathcal{A} \to \mathcal{B}$, such that $\mathrm{Fib}_{\overline{x}}(u)$ is an isomorphism. We have $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1 \to \mathcal{A}_1 \to \mathcal{B}$, with the first morphism surjective and the second one injective. Since $\mathrm{Fib}_{\overline{x}}$ sends direct products in $\mathbf{Qcoh}^{\mathrm{fet}}(X)$ to disjoint unions, the hypothesis implies that $\mathrm{Fib}_{\overline{x}}(\mathcal{A}_1) \to \mathrm{Fib}_{\overline{x}}(\mathcal{A}_0) \amalg \mathrm{Fib}_{\overline{x}}(\mathcal{A}_1)$ is surjective; hence, $\mathrm{Fib}_{\overline{x}}(\mathcal{A}_0) = \emptyset$. Since the degree of $\mathcal{A}$ is constant, $\mathcal{A}_0 = 0$ and, in particular, $u : \mathcal{A} = \mathcal{A}_1 \to \mathcal{B}$ is injective. Now, since $\mathrm{Fib}_{\overline{x}}(u)$ is an isomorphism, $\mathrm{Fib}_{\overline{x}}(\mathcal{A})$ and $\mathrm{Fib}_{\overline{x}}(\mathcal{B})$ have the same number of elements, i.e., $\deg(\mathcal{A}) = \deg(\mathcal{B})$ as $\mathcal{O}_X$-modules. We conclude by Lemma 7.15. □

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