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On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term

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ABSTRACT

In this paper, we deal with the asymptotic and oscillatory behavior of the solutions of higher-order differential equations with middle term of the form $L_w' + p(\nu) f\left(w^{(m-1)}(\nu)\right) + q(\nu) g(w(\sigma(\nu))) = 0$, where $L_w := r(\nu)\left(w^{(m-1)}(\nu)\right)^{\alpha}$. By using generalized Riccati transformations we study the asymptotic behavior and derive a new oscillation criterion. The results obtained here extend and improve some well-known results which have been published recently in the literature. An example is given to illustrate the applicability of the obtained results.

1. Introduction

Lately, the use of higher order delay differential equations has been considered to describe many real life applications, as models concerning physical, biological, or chemical phenomena applications in dynamical systems (see [1–4]). As a result of the interest in this kind of equations, the qualitative behavior of their solutions has been the subject of study for many scholars in the past years [5–15]. Particular emphasis has been given to the study of oscillations and the oscillatory behavior of these equations, which have been investigated using different methods and various techniques; we refer the interested reader to the papers [16–25]. In this paper, we will establish an oscillation criterion for the higher order differential equation with middle term of the form

$$\left(r(\nu)\left(w^{(m-1)}(\nu)\right)^{\alpha}\right)' + p(\nu) f\left(w^{(m-1)}(\nu)\right) + q(\nu) g(w(\sigma(\nu))) = 0, \tag{1}$$

where α is a quotient of odd positive integers and $m \geq 2$ is an even integer. Throughout this work, we suppose that:

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(P_1) $r,\ p,\ q \in C([\nu_0,\infty),[0,\infty))$, $r(\nu)>0,\ q(\nu)>0, r'(\nu)+p(\nu)\geq 0$,
(P_2) $f,\ g\in C(\mathbb{R},\mathbb{R})$, $f(u)\geq k_f u^{\alpha}>0,\ g(u)\geq k_g u^{\alpha}>0$ for $u\neq 0,\ k_f\geq 1$ and $k_g>0$,
(P_3) $\sigma\in C([\nu_0,\infty),(0,\infty))$, $\sigma(\nu)\leq\nu$ and $\lim_{\nu\to\infty}\sigma(\nu)=\infty$ and under the condition

$$\int_{\nu_0}^{\infty}\left[\frac{1}{r(s)}\exp\left(-\int_{\nu_0}^{s}\frac{p(u)}{r(u)}du\right)\right]^{1/\alpha}ds<\infty. \tag{2}$$

Definition 1.1. A function $w\in C^{m-1}[\nu_w,\infty),\ \nu_w\geq\nu_0,$ is called a solution of (1), if $r(\nu)\left(w^{(m-1)}(\nu)\right)^{\alpha}\in C^1[\nu_w,\infty),$ and $w(\nu)$ satisfies (1) on $[\nu_w,\infty)$.

Definition 1.2. If a solution of (1) has arbitrarily large zeros on $[\nu_w,\infty)$, then it is called oscillatory, and otherwise is called to be nonoscillatory.

Definition 1.3. Eq. (1) is called to be oscillatory if all its solutions are oscillatory.

In what follows, we present a brief review of some results that have provided the background and the motivation for the present work.

Elabbasy et al. [20] studied the oscillation of solutions of the fourth-order equation

$$[r(\nu)w'''(\nu)]'+p(\nu)w'''(\nu)+q(\nu)w(\tau(\nu))=0,$$

under the condition

$$\int_{\nu_0}^{\infty}\left[\frac{1}{r(s)}\exp\left(-\int_{\nu_0}^{s}\frac{p(u)}{r(u)}du\right)\right]^{1/\alpha}ds=\infty.$$

In [7], Liu et al. discussed the equation

$$\left(r(\nu)\varphi\left(w^{(m-1)}(\nu)\right)\right)'+p(\nu)\varphi\left(w^{(m-1)}(\nu)\right)+q(\nu)\varphi(w(g(\nu)))=0,$$

with $\varphi(\nu)=|\nu|^{p-2}\nu$ and

$$\int_{\nu_0}^{\infty}\left[\frac{1}{r(s)}\exp\left(-\int_{\nu_0}^{s}\frac{p(u)}{r(u)}du\right)\right]^{1/(p-1)}ds=\infty,$$

where p is a real number satisfying $p>1$.

Elabbasy et al. [21] consider the oscillatory behavior of the equation

$$\left[r(\nu)\left|\left(w^{(m-1)}(\nu)\right)\right|^{p-2}w^{(m-1)}(\nu)\right]'+q(\nu)f(w(\tau(\nu)))=0,$$

under the condition

$$\int_{\nu_0}^{\infty}\frac{1}{r^{1/(p-1)}(\nu)}d\nu=\infty,$$

where p is a real number satisfying $p>1$.

Zhang et al. in [24] considered a higher-order differential equation

$$L_w'+p(\nu)\left|\left(w^{(m-1)}(\nu)\right)\right|^{p-2}w^{(m-1)}(\nu)+q(\nu)\left|(w(\tau(\nu)))\right|^{p-2}w(\tau(\nu))=0,$$

where

$$L_w=r(\nu)\left|\left(w^{(m-1)}(\nu)\right)\right|^{p-2}w^{(m-1)}(\nu)$$

and p is a real number satisfying $p>1$.

The aim of this paper is to establish new oscillation results of solutions to a class of higher-order differential equations with delayed arguments by using a Riccati technique, the results obtained here essentially complement some well-known results which have been published recently in the literature. In order to discuss our results, we need the following lemmas.

Lemma 1.1 *([26], Lemma 2.2.3).* *Let $u \in C^m([\nu_0,\infty),(0,\infty))$. Assume that $u^{(m)}(\nu)$ is of a fixed sign on $[\nu_0,\infty)$, m a positive integer, $u^{(m)}(\nu)$ not identically zero and that there exists $\nu_1 \geq \nu_0$ such that, for all $\nu \geq \nu_1$ it is*

$$u^{(m-1)}(\nu)\,u^{(m)}(\nu) \leq 0.$$

If we have $\lim_{\nu\to\infty} u(\nu) \neq 0$, then there exists $\nu_\lambda \geq \nu_0$ such that

$$u(\nu) \geq \frac{\lambda}{(m-1)!}\nu^{m-1}\left|u^{(m-1)}(\nu)\right|,$$

for every $\lambda \in (0,1)$ and $\nu \geq \nu_\lambda$.

Lemma 1.2 *([27], Lemma 2.1).* *Let $\alpha \geq 1$ be a ratio of two odd numbers, $C > 0$ and D are constants. Then*

$$Dw - Cw^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{D^{\alpha+1}}{C^\alpha}.$$

Lemma 1.3 *([26], Lemma 2.2.2).* *Let $u \in C^m([\nu_0,\infty),(0,\infty))$ with m a positive integer such that it and its derivatives up to order $(m-1)$ are absolutely continuous and of constant sign in an interval (ν_0,∞). If $u^{(m-1)}(\nu)\,u^{(m)}(\nu) \leq 0$ for all $\nu \geq \nu_u$, then for every $\theta \in (0,1)$ there exists a constant $M > 0$ such that*

$$u(\theta\nu) \geq M\nu^{m-1}u^{(m-1)}(\nu),$$

for all sufficient large ν.

Lemma 1.4 *([28], Theorem 2.1).* *Assume that (2) is satisfied and let $w(\nu)$ be an eventually positive solution of (1) such that $\lim_{\nu\to\infty} w(\nu) \neq 0$. Then there exists a sufficiently large $\nu_1 \geq \nu_0$ such that for all $\nu \geq \nu_1$ is verified one of the two possible cases*

$$(I_1)\;\; w(\nu) > 0,\; w'(\nu) > 0,\; w^{(m-1)}(\nu) > 0,\; w^{(m)}(\nu) < 0.$$
$$(I_2)\;\; w(\nu) > 0,\; w^{(m-2)}(\nu) > 0,\;\; w^{(m-1)}(\nu) < 0.$$

2. Main results

We use the Riccati transformation to prove some necessary lemmas before establishing an oscillation criterion for (1). For convenience, we use the following notations:

$$\begin{aligned}
\zeta(\nu_0,\nu) &:= \exp\left(\int_{\nu_0}^{\nu}\frac{p(u)}{r(u)}du\right),\\
\vartheta(\nu) &:= \int_{\nu}^{\infty}\frac{ds}{(r(s)\,\zeta(\nu_0,s))^{\frac{1}{\alpha}}},\\
\phi(\nu) &:= \frac{\delta'(\nu)}{\delta(\nu)} - \frac{k_f p(\nu)}{r(\nu)},\;\; \phi_+(\nu) := \max(0,\phi(\nu))\\
\varphi(\nu) &:= \frac{1}{\zeta^{\frac{1}{\alpha}}(\nu_0,\nu)} - \frac{\vartheta(\nu)\,p(\nu)\,r^{(1-\alpha)/\alpha}(\nu)}{\alpha},\;\; \varphi_+(\nu) := \max(0,\varphi(\nu))\\
\tilde{\varphi}(\nu) &:= \frac{p(\nu)}{r(\nu)} + \frac{\alpha^{(\alpha+1)}\delta(\nu)\,\varphi^{\alpha+1}(\nu)\,\zeta(\nu_0,\nu)}{\vartheta(\nu)\,r^{\frac{1}{\alpha}}(\nu)}.
\end{aligned}$$

Lemma 2.1. *Let $w(\nu)$ be an eventually positive solution of Eq. (1) and assume that (I_1) holds. If $\psi(\nu)$ is a Riccati transformation defined by*

$$\psi(\nu) := \delta(\nu)\frac{r(\nu)\left(w^{(m-1)}\right)^{\alpha}(\nu)}{w^{\alpha}(\nu/2)}, \tag{3}$$

where $\delta \in C^1([\nu_0,\infty))$, then there exists a constant $M > 0$ such that

$$\psi'(\nu) \le -k_g\delta(\nu)q(\nu) + \phi_+(\nu)\psi(\nu) - \frac{\alpha M\nu^{m-2}}{2(r(\nu)\delta(\nu))^{1/\alpha}}\psi^{\frac{\alpha+1}{\alpha}}(\nu). \tag{4}$$

Proof. Assume that $w(\nu)$ is an eventually positive solution of (1) and from Lemma 1.4, we see (I_1) holds. By Lemma 1.3, we get

$$w'(\nu/2) \ge M\nu^{m-2}w^{(m-1)}(\nu). \tag{5}$$

From the definition of $\psi(\nu)$, we see that $\psi(\nu) > 0$ for $\nu \ge \nu_1$, and

$$\psi'(\nu) = \delta'(\nu)\frac{r(\nu)\left(w^{(m-1)}\right)^{\alpha}(\nu)}{w^{\alpha}(\nu/2)} + \delta(\nu)\frac{\left(r\left(w^{(m-1)}\right)^{\alpha}\right)'(\nu)}{w^{\alpha}(\nu/2)} - \alpha\delta(\nu)\frac{w'(\nu/2)r(\nu)\left(w^{(m-1)}\right)^{\alpha}(\nu)}{2w^{\alpha+1}(\nu/2)}.$$

Using (3) and (5), we obtain

$$\psi'(\nu) \le \frac{\delta'_+(\nu)}{\delta(\nu)}\psi(\nu) + \delta(\nu)\frac{\left(r\left(w^{(m-1)}\right)^{\alpha}\right)'(\nu)}{w^{\alpha}(\nu/2)} - \alpha M\nu^{m-2}\delta(\nu)\frac{r(\nu)\left(w^{(m-1)}\right)^{\alpha+1}(\nu)}{2w^{\alpha+1}(\nu/2)}.$$

From (1), we get

$$\begin{aligned}\psi'(\nu) &\le \frac{\delta'_+(\nu)}{\delta(\nu)}\psi(\nu) - k_f p(\nu)\frac{\psi(\nu)}{r(\nu)} - k_g\delta(\nu)q(\nu)\frac{w^{\alpha}(\sigma(\nu))}{w^{\alpha}(\nu/2)} - \alpha M\nu^{m-2}\frac{\psi^{\frac{\alpha+1}{\alpha}}(\nu)}{2(\delta(\nu)r(\nu))^{1/\alpha}} \\ &\le -k_g\delta(\nu)q(\nu) + \left(\frac{\delta'_+(\nu)}{\delta(\nu)} - k_f\frac{p(\nu)}{r(\nu)}\right)\psi(\nu) - \alpha M\nu^{m-2}\frac{\psi^{\frac{\alpha+1}{\alpha}}(\nu)}{2(\delta(\nu)r(\nu))^{1/\alpha}}.\end{aligned}$$

Hence, we find

$$\psi'(\nu) \le -k_g\delta(\nu)q(\nu) + \phi_+(\nu)\psi(\nu) - \alpha M\nu^{m-2}\frac{\psi^{\frac{\alpha+1}{\alpha}}(\nu)}{2(\delta(\nu)r(\nu))^{1/\alpha}}.$$

The proof is complete. □

Lemma 2.2. *Let $w(\nu)$ be an eventually positive solution of Eq. (1) and assume that (I_2) holds. If $\varpi(\nu)$ is a Riccati transformation defined by*

$$\varpi(\nu) := -\frac{r(\nu)\left(-w^{(m-1)}\right)^{\alpha}(\nu)}{\left(w^{(m-2)}\right)^{\alpha}(\nu)}, \tag{6}$$

then there exists a constant $\mu \in (0,1)$ such that

$$\varpi'(\nu) \le \frac{k_f p(\nu)}{r(\nu)\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)} - k_g q(\nu)\left(\frac{\mu}{(m-2)!}\sigma^{m-2}(\nu)\right)^{\alpha} - \alpha\frac{\varpi^{\frac{\alpha+1}{\alpha}}(\nu)}{r^{\frac{1}{\alpha}}(\nu)}. \tag{7}$$

Proof. Assume that $w(\nu)$ be an eventually positive solution of (1) and (I_2) holds. Since

$$\left(-r(\nu)\left(-w^{(m-1)}(\nu)\right)^{\alpha}\zeta(\nu_0,\nu)\right)' \le -k_g q(\nu)w^{\alpha}(\sigma(\nu))\zeta(\nu_0,\nu) < 0,$$

we deduce that $-r(\nu)\left(-w^{(m-1)}(\nu)\right)^{\alpha}\zeta(\nu_0,\nu)$ is decreasing. Thus, for $s \geq \nu \geq \nu_1$ it is

$$(r(s)\zeta(\nu_0,s))^{1/\alpha} w^{(m-1)}(s) \leq (r(\nu)\zeta(\nu_0,\nu))^{1/\alpha} w^{(m-1)}(\nu). \tag{8}$$

Dividing both sides of (8) by $(r(s)\zeta(\nu_0,s))^{1/\alpha}$ and integrating the resulting inequality from ν to u, we get

$$w^{(m-2)}(u) \leq w^{(m-2)}(\nu) + (r(\nu)\zeta(\nu_0,\nu))^{1/\alpha} w^{(m-1)}(\nu) \int_{\nu}^{u} \frac{ds}{(r(s)\zeta(\nu_0,s))^{1/\alpha}}.$$

Letting $u \to \infty$, we arrive that

$$0 \leq w^{(m-2)}(\nu) + (r(\nu)\zeta(\nu_0,\nu))^{1/\alpha} w^{(m-1)}(\nu)\vartheta(\nu),$$

which yields

$$-\frac{w^{(m-1)}(\nu)}{w^{(m-2)}(\nu)}\vartheta(\nu)(r(\nu)\zeta(\nu_0,\nu))^{1/\alpha} \leq 1.$$

Hence,

$$\frac{r(\nu)\left(w^{(m-1)}(\nu)\right)^{\alpha}}{\left(w^{(m-2)}(\nu)\right)^{\alpha}} \geq \frac{-1}{\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)}.$$

From (6), we have

$$\varpi(\nu) \geq \frac{-1}{\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)}. \tag{9}$$

From the definition of $\omega(\nu)$, we see that $\omega(\nu) < 0$ for $\nu \geq \nu_1$, and

$$\varpi'(\nu) = \frac{\left(-r(\nu)\left(-w^{(m-1)}(\nu)\right)^{\alpha}\right)'}{\left(w^{(m-2)}(\nu)\right)^{\alpha}} - \alpha\frac{-r(\nu)\left(-w^{(m-1)}(\nu)\right)^{\alpha+1}}{\left(w^{(m-2)}(\nu)\right)^{\alpha+1}}.$$

From (1) and (6), we obtain

$$\begin{aligned}\varpi'(\nu) &\leq -k_f\frac{p(\nu)}{r(\nu)}\varpi(\nu) - k_g q(\nu)\frac{w^{\alpha}(\sigma(\nu))}{\left(w^{(m-2)}(\nu)\right)^{\alpha}} - \alpha\frac{\varpi^{\frac{\alpha+1}{\alpha}}(\nu)}{r^{\frac{1}{\alpha}}(\nu)} \\ &= -k_f\frac{p(\nu)}{r(\nu)}\varpi(\nu) - k_g q(\nu)\frac{w^{\alpha}(\sigma(\nu))}{\left(w^{(m-2)}(\sigma(\nu))\right)^{\alpha}}\frac{\left(w^{(m-2)}(\sigma(\nu))\right)^{\alpha}}{\left(w^{(m-2)}(\nu)\right)^{\alpha}} - \alpha\frac{\varpi^{\frac{\alpha+1}{\alpha}}(\nu)}{r^{\frac{1}{\alpha}}(\nu)}.\end{aligned} \tag{10}$$

By Lemma 1.1, we get, for some constant $\mu \in (0,1)$

$$w(\nu) \geq \frac{\mu}{(m-2)!}\nu^{m-2}w^{(m-2)}(\nu). \tag{11}$$

Thus, from (9) and (11), we get

$$\varpi'(\nu) \leq \frac{k_f p(\nu)}{r(\nu)\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)} - k_g q(\nu)\left(\frac{\mu}{(m-2)!}\sigma^{m-2}(\nu)\right)^{\alpha} - \alpha\frac{\varpi^{\frac{\alpha+1}{\alpha}}(\nu)}{r^{\frac{1}{\alpha}}(\nu)}.$$

The proof is complete. □

Theorem 2.1. *Assume that (2) holds. If there exist constants $M > 0$, $\mu \in (0,1)$, and positive functions $\delta, \vartheta \in C^1([\nu_0,\infty),(0,\infty))$ such that*

$$\lim_{\nu\to\infty}\sup\int_{\nu_0}^{\nu}\left(k_g\delta(s)q(s) - \left(\frac{2}{Ms^{m-2}}\right)^{\alpha}\frac{r(s)\delta(s)(\phi_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds = \infty \tag{12}$$

and

$$\lim_{\nu\to\infty}\sup\int_{\nu_0}^{\nu}\left(k_g q(s)\left(\frac{\mu\sigma^{m-2}(s)}{(m-2)!}\vartheta(s)\right)^{\alpha}\zeta(\nu_0,s) - \tilde{\varphi}(s)\right)ds = \infty, \tag{13}$$

then every solution of (1) is oscillatory.

Proof. Let w be a nonoscillatory solution of Eq. (1). The proof will be completed by showing that this leads to a contradiction. Without loss of generality, we can assume that $w(\nu) > 0$. From Lemma 2.1, we get that (4) holds. Using Lemma 1.2, we set

$$D = \phi_+(\nu), \quad C = \alpha M \nu^{m-2} / \left(2 (r(\nu)\delta(\nu))^{1/\alpha}\right) \quad \text{and} \quad w = \psi.$$

We have

$$\psi'(\nu) \le -k_g \delta(\nu) q(\nu) + \left(\frac{2}{M\nu^{m-2}}\right)^{\alpha} \frac{r(\nu)\delta(\nu)(\phi_+(\nu))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{14}$$

Integrating from ν_1 to ν, we find

$$\int_{\nu_1}^{\nu} \left(k_g \delta(s) q(s) - \left(\frac{2}{M s^{m-2}}\right)^{\alpha} \frac{r(s)\delta(s)(\phi_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right) ds \le \psi(\nu_1),$$

which contradicts (12).

From Lemma 2.2, we get that (7) holds. Multiplying (7) by $\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)$ and integrating the resulting inequality from ν_1 to ν, we get

$$\begin{aligned}
&\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)\varpi(\nu) - \vartheta^{\alpha}(\nu_1)\zeta(\nu_0,\nu_1)\varpi(\nu_1) - \int_{\nu_1}^{\nu} \frac{p(s)}{r(s)} ds \\
&+ \alpha \int_{\nu_1}^{\nu} r^{\frac{-1}{\alpha}}(s)\vartheta^{\alpha-1}(s)\zeta(\nu_0,s)\varphi_+(s)\varpi(s) ds \\
&+ \int_{\nu_1}^{\nu} k_g q(s) \left(\frac{\mu}{(m-2)!}\sigma^{m-2}(s)\right)^{\alpha} \vartheta^{\alpha}(s)\zeta(\nu_0,s) ds \\
&+ \alpha \int_{\nu_1}^{\nu} \frac{\varpi^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)} \vartheta^{\alpha}(s)\zeta(\nu_0,s) ds \le 0.
\end{aligned}$$

Using Lemma 1.2 we set

$$C = \vartheta^{\alpha}(s)\zeta(\nu_0,s)/r^{\frac{1}{\alpha}}(s), \; D = \int_{\nu_1}^{\nu} r^{\frac{-1}{\alpha}}(s)\vartheta^{\alpha-1}(s)\zeta(\nu_0,s)\varphi_+(s), \; w = \varpi(\nu).$$

Thus, we get

$$\begin{aligned}
&\vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)\varpi(\nu) - \vartheta^{\alpha}(\nu_1)\zeta(\nu_0,\nu_1)\varpi(\nu_1) - \int_{\nu_1}^{\nu} \frac{p(s)}{r(s)} ds \\
&+ \int_{\nu_1}^{\nu} k_g q(s) \left(\frac{\mu}{(m-2)!}\sigma^{m-2}(s)\right)^{\alpha} \vartheta^{\alpha}(s)\zeta(\nu_0,s) ds \\
&+ \int_{\nu_1}^{\nu} \frac{\alpha^{(\alpha+1)}\delta(s)\varphi^{\alpha+1}(s)\zeta(\nu_0,s)}{\vartheta(s) r^{\frac{1}{\alpha}}(\nu)} ds \le 0.
\end{aligned}$$

Hence, by (9), we obtain

$$\int_{\nu_1}^{\nu} \left(k_g q(s) \left(\frac{\mu\sigma^{m-2}(s)}{(m-2)!}\vartheta(s)\right)^{\alpha} \zeta(\nu_0,s) - \tilde{\varphi}(s)\right) ds \le \vartheta^{\alpha}(\nu)\zeta(\nu_0,\nu)\varpi(\nu_1) + 1,$$

which contradicts (13).

Theorem 2.1 is proved. □

Example 2.1. For $\nu \ge 1$, consider the differential equation

$$\left(\nu^2 (w'(\nu))\right)' + \frac{\nu}{2} w'(\nu) + \upsilon w\left(\frac{\nu}{2}\right) = 0, \; \nu \ge 1, \tag{15}$$

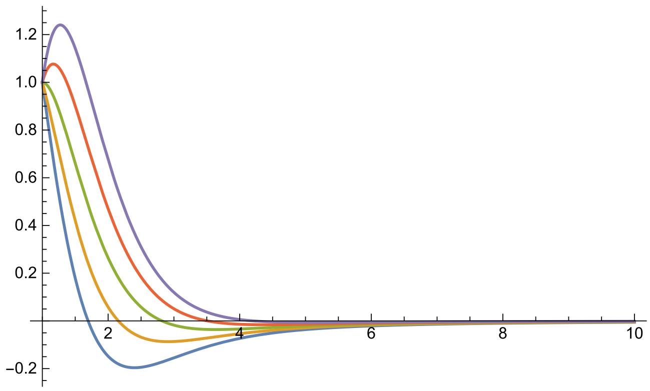

Fig. 1. Some solutions of the equation in (15) taking $\upsilon = 10$.

where $\upsilon > 0$ is a constant. Note that $\alpha = 1,\ m = 2,\ \nu_0 = 1,\ r(\nu) = \nu^2,\ p(\nu) = \nu/2,\ q(\nu) = \upsilon,\ \sigma(\nu) = \nu/2$. We now set $\delta(\nu) = k_f = k_g = 1$, then

$$\zeta(\nu_0, \nu) : = \exp\left(\int_{\nu_0}^{\nu} \frac{p(u)}{r(u)} du\right) = \nu^{1/2},\ \vartheta(\nu) := \int_{\nu}^{\infty} \frac{ds}{(r(s)\zeta(\nu_0, s))^{\frac{1}{\alpha}}} = \frac{2}{3\nu^{3/2}},$$

$$\phi(\nu) : = \frac{\delta'(\nu)}{\delta(\nu)} - \frac{k_f p(\nu)}{r(\nu)} = \frac{-1}{2\nu},\ \varphi(\nu) := \frac{1}{\zeta^{\frac{1}{\alpha}}(\nu_0, \nu)} - \frac{\vartheta(\nu) p(\nu) r^{(1-\alpha)/\alpha}(\nu)}{\alpha} = \frac{2}{3\nu^{1/2}}$$

$$\tilde{\varphi}(\nu) : = \frac{p(\nu)}{r(\nu)} + \frac{\alpha^{(\alpha+1)} \delta(\nu) \varphi^{\alpha+1}(\nu) \zeta(\nu_0, \nu)}{\vartheta(\nu) r^{\frac{1}{\alpha}}(\nu)} = \frac{7}{6\nu},$$

thus, we get

$$\lim_{\nu\to\infty} \sup \int_{\nu_0}^{\nu} \left(k_g \delta(s) q(s) - \left(\frac{2}{M s^{m-2}}\right)^{\alpha} \frac{r(s)\delta(s)(\phi_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right) ds = \infty$$

and, for some $\mu \in (0, 1)$,

$$\lim_{\nu\to\infty} \sup \int_{\nu_0}^{\nu} \left(k_g q(s) \left(\frac{\mu \sigma^{m-2}(s)}{(m-2)!} \vartheta(s)\right)^{\alpha} \zeta(\nu_0, s) - \tilde{\varphi}(s)\right) ds$$
$$= \lim_{\nu\to\infty} \sup \int_{\nu_0}^{\nu} \left(\frac{\upsilon\mu}{s} - \frac{7}{6s}\right) ds.$$

Therefore, by Theorem 2.1, all the solutions of (15) are oscillatory if $\upsilon > \frac{7}{6\mu}$.

Fig. 1 presents different solutions of the equation in (15) taking $\upsilon = 10$ for $w(1) = 1, w'(1) = -2, -1, \ldots, 2$, showing their oscillatory behavior. In Fig. 2 we show different solutions of the equation in (15) taking $\upsilon = 0.5$ for $w(1) = 1, w'(1) = -2, -1, \ldots, 2$, where a non-oscillatory behavior can be observed.

3. Conclusion

The aim of this article was to provide a study of the asymptotic nature for a class of higher-order delay differential equations with middle term. We use Riccati substitutions to prove that under certain assumptions, every solution of the studied equation is oscillatory. The results presented here complement some of the known results reported in the literature, as the ones in [7] and references contained therein.

A further extension of this article is to use our results to study a class of systems of higher order neutral differential equations as well as equations of fractional order. For all these classes of equations, there are already some researches in progress.

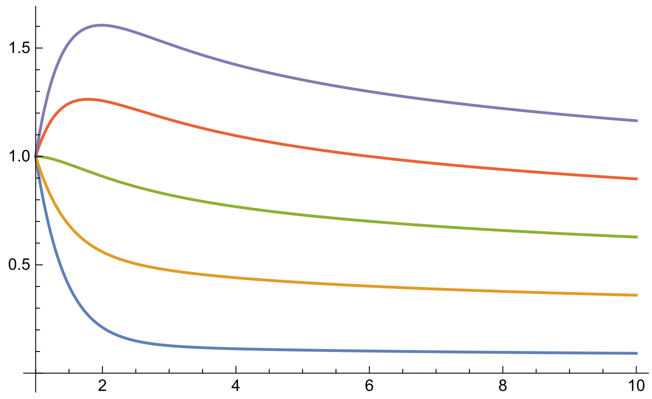

Fig. 2. Some solutions of the equation in (15) taking $\upsilon = 0.5$.

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CRediT authorship contribution statement

Omar Bazighifan: Conceptualization, Methodology, Investigation, Writing - original draft. **Higinio Ramos:** Formal analysis, Visualization, Investigation, Supervision, Writing - review & editing.

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