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Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data

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In the present work, we consider a parabolic convection-diffusion-reaction problem where the diffusion and convection terms are multiplied by two small parameters, respectively. In addition, we assume that the convection coefficient and the source term of the partial differential equation have a jump discontinuity. The presence of perturbation parameters leads to the boundary and interior layers phenomena whose appropriate numerical approximation is the main goal of this paper. We have developed a uniform numerical method, which converges almost linearly in space and time on a piecewise uniform space adaptive Shishkin-type mesh and uniform mesh in time. Error tables based on several examples show the convergence of the numerical solutions. In addition, several numerical simulations are presented to show the effectiveness of resolving layer behavior and their locations.

KEYWORDS

initial-boundary value problem, interior and boundary layer phenomena, non-smooth data, parabolic convection-diffusion problem, parameter uniformly convergent method, Shishkin-type mesh, singular perturbation, two-parameter

1 | INTRODUCTION

Boundary and interior layers originated by singularly perturbed problems (SPPs) are very frequent in several fields of engineering like drift diffusion equation of semi conductor modeling,[1] chemical reactor model,[2] and fluid dynamics.[3] In the present work, we consider the following two-parameter parabolic initial-boundary value problem (IBVP) on the domain $\overline{\Gamma} = \overline{\Omega}_x \times \Omega_t$, which combines the reaction-diffusion and convection-diffusion forms:

$$L_{\varepsilon,\mu} y(x,t) \equiv (\varepsilon y_{xx} + \mu a y_x - by - cy_t)(x,t) = f(x,t), \quad (x,t) \in (\Gamma^- \cup \Gamma^+), \tag{1}$$

$$y(x,t) = p(x,t),\ (x,t) \in \Gamma_c,\ y(x,t) = q(x,t),\ (x,t) \in \Gamma_l,\ y(x,t) = r(x,t),\ (x,t) \in \Gamma_r. \tag{2}$$

Here, $0 < \varepsilon \ll 1, 0 \le \mu \le 1$ are two singular perturbation parameters. The coefficient functions $b(x,t), c(x,t)$ are assumed to be sufficiently smooth functions on Γ such that $b(x,t) \ge \beta > 0, c(x,t) \ge \nu > 0$. In addition, we assume $a(x,t), f(x,t)$ are sufficiently smooth on $(\Gamma^- \cup \Gamma^+)$ such that

$$a(x,t) \le -\alpha_1 < 0,\ (x,t) \in \Gamma^- \quad \text{and} \quad a(x,t) \ge \alpha_2 > 0,\ (x,t) \in \Gamma^+. \tag{3}$$

Here, α_1, α_2 are positive constants. Let $\alpha = \min\{\alpha_1, \alpha_2\}$. In addition, we assume the jumps of $a(x,t)$ and $f(x,t)$ at (d,t) satisfying $|[a](d,t)| \le C, |[f](d,t)| \le C$, where the jump of ω at (d,t) is defined as $[\omega](d,t) = \omega(d+,t) - \omega(d-,t)$.

In general, the presence of discontinuity in $a(x,t)$ and $f(x,t)$ leads to an interior layer in the neighborhood of the point of discontinuity of $y(x,t)$ in addition to the boundary layer phenomena due to the presence of perturbation parameters. Under these assumptions, the problem (1)-(2) has a continuous unique solution $y(x,t)$ on the whole domain.

There can be two special cases: reaction-diffusion ($\mu = 0$) and convection-diffusion ($\mu = 1$), which can appear corresponding to a general two-parameter SPPs. When the parameter $\mu = 0$, the boundary layers appear in both sides of the boundary points on the domain with an approximate width $O(\sqrt{\epsilon})$. When $\mu = 1$, the boundary layers appear in the neighborhood of either left or right boundary point of width $O(\epsilon)$ and mainly depend on the sign of the convection coefficient. The solution of this problem with only discontinuous source term contains weak interior layers when $a(x,t) > 0$ (or alternatively $a(x,t) < 0$) for any value of ϵ. However, the discontinuity in the convection coefficient ($a(x,t) < 0, x < d$ with $a(x,t) > 0, x > d$) can lead to strong interior layers in the solution. A related discussion can be seen in Mukherjee and Natesan[4] for the particular case $\mu = 1$. In this case, the sign of convection coefficient shifts the layers position in the solution. The detailed discussion on this matter is given in the numerical section of the present article by considering several examples.

In general, the discontinuity in the convection coefficient and source term for two-parameter SPPs can give rise to interior layers along with boundary layers. It is observed that the sign of the convection coefficient and the magnitude of the perturbation parameters have many influence on determining the nature of the layer. A discussion on the effect of discontinuity at convection coefficient and source term for singularly perturbed convection-diffusion problems can be seen in Farrell et al.[5] for ordinary differential equations and in O'Riordan and Shishkin and Clavero et al.[6,7] for partial differential equations.

In the present work, we have presented in the numerical section these cases by taking several examples which satisfy the condition (3) and also the reverse situation, ie, $a(x,t) \geq \alpha_1 > 0, (x,t) \in \Gamma^-$, and $a(x,t) \leq -\alpha_2 < 0, (x,t) \in \Gamma^+$. Let $\gamma = \min_{(\Gamma^- \cup \Gamma^+)} \left\{ \frac{b(x,t)}{\alpha^*(x,t)} \right\}$, where $\alpha^*(x,t) = \alpha_1, (x,t) \in \Gamma^-$, and $\alpha^*(x,t) = \alpha_2, x \in \Gamma^+$. We divide the convergence analysis for (1) to (2) into 2 cases: $\alpha\mu^2 \leq \gamma\epsilon$ and $\alpha\mu^2 \geq \gamma\epsilon$, ie, depending upon the ratio of the perturbation parameters μ^2 to ϵ. In the first case, the analysis follows closely that of parabolic reaction-diffusion type when $\mu = 0$[8] ; however, in the second case, the analysis is comparatively more difficult. This problem is well studied for parabolic convection-diffusion case with smooth data in O'Riordan et al.[9]

For SPPs, the existence and uniqueness of classical solutions for steady state problems are well established in several papers, say Feckan.[10] The analysis for two-parameter problems mainly started by O'Malley, based on asymptotic expansion methods.[11] The numerical solution of these problems with smooth data[12-15] and nonsmooth data[16-19] are also considered in several journals and proceedings. Singularly perturbed parabolic problems are also considered for numerical analysis in Bansal et al. and Munyakazi and Patidar.[20,21] An almost first-order accurate solution for two-parameter singulary perturbed time-dependent problems with smooth data can be seen in O'Riordan et al.[9] on a piecewise uniform mesh. In Kadalbajoo and Yadaw,[22] the authors proposed a combination of finite element method in space with implicit Euler method in temporal direction using Rothe's method and obtained an almost second-order accuracy in space. First-order accurate methods for smooth data are also observed in other studies.[23-26] Analysis on different meshes like Shishkin-Bakhvalov meshes can be also seen in Jha and Kadalbajoo.[27] The a posteriori based convergence analysis for singularly perturbed parameterized problems is also carried out to get optimal order parameter uniform accuracy in Das,[28] which avoids the requirement of a priori derivative bound. An equidistribution based new adaptive mesh is proposed in Das and Mehrmann[29] for parabolic problems and in Das and Vigo-Aguiar[30] for systems of reaction-diffusion problems with smooth data. The first-order space time uniform convergence obtained in Das and Mehrmann[29] is also enhanced to higher order accuracy in Das[31] by extrapolation methods. This mesh is generated by the equidistribution of a special positive monitor function. Clavero et al.[32] discussed the numerical analysis of a parabolic singularly perturbed reaction-convection-diffusion problem where the source term has a discontinuity of first kind on the degeneration line and obtained an almost first-order accuracy on Shishkin-type mesh when the convection parameter is less than the diffusion parameter.[7] In Clavero et al.,[32] the authors have examined a parameter uniform numerical method for singularly perturbed one-dimensional parabolic convection-diffusion problem, with degenerate convection term and discontinuous source term at the same point inside the domain. A recent appraisal on different classes of boundary and interior layers can be noticed in O'Riordan.[33] In Chandru et al. and Cen,[16,34] the authors have developed a higher order numerical scheme for singularly perturbed ordinary differential equations with discontinuous convection coefficient and source term. In the present work, we consider a parameter uniform numerical method for parabolic convection-diffusion-reaction problems where both the convection coefficient and source term have discontinuity inside the domain, which leads to an interior layer in addition to the boundary layers.

Throughout the paper, C will be denoted as a generic positive constant, which is independent of the parameters ε, μ, N_x, and N_t. Here, N_x, N_t are the number of mesh intervals used in space and time direction, respectively. We use the maximum norm for our analysis, which is defined as: $||w||_{\overline{G}} = \max_{(x,t)\in\overline{G}}|w(x,t)|$, where $\overline{G}$ is a closed bounded region. The discrete maximum norm is denoted as $||W||_{\overline{G}^N} = \max\limits_{0\le i\le N_x, 0\le j\le N_t} |W(x_i, t_j)|$, where $\overline{G}^N$ denotes the discretized version of $\overline{G}$ in space and time. To simplify the notation, we use $||.||$ instead of $||.||_{\overline{G}}$ or $||.||_{\overline{G}^N}$. The following operators will be frequently used in the later analysis

$$L_0 y(x,t) = (-by - cy_t)(x,t), \quad L_\mu y(x,t) = (a\mu y_x + L_0 y)(x,t), \quad L_{\varepsilon,\mu} y(x,t) = \left(\varepsilon y_{xx} + L_\mu y\right)(x,t).$$

In addition, a function $g(x,t)$ is said to be in class C^p if the partial derivatives of all order up to p are continuous with respect to its independent variables.

This article is sequentially divided as follows. We discuss the existence, uniqueness, and stability properties of the continuous solution and its decomposition into regular (smooth) and singular components in Section 2. Their derivative bounds are also mentioned here. The discretization of the continuous problem is presented in Section 3. In Section 4, the truncation error analysis and stability of the numerical solution based on barrier function technique are considered. The main result is highlighted in Section 5. Numerical examples are provided to show the convergent solution and its expected rate of accuracy in Section 6, which indicates that the convergence does not depend on the parameters size. The overall conclusion is mentioned in Section 7. In addition, for ease of reading, we provide a nomenclature table for frequently used notations.

2 | ANALYTICAL PROPERTIES OF THE SOLUTION

For the numerical analysis, the derivative bounds of (1) to (2) are useful, which will be addressed in this section. In this paper, we consider that the solution of (1) to (2) is parameter uniformly bounded. Equations 1 and 2 are considered for the parabolic case with $\mu = 1$ in Farrell et al.,[5] and the existence of the solution is considered for steady state case with $\mu = 0$ in O'Riordan and Shishkin.[6] Now, let us consider the following lemma, which shows the existence of solution.

Lemma 1. *The problem (1) to (2) has a solution* $y(x,t) \in C^0(\overline{\Gamma}) \cap C^1(\Gamma) \cap C^2(\Gamma^- \cup \Gamma^+)$.

Proof. The Lemma can be proved based on the constructive method by following the procedures given in Farrell et al.[5] Consider the functions $y_1(x,t)$ and $y_2(x,t)$, which satisfy, respectively, the following singularly perturbed differential equations:

$$(\varepsilon y_{1xx} + \mu a_1 y_{1x} - by_1 - cy_{1t})(x,t) = f(x,t), \quad (x,t) \in \Gamma^-$$

and

$$(\varepsilon y_{2xx} + \mu a_2 y_{2x} - by_2 - cy_{2t})(x,t) = f(x,t), \quad (x,t) \in \Gamma^+,$$

where $a_1(x,t), a_2(x,t) \in C^2(\Gamma)$ are such that these extended functions $a_1(x,t), a_2(x,t)$ in Γ satisfy

$$\begin{aligned} a_1(x,t) &= a(x,t), \quad (x,t)\in\Gamma^-, \quad \text{such that} \quad a_1(x,t) < 0, \quad (x,t)\in\Gamma,\\ a_2(x,t) &= a(x,t), \quad (x,t)\in\Gamma^+, \quad \text{such that} \quad a_2(x,t) > 0, \quad (x,t)\in\Gamma. \end{aligned}$$

Let us consider the function

$$y(x,t) = \begin{cases} y_1(x,t) + (y(0,t) - y_1(0,t))\,\phi_1(x,t) + A_1\phi_2(x,t), & (x,t)\in\Gamma^-,\\ y_2(x,t) + A_2\phi_1(x,t) + (y(1,t) - y_2(1,t))\,\phi_2(x,t), & (x,t)\in\Gamma^+, \end{cases}$$

where $\phi_1(x,t)$ and $\phi_2(x,t)$ are, respectively, the solutions of the following two-parameter singularly perturbed boundary value problems:

$$\varepsilon\phi_{1xx} + \mu a_1\phi_{1x} - b\phi_1 - c\phi_{1t} = 0, \quad (x,t)\in\Gamma,\ \phi_1(0,t) = 1,\ \phi_1(1,t) = 0,\ \phi_1(x,0) = 0$$

and

$$\varepsilon\phi_{2xx} + \mu a_2\phi_{2x} - b\phi_2 - c\phi_{2t} = 0, \quad (x,t)\in\Gamma,\ \phi_2(0,t) = 0,\ \phi_2(1,t) = 1,\ \phi_2(x,0) = 0.$$

Note that $y(x,t)$ satisfies (1) to (2) in $(\Gamma^- \cup \Gamma^+)$ for two properly chosen constants A_1 and A_2, so that $y(x,t) \in C^1(\Gamma)$. Observe that $0 < \phi_i(x,t) < 1$, for $i = 1,2$ on Γ.[35] Thus, ϕ_1 and ϕ_2 cannot have a maximum or minimum at the interior points of the domain, and therefore,

$$\phi_1'(x,t) < 0,\ \phi_2'(x,t) > 0, \quad (x,t) \in \Gamma.$$

Now, we impose

$$y(d-,t) = y(d+,t) \text{ and } y'(d-,t) = y'(d+,t).$$

Therefore, we need the following relation for the existence of the constants A_1, A_2,

$$\begin{vmatrix} \phi_2(d,t) & -\phi_1(d,t) \\ \phi_2'(d,t) & -\phi_1'(d,t) \end{vmatrix} \neq 0.$$

The proof follows from $\phi_2'(d,t)\phi_1(d,t) - \phi_1'(d,t)\phi_2(d,t) > 0$.

The differential operator $L_{\varepsilon,\mu}$ also satisfies the following continuous comparison principle on $\overline{\Gamma} = \overline{\Omega}_x \times \Omega_t$. □

Lemma 2. *Assuming* $y(x,t) \in C^0(\overline{\Gamma}) \cap C^1(\Gamma) \cap C^2(\Gamma^- \cup \Gamma^+)$, *which satisfies*

$$y(x,t) \le 0,\ \forall\ (x,t) \in \Gamma_0,\ L_{\varepsilon,\mu}y(x,t) \ge 0,\ \forall\ (x,t) \in (\Gamma^- \cup \Gamma^+)$$
$$\text{and } [y_x](d,t) \ge 0,\ t > 0,\ \text{we get } y(x,t) \le 0,\ \forall\ (x,t) \in \overline{\Gamma}.$$

Proof. Let us assume that $y(x,t)$ is positive for some point $(x,t) \in \overline{\Gamma}$. Now, introduce the continuous function $\Lambda(x,t)$, defined by

$$y(x,t) = \begin{cases} e^{-\alpha_1\mu(d-x)/(2\varepsilon)}\Lambda(x,t), & \text{for } (x,t) < (d,t), \\ e^{-\alpha_2\mu(x-d)/(2\varepsilon)}\Lambda(x,t), & \text{for } (x,t) > (d,t). \end{cases}$$

Let Λ has its maximum value at a point (x,t) in $\overline{\Gamma}$. Then, the above result becomes true when $\Lambda(x,t) \le 0$. Now assume $\Lambda(x,t) > 0$. We will get a contradiction. With this assumption on the boundary points, we have either $(x,t) = (d,t)$ or $(x,t) \in (\Gamma^- \cup \Gamma^+)$.

Case 1: Assume $(x,t) = (d,t)$. Then, $[y_x](d,t) = [\Lambda_x] - [2(\alpha_1 + \alpha_2)/\varepsilon]\Lambda(d,t) < 0$ since $[\Lambda](d,t) = [y](d,t) = 0$ and $[\Lambda_x] \le 0$. This gives a contradiction.

Case 2: Let $(x,t) \in (\Gamma^- \cup \Gamma^+)$.
If $(x,t) \in \Gamma^-$, then

$$L_{\varepsilon,\mu}y(x,t) = e^{-\alpha_1\mu(d-x)/(2\varepsilon)}\left(\varepsilon\Lambda_{xx} + (a+\alpha_1)\mu\Lambda_x + \left(\frac{\alpha_1\mu^2}{2\varepsilon}\left(\frac{\alpha_1}{2\varepsilon} + a\right) - b\right)\Lambda - c\Lambda_t\right)(x,t) < 0,$$

which is a contradiction. Similarly, the operator $L_{\varepsilon,\mu}$ gives a contradiction for $y(x,t)$ in Γ^+. Therefore, we obtain the required result.

The consequence of the above comparison principle is the following stability estimate from which the uniqueness of the solution can be established. □

Lemma 3. *Let* $y(x,t)$ *be the solution of (1) to (2) then*

$$\|y\|_{\overline{\Gamma}} \le C \quad \max\left\{\|p\|_{\Gamma_c}, \|q\|_{\Gamma_l}, \|r\|_{\Gamma_r}\right\} + \frac{1}{\eta}\|L_{\varepsilon,\mu}y\|_{(\Gamma^-\cup\Gamma^+)},$$

where $\eta = \min\{\alpha_1/d, \alpha_2/(1-d)\}$.

Proof. Consider the barrier functions

$$\psi_\pm(x,t) = CK \pm y(x,t), \quad \text{where} \quad K = \begin{cases} -\chi - \frac{x\|L_{\varepsilon,\mu}y\|}{\eta d}, & \text{for } (x,t) \in (\Gamma^- \cup (d,t)), \\ -\chi - \frac{(1-x)\|L_{\varepsilon,\mu}y\|}{\eta(1-d)}, & \text{for } (x,t) \in \Gamma^+, \end{cases}$$

where $\chi = \max\left\{\|p\|_{\Gamma_c}, \|q\|_{\Gamma_l}, \|r\|_{\Gamma_r}\right\}$.

Then, $\psi_{\pm}(x,t) \in C^0(\overline{\Gamma})$, $\psi_{\pm}(x,t) \le 0$, $(x,t) \in \Gamma_0$ for sufficiently chosen C and for each $(x,t) \in (\Gamma^- \cup \Gamma^+)$, we have

$$L_{\varepsilon,\mu}\psi_{\pm}(x,t) \ge 0.$$

Also, since $\psi_{\pm}(x,t) \in C^1(\Gamma)$, we have

$$[\psi_{\pm}](d,t) = \pm[y](d,t) = 0 \text{ and } [\psi'_{\pm}](d,t) = \frac{\|L_{\varepsilon,\mu}y\|}{\eta d} + \frac{\|L_{\varepsilon,\mu}y\|}{\eta(1-d)} \ge 0.$$

It follows from the comparison principle that $\psi_{\pm}(x,t) \le 0\ \forall\ (x,t) \in \overline{\Gamma}$, which completes the proof. □

Lemma 4. *For $1 \le k + 2m \le 3$, the solution $y(x,t)$ of (1) to (2) and its derivatives satisfy:*
if $\alpha\mu^2 \le \gamma\varepsilon$, then

$$\left\|\frac{\partial^{k+m}y}{\partial x^k \partial t^m}\right\|_{\Gamma^-\cup\Gamma^+} \le C\varepsilon^{-k/2} \max\left\{\|y\|_{\overline{\Gamma}}, \sum_{i+2j=0}^{2} \varepsilon^{i/2}\left\|\frac{\partial^{i+j}f}{\partial x^i \partial t^j}\right\|, \sum_{i=0}^{4}\left[\varepsilon^{i/2}\left\|\frac{d^ip}{dx^i}\right\|_{\Gamma_c} + \left\|\frac{d^iq}{dt^i}\right\|_{\Gamma_l} + \left\|\frac{d^ir}{dt^i}\right\|_{\Gamma_r}\right]\right\},$$

and if $\alpha\mu^2 \ge \gamma\varepsilon$, then

$$\left\|\frac{\partial^{k+m}y}{\partial x^k \partial t^m}\right\|_{\Gamma^-\cup\Gamma^+} \le C\frac{\mu^{k+2m}}{\varepsilon^{k+m}} \max\left\{\|y\|_{\overline{\Gamma}}, \sum_{i+2j=0}^{2} \frac{\varepsilon^{i+j+1}}{\mu^{i+2j+2}}\left\|\frac{\partial^{i+j}f}{\partial x^i \partial t^j}\right\|, \sum_{i=0}^{4}\left[\frac{\varepsilon^i}{\mu^i}\left\|\frac{d^ip}{dx^i}\right\|_{\Gamma_c} + \frac{\varepsilon^i}{\mu^{2i}}\left\|\frac{d^iq}{dt^i}\right\|_{\Gamma_l} + \frac{\varepsilon^i}{\mu^{2i}}\left\|\frac{d^ir}{dt^i}\right\|_{\Gamma_r}\right]\right\},$$

where C is independent of ε and μ.

Proof. We follow the technique given in O'Riordan et al.[9] The bounds of the solution and its derivatives can be derived by splitting the arguments into two cases $\alpha\mu^2 \le \rho\varepsilon$ and $\alpha\mu^2 \ge \rho\varepsilon$.

Case 1: Let $\alpha\mu^2 \le \rho\varepsilon$. Now, consider the stretching variable $\xi = x/\sqrt{\varepsilon}$, $\tau = t$. The transformed domain is given by $\tilde{G} = \left((0,d) \cup (d, 1/\sqrt{\varepsilon})\right) \times (0,T]$. On the domain $\tilde{G}$, the transformed functions are defined as $\tilde{y}(\xi,\tau) = y(x,t)$, $\tilde{a}(\xi,\tau) = a(x,t)$, $\tilde{b}(\xi,\tau) = b(x,t)$, $\tilde{c}(\xi,\tau) = c(x,t)$, and $\tilde{f}(\xi,\tau) = f(x,t)$. Then, we apply the above transformation for the considered problem (1) to (2), from transformed equation we obtain the solution $\tilde{y}(\xi,\tau)$.
Now, we denote the rectangle $R_{\tilde{\kappa},\tilde{\delta}} = ((\tilde{\kappa}-\tilde{\delta}, \tilde{\kappa}+\tilde{\delta}) \times \Omega_t) \cap \tilde{G}$, and $\tilde{R}_{\tilde{\kappa},\tilde{\delta}}$ is a closure of $R_{\tilde{\kappa},\tilde{\delta}}$, where $\tilde{\kappa} \in \left((0,d) \cup (d, 1/\sqrt{\varepsilon})\right)$ and $\tilde{\delta} > 0$. For every $(\tilde{\kappa},\tau)$, the above classical differential equation satisfies the following estimate[36] for $1 \le k + 2m \le 3$:

$$\left\|\frac{\partial^{k+m}\tilde{y}}{\partial \xi^k \partial \tau^m}\right\|_{R_{\tilde{\kappa},\tilde{\delta}}} \le C\max\left\{\|\tilde{y}\|, \sum_{i+2j=0}^{2}\left\|\frac{\partial^{i+j}\tilde{f}}{\partial \xi^i \partial \tau^j}\right\|, \sum_{i=0}^{4}\left[\left\|\frac{d^i\tilde{p}}{d\xi^i}\right\|_{\tilde{\Gamma}_c} + \left\|\frac{d^i\tilde{q}}{d\tau^i}\right\|_{\tilde{\Gamma}_l} + \left\|\frac{d^i\tilde{r}}{d\tau^i}\right\|_{\tilde{\Gamma}_r}\right]\right\},$$

where $\tilde{\Gamma}_c = \tilde{R}_{\tilde{\kappa},2\tilde{\delta}} \cap \Gamma_c$, $\tilde{\Gamma}_l = \tilde{R}_{\tilde{\kappa},2\tilde{\delta}} \cap \Gamma_l$, $\tilde{\Gamma}_r = \tilde{R}_{\tilde{\kappa},2\tilde{\delta}} \cap \Gamma_r$, and C is independent of $\tilde{R}_{\tilde{\kappa},\tilde{\delta}}$. Hence, these estimates hold for any point $(\xi,\tau) \in \tilde{G}$. Finally, if we convert the variables (ξ,τ) into the original variables (x,t), we obtain the required result.

Case 2: If $\alpha\mu^2 \ge \rho\varepsilon$, then consider two stretching variables $\zeta = \frac{\mu x}{\varepsilon}$ and $\varsigma = \frac{\mu^2 t}{\varepsilon}$. The transformed domain is given by $\hat{G} = ((0,d) \cup (d, \mu/\varepsilon)) \times \left(0, \frac{\mu^2 T}{\varepsilon}\right)$, and the transformed functions are $\hat{a}(\zeta,\varsigma) = a(x,t)$, $\hat{b}(\zeta,\varsigma) = b(x,t)$, $\hat{c}(\zeta,\varsigma) = c(x,t)$, $\hat{f}(\zeta,\varsigma) = f(x,t)$, and $\hat{y}(\zeta,\varsigma) = y(x,t)$. Substituting the above defined transformation in (1) to (2), we get the solution $\hat{y}(\zeta,\varsigma)$ from the transformed equation.

Again, we denote another rectangle $R_{\hat{\kappa},\hat{\delta}} = \left((\hat{\kappa}-\hat{\delta}, \hat{\kappa}+\hat{\delta}) \times \left(0, \frac{\mu^2 T}{\varepsilon}\right]\right) \cap \hat{G}$ and $\hat{R}_{\hat{\kappa},\hat{\delta}}$ as the closure of $R_{\hat{\kappa},\hat{\delta}}$, where $\hat{\kappa} \in ((0,d) \cup (d, \mu/\varepsilon)$ and $\hat{\delta} > 0$. For each $(\hat{\kappa},\hat{\delta}) \in \hat{G}$ and with the help of O'Riordan et al.,[9] we get

$$\left\|\frac{\partial^{k+m}\hat{y}}{\partial \zeta^k \partial \varsigma^m}\right\|_{R_{\hat{\kappa},\hat{\delta}}} \le C\max\left\{\|\hat{y}\|, \sum_{i+2j=0}^{2}\left\|\frac{\partial^{i+j}\hat{f}}{\partial \zeta^i \partial \varsigma^j}\right\|, \sum_{i=0}^{4}\left[\left\|\frac{d^i\hat{p}}{d\zeta^i}\right\|_{\hat{\Gamma}_c} + \left\|\frac{d^i\hat{q}}{d\varsigma^i}\right\|_{\hat{\Gamma}_l} + \left\|\frac{d^i\hat{r}}{d\varsigma^i}\right\|_{\hat{\Gamma}_r}\right]\right\},$$

where, $\hat{\Gamma}_c = \hat{R}_{\hat{\kappa},2\hat{\delta}} \cap \Gamma_c$, $\hat{\Gamma}_l = \hat{R}_{\hat{\kappa},2\hat{\delta}} \cap \Gamma_l$, $\hat{\Gamma}_r = \hat{R}_{\hat{\kappa},2\hat{\delta}} \cap \Gamma_r$, and C is independent of $\hat{R}_{\hat{\kappa},\varsigma}$. Hence, these estimates hold for any point $(\zeta, \varsigma) \in \hat{G}$. Finally, the transformed variables (ζ, ς) are replaced by the original variables (x, t) to get the desired result. □

Corollary 1. *The solution $y(x, t)$ of (1) to (2) satisfies the bounds for second-order time derivative*

$$\|y_{tt}\|_{\overline{\Gamma}} \leq \begin{cases} C, & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \\ C\mu^4\varepsilon^{-2}, & \text{if } \alpha\mu^2 \geq \gamma\varepsilon. \end{cases}$$

Proof. This follows from the argument given in Lemma 4 and the techniques in O'Riordan et al. and Gracia et al.[9,12]

Now, we decompose the solution $y(x, t)$ into a regular component $v(x, t)$ and a singular component $w(x, t)$. The regular component $v(x, t)$ will be defined as the solution of the problem

$$\begin{aligned} &L_{\varepsilon,\mu}v(x,t) = f(x,t), \text{ for } (x,t) \in (\Gamma^- \cup \Gamma^+), \\ &v(0,t) = y(0,t),\ v(d-,t) = u_1(t),\ v(d+,t) = u_2(t), \\ &v(1,t) = y(1,t),\ v(x,0) = y(x,0), \end{aligned} \tag{4}$$

where $v(d+,t) = \lim_{x\to d+0} v(x,t)$, $v(d-,t) = \lim_{x\to d-0} v(x,t)$, and $u_1(t)$ and $u_2(t)$ are suitably chosen.

Case 1: $\alpha\mu^2 \leq \gamma\varepsilon$. Let $\Gamma^{-*} = [0, d) \times \Omega_t, \Gamma^{+*} = (d, 1] \times \Omega_t$. Now, we decompose $v(x, t)$ by

$$v(x,t;\varepsilon,\mu) = v_0(x,t) + \sqrt{\varepsilon}v_1(x,t;\varepsilon,\mu) + \varepsilon v_2(x,t;\varepsilon,\mu), \tag{5}$$

where $v_0(x, t), v_1(x, t; \varepsilon, \mu)$ and $v_2(x, t; \varepsilon, \mu)$ are the solutions of the following problems:

$$L_0v_0(x,t) = f(x,t),\ v_0(x,0) = y(x,0),\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+*}), \tag{6}$$

$$\sqrt{\varepsilon}L_0v_1(x,t) = (L_0 - L_{\varepsilon,\mu})v_0(x,t),\ v_1(x,0) = 0, \quad (x,t) \in (\Gamma^{-*} \cup \Gamma^{+*}), \tag{7}$$

$$\begin{aligned} &\varepsilon L_{\varepsilon,\mu}v_2(x,t;\varepsilon,\mu) = \sqrt{\varepsilon}(L_0 - L_{\varepsilon,\mu})v_1(x,t;\varepsilon,\mu),\ (x,t) \in (\Gamma^- \cup \Gamma^+), \\ &v_2(x,0;\varepsilon,\mu) = v_2(0,t;\varepsilon,\mu) = v_2(1,t;\varepsilon,\mu) = 0, \\ &\text{and } v_2(d-,t;\varepsilon,\mu),\ v_2(d+,t;\varepsilon,\mu) \text{ are chosen appropriately.} \end{aligned} \tag{8}$$

Again,

$$\begin{cases} v(0,t;\varepsilon,\mu) = v_0(0,t) + \sqrt{\varepsilon}v_1(0,t;\varepsilon,\mu),\ t \in \Omega_t, \\ v(d-,t;\varepsilon,\mu) = v_0(d-,t) + \sqrt{\varepsilon}v_1(d-,t;\varepsilon,\mu),\ t \in \Omega_t, \\ v(d+,t;\varepsilon,\mu) = v_0(d+,t) + \sqrt{\varepsilon}v_1(d+,t;\varepsilon,\mu),\ t \in \Omega_t, \\ v(1,t;\varepsilon,\mu) = v_0(1,t) + \sqrt{\varepsilon}v_1(1,t;\varepsilon,\mu),\ t \in \Omega_t, \\ v(x,t;\varepsilon,\mu) = r(x,t),\ (x,t) \in \Gamma_c. \end{cases} \tag{9}$$

Now, the following lemma provides the derivative bounds of $v(x, t)$ with respect to x and t. □

Lemma 5. *The regular component solution $v(x, t)$ and its derivatives of (4) satisfy the following bounds*

$$\left\|\frac{\partial^{k+m}v}{\partial x^k\partial t^m}\right\|_{\Gamma^-\cup\Gamma^+} \leq C\left(1 + \varepsilon^{1-k/2}\right),\ \textit{for } 0 \leq k + 2m \leq 3 \textit{ and } \|v_{tt}\|_{\Gamma^-\cup\Gamma^+} \leq C.$$

Proof. The above bounds of the derivatives can be derived by following the techniques given in O'Riordan et al. and Gracia et al.[9,12] Here, $v_0(x, t)$ and $v_1(x, t)$ are solutions of the first-order differential Equations 6 and 7, and the bound for $v_2(x, t)$ of (8) can be obtained by using the techniques used in Lemmas 3 and 4. Next, by using the following relation

$$\frac{\partial^{k+m}v}{\partial x^k\partial t^m} = \frac{\partial^{k+m}v_0}{\partial x^k\partial t^m} + \sqrt{\varepsilon}\frac{\partial^{k+m}v_1}{\partial x^k\partial t^m} + \varepsilon\frac{\partial^{k+m}v_2}{\partial x^k\partial t^m},\ \text{for } 0 \leq k + 2m \leq 3,$$

we obtain the required derivative bounds of $v(x, t)$. Next, we can use the Corollary 1 to obtain

$$\|v_{tt}\|_{\Gamma^-\cup\Gamma^+} \leq C.$$

Case 2: For $\alpha\mu^2 \geq \gamma\epsilon$. Let $\Gamma^{-*} = [0,d) \times \Omega_t, \Gamma^{+*} = (d,1] \times \Omega_t$. Now, we decompose $v(x,t)$ as follows:

$$v(x,t;\epsilon,\mu) = v_0(x,t;\mu) + \epsilon v_1(x,t;\mu) + \epsilon^2 v_2(x,t;\epsilon,\mu), \tag{10}$$

where $v_0(x,t;\mu), v_1(x,t;\mu)$ and $v_2(x,t;\epsilon,\mu)$ are the solutions of

$$\begin{aligned} &L_\mu v_0(x,t;\mu) = f(x,t),\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+}), \\ &v_0(x,0;\mu) = u(x,0),\ v_0(1,t;\mu) \text{ are chosen appropriately,} \end{aligned} \tag{11}$$

$$\begin{aligned} &\epsilon L_\mu v_1(x,t;\mu) = (L_\mu - L_{\epsilon,\mu}) v_0(x,t;\mu),\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+}), \\ &v_1(x,0;\mu) = v_1(1,t;\mu) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\epsilon^2 L_{\epsilon,\mu} v_2(x,t;\epsilon,\mu) = \epsilon(L_\mu - L_{\epsilon,\mu}) v_1(x,t;\mu),\ (x,t) \in (\Gamma^{-} \cup \Gamma^{+}), \\ &v_2(x,0;\epsilon,\mu) = v_2(0,t;\epsilon,\mu) = v_2(1,t;\epsilon,\mu) = 0, \\ &\text{and } v_2(d-,t;\epsilon,\mu),\ v_2(d+,t;\epsilon,\mu) \text{ are chosen accordingly.} \end{aligned} \tag{13}$$

We see that

$$\begin{cases} v(0,t;\epsilon,\mu) = v_0(0,t;\mu) + \epsilon v_1(0,t;\mu),\ t \in \Omega_t, \\ v(d-,t;\epsilon,\mu) = v_0(d-,t;\mu) + \epsilon v_1(d-,t;\mu),\ t \in \Omega_t, \\ v(d+,t;\epsilon,\mu) = v_0(d+,t;\mu) + \epsilon v_1(d+,t;\mu),\ t \in \Omega_t, \\ v(1,t;\epsilon,\mu) = v_0(1,t;\mu),\ t \in \Omega_t, \\ v(x,t;\epsilon,\mu) = r(x,t),\ (x,t) \in \Gamma_c. \end{cases} \tag{14}$$

Now, consider the following first-order IBVP:

$$L_\mu y^*(x,t) = (a\mu y_x^* - by^* - cy_t^*)(x,t) = f(x,t), \quad (x,t) \in (\Gamma^{-*} \cup \Gamma^{+}) \tag{15}$$

$$y^*(x,t) = p_1(x,t),\ (x,t) \in \Gamma_c,\ y^*(x,t) = r_1(x,t), \quad (x,t) \in \Gamma_r. \tag{16}$$

Note that L_μ satisfies the following comparison principle. □

Lemma 6. *Assume in (15) and (16) $y^*(x,t) \in C^0(\Gamma^{-*} \cup \Gamma^{+*}) \cup C^1(\Gamma^{-*} \cup \Gamma^{+})$ then*

$$y^*(x,t) \leq 0,\ \forall\ (x,t) \in (\Gamma_c \cup \Gamma_r),\ L_\mu y^*(x,t) \geq 0,\ \forall\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+}), [y_x^*](d,t) \geq 0$$
for $t > 0$. We can conclude that $y^*(x,t) \leq 0,\ \forall\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+*})$.

Proof. This lemma can be derived by following the procedure given in Lemma 2. □

Lemma 7. *The solution $y^*(x,t)$ of (15) to (16) is stable, since it satisfies*

$$\|y^*\|_{\Gamma^{-*} \cup \Gamma^{+*}} \leq C \max\left\{ \|y^*\|_{(\Gamma_c \cup \Gamma_r)}, \frac{1}{\eta} \|L_\mu y^*\|_{(\Gamma^{-*} \cup \Gamma^{+})} \right\},$$

where $\eta = \min\{\alpha_1/d, \alpha_2/(1-d)\}$.

Proof. Consider the following barrier function

$$\psi_\pm(x,t) = C \begin{cases} \left(-\tilde{\chi} - \frac{x\|L_\mu y^*\|}{\eta d}\right) \pm y(x,t), \text{for } (x,t) \in (\Gamma^{-*} \cup (d,t)) \text{ and} \\ \left(-\tilde{\chi} - \frac{(1-x)\|L_\mu y^*\|}{\eta(1-d)}\right) \pm y(x,t), \text{for } (x,t) \in \Gamma^{+}, \end{cases}$$

where $\tilde{\chi} = \max\left\{\|p\|_{\Gamma_c}, \|r\|_{\Gamma_r}\right\}$. It satisfies $\psi_\pm(x,t) \leq 0,\ \forall\ (x,t) \in (\Gamma_c \cup \Gamma_r)$ and also

$$L_\mu \psi_\pm(x,t) = \begin{cases} -\alpha_1 \mu \frac{\|L_\mu \psi_\pm\|}{\eta d} - b\left(-\chi - \frac{x\|L_\mu y^*\|}{\eta d}\right) \pm L_\mu \psi_\pm \geq 0,\ \forall\ (x,t) \in \Gamma^{-*}, \\ \alpha_2 \mu \frac{\|L_\mu \psi_\pm\|}{\eta d} - b\left(-\chi - \frac{(1-x)\|L_\mu y^*\|}{\eta(1-d)}\right) \pm L_\mu \psi_\pm \geq 0,\ \forall\ (x,t) \in \Gamma^{+}. \end{cases}$$

Furthermore, at discontinuity point $[(\psi_\pm)_x](d,t) \geq 0$. Then, by applying Lemma 2, we obtain

$$\psi_\pm(x,t) \leq 0,\ \forall\ (x,t) \in (\Gamma^{-*} \cup \Gamma^{+*}),$$

which leads to obtain the bound on $y^*(x,t)$.

Following the analysis given in O'Riordan et al.[9] and Gracia et al.[12] for smooth data, we can derive the following lemmas. □

Lemma 8. *Suppose $y^*(x,t) \in C^{(k+m)}(\Gamma^{-*} \cup \Gamma^{+*})$ satisfies (15) to (16). Then, its derivatives satisfy*

$$\left\| \frac{\partial^{k+m} y^*}{\partial x^k \partial t^m} \right\|_{\Gamma^{-*}\cup\Gamma^{+*}} \leq \frac{C}{\mu^k} \left(\left\| \frac{\partial^{k+m} f}{\partial t^{k+m}} \right\| + \sum_{k_0+m_0=0}^{k+m-1} \mu^r \left\| \frac{\partial^{k_0+m_0} f}{\partial x^{k_0} \partial t^{m_0}} \right\| + \sum_{j=0}^{k+m} \left\| \frac{d^j p_1}{dx^j} \right\| + \sum_{j=0}^{k+m} \left\| \frac{d^j r_1}{dt^j} \right\| + \|y^*\| \right) e^{-(k+m)AT},$$

where $A = \min\left\{0, a(x,t)\left(\frac{1}{a}\right)_t (x,t)\right\}$, and the constant C is independent of μ.

Lemma 9. *The regular component $v(x,t)$ at (10) and its derivatives satisfy the following bounds:*

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\|_{\Gamma^- \cup \Gamma^+} \leq C\left(1 + \left(\frac{\mu}{\varepsilon}\right)^{k-2}\right),\ \textit{for}\ 0 \leq k+2m \leq 3 \quad \textit{and} \quad \|v_{tt}\|_{\Gamma^-\cup\Gamma^+} \leq C.$$

Proof. The first bound follows from the techniques given in O'Riordan et al. and Gracia et al.[9,12] and the procedure given in Lemma 3 and Lemma 4 for the domain $\Gamma^- \cup \Gamma^+$. For the second part, we can extend the argument from Corollary 1 to obtain

$$\|v_{tt}\| \leq C\left(1 + \frac{\varepsilon^2\mu^4}{\varepsilon^2\mu^2}\right) \leq C(1+\mu^2) \leq C.$$

Now, we decompose the solution $u(x,t)$ as $v(x,t)$ and $w(x,t)$ for both cases. Then, the regular component satisfies

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \leq C\left(1 + \varepsilon^{2-k}\right),\ \text{for}\ 0 \leq k+2m \leq 3\ \text{and}\ \|v_{tt}\| \leq C. \tag{17}$$

Now, we decompose the singular component $w(x,t)$ as $w_l(x,t)$ and $w_r(x,t)$, which are defined as follows:

$$\begin{aligned} &L_{\varepsilon,\mu} w_l(x,t) = 0,\ \text{for}\ (x,t) \in (\Gamma^- \cup \Gamma^+), \\ &w_l(x,0) = 0,\ w_l(0,t) = y(0,t) - v(0,t) - w_r(0,t),\ w_l(1,t) = 0. \end{aligned} \tag{18}$$

$$\begin{aligned} &L_{\varepsilon,\mu} w_r(x,t) = 0,\ \text{for}\ (x,t) \in (\Gamma^- \cup \Gamma^+), \\ &w_r(0,t)\ \text{is suitably chosen},\ w_r(1,t) = y(1,t) - v(0,t),\ w_r(x,0) = 0, \\ &[w_r](d,t) = -\left([v] + [w_l]\right)(d,t),\ [(w_r)_x](d,t) = -\left([v_x] + [(w_l)_x]\right)(d,t). \end{aligned} \tag{19}$$

Note that $w(d-,t) = y(d-,t) - v(d-,t)$ and $w(d+,t) = y(d+,t) - v(d+,t)$.

For $\alpha\mu^2 \leq \gamma\varepsilon$, the singular components $w_l(x,t)$ and $w_r(x,t)$ satisfy the derivative bounds given in Lemma 4 and Corollary 1. When $\alpha\mu^2 \geq \gamma\varepsilon$, the following decomposition helps us to find $w_r(0,t)$:

$$w_r(x,t;\varepsilon,\mu) = w_0(x,t;\mu) + \varepsilon w_1(x,t;\mu) + \varepsilon^2 w_2(x,t;\varepsilon,\mu), \tag{20}$$

where

$$\begin{aligned} &L_\mu w_0(x,t) = 0,\ (x,t) \in (\Gamma^{-*} \cup \Gamma^+), \\ &w_0(x,t) = 0, (x,t) \in \Gamma_c,\ w_0(x,t) = r(1,t) - v_0(1,t), \end{aligned} \tag{21}$$

$$\begin{aligned} &\varepsilon L_\mu w_1(x,t) = (L_\mu - L_{\varepsilon,\mu}) w_0(x,t), \quad (x,t) \in (\Gamma^{-*} \cup \Gamma^+), \\ &w_1(x,t) = 0, (x,t) \in (\Gamma_c \cup \Gamma_r), \end{aligned} \tag{22}$$

$$\begin{aligned} &\varepsilon^2 L_{\varepsilon,\mu} w_2(x,t) = \varepsilon(L_\mu - L_{\varepsilon,\mu}) w_1(x,t),\ (x,t) \in (\Gamma^- \cup \Gamma^+), \\ &w_2(x,t) = 0,\ (x,t) \in \Gamma_0. \end{aligned} \tag{23}$$

□

Lemma 10. *From the above decomposition, it follows that $w_r(x,t)$ satisfies the following bound*

$$|w_r(0,t)|_{\Gamma^- \cup \Gamma^+} \le e^{-2Bte^{-\gamma/\mu}},$$

where $B < A = \min\left\{0, a(x,t)\left(\frac{1}{a}\right)_t(x,t)\right\}$.

Proof. It follows from O'Riordan et al.[9]

The following lemmas provide the derivative bounds of $w_l(x,t)$ and $w_r(x,t)$, which will be required for the convergence analysis. □

Lemma 11. *When $\alpha\mu^2 \le \gamma\varepsilon$, the singular components $w_l(x,t)$ and $w_r(x,t)$ of (18) to (19) satisfy the following bounds*

$$|w_l(x,t)|_{\Gamma^- \cup \Gamma^+} \le C \begin{cases} e^{-\theta_2 x}, & (x,t) \in \Gamma^-, \\ e^{-\theta_1 (x-d)}, & (x,t) \in \Gamma^+, \end{cases} \qquad |w_r(x,t)|_{\Gamma^- \cup \Gamma^+} \le C \begin{cases} e^{-\theta_1 (d-x)}, & (x,t) \in \Gamma^-, \\ e^{-\theta_2 (1-x)}, & (x,t) \in \Gamma^+, \end{cases}$$

where

$$\theta_1 = \frac{\sqrt{\alpha\gamma}}{\sqrt{\varepsilon}} \quad \textit{and} \quad \theta_2 = \frac{\sqrt{\alpha\gamma}}{2\sqrt{\varepsilon}}. \tag{24}$$

Proof. To find the bound for $w_l(x,t)$, let us first consider the following barrier functions:

$$\psi_1^{\pm}(x,t) = Ce^{-\theta_2 x} \pm w_l(x,t).$$

For the domain Γ^-, it can be written as

$$\psi_1^{\pm}(x,t) = Ce^{-\sqrt{\alpha\gamma}x/(2\sqrt{\varepsilon})} \pm w_l(x,t).$$

Applying the operator $L_{\varepsilon,\mu}$ to the above equation, we obtain

$$\begin{aligned} L_{\varepsilon,\mu}\psi_l^{\pm}(x,t) &= C\left[\frac{\alpha\gamma}{4} - a\mu\frac{\sqrt{\alpha\gamma}}{2\sqrt{\varepsilon}} - b\right] e^{-\sqrt{\alpha\gamma}x/(2\sqrt{\varepsilon})} \pm L_{\varepsilon,\mu} w_l(x,t) \\ &\le C\left[\frac{\alpha\gamma}{4} - \frac{a\gamma}{2} - b\right] e^{-\sqrt{\alpha\gamma}x/(2\sqrt{\varepsilon})} \pm L_{\varepsilon,\mu} w_l(x,t) \ge 0. \end{aligned}$$

Let us define the barrier functions $\psi_2^{\pm}$ on Γ^+ as

$$\psi_2^{\pm}(x,t) = Ce^{-\sqrt{\alpha\gamma}(x-d)/(\sqrt{\varepsilon})} \pm w_l(x,t).$$

Then, we have

$$\begin{aligned} L_{\varepsilon,\mu}\psi_2^{\pm}(x,t) &= C\left[\alpha\gamma - a\mu\frac{\sqrt{\alpha\gamma}}{\sqrt{\varepsilon}} - b\right] e^{-\sqrt{\alpha\gamma}(x-d)/\sqrt{\varepsilon}} \pm L_{\varepsilon,\mu} w_l(x,t) \\ &\le C\left[\alpha\gamma - a\gamma - b\right] e^{-\sqrt{\alpha\gamma}(x-d)/\sqrt{\varepsilon}} \pm L_{\varepsilon,\mu} w_l(x,t) \ge 0. \end{aligned}$$

The above comparisons can be also developed for the right singular component $w_r(x,t)$ in Γ^- and Γ^+. Hence, by using Lemma 2, we obtain the required bound. □

Lemma 12. *For $\alpha\mu^2 \ge \gamma\varepsilon$, the singular components $w_l(x,t)$ and $w_r(x,t)$ of (18) to (19) satisfy*

$$|w_l(x,t)|_{\Gamma^- \cup \Gamma^+} \le C \begin{cases} e^{-\theta_2 x}, & (x,t) \in \Gamma^-, \\ e^{-\theta_1 (x-d)}, & (x,t) \in \Gamma^+, \end{cases} \qquad |w_r(x,t)|_{\Gamma^- \cup \Gamma^+} \le C \begin{cases} e^{-\theta_1 (d-x)}, & (x,t) \in \Gamma^-, \\ e^{-\theta_2 (1-x)}, & (x,t) \in \Gamma^+, \end{cases}$$

where

$$\theta_1 = \frac{\alpha\mu}{\varepsilon} \quad \textit{and} \quad \theta_2 = \frac{\gamma}{2\mu}. \tag{25}$$

Proof. We follow the idea given in O'Riordan et al.[9] for parabolic problems with smooth data. For the case $\alpha\mu^2 \geq \gamma\varepsilon$, a careful examination will be needed since $w_r(0,t) \neq 0$. Consider the following barrier functions of $w_l(x,t)$

$$\psi_3^{\pm}(x,t) = Ce^{-2At}e^{-\theta_2 x} \pm w_l(x,t) = Ce^{-2At}e^{-\gamma x/(2\mu)} \pm w_l(x,t),\ (x,t) \in \Gamma^-,$$
$$\psi_4^{\pm}(x,t) = Ce^{-2At}e^{-\theta_1 (x-d)} \pm w_l(x,t) = Ce^{-2At}e^{-\alpha\mu x/\varepsilon} \pm w_l(x,t),\ (x,t) \in \Gamma^+,$$

where A is previously defined. From Lemma 10 with sufficiently large C, we have $\psi_3^{\pm}(x,t) \geq 0,\ \psi_4^{\pm}(x,t) \geq 0,\ (x,t) \in \Gamma_0$ and $L_{\varepsilon,\mu}\psi_3^{\pm}(x,t) \leq 0,\ L_{\varepsilon,\mu}\psi_4^{\pm}(x,t) \leq 0$. Now, by using Lemma 2, we get the required bound. Similarly, we can derive the corresponding bound for the right singular component $w_r(x,t)$ in both domain Γ^- and Γ^+. □

Lemma 13. *When $\alpha\mu^2 \geq \gamma\varepsilon$, then $w_r(x,t)$ and $w_l(x,t)$ of (18) to (19) satisfy the bounds*

$$\left\| \frac{\partial^k w_r}{\partial x^k} \right\|_{\Gamma^- \cup \Gamma^+} \leq C\left(\mu^{-k} + \mu^{-1}\varepsilon^{2-k}\right),\ 1 \leq k \leq 3 \text{ and } \left\| \frac{\partial^m w_r}{\partial t^m} \right\|_{\Gamma^- \cup \Gamma^+} \leq C, \quad m = 1, 2$$

and

$$\left\| \frac{\partial^k w_l}{\partial x^k} \right\|_{\Gamma^- \cup \Gamma^+} \leq C\left(\mu^k \varepsilon^{-k}\right),\ 1 \leq k \leq 3 \text{ and } \left\| \frac{\partial^2 w_l}{\partial t^2} \right\|_{\Gamma^- \cup \Gamma^+} \leq C\left(1 + \mu^2\varepsilon^{-1}\right).$$

Now, we write the solution of (1) to (2) as $y(x,t) = v(x,t) + w(x,t)$. Note that both $v(x,t)$ and $w(x,t)$ are discontinuous at $(d,t), t > 0$, but their sum is in $C^1(\Gamma)$. Note

$$v(x,t) = \begin{cases} v^-(x,t), \text{ for } (x,t) \in \Gamma^-, \\ v^+(x,t), \text{ for } (x,t) \in \Gamma^+, \end{cases} w_l(x,t) = \begin{cases} w_l^-(x,t), \text{ for } (x,t) \in \Gamma^-, \\ w_l^+(x,t), \text{ for } (x,t) \in \Gamma^+, \end{cases} \text{and } w_r(x,t) = \begin{cases} w_r^-(x,t), \text{ for } (x,t) \in \Gamma^-, \\ w_r^+(x,t), \text{ for } (x,t) \in \Gamma^+. \end{cases}$$

3 | DISCRETIZATION AND THE STABILITY OF THE DISCRETE PROBLEM

Now, we define an a priori defined space adaptive mesh on which the convergence of the solution will be discussed. To obtain this, we first split $\overline{\Omega}_x$ as follows:

$$\overline{\Omega}_x = [0,\tau_1] \cup [\tau_1, d-\tau_2] \cup [d-\tau_2, d] \cup [d, d+\tau_3] \cup [d+\tau_3, 1-\tau_4] \cup [1-\tau_4, 1].$$

Here, τ_1, τ_2, τ_3 and τ_4 are transition parameters. The subintervals $[0,\tau_1], [d-\tau_2, d], [d, d+\tau_3]$ and $[1-\tau_4, 1]$ contain $N_x/8$ mesh intervals, and remaining are coarse regions with $N_x/4$ mesh intervals. Here, N_x denotes the number of mesh intervals used in the x-direction. The interior mesh points are denoted by

$$\Omega_x^{N_x} \equiv \left\{ x_i : 1 \leq i \leq \frac{N_x}{2} - 1 \right\} \cup \left\{ x_i : \frac{N_x}{2} + 1 \leq i \leq N_x - 1 \right\}.$$

We choose $x_{N_x/2} = d$ and $\overline{\Omega}_x^{N_x} = \{x_i\}_0^{N_x} \cup \{d\}$. The transition parameters are chosen as follows:

$$\tau_1 = \min\left\{ \frac{d}{4}, \frac{2}{\theta_2}\ln N_x \right\},\ \tau_2 = \min\left\{ \frac{d}{4}, \frac{2}{\theta_1}\ln N_x \right\},\ \tau_3 = \min\left\{ \frac{1-d}{4}, \frac{2}{\theta_1}\ln N_x \right\},\ \text{and } \tau_4 = \min\left\{ \frac{1-d}{4}, \frac{2}{\theta_2}\ln N_x \right\}, \tag{26}$$

where θ_1 and θ_2 are defined in the earlier section. The above piecewise uniform space mesh for $\overline{\Omega}_x^{N_x}$ on $\overline{\Omega}_x$ is containing N_x mesh elements. We use the uniform mesh to discretize $\Omega_t^{N_t}$ on Ω_t with N_t mesh elements, ie, step-size is $\Delta t = T/N_t$. In practice, we use N_x and N_t are assumed to be of same order to reduce the computational cost. Let us denote $N = (N_x, N_t)$.

Now, we discretize (1) to (2) as

$$L_{\varepsilon,\mu}^{N_x,N_t} Y(x_i,t_j) = (\varepsilon\delta_x^2 + a\mu D_x^* - b - cD_t^-)Y(x_i,t_j) = f(x_i,t_j), \quad (x_i,t_j) \in (\Gamma^{N-} \cup \Gamma^{N+}), \tag{27}$$

$$Y(0,t_j) = y(0,t_j),\ Y(1,t_j) = y(1,t_j),\ j > 0,\ Y(x_i,0) = y(x_i,0),\ i = 0,\cdots,N_x,\ \text{and } j = 1,\cdots,N_t, \tag{28}$$

$$D_x^- Y(x_{N_x/2},t_j) = D_x^+ Y(x_{N_x/2},t_j),\ j = 1,\cdots,N_t, \tag{29}$$

where $\delta_x^2 Y(x_i,t_j) = \frac{2(D_x^+ - D_x^-)Y(x_i,t_j)}{x_{i+1}-x_{i-1}}$, $D_x^* Y(x_i,t_j) = \begin{cases} D_x^- Y(x_i,t_j), & i < N_x/2 \\ D_x^+ Y(x_i,t_j), & i > N_x/2, \end{cases}$ with $D_x^+ Y(x_i,t_j) = \frac{Y(x_{i+1},t_j) - Y(x_i,t_j)}{x_{i+1}-x_i}$, $D_x^- Y(x_i,t_j) = \frac{Y(x_i,t_j) - Y(x_{i-1},t_j)}{x_i - x_{i-1}}$, and $D_t^- Y(x_i,t_j) = \frac{Y(x_i,t_j) - Y(x_i,t_{j-1})}{t_j - t_{j-1}}$.

The following lemmas show that the above discrete operator leads to a stable numerical solution.

Lemma 14. *Suppose that a mesh function $Y(x_i,t_j)$ satisfies*

$$Y(x_i,t_j) \le 0,\ \forall\ (x_i,t_j) \in \Gamma_0^N,\ L_{\varepsilon,\mu}^{N_x,N_t} Y(x_i,t_j) \ge 0,\ \forall\ (x_i,t_j) \in (\Gamma^{N-} \cup \Gamma^{N+}),\ \textit{and}$$
$$\left(D_x^+ Y(x_i,t_j) - D_x^- Y(x_i,t_j)\right) \ge 0,\ \forall\ (x_i,t_j) \in \Gamma^{N\pm},\ \textit{then}\ Y(x_i,t_j) \le 0,\ \forall\ (x_i,t_j) \in \overline{\Gamma}^N.$$

Proof. Let us assume that the maximum value of $Y(\tilde{x}_i,\tilde{t}_j)$ attains at $(\tilde{x}_i,\tilde{t}_j)$ in $\overline{\Gamma}^N$. Note that the result is obvious if $Y(\tilde{x}_i,\tilde{t}_j) \le 0$. Now, consider $Y(\tilde{x}_i,\tilde{t}_j) < 0$. An easy calculation shows that this leads to a contradiction for two separate cases $(\tilde{x}_i,\tilde{t}_j) \in (\Gamma^{N-} \cup \Gamma^{N+})$ and $(\tilde{x}_i,\tilde{t}_j) = (d,t_j)$, since either $L_{\varepsilon,\mu}^{N_x,N_t} Y(x_i,t_j) \ge 0$ or $\left(D_x^+ - D_x^-\right) Y(d,t_j) \ge 0$. Therefore, the desired result follows. □

Lemma 15. *Let $Y(x_i,t_j)$ be a discrete solution of (27) to (29), then*

$$\|Y\|_{\overline{\Gamma}^N} \le C \max\left\{ \|Y\|_{\Gamma_0^N}, \frac{1}{\xi} \|L_{\varepsilon,\mu}^{N_x,N_t} Y\|_{(\Gamma^{N-}\cup\Gamma^{N+})} \right\},$$

where $\xi = \min\{\alpha_1/d, \alpha_2/(1-d)\}$.

Proof. The proof of the discrete stability result is analogues to the continuous stability result. Consider $\Psi_\pm(x_i,t_j) = \left(-\|Y\|_{\Gamma_0^N} - \frac{x_i \|L_{\varepsilon,\mu}^{N_x,N_t} Y\|}{\xi d} \right) \pm Y(d,t_j)$, where we have considered (x_d,t_j) as the point of discontinuity in $\overline{\Gamma}$. Note that $\Psi_\pm(x_i,t_j) > 0$ and the discrete operator $L_{\varepsilon,\mu}^{N_x,N_t} \Psi_\pm(x_i,t_j) \ge 0$ when $(x_i,t_j) \in (\Gamma^{N-} \cup \Gamma^{N+})$. Then, we have

$$\left(D_x^+ - D_x^-\right) \Psi_\pm(d,t_j) = \frac{\|L_{\varepsilon,\mu}^{N_x,N_t} Y\|}{\xi d} + \frac{\|L_{\varepsilon,\mu}^{N_x,N_t} Y\|}{\xi(1-d)} \geqslant 0,$$

and $\left(D_x^+ - D_x^-\right) Y(d,t_j) = 0$. Hence, by using the discrete comparison principle at Lemma 14, we have $\Psi_\pm(x_i,t_j) \le 0, \forall\ (x_i,t_j) \in \overline{\Gamma}^N$. Therefore, the required result follows. □

4 | ERROR ANALYSIS

The convergence of the present method will be considered by combining the consistency analysis based on truncation error and stability analysis. We decompose the discrete solution $Y(x_i,t_j)$ into discrete regular component $V(x_i,t_j)$ and discrete singular component $W(x_i,t_j)$ as $Y(x_i,t_j) = V(x_i,t_j) + W(x_i,t_j)$. Now, we define the discrete problems corresponding to $V^-(x_i,t_j)$ and $V^+(x_i,t_j)$, which approximate $V(x_i,t_j)$ to the left and right side of discontinuity point (d,t_j) as follows:

$$L_{\varepsilon,\mu}^{N_x,N_t} V^-(x_i,t_j) = f(x_i,t_j),\ \forall\ (x_i,t_j) \in \Gamma^{N-},$$
$$V^-(0,t_j) = v(0,t_j),\ V^-(x_{N_x/2},t_j) = v(d-,t_j), \quad v^-(x_i,0) = \tilde{v}, \quad \text{for some } \tilde{v},$$

and

$$L_{\varepsilon,\mu}^{N_x,N_t} V^+(x_i,t_j) = f(x_i,t_j),\ \forall\ (x_i,t_j) \in \Gamma^{N+},$$
$$V^+(x_{N_x/2},t_j) = v(d+,t_j),\ V^+(x_{N_x},t_j) = v(1,t_j) \quad v^+(x_i,0) = u(x,0) - \tilde{v}.$$

Further, we decompose the discrete singular component as $W(x_i,t_j) = W_l(x_i,t_j) + W_r(x_i,t_j)$. Then, we construct the mesh functions $W_l^-(x_i,t_j)$, $W_l^+(x_i,t_j)$, and $W_r^-(x_i,t_j)$, $W_r^+(x_i,t_j)$ to approximate $W_l(x_i,t_j)$ and $W_r(x_i,t_j)$ on left and right sides of discontinuity point (d,t_j). Here, $W_l^-(x_i,t_j)$ and $W_r^-(x_i,t_j)$ correspond to the left boundary layer and right interior layer part of the solution, respectively. Similarly, $W_l^+(x_i,t_j)$ and $W_r^+(x_i,t_j)$ are the solution parts of left interior layer and right boundary layer. The problems of the singular component solutions are defined as follows:

$$\begin{aligned} L_{\varepsilon,\mu}^{N_x,N_t} W_l^-(x_i,t_j) &= 0 \ \forall\ (x_i,t_j) \in \Gamma^{N-}, \\ W_l^-(0,t_j) = w_l^-(0,t_j), &\quad W_l^-(d,t_j) = w_l^-(d,t_j), \end{aligned} \tag{30}$$

$$\begin{aligned} L_{\varepsilon,\mu}^{N_x,N_t} W_l^+(x_i,t_j) &= 0 \ \forall\ (x_i,t_j) \in \Gamma^{N+}, \\ W_l^+(d,t_j) = w_l^+(d,t_j), &\quad W_l^+(1,t_j) = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} L_{\varepsilon,\mu}^{N_x,N_t} W_r^-(x_i,t_j) &= 0 \ \forall\ (x_i,t_j) \in \Gamma^{N-}, \\ W_r^-(0,t_j) = 0, &\quad W_r^-(d,t_j) = w_r^-(d,t_j), \end{aligned} \tag{32}$$

$$\begin{aligned} & L_{\varepsilon,\mu}^{N_x,N_t} W_r^+(x_i,t_j) = 0 \ \forall\ (x_i,t_j) \in \Gamma^{N+}, \\ & W_r^+(d,t_j) = 0,\ W_r^+(1,t_j) = w_r^+(1,t_j), \\ & \left(V^- + W_l^- + W_r^-\right)(d,t_j) = \left(V^+ + W_l^+ + W_r^+\right)(d,t_j), \end{aligned} \tag{33}$$

where $W_l^-(x_i,0)$, $W_l^+(x_i,0)$, $W_r^-(x_i,0)$, and $W_r^+(x_i,0)$ are chosen suitably. Hence, the discrete solution $Y(x_i,t_j)$ can be written as

$$Y(x_i,t_j) = \begin{cases} \left(V^- + W_l^- + W_r^-\right)(x_i,t_j), & (x_i,t_j) \in \Gamma^{N-}, \\ \left(V^- + W_l^- + W_r^-\right)(d,t_j) = \left(V^+ + W_l^+ + W_r^+\right)(d,t_j), & \\ \left(V^+ + W_l^+ + W_r^+\right)(x_i,t_j), & (x_i,t_j) \in \Gamma^{N+}. \end{cases}$$

Lemma 16. [12] *The bounds for the singular components $W_l^-(x_i,t_k)$, $W_l^+(x_i,t_k)$, $W_r^-(x_i,t_k)$, and $W_r^+(x_i,t_k)$ are*

$$|W_l^-(x_i,t_k)| \le C\prod_{j=1}^{i}(1+\theta_2 h_j)^{-1} = \psi_{li}^-, \quad 1 \le i \le N_x/2,\ \psi_{l0}^- = C, \tag{34}$$

$$|W_l^+(x_i,t_k)| \le C\prod_{j=N_x/2+1}^{i}(1+\theta_1 h_j)^{-1} = \psi_{li}^+, \quad i \le N_x,\ \psi_{lN_x/2}^+ = C, \tag{35}$$

$$|W_r^-(x_i,t_k)| \le C\prod_{j=i+1}^{N_x/2}(1+\theta_1 h_j)^{-1} = \psi_{ri}^-, \quad 0 \le i \le N_x/2,\ \psi_{rN_x/2}^- = C, \tag{36}$$

$$|W_r^+(x_i,t_k)| \le C\prod_{j=i+1}^{N_x}(1+\theta_2 h_j)^{-1} = \psi_{ri}^+, \quad i \ge N_x/2,\ \psi_{rN_x}^+ = C, \tag{37}$$

where θ_1 and θ_2 are defined as follows:

$$\theta_1 = \begin{cases} \frac{\sqrt{\alpha\gamma}}{2\sqrt{\varepsilon}}, & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ \frac{\alpha\mu}{2\varepsilon}, & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases} \quad \text{and } \theta_2 = \begin{cases} \frac{\sqrt{\alpha\gamma}}{2\sqrt{\varepsilon}}, & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ \frac{\gamma}{2\mu}, & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases}$$

and $h_j = x_j - x_{j-1}$ and $W_l^-(x_i,t_j)$, $W_l^+(x_i,t_j)$, $W_r^-(x_i,t_j)$, and $W_r^+(x_i,t_j)$ are solutions of (30) to (33), respectively.

Lemma 17. *The truncation error of the regular component satisfies*

$$\|V - v\|_{\Gamma^{N-}\cup\Gamma^{N+}} \le C\left(N_x^{-1} + N_t^{-1}\right).$$

Proof. The truncation error bound for regular component is estimated using the following classical argument at the domain Γ^{N-} by a similar technique provided in O'Riordan et al.[9] Observe from (17)

$$\begin{aligned} \left|L_{\varepsilon,\mu}^{N_x,N_t}(V^- - v^-)(x_i,t_j)\right| &\le \left|L_{\varepsilon,\mu}^{N_x,N_t} V^-(x_i,t_j) - L_{\varepsilon,\mu} v^-(x_i,t_j)\right| \\ &\le \varepsilon\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right)\left|v^-(x_i,t_j)\right| + a(x_i,t_j)\mu\left(D_x^- - \frac{\partial}{\partial x}\right)\left|v^-(x_i,t_j)\right| + c(x_i,t_j)\left(D_t^- - \frac{\partial}{\partial t}\right)\left|v^-(x_i,t_j)\right| \\ &\le C\max_i h_i\left(\varepsilon\|v_{xxx}^-\| + \mu\|v_{xx}^-\|\right) + CN_t^{-1}\|v_{tt}^-\| \\ &\le C(N_x^{-1} + N_t^{-1}). \end{aligned}$$

Similarly, we can derive the same error estimate at Γ^{N+}. Hence from Lemma 14, we have

$$\|V - v\|_{\Gamma^{N-}\cup\Gamma^{N+}} \le C\left(N_x^{-1} + N_t^{-1}\right).$$

□

Lemma 18. *The truncation error of the left singular component satisfies*

$$\|W_l^- - w_l^-\|_{\Gamma^{N-}} \le \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases} \text{ and } \|W_l^+ - w_l^+\|_{\Gamma^{N+}} \le \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon. \end{cases}$$

Proof. On Γ^{N-}, we have

$$\begin{aligned} \left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| &\le \left|L_{\varepsilon,\mu}^{N_x,N_t}W_l^-(x_i,t_j) - L_{\varepsilon,\mu}w_l^-(x_i,t_j)\right| \\ &\le \varepsilon\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right)\left|w_l^-(x_i,t_j)\right| + a(x_i,t_j)\mu\left(D_x^- - \frac{\partial}{\partial x}\right)\left|w_l^-(x_i,t_j)\right| + c(x_i,t_j)\left(D_t^- - \frac{\partial}{\partial t}\right)\left|w_l^-(x_i,t_j)\right| \\ &\le C\max_i h_i\left(\varepsilon\|(w_l^-)_{xxx}\| + \mu\|(w_l^-)_{xx}\|\right) + CN_t^{-1}\|(w_l^-)_{tt}\|, \end{aligned} \tag{38}$$

and similarly for Γ^{N+}, we have

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \le C\max_i h_i\left(\varepsilon\|(w_l^+)_{xxx}\| + \mu\|(w_l^+)_{xx}\|\right) + CN_t^{-1}\|(w_l^+)_{tt}\|. \tag{39}$$

Now we consider two cases (1) $\tau_1 < \frac{1}{8}$, and (2) $\tau_1 = \frac{1}{8}$ for the singular component analysis.

Case 2: $\tau_1 < \frac{1}{8}$ implies mesh is piecewise uniform. In Γ^{N-}, let us first consider $[\tau_1, d)\times\Omega_t^{N_t}$ to find the required bound from Lemma 11 and (34). Note

$$\begin{aligned} \left|W_l^-(x_{N_x/8,t_j})\right| &\le C\left(1+\theta_2 h_j\right)^{-N_x/8} \\ &\le C\left(1+\theta_2\left(\frac{8\tau_1}{N_x}\right)\right)^{-N_x/8} \\ &\le C\left(1+16N_x^{-1}\ln N_x\right)^{-N_x/8}, \end{aligned}$$

where we have used the inequality $\ln(1+x) > x(1-x/2)$ for $x > 0$ with $x = 16N_x^{-1}\ln N_x$, to get the last inequality. Hence,

$$\left|W_l^-(x_{N_x/8},t_j)\right| \le CN_x^{-1}, \text{ for } (x_i,t_j) \in [\tau_1,d)\times\Omega_t^{N_t}.$$

From the Lemmas 11 and 12, we have

$$\left|w_l^-(x,t)\right| \le Ce^{-\theta_2 x} \le Ce^{-\theta_2\tau_1} \le Ce^{-2\ln N_x} \le CN_x^{-2}.$$

Now, combining the results of $W_l^-(x_i,t_j)$ and $w_l^-(x,t)$, we obtain

$$\left|(W_l^- - w_l^-)(x_i,t_j)\right| \le CN_x^{-1}, \text{ for } (x_i,t_j) \in [\tau_1,d)\times\Omega_t^{N_t}.$$

Now, consider the fine mesh region $(0,\tau_1)\times\Omega_t^{N_t}$ in Γ^{N-}. First, consider the case $\alpha\mu^2 \le \gamma\varepsilon$. Observe

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| \le \frac{C}{\sqrt{\varepsilon}}(h_{i+1}+h_i) + CN_t^{-1}.$$

Since $\tau_1 < \frac{1}{8}$, therefore $h_{i+1} = h_i = \frac{16\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}}N_x^{-1}\ln N_x$ and hence

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

The required bound follows from the discrete comparison principle. Now, consider $\alpha\mu^2 \geq \gamma\varepsilon$. Then (38) reduces to

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| \leq C\mu^{-1}(h_{i+1}+h_i) + CN_t^{-1} \leq CN_x^{-1}\ln N_x + CN_t^{-1},$$

since $h_{i+1} = h_i = \frac{16\mu}{\gamma}N_x^{-1}\ln N_x$. Hence, we have

$$\left|(W_l^- - w_l^-)(x_i,t_j)\right| \leq CN_x^{-1}\ln N_x + CN_t^{-1} \quad (x_i,t_j) \in \Gamma^{N-},$$

from Lemma 14. To find the bounds for $W_l^+(x_i,t_j)$ and $w_l^+(x,t)$ on the domain Γ^{N+}, we first divide it into $[d, d+\tau_3)\times\Omega_t^{N_t}$ and $[d+\tau_3, 1)\times\Omega_t^{N_t}$. Let us consider $[d+\tau_3, 1)\times\Omega_t^{N_t}$. From Lemmas 11 and 12, we have in Γ^+,

$$\left|w_l^+(x,t)\right| \leq Ce^{-\theta_1 x} \leq Ce^{-\theta_1\tau_1} \leq Ce^{-2\ln N_x} \leq CN_x^{-2}.$$

From the bounds at (35), it follows at $[d+\tau_3, 1)\times\Omega_t^{N_t}$ that

$$\begin{aligned}\left|W_l^+(x_{5N_x/8},t_j)\right| &\leq C\left(1+\theta_1 h_j\right)^{-N_x/8}\\ &\leq C\left(1+\theta_1\left(\frac{8\tau_3}{N_x}\right)\right)^{-N_x/8}\\ &\leq C\left(1+16N_x^{-1}\ln N_x\right)^{-N_x/8}\\ &\leq CN_x^{-1}.\end{aligned}$$

Combining the bounds of $W_l^+(x_i,t_j)$ and $w_l^+(x,t)$, we have

$$\left|(W_l^+ - w_l^+)(x_i,t_j)\right| \leq CN_x^{-1}, \qquad \text{for} \quad (x_i,t_j) \in [d+\tau_3, 1)\times\Omega_t^{N_t}.$$

Let us start with the case $\alpha\mu^2 \leq \gamma\varepsilon$ on the fine region $[d, d+\tau_3)\times\Omega_t^{N_t}$ in Γ^{N+}. Here,

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \leq \frac{C}{\sqrt{\varepsilon}}(h_{i+1}+h_i) + CN_t^{-1}.$$

As $\tau_3 < 5/8$ and $h_{i+1} = h_i = \frac{16\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}}N_x^{-1}\ln N_x$, therefore

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \leq C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

The required result follows from the discrete comparison principle. Now, consider the second case $\alpha\mu^2 \geq \gamma\varepsilon$ and $h_{i+1} = h_i = \frac{16\varepsilon}{\alpha\mu}N_x^{-1}\ln N_x$. The truncation error bound given in (39) reduces to

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \leq CN_x^{-1}\ln N_x + CN_x^{-1}\frac{\mu^2}{\varepsilon}\ln N_x + CN_t^{-1}\left(1+\frac{\mu^2}{\varepsilon}\right).$$

Consider the following barrier functions

$$\psi_1^{\pm} = C\left(N_x^{-1}\ln N_x + N_t^{-1} + \left((\tau_3 - x_i)\frac{\mu}{\varepsilon}\right)(N_x^{-1}\ln N_x + N_t^{-1})\right) \pm (W_l^+ - w_l^+)(x_i,t_j),$$

which becomes nonnegative for sufficiently large C at all points of Γ_0^N and $L_{\varepsilon,\mu}^{N_x,N_t}\psi_1^{\pm}(x_i,t_j) \leq 0$, $(x_i,t_j) \in \Gamma^N$. Therefore, using $\tau_3 = \frac{2\varepsilon}{\alpha\mu}\ln N_x$ in Γ^{N+}, we have from Lemma 14

$$\left|(W_l^+ - w_l^+)(x_i,t_j)\right| \leq C\left(N_x^{-1}\ln N_x + N_t^{-1} + \left((\tau_3 - x_i)\frac{\mu}{\varepsilon}\right)(N_x^{-1}\ln N_x + N_t^{-1})\right) \leq C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right).$$

Case 2: In the first case, the mesh is uniform. Now, assume $\alpha\mu^2 \leq \gamma\varepsilon$. If $\tau_1 = \frac{1}{8}$ then $\frac{2}{\theta_2}\ln N_x \leq \frac{1}{8}$, where $\theta_2 = \frac{\sqrt{\alpha\gamma}}{2\sqrt{\varepsilon}}$ from (24). By the classical arguments given in (38) on Γ^{N-}, it follows

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| \leq C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

Now, consider the condition $\alpha\mu^2 \geq \gamma\varepsilon$. Here, $\theta_2 = \frac{\gamma}{2\mu}$ from (25). Hence from (38), we can write

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^- - w_l^-)(x_i,t_j)\right| \leq C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

If $\tau_3 = \frac{1}{8}$ then $\frac{2}{\theta_1}\ln N_x \leq \frac{1}{8}$, where $\theta_1 = \frac{\sqrt{\alpha\gamma}}{\sqrt{\varepsilon}}$ from (24) for $\alpha\mu^2 \leq \gamma\varepsilon$. Using (39) in Γ^{N+}, we have

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \leq C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

Now, for $\alpha\mu^2 \geq \gamma\varepsilon$, we use $\theta_1 = \frac{\alpha\mu}{\varepsilon}$ from (24), to get

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_l^+ - w_l^+)(x_i,t_j)\right| \leq C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right).$$

From the above results and the discrete comparison principle Lemma 14, we can obtain the desired results

$$\|W_l^- - w_l^-\|_{\Gamma^{N-}} \leq \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \\ C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \geq \gamma\varepsilon, \end{cases}$$

and

$$\|W_l^+ - w_l^+\|_{\Gamma^{N+}} \leq \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \\ C(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x), & \text{if } \alpha\mu^2 \geq \gamma\varepsilon. \end{cases}$$

□

Lemma 19. *The truncation error of the right singular component satisfies*

$$\|W_r^- - w_r^-\|_{\Gamma^{N-}} \leq \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \\ C(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x), & \text{if } \alpha\mu^2 \geq \gamma\varepsilon, \end{cases}$$

and

$$\|W_r^+ - w_r^+\|_{\Gamma^{N+}} \leq \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \leq \gamma\varepsilon, \\ C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \geq \gamma\varepsilon. \end{cases}$$

Proof. We follow a similar procedure provided in Lemma 18 to find the error estimate for the right singular component. On Γ^{N-}, we have

$$\begin{aligned} \left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i,t_j)\right| &\leq \left|L_{\varepsilon,\mu}^{N_x,N_t}W_r^-(x_i,t_j) - L_{\varepsilon,\mu}w_r^-(x_i,t_j)\right| \\ &\leq \varepsilon\left(\delta_x^2 - \frac{\partial^2}{\partial x^2}\right)\left|w_r^-(x_i,t_j)\right| + a(x_i,t_j)\mu\left(D_x^- - \frac{\partial}{\partial x}\right)\left|w_r^-(x_i,t_j)\right| + c(x_i,t_j)\left(D_t^- - \frac{\partial}{\partial t}\right)\left|w_r^-(x_i,t_j)\right| \\ &\leq C\max_i h_i\left(\varepsilon\|(w_r^-)_{xxx}\| + \mu\|(w_r^-)_{xx}\|\right) + CN_t^{-1}\|(w_r^-)_{tt}\|. \end{aligned} \tag{40}$$

Similarly for Γ^{N+}, we have

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i,t_j)\right| \leq C\max_i h_i\left(\varepsilon\|(w_r^+)_{xxx}\| + \mu\|(w_r^+)_{xx}\|\right) + CN_t^{-1}\|(w_r^+)_{tt}\|. \tag{41}$$

Here, we consider two cases (1) $d - \tau_2 < \frac{3}{8}$ and (2) $d - \tau_2 = \frac{3}{8}$ for the singular component analysis.

Case 1: The condition $d - \tau_2 < \frac{3}{8}$ gives the piecewise uniform mesh. In Γ^{N-}, first consider $[0, d-\tau_2) \times \Omega_t^{N_t}$ to find the required bound from Lemma 11 and (36). Note

$$\left|W_r^-(x_{3N_x/8,t_j})\right| \leq C\left(1 + \theta_1 h_j\right)^{-N_x/8} \leq C\left(1 + \theta_1\left(\frac{8\tau_2}{N_x}\right)\right)^{-N_x/8} \leq C\left(1 + 16N_x^{-1}\ln N_x\right)^{-N_x/8},$$

where we have used the inequality $\ln(1+x) > x(1 - x/2)$ with $x = 16N_x^{-1}\ln N_x$, to get the last inequality. Hence

$$\left|W_r^-(x_{3N_x/8}, t_j)\right| \leq CN_x^{-1}, \text{ for } (x_i, t_j) \in [0, d-\tau_2) \times \Omega_t^{N_t}.$$

From the Lemmas 11 and 12, we have

$$|w_r^-(x,t)| \leq Ce^{-\theta_1(d-x)} \leq Ce^{-\theta_1\tau_2} \leq Ce^{-2\ln N_x} \leq CN_x^{-2}.$$

Now, combining the bounds of $W_r^-(x_i,t_j)$ and $w_r^-(x,t)$, we obtain

$$\left|(W_r^- - w_r^-)(x_i,t_j)\right| \le CN_x^{-1}, \text{ for } (x_i,t_j) \in [0, d-\tau_2) \times \Omega_t^{N_t}.$$

Now, consider the fine mesh region $(d-\tau_2, d) \times \Omega_t^{N_t}$ in Γ^{N-}. First, consider the case $\alpha\mu^2 \le \gamma\varepsilon$. Note

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i,t_j)\right| \le C\varepsilon^{-1/2}(h_{i+1}+h_i) + CN_t^{-1}.$$

Since $d-\tau_2 < \frac{3}{8}$, therefore, $h_{i+1} = h_i = \frac{16\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}} N_x^{-1} \ln N_x$. So,

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i,t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

The expected bound follows from the comparison principle. Now, consider $\alpha\mu^2 \ge \gamma\varepsilon$ with $h_{i+1} = h_i = \frac{16\mu}{\gamma} N_x^{-1} \ln N_x$. Then, (40) reduces to

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i,t_j)\right| \le \frac{C}{\mu}(h_{i+1}+h_i) + CN_t^{-1} \le CN_x^{-1}\ln N_x + CN_t^{-1}.$$

Using Lemma 14, we obtain $||W_r^- - w_r^-|| \le CN_x^{-1}\ln N_x + CN_t^{-1}$. To find the bounds for $W_r^+(x_i,t_j)$ and $w_r^+(x,t)$ on the domain Γ^{N+}, we first divide it into $[d, 1-\tau_4) \times \Omega_t^{N_t}$ and $[1-\tau_4, 1) \times \Omega_t^{N_t}$. Let us begin with the domain $[d, 1-\tau_4) \times \Omega_t^{N_t}$ to obtain the bounds from (35).

$$\left|W_r^+(x_{7N_x/8}, t_j)\right| \le C\big(1+\theta_2 h_j\big)^{-N_x/8} \le C\left(1+\theta_2\left(\frac{8\tau_4}{N_x}\right)\right)^{-N_x/8} \le C\big(1+16N_x^{-1}\ln N_x\big)^{-N_x/8} \le CN_x^{-1},$$

for $(x_i,t_j) \in [d, d-\tau_4) \times \Omega_t^{N_t}$. Again from Lemmas 11 and 12, we have in Γ^+

$$\left|w_r^+(x,t)\right| \le Ce^{-\theta_2(1-x)} \le Ce^{\theta_2\tau_4} \le Ce^{-2\ln N_x} \le CN_x^{-2}.$$

Therefore, by combining the above bounds of $W_r^+(x_i,t_j)$ and $w_r^+(x,t)$, we obtain

$$\left|(W_r^+ - w_r^+)(x_i,t_j)\right| \le CN_x^{-1}, \text{ for } (x_i,t_j) \in [d, 1-\tau_4) \times \Omega_t^{N_t}.$$

Now consider the case $\alpha\mu^2 \le \gamma\varepsilon$ on $[d, 1-\tau_4) \times \Omega_t^{N_t}$ in Γ^{N+}. Here

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i,t_j)\right| \le \frac{C}{\sqrt{\varepsilon}}(h_{i+1}+h_i) + CN_t^{-1}.$$

Since $1-\tau_4 < \frac{7}{8}$ and $h_{i+1} = h_i = \frac{16\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}} N_x^{-1} \ln N_x$, therefore

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i,t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

The required result follows from the discrete comparison principle. Now, consider the second case $\alpha\mu^2 \ge \gamma\varepsilon$ and $h_{i+1} = h_i = \frac{16\varepsilon}{\alpha\mu} N_x^{-1} \ln N_x$. The error in (41) reduces to

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i,t_j)\right| \le CN_x^{-1}\ln N_x + CN_x^{-1}\frac{\mu^2}{\varepsilon}\ln N_x + CN_t^{-1}\left(1+\frac{\mu^2}{\varepsilon}\right).$$

Consider the following barrier functions:

$$\psi_1^{\pm} = C\left(N_x^{-1}\ln N_x + N_t^{-1} + \left(((1-\tau_4)-x_i)\frac{\mu}{\varepsilon}\right)(N_x^{-1}\ln N_x + N_t^{-1})\right) \pm (W_r^+ - w_r^+)(x_i,t_j).$$

For sufficiently large C, we have $\psi_1^{\pm}(x_i,t_j)$ is nonnegative at all points Γ_0^N and $L_{\varepsilon,\mu}^{N_x,N_t}\psi_1^{\pm}(x_i,t_j) \le 0$, $(x_i,t_j) \in \Gamma^N$, and so by discrete comparison principle, we have

$$\left|(W_r^+ - w_r^+)(x_i,t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1} + \left(((1-\tau_4)-x_i)\frac{\mu}{\varepsilon}\right)(N_x^{-1}\ln N_x + N_t^{-1})\right).$$

Since $\tau_4 = \frac{4\mu}{\gamma} \ln N_x$, we obtain

$$\left|(W_r^+ - w_r^+)(x_i, t_j)\right| \le C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right), \text{ for } (x_i, t_j) \in \Gamma^{N+}.$$

Case 2: Now, consider $\alpha\mu^2 \le \gamma\varepsilon$. If $d-\tau_2 = \frac{3}{8}$ then $\frac{2}{\theta_1}\ln N_x \le \frac{3}{8}$, where $\theta_1 = \sqrt{\frac{\alpha\gamma}{2\varepsilon}}$ from (24). By the classical arguments for (40) in Γ^{N-}, it follows

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i, t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

Now, consider the condition $\alpha\mu^2 \ge \gamma\varepsilon$. Here, $\theta_1 = \frac{\alpha\mu}{\varepsilon}$ from (25). Hence, from (40), we can write

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^- - w_r^-)(x_i, t_j)\right| \le C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right).$$

For $1-\tau_4 = \frac{7}{8}$, we have $\frac{2}{\theta_2}\ln N_x \le \frac{1}{8}$, where $\theta_2 = \frac{\sqrt{\alpha\gamma}}{2\varepsilon}$ from (24) for $\alpha\mu^2 \le \gamma\varepsilon$. Using (41) in Γ^{N+}, we get

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i, t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

For $\alpha\mu^2 \ge \gamma\varepsilon$, we take $\theta_2 = \frac{\gamma}{2\mu}$ from (25) and get

$$\left|L_{\varepsilon,\mu}^{N_x,N_t}(W_r^+ - w_r^+)(x_i, t_j)\right| \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right).$$

Therefore, from the above results and the discrete comparison principle, we can write

$$\|W_r^- - w_r^-\|_{\Gamma^{N-}} \le \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases}$$

and

$$\|W_r^+ - w_r^+\|_{\Gamma^{N+}} \le \begin{cases} C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C(N_x^{-1}\ln N_x + N_t^{-1}), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases}$$

which is the required bound.
At the point $(x_i, t_j) = (d, t_j)$, we have $\left(D_x^+ - D_x^-\right)Y(d, t_j) = 0$. Therefore,

$$\left|\left(D_x^+ - D_x^-\right)(Y-y)(d, t_j)\right| = \left|\left(D_x^+ - D_x^-\right)Y(d, t_j) - \left(D_x^+ - D_x^-\right)y(d, t_j)\right| \le \left|\left(D_x^+ - D_x^-\right)y(d, t_j)\right|.$$

Now, note that $h_3 = \frac{8\tau_2}{N_x}$ and $h_4 = \frac{8\tau_3}{N_x}$ on either side of (d, t_j). Therefore,

$$\begin{aligned}\left|\left(D_x^+ - D_x^-\right)(Y-y)(d, t_j)\right| &\le \left|\left(D_x^+ - D_x^-\right)y(d, t_j)\right| \\ &\le \left|\left(D^+ - \frac{d}{dx}\right)y(d+, t_j)\right| + \left|\left(D^- - \frac{d}{dx}\right)y(d-, t_j)\right| \\ &\le C\frac{h_3}{2}|y_{xx}|_{\Gamma^{N-}} + C\frac{h_4}{2}|y_{xx}|_{\Gamma^{N+}} \\ &\le C\frac{(h_3+h_4)}{2}|y_{xx}|_{\Gamma^{N-}\cup\Gamma^{N+}}, \end{aligned}$$

$$\left|\left(D_x^+ - D_x^-\right)(Y-y)(d, t_j)\right| \le C\begin{cases} \frac{(h_3+h_4)}{2}\left(\frac{1}{\varepsilon}\right), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ \frac{(h_3+h_4)}{2}\left(\frac{\mu}{\varepsilon}\right)^2, & \text{if } \alpha\mu^2 \ge \gamma\varepsilon. \end{cases}$$

□

5 | ERROR ESTIMATE

Theorem 1. *The continuous solution $y(x,t)$ of (1) to (2) and the numerical solution $Y(x_i, t_j)$ of (27) to (29) satisfy the following error estimate:*

$$\|Y-y\|_{\overline{\Gamma}^N} \le \begin{cases} C\left(N_x^{-1}\ln N_x + N_t^{-1}\right), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon, \end{cases}$$

ie, the difference scheme (27) to (29) converges with almost first-order accuracy in space and time.

Proof. Combining the Lemmas 17, 18, and 19, we obtain the following error bound for $(x_i, t_j) \neq (d, t_j)$

$$\left|(Y-y)(x_i,t_j)\right| \le \begin{cases} C\left(N_x^{-1}\ln N_x + N_t^{-1}\right), & \text{if } \alpha\mu^2 \le \gamma\varepsilon, \\ C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right), & \text{if } \alpha\mu^2 \ge \gamma\varepsilon. \end{cases} \tag{42}$$

Using the techniques given in Clavero et al. and Chandru et al.,[7,16] we can also obtain the error for $(x_i, t_j) = (d, t_j)$. First, we obtain the result for the case $\alpha\mu^2 \le \gamma\varepsilon$. Consider the following discrete barrier function:

$$\omega_1^{\pm}(x_i,t_j) = \begin{cases} C_1\left(N_x^{-1}\ln N_x + N_t^{-1}\right) + C_2\frac{h}{\varepsilon}\tau_2^2((d-\tau_2)-x) \pm e(x_i), & \text{for } x_i \in (d-\tau_2, d)\times\Omega_t^{N_t}, \\ C_3\left(N_x^{-1}\ln N_x + N_t^{-1}\right) + C_4\frac{h}{\varepsilon}\tau_2^2(x-(d+\tau_3)) \pm e(x_i), & \text{for } x_i \in (d, d+\tau_3)\times\Omega_t^{N_t}, \end{cases}$$

where $h_3 = \frac{8\tau_2}{N}$, $h_4 = \frac{8\tau_3}{N}$ and $h = \max\{h_3, h_4\}$. Then, it is easy to verify that

$$\omega_1^{\pm}(x_i,t_j) \le 0, \ \forall\ (x_i,t_j) \in \overline{\Gamma}_0^N,$$

for sufficiently large C_1. We also have

$$L_{\varepsilon,\mu}^{N_x,N_t}\omega_1^{\pm}(x_i,t_j) \ge 0, \ \forall\ (x_i,t_j) \in (\Gamma^{N-} \cup \Gamma^{N+})$$

and

$$\left(D_x^+ - D_x^-\right)\omega_1^{\pm}(d,t_j) \ge 0,$$

for suitably large C_2. Thus, from the discrete comparison principle, we get

$$\omega_1^{\pm}(x_i,t_j) \le 0, \ \forall\ (x_i,t_j) \in \overline{\Gamma}^N.$$

Therefore, for sufficiently larger N_x, we obtain the following estimate

$$\left|(Y-y)(x_i,t_j)\right|_{\overline{\Gamma}^N} \le C\left(N_x^{-1}\ln N_x + N_t^{-1}\right), \ \text{if } \alpha\mu^2 \le \gamma\varepsilon. \tag{43}$$

In the second case $\alpha\mu^2 \ge \gamma\varepsilon$, consider the discrete barrier function

$$\omega_2^{\pm}(x_i,t_j) = \begin{cases} C_5\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right) + \left(C_6\left(N_x^{-1}\ln N_x + N_t^{-1}\right)\frac{\mu}{\varepsilon} + C_7\frac{h\mu}{\varepsilon^2}\tau_2^2\right) \\ \quad (d-\tau_2-x_i) \pm e(x_i), \ x_i \in (d-\tau_2, d)\times\Omega_t^{N_t}, \\ C_8\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right) + \left(C_9\left(N_x^{-1}\ln N_x + N_t^{-1}\right)\frac{\mu}{\varepsilon} + C_{10}\frac{h\mu}{\varepsilon^2}\tau_2^2\right) \\ \quad (x_i-(d+\tau_3)) \pm e(x_i), \ x_i \in (d, d+\tau_3)\times\Omega_t^{N_t}. \end{cases}$$

Using a similar procedure from the above technique based on discrete comparison principle, we also obtain the error estimate

$$\left|(Y-y)(x_i,t_j)\right|_{\overline{\Gamma}^N} \le C\left(N_x^{-1}(\ln N_x)^2 + N_t^{-1}\ln N_x\right), \quad \text{if} \quad \alpha\mu^2 \ge \gamma\varepsilon. \tag{44}$$

Hence, we have the required result. □

The above bound provides the parameter uniform estimate as the constant C is independent of singular perturbation parameters ε and μ. In practice, we will only take $N_t = O(N_x)$ to reduce the computational cost. Hence, the numerical solution will converge to the continuous solution almost linearly.

6 | NUMERICAL EXAMPLES

Here, we consider four test problems with discontinuous convection coefficient and discontinuous source term to show that our estimated rate of convergence is true in practice. Motivated from the examples and their error analysis given in

Das and Mehrmann,[29] we construct the following examples. Here, we only mention the coefficients and source terms with initial data to define the parabolic IBVP in (1).

Example 1.

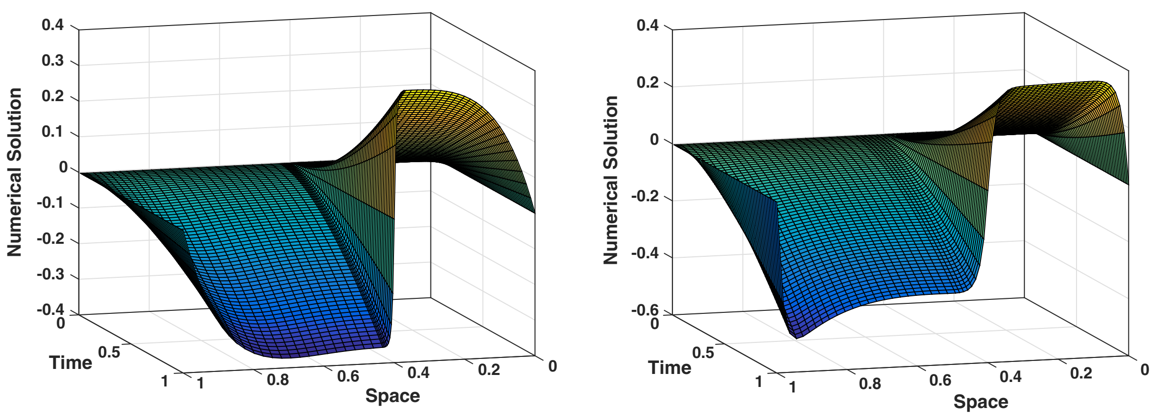
$$a(x,t) = \begin{cases} -(1+\exp(x)), & 0 \le x \le 0.4, \\ (1+\exp(x)), & 0.4 < x \le 1, \end{cases} \quad f(x,t) = \begin{cases} -(1+x^4)t, & 0 \le x \le 0.4, \\ (1+x^6)t, & 0.4 < x \le 1, \end{cases}$$
$$b(x,t) = 1 + x^2, \; c(x,t) = 1, \text{ and } y(0,t) = y(1,t) = y(x,0) = 0.$$

Example 2.

$$a(x,t) = \begin{cases} x+2, & 0 \le x \le 0.5, \\ -(2x+3), & 0.5 < x \le 1, \end{cases} \quad f(x,t) = \begin{cases} (2x+1)t, & 0 \le x \le 0.5, \\ -(3x+4)t, & 0.5 < x \le 1, \end{cases}$$
$$b(x,t) = 1 + \exp(x), \; c(x,t) = 1, \text{ and } y(0,t) = y(1,t) = y(x,0) = 0.$$

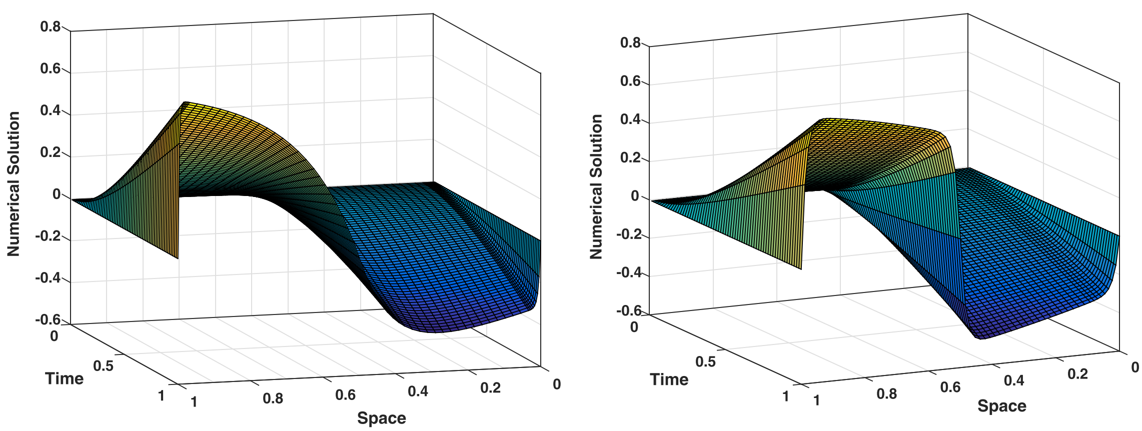
Example 3.

$$a(x,t) = \begin{cases} -(1+e^{-xt}), & 0 \le x \le 0.5, \\ 2+x+t, & 0.5 < x \le 1, \end{cases} \quad f(x,t) = \begin{cases} (e^{t^2}-1)(1+xt), & 0 \le x \le 0.5, \\ -(2+x)t^2, & 0.5 < x \le 1, \end{cases}$$
$$b(x,t) = 2 + xt, \; c(x,t) = 1, \text{ and } y(0,t) = y(1,t) = y(x,0) = 0.$$



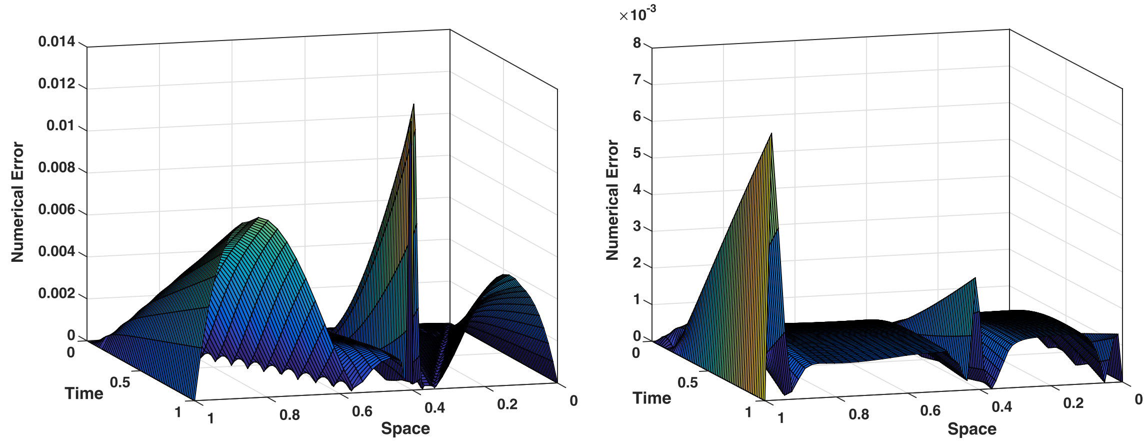
(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$. (B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

FIGURE 1 Numerical solutions for $N_x = 64$ for Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$. (B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

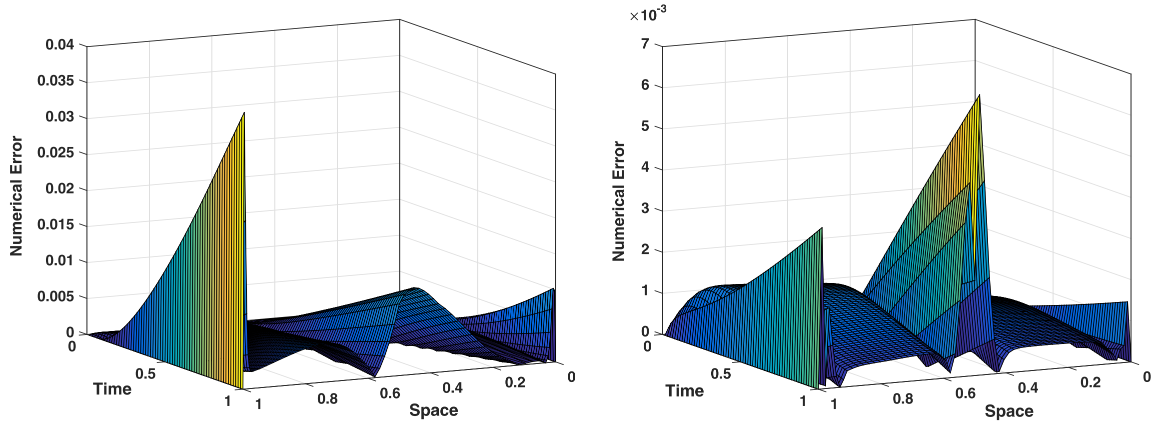
FIGURE 2 Numerical solutions for $N_x = 64$ for Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$.

(B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

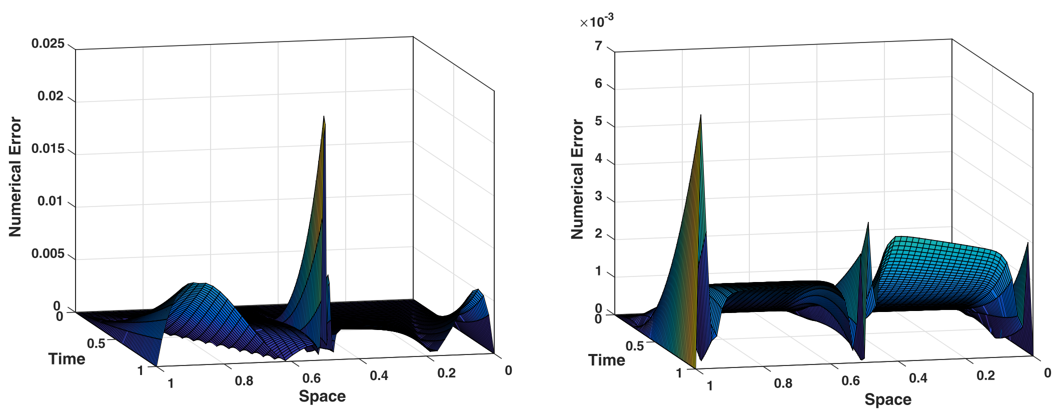
FIGURE 3 Maximum point-wise errors for $N_x = 64$ for Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$.

(B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

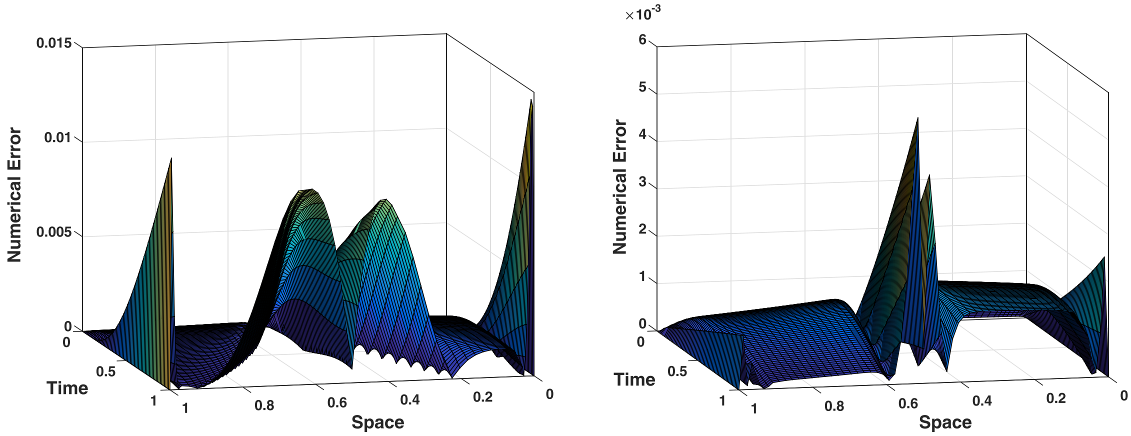
FIGURE 4 Maximum point-wise errors for $N_x = 64$ for Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$.

(B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

FIGURE 5 Maximum point-wise errors of Example 3 for $N_x = 64$ [Colour figure can be viewed at wileyonlinelibrary.com]



(a) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$. (b) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

FIGURE 6 Maximum point-wise errors of Example 4 for $N_x = 64$ [Colour figure can be viewed at wileyonlinelibrary.com]

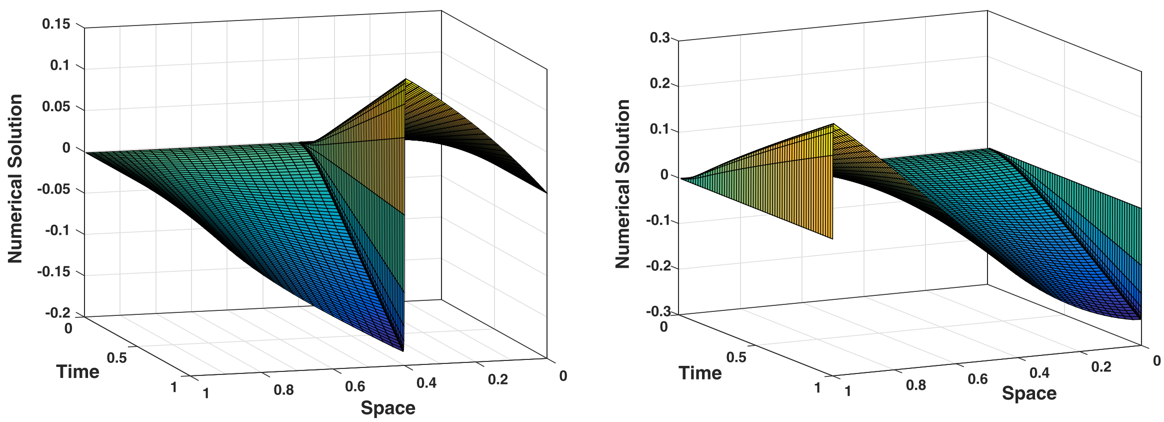


FIGURE 7 Numerical solution with $N_x = 64, \varepsilon = 2^{-20}$, and $\mu = 1$ of Examples 1 and 2, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

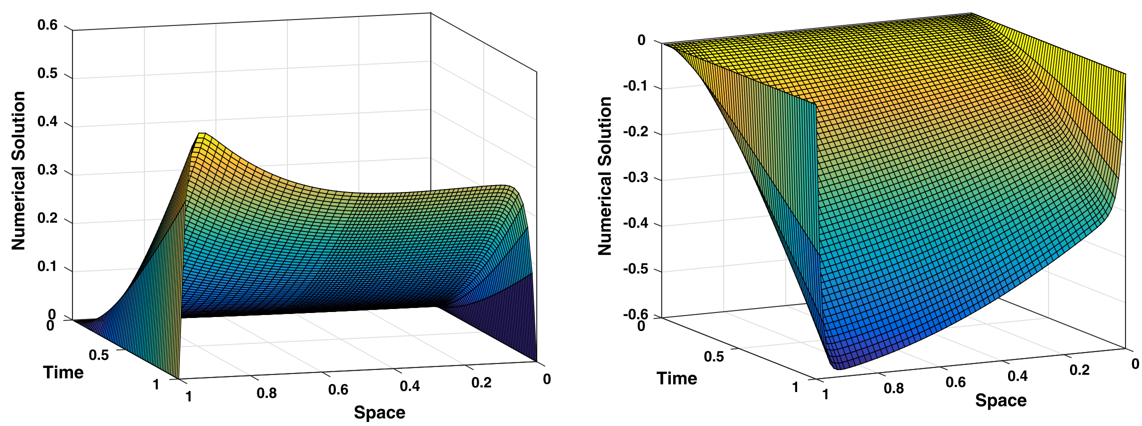
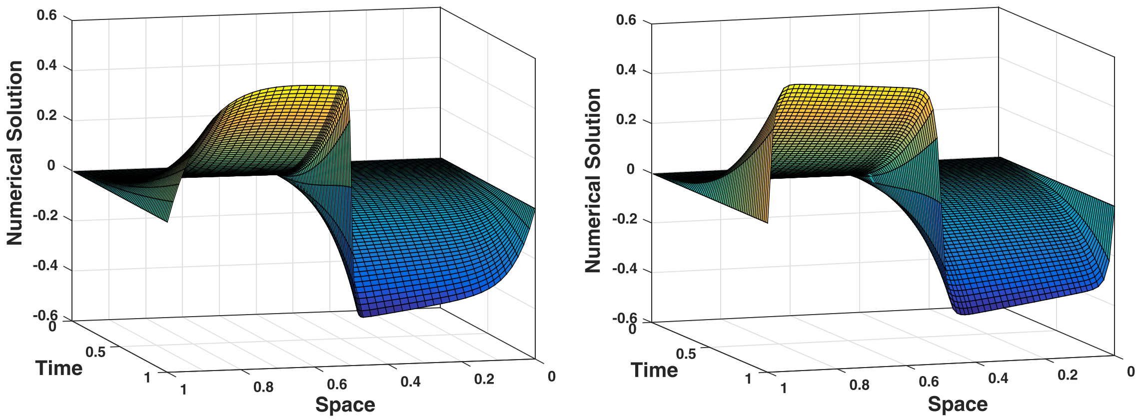
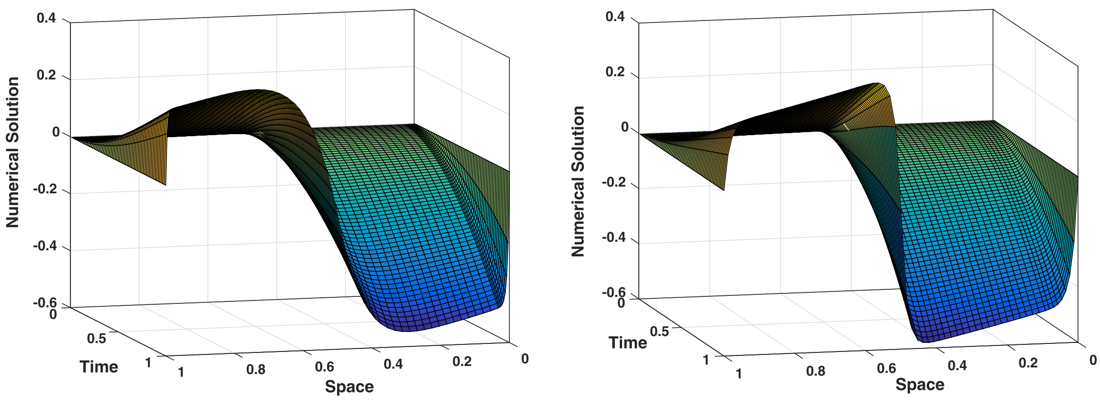


FIGURE 8 Numerical solutions with $N_x = 64, \varepsilon = 2^{-10}$ and $\mu = 2^{-20}$ of the Example 1 whose source term is replaced by $f(x, t) = (1 + x^4)t$ and Example 2 whose source term is replaced by $f(x, t) = (2x + 1)t$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$. (B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

FIGURE 9 Numerical solutions of Example 3 for $N_x = 64$ [Colour figure can be viewed at wileyonlinelibrary.com]



(A) $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}$. (B) $\varepsilon = 2^{-10}$ and $\mu = 2^{-20}$.

FIGURE 10 Numerical solutions of Example 4 for $N_x = 64$ [Colour figure can be viewed at wileyonlinelibrary.com]

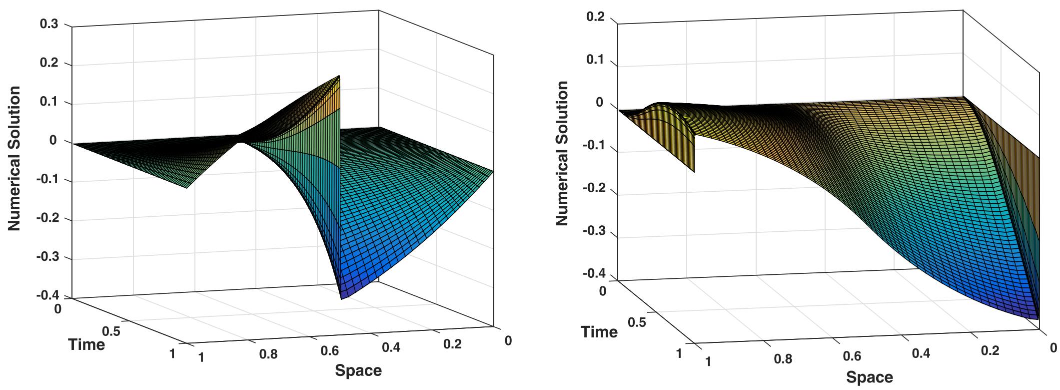


FIGURE 11 Numerical solution for $N_x = 64, \varepsilon = 2^{-20}$ and $\mu = 1$ of Examples 3 and 4, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4.

$$a(x,t) = \begin{cases} 1 + x(1-x) + t, & 0 \le x \le 0.5, \\ -(1+3xt), & 0.5 < x \le 1, \end{cases} \quad f(x,t) = \begin{cases} (1+x)(e^t - 1), & 0 \le x \le 0.5, \\ (-2+x)t, & 0.5 < x \le 1, \end{cases}$$

$$b(x,t) = 1 + x + t, \; c(x,t) = 1, \text{ and } y(0,t) = y(1,t) = y(x,0) = 0.$$

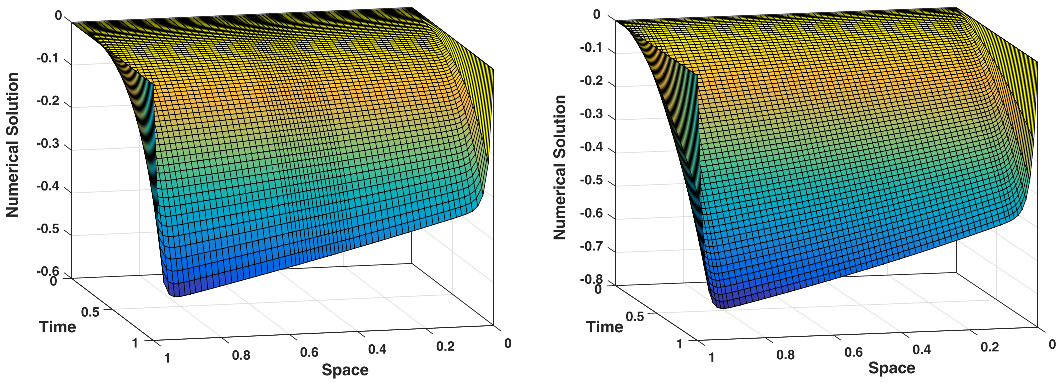


FIGURE 12 Numerical solution with $N_x = 64, \varepsilon = 2^{-10}, \mu = 2^{-20}$ of Example 3 whose source function is replaced by $f(x,t) = (e^{t^2} - 1)(1 + xt)$ and Example 4 whose source function is replaced by $f(x,t) = (1 + x)(e^t - 1)$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Maximum pointwise error E_μ^N and order of convergence ρ_μ^N for $\varepsilon \in S_\varepsilon$ and different values of μ for Example 1

$\mu\downarrow$	$N_x \rightarrow$	64	128	256	512	1024
1	E_μ^N	7.88687E-03	6.50604E-03	4.72305E-03	3.12290E-03	1.88771E-03
	ρ_μ^N	2.77675E-01	4.62061E-01	5.96834E-01	7.26247E-01	
2^{-2}	E_μ^N	1.72968E-02	1.40835E-02	1.00169E-02	6.64725E-03	3.93049E-03
	ρ_μ^N	2.96503E-01	4.91576E-01	5.91600E-01	7.58048E-01	
2^{-4}	E_μ^N	1.44482E-02	1.24223E-02	9.20659E-03	6.23998E-03	3.73672E-03
	ρ_μ^N	2.17959E-01	4.32192E-01	5.61125E-01	7.39769E-01	
2^{-6}	E_μ^N	1.78930E-02	1.25187E-02	9.26266E-03	6.27187E-03	3.74868E-03
	ρ_μ^N	5.15312E-01	4.34585E-01	5.62531E-01	7.42515E-01	
2^{-8}	E_μ^N	1.74626E-02	1.25269E-02	9.27471E-03	6.27892E-03	3.75345E-03
	ρ_μ^N	4.79237E-01	4.33654E-01	5.62785E-01	7.42299E-01	
2^{-10}	E_μ^N	1.73432E-02	1.25281E-02	9.27763E-03	6.28112E-03	3.75437E-03
	ρ_μ^N	4.69204E-01	4.33336E-01	5.62734E-01	7.42453E-01	
2^{-12}	E_μ^N	1.73126E-02	1.25282E-02	9.27834E-03	6.28158E-03	3.75468E-03
	ρ_μ^N	4.66637E-01	4.33243E-01	5.62740E-01	7.42438E-01	
2^{-14}	E_μ^N	1.73048E-02	1.25281E-02	9.27842E-03	6.28166E-03	3.75473E-03
	ρ_μ^N	4.65997E-01	4.33219E-01	5.62733E-01	7.42436E-01	
2^{-16}	E_μ^N	1.73010E-02	1.25258E-02	9.27711E-03	6.28092E-03	3.75438E-03
	ρ_μ^N	4.65954E-01	4.33158E-01	5.62700E-01	7.42399E-01	
2^{-18}	E_μ^N	1.73024E-02	1.25282E-02	9.27838E-03	6.28181E-03	3.75451E-03
	ρ_μ^N	4.65795E-01	4.33236E-01	5.62691E-01	7.42557E-01	
2^{-20}	E_μ^N	1.72721E-02	1.24888E-02	9.25542E-03	6.26846E-03	3.74836E-03
	ρ_μ^N	4.67810E-01	4.32259E-01	5.62187E-01	7.41851E-01	
2^{-22}	E_μ^N	1.72947E-02	1.25184E-02	9.27278E-03	6.27843E-03	3.75317E-03
	ρ_μ^N	4.66282E-01	4.32974E-01	5.62598E-01	7.42296E-01	
2^{-24}	E_μ^N	1.73022E-02	1.25292E-02	9.27766E-03	6.28254E-03	3.75430E-03
	ρ_μ^N	4.65666E-01	4.33456E-01	5.62412E-01	7.42804E-01	

Since exact solutions of these examples are not available, the maximum point-wise errors and rates of convergence will be calculated using the double mesh principle (see Das and Mehrmann and Das[29,31]). We define the double mesh error and corresponding order of convergence for fixed values of ε and μ as follows:

$$E^N_{\varepsilon,\mu} = \max_{(x_i,t_j)\in\overline{\Gamma}^N} |Y^N - \overline{Y}^{2N}| \quad \text{and} \quad \rho^N_{\varepsilon,\mu} = \log_2\left(\frac{E^N_{\varepsilon,\mu}}{E^{2N}_{\varepsilon,\mu}}\right),$$

where Y^N defines the numerical solution by taking $N = N_x = N_t$, (here, N_x is the number of partitions in space, and N_t is the number of partitions in time), and $\overline{Y}^{2N}$ is the piecewise linear interpolant of the solution Y^{2N} on the same mesh points, obtained with $2N = 2N_x = 2N_t$. In addition, we also compute the error and order of convergence by fixing μ and varying ε from a larger set, say $\varepsilon \in S_\varepsilon$ and name them as

$$E^N_\mu = \max_{\varepsilon\in S_\varepsilon} E^N_{\varepsilon,\mu} \quad \text{and} \quad \rho^N_\mu = \log_2\left(\frac{E^N_\mu}{E^{2N}_\mu}\right).$$

The parameter uniform error and order of convergence will be calculated as

$$E^N = \max_{(\varepsilon,\mu)\times\in S_\varepsilon\times S_\mu} E^N_{\varepsilon,\mu} \quad \text{and} \quad \rho^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

The solution surface plots over all time level in Figures 1 and 2 show that the boundary and interior layers inside the domain appears because of the presence of perturbation parameters and discontinuous data. These layers become sharper as μ decreases. Here, we have taken $\alpha = 2, \beta = 1$, and $\gamma = 0.5$ for the Examples 1 and 2. Note that the error plots in

TABLE 2 Maximum point-wise error E^N_μ and order of convergence ρ^N_μ for $\varepsilon \in S_\varepsilon$ and various values of μ for Example 2

$\mu\downarrow$	$N_x \rightarrow$	64	128	256	512	1024
1	E^N_μ	1.29224E-02	9.03664E-03	6.20094E-03	3.84505E-03	2.26100E-03
	ρ^N_μ	5.16012E-01	5.43300E-01	6.89486E-01	7.66040E-01	
2^{-2}	E^N_μ	1.51852E-02	1.12136E-02	7.91271E-03	5.01053E-03	2.95183E-03
	ρ^N_μ	4.37411E-01	5.03009E-01	6.59210E-01	7.63355E-01	
2^{-4}	E^N_μ	2.17182E-02	1.20962E-02	6.46309E-03	4.17747E-03	2.45903E-03
	ρ^N_μ	8.44352E-01	9.04253E-01	6.29594E-01	7.64543E-01	
2^{-6}	E^N_μ	2.37882E-02	1.83583E-02	1.26131E-02	8.07884E-03	4.72060E-03
	ρ^N_μ	3.73812E-01	5.41508E-01	6.42704E-01	7.75178E-01	
2^{-8}	E^N_μ	2.31625E-02	1.80327E-02	1.24177E-02	7.96677E-03	4.65211E-03
	ρ^N_μ	3.61175E-01	5.38221E-01	6.40330E-01	7.76109E-01	
2^{-10}	E^N_μ	2.30061E-02	1.79514E-02	1.23689E-02	7.93878E-03	4.63500E-03
	ρ^N_μ	3.57921E-01	5.37382E-01	6.39725E-01	7.76348E-01	
2^{-12}	E^N_μ	2.29670E-02	1.79311E-02	1.23567E-02	7.93178E-03	4.63072E-03
	ρ^N_μ	3.57102E-01	5.37172E-01	6.39573E-01	7.76409E-01	
2^{-14}	E^N_μ	2.29572E-02	1.79259E-02	1.23536E-02	7.93000E-03	4.62963E-03
	ρ^N_μ	3.56897E-01	5.37120E-01	6.39535E-01	7.76425E-01	
2^{-16}	E^N_μ	2.29534E-02	1.79234E-02	1.23518E-02	7.92886E-03	4.62890E-03
	ρ^N_μ	3.56862E-01	5.37126E-01	6.39538E-01	7.76445E-01	
2^{-18}	E^N_μ	2.29542E-02	1.79244E-02	1.23527E-02	7.92949E-03	4.62932E-03
	ρ^N_μ	3.56832E-01	5.37102E-01	6.39523E-01	7.76428E-01	
2^{-20}	E^N_μ	2.29540E-02	1.79243E-02	1.23526E-02	7.92947E-03	4.62930E-03
	ρ^N_μ	3.56829E-01	5.37101E-01	6.39522E-01	7.76429E-01	
2^{-22}	E^N_μ	2.29540E-02	1.79243E-02	1.23526E-02	7.92945E-03	4.62929E-03
	ρ^N_μ	3.56828E-01	5.37101E-01	6.39522E-01	7.76429E-01	
2^{-24}	E^N_μ	2.29536E-02	1.79240E-02	1.23524E-02	7.92927E-03	4.62918E-03
	ρ^N_μ	3.56832E-01	5.37106E-01	6.39525E-01	7.76433E-01	

Figures 3 and 4 depict that the errors are mainly dominating from the boundary and interior layer regions for Examples 1 and 2. The same behavior can be also noticed for Examples 3 and 4 from Figures 5 and 6, respectively. In the subsequent figures, we discus the behavior of the solution for various values of ε and μ, which helps us to understand the impact of parameter values and their effect in the layer appearance due to the discontinuity in convection coefficient as well as source term.

To show the nature of the layer phenomena for all possible variation of signs between $a(x, t)$ and $f(x, t)$, we consider four different examples. Figure 7 shows the layer behavior for two possible sign changes of discontinuous convection-coefficient at the interior part of the domain for the Examples 1 and 2 with $\mu = 1$. One can see that the solution of Example 1 has only interior layer and the solution of Example 2 has only boundary layers. Note also that it is numerically observed in O'Riordan and Shishkin[6] that the discontinuity in convection coefficient can lead to only interior layer, when the source function is continuous. The layer shifting towards an interior point can be related with the sign of convection-coefficient inside the domain. Note also that, the interior layer(s) may not appear even if the convection-coefficient is discontinuous, which can be clarified from Figure 8 where the boundary layers are observed corresponding to smooth source functions. To develop this figure, we only replace the source functions in Examples 1 and 2 by smooth functions which are $f(x, t) = (1 + x^4)t$ and $f(x, t) = (2x + 1)t$, respectively. In this case, we have unchanged $a(x, t), b(x, t), c(x, t)$ and the initial data to modify the Examples 1 and 2.

The solution surface plot at Figure 8 observes only boundary layers for both of the modified examples. Therefore, the sign of the convection coefficient also has an influence in generating interior and boundary layers in addition to the presence of perturbation parameters ε and μ.

In Figures 9 and 10, we have shown the solution and error plots for Examples 3 and 4 with variable coefficient functions. For these two examples, we take $\alpha = 2, \beta = 2, \gamma = 1$ and $\alpha = 1, \beta = 1, \gamma = 1$, respectively. Example 3 is chosen to

TABLE 3 Maximum point-wise error E_μ^N and order of convergence ρ_μ^N for $\varepsilon \in S_\varepsilon$ and various values of μ for Example 3

$\mu \downarrow$	$N_x \rightarrow$	64	128	256	512	1024
1	E_μ^N	9.96942E-03	6.90947E-03	3.92427E-03	2.38737E-03	2.20056E-03
	ρ_μ^N	5.28935E-01	8.16149E-01	7.17003E-01	1.17553E-01	
2^{-2}	E_μ^N	1.16972E-02	9.19025E-03	5.44939E-03	3.33415E-03	2.26340E-03
	ρ_μ^N	3.47992E-01	7.54010E-01	7.08775E-01	5.58827E-01	
2^{-4}	E_μ^N	1.33814E-02	9.06223E-03	5.54733E-03	3.41629E-03	1.98305E-03
	ρ_μ^N	5.62293E-01	7.08073E-01	6.99363E-01	7.84711E-01	
2^{-6}	E_μ^N	1.32785E-02	1.15506E-02	8.62831E-03	5.85880E-03	3.47676E-03
	ρ_μ^N	2.02385E-01	4.21979E-01	5.59037E-01	7.52707E-01	
2^{-8}	E_μ^N	1.33602E-02	1.16115E-02	8.66681E-03	5.88265E-03	3.49129E-03
	ρ_μ^N	2.02385E-01	4.21979E-01	5.59037E-01	7.52707E-01	
2^{-10}	E_μ^N	1.33806E-02	1.16268E-02	8.67646E-03	5.88862E-03	3.49492E-03
	ρ_μ^N	2.02697E-01	4.22270E-01	5.59177E-01	7.52669E-01	
2^{-12}	E_μ^N	1.33857E-02	1.16306E-02	8.67887E-03	5.89011E-03	3.49583E-03
	ρ_μ^N	2.02774E-01	4.22343E-01	5.59212E-01	7.52660E-01	
2^{-14}	E_μ^N	1.33870E-02	1.16315E-02	8.67947E-03	5.89048E-03	3.49606E-03
	ρ_μ^N	2.02794E-01	4.22361E-01	5.59221E-01	7.52658E-01	
2^{-16}	E_μ^N	1.33873E-02	1.16318E-02	8.67961E-03	5.89056E-03	3.49610E-03
	ρ_μ^N	2.02799E-01	4.22367E-01	5.59224E-01	7.52658E-01	
2^{-18}	E_μ^N	1.33871E-02	1.16315E-02	8.67934E-03	5.89035E-03	3.49595E-03
	ρ_μ^N	2.02808E-01	4.22379E-01	5.59232E-01	7.52670E-01	
2^{-20}	E_μ^N	1.33825E-02	1.16266E-02	8.67456E-03	5.88660E-03	3.49324E-03
	ρ_μ^N	2.02923E-01	4.22563E-01	5.59358E-01	7.52868E-01	
2^{-22}	E_μ^N	1.33874E-02	1.16318E-02	8.67967E-03	5.89061E-03	3.49613E-03
	ρ_μ^N	2.02800E-01	4.22367E-01	5.59224E-01	7.52657E-01	
2^{-24}	E_μ^N	1.33874E-02	1.16318E-02	8.67966E-03	5.89059E-03	3.49612E-03
	ρ_μ^N	2.02801E-01	4.22368E-01	5.59225E-01	7.52658E-01	

be different from other problems by alternatively changing the sign of the function values $a(x,t), f(x,t)$ at each partition of the domain. For Examples 3 and 4, we again note from Figure 11 with $\mu = 1$ that the discontinuity in the source term and convection term can lead to any of the following cases: a) the appearance of only interior layer b) the appearance of only boundary layers.

In addition, we again consider Examples 3 and 4 whose source terms are replaced by smooth functions: $f(x,t) = (e^{t^2} - 1)(1 + xt)$ and $f(x,t) = (1 + x)(e^t - 1)$, respectively. In this case, Figure 12 shows that the interior layer phenomena may not appear when the source functions are considered to be smooth even when the convection coefficient is discontinuous. From the Figures 7 - 8 and 11 - 12, we can say that the discontinuity in the source term can create the interior layer(s) and the discontinuity in convection coefficient can only produce the layer shifting in the solution.

TABLE 4 Maximum point-wise error E_μ^N and order of convergence ρ_μ^N for $\varepsilon \in S_\varepsilon$ and various values of μ for Example 4

$\mu \downarrow$	$N_x \rightarrow$	64	128	256	512	1024
1	E_μ^N	1.67332E-02	1.23727E-02	8.72716E-03	5.54483E-03	3.26259E-03
	ρ_μ^N	4.35551E-01	5.03580E-01	6.54371E-01	7.65123E-01	
2^{-2}	E_μ^N	1.88593E-02	1.45140E-02	1.03177E-02	6.68495E-03	3.93497E-03
	ρ_μ^N	3.77833E-01	4.92320E-01	6.26136E-01	7.64563E-01	
2^{-4}	E_μ^N	1.99440E-02	1.36983E-02	9.77397E-03	6.33137E-03	3.72655E-03
	ρ_μ^N	5.41964E-01	4.86975E-01	6.26428E-01	7.64676E-01	
2^{-6}	E_μ^N	2.18762E-02	1.79030E-02	1.28458E-02	7.85065E-03	4.65296E-03
	ρ_μ^N	2.89159E-01	4.78910E-01	7.10411E-01	7.54662E-01	
2^{-8}	E_μ^N	2.18515E-02	1.78774E-02	1.28332E-02	7.84020E-03	4.64661E-03
	ρ_μ^N	2.89595E-01	4.78257E-01	7.10915E-01	7.54714E-01	
2^{-10}	E_μ^N	2.18451E-02	1.78709E-02	1.28300E-02	7.83758E-03	4.64530E-03
	ρ_μ^N	2.89699E-01	4.78091E-01	7.11042E-01	7.54637E-01	
2^{-12}	E_μ^N	2.18435E-02	1.78693E-02	1.28292E-02	7.83691E-03	4.64496E-03
	ρ_μ^N	2.89726E-01	4.78049E-01	7.11074E-01	7.54617E-01	
2^{-14}	E_μ^N	2.18427E-02	1.78684E-02	1.28286E-02	7.83651E-03	4.64475E-03
	ρ_μ^N	2.89745E-01	4.78044E-01	7.11080E-01	7.54610E-01	
2^{-16}	E_μ^N	2.18430E-02	1.78687E-02	1.28289E-02	7.83670E-03	4.64486E-03
	ρ_μ^N	2.89734E-01	4.78036E-01	7.11083E-01	7.54611E-01	
2^{-18}	E_μ^N	2.18430E-02	1.78688E-02	1.28290E-02	7.83672E-03	4.64487E-03
	ρ_μ^N	2.89733E-01	4.78036E-01	7.11084E-01	7.54611E-01	
2^{-20}	E_μ^N	2.18430E-02	1.78687E-02	1.28289E-02	7.83669E-03	4.64486E-03
	ρ_μ^N	2.89734E-01	4.78036E-01	7.11084E-01	7.54611E-01	
2^{-22}	E_μ^N	2.18426E-02	1.78682E-02	1.28285E-02	7.83646E-03	4.64473E-03
	ρ_μ^N	2.89747E-01	4.78041E-01	7.11083E-01	7.54608E-01	
2^{-24}	E_μ^N	2.18430E-02	1.78687E-02	1.28289E-02	7.83669E-03	4.64485E-03
	ρ_μ^N	2.89734E-01	4.78036E-01	7.11084E-01	7.54611E-01	

TABLE 5 Uniform error E^N and order of convergence ρ^N for $\varepsilon \in S_\varepsilon$, $\mu \in S_\mu$ for Examples 1 to 4

Example	$N \rightarrow$	64	128	256	512	1024
Example 1	E^N	1.78930E-02	1.40835E-02	1.00169E-02	6.64725E-03	3.93049E-03
	ρ^N	3.45390E-01	4.91580E-01	5.91600E-01	7.58050E-01	
Example 2	E^N	2.37882E-02	1.83583E-02	1.26131E-02	8.07884E-03	4.72060E-03
	ρ^N	3.73810E-01	5.41510E-01	6.42700E-01	7.75180E-01	
Example 3	E^N	1.33874E-02	1.16318E-02	8.67966E-03	5.89059E-03	3.49612E-03
	ρ^N	2.02801E-01	4.22368E-01	5.59225E-01	7.52658E-01	
Example 4	E^N	2.18430E-02	1.78687E-02	1.28289E-02	7.83669E-03	4.64485E-03
	ρ^N	2.89734E-01	4.78036E-01	7.11084E-01	7.54611E-01	

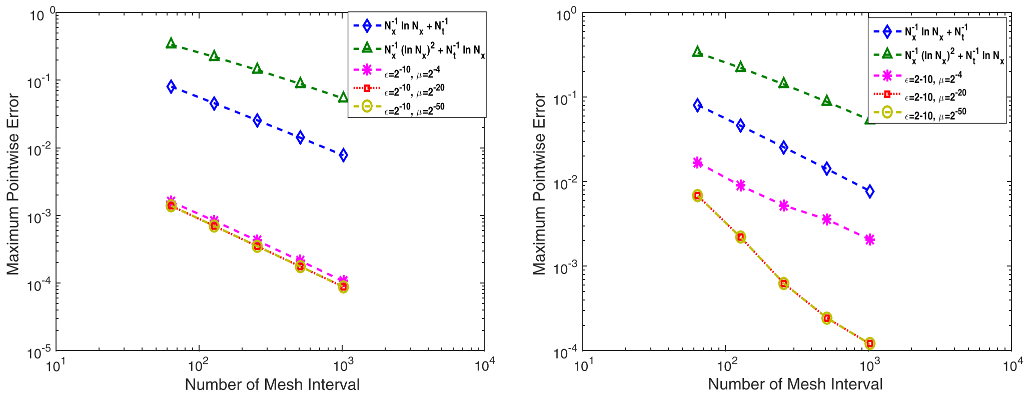


FIGURE 13 Loglog plot of the maximum point-wise errors of Examples 1 and 2 , respectively [Colour figure can be viewed at wileyonlinelibrary.com]

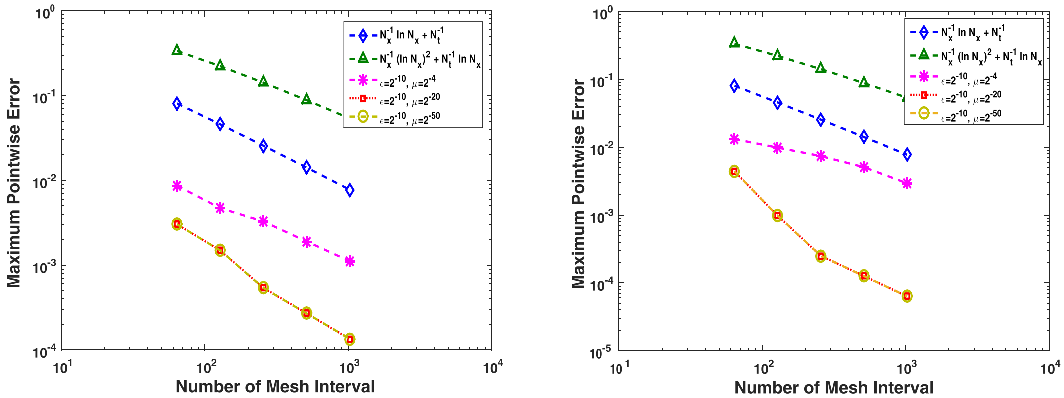


FIGURE 14 Loglog plot of the maximum pointwise errors of Examples 3 and 4, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

Tables 1 to 4 show that the maximum point-wise errors are converging with almost first-order accuracy in space and time for all the test problems, as expected from the analysis. These errors do not depend on the magnitude of the convection and diffusion parameters and hence, they are parameter uniform. To generate these tables, ε is considered from a set $S_\varepsilon = \{1, 2^{-2}, \cdots, 2^{-50}\}$ for each fixed value of μ. Table 5 shows the uniform error and corresponding order of convergence for various values of ε and μ, which are taken from the set $\varepsilon \in S_\varepsilon$ and $\mu \in S_\mu$, where $S_\mu = \{1, 2^{-2}, \cdots 2^{-24}\}$. In addition, we have plotted the errors on loglog scale in Figures 13 and 14 for Examples 1 to 2 and 3 to 4, respectively for $\varepsilon = 2^{-10}$ and $\mu = 2^{-4}, 2^{-20}, 2^{-50}$ to show that the expected rate of convergence is true in practice.

7 | CONCLUSIONS

A convergent numerical method is presented for a two-parameter singularly perturbed parabolic problem, which has discontinuous convection-coefficient and source term. In general, the solution of this kind of problems have both boundary and interior layers, which makes the numerical analysis different. Here, we provide a convergent solution by discretizing the continuous problem with backward Euler scheme for time variable on uniform mesh and an upwind scheme on an a priori layer adaptive piecewise uniform mesh for the spatial variable. The theoretical analysis shows that the numerical method is almost first-order accurate in space and time, which is also validated by several numerical experiments. Note that boundary layers appear because of the presence of perturbation parameters. However, it is observed that the interior layers can appear because of the discontinuity in the source term. In addition, we observe that the discontinuity in the convection coefficient can shift the layer position. Several simulations depict the layer appearances and their behavior in different locations.

NOMENCLATURE

Notations	Explanations
Ω_x and $\overline{\Omega}_x$	$(0,1)$ and $[0,1]$
$\Omega_x^{N_x}$ and $\overline{\Omega}_x^{N_x}$	$\{x_i : 1 \le i \le N_x - 1\}$ and $\{x_i : 0 \le i \le N_x\}$
Ω_x^- and Ω_x^+	$(0,d)$ and $(d,1)$
$\Omega_x^{N_x-}$ and $\Omega_x^{N_x+}$	$\left\{x_i : 1 \le i \le \frac{N_x}{2} - 1\right\}$ and $\left\{x_i : \frac{N_x}{2} + 1 \le i \le N_x - 1\right\}$
Ω_t and $\Omega_t^{N_t}$	$(0,T]$ and $\left\{t_j, t_j = jT/N_t \text{ for } 1 \le j \le N_t\right\}$
Γ and Γ^N	$\Omega_x \times \Omega_t$ and $\left\{(x_i,t_j) : 1 \le i \le N_x - 1, 1 \le j \le N_t\right\}$
$\overline{\Gamma}$ and $\overline{\Gamma}^N$	$\overline{\Omega}_x \times \Omega_t$ and $\left\{(x_i,t_j) : 0 \le i \le N_x, 1 \le j \le N_t\right\}$
Γ^- and Γ^+	$\Omega_x^- \times \Omega_t$ and $\Omega_x^+ \times \Omega_t$
Γ^{-*} and Γ^{+*}	$[0,d) \times \Omega_t$ and $(d,1] \times \Omega_t$
Γ^{N-} and Γ^{N+}	$\left\{(x_i,t_j) : 1 \le i \le \frac{N_x}{2} - 1, 1 \le j \le N_t\right\}$ and $\left\{(x_i,t_j) : \frac{N_x}{2} + 1 \le i \le N_x - 1, 1 \le j \le N_t\right\}$
$\Gamma^{\pm}$ and $\Gamma^{N\pm}$	$\{(d,t) : 0 < t \le T\}$ and $\{(d,t_j) : 1 \le j \le N_t\}$
Γ_c^- and Γ_c^+	$[0,d] \times \{t = 0\}$ and $[d,1] \times \{t = 0\}$
Γ_c and Γ_c^N	$\Gamma_c^- \cup \Gamma_c^+$ and $\left\{(x_i,t_j) : 0 \le i \le N_x, j = 0\right\}$
Γ_l, Γ_r and Γ_c	$\{(0,t) : 0 < t \le T\}, \{(1,t) : 0 < t \le T\}$ and $[0,1] \times \{t = 0\}$
$\Gamma_l^N,\ \Gamma_r^N$ and Γ_c^N	$\{(x_i,t_j) : i = 0, 1 \le j \le N_t\}, \{(x_i,t_j) : i = N_x, 1 \le j \le N_t\}$ and $\{(x_i,t_j) : 0 \le i \le N_x, j = 0\}$
Γ_0 and Γ_0^N	$(\Gamma_l \cup \Gamma_r \cup \Gamma_c)$ and $(\overline{\Gamma}^N \cap \Gamma_0)$
$y(x,t)$	Continuous solution of (1)-(2) on $(\Gamma^- \cup \Gamma^+)$
$v(x,t)$	Regular component (continuous) of $y(x,t)$ on $(\Gamma^- \cup \Gamma^+)$
$w(x,t)$	Singular component (continuous) of $y(x,t)$ on $(\Gamma^- \cup \Gamma^+)$
$y^*(x,t)$	Solution of the initial-boundary value problem (15)-(16) on $(\Gamma^{-*} \cup \Gamma^{+*})$
$Y(x_i,t_j)$	Discrete solution of fully discretized problem (27)-(29) on $(\Gamma^{N-} \cup \Gamma^{N+})$
$V(x_i,t_i)$	Regular component of discrete solution $Y(x_i,t_i)$ on $(\Gamma^{N-} \cup \Gamma^{N+})$
$W(x_i,t_i)$	Singular component of discrete solution $Y(x_i,t_i)$ on $(\Gamma^{N-} \cup \Gamma^{N+})$

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