An N-Soft Set Approach to Rough Sets

José Carlos R. Alcantud, Feng Feng, and Ronald R. Yager

***Abstract*—The philosophy of soft sets is founded on the fundamental idea of parameterization, while Pawlak's rough sets put more emphasis on the importance of granulation. As a multi-valued extension of soft sets, the newly emerging concept called N-soft sets can provide a finer granular structure with higher distinguishable power. This study offers a fresh insight into rough set theory from the perspective of N-soft sets. We reveal a close connection between N-soft sets and rough structures of various types. First, we show how the corresponding structures of Pawlak's rough sets, tolerance rough sets and multigranulation rough sets can be derived from a given N-soft set. Conversely, we investigate the representation of these distinct rough structures using the corresponding notions derived from suitable N-soft sets. The applicability of these theoretical results is highlighted with a case study using real data regarding hotel rating. The established two-way correspondences between N-soft sets and diverse rough structures are constructive, which can bridge the gap between seemingly disconnected disciplines, and hopefully nourish the development of both rough sets and soft sets.**

***Index Terms*—Rough set, soft set, N-soft set, tolerance relation, tolerance rough set, multigranulation rough set.**

I. Introduction

ROUGH set theory has come a long way since Pawlak published his pioneering work [1] in 1982. The indiscernibility relation, which is an equivalence relation generated from the collected data, serves as the basis of Pawlak's rough sets. Objects taking identical values are indiscernible in view of the available information about them. In fact, this reveals a fundamental fact that human knowledge and conceptualization of objects is based on granulation of the universe of discourse. In the literature, some developments of the classical rough set model have been made by virtue of various granulation structures, which are induced by customary extensions of the indiscernibility relation. For instance, instead of using an equivalence relation as in [1], a similarity relation [2], a tolerance relation [3], or a general binary relation [4]–[6] can serve for the original purpose of granulation. Furthermore, a multiplicity of binary relations or partitions can do this work even better, which gives rise to multigranulation rough sets [7], [8].

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J. C. R. Alcantud is with BORDA Research Unit and Multidisciplinary Institute of Enterprise (IME), University of Salamanca, 37007 Salamanca, Spain (e-mail: jcr@usal.es).

Feng Feng is with the School of Science, and Shaanxi Key Laboratory of Network Data Analysis and Intelligent Processing, Xi'an University of Posts and Telecommunications, Xi'an 710121, China (e-mail: fengnix@hotmail.com).

Ronald R. Yager is with the Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA (e-mail: yager@panix.com).

Later on, several more challenging extensions of Pawlak's rough sets have been put forward. The concept of covering based rough sets, an important type of generalized rough sets, relies on the granulation structure provided by coverings of the universe of discourse [9]–[16]. The variable precision rough set model [17] permits to represent and address classification problems with imprecise or uncertain information. The emergence of soft set theory [18] further boosted the development of rough sets. Using a soft set as the underlying granulation structure, Feng et al. [19] initiated soft approximation spaces and soft rough sets. A preliminary investigation regarding the combination of soft sets, fuzzy sets and rough sets was first reported in [20]. More complicated hybridizations such as multigranulation fuzzy rough sets [21] and other generalizations of fuzzy rough sets [22], rough soft sets [23], [24], multigranulation covering rough sets [25], or fuzzy covering-based rough set models [26] have been proposed, promoting both theories of rough sets and fuzzy sets. Some researchers explored the fundamental relationships among existing extensions of rough sets. For example, Zhu [27] ascertain a connection between covering based rough sets and generalized rough sets based on a binary relation, whereas D'eer and Cornelis [28] are concerned with interrelationships among a wide variety of fuzzy covering-based rough set models.

The present study belongs to a different strand of literature that explores the connections between rough sets and soft sets, two different mathematical theories for modeling uncertainty. As mentioned above, rough set approach deals with uncertainly from the perspective of granularity, which uses lower and upper approximations to capture the uncertainty caused by indiscernibility. In contrast, soft set theory is built up on the basis of parameterization. It relies on set-valued approximate functions to model uncertain concepts by jointly considering a variety of different aspects expressed by parameters. Each parameter can only give an approximate or partial description of the entire concept. A simple illustration of the difference between these two theories can be found in photography. In a nutshell, rough set theory concerns the size of pixels (i.e., granulation of the universe of objects), whereas soft set theory emphasizes the camera angle from which a picture is taken (i.e., parameterized description of the universe). Our study does not aim at extending or hybridizing some of the existing models as well done in [20], [25], [29], [30]. In contrast, we endeavor to give a fresh insight into rough set theory from the perspective of N-soft sets [31]. This idea is motivated in spirit by Zhu's inspiring work [27]. Note also that relationships among several basic concepts in the theory of covering based rough sets were examined in [32]. A more distant reference is Bustince and Burillo's important work [33], which shows that vague sets can be identified with intuitionistic fuzzy sets.

The novelty of all these influential studies does not lie in their direct real world applications, which the authors disregard in their papers. Instead, these theoretical works provide novel or deeper insight into the internal configuration of existing models in the literature.

Similarly, our paper concentrates on issues that are at the core of theoretical foundations of various models although we demonstrate them with real-life data. To the best of our knowledge, there are very limited research works investigating explicit connections between rough sets and soft sets, except for a few earlier studies [19], [20], [34], [35]. This might be due to the fact that the rationale of soft sets is founded on binary divisions of the universe of discourse derived from all relevant parameters, which falls short of the differentiation ability required by rough sets. Precisely because many social and decisional contexts require finer distinctions too [36]–[39], Fatimah et al. [31] put forward N-soft sets. The distinguishable power provided by this model is exactly what is needed in order to guarantee a correspondence between Pawlak's spaces, tolerance rough structures and even multigranulation rough structures, with comparable notions derived from the theory of N-soft sets. Our results mean a crossroads in the literature that paves the way to another innovative view on rough set theory.

We organize our research as follows. Section II recalls some basic terminologies. Section III investigates one direction of the interaction between rough structures and N-soft sets, namely, how N-soft sets produce rough structures. Section IV explains the relationship in the converse direction and therefore, how N-soft sets can act as a representation of any given rough structure. The techniques of proof are constructive and both cases are illustrated with examples. At the end of Section IV we summarize our main findings and provide a list of examples that help us understand their operations in various scenarios. Section V applies the new concepts in Sections III and IV to a real case with data from characteristics of lodging facilities. Section VI concludes the main contributions of this study. It also points out the limitations of our work and indicates some possible directions for future research.

II. Preliminaries

This section reviews some terminologies and results related to soft sets and rough sets. Along this article, U denotes a nonempty finite set of options (namely the universe of discourse), and E_U (or simply E) denotes the set of all parameters (associated with options in U), called the parameter space. The pair (U, E) is also known as a soft universe. By $\sim B$ we mean the complement set of B in U. For any set U, $\mathscr{P}(U)$ denotes its power set and $|U|$ denotes the cardinality of U. When ρ is an equivalence relation on U, the equivalence class of $u \in U$ under ρ is denoted by $[u]_\rho$, and U/ρ is the quotient set of U by ρ.

Definition II.1. [18] *A pair $S = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping, called the approximate function of the soft set S.*

Definition II.2. [20] *Let $S = (F, A)$ be a soft set over U. If $\bigcup_{a\in A} F(a) = U$, then S is said to be a full soft set.*

Definition II.3. [20] *A soft set $S = (F, A)$ over U is called a partition soft set if $\{F(a) : a \in A\}$ forms a partition of U.*

In what follows, the collection of all soft sets over U with parameter sets from E is denoted by $\mathscr{S}^E(U)$. Moreover, we denote by $\mathscr{S}_A(U)$ the collection of all soft sets over U with a fixed parameter set $A \subseteq E$. For more details regarding soft sets, their extensions and related applications, we refer to [34], [40]–[48].

The parameterized binary divisions of the universe of discourse provided by a soft set are insufficient in some daily situations. Here are a few examples:

i) Real evidences that appear in [31] include online mobile comparators, evaluations of manuscripts submitted to a journal (namely, the American Sociological Review), or rankings of movies by cinema critics.

ii) It is easy to find similar utilizations of real data that demand a finer parameterized division of the universe of discourse.
Figure 1 is a screen capture from the website Metacritic, which shows real reviews for a film (both from professional critics and users). Figure 2 shows how the users of that website can give approximate descriptions of the films as well as written reviews. In both cases, the parameterization requires more than a binary distinction.
Figure 3 is a screen capture from the Amazon website, which shows the opinions of a set of customers on a given product. The data are retrieved by asking the users to give an approximate description of the products that they have purchased, which must belong to a set of five grades as shown in Figure 4.

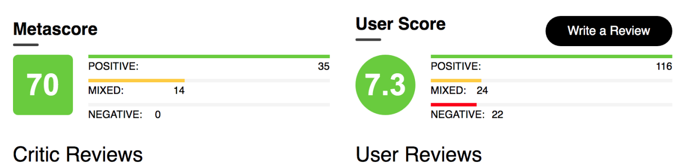

Fig. 1. Screen capture of data on a film at the website Metacritic (https://www.metacritic.com/).

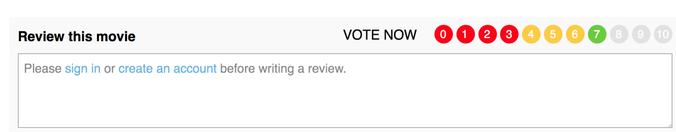

Fig. 2. Screen capture of data requested from the users at the website Metacritic (https://www.metacritic.com/).

In order to investigate data with the structure of these examples, Fatimah et al. [31] define a more powerful tool as follows:

Definition II.4. [31] *Let $G_N = \{0, 1, \ldots, N-1\}$ be a set of ordered grades where $N > 1$ is a natural number. We say that (F, A, N) is an N-soft set on U if $F : A \to 2^{U\times G_N}$ satisfies the condition that for each $a \in A$ and $u \in U$ there exists a unique $r \in G_N$ such that $(u, r) \in F(a)$.*

Fig. 3. Screen capture of reviews of a product from the customers of Amazon (https://www.amazon.com/).

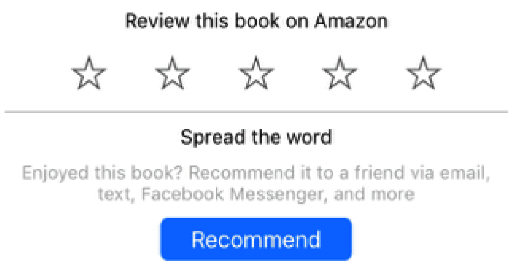

Fig. 4. Screen capture of data requested from the readers of a book at the Amazon website (https://www.amazon.com/).

Note that N-soft sets extend soft sets in a natural way, since the parameterization of the universe of options is N-ary. An N-soft set reduces to a soft set if $N = 2$. Relatedly, although we give a mathematically unequivocal formulation that avoids imprecisions, the set of grades can be made of abstract symbols as in the examples in this section (see also Section V below).

This concept is less obviously related to other models:

$i)$ N-soft sets can be regarded as the soft set-theoretic counterpart of multi-valued information systems (also known as knowledge representation systems). Thus the condition in Definition II.4 is logically equivalent to the requirement that F is a mapping (called information function) from $A \times U$ to G_N. We return to this issue in Section V.

$ii)$ From the perspective of L-fuzzy sets, N-soft sets whose grades lie in $G_N = \{0, 1, \dots, N-1\}$ can be viewed as a special case of L-fuzzy soft sets with the valuation lattice G_N, which is indeed a chain with the natural order.

In addition, multi-valued parameterizations are naturally compatible with hesitancy. Their combination produces the notion of hesitant N-soft sets [49]. Hybridization with hesitancy is not truly beneficial in the case of soft sets, for which hesitancy is equivalent to incompleteness or full uncertainty [43]. Note also that fuzzy N-soft sets were investigated in [50].

In contrast to the above case, rough set theory investigates vagueness from a different perspective. It does not aim at producing approximate descriptions based on parameterization of the universe of discourse as in the case of soft sets and related extensions. In fact, rough sets put more emphasis on granulation of the universe of discourse, which is typically caused by indiscernibility. In Pawlak's classical rough set model, an approximation space consists of the universe of discourse U and an equivalence relation on it. Using the granulation structure derived from the given approximation space, an arbitrary subset X of U can be described in terms of two definable subsets, called lower and upper approximations of X. More specifically, the formal definition of Pawlak's rough sets is given in the following way:

Definition II.5. [1] *Let ρ be an equivalence relation on U. The ordered pair (U, ρ) defines a Pawlak approximation space. When $X \subseteq U$ we define $\underline{\rho}(X) = \{u \in U | [u]_\rho \subseteq X\}$, the lower approximation of X, and $\overline{\rho}(X) = \{u \in U | [u]_\rho \cap X \neq \emptyset\}$, the upper approximation of X.*

Then $X \subseteq U$ is said to be definable if $\underline{\rho}(X) = \overline{\rho}(X)$, otherwise it is called a rough set.

Remark II.6. *Rough sets are often described from the previous concept of an information system, that defines an indiscernibility relation I_B associated with any subset B of the attributes sets. From the equivalence relation I_B, the B-approximations of all subsets of the universe of discourse are derived. This additional ingredient of the model is explained for example, in [51], [52], and from a practical perspective we consider this issue in the case study of Section V.*

The rough set approach is notable in the granular computing realm due to its feasibility and flexibility. When the universe of discourse has an algebraic structure, we can resort to strengthened forms of equivalence relations. For example, (full) congruence relations improve their performance when U is a lattice [53], [54]. But one can also generalize Definition II.5 easily. In the following, we proceed to list some remarkable extensions of Pawlak's rough sets, which have further boosted the application of Pawlak's original idea to various scenarios.

Tolerance relations are binary relations which are reflexive and symmetric, but not necessarily transitive. They have been used to define generalized approximations spaces and tolerance relation based rough sets since [29]. Their original motivation comes from incomplete symbolic information systems in which attribute values may be either unknown or missing. The need for avoiding transitivity has a long tradition in other disciplines like behavioral decision making [55]–[57]. Generalized approximation spaces can be defined from tolerance relations by the expressions given in Definition II.5.

Multigranularity stands out among the most valuable characteristics that have been added to Pawlak's original concept. Qian et al. [7] developed this successful model in order to account for the cases where a variety of sources furnish granulations. Their extension is twofold: an optimistic and a pessimistic model reduce to the standard Pawlak's model when there is one source of granularity. We recall them in Definitions II.7 and II.9, respectively.

Definition II.7. [7], [8] *Let $\{\hat{P}_1, \dots, \hat{P}_m\}$ be a collection of partitions of U. For each subset X of U we define*

$\underline{X}^O_{\sum \hat{P}_i} = \cup\{P \subseteq X | P \in \hat{P}_i \text{ for some } i\}$,

$\overline{X}^O_{\sum \hat{P}_i} = \sim (\underline{\sim X})^O_{\sum \hat{P}_i}$.

Then $\underline{X}^O_{\sum \hat{P}_i}$ *and* $\overline{X}^O_{\sum \hat{P}_i}$ *are called the optimistic multi-lower approximation and the optimistic multi-upper approximation of* X*, respectively.*

When $\underline{X}^O_{\sum \hat{P}_i} = X$ *(resp.* $\overline{X}^O_{\sum \hat{P}_i} = X$*), we say that* X *is lower definable, (resp., upper definable), in the optimistic multigranulation rough set model.*

Finally, X *is a definable set in the optimistic multigranulation rough set model when* X *is both lower definable and upper definable in the optimistic multigranulation rough set model.*

By inspiration of Definition II.5, Definition II.7 can be rewritten by way of a collection of equivalence relations as well. In order to do that, we just need to redefine optimistic multi-lower and multi-upper approximations in the following terms:

Definition II.8. *Let* $\{\rho_1, \ldots, \rho_m\}$ *be a collection of equivalence relations over* U*. For each subset* X *of* U *we define*

$\underline{X}^O_{\sum \rho_i} = \cup\{[u]_{\rho_i} \,|\, [u]_{\rho_i} \subseteq X \text{ for some } i\}$,

$\overline{X}^O_{\sum \rho_i} = \sim (\underline{\sim X})^O_{\sum \rho_i}$.

Definition II.9. [7], [8] *Let* $\{\hat{P}_1, \ldots, \hat{P}_m\}$ *be a collection of partitions of* U*. For each subset* X *of* U *we define*

$\underline{X}^P_{\sum \hat{P}_i} = \cup\{P \subseteq X | P \in \hat{P}_i \text{ for every } i\}$,

$\overline{X}^P_{\sum \hat{P}_i} = \sim (\underline{\sim X})^P_{\sum \hat{P}_i}$

Then $\underline{X}^P_{\sum \hat{P}_i}$ *and* $\overline{X}^P_{\sum \hat{P}_i}$ *are called the pessimistic multi-lower approximation and the pessimistic multi-upper approximation of* X*, respectively.*

When $\underline{X}^P_{\sum \hat{P}_i} = X$ *(resp.* $\overline{X}^P_{\sum \hat{P}_i} = X$*), we say that* X *is lower definable, (resp., upper definable), in the pessimistic multigranulation rough set model.*

Finally, X *is a definable set in the pessimistic multigranulation rough set model when* X *is both lower definable and upper definable in the pessimistic multigranulation rough set model.*

Definition II.9 can also be rewritten if we define pessimistic multi-lower and multi-upper approximations in terms of a collection of equivalence relations as follows:

Definition II.10. *Let* $\{\rho_1, \ldots, \rho_m\}$ *be a collection of equivalence relations over* U*. For each subset* X *of* U *we define*

$\underline{X}^P_{\sum \rho_i} = \cup\{[u]_{\rho_i} \,|\, [u]_{\rho_i} \subseteq X \text{ for every } i\}$,

$\overline{X}^P_{\sum \rho_i} = \sim (\underline{\sim X})^P_{\sum \rho_i}$.

Soft sets are already linked to rough structures by concepts like the next two definitions:

Definition II.11. [19], [20] *Let* $S = (f, A)$ *be a soft set over* U*. Then the pair* $P = (U, S)$ *is called a soft approximation space, which induces the following two operations:*

$$\underline{apr}_P(B) = \{u \in U \mid \exists e \in A\,(u \in f(e) \subseteq B)\},$$

$$\overline{apr}_P(B) = \{u \in U \mid \exists e \in A\,(u \in f(e), f(e) \cap B \neq \varnothing)\},$$

where B *is a subset of* U*. We refer to* $\underline{apr}_P(B)$ *and* $\overline{apr}_P(B)$ *as the lower and upper soft rough approximations of* B *in* P*, respectively. If* $\underline{apr}_P(B) = \overline{apr}_P(B)$*, the set* B *is said to be soft definable; otherwise it is called a Feng-soft rough set (briefly, FSRS).*

Definition II.12. [58] *Let* $\mathfrak{S} = (F, A) \in \mathscr{S}^E(U)$*. Define the mapping* $\eta : U \to \mathscr{P}(A)$ *as*

$$\eta(y) = \{e \in A \mid y \in F(e)\}$$

for all $y \in U$*. For any* $B \subseteq U$*, we refer to*

$$\underline{B}_\eta = \{x \in B \mid \eta(x') \neq \eta(x) \text{ for all } x' \in \sim B\},$$

$$\overline{B}_\eta = \{x \in U \mid \eta(x') = \eta(x) \text{ for some } x' \in B\}$$

as the lower and upper modified soft rough approximations of B*, respectively. If* $\underline{B}_\eta = \overline{B}_\eta$*,* B *is said to be modified soft-definable; otherwise it is called a modified soft rough set (briefly, MSRS).*

III. Rough structures induced by N-soft sets

The links that we show in this section will be better understood if we use N-soft sets in their tabular form. The reader is referred to [31], [50] for details about the intuitive process that transforms N-soft sets into tables and vice versa. In order to make this paper self-contained, we refer to the following example concerning faculty appointments.

Example III.1. *Faculty appointments to senior positions in research institutions involve careful evaluations and decision making. Candidates can be judged by various attributes such as "research productivity" (*a *for brevity), "managerial skills" (*b *for brevity) and "academic leadership qualities" (*c *for brevity).*

Suppose that $O = \{o_1, o_2, o_3, o_4, o_5\}$ *are the final candidates who apply to a senior faculty position at X University. Among the set of attributes* P *or "evaluation of candidates by selection panel", a subset* $A \subseteq P$ *consisting of* a, b, c *described above determines the choice. As a result, the recruiting panel produces a* 5*-soft set* $S = (F, A, 5)$ *with the graded parameterized description of the candidates as shown in Table I.*

TABLE I
THE 5-SOFT SET IN SECTION III

$(F, A, 5)$	a	b	c
o_1	2	2	2
o_2	1	2	3
o_3	3	3	3
o_4	4	1	2
o_5	1	2	3

This description can be easily matched with Definition II.4. For example, we interpret the elements in the first line as $(o_1, 2) \in F(a)$*,* $(o_1, 2) \in F(b)$ *and* $(o_1, 2) \in F(c)$*. Similarly,* $(o_2, 1) \in F(a)$ *and so forth.*

Now we are ready to define standard rough structures as well as multigranulation rough structures associated with any N-soft set.

A. Pawlak's rough structures induced by N-soft sets

Suppose that (F, A, N) is an N-soft set on U. Each attribute a in the set of parameters A induces an equivalence relation $\sim^a$ by

$$u \sim^a v \Leftrightarrow \exists r \in G_N \text{ such that } \{(u,r),(v,r)\} \subseteq F(a) \quad (1)$$

Therefore, a induces a Pawlak approximation space $(U, \sim^a)$ as in Definition II.5. In intuitive terms, we can derive this equivalence relation (or the partition that it induces) from the column corresponding to a in the tabular representation: two alternatives are related (or belong to the same equivalence class in the partition) if and only if their grade under that column is the same. For illustration, let us reconsider the running Example III.1.

Example III.2. *In the situation of Example III.1,*

- *attribute a induces the equivalence relation $\sim^a$ whose equivalence classes form the partition* $\hat{P}_1 = \{\{o_1\},\{o_3\},\{o_4\},\{o_2,o_5\}\}$,
- *attribute b induces the equivalence relation $\sim^b$ whose equivalence classes form the partition* $\hat{P}_2 = \{\{o_3\},\{o_4\},\{o_1,o_2,o_5\}\}$, *and*
- *attribute c induces the equivalence relation $\sim^c$ whose equivalence classes form the partition* $\hat{P}_3 = \{\{o_1,o_4\},\{o_2,o_3,o_5\}\}$.

If we look at the joint information in (F, A, N), then we are endowed with a collection of equivalence relations (or partitions), one for each attribute. This family of relations $\{\sim^a\}_{a\in A}$ produces the equivalence relation defined as

$$u \sim v \Leftrightarrow u \sim^a v \text{ for every } a \in A \quad (2)$$

Therefore (F, A, N) itself defines a Pawlak approximation space $(U, \sim)$ in a uniquely determined fashion. We call it the *Pawlak approximation space derived from* (F, A, N). The following example illustrates this procedure.

Example III.3. *In the running Example III.1, the construction in* (2) *allows one to canonically associate a Pawlak's rough set $(O, \sim)$ with the* 5*-soft set $(F, A, 5)$ on O. The relation $\sim$ that we obtain coincides with $\sim^a$, because we know $o_2 \sim^a o_5$, $o_2 \sim^b o_5$, $o_2 \sim^c o_5$ and no other pair of options verifies this triple relationship.*

More generally, every subset B of A defines an equivalence relation I_B that we call the B-indiscernibility relation induced by (F, A, N):

$$uI_Bv \Leftrightarrow u \sim^a v \text{ for every } a \in B \quad (3)$$

With the assistance of I_B, for every subset of U we can define its *B-lower and B-upper approximation derived from* (F, A, N). Section V illustrates the computation of indiscernibilities and rough approximations derived from N-soft sets.

It is worth noting that the above ideas are also associated with the following notions initiated in [59].

Definition III.4. [59] *A soft set (σ, A) over $U \times U$ is called a soft binary relation on U.*

Definition III.5. [59] *A soft binary relation (σ, A) is called a soft equivalence relation on U if for every $a \in A$, $\sigma(a)$ is an equivalence relation on U whenever it is nonempty.*

Soft equivalence relations are helpful for the exploration of algebraic structures. For instance, Bera et al. investigated soft congruence relations over lattices in [54].

Remark III.6. *It is clear that any N-soft set $S = (F, A, N)$ on U induces a soft equivalence relation (σ_S, A) on U, with σ_S given by*

$$\sigma_S(a) = \sim^a = \{(u,v) \in U \times U \mid u \sim^a v\}, \text{ for all } a \in A.$$

For instance, the 5-soft set $S = (F, A, 5)$ in Example III.1 produces in a natural way a soft equivalence relation (σ_S, A) on the set O, with σ_S defined as

$$\sigma_S(a) = \triangle_O \cup \{(o_2,o_5),(o_5,o_2)\},$$
$$\sigma_S(b) = \triangle_O \cup \{(o_1,o_2),(o_2,o_1),(o_1,o_5),(o_5,o_1),(o_2,o_5),(o_5,o_2)\},$$
$$\sigma_S(c) = \triangle_O \cup \{(o_1,o_4),(o_4,o_1),(o_2,o_3),(o_3,o_2),(o_2,o_5),(o_5,o_2),(o_3,o_5),(o_5,o_3)\},$$

where $\triangle_O = \{(o_i,o_i)\}_{i=1}^5$ is the identity relation on O.

Conversely, we can associate an N-soft set (which is in general not unique) with each soft equivalence relation on U, and then obtain its Pawlak approximation space as shown above in this section. In conclusion, Pawlak's approximation spaces can also be constructed from each soft equivalence relation on a set.

B. Tolerance based rough structures induced by N-soft sets

When (F, A, N) is an N-soft set on U, the relation

$$u\tau v \Leftrightarrow u \sim^a v \text{ for some } a \in A \quad (4)$$

is obviously reflexive and symmetric. It may fail to be transitive, hence we can only assure that it is a tolerance relation. Every option defines a class of referent options to which it is similar [29]: for each $u \in U$, $R(u) = \{v \in U : u\tau v\} = \{v \in U : v\tau u\}$. When we use these classes the expressions in Definition II.5 produce the tolerance-based lower and upper approximations derived from (F, A, N), for the subsets of U. Consequently they define the *tolerance rough structure induced by* (F, A, N).

Example III.7. *In the situation of Example III.1, the tolerance relation defined by $S = (F, A, 5)$ is the reflexive and symmetric relation that satisfies*

$$o_1\tau o_2,\ o_1\tau o_4,\ o_1\tau o_5,\ o_2\tau o_3,\ o_2\tau o_5,\ \textit{and}\ o_3\tau o_5.$$

We can observe that it is intransitive, because $o_1\tau o_2$, $o_2\tau o_3$, but $o_1\tau o_3$ is false. Or alternatively, because $o_1\tau o_5$, $o_5\tau o_3$, but $o_1\tau o_3$ is false.

The classes of referent options to which every option is similar are

- $R(o_1) = \{o_1, o_2, o_4, o_5\}$.
- $R(o_2) = \{o_1, o_2, o_3, o_5\}$.
- $R(o_3) = \{o_2, o_3, o_5\}$.

- $R(o_4) = \{o_1, o_4\}$.
- $R(o_5) = \{o_1, o_2, o_3, o_5\}$.

Let us consider $X = \{o_2, o_3, o_5\}$*. Then its tolerance-based lower approximation derived from* S *is*

$$\{x \in U \mid R(x) \subseteq X\} = \{o_3\}$$

and its tolerance-based upper approximation derived from S *is*

$$\{x \in U \mid R(x) \cap X \neq \varnothing\} = \bigcup_{o_i \in X} R(o_i) = \{o_1, o_2, o_3, o_5\}.$$

C. Multigranular rough structures induced by N-soft sets

Section III-A explains how the N-soft set (F, A, N) induces a collection of equivalence relations (or a soft equivalence relation). This collection has a cardinality which is less than or equal to $|A|$. It is only natural to use them in order to define optimistic multi-lower and multi-upper approximations associated with (F, A, N) in terms of Definition II.8. And of course, (F, A, N) also defines pessimistic multi-lower and multi-upper approximations in terms of Definition II.10.

Definition III.8. *Let* $S = (F, A, N)$ *be an* N*-soft set on* U *which induces the soft equivalence relation* (σ_S, A) *on* U *such that* $\sigma_S(a) = \sim^a$ *for all* $a \in A$.

The optimistic multi-lower and multi-upper approximations associated with the N*-soft set* $S = (F, A, N)$ *are constructed from the soft equivalence relation* (σ_S, A) *by recourse to the corresponding expressions in Definition II.8 as follows:*

$$\underline{X}_S^O = \cup\{[u]_{\sigma_S(a)} \mid \exists a \in A\,([u]_{\sigma_S(a)} \subseteq X)\}$$

and

$$\overline{X}_S^O = \sim (\underline{\sim X})_S^O.$$

In a similar fashion, the pessimistic multi-lower and multi-upper approximations associated with $S = (F, A, N)$ *can be constructed from* (σ_S, A) *by recourse to the corresponding expressions in Definition II.10.*

Let us show how these elements operate in the following example.

Example III.9. *In the running Example III.1, let us consider* $X = \{o_2, o_3, o_5\}$ *as in Example III.7. Then* X *is definable in the optimistic multigranulation rough set model derived from* $(F, A, 5)$ *because*
$\underline{X}_S^O = \cup\{P \subseteq X | P \in \hat{P}_1 \text{ or } P \in \hat{P}_2 \text{ or } P \in \hat{P}_3\} = X$,
$O - X = \{o_1, o_4\}$ *thus* $(\underline{O - X})_S^O = \{o_1, o_4\}$*, and*
$\overline{X}_S^O = O - (\underline{O - X})_S^O = O - \{o_1, o_4\} = X$.

However X *is not definable in the pessimistic multigranulation rough set model derived from the* 5*-soft set* $(F, A, 5)$ *since* $\underline{X}_S^P = \cup\{P \subseteq X | P \in \hat{P}_1 \text{ and } P \in \hat{P}_2 \text{ and } P \in \hat{P}_3\} = \varnothing$. *In fact, there is no definable set of options in the pessimistic multigranulation rough set model derived from* $(F, A, 5)$*: no subset belongs to all the partitions derived from the* 5*-soft set* $(F, A, 5)$ *hence for all* $Y \subseteq O$, $\underline{Y}_S^P = \varnothing$ *and* $\overline{Y}_S^P = Y$.

IV. N-SOFT SETS AS REPRESENTATIONS OF ROUGH STRUCTURES

This section explores the reverse processes to the constructions in section III. We show that Pawlak's spaces, tolerance rough structures and multigranulation rough structures can be explained by corresponding notions derived from suitable N-soft sets over the universe of discourse.

It is worth noting that soft sets and binary relations are closely related as shown by the following theorem.

Theorem IV.1. [19] *Let* $S = (F, A)$ *be a soft set over* U*. Then* S *induces a binary relation* $\rho_S \subseteq A \times U$*, which is defined by*

$$(x, y) \in \rho_S \Leftrightarrow y \in F(x)$$

for all $x \in A$ *and* $y \in U$.

Conversely, let ρ *be a binary relation from* A *to* U*. Define a set-valued mapping* $F_\rho : A \to \mathscr{P}(U)$ *by*

$$F_\rho(x) = \{y \in U : (x, y) \in \rho\}$$

for all $x \in A$*. Then* $S_\rho = (F_\rho, A)$ *is a soft set over* U*. Moreover, we have that* $S_{\rho_S} = S$ *and* $\rho_{S_\rho} = \rho$.

The relation ρ_S is called the canonical relation of the soft set S, and the soft set S_ρ is called the canonical soft set of the binary relation ρ.

Theorem IV.2. [19] *Let* R *be an equivalence relation on* U*,* $S_R = (F_R, U)$ *the canonical soft set of* R *and* $P = (U, S_R)$ *a soft approximation space. Then for all* $X \subseteq U$*,*

$$R_* X = \underline{apr}_P(X) \quad \text{and} \quad R^* X = \overline{apr}_P(X).$$

Thus in this case, $X \subseteq U$ *is a (Pawlak) rough set if and only if* X *is a soft* P*-rough set.*

The following statement can be derived from Theorem IV.2, which puts Pawlak's rough sets in relation with very simple N-soft sets. We give an illustrative example below in Example IV.7.

Theorem IV.3. *Let* (U, ρ) *be a Pawlak approximation space. Let* $A = \{\rho\}$ *and* $N = |U/\rho|$*. Then there exists an* N*-soft set* (F, A, N) *on* U *such that* (U, ρ) *is the Pawlak approximation space derived from* (F, A, N).

Proof: Let $U/\rho = \{[u_0]_\rho, [u_1]_\rho, \ldots, [u_{N-1}]_\rho\}$ and $G_N = \{0, 1, \ldots, N-1\}$. Then we define $F : A \to 2^{U \times G_N}$ by

$$(u, r_i) \in F(\rho) \Leftrightarrow u \in [u_{r_i}]_\rho,$$

where $u \in U$ and $r_i \in G_N$. It is routine to check that the procedure explained in section III-A produces (U, ρ) from (F, A, N). ■

And in turn, the next theorem generalizes the aforementioned [19, Theorem 4.4] to tolerance rough sets:

Theorem IV.4. *Let* τ *be a tolerance relation on* U*. Then there is an* N*-soft set* (F, A, N) *on* U *such that the upper and lower approximations defined by* τ *coincide with the corresponding approximations in the tolerance rough set model derived from* (F, A, N).

Proof: We need to justify that the process in section III-B generates the upper and lower approximations defined by τ. It is known that we can write τ as a union of a finite number of equivalence relations on U (cf., [60] for a statement of the problem of finding the minimal number of equivalence relations that are needed). From this representation, the conclusion is trivial by an appeal to the technique of proof of Theorem IV.3.

Indeed if $\tau = \cup_{i\in I}\rho_i$ and ρ_i is an equivalence relation for each i, then we take $A = I$ and for each $i \in A$ we order $U/\rho_i = \{[u_0]_{\rho_i}, [u_1]_{\rho_i}, \ldots, [u_{N_i-1}]_{\rho_i}\}$ in any arbitrary manner. Let $N = \max\{N_i : i \in I\}$ and $R = \{0, 1, \ldots, N-1\}$. Then we define $F : A \to 2^{U\times R}$, where $R = \{0, 1, \ldots, N-1\}$, by the expression: $(u, r_i) \in U \times R$ is such that $(u, r_i) \in F(a)$ if and only if $u \in [u_i]_{\rho_a}$ (or alternatively, $u\rho_a u_i$). It is routine to check that the procedure explained in section III-B produces the upper and lower approximations of (U, τ) from (F, A, N). ■

In order to make this paper self-contained, we now proceed to give two explicit (although not necessarily efficient) solutions to the problem solved in Theorem IV.4. We do this in the following constructive proof of this result.

Proof: (*An alternative proof of Theorem IV.4*). We select the set of subsets of U defined as

$E = \{Y \subseteq U : Y$ maximal w.r.t. the property $\{u, v\} \subseteq Y \Rightarrow u\tau v\}$

or alternatively the larger set of subsets of U

$E = \{\{u, v\} \subseteq U : u\tau v, u \neq v\}$.

Now each element $e \in E$ defines an equivalence relation R^e by the expression: for all $x, y \in U$,

$$xR^e y \Leftrightarrow \text{ either } x = y \text{ or } \{x, y\} \subseteq E.$$

A routine checking ensures that for all $x, y \in U$, xRy if and only if there exists $e \in E$ such that $xR^e y$. With this representation of τ we proceed as in the original proof. ■

We can gain more intuition if we consider the problem in graph-theoretical terms. Then any tolerance relation τ can be visualized as an undirected graph. Its vertices are the elements of U and an edge joins u and v (we also say that u and v are adjacent) if and only if $u\tau v$. The tolerance relation is transitive (i.e., an equivalence relation) if and only if its graph is a vertex disjoint union of complete graphs (i.e., of graphs for which every two distinct vertices are adjacent). A clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent. A clique that cannot be extended by adding more adjacent vertices is a maximal clique. For more on cliques and their interaction with fuzzy structures, see the application to political sciences in [61]. Based on the above observation, the fact that a tolerance relation τ can be written as a union of a finite number of equivalence relations on U reduces to the intuitive observation that we can find graphs that are a vertex disjoint union of complete graphs, with the property that every edge of the graph associated with τ is an edge of at least one of these complete graphs. For illustration, we consider the following example.

Example IV.5. *Let us consider $U = \{x, y, z, t, u\}$. A tolerance relation τ is defined on U as follows: $x\tau y\tau x$, $z\tau y\tau z$, $z\tau u\tau z$, $z\tau t\tau z$, $t\tau u\tau t$, plus $w\tau w$ for each $w \in U$. We can visualize it as an undirected graph. Consider*

$E = \{Y \subseteq U : Y$ maximal w.r.t. the property $\{u, v\} \subseteq Y \Rightarrow u\tau v\}$,

then $E = \{\{x, y\}, \{y, z\}, \{z, t, u\}\}$ is identified with the maximal cliques of the graph, and Figure 5 shows their visual representation (however for the sake of simplicity, all loops like $x\tau x$ are omitted).

Corresponding to these cliques we define three respective equivalence relations R_1, R_2, R_3, namely:

1) *$wR_1w' \Leftrightarrow w = w' \vee \{w, w'\} = \{x, y\}$, for each $w, w' \in U$.*
2) *$wR_2w' \Leftrightarrow w = w' \vee \{w, w'\} = \{y, z\}$, for each $w, w' \in U$.*
3) *$wR_3w' \Leftrightarrow w = w' \vee \{w, w'\} \subseteq \{z, t, u\}$, for each $w, w' \in U$.*

These equivalence relations ensure that for all $w, w' \in U$, $w\tau w'$ if and only if there is $i \in \{1, 2, 3\}$ with wR_iw'.

Finally, from these relations we define $(G, B, 4)$ where $A = \{a_1, a_2, a_3\}$ according to Table II. Then the process in section III-B produces the upper and lower approximations defined by τ, because for each $w, w' \in U$, $w\tau w'$ if and only if $w \sim^{a_i} w'$ for some $a_i \in A$. That is, we retrieve the original tolerance relation from the 4-soft set that we have constructed.

To conclude this section, we show how N-soft sets can be used to represent any given multigranulation rough structure in the sense of Section III-C.

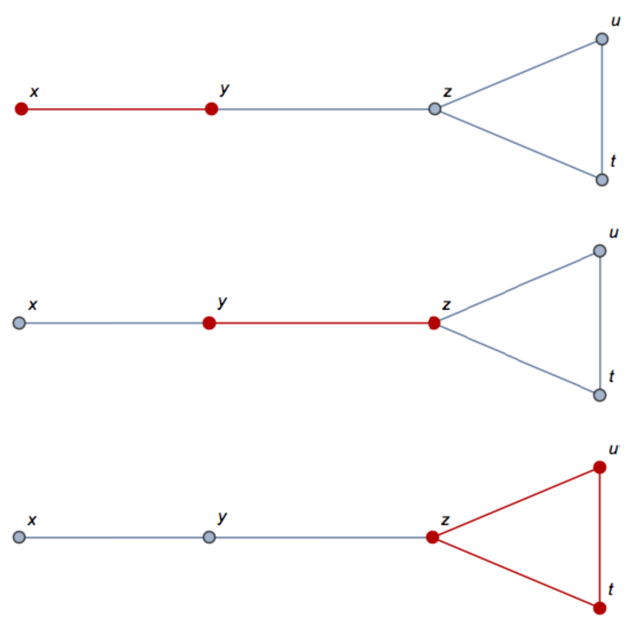

Fig. 5. Cliques of the graph associated with the tolerance relation τ in Example IV.5.

Theorem IV.6. *Let $\{\rho_1, \ldots, \rho_m\}$ be a collection of equivalence relations over U. Then there is an N-soft set (F, A, N) on U such that the optimistic, respectively pessimistic, multi-lower and multi-upper approximations derived from $\{\rho_1, \ldots, \rho_m\}$ coincide with the corresponding optimistic,*

TABLE II
THE 4-SOFT SET IN EXAMPLE IV.5

$(G, B, 4)$	a_1	a_2	a_3
x	0	0	0
y	0	1	1
z	1	1	2
t	2	2	2
u	3	3	2

respectively pessimistic, multi-lower and multi-upper approximations associated with (F, A, N).

Proof: We need to justify that the process in Section III-C generates the optimistic, respectively pessimistic, multi-lower and multi-upper approximations derived from $\{\rho_1, \ldots, \rho_m\}$ for a suitably chosen N-soft set on U. The argument benefits from the proof of Theorem IV.3.

Let us define $A = \{a(1), \ldots, a(m)\}$ and for each $i \in \{1, \ldots, m\}$ we order $U/\rho_i = \{[u_0]_{\rho_i}, [u_1]_{\rho_i}, \ldots, [u_{N_i-1}]_{\rho_i}\}$ in any arbitrary manner. Let $N = \max\{N_i : i \in I\}$ and $G_N = \{0, 1, \ldots, N-1\}$. Then we define $F : A \to 2^{U \times G_N}$ by the expression: $(u, r_i) \in U \times G_N$ is such that $(u, r_i) \in F(a(j))$ if and only if $u \in [u_i]_{\rho_j}$ (or alternatively, $u\rho_j u_i$). It is routine to check that the procedure detailed in Definition III.8 produces the optimistic, respectively pessimistic, multi-lower and multi-upper approximations derived from $\{\rho_1, \ldots, \rho_m\}$. ■

Example IV.7. *Let us consider* $U = \{x, y, z, t, u\}$. *The equivalence relation* ρ_1 *is defined on* U *as follows:*

$U/\rho_1 = \{[u_0]_{\rho_1} = \{x, y\}, [u_1]_{\rho_1} = \{z\}, [u_2]_{\rho_1} = \{t, u\}\}$
where we choose $u_0 = x$, $u_1 = z$, $u_2 = t$ *as representatives of the equivalence classes by* ρ_1.

Then (U, ρ_1) *is the Pawlak approximation space derived from* $(F_1, \{\rho_1\}, 3)$ *where* $F_1 : \{\rho_1\} \to 2^{U \times G_3}$ *is defined as* $F_1(\rho_1) = \{(x, 0), (y, 0), (z, 1), (t, 2), (u, 2)\}$.

Consider now the equivalence relation ρ_2 *defined on* U *as follows:*

$U/\rho_2 = \{[u'_0]_{\rho_2} = \{x, u\}, [u'_1]_{\rho_2} = \{y, z, t\}\}$
where we choose $u'_0 = x$, $u'_1 = y$ *as representatives of the equivalence classes by* ρ_2.

Then (U, ρ_2) *is the Pawlak approximation space derived from* $(F_2, \{\rho_2\}, 2)$ *where* $F_2 : \{\rho_2\} \to 2^{U \times G_2}$ *is defined as* $F(\rho_2) = \{(x, 0), (u, 0), (y, 1), (z, 1), (t, 1)\}$.

Let $A = \{a(1), a(2)\}$ *and* $S = (F_1, A_1, 3)$ *be a 3-soft set as shown in Table III.*

TABLE III
THE 3-SOFT SET IN EXAMPLE IV.7

$(F_1, A_1, 3)$	$a(1)$	$a(2)$
x	0	0
y	0	1
z	1	1
t	2	1
u	2	0

Observe that S *produces the soft equivalence relation* (σ_S, A) *on the set* U *given by*

$$\sigma_S(a(1)) = \triangle_U \cup \{(x, y), (y, x), (t, u), (u, t)\},$$
$$\sigma_S(a(2)) = \triangle_U \cup \{(x, u), (u, x), (y, z), (z, y), (y, t), (t, y), (z, t), (t, z)\},$$

Then the optimistic, respectively pessimistic, multi-lower and multi-upper approximations derived from $\{\rho_1, \rho_2\}$ *coincide with the corresponding optimistic, respectively pessimistic, multi-lower and multi-upper approximations associated with* $S = (F_1, A_1, 3)$. *For example, let* $X = \{x, y, u\}$. *Then a routine application of Definition II.8 yields*

$\underline{X}^{O}_{\sum \rho_i} = \cup\{[w]_{\rho_i} | [w]_{\rho_i} \subseteq X \text{ for some } i\} = \{x, y\} \cup \{x, u\} = X$, *and*

$\overline{X}^{O}_{\sum \rho_i} = \sim (\underline{\sim X})^{O}_{\sum \rho_i} = \{x, y, t, u\}$.

We obtain the same conclusion if we use Definition III.8 instead: $\underline{X}^{O}_{S} = X$, *and* $\overline{X}^{O}_{S} = \sim (\underline{\sim X})^{O}_{S} = \{x, y, t, u\}$.

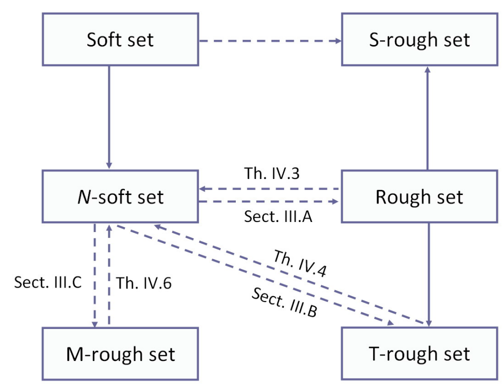

Fig. 6. Relationships among soft sets, rough sets and their extensions. The connections without a legend are known from the existing literature.

To summarize the main results in Section III and Section IV, we inventory the known and newly presented relationships among soft sets, rough sets and their extensions with Figure 6. The meaning of the symbols or abbreviations used in Figure 6 is as follows:

- $A \longrightarrow B$ means "B is a generalization of A";
- $A \dashrightarrow B$ means "B can be derived from A";
- "M-rough set" stands for "multigranulation rough sets";
- "S-rough set" stands for "soft rough sets";
- "T-rough set" stands for "tolerance rough sets".

It is also worth listing the practical situations from previous sections that put these elements into action. Example III.2 and Example III.3 clarify the procedures in Section III-A. Example III.7 sheds light on the purpose of Section III-B. Then Example III.9 shows how the elements of Section III-C take effect. In the opposite direction, Example IV.5 demonstrates the technique in Theorem IV.4, which is more general than that of Theorem IV.3. Example IV.7 illustrates both Theorem IV.3 and Theorem IV.6. In addition, the first part of Section V below refers to Sections III-A and III-B, and its second part revisits the performance of Theorem IV.4. Let us therefore present a fully developed case study in Section V.

V. A CASE STUDY

This section is intended to emphasize the applicability of our results with real data. We illustrate the concepts that we have developed in Sections III and IV with a real-life example incorporating data from the website of Trivago (https://www.trivago.co.uk).

Trivago N.V. is a transnational technology company offers internet-related services in the hotel field. Hotels are categorized according to up to 10 attributes (a shorter description with 5 attributes is available too). For the sake of conciseness we will develop our example with respect to six lodging facilities in a North European capital. We have extracted data from the website of Trivago, and the hotel name information is anonymized in Figure 7 for privacy protection.

The attributes produce parameterized descriptions of the universe of hotels $U = \{h_1, \ldots, h_6\}$. These hotels are evaluated in terms of linguistic terms in $L = \{OK, F, G, VG, E\}$. These non-numerical values stand for “Okay”, “Fair”, “Good”, “Very Good”, and “Excellent”, as provided by the recommender of Trivago.

The starting point of rough set theory is usually the indiscernibility relation. However it is also common to define the fundamental concepts of the theory from data like those we have succinctly described above. In that case we input an information system which is a pair $I = (U, \tilde{A})$ where $\tilde{A}$ is the set of attributes [51]. Every attribute is identified with a mapping $\tilde{a} : U \longrightarrow V_{\tilde{a}}$ and $V_{\tilde{a}}$ is the domain of attribute $\tilde{a}$ (in our case study we can fix $V_{\tilde{a}} = L$ for every $\tilde{a}$ in the set of ten attributes). Now for each subset A of $\tilde{A}$, the information system induces an equivalence relation on U called A-indiscernibility relation [8], [29], [51]. Informally, elements are indiscernible with respect to a list of available attributes A (hence belong to the same equivalence class) when their descriptions in terms of the attributes in A coincide.

For example, let us select the attributes “Service” (a), “Value for money” (b), “Facilities” (c), and “Food” (d) to form $A = \{a, b, c, d\}$. The equivalence classes derived from A (also called A-elementary granules) are

$$I_A = \{\{h_1, h_6\}, \{h_2\}, \{h_3\}, \{h_4\}, \{h_5\}\}.$$

Their unions are the A-definable sets. And they produce the A-lower and A-upper approximations of subsets of U.

Similarly, if $B = \{a, c\}$ then the corresponding indiscernibility relation produces the equivalence clases (or B-elementary granules)

$$I_B = \{\{h_1, h_6\}, \{h_2, h_5\}, \{h_3\}, \{h_4\}\}.$$

In what follows, the main contribution in Sections III and IV will be illustrated.

1) Illustration of Section III: Table IV summarizes the information system $I_1 = (U, A)$ coded from the data about the six hotels, when the attributes are restricted to $A = \{a, b, c, d\}$ as mentioned above.

The information system $I_1 = (U, A)$ in turn can be associated with a 5-soft set $(H, A, 5)$. To that purpose we codify the information in the standard notation for the set of grades, by replacing “OK” with 0, “F” with 1, and so forth. The result

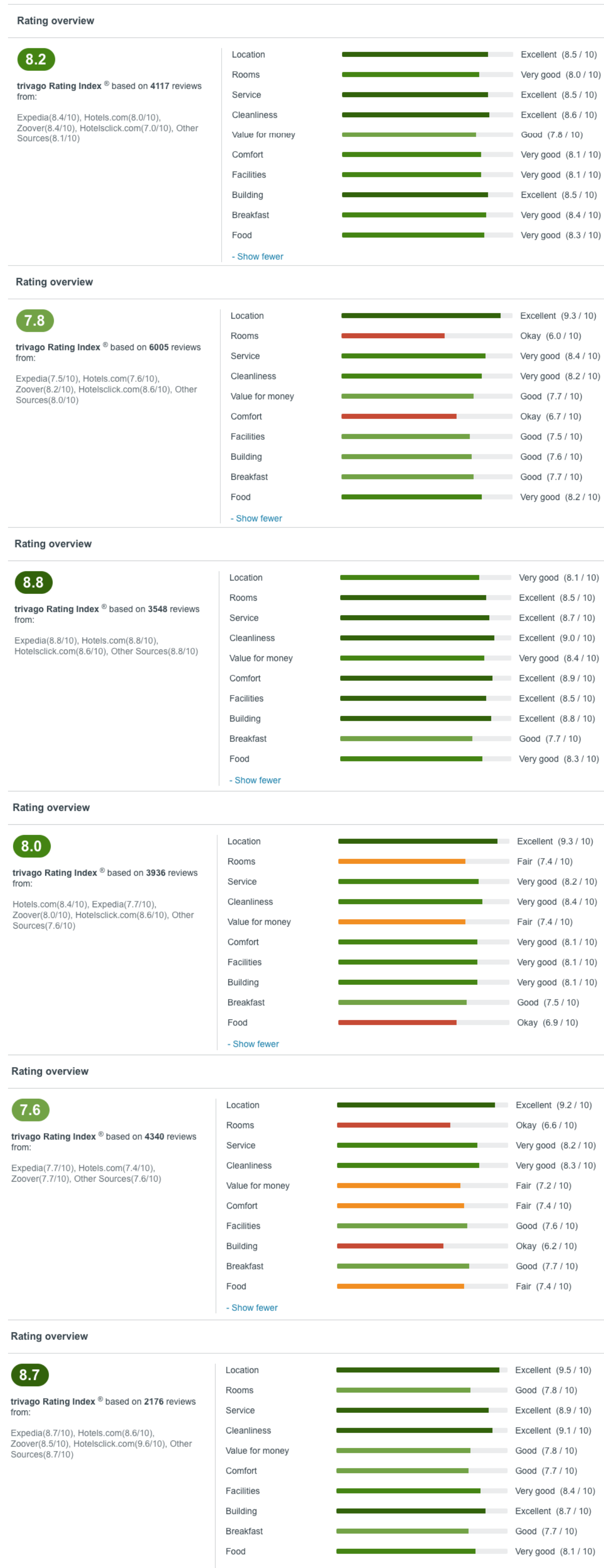

Fig. 7. Screen capture of data about hotels in a North European capital from the website Trivago (https://www.trivago.co.uk).

TABLE IV
AN INFORMATION SYSTEM $I_1 = (U, A)$ OBTAINED FROM A REAL-LIFE DATA SET REGARDING SIX HOTELS.

Hotels	Parameters			
	Service	Value for money	Facilities	Food
h_1	E	G	VG	VG
h_2	VG	G	G	VG
h_3	E	VG	E	VG
h_4	VG	F	VG	OK
h_5	VG	F	G	F
h_6	E	G	VG	VG

appears in Table V. We are ready to compute the elements defined in Section III for this 5-soft set. Section III-A defines four Pawlak's rough structures associated with $(H, A, 5)$, one for each attribute:

- attribute a induces the equivalence relation $\sim^a$ whose equivalence classes form the partition $\hat{P}_1 = \{\{h_1, h_3, h_6\}, \{h_2, h_4, h_5\}\}$,
- attribute b induces the equivalence relation $\sim^b$ whose equivalence classes form the partition $\hat{P}_2 = \{\{h_1, h_2, h_6\}, \{h_3\}, \{h_4, h_5\}\}$,
- attribute c induces the equivalence relation $\sim^c$ whose equivalence classes form the partition $\hat{P}_3 = \{\{h_3\}, \{h_2, h_5\}, \{h_1, h_4, h_6\}\}$, and
- attribute d induces the equivalence relation $\sim^d$ whose equivalence classes form the partition $\hat{P}_4 = \{\{h_4\}, \{h_5\}, \{h_1, h_2, h_3, h_6\}\}$.

TABLE V
THE 5-SOFT SET ASSOCIATED WITH $I_1 = (U, A)$.

$(H, A, 5)$	a	b	c	d
h_1	4	2	3	3
h_2	3	2	2	3
h_3	4	3	4	3
h_4	3	1	3	0
h_5	3	1	2	1
h_6	4	2	3	3

In order to obtain the Pawlak approximation space $(U, \sim)$ derived from the 5-soft set that describes our problem, we set $\sim$ by the expression in (2). The definition $h_i \sim h_j$ if and only if $h_i \sim^e h_j$ for each $e \in A$ produces an equivalence relation whose equivalence classes are

$$\{\{h_1, h_6\}, \{h_2\}, \{h_3\}, \{h_4\}, \{h_5\}\}.$$

which as the theory prescribes, coincide with the equivalence classes derived from A in the original information system I.

If $B = \{a, c\}$ then (3) produces the B-indiscernibility relation induced by $(H, A, 5)$. One readily checks that its equivalence classes coincide with the B-elementary granules computed above from the original information system. Let us fix $X = \{h_1, h_2, h_5\}$. Then the B-lower approximation of X induced by $(H, A, 5)$ is $\{h_2, h_5\}$ and the B-upper approximation of X induced by $(H, A, 5)$ is $\{h_1, h_2, h_5, h_6\}$.

Section III-B defines the tolerance-based rough structure derived from the 5-soft set $(H, A, 5)$. It arises from the tolerance relation τ such that $h_i \tau h_j$ if and only if $h_i \sim^e h_j$ for some $e \in A$. Simple computations show that τ is the reflexive and symmetric relation that satisfies

$$h_1\tau h_2,\ h_1\tau h_3,\ h_1\tau h_4,\ h_1\tau h_6;\ h_2\tau h_j\ (j = 1, \ldots, 6);$$
$$h_3\tau h_6,\ h_4\tau h_5,\ \text{and}\ h_4\tau h_6.$$

2) Illustration of Section IV: Now suppose that we are in possession of information relating to which options are similar to others [62, Section 3.1]. In particular, we know that when we represent this relationship by a tolerance relation T, one has the non-trivial similarities

$$h_1\,T\,h_3, h_1\,T\,h_4, h_1\,T\,h_6, h_2\,T\,h_4, h_2\,T\,h_5, h_3\,T\,h_6, h_4\,T\,h_5$$

in addition to its reverse similarities

$$h_3\,T\,h_1, h_4\,T\,h_1, h_6\,T\,h_1, h_4\,T\,h_2, h_5\,T\,h_2, h_6\,T\,h_3, h_5\,T\,h_4$$

and the diagonal part $h_i\,T\,h_i$ $(i = 1, \ldots, 6)$.

Using the techniques developed in Section IV, we can represent T by an undirected graph whose maximal cliques are shown in Figure 8. From these maximal cliques, we can define three equivalence relations, whose partitions are as follows:

$$\{\ \{h_1, h_3, h_6\}, \{h_2\}, \{h_4\}, \{h_5\}\ \},$$
$$\{\ \{h_1, h_4\}, \{h_2\}, \{h_3\}, \{h_5\}, \{h_6\}\ \},$$
$$\{\ \{h_2, h_4, h_5\}, \{h_1\}, \{h_3\}, \{h_6\}\ \}.$$

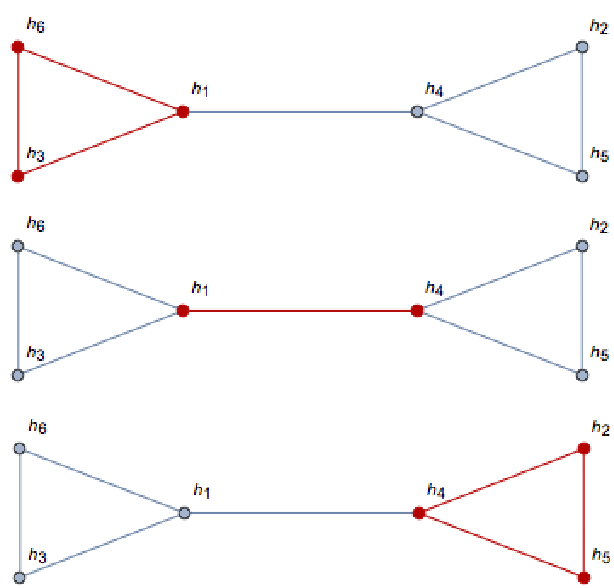

Fig. 8. Cliques of the graph associated with the tolerance relation T.

Then it can be easily checked that the union of these three relations results in the given tolerance relation T. To conclude, these equivalence relations in turn define a 5-soft set $(H', A', 5)$ with $A' = \{a_1, a_2, a_3\}$ as shown in Table VI.

Moreover, it is worth noting that we can retrieve the original tolerance relation T when we apply the concepts of section III-B to the 5-soft set$(H', A', 5)$.

TABLE VI
THE 5-SOFT SET $(H', A', 5)$ IN SECTION V

$(H', A', 5)$	a_1	a_2	a_3
h_1	0	0	0
h_2	1	1	1
h_3	0	2	2
h_4	2	0	1
h_5	3	3	1
h_6	0	4	3

Remark V.1. *It should be noted that the above-mentioned binary relation T is not artificially given. In fact, it is the tolerance relation derived from another information system similar to $I_1 = (U, A)$, by considering the attributes "Rooms", "Cleanliness", "Comfort" and "Building", which are actually used by the recommender of Trivago (see the screen capture in Figure 7). Specifically, $h_i T h_j$ means that the descriptions of hotels h_i and h_j by at least one of these attributes are identical. Nevertheless, the observer does not need to know whether similarities come from this source of information, or are a primitive concept.*

As emphasized in Section III-B, uniqueness of the representation of rough structures by N-soft sets is not guaranteed. Actually, two different proofs of Theorem IV.4 give rise to two distinct procedures for producing representations of tolerance-based rough structures. The construction based on maximal cliques is more efficient. In particular, the above calculation shows that the tolerance-based rough structure, originally derived from four real attributes (i.e., "Rooms", "Cleanliness", "Comfort" and "Building"), can be equivalently represented by only three artificial parameters a_1, a_2 and a_3.

Lastly, since the relationships between multigranulation rough structures and N-soft sets have been well illustrated by Example III.9 and Example IV.7, we omit similar discussions in this case study.

VI. CONCLUSION

This study has shown that abundant mutual connections exist between various rough structures and N-soft sets. The methods used for establishing these interrelationships are constructive; hence we can move from one setting to the other in a definite manner. Each of the methods is applicable when the structure of the uncertain data corresponds to its input. In a concrete situation, Section III provides methodologies that apply to data in the form of an N-soft set, whereas the methods of Section IV are applicable when the information pertains to the realm of granular computing. The precise details of these dual ideas are as follows:

$i)$ Various types of rough structures can be derived from a given N-soft set with the approach proposed in Section III. We give three detailed constructions that are applicable when the information takes the form of an N-soft set. They produce Pawlak's rough sets, tolerance rough sets, and multigranulation rough sets, respectively. In this way we have established a blueprint for the subsequent analysis of rough sets in terms of multi-valued approximate descriptions.

$ii)$ A more challenging task is the converse problem, that is, the possibility of employing N-soft sets as a uniform representation of several granular knowledge structures. Results from [19] help anticipate the ability of N-soft sets with a single attribute to represent rough sets (cf., Theorem IV.3). Here we made further improvement upon this insight, and overcame the limitations of soft sets as a tool for giving approximate descriptions. We benefited from an integrated use of combinatorial and graph-theoretic techniques in order to represent tolerance rough structures by means of N-soft sets (cf., Theorem IV.4). In particular, two feasible ways to achieve this goal derive from the alternative proofs of Theorem IV.4. The first construction takes advantage of the maximal cliques of the graph of the relation and is more efficient. The second one is simpler to understand. It shows that we can take all the vertices separately, and then each edge induces one equivalence relation. At the same time, the fundamental relationship that Theorem IV.3 embodies is at the core of our representation of multigranulation rough structures by virtue of N-soft sets. To be precise, Theorem IV.6 is applicable when the data take the form of a multigranulation rough structure. It assures the existence of an N-soft set that produces this configuration by the routine implementation of the techniques in Section III-C.

These results are helpful for bridging the gap between diverse types of rough structures and N-soft sets. Hopefully they can also help to visualize rough structures in a new intuitive manner. In addition, we can safely claim that more relationships are yet to come because the theory of N-soft sets is complemented with other models with a richer structure [49], [50]. Meanwhile, it must be pointed out that our work has some limitations which should be overcome in ensuing studies. Our investigation is mainly theoretical and it does not produce new techniques by itself for promoting practical applications, such as decision making or clustering analysis. Despite the weakness of not being directly applicable to solve practical problems, it could be used to pass on useful techniques or tools from one setting to another. In that case, the feasibility of the transformed mechanisms should be further verified by conducting comparative analysis or cross-validation. Admittedly, these challenges are beyond the scope of our theoretical investigation. In the future, one can try to overcome these difficulties or establish more abundant connections between N-soft sets and other soft computing models such as fuzzy sets, intuitionistic fuzzy sets or neutrosophic sets [63].

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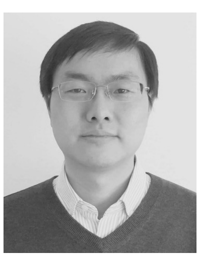
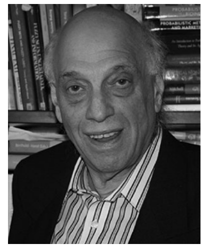
José Carlos R. Alcantud received his M.Sc. in Mathematics in 1991 from the University of Valencia, Spain and Ph.D. in Mathematics in 1996 from the University of Santiago de Compostela, Spain. He is currently a professor and the director of the Department of Economics and Economic History, with the University of Salamanca, Spain. He has authored over 70 papers on topics including mathematical economics, social choice, fuzzy mathematics and soft computing. He serves on the editorial board of *Symmetry* and *Global & Local Economic Review*, and the advisory board of *Sci*. He has been a reviewer for more than 50 SCI journals, including *IEEE Transactions on Fuzzy Systems*, *Information Sciences*, *Information Fusion*, *Expert Systems with Applications*, *Fuzzy Sets and Systems*, *IEEE Access*, etc. As the project leader, he has directed 7 research projects at the national or provincial level. He has also been a member of a EU-funded reseach project.



Feng Feng received the B.S. and M.S. degrees in mathematics from Northwest University, Xi'an, China in 2003 and 2006, respectively, and the Ph.D. degree in mathematics from Shaanxi Normal University, Xi'an, China in 2012.

From 2015 to 2016, he was a visiting research faculty with the University of California, Los Angeles, USA. He is currently a professor and the director of the Department of Applied Mathematics, with Xi'an University of Posts and Telecommunications, Xi'an, China. He has authored over 50 papers (including 1 ESI Hot Paper and 6 ESI Highly Cited Papers), with over 2400 citations in Google Scholar. His research interests include soft computing, soft sets, granular computing, rough sets, fuzzy mathematics, data mining, decision analysis, etc. He serves on the editorial board of *Journal of Mathematics*, *Annals of Fuzzy Mathematics and Informatics*, etc. He has been a reviewer for more than 40 SCI journals, including *IEEE Transactions on Fuzzy Systems*, *European Journal of Operational Research*, *IEEE Transactions on Systems, Man and Cybernetics: Systems*, *Information Sciences*, *Knowledge-Based Systems*, *IEEE Access*, *Applied Mathematical Modeling*, *Journal of Computational and Applied Mathematics*, etc. As the project leader, he has directed 7 research projects at the national or provincial level.

Dr. Feng was awarded the University Sci-Tech Prize of Shaanxi Province in 2013, and the Sci-Tech Prize of Shaanxi Province in 2014. He was honored as the Young Sci-Tech New Star of Shaanxi Province in 2014.



Ronald R. Yager (S'66–M'68–SM'93–F'97–LF'11) received the B.E. degree from the City College of New York, New York, USA in 1963 and the Ph.D. degree in systems science from the Polytechnic University of New York, New York, USA in 1968.

He is currently the Director of the Machine Intelligence Institute and a Professor of Information Systems at Iona College, New York, USA. He has worked in the area of machine intelligence for over 25 years. He served at the National Science Foundation as a Program Director in the Information Sciences program. He was a NASA/Stanford Visiting Fellow and a Research Associate at the University of California, Berkeley, USA. He has been a Lecturer at NATO Advanced Study Institutes. He is a Distinguished Honorary Professor with Aalborg University, Aalborg, Denmark. He is a Distinguished Visiting Scientist at King Saud University, Riyadh, Saudi Arabia. He has published over 500 papers and 15 books in areas related to fuzzy sets, decision making under uncertainty, and the fusion of information. He is among the world's top 1% most highly cited researchers with nearly 70000 citations.

Dr. Yager was a recipient of the IEEE Computational Intelligence Society Pioneer Award in Fuzzy Systems and the Lifetime Achievement Award by the Polish Academy of Sciences for his contributions. He is an Editor-in-Chief of the *International Journal of Intelligent Systems*. He serves on the editorial board of numerous technology journals. He is a fellow of the New York Academy of Sciences and the Fuzzy Systems Association.