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Decomposition theorems and extension principles for hesitant fuzzy sets

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ABSTRACT

We prove a decomposition theorem for hesitant fuzzy sets, which states that every typical hesitant fuzzy set on a set can be represented by a well-structured family of fuzzy sets on that set. This decomposition is expressed by the novel concept of hesitant fuzzy set associated with a family of hesitant fuzzy sets, in terms of newly defined families of their cuts. Our result supposes the first representation theorem of hesitant fuzzy sets in the literature. Other related representation results are proven. We also define two novel extension principles that extend crisp functions to functions that map hesitant fuzzy sets into hesitant fuzzy sets.

1. Introduction

This paper investigates some decomposition results for hesitant fuzzy sets that permit to replicate the considerable significance of classical decomposition theorems for fuzzy sets stated in terms of their α-cuts. And it provides novel extension principles that generalize the important principles in fuzzy set theory to hesitant fuzzy set theory.

Klir and Yuan [34, Section 2.2] explain that α-cuts and strong α-cuts of a fuzzy set have an important role in fuzzy set theory because they are capable of representing fuzzy sets. And by doing this, one has a tool to extend some properties of crisp sets and operations on crisp sets to their fuzzy counterparts. The classical representations by (strong) α-cuts are universally applicable to all fuzzy sets. Moreover they permit to state decompositions of any fuzzy set in terms of special fuzzy sets associated with either its α-cuts or its strong α-cuts (cf., Dubois and Prade [22], Negoita and Ralescu [41]).

The extension principle is another long-established contribution to the algebraic theory of fuzzy sets. It was introduced by Zadeh [58] and further elaborated by Yager [55], Nguyen [43] or Bzowski and Urbański [14] among others. Classical mathematical theories can be fuzzified thanks to this principle (see e.g., Bednar [8], Gerla [25], Gerla and Scarpati [26] or Kaleva and Seikkala [32]).

In our analysis we extend these developments by permitting hesitancy. Our paper complements previous successful contributions by scholars like the following short sample. Negoita and Ralescu [41] actually prove representation theorems for the lattice of L-sets (cf., Goguen [28]) of a set, from which they deduce representation theorems for the lattice of fuzzy sets. Couso et al. [18] derive a new interpretation of strong α-cuts of a normalized fuzzy set. Li, Yuan and Lee [36] introduce three-dimensional fuzzy sets, a special class of L-fuzzy sets for which decomposition and representation theorems are given. These authors also claim that the cut sets, decomposition theorems and representation theorems of the left, resp. right interval-valued intuitionistic fuzzy sets can be easily derived from their results. Li [35] represents intuitionistic fuzzy sets by level sets and Yuan, Li and Sun [56] prove some decomposition theorems and representation theorems on intuitionistic fuzzy sets and interval valued fuzzy sets (see also Martinetti, Janiš and Montes [37] for an investigation of cuts of intuitionistic fuzzy sets respecting fuzzy connectives, and Rahman [45] for a definition of t-norm and t-conorm based cuts of intuitionistic fuzzy sets and their generalised intuitionistic fuzzy operations). Ngan [42] gives a unified representation of intuitionistic fuzzy sets, hesitant fuzzy sets and generalized hesitant fuzzy sets based on their u-maps. Mendel, John and Liu [39] prove that all discrete type-2 fuzzy sets can be expressed as a union of simpler type-2 fuzzy sets (see also Mendel and John [38]). Torra [49] or Akram and Nawaz [1] belong to different lines of works. Torra shows that all hesitant fuzzy sets can be represented as fuzzy multisets (Lemma 14) and as type-2 fuzzy sets (Lemma 16),

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whereas Akram and Nawaz provide tabular representations for fuzzy soft graphs. The reader may consult Bustince et al. [13] for a historical survey of types of fuzzy sets and their relationships.

Zadeh's [57] introduction of fuzzy set theory was subsequently followed by extended theories that attempt to better capture the possible subjectivity, uncertainty, imprecision of the evaluations, et cetera, that are usual in applications. In particular, Torra [49] introduced hesitant fuzzy sets which coincide with set-valued fuzzy sets in Grattan-Guinness [29] (cf., Bustince et al. [13]). They are useful to model imprecise human knowledge (particularly collective knowledge, e.g., Alcantud et al. [4]) which cannot be correctly captured by fuzzy sets. Relatedly, Zhu et al. [60] define dual hesitant fuzzy sets, and Zhu et al. [59] define extended hesitant fuzzy sets. As to hybridization, Wang, Li and Chen [52] introduced hesitant fuzzy soft sets by combining the notion of hesitancy with Molodtsov's [40] soft sets.

In this paper we define hesitant fuzzy sets associated with (possibly infinite) families of hesitant fuzzy sets. Then we introduce uniformly typical hesitant fuzzy sets, an apposite notion that we relate with existing notions. We define the characteristic of a hesitant fuzzy set and prove that it is an useful notion for the purpose of identifying certain special types of hesitant fuzzy sets. We also define a new notion of cuts for hesitant fuzzy sets that enables us to prove decomposition theorems for typical hesitant fuzzy sets. In a similar vein, we prove some properties of these cuts in the context of uniformly typical hesitant fuzzy sets. Finally, we define two new extension principles that extend crisp functions to functions that map (uniformly typical) hesitant fuzzy sets into (uniformly typical) hesitant fuzzy sets. We argue that these extension principles are generalizations of the standard principles for fuzzy sets, and we prove some additional properties.

This paper is organized as follows. Section 2 recalls some notation and definitions. Here we introduce hesitant fuzzy sets associated with arbitrary families of hesitant fuzzy sets. Section 3 presents the main new notions in this paper, namely, uniformly typical hesitant fuzzy soft sets, characteristic of a hesitant fuzzy set, and (α, k)-cuts. We also prove some useful fundamental properties of these concepts. In Section 4 we present our main results, which prove that (α, k)-cuts are a fitting tool for providing decompositions of typical hesitant fuzzy sets. Then we introduce and illustrate our new extension principles as well as some properties of them. And we also discuss the relationships of our results with existing literature and their implications for decision making. We conclude in Section 5.

2. Notation and definitions

For any set X, $\mathcal{P}^*(X)$ denotes the set of non-empty subsets of X, $\mathcal{P}(X)$ denotes the set of all subsets of X. Furthermore, $\mathcal{F}^*(X)$ denotes the set of non-empty finite subsets of X and for each $N \in \mathbb{N}$, $\mathcal{F}^*_N(X)$ denotes the set of non-empty subsets of X with N or fewer elements.

Now we recall concepts from fuzzy sets and hesitant fuzzy sets. Throughout the remaining of this Section we refer to a fixed non-empty set of alternatives X.

2.1. Fuzzy and hesitant fuzzy sets

A fuzzy subset (FS) A of X is characterized by a function μ_A: $X \to [0,\ 1]$. When $x \in X$, the number $\mu_A(x) \in [0,\ 1]$ is called the degree of membership of x in the subset. It represents the degree of truth of the statement "x belongs to A". Zadeh's fuzzy subsets of X are denoted by $\mathbf{FS}(X)$.

The following notion of hesitant fuzzy element is extensively used in this paper:

Definition 1 (Xia and Xu [53]). A hesitant fuzzy element (HFE) is a non-empty, finite subset of [0, 1]. The set of HFEs is denoted by $\mathcal{F}^*([0,1])$.

Generic HFEs are expressed as $h = \{h^1, \ldots, h^{l_h}\}$, where $h^1 < \ldots < h^{l_h}$ and $l_h = |h|$ is the cardinality of the HFE h. In particular, $h = \{1\}$ is usually called the full HFE, and $h = \{0\}$ is usually called the empty HFE.

We now recall the definition of hesitant fuzzy set and typical hesitant fuzzy set:

Definition 2 (Torra [49]). A *hesitant fuzzy set* (HFS) on X is a function $h_M : X \longrightarrow \mathcal{P}([0,1])$. Henceforth $\mathbf{HFS}(X)$ means the set of HFSs on X.

Definition 3 (Bedregal. et al. [9]). A *typical hesitant fuzzy set* (THFS) on X is $h_M : X \longrightarrow \mathcal{F}^*([0,1])$. Henceforth $\mathrm{HFS}^t(X)$ denotes the set of all THFSs on X.

Clearly, $\mathrm{HFS}^t(X) \subseteq \mathbf{HFS}(X)$. Each HFS on X defines a set of membership values for each element of X, and in the case that the HFS is typical such set is always finite and non-empty.

HFEs represent the set of possible membership values of a typical hesitant fuzzy set at an alternative.

Regarding Definitions 1 and 3, observe that on each alternative, at least one assessment must be made because the respective codomains in these definitions are $\mathcal{P}^*([0,1])$ and $\mathcal{F}^*([0,1])$.

In formal terms, the notions in Definitions 2 and 3 can be stated as follows. A THFS is a subset $M \subseteq X \times \mathcal{F}^*([0,1])$ such that for each $x \in X$, there is exactly one element $h_M(x) \in \mathcal{F}^*([0,1])$ with the property $(x,\ h_M(x)) \in M$. Here $h_M(x) \neq \varnothing$. And HFSs are characterized as subsets $M \subseteq X \times \mathcal{P}([0,1])$ such that for each $x \in X$, there is exactly one element $h_M(x) \in \mathcal{P}([0,1])$ with the property $(x,\ h_M(x)) \in M$, which may be $\varnothing$.

Therefore for practical purposes, any hesitant fuzzy set h_M can be represented as $M = \{(x, h_M(x)) \mid x \in X\}$. For example, Torra [49] defines the *ideal* or *full* HFS on X by $M^* = \{(x, \{1\}) \mid x \in X\}$, and the *anti-ideal* or *empty* HFS on X by $M^- = \{(x, \{0\}) \mid x \in X\}$.

Remark 1. Any FS on X with membership function $\mu_M : X \longrightarrow [0,1]$ such that $\mu_M(x) = M_x$ can be identified with the THFS h_M described as $M = \{(x, h_M(x)) \mid x \in X, h_M(x) = \{M_x\}\}$. In this fashion we can naturally embed $\mathbf{FS}(X)$ into $\mathrm{HFS}^t(X)$ and therefore into $\mathbf{HFS}(X)$.

In other words, FSs are special THFSs with the natural identification explained above.

For each typical hesitant fuzzy set h_M on X we let

$$h_M(x) = \{h_M^1(x), \ldots, h_M^{l_M(x)}(x)\}$$

where $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$ and $l_M(x) = |h_M(x)|$ is the cardinality of the HFE $h_M(x)$. Since $h_M(x)$ is a set, repetitions are excluded by definition.

Torra [49, Definition 9] and Torra and Narukawa [50, Definition 11] define the union of two HFSs h_1 and h_2 on X, denoted $h_1 \cup h_2$, by reference to the "lower bounds" of HFEs. The expression is: for each $x \in X$,

$$(h_1 \cup h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \geqslant \max\{\inf h_1(x), \inf h_2(x)\}\}$$

When h_1 and h_2 are typical HFSs we obtain

$$(h_1 \cup h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \geqslant \max\{h_1^1(x), h_2^1(x)\}\}$$

And if h_1 and h_2 are FSs on X then $(h_1 \cup h_2)(x) = \{\max\{h_1^1(x), h_2^1(x)\}\}$ which produces the standard union of fuzzy set theory under the identification in Remark 1.

Following the usual rationale in set theory, we naturally define inclusion for HFSs as follows:

Definition 4. Let h_1 and h_2 be HFSs on X, then

$h_1 \subseteq h_2$ if and only if $h_1 \cup h_2 = h_2$

When we compare two typical hesitant fuzzy sets h_1 and h_2 on X by inclusion, we deduce $h_1 \subseteq h_2$ implies $h_1^1(x) \leqslant h_2^1(x)$ for all $x \in X$. In particular, if h_1 and h_2 are fuzzy sets on X then $h_1 \subseteq h_2$ if and only if $h_1(x) \leq h_2(x)$ for all $x \in X$. This is the standard subsethood of fuzzy set theory.

Remark 2. Equivalently, one can define $h_1 \subseteq h_2$ if and only if $h_1 \cap h_2 = h_1$ where $(h_1 \cap h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \leqslant \min\{\sup h_1(x), \sup h_2(x)\}\}$ and $\sup h_1(x), \sup h_2(x)$ respectively represent the "upper bounds" of $h_1(x)$, $h_2(x)$, for each $x \in X$. The definition of intersection of two HFSs is first given by Torra [49, Definition 10]. When specialized to FSs, it produces the standard intersection of fuzzy set theory.

Extensive recent surveys of HFSs and their applications include Rodríguez et al. [46], Rodríguez et al. [47], and Xu [54], which justify the importance of hesitant fuzzy elements and sets from the perspective of theoretical and applied approaches. Alcantud [2] relates hesitant fuzzy sets with other soft computing models, and Alcantud and de Andrés [3] suggest a new approach to analyze projects characterized by hesitant fuzzy sets.

2.2. HFSs associated with families of hesitant fuzzy sets

Since [49, Definition 5], a notion of HFS associated with a *finite* family of membership functions or fuzzy sets $F = \{\mu_1, \dots, \mu_n\}$ has been present in the literature about hesitancy (see also Rodríguez et al. [46, Definition 2], Rodríguez et al. [47, Definition 2]). The novel Definition 5 below shows that we can generalize this construction because a similar approach can be used to define HFSs from *possibly infinite* families of *hesitant* fuzzy sets:

Definition 5. Let $\mathcal{F} = \{h_{M(i)}\}_{i \in J}$ be a family of hesitant fuzzy sets on X, indexed by the set of indices J. Then the HFS associated with $\mathcal{F}$, denoted by either $h_{\mathcal{F}}$ or $\bigcup_{i \in J} h_{M(i)}$, is defined as:

$$h_{\mathcal{F}} : X \longrightarrow \mathcal{P}([0,1])$$
$$x \quad \bigcup_{i \in J} h_{M(i)}(x)$$

Through the standard identification of FSs with special types of THFSs (cf., Remark 1), Definition 5 generalizes [49, Definition 5].

Clearly, the HFS associated with a finite family of THFSs is a THFS too. In particular, the HFS associated with a finite family of FSs is a THFS.

3. Some novel concepts related to hesitant fuzzy sets

In this Section we introduce the main new notions in this paper, namely, uniformly typical hesitant fuzzy sets, characteristic of a hesitant fuzzy sets, and (α, k)-cuts and strong (α, k)-cuts. We also prove some particular properties that are needed in our subsequent decomposition theorems.

3.1. Uniformly typical hesitant fuzzy sets

In applications virtually all HFSs verify the following novel concept:

Definition 6. A typical hesitant fuzzy set h_M on X is *uniformly typical* if there is N such that $l_M(x) \leq N$ for each $x \in X$. We abbreviate uniformly typical HFS by UHFS. Henceforth **UHFS**(X) means the set of UHFSs on X

We have introduced Definition 6 because typical and uniformly typical HFS are related but different notions, as the following Proposition proves:

Proposition 1. *Every uniformly typical HFS is a typical HFS, but the converse is not true. Any typical HFS on a finite set is uniformly typical.*

Proof. Clearly, UHFSs are THFSs. We proceed by example to prove that there are typical HFSs that are not uniformly typical.

Let $I = [0, 1]$ and define for each $y \in I$,

$$h_M(y) = \begin{cases} \{\frac{1}{n}, \frac{1}{n-1}, \dots, 1\}, & \text{if } y = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then $M = \{(y, h_M(y))$ such that $y \in I\}$ defines a typical HFS on I because each $h_M(y)$ is a finite subset of membership values. However there is no bounding number N with the property that every $h_M(y)$ contains at most N membership values. Hence M is not a UHFS.

Let us now prove the second assertion. Fix a typical hesitant fuzzy set h_M on a finite X. Then it is immediate to check that h_M is uniformly typical because the number $char(h_M) = \max\{l_M(x) : x \in X\} \in \mathbb{N}$ is well defined due to finiteness of X, and it verifies the required property. □

We now introduce the characteristic of a HFS. This is an element from $\mathbb{N} \cup \{+\infty\}$ defined as follows:

Definition 7. For each hesitant fuzzy set h_M on X,

$$char(h_M) = \begin{cases} \min\{N \in \mathbb{N} : |h_M(x)| \leqslant N, \forall x \in X\}, & \text{if there is } N \in \mathbb{N} \text{ with } |h_M(x)| \leqslant N \text{ for all } x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

The following straightforward Lemma shows the usefulness of this new notion in order to identify special classes of hesitant fuzzy sets:

Lemma 1. *For each hesitant fuzzy set h_M on X,*

(1) *h_M is a UHFS if and only if $char(h_M) < +\infty$ and h_M is a THFS.*
(2) *h_M is a FS if and only if $char(h_M) = 1$ and h_M is a THFS.*
(3) *If X is finite then $char(h_M) < +\infty$ when h_M is a THFS.*

In any uniformly typical HFS, there is a number N with the property that every HFE of the form $h_M(x)$ has N or fewer elements. In that case, $char(h_M)$ is the smallest number with such feature. Corollary 1 and Section 4.3 below are additional evidence of the usefulness of that concept.

A uniformly typical HFS h_M with characteristic $N = char(h_M)$ can be formally defined as a subset $M \subseteq X \times \mathcal{F}_N^*([0,1])$ such that for each $x \in X$, there is exactly one element $h_M(x) \in \mathcal{F}_N^*([0,1])$ with the property $(x, h_M(x)) \in M$.

Remark 3. In relation to Definition 5, we observe that if $\mathcal{F} = \{h_{M(i)}\}_{i \in J}$ is a finite family of UHFSs on X, then $h_{\mathcal{F}}$ associated with $\mathcal{F}$ produces a UHFS too. In that case, $char(h_{\mathcal{F}}) \leqslant \sum_{i \in J} char(h_{M(i)})$.

Therefore in particular, if $\mathcal{F} = \{h_{M(i)}\}_{i=1,\dots,N}$ is a finite family of FSs then $h_{\mathcal{F}}$ associated with $\mathcal{F}$ produces a UHFS with characteristic at most N.

3.2. A new notion of cuts for hesitant fuzzy sets

It is known that α-cuts and strong α-cuts are important notions that draw a bridge between classical and fuzzy set theory. Here we define new generalized concepts that permit to extend the decomposition results for fuzzy sets to hesitant fuzzy sets.

Let us fix a hesitant fuzzy set h_M on X. Then for any $\alpha \in [0, 1]$ and any $k \in \{1, 2, \dots, \}$, let

$${}_{\alpha,k}A = \{x \in X : |\{a \in h_M(x) : a \geqslant \alpha\}| \geqslant k\}$$

$_{\alpha+,k}A = \{x \in X : |\{a \in h_M(x) : a > \alpha\}| \geqslant k\}$

be the (α, k)-cut, resp. strong (α, k)-cut associated with h_M.

The (α, k)-cut, resp. strong (α, k)-cut, associated with h_M is the set of elements from X such that at least k membership values of h_M at x are higher or equal, resp. strictly higher, than α. Hence the following properties are immediate:

Proposition 2. *Let h_M be a HFS on X. Fix $x \in X$, and let $L(x) = |h_M(x)|$ be the cardinality of $h_M(x)$, which may be 0 or $+\infty$. Let $\alpha \in [0, 1]$ and $k \in \{1, 2, \ldots, \}$. Then:*

(a) *If $k > L(x)$ then $x \notin_{\alpha,\, k} A$.*
(b) *If $\alpha \leqslant \inf h_M(x)$ and $k \leq L(x)$ then $x \in_{\alpha,\, k} A$.*
(c) *If $h_M(x)$ is a typical HFE and $\alpha \leqslant h_M^{L(x)-k}(x)$ then $x \in_{\alpha,k+1} A$.*

Proof. Statement (a) follows from $|\{a \in h_M(x)\colon\ a \geq \alpha\}| \leq L(x) < k$. Statement (b) follows from $|\{a \in h_M(x) : a \geqslant \alpha\}| = L(x) \geqslant k$. And statement (c) follows from $|\{a \in h_M(x) : a \geqslant \alpha\}| \geqslant |\{h_M^{L(x)-k}(x), \ldots, h_M^{L(x)}(x)\}| \geqslant k+1$. □

The (α, k)-cuts, resp. strong (α, k)-cuts, extend the standard α-cuts, resp. strong α-cuts, from fuzzy set theory. When μ_M is a fuzzy set on X, Remark 1 and Lemma 1 identifies it with a UHFS h_M with characteristic 1. Then with respect to the (α, k)-cuts and strong (α, k)-cuts of h_M, we note that $_{\alpha,1}A = \{x \in X : |\{a \in h_M(x) : a \geqslant \alpha\}| \geqslant 1\}$ returns the α-cut of μ_M. Similarly, $_{\alpha+,1}A$ returns its strong α-cut. Proposition 2(a) proves that $_{\alpha,k}A =_{\alpha+,k} A = \varnothing$ when $k > 1$.

We now introduce a running example that aims at clarifying the notion of (α, k)-cut and will be referred to hereafter:

Example 1. Let $Y = \{x, y\}$ and $h_M = \{(x, \{0.2, 0.5\}), (y, \{0.4\})\}$ be a uniformly typical HFS on Y. Then the characteristic of h_M is 2 because $l_M(x) = 2 > l_M(y)$. We can compute

$$\begin{array}{llll} _{0.2,1}A = & \{x, y\} & = {}_{0.4,1}A & \\ _{0.2,2}A = & \{x\} & & = {}_{0.5,1}A \\ _{0.2,3}A = & \varnothing & = {}_{0.4,2}A & = {}_{0.5,2}A \end{array}$$

These are all distinct (α, k)-cuts associated with h_M. In precise terms:

$$_{\alpha,1}A = \begin{cases} \{x, y\}, & \text{if } \alpha \leqslant 0.4, \\ \{x\}, & \text{if } 0.4 < \alpha \leqslant 0.5, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$_{\alpha,2}A = \begin{cases} \{x\}, & \text{if } \alpha \leqslant 0.2, \\ \varnothing & \text{otherwise.} \end{cases}$$

$_{\alpha,3}A = \varnothing$ for each $\alpha \in [0, 1]$.

The behavior of (α, k)-cuts in Example 1 derives from the following universal property, whose proof consists of a direct checking:

Lemma 2. *Let h_M be a hesitant fuzzy set on X. Then:*

(1) *For any $\alpha, \alpha' \in [0, 1]$ and any $k, k' \in \{1, 2, \ldots, \}$, if $\alpha \geq \alpha'$ and $k \geq k'$ then $_{\alpha,k}A \subseteq_{\alpha',k'} A$.*
(2) *If h_M is uniformly typical then $_{\alpha,k}A = \varnothing$ for each $k > char(h_M)$.*

Lemma 2 in particular assures that $_{\alpha,k}A \subseteq_{\alpha',k} A$ when $\alpha \geq \alpha'$ irrespective of k, and $_{\alpha,k}A \subseteq_{\alpha,k'} A$ when $k \geq k'$ irrespective of α.

Proposition 2 and Lemma 2 are concerned with (α, k)-cuts. It is trivial to derive related statements for strong (α, k)-cuts, whose proofs are closely linked to those that prove these results.

4. Results

We proceed to prove results of two kinds. In Section 4.1 we show that we can represent HFSs by either finite or infinite families of membership functions. Then in Section 4.2 we prove a decomposition theorem for typical HFSs in terms of cut sets as defined in Section 3.2, which by contrast with the latter representation theorem, provides a bridge between crisp and hesitant fuzzy concepts. Finally, Section 4.3 proves that the notion of UHFS permits to define a new extension principle in the setting of hesitant fuzzy sets. Such principle can be defined in more general settings without effort.

4.1. A direct representation result by membership functions

Torra [49, Definition 5] defined the HFS associated with a *finite* family of membership functions $F = \{\mu_1, \ldots, \mu_n\}$, which can be extended to infinite families (cf., Definition 5). The latter construction enables us to state the following representation result:

Theorem 1. *Let h_M be a hesitant fuzzy set on X. Then h_M is the hesitant fuzzy set associated with a (possibly infinite) family of membership functions.*

Proof. Consider the family

$$\mathcal{F} = \{\mu : X \longrightarrow [0, 1] \text{ such that for each } x \in X, \mu(x) \in h_M(x)\}$$

of membership functions. Let us check that h_M is the hesitant fuzzy set associated with $\mathcal{F}$ through Definition 5.

It is clear that $\bigcup_{\mu\in\mathcal{F}} \mu(x) \subseteq h_M(x)$ for each $x \in X$, by construction of $\mathcal{F}$.

In order to prove $h_M(x) \subseteq \bigcup_{\mu\in\mathcal{F}} \mu(x)$ for each $x \in X$, we proceed to chek that when $x \in X$ and $\alpha \in h_M(x)$ with $\alpha \in [0, 1]$ we can assure the existence of $\mu_x^\alpha : X \longrightarrow [0, 1]$ such that $\mu_x^\alpha \in \mathcal{F}$ and $\mu_x^\alpha(x) = \alpha$. We accomplish that aim by the recourse of the Axiom of Choice, which permits to associate an arbitrary $\alpha_y \in \mu(y)$ with each $y \neq x$ and then define

$$\mu_x^\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ \alpha_y & \text{otherwise.} \end{cases}$$

This proves our claim. □

Theorem 1 bears comparison with representation results for other notions like Mendel, John and Liu [39, Theorem 1], in that this result also expresses its focal notion (interval type-2 fuzzy set) as a union of simpler type-2 fuzzy sets where their secondary type-1 membership functions are singletons.

We have made explicit use of the Axiom of Choice in the proof of Theorem 1. Proponents of limited forms of constructive mathematics deny the validity of the Axiom of Choice, even though to most mathematicians it seems quite plausible. Thus it is often pertinent to know whether mathematical statements can be proven without invoking it (cf., Jech [31, p. 47]).

We do not know if Theorem 1 can be proven without the recourse to the Axiom of Choice. Nevertheless we do not need to use it in order to prove the following particular instance:

Corollary 1. *Let h_M be a UHFS on X with characteristic N. Then h_M is the hesitant fuzzy set associated with a family of N membership functions. Furthermore, h_M is not the hesitant fuzzy set associated with any family of fewer than N membership functions.*

Proof. The second claim is trivial, since HFSs associated with families of $N-1$ membership functions have characteristic at most $N-1$ (cf., Remark 3).

To prove the first claim, observe that we can describe

$$h_M(x) = \{h_M^1(x), \ldots, h_M^{l_M(x)}(x)\}$$

where $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$ and $l_M(x) \le N$. For $i = 1, \ldots, N$ we define $\mu_i : X \longrightarrow [0,1]$ by the constructive expression

$$\mu_i(x) = \begin{cases} h_M^i(x), & \text{if } i \leqslant l_M(x), \\ h_M^{l_M(x)} & \text{otherwise.} \end{cases}$$

It is straightforward to check that h_M is the hesitant fuzzy set associated with $\mathcal{F} = \{\mu_i\}_{i=1,\ldots,N}$. □

4.2. A decomposition theorem for typical hesitant fuzzy sets

Just like α-cuts and strong α-cuts are the main tools to represent fuzzy sets, we proceed to show that we can benefit from (α, k)-cuts to propose a decomposition result for THFSs. To that purpose, suppose that for a given THFS h_M, the collection $\{_{\alpha,k}A : \alpha \in [0,1], k \in \{1,2,\ldots,\}\}$ of subsets of X is known. Let us define the fuzzy subsets $_tH$ of X $(t = 1, 2, \ldots,)$ by the following recursive expressions:

$_1H(x) = \max\{\alpha \in [0,1] : x \in_{\alpha,1} A\} = h_M^{l_M(x)}(x)$ for each $x \in X$.

If $_1H, \ldots, {_tH}$ are known then:

$$_{t+1}H(x) = \begin{cases} \max\{\alpha \in [0,1] : x \in_{\alpha,t+1} A\}, & \text{if } x \in_{\alpha,t+1} A \text{ some } \alpha \in [0,1], \\ _tH(x) & \text{otherwise.} \end{cases}$$

Example 2. In the situation of Example 1,

$$\begin{array}{llll} _1H: & Y & \longrightarrow & [0,1] \\ & x & & 0.5 \\ & y & & 0.4 \\ _2H: & Y & \longrightarrow & [0,1] \\ & x & & 0.2 \\ & y & & 0.4 \end{array}$$

and $_2H = {_3H} = {_4H} = \ldots$

The last statement of Example 2 holds for the $_tH$ fuzzy subsets associated with (α, k)-cuts derived from UHFSs. In such case, the following Lemma applies:

Lemma 3. *Let h_M be a uniformly typical hesitant fuzzy set on X and $N = char(h_M)$. Then the $_tH$ fuzzy subsets associated with (α, k)-cuts verify:*

(1) *For each $x \in X$, $_{l_M(x)}H(x) = h_M^1(x)$.*
(2) *$_NH(x) = h_M^1(x)$ for each $x \in X$.*
(3) *If $k > N$ then $_kH = {_NH}$.*

We are ready to prove our first decomposition theorem for HFSs, where we apply Definition 5 after the standard convention that regards every FS as a HFS:

Theorem 2. *Let h_M be a typical hesitant fuzzy set on X. Then h_M is the HFS associated with the family of fuzzy sets $\mathcal{F} = \{_kH\}_{k\in\mathbb{N}}$, i.e.,*

$$h_M = \bigcup_{k=1,2,\ldots} {_kH}.$$

Proof. For every $x \in X$, we proceed to check two set inclusions.

Let us first prove that $\alpha \in h_M(x)$ implies $\alpha \in \bigcup_{k=1,2,\ldots} {_kH}(x)$, i.e., $\alpha \in {_kH}(x)$ for some k.

If $\alpha = h_M^{l_M(x)}(x)$ then $\alpha \in {_1H}(x)$ and we are done.

If $\alpha = h_M^{l_M(x)-1}(x)$ then $x \in_{\alpha,2} A$ because $\{a \in h_M(x) : a \geqslant \alpha\} = \{h_M^{l_M(x)-1}(x), h_M^{l_M(x)}(x)\}$ has exactly 2 elements. Furthermore, $\alpha' > \alpha$ implies that $\{a \in h_M(x) : a \geqslant \alpha'\} = \{h_M^{l_M(x)}(x)\}$ has only 1 element. Therefore $\alpha \in {_2H}(x)$ and we are done.

A direct recursive argument completes this part of the proof: when $\alpha = h_M^{l_M(x)-t}(x)$ then $x \in_{\alpha,t+1} A$ for $t = 2, \ldots, l_M(x) - 1$.

Let us now prove that $\alpha \in h_M(x)$ when $\alpha \in \bigcup_{k=1,2,\ldots,} {_kH}(x)$, i.e., when $\alpha \in {_kH}(x)$ for some $k = 1, 2, \ldots,$ Hence we assume $\alpha \in {_{k'}H}(x)$ for some k', and we let k be the smallest index with that property.

If $k = 1$ then we are done because $\alpha \in {_1H}(x)$ means $\alpha = h_M^{l_M(x)}(x)$. Therefore we proceed with the case $k > 1$.

One must conclude that for some $\alpha' \in [0, 1]$, $x \in_{\alpha',k} A$ because otherwise $\alpha = {_kH}(x) = {_{k-1}H}(x)$ which contradicts the choice of the k index. Therefore $\alpha = {_kH}(x) = \max\{\alpha' \in [0,1] : x \in_{\alpha',k} A\}$.

Now the latter fact boils down to $|\{a \in h_M(x):\ a \ge \alpha'\}| < k$ if $\alpha' > \alpha$, and $|\{a \in h_M(x):\ a \ge \alpha\}| \ge k$. The combination of both properties produce the desired conclusion $\alpha \in h_M(x)$. □

Theorem 2 produces a decomposition of any THFS in terms of the simplest THFSs, which are the fuzzy sets.

Example 3. In the situation of Example 1, $h_M = \bigcup_{k=1,2,\ldots} {_kH} = {_1H} \cup {_2H}$ because $_2H = {_3H} = {_4H} = \ldots$ This equality is simple to check with our data.

4.3. New extension principles

Principles for fuzzifying crisp functions are called extension principles (cf., Klir and Yuan [34, Section 2.3]). Concerning hesitant fuzzy sets, to the best of our knowledge there are no similar extension principles in the literature.

We proceed to define a new principle that extends crisp functions (say, from X to Y) to functions defined on typical hesitant fuzzy sets. The argument will be more clear if we restrict ourselves to UHFSs, which are the most relevant cases for possible applications. Afterwards we establish properties that prove that our principle indeed generalizes the aforementioned extension principle for fuzzy sets.

Subsequently we define an extended version of the inverse of that crisp mapping to functions defined on hesitant fuzzy sets. And then we investigate some of its main fudamental properties.

In the remaining of this section we fix a crisp mapping $f : X \longrightarrow Y$. Unless otherwise stated, we assume that f is surjective.

4.3.1. First extension principle

The mapping f can be extended to $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ through the following expression. For each $h_M \in \mathbf{UHFS}(X)$ with characteristic N, we decompose $h_M(x) = \{h_M^1(x), \ldots, h_M^{l_M(x)}(x)\}$ where $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$ and $l_M(x) \le N$. There must be $x \in X$ with $l_M(x) = N$. Then we define

$$\begin{array}{llll} \bar{f}(h_M): & Y & \longrightarrow & \mathcal{F}^*([0,1]) \\ & y & & \bigcup_{i=1,\ldots,N} \{\sup\{h_M^i(x) : f(x) = y, x \in X\}\} \end{array}$$

with the natural convention $\{\sup \varnothing\} = 0$ when there is no $x \in X$ such that $f(x) = y$. Clearly $\bar{f}(h_M)$ is another UHFS with characteristic N. Observe that surjectivity of f guarantees the key fact $\bar{f}(h_M)(y) \neq \varnothing$, because $\{x \in X : f(x) = y\} \neq \varnothing$, for each $y \in Y$.

By notational convenience we also denote the standard decomposition of the typical HFE $\bar{f}(h_M)(y)$ as

$$\bar{f}(h_M)(y) = \{\bar{h}_M^1(y), \ldots, \bar{h}_M^{\bar{l}_M(y)}(y)\} \quad (1)$$

and then $\bar{h}_M^i(y) = \sup\{h_M^i(x) : f(x) = y, x \in X\}$ for each $i = 1, \ldots, \bar{l}_M(y)$. Implicit in this expression is the fact that $\bar{h}_M^1(y) = (\bar{f}(h_M))^-(y)$, the lower bound of the HFE $\bar{f}(h_M)(y)$ as defined in Torra and Narukawa [50, Definition 11].

The following example illustrates the application of our new extension principle.

Example 4. Let $X = \{x, y, z, t\}$ and let

$h_M = \{(x, \{0.2, 0.5\}), (y, \{0.4\}), (z, \{0.2, 0.4, 0.6\}), (t, \{0.1, 0.3, 0.7\})\}$

be a UHFS on X. The characteristic of h_M is 3 because $l_M(z) = l_M(t) = 3 > l_M(x) = 2 > l_M(y) = 1$.

Define $Y = \{a, b\}$ and let $f : X \longrightarrow Y$ be the surjective mapping $f(x) = f(y) = a,\ f(z) = f(t) = b$.

We can compute $\bar{f}(h_M) : Y \longrightarrow \mathcal{F}^*([0,1])$ as follows. In order to calculate $\bar{f}(h_M)(a)$ we use

$\bar{f}(h_M)(a) = \bigcup_{i=1,2,3}\{\sup\{h_M^i(\bar{x}) : f(\bar{x}) = a, \bar{x} \in X\}$

$\bar{f}(h_M)(a) = \bigcup_{i=1,2,3}\{\sup\{h_M^i(x), h_M^i(y)\}\}$

$\bar{f}(h_M)(a) = \{\sup\{h_M^1(x), h_M^1(y)\}\} \cup \{\sup\{h_M^2(x), h_M^2(y)\}\} \cup \{\sup\{h_M^3(x), h_M^3(y)\}\}$

$\bar{f}(h_M)(a) = \{\sup\{0.2, 0.4\}\} \cup \{\sup\{0.5\}\} \cup \{\sup\{\varnothing\}\} = \{0.4, 0.5\}$.

In order to calculate $\bar{f}(h_M)(b)$ we use a similar methodology, which produces $\bar{f}(h_M)(b) = \{0.2, 0.4, 0.7\}$.

In conclusion, f can be extended to $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ in such way that its application to h_M is

$$\begin{array}{llll} \bar{f}(h_M): & Y & \longrightarrow & \mathcal{F}^*([0,1]) \\ & a & & \{0.4, 0.5\} = \{\bar{h}_M^1(a), \bar{h}_M^2(a)\}\ (\bar{l}_M(a) = 2) \\ & b & & \{0.2, 0.4, 0.7\} = \{\bar{h}_M^1(b), \bar{h}_M^2(b), \bar{h}_M^3(b)\}\ (\bar{l}_M(b) = 3) \end{array}$$

Fuzzy sets are particular UHFSs with characteristic 1 by Lemma 1. Thus our extension principle generalizes the classical extension principle for fuzzy sets (cf. Klir and Yuan [34, Section 2.3]), in the sense that if h_M is a FS on X then our formula for $\bar{f}(h_M)$ boils down to

$$\begin{array}{llll} \bar{f}(h_M): & Y & \longrightarrow & \mathcal{P}^*([0,1]) \\ & y & & \{\sup\{h_M(x) : f(x) = y, x \in X\}\} \end{array}$$

which is the classical expression defining the extension principle in fuzzy set theory. In particular, if h_M is a FS on X then $\bar{f}(h_M)$ is a FS on Y.[1]

In order for our construction to be fully acceptable, the standard properties of that extension principle should be preserved. For expositional purposes here we only prove one key proposition, that will be subsequently generalized.

Klir and Yuan [34, Theorem 2.8 (ii)] demonstrate that if A_1, A_2 are fuzzy sets on X with $A_1 \subseteq A_2$ then $f(A_1) \subseteq f(A_2)$ when f is defined by the application of the extension principle for fuzzy sets. Our next result proves that this property holds true in our model of extension principle for UHFSs too.

Proposition 3. *Let $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ be the extension of the surjective mapping $f : X \longrightarrow Y$. Then $h_1 \subseteq h_2$ and $h_1, h_2 \in \mathbf{FS}(X)$ imply $\bar{f}(h_1) \subseteq \bar{f}(h_2)$.*

Proof. We use the comments on the application of Definition 4 to FSs. We know $\bar{f}(h_1)(y) = \{\sup\{h_1(x) : f(x) = y, x \in X\}\}$ and $\bar{f}(h_2)(y) = \{\sup\{h_2(x) : f(x) = y, x \in X\}\}$ for each $y \in Y$. The fact $h_1 \subseteq h_2$ reduces to $h_1(x) \leq h_2(x)$ for all $x \in X$. Therefore $\bar{f}(h_1)(y) \leqslant \bar{f}(h_2)(y)$ for each $y \in Y$ and we conclude $\bar{f}(h_1) \subseteq \bar{f}(h_2)$ because both $\bar{f}(h_1)$ and $\bar{f}(h_2)$ are FSs on Y. □

What further behavior can we assure for the extension principle with respect to inclusion of HFSs? In order to derive a more general property, we next prove a technical lemma that exploits the structure of the inclusion of HFSs.

Lemma 4. *Suppose that $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ is the extension of the surjective mapping $f : X \longrightarrow Y$. Then $h_1 \subseteq h_2$ and $h_1, h_2 \in \mathbf{UHFS}(X)$ with X finite imply $\bar{h}_1^1(y) \leqslant \bar{h}_2^1(y)$ for each $y \in Y$.*

Proof. We recall that $\bar{h}_1^1(y)$ denotes the lower bound of $\bar{f}(h_1)(y)$ and $\bar{h}_2^1(y)$ is the lower bound of $\bar{f}(h_2)(y)$. Hence $\bar{h}_2^1(y) = \sup\{h_2^1(x) : f(x) = y\}$. By the finiteness assumption, the supremum is attained at a point x_0. Hence there is $x_0 \in X$ with $f(x_0) = y$, $\bar{h}_2^1(y) = h_2^1(x_0) \in h_2(x_0) = h_1(x_0) \cup h_2(x_0)$ because $h_1 \subseteq h_2$, and also $f(x) = y$ implies $h_2^1(x_0) \geqslant h_2^1(x)$ whenever $x \in X$. d Similarly, $\bar{h}_1^1(y) = \sup\{h_1^1(x) : f(x) = y\}$ guarantees the existence of x'_0 with $f(x'_0) = y$ and $\bar{h}_1^1(y) = h_1^1(x'_0)$.

We deduce from the properties of x_0 that $\bar{h}_2^1(y) = h_2^1(x_0) \geqslant h_2^1(x'_0)$.

Now the fact $h_2^1(x'_0) \in h_2(x'_0) = h_1(x'_0) \cup h_2(x'_0)$ guarantees by definition that $h_2^1(x'_0) \geqslant \max\{h_1^1(x'_0), h_2^1(x'_0)\} \geqslant h_1^1(x'_0)$.

These inequalities assure $\bar{h}_2^1(y) \geqslant h_2^1(x'_0) \geqslant h_1^1(x'_0) = \bar{h}_1^1(y)$. □

Proposition 4 and Example 5 below investigate if Proposition 3 can be extended to general UHFSs. Example 5 shows that the answer is negative. Although Proposition 4 applies when X is finite, the case of a general X is similar but requires a longer argument.

Proposition 4. *Let $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ be the extension of the surjective mapping $f : X \longrightarrow Y$ with X finite. Then $h_1 \subseteq h_2$ and $h_1, h_2 \in \mathbf{UHFS}(X)$ imply $\bar{f}(h_2)(y) \subseteq (\bar{f}(h_1) \cup \bar{f}(h_2))(y)$ for each $y \in Y$.*

Proof. We use that $\max\{\bar{h}_1^1(y), \bar{h}_2^1(y)\} = \bar{h}_2^1(y)$ by Lemma 4. As recalled in the proof of this Lemma, $\bar{h}_1^1(y)$ and $\bar{h}_2^1(y)$ are the lower bounds of $\bar{f}(h_1)(y)$ and $\bar{f}(h_2)(y)$ respectively. Fix any $y \in X$.

Suppose $h \in \bar{f}(h_2)(y)$. Then it is obvious that $h \in \bar{f}(h_1)(y) \cup \bar{f}(h_2)(y)$. Clearly it is also the case that $h \geqslant \bar{h}_2^1(y) = \max\{\bar{h}_1^1(y), \bar{h}_2^1(y)\}$. Hence we have checked the claim $h \in (\bar{f}(h_1) \cup \bar{f}(h_2))(y)$. □

In order to clarify the definitions and relationships above, the following example is useful.

Example 5. In the conditions of Proposition 4, the property $\bar{f}(h_1) \subseteq \bar{f}(h_2)$ when $h_1 \subseteq h_2$ is not universally true.

Consider the situation of Example 4. Define

$$h_R = \{(x, \{0.2, 0.5\}), (y, \{0.4, 0.6\}), (z, \{0.2, 0.4, 0.6\}), (t, \{0.1, 0.3, 0.7\})\}.$$

It is a UHFS on X with characteristic 3 because $3 = l_R(z) = l_R(t) > l_R(x) = l_R(y) = 2$. The reader can easily check that $h_M \subseteq h_R$, i.e., that $h_M \cup h_R = h_R$.

We can compute $\bar{f}(h_R) : Y \longrightarrow \mathcal{F}^*([0,1])$ as in Example 4. We conclude that the application of $\bar{f} : \mathbf{UHFS}(X) \longrightarrow \mathbf{UHFS}(Y)$ to h_R is

$$\begin{array}{llll} \bar{f}(h_R): & Y & \longrightarrow & \mathcal{F}^*([0,1]) \\ & a & & \{0.4, 0.6\} = \{\bar{h}_R^1(a), \bar{h}_R^2(a)\}\ (\bar{l}_R(a) = 2) \\ & b & & \{0.2, 0.4, 0.7\} = \{\bar{h}_R^1(b), \bar{h}_R^2(b), \bar{h}_R^3(b)\}\ (\bar{l}_R(b) = 3) \end{array}$$

It is now apparent that $\bar{f}(h_M) \subseteq \bar{f}(h_R)$ is false, therefore $\bar{f}(h_M) \cup \bar{f}(h_R) = \bar{f}(h_R)$ cannot be proven.

Nevertheless $\bar{f}(h_R)(a) \subseteq (\bar{f}(h_M) \cup \bar{f}(h_R))(a)$ and $\bar{f}(h_R)(b) = (\bar{f}(h_M) \cup \bar{f}(h_R))(b)$, in agreement with Proposition 4.

4.3.2. Second extension principle

It is also possible to define an extended version of the inverse of the crisp mapping f that generalizes the standard construction in Klir and Yuan [34, Equation (2.11)]. This extension should map HFSs on Y into HFSs on X. We proceed to do this and then we investigate its main properties.

The (*not necessarily surjective*) mapping $f : X \longrightarrow Y$ generates a mapping $\bar{f}^{-1} : \mathbf{HFS}(Y) \longrightarrow \mathbf{HFS}(X)$ by the expression: for each $h_M \in \mathbf{HFS}(Y)$,

$$\begin{array}{llll} \bar{f}^{-1}(h_M): & X & \longrightarrow & \mathcal{P}([0,1]) \\ & x & & h_M(f(x)) \end{array}$$

Clearly, this construction is the natural generalization of the usual definition for FSs [34, Equation (2.11)] to HFSs.

[1] The reader is reminded that Remark 1 formally identifies FSs with adequate HFSs. For this reason the codomain here is $\mathcal{P}^*([0,1])$ instead of [0, 1]. We insist that surjectivity of f permits to ensure that the supremum is taken in a non-empty set of numbers.

Remark 4. We have stated that our construction of the second extension principle is valid when f is not surjective. Nevertheless the reader should be aware that some of its properties crucially depend on surjectivity. We are explicit in explaining when we impose this requirement in the remaining of this section.

Now we present some immediate properties of our second extension principle and its relationships with the first extension principle in Section 4.3.1.

1. For each typical $h_M \in \mathbf{HFS}(Y)$, $\bar{f}^{-1}(h_M)(x)$ is a typical HFS, and
$$\bar{f}^{-1}(h_M)(x) = \{h_M^1(f(x)), \dots, h_M^{l_M(f(x))}(f(x))\}$$
Furthermore, $\bar{f}^{-1}(h_M)$ is a UHFS on X when h_M is uniformly typical.
2. For each $h_M \in \mathbf{UHFS}(Y)$, if f is surjective then $\bar{f}(\bar{f}^{-1}(h_M)) = h_M$. In particular, this is true when h_M is a FS on Y.
To check this claim we only need to recall that the standard decomposition of the typical HFE $\bar{f}(\bar{f}^{-1}(h_M))(y) = \{\tilde{h}_M^1(y), \dots, \tilde{h}_M^{l_M(y)}(y)\}$ verifies $\tilde{h}_M^i(y) = \sup\{[\bar{f}^{-1}(h_M)]^i(x) : f(x) = y, x \in X\}$. The assumption that f is surjective ensures that we obtain a number (i.e., we do not take the supremum of a void set). This figure is $h_M^i(y)$ because $[\bar{f}^{-1}(h_M)]^i(x) = h_M^i(f(x))$ for each i by definition.
3. For each $h_M \in \mathbf{UHFS}(X)$, and each $x \in X$,
$$[\bar{f}^{-1}(\bar{f}(h_M))](x) = \{\sup\{h_M^1(x) : x \in X\}, \dots, \sup\{h_M^N(x) : x \in X\}\}$$
because $[\bar{f}^{-1}(\bar{f}(h_M))](x) = [\bar{f}(h_M)](f(x)) \neq \varnothing$.
The standard decomposition $[\bar{f}(h_M)](f(x)) = \{\hat{h}_M^1(f(x)), \dots, \hat{h}_M^N(f(x))\}$ is computed by replacing $y = f(x)$ in equation (1), which produces
$$\hat{h}_M^i(x) = \sup\{h_M^i(x) : f(x) = f(x), x \in X\}.$$
This formula gives the expression above, since all $x \in X$ verify the property $f(x) = f(x)$.
We deduce that when h_M is a FS on Y then $h_M \subseteq \bar{f}^{-1}(\bar{f}(h_M))$.
4. If $h_1, h_2 \in \mathbf{HFS}(Y)$ are typical and $h_1 \subseteq h_2$ then $\bar{f}^{-1}(h_1) \subseteq \bar{f}^{-1}(h_2)$.
To prove this claim we need to check $\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2) = \bar{f}^{-1}(h_2)$. Let us fix $x \in X$. We need to verify the equality $[\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x) = (\bar{f}^{-1}(h_2))(x)$.
Firstly we observe that $h \in [\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x)$ implies $h \in \bar{f}^{-1}(h_2)(x)$ by definition of the union of HFSs.
Secondly we select an arbitrary $h \in \bar{f}^{-1}(h_2)(x) = h_2(f(x))$. Since h_1, h_2 are typical and $h_1 \subseteq h_2$ we can assure $\inf h_1(f(x)) = h_1^1(f(x)) \leqslant h_2^1(f(x)) = \inf h_2(f(x))$. Therefore $h \in \bar{f}^{-1}(h_1)(x) \cup \bar{f}^{-1}(h_2)(x)$, and $h \geqslant \max\{h_1^1(f(x)), h_2^1(f(x))\} = h_2^1(f(x))$ because $h \in h_2(f(x))$. By definition we have stated that $h \in [\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x)$, because $h_1^1(f(x))$, resp. $h_2^1(f(x))$, is the lower bound of $\bar{f}^{-1}(h_1)(x) = h_1(f(x))$, resp. $\bar{f}^{-1}(h_2)(x) = h_2(f(x))$ due to Property 1 in this list.
5. If $h_1, h_2 \in \mathbf{HFS}(Y)$ are typical then $\bar{f}^{-1}(h_1 \cup h_2) = \bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)$.
To prove this claim we fix $x \in X$ in order to check the set equality $[\bar{f}^{-1}(h_1 \cup h_2)](x) = [\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x)$.
By definition, $h \in \bar{f}^{-1}(h_1 \cup h_2)(x)$ is equivalent to $h \in (h_1 \cup h_2)(f(x))$, which is equivalent to $h \in h_1(f(x)) \cup h_2(f(x))$ and $h \geqslant \max\{h_1^1(f(x)), h_2^1(f(x))\}$ by the definition of union of HFSs.
This definition also shows that $h \in [\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x)$ is equivalent to $h \in \bar{f}^{-1}(h_1)(x) \cup \bar{f}^{-1}(h_2)(x)$ and $h \geqslant \max\{h_1^1(f(x)), h_2^1(f(x))\}$. The latter inequality uses Property 1 in this list in order to identify the lower bounds of $\bar{f}^{-1}(h_1)(x)$ and $\bar{f}^{-1}(h_2)(x)$.
Now we can observe that the respective equivalent statements coincide.
Put shortly, we have checked that $h \in [\bar{f}^{-1}(h_1 \cup h_2)](x)$ holds if and only if $h \in [\bar{f}^{-1}(h_1) \cup \bar{f}^{-1}(h_2)](x)$ holds.
6. If $h_1, h_2 \in \mathbf{HFS}(Y)$ are typical then $\bar{f}^{-1}(h_1 \cap h_2) = \bar{f}^{-1}(h_1) \cap \bar{f}^{-1}(h_2)$.
The proof of this claim is an immediate modification of the proof of the previous property.

4.3.3. Relationship with the literature and decision making

Our novel extension principles are new in the literature on HFSs. Having said that, it is worth mentioning that an altogether different extension principle had been stated before, and that our results are related to existing principles in fuzzy set theory.

Torra and Narukawa [50, Section IV] already refer to an extension principle for extending crisp functions to hesitant fuzzy sets. Nonetheless their idea originates from a completely independent motivation. These authors extend operators $\mathbb{O} : [0, 1]^N \longrightarrow [0, 1]$ (like the arithmetic mean) in such way that they can operate on HFSs, by considering all values in such sets and the application of $\mathbb{O}$ on them. This practical position is unrelated to our perspective in this paper, which in fact owes to the standard viewpoint on extension principles for fuzzy sets. Let us recap these developments.

Extension principles were introduced by Zadeh [58] in the fundamental theory of fuzzy sets, and their relevance is highlighted in many textbooks and articles [10,14,34,43,55]. They allow us to compute an (approximate) functional dependence among variables even when the argument of a given precise mapping is only approximately known as a fuzzy set. For example, one may define arithmetic operations for fuzzy numbers from the application of the extension principle to the standard operations for real numbers. In connection with applications, Bělohlávek [10] explains that "[t]he extension principle is used mainly in situations where no precise description of the input data is available, e.g., if a linguistic variable is used to describe the inputs". Dubois [21, section 4] assures that fuzzy intervals have been widely used in fuzzy decision analysis, in a way that comes down to applying the extension principle to existing evaluation tools. Recently de Barros et al. [19, Example 2.10] illustrate this point with a practical transport situation. In addition, [19, Chapter 2] concludes with a relationship of the extension principle and problems with probabilities. Clearly these remarks can be exported to our setting too, since our results generalize the original statements in fuzzy set theory. Our extension principles are therefore directly applicable in decision making.

However the extension principle applied to arithmetic operators yields an unwieldy nonlinear programming problem hence it is unfeasible for real time calculation in many applications [5]. Authors like Kaufmann and Gupta [33] or Giachetti and Young [27] showed that using α-cuts to represent fuzzy numbers by crisp intervals, one can apply interval arithmetic operations (e.g., addition and subtraction, multiplication and division, power) in order to perform fuzzy arithmetic in a computationally efficient manner (see also [23]). Such parametric representation is easily understood by practitioners, and provides accuracy and efficiency at a time. The fuzzy arithmetic that arises has been used in applications to engineering (e.g., industrial machining processes [20], classical control design [51]), experimental measurement in physics [48], mining investments [6,7], management sciences [30,33], nutrition [11,12], ... In addition, Chen and Lu [15,16] use several α-cuts of fuzzy numbers in order to incorporate quality factors for improving decision-making solutions based on fuzzy number rankings, which they illustrate with various examples. Quality is defined by a signal/noise ratio which refers to middle-point and spread of each α-cut. Therefore the new notion of α-cuts is also at the core of potential applications of HFSs by assimilation to the fundamental techniques in fuzzy set theory.

5. Conclusions

In this paper we have introduced some new notions in the fundamental theory of hesitant fuzzy sets. The novel notion of uniformly typical HFS simplifies many theoretical and practical arguments. It is a particular case of typical HFSs which remained undefined albeit it is thoroughly used in real world applications, where both the number of alternatives and attributes are finite (Proposition 1). Uniformly typical HFSs are HFSs for which not only all HFEs that define it are typical, but also their cardinality is bounded by some fixed number. The characteristic of a HFS is a related operator on HFSs. When a HFS is uniformly typical, its characteristic is the smallest integer with the above mentioned bounding property. Otherwise its characteristic is infinite. This novel operator facilitates the analysis of computational complexity of algorithmic solutions hence it is relevant for the discussion of feasibility and implementability issues in decision making.

The construction of HFSs from a finite family of membership functions has been extended to arbitrary families. This enabled us to prove a representation result that resembles other approaches for interval type-2 fuzzy sets (see Section 4.1).

We have also defined (α, k)-cuts for HFSs. These are the HFS counterpart of α-cuts in fuzzy set theory which were missing in the literature. With these new elements we have proved a decomposition theorem for typical HFSs in Section 4.2. In addition we have defined two novel extension principles in section 4.3, the first of which applies to uniformly typical HFSs and the second to generic HFSs. We have proved several properties of these principles. Here some explanatory examples help the reader to understand their application, which may underlie corresponding decision making mechanisms (cf., section section 4.3.3). We have also studied their relationships with other methods from the literature

These results show that the novel notion of uniformly typical HFSs is very promising and deserves careful consideration. They also demonstrate that the fundamental theory of hesitant fuzzy sets is still open to novel contributions.

Some of the ideas that we present can be developed further. For example, the interaction of the extension principles with unions and intersections may be the subject of an additional analysis. Their relationship with (α, k)-cuts can be discussed too.

In addition, it is plausible to extend the present study to the case of Generalized hesitant fuzzy sets (GHF-sets), introduced in Qian et al. [44], and subsequently studied e.g., by Farhadinia [24] and N. Chen et al. [17].

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