Journal of Computational and Applied Mathematics 404 (2022) 113116

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the approximate solutions of a class of fractional order nonlinear Volterra integro-differential initial value problems and boundary value problems of first kind and their convergence analysis

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A R T I C L E I N F O

Article history:
Received 6 January 2020
Received in revised form 1 July 2020

MSC:
26A33
34K37
34K06
34K09
34K28
34A08
34A10
34A12
41A58

Keywords:
Fractional integro differential equation
Volterra differential equation of first kind
Existence and uniqueness
Perturbation based approximation
Homotopy perturbation
Convergence analysis

A B S T R A C T

In this work we consider a class of fractional order Volterra integro-differential equations of first kind where the fractional derivative is considered in the Caputo sense. Here, we consider the initial value problem and the boundary value problem separately. For simplicity of the analysis, we reduce each of these problems to the fractional order Volterra integro-differential equation of second kind by using the Leibniz's rule. We have obtained sufficient conditions for the existence and uniqueness of the solutions of initial and the boundary value problems. An operator based method has been considered to approximate their solutions. In addition, we provide a convergence analysis of the adopted approach. Several numerical experiments are presented to support the theoretical results.

1. Introduction

The concept of the fractional calculus dates back to the times of Newton and Liouville. Since that time, the theoretical development of fractional calculus was in the mind of several mathematicians. In recent years, it has experienced a growing focus because of its application in real world problems. See for e.g., [1,2]. On the other hand, the problems related to the integro differential equations have also obtained interest by several mathematicians. The question of existence and uniqueness of solutions of fractional differential equations has been investigated in several papers [2–4]. But, most of the works deal with the numerical analysis of fractional integro differential equations without addressing the existence and uniqueness of the solution inside its domain of definition. In the present work, we consider a

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https://doi.org/10.1016/j.cam.2020.113116

class of fractional order Volterra integro-differential equations and present a result about existence and uniqueness of solutions inside their domain of definition. Different problems of physics, mechanics, and engineering can be modelled by fractional integral equations (see [5,6]). As like most of the differential equations, several fractional integral models do not have a direct way of finding analytical solutions. Thus, there has been an increasing interest in developing numerical approaches for the solution of fractional-order integro differential equations. Recently, several methods for obtaining such approximate solutions have appeared. Among these methods, we can mention, Adomian decomposition method [7], variational iteration method [8], reproducing kernel method [9], fractional differential transform method [10], collocation method [11], or wavelet method [12].

Volterra integral equations of first kind, appear in several real life situations, like Steady state heat distribution, Biological immunology model [13], Dirichlet problems in plane elasticity [14], etc. In general, two types (based on their kernels) of first kind integral equations, are popular in literature. The first type comprises the well-behaved kernels, i.e., for sufficiently smooth functions defined on their domain of definition. The second one is for unbounded kernels, i.e, singular kernels. Since small changes in the kernels or in the given functions, can make a huge effect on the solutions, the first kind integral equations lead to ill-posed problems, in general. More details on these type of problems can be seen in [15]. Therefore, the theories on first kind integral equations are not much rich as compared to integral equations of the second kind. Hence, we provide the existence and uniqueness properties of the present type of first kind integral equations by reformulating them as a second kind form. In the literature, the approximate solutions of Volterra integral equations of first kind, have achieved attentions by several researchers. Direct methods, including, quadrature method [16], operational matrix method with block-pulse functions [17], Modified homotopy perturbation method [18,19] and Adomian decomposition method [20] are some of the most popular methods which are used after reducing the first kind problem into the second kind form. One can also observe the Laplace transform method with special type of kernels, and variational iteration methods in [21] for approximating solutions of Volterra integro-differential equations of first kind. In the present research, we propose an approximation method based on Homotopy perturbation [18,19] for solving the fractional differential integral equations of first kind and provide the convergence analysis of this approach.

The paper is organized as follows. First, we define the fractional derivatives and integrals with some of their properties, to set our model problems at Section 2. In Section 3, we consider the nonlinear fractional Volterra integro-differential equation of first kind with initial value problem structure and boundary value problem structure. Here, we study the existence and uniqueness of the solution of first kind equation by reformulating it into second kind form. In Section 4, a homotopy perturbation based method is discussed for the solution approximation of the present model. In addition, we provide the convergence of the adopted strategy and its error analysis, here. A concrete computational algorithm is provided with numerical experiments in Section 5. For polynomial approximation, we use the Chebyshev polynomials, whose details are given in Appendix. In Section 6, conclusions of the present work, are summarized.

Notations: For a domain Ω, we define, $\bar{\Omega}$ as the closure of Ω. $C(\Omega)$ denotes the set of all continuous functions on Ω. In addition, the set of all continuous function from $[0,T]$ to $\mathbb{R}$ is noted as $C([0,T],\mathbb{R})$. Define $C^n(\Omega)=\{g(x)|g^n(x)$ exists and $g^n(x)\in C(\Omega)\}$. For a function $g(x)$, defined on Ω, we define $\|g(x)\|=\|g(x)\|_\infty=\max_{x\in\Omega}|g(x)|$. Let $u=\sum_{i=0}^{\infty}p^iu_i$ where $p\in[0,1]$, then the non-linear term $N(u)$ can be approximated by He's polynomial H_n, which is defined by

$$H_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^ku_k\right)_{p=0},\quad n=0,1,2,\dots. \tag{1.1}$$

2. Preliminaries

This section defines the Liouville–Caputo fractional derivative and the Riemann–Liouville fractional integral [1,4] which will be used to define our present problems and their approximate solutions. Additionally, some basic properties of these fractional operators are listed below.

2.1. Riemann–Liouville fractional integral

The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $f(x)$ is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}f(\tau)d\tau,\quad x>0. \tag{2.1}$$

The above integral exists almost everywhere for any absolutely integrable function $f(x)$ (for more details see [1]).

2.2. Liouville-Caputo fractional derivative

The Liouville–Caputo fractional derivative of a function f is defined as

$$D^\alpha f(x)=J^{m-\alpha}D^mf(x)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x\frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}\,d\tau, & \text{if } m-1<\alpha<m,\\ \dfrac{d^mf(x)}{dx^m}, & \alpha=m,\end{cases} \tag{2.2}$$

where $m \in \mathbb{N}$, and α is a positive real number called the order of the fractional derivative. Here, $\mathbb{N}$ defines the set of all natural numbers. The above integral exists almost everywhere if a function $f(x)$ and its derivatives up to order $m-1$ are absolutely continuous on $\bar{\Omega}$ (for more details see [1]).

Properties:

1. The Liouville–Caputo derivative of order α of a constant function $f(x) = c$ satisfies $D^\alpha c = 0$.
2. The Liouville–Caputo derivative and the Riemann–Liouville integral are linear operators. This means, if γ_1 and γ_2 are real constants, then

$$D^\alpha(\gamma_1 f(x) + \gamma_2 g(x)) = \gamma_1 D^\alpha f(x) + \gamma_2 D^\alpha g(x),$$

$$J^\alpha(\gamma_1 f(x) + \gamma_2 g(x)) = \gamma_1 J^\alpha f(x) + \gamma_2 J^\alpha g(x).$$

3. If $n-1 < \alpha < n,\ n \in \mathbb{N}$, then

$$D^\alpha J^\alpha g(x) = g(x), \tag{2.3}$$

and

$$J^\alpha D^\alpha g(x) = g(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} g^{(k)}(0), \quad x > 0. \tag{2.4}$$

In particular, for $0 < \alpha < 1$, we have

$$J^\alpha D^\alpha g(x) = g(x) - g(0)\,. \tag{2.5}$$

4. For $\beta > -1$ and $x > 0$, we have the following result

$$J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, x^{\beta+\alpha}.$$

3. Fractional order Volterra integro differential equation of first kind model

3.1. Initial value problem

We consider the following first kind of nonlinear fractional order Volterra integro-differential initial value problem, for $0 < \alpha \le 1$:

$$\begin{aligned} &\int_0^x k_1(x,t) F(u(t))\,dt + \int_0^x k_2(t) D^\alpha u(t)\,dt \ = f(x), \quad x \in \Omega = (0,1], \\ &u(0) = c_0. \end{aligned} \tag{3.1}$$

Here, $f(x)$ is a sufficiently smooth function on $\bar{\Omega}$ and $F(u(x))$ is a nonlinear function of $u(x)$. In addition, we assume that the kernels $k_1(x,t)$ and $k_2(t)$ are also sufficiently smooth on $\bar{\Omega} \times \bar{\Omega}$ with $k_2(x) \neq 0$ on $\bar{\Omega}$. We are interested to find a sufficiently smooth solution $u(x)$, defined on $\bar{\Omega}$.

By using Leibniz's rule, Eq. (3.1) can be written as

$$\begin{aligned} &D^\alpha u(x) = P(x) + Q(x) F(u(x)) + \int_0^x K(x,t) F(u(t)) dt, \\ &u(0) = c_0, \end{aligned} \tag{3.2}$$

where $P(x)$, $Q(x)$ and $K(x,t)$ are defined by

$$P(x) = \frac{f'(x)}{k_2(x)}, \quad Q(x) = -\frac{k_1(x,x)}{k_2(x)}, \quad K(x,t) = -\frac{1}{k_2(x)}\left[\frac{\partial}{\partial x} k_1(x,t)\right]. \tag{3.3}$$

By assuming $f(0) = 0$ and integrating (3.2), we can obtain (3.1). In addition, the following assumptions are required for the rest of the analysis.

(I) The nonlinear function $F(u(x))$ satisfies the Lipschitz condition with respect to $u(x)$, with Lipschitz constant $L(>0)$, and $F(0) = 0$ for all $x \in \bar{\Omega}$.
(II) The kernel $K(x,t)$ at (3.2) is continuous and bounded by a positive real number M_1 on $\bar{\Omega} \times \bar{\Omega}$. In addition, we also assume $Q(x)$ and $P(x)$ are continuous functions on $\bar{\Omega}$ and bounded by $M_2(>0)$ and $M_3(>0)$ respectively.

Now, we derive a sufficient condition for the existence and uniqueness of the solution.

3.1.1. Existence and uniqueness of solution

The following theorem uses the contraction mapping principle to obtain a sufficient condition on the existence and uniqueness of the solution.

Theorem 3.1. *Under the assumptions given in (I)–(II), the Volterra fractional integro differential initial value problem of first kind (3.1) (equivalently (3.2)) has a unique solution $u(x)$ for all $x \in \overline{\Omega}$, if the condition $(M_2(\alpha+1)+M_1)L < \Gamma(\alpha+2)$ is satisfied.*

Proof. We apply J^α on both sides of (3.2) and obtain

$$u(x) = c_0 + J^\alpha(P(x)) + J^\alpha(Q(x)F(u(x))) + J^\alpha\left[\int_0^x K(x,t)F(u(t))dt\right]. \tag{3.4}$$

Now, we write the above equation in the form $\Lambda u(x) = u(x)$, where the operator Λ is defined as

$$\Lambda u(x) = c_0 + J^\alpha(P(x)) + J^\alpha(Q(x)F(u(x))) + J^\alpha\left[\int_0^x K(x,t)F(u(t))dt\right]. \tag{3.5}$$

Let $u_1(x), u_2(x) \in C[0,1]$. Then, for every $x \in [0,1]$, we have

$$\begin{aligned}|\Lambda u_1(x) - \Lambda u_2(x)| &\le J^\alpha(|Q(x)||F(u_1(x)) - F(u_2(x))|) + J^\alpha\left[\int_0^x |K(x,t)||F(u_1(t)) - F(u_2(t))|dt\right]\\ &\le \frac{M_2 L}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}|u_1(t)-u_2(t)|dt + \frac{M_1 L}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_0^t |u_1(s)-u_2(s)|ds\right]dt\\ &\le \frac{(M_2(\alpha+1)+M_1)L}{\Gamma(\alpha+2)}\|u_1-u_2\|.\end{aligned}$$

This implies

$$\|\Lambda u_1(x) - \Lambda u_2(x)\| \le \frac{(M_2(\alpha+1)+M_1)L}{\Gamma(\alpha+2)}\|u_1-u_2\|.$$

Note that $(C[0,1], \|.\|)$ is a Banach space. Hence, by Banach's Fixed-Point Theorem, we can conclude that the initial value problem (3.2) has a unique solution in $C[0,1]$ when

$$\theta = \frac{(M_2(\alpha+1)+M_1)L}{\Gamma(\alpha+2)} < 1. \tag{3.6}$$

This completes the proof. ■

3.2. Boundary value problem

We consider the following fractional order nonlinear Volterra integro-differential boundary value problem (BVP) of first kind, for $0 < \alpha \le 1$:

$$\begin{aligned}&\int_0^x k_1(x,t)F(u(t))\,dt + \int_0^x k_2(t)D^\alpha u(t)\,dt = f(x), \quad x \in (0,T],\\ &a_0 u(0) + b_0 u(T) = c_0.\end{aligned} \tag{3.7}$$

Let us reformulate the nonlinear fractional order Volterra integro-differential boundary value problem (3.7) as

$$\begin{aligned}&D^\alpha u(x) = P(x) + Q(x)F(u(x)) + \int_0^x K(x,t)F(u(t))dt, \quad \text{on} \quad (0,T],\\ &a_0 u(0) + b_0 u(T) = c_0,\end{aligned} \tag{3.8}$$

where $a_0, b_0, c_0 \in \mathbb{R}$ with $a_0 + b_0 \neq 0$ and $P(x), Q(x)$ and $K(x,t)$ are as in (3.3).

Theorem 3.2. *Let $0 < \alpha < 1$ and $a_0 + b_0 \neq 0$. Assume that $P(x), Q(x)$ and $K(x,t)$ are sufficiently smooth on $[0,T]$. Then, the boundary value problem (3.8) is equivalent to the following integral equation of Volterra–Fredholm type*

$$\begin{aligned}u(x) = h(x) &- \frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt,\end{aligned} \tag{3.9}$$

where $h(x) = \frac{c_0}{a_0+b_0} - \frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}P(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}P(t)dt.$

Proof. We apply J^α on both sides of (3.8) to get

$$u(x)=u(0)+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[P(t)+Q(t)F(u(t))+\int_0^t K(t,s)F(u(s))ds\right]dt. \tag{3.10}$$

From the above equation, we readily get at $x=T$,

$$u(T)=u(0)+\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left[P(t)+Q(t)F(u(t))+\int_0^t K(t,s)F(u(s))ds\right]dt.$$

From the boundary condition and the above identity, we obtain

$$u(0)=\frac{c_0}{a_0+b_0}-\frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left[P(t)+Q(t)F(u(t))+\int_0^t K(t,s)F(u(s))ds\right]dt.$$

Hence, substituting the above value of $u(0)$ in (3.10), we obtain the desired equivalent form.

Now, we use the Schaefer's fixed point theorem to show the existence of the solution of the boundary value problem in (3.8). This is as follows.

Theorem 3.3 (*Schaefer's Fixed Point Theorem [22]*)**.** *Let $\mathcal{X}$ be a Banach space. Consider, a continuous mapping $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ which is compact on each bounded subset A of $\mathcal{X}$. Then, either*

(a) $\mathcal{F}$ has a fixed point, or
(b) the set $\{u\in\mathcal{X}: u=\tau\mathcal{F}(u) \text{ for } 0<\tau<1\}$ is unbounded.

Remark 3.4. Note that in the context of the BVP, when we make any reference to assumptions (I)–(II), the set $\bar{\Omega}=[0,1]$ must be replaced by $[0,T]$.

Theorem 3.5. *Let the assumptions given in (I)–(II) be satisfied. Additionally, we assume that $|F(u(x))|\le M^*$ for all $x\in[0,T]$ and for all $u(x)\in\mathbb{R}$. Under these assumptions, the Volterra integro differential equation of first kind (3.7) (equivalently (3.8)) has at least one solution in $[0,T]$.*

Proof. Let us consider the operator $\Upsilon: C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$, defined by

$$\begin{aligned}\Upsilon(u(x))&=h(x)-\frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left[Q(t)F(u(t))+\int_0^t K(t,s)F(u(s))ds\right]dt\\&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[Q(t)F(u(t))+\int_0^t K(t,s)F(u(s))ds\right]dt,\end{aligned} \tag{3.11}$$

where $h(x)$ is defined in Theorem 3.2. Now we show that the operator Υ has a fixed point, which follows by proving that Υ is continuous on $C([0,T],\mathbb{R})$ and compact on each bounded subset of $C([0,T],\mathbb{R})$, and the statement given in Theorem 3.3(b) is not true. This will imply that Theorem 3.3(a) must be true. This result will be shown through several steps.

First, we show that the operator Υ is continuous. To do this, consider a sequence of functions $\{u_n\}$ such that $u_n\to u$ in $C([0,T],\mathbb{R})$ as $n\to\infty$. Therefore, for every $x\in[0,T]$

$$\begin{aligned}&|\Upsilon(u_n(x))-\Upsilon(u(x))|\\&\le\left|\frac{b_0}{a_0+b_0}\right|\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left[|Q(t)||F(u_n(t))-F(u(t))|+\int_0^t |K(t,s)||F(u_n(s))-F(u(s))|ds\right]dt\\&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[|Q(t)||F(u_n(t))-F(u(t))|+\int_0^t |K(t,s)||F(u_n(s))-F(u(s))|ds\right]dt\\&\le\left(1+\frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2LT^\alpha}{\Gamma(\alpha+1)}+\frac{M_1LT^{\alpha+1}}{\Gamma(\alpha+2)}\right)\|u_n-u\|.\end{aligned}$$

For $n\to\infty$, we have

$$\|\Upsilon(u_n)-\Upsilon(u)\|\le\left(1+\frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2LT^\alpha}{\Gamma(\alpha+1)}+\frac{M_1LT^{\alpha+1}}{\Gamma(\alpha+2)}\right)\|u_n-u\|\to 0.$$

This implies that Υ is continuous.

In a second step, we will show that the operator Υ maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$, i.e., for any $\kappa>0$, there exists a $m>0$ such that for every $u\in B_\kappa$, we have $\|\Upsilon(u)\|\le m$, where B_κ is defined by $B_\kappa=\{u\in C([0,T],\mathbb{R}):\|u\|\le\kappa\}$.

For every $x \in [0, T]$,

$$\begin{aligned}
|\Upsilon(u(x))| &\leq \left|\frac{b_0}{a_0+b_0}\right| \frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1}\left[|Q(t)||F(u(t))| + \int_0^t |K(t,s)||F(u(s))|ds\right]dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\left[|Q(t)||F(u(t))| + \int_0^t |K(t,s)||F(u(s))|ds\right]dt \\
&\leq \left(1+\frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2(\alpha+1)+M_1T}{\Gamma(\alpha+2)}\right) LT^{\alpha}\|u\| \\
&\leq \left(1+\frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2(\alpha+1)+M_1T}{\Gamma(\alpha+2)}\right) \kappa LT^{\alpha}.
\end{aligned}$$

By choosing $m = \left(1+\frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2(\alpha+1)+M_1T}{\Gamma(\alpha+2)}\right)\kappa LT^{\alpha}$, we have $\|\Upsilon(u(x))\| \leq m$.

Finally, for the third step, let $x_1, x_2 \in (0, T]$, with $x_1 < x_2$. For $u \in B_\kappa$, (here B_κ is a bounded set and is defined in second step) we have

$$\begin{aligned}
|\Upsilon(u(x_2)) - \Upsilon(u(x_1))| &= \left|\frac{1}{\Gamma(\alpha)} \int_0^{x_2} (x_2-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right| \\
&= \left|\frac{1}{\Gamma(\alpha)} \int_0^{x_1} \left((x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1}\right)\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right| \\
&\leq \frac{M_2L\|u\|}{\Gamma(\alpha+1)}|2(x_2-x_1)^{\alpha} + (x_1^{\alpha} - x_2^{\alpha})| + \frac{M_1L\|u\|}{\Gamma(\alpha+2)}|2(x_2-x_1)^{\alpha+1} + (x_1^{\alpha+1} - x_2^{\alpha+1})|.
\end{aligned}$$

From the above inequality, it holds that $|\Upsilon(u(x_2)) - \Upsilon(u(x_1))| \to 0$ as $x_1 \to x_2$. This shows that the operator maps a bounded set into an equicontinuous set of $C([0, T], \mathbb{R})$. Therefore, by the Arzelà–Ascoli theorem, the operator Υ is compact.

For the last step, let us consider the set ω, which is defined by

$$\omega = \{u \in C([0, T], \mathbb{R}) : u = \tau\Upsilon(u) \text{ for } 0 < \tau < 1\}.$$

Now we show that the above set is bounded.

Let us consider $u \in \omega$. For every $x \in [0, T]$, we have from (3.11)

$$\begin{aligned}
u(x) &= \tau\left(h(x) - \frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt\right),
\end{aligned}$$

where $h(x)$ is defined in Theorem 3.2.

Note that, $|P(x)| \leq M_3$ from assumption (II). Hence, we have

$$|h(x)| \leq \frac{|c_0|}{|a_0+b_0|} + \left(1+\frac{|b_0|}{|a_0+b_0|}\right)\frac{M_3}{\Gamma(\alpha+1)} = M_4.$$

Therefore, for every $x \in [0, T]$, using $|F(u(x))| \leq M^*$, we obtain

$$\begin{aligned}
|u(x)| &= \Bigg| \tau \Bigg(h(x) - \frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\Bigg[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\Bigg]dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Bigg[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\Bigg]dt \Bigg) \Bigg| \\
&\leq |h(x)| + \frac{|b_0|}{|a_0+b_0|}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\Bigg[|Q(t)||F(u(t))| + \int_0^t |K(t,s)||F(u(s))|ds\Bigg]dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Bigg[|Q(t)||F(u(t))| + \int_0^t |K(t,s)||F(u(s))|ds\Bigg]dt \\
&\leq M_4 + \left(1 + \frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2(\alpha+1)+M_1T}{\Gamma(\alpha+2)}\right)M^*T^\alpha.
\end{aligned}$$

Define $m^* = M_4 + \left(1 + \frac{|b_0|}{|a_0+b_0|}\right)\left(\frac{M_2(\alpha+1)+M_1T}{\Gamma(\alpha+2)}\right)M^*T^\alpha$. Therefore $\|u\| \leq m^*$. This shows that any $u \in \omega$ is bounded. Therefore, the set ω is bounded.

Hence, the operator Υ has a fixed point which follows from Schaefer's fixed point theorem. This implies that Eq. (3.8) has at least one solution $u(x)$ for all $x \in [0, T]$. ∎

In addition, with the assumptions in (I) and (II), we can show that the boundary value problem (3.8) has a unique continuous solution on $[0, T]$, if the condition

$$\Theta = \frac{(M_2(\alpha+1)+M_1T)}{\Gamma(\alpha+2)}LT^\alpha\left(1 + \frac{|b_0|}{|a_0+b_0|}\right) < 1, \tag{3.12}$$

is satisfied. This can be derived by a similar fashion to that described in the proof of Theorem 3.1. Note that the above bound keeps a restriction on T depending on the given data.

4. Approximation of the solutions of IVP and BVP

In this section, we provide a method to approximate the solutions of (3.2) and (3.8) by homotopy based perturbation strategy (see [18,19,23]). First, we discuss the approximation of the solution for the initial value problem. For this, we construct the following homotopy equation corresponding to (3.2)

$$H(v,p) \equiv D^\alpha(v(x,p)) + p\Big(Q(x)F(v(x,p)) + \int_0^x K(x,t)F(v(t,p))dt\Big) - P(x) = 0. \tag{4.1}$$

Here, p is a small parameter such that $0 \leq p \leq 1$ and $v(x, p)$ is a generic function, defined on an appropriate domain according to the type of the problem considered. We consider that the solution of (4.1) can be written as a power series of p, of the form

$$v(x,p) = \sum_{i=0}^{\infty} v_i(x)p^i. \tag{4.2}$$

By substituting (4.2) in (4.1) and comparing the coefficients of the like powers of p, we obtain the following recurrence relations

$$\begin{cases} D^\alpha v_0(x) = P(x), \\ D^\alpha v_{i+1}(x) = Q(x)F(v_i(x)) + \displaystyle\int_0^x K(x,t)F(v_i(t))dt, \quad i \geq 0. \end{cases} \tag{4.3}$$

For solution approximation of IVP, we solve the above relation with initial condition. We choose the initial conditions for (4.3) in such a way that the initial conditions play an important role to construct the solution and the recurrence relations can be solved easily.

Now, for the boundary value problem, we construct a homotopy equation corresponding to (3.9) as follows

$$\begin{aligned}
H(v,p) \equiv v(x,p) + p\Bigg(&\frac{b_0}{a_0+b_0}\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\Bigg[Q(t)F(v(t,p)) + \int_0^t K(t,s)F(v(s,p))ds\Bigg]dt \\
-&\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Bigg[Q(t)F(v(t,p)) + \int_0^t K(t,s)F(v(s,p))ds\Bigg]dt\Bigg) - h(x) = 0.
\end{aligned} \tag{4.4}$$

We construct this homotopy equation in such a way that the obtaining solution satisfy the boundary condition automatically. Similarly as for IVPs, we consider that the solution of the BVP can be expressed as in (4.2). By substituting (4.2) into the above equation and comparing the coefficients of the like powers of p, we obtain the following recurrence relations

$$\begin{cases} v_0(x) = h(x), \\ v_{i+1}(x) = -\Big(\dfrac{b_0}{a_0+b_0}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^T (T-t)^{\alpha-1}\Big[Q(t)F(v_i(t)) + \int_0^t K(t,s)F(v_i(s))ds\Big]dt \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1}\Big[Q(t)F(v_i(t)) + \int_0^t K(t,s)F(v_i(s))ds\Big]dt\Big), \quad i \geq 0. \end{cases} \tag{4.5}$$

For BVP, we obtain the solution by solving the above relation only.

On each case, the solution $u(x)$, for the IVP or for the BVP, can be obtained as

$$u(x) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i v_i(x) = \sum_{i=0}^{\infty} v_i(x), \tag{4.6}$$

if the series (4.2) is uniformly convergent for all p. Now, by solving (4.3) and using the initial condition, or by solving (4.5), we obtain the approximate solution of (3.1) or of (3.8) respectively by taking the partial sum of the series with N terms, as

$$\Phi_N(x) \equiv \sum_{i=0}^{N-1} v_i(x). \tag{4.7}$$

4.1. Convergence analysis

Here, we discuss about the convergence of the above approximate solution for the IVP analogous to [18,19].

Theorem 4.1. *Assume that the conditions (I) and (II) hold. In addition, consider $0 < \theta < 1$ as described in (3.6). Then, the series (4.6) is uniformly convergent on $\bar{\Omega}$ to the solution $u(x)$ of the IVP in (3.1). Furthermore, an approximate solution of $u(x)$ is given by the partial sum (4.7).*

Proof. Observe that $v_0(x) \in C(\bar{\Omega})$, since $P(x) \in C(\bar{\Omega})$. Hence, there exist $M \in \mathbb{R}$ and $M > 0$ such that $|v_0(x)| \leq M$ for all $x \in \bar{\Omega}$. Now, we show that the ith term of the series (4.6) satisfies the following bound

$$|v_i(x)| \leq M\theta^i \quad \text{on} \quad \bar{\Omega}, \tag{4.8}$$

where θ was defined in (3.6). We use induction on $i \in \mathbb{N}$. For $i = 1$, we have

$$\begin{aligned} |v_1(x)| &= \left|J^\alpha\left[Q(x)F(v_0(x)) + \int_0^x K(x,t)F(v_0(t))\,dt\right]\right| \\ &\leq M_2L|v_0(x)|J^\alpha(1) + M_1L|v_0(x)|J^\alpha(x) \\ &= \frac{M_2L}{\Gamma(\alpha+1)}|v_0(x)|x^\alpha + \frac{M_1L}{\Gamma(\alpha+2)}|v_0(x)|x^{\alpha+1} \\ &\leq \theta|v_0(x)| \leq M\theta. \end{aligned}$$

Now we assume that (4.8) is true for $i = k-1$, i.e., $|v_{k-1}(x)| \leq M\theta^{k-1}$. Proceeding as before, for $i = k$, we have

$$\begin{aligned} |v_k(x)| = &\left|J^\alpha\left[Q(x)F(v_{k-1}(x)) + \int_0^x K(x,t)F(v_{k-1}(t))\,dt\right]\right|. \\ &\leq \theta|v_{k-1}(x)| \leq M\theta^k. \end{aligned}$$

Hence, we get the desired result at (4.8).

Therefore, for all $x \in \bar{\Omega}$,

$$\sum_{i=0}^{\infty}|v_i(x)| \leq \sum_{i=0}^{\infty} M\theta^i. \tag{4.9}$$

For $0 < \theta < 1$, $\sum_{i=0}^{\infty} M\theta^i$ is a convergent geometric series. Therefore, by the Weierstrass M-test, we conclude that $\sum_{i=0}^{\infty} v_i(x)$ converges uniformly on $\bar{\Omega}$.

Note that for all $p \in [0, 1]$ and for all $x \in \bar{\Omega}$, we have

$$\sum_{i=0}^{\infty} p^i v_i(x) \leq \sum_{i=0}^{\infty}|v_i(x)| \leq \sum_{i=0}^{\infty} M\theta^i.$$

Therefore, again by the Weierstrass M-test, the series (4.2) is uniformly convergent on $\bar{\Omega}$. Hence, the partial sum in (4.7) is an approximate solution of (3.2). ∎

The convergence of the approximate solution for the boundary value problem can be proved similarly, as was discussed in Theorem 4.1. This is also given below.

Theorem 4.2. *Assume that the conditions (I) and (II) hold on* $[0, T]$. *Let* $0 < \Theta < 1$ *be as defined in* (3.12)*. Then, the series* (4.6) *is uniformly convergent on* $[0, T]$ *to* $u(x)$*, the true solution of the boundary value problem in* (3.7)*. Furthermore, an approximate solution of* (3.8) *can be obtained from the partial sum* (4.7)*.*

Here, we discuss error analysis and required minimum number of terms for tolerance for initial value problem.

4.2. Error analysis

Let $u(x) = \sum_{i=0}^{\infty} v_i(x)$ be the solution of (3.2). Now consider the approximate solution $\Phi_N(x)$ in (4.7), where N is the number of terms in the partial sum. In this case, for the initial value problem, an upper bound of the approximate solution will be given by $\dfrac{M\theta^N}{1-\theta}$ (see (4.9) of Theorem 4.1), where θ was defined in (3.6) and M was the bound described in Theorem 4.1.

Thus, a lower bound of N for a given error tolerance ϵ can be estimated as follows.

Remark 4.3. Given an error tolerance ϵ of the absolute error of the approximate solution $\Phi_N(x)$ of (3.2), to reach this error the number of terms N in (4.7) must verify

$$N \geq \left\lfloor \frac{\ln(\epsilon(1-\theta)/M)}{\ln(\theta)} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ defines the floor function, which gives the greatest integer less than or equal to x. Here θ is taken from (3.6) and M is defined in the proof of Theorem 4.1.

In a similar way, we can obtain an upper bound of the absolute error of the approximate solution $\Phi_N(x)$ of the boundary value problem, which will be given by $\dfrac{M_4\Theta^N}{1-\Theta}$. Now, to reach a given error tolerance ϵ of the approximate solution, the number of terms N must verify

$$N \geq \left\lfloor \frac{\ln(\epsilon(1-\Theta)/M_4)}{\ln(\theta)} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is again the floor function. Here Θ is given in (3.12) and M_4 is defined in Theorem 3.5.

5. Numerical experiments

This section presents some examples to show that the present approach is very effective for approximating the solution of fractional order nonlinear Volterra integro differential equations of first kind. To solve each problem, we convert it into an integral equation of second kind, as explained in (3.2) and (3.8). The examples considered are required to satisfy the existence and uniqueness condition in (3.6) for IVPs, or the condition in (3.12) for BVPs. The following algorithm will be used to approximate the exact solution.

5.1. Computational algorithm

Step 1. Fix ϵ as an user chosen desired tolerance, and find N from Remark 4.3.
Step 2. Obtain v_i from (4.3) for the initial value problem (3.1) or from (4.5) for the boundary value problem (3.7), $i = 0, 1, \ldots, N-1$.
Step 3. Consider $\Phi_0(x) = 0$. Now, compute $\Phi_i(x) = \Phi_{i-1}(x) + v_{i-1}(x)$ for $i = 1, \ldots, N$. Define $\Phi_N(x)$ as an approximate solution.

Example 5.1. Let us consider the following non-linear fractional order Volterra integro-differential equation of first kind:

$$\begin{cases} \dfrac{1}{10}\displaystyle\int_0^x (1+2x-t)u^2(t)\,dt + \int_0^x (8/(2+3t))D^{1/2}u(t)\,dt = f(x), \quad x \in (0, 1], \\ u(0) = 1, \end{cases} \tag{5.1}$$

where $f(x) = \dfrac{x}{5}\left(\dfrac{1}{2} + \dfrac{3x}{4} + \dfrac{x^2}{3} + \dfrac{5x^3}{12} + \dfrac{x^4}{10} + \dfrac{7x^5}{60}\right) - \dfrac{256x^{1/2}}{27\sqrt{\pi}} + \dfrac{128x^{3/2}}{27\sqrt{\pi}} + \dfrac{256}{27}\sqrt{\dfrac{2}{3\pi}}\tan^{-1}(\sqrt{3x/2})$.

For this choice of $f(x)$, the exact solution of Example 5.1 is $u(x) = x^2 + 1$.

Table 1
Absolute point-wise errors of Example 5.1.

x	$E_6^\infty(x)$	$E_7^\infty(x)$	x	$E_6^\infty(x)$	$E_7^\infty(x)$
0.1	3.38896E−11	7.18757E−13	0.6	2.88031E−06	3.78431E−07
0.2	9.24018E−10	3.34467E−11	0.7	1.49243E−05	2.54891E−06
0.3	1.03652E−08	5.53495E−10	0.8	7.30229E−05	1.60628E−05
0.4	8.05188E−08	5.97289E−09	0.9	3.42636E−04	9.64305E−05
0.5	5.12190E−07	5.10794E−08	1	1.55743E−03	5.58302E−04

Table 2
L^2 norm errors of Example 5.1.

x	$n=3$	$n=4$	$n=5$	$n=6$
1	9.83216E−02	2.93216E−02	9.26138E−03	3.03772E−03

Applying Leibniz rule to the above equation, we obtain

$$\begin{cases} D^{\frac{1}{2}}u(x) = P(x) + Q(x)u^2(x) + \int_0^x K(x,t)\, u^2(t)dt, \\ u(0) = 1, \end{cases} \tag{5.2}$$

where $P(x) = \frac{1}{40}\left(1 + \frac{9x}{2} + \frac{13x^2}{2} + \frac{19x^3}{3} + 6x^4 + \frac{29x^5}{10} + \frac{21x^6}{10}\right) + \frac{8x^{3/2}}{3\sqrt{\pi}}$, $Q(x) = -\frac{2+5x+3x^2}{80}$ and $K(x,t) = -\frac{2+3x}{40}$.

Now, we consider the following homotopy

$$H(v,p) \equiv D^{\frac{1}{2}}v(x) - P(x) + p\left(-Q(x)v^2(x) - \int_0^x K(x,t)\, v^2(t)dt\right) = 0\,. \tag{5.3}$$

Substituting (4.2) into the above equation and equating the identical powers of p, we obtain

$$D^{\frac{1}{2}}v_0 = P(x).$$

Now applying (2.1) on both sides of the above equation and using the initial condition, we obtain $v_0(x) = 1 + \frac{\sqrt{x}}{20\sqrt{\pi}} + \frac{3}{20\sqrt{\pi}}x^{3/2} + x^2 + \frac{13}{75\sqrt{\pi}}x^{5/2} + \frac{4x^{7/2}}{25\sqrt{\pi}}\left(\frac{19}{21} + \frac{16}{21}x + \frac{132}{693}x^2 + \frac{32}{143}x^3\right)$. Similarly, by equating the identical powers of p we obtain

$$D^{\frac{1}{2}}v_1(x) = Q(x)v_0^2(x) + \int_0^x K(x,t)\, v_0^2(t)dt.$$

Applying (2.1) on both sides of the above equation and using initial condition, we obtain $v_1(x)$. Now we use the following relation to obtain the next terms of the series in (4.7). For $i \geq 1$,

$$D^{\frac{1}{2}}v_{i+1}(x) = Q(x)H_i(v_0, v_1, \ldots, v_i) + \int_0^x K(x,t)\, H_i(v_0, v_1, \ldots, v_i)dt, \tag{5.4}$$

where $H_i(v_0, v_1, \ldots, v_i)$ is the He's polynomial and can be obtained by (1.1), i.e., $H_1(v_0, v_1) = 2v_0v_1$.

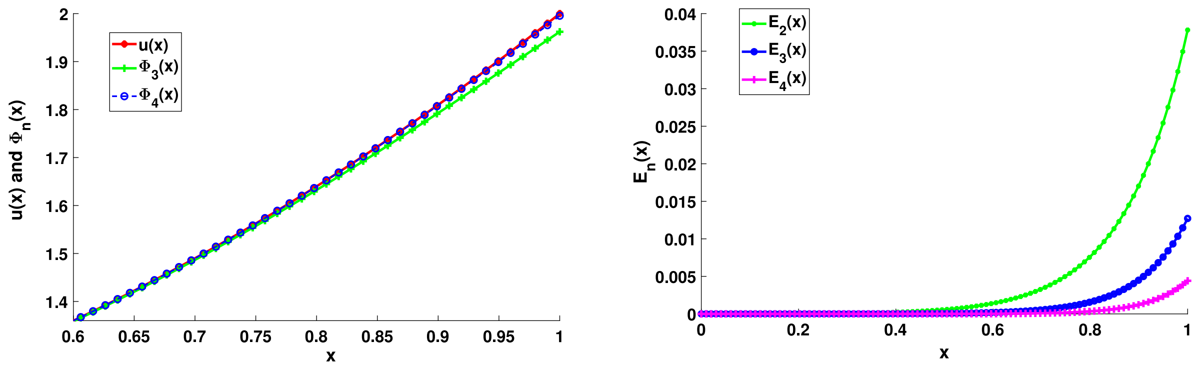
Let us define the approximate solution $\Phi_n(x)$ by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$ for all the numerical examples. Using this approximation, we produce the absolute error $E_n^\infty(x)$ to show the effectiveness of our present method. These point-wise errors are obtained as follows [24–27]:

$$E_n^\infty(x) = |u(x) - \Phi_n(x)| = \left| u(x) - \sum_{m=0}^{n-1} v_m(x) \right|. \tag{5.5}$$

For Example 5.1, we took $n = 6$ and $n = 7$ respectively, at (4.2) to produce the errors which are given in Table 1. In addition, we produce the error with respect to L^2 norm over $\overline{\Omega}$ as

$$E_n^2 = \left(\int_0^1 (u(x) - \Phi_n(x))^2\, dx \right)^{1/2}, \tag{5.6}$$

at Table 2 to show the effectiveness of our present approach. These results show that the present perturbation approach can be considered as an alternative approach compared to adaptive discretization methods [28,29] (see Fig. 1). Here Fig. 1 (a) shows the convergence of the computed solutions to exact solution as the number of terms in the series approximation increases. This convergence can also be pointed out from the absolute errors graph in Fig. 1 (b). It shows that the errors are gradually decreasing as the number of term in the series increases.



(a) Comparison of exact and approximate solutions (b) Absolute point-wise errors

Fig. 1. Solution approximations and error plots of Example 5.1 for different values of n.

Example 5.2. Now consider the following fractional order Volterra integro-differential equation:

$$\begin{cases} \dfrac{2}{15}\displaystyle\int_0^x (1+xt)u^2(t)\,dt + \int_0^x (4/(1+2t))D^{3/4}u(t)\,dt = 3x^2 - 2x^3, \quad x \in (0,1], \\ u(0) = 0. \end{cases} \tag{5.7}$$

The exact solution of Example 5.2 is unknown.

Using Leibniz rule, we obtain

$$\begin{cases} D^{\frac{3}{4}}u(x) = P(x) + Q(x)u^2(x) + \displaystyle\int_0^x K(x,t)\,u^2(t)dt, \\ u(0) = 0, \end{cases} \tag{5.8}$$

where $P(x) = \dfrac{3x}{2}\left(1 + x - 2x^2\right)$, $Q(x) = -\dfrac{1+2x+x^2+2x^3}{30}$ and $K(x,t) = -\dfrac{t+2xt}{30}$.

Now, let us construct the homotopy as follows

$$H(v,p) \equiv D^{\frac{3}{4}}v(x) - P(x) + p\left(-Q(x)v^2(x) - \int_0^x K(x,t)\,v^2(t)dt\right) = 0\,. \tag{5.9}$$

Substituting (4.2) into the above equation and equating the identical powers of p, we obtain

$$D^{\frac{3}{4}}v_0(x) = P(x).$$

Now applying J^{α} from (2.1) on both sides of the above equation and using the initial condition, we obtain

$$v_0(x) = \frac{8x^{7/4}}{7\Gamma(3/4)} + \frac{64x^{11/4}}{77\Gamma(3/4)} - \frac{512x^{15/4}}{385\Gamma(3/4)}.$$

Now, we use the following relation to obtain the next terms of the series for $i \geq 0$:

$$D^{\frac{3}{4}}v_{i+1}(x) = Q(x)H_i(v_0, v_1, \ldots, v_i)(x) + \int_0^x K(x,t)\,H_i(v_0, v_1, \ldots, v_i)(t)dt, \tag{5.10}$$

where $H_i(v_0, v_1, \ldots, v_i)$ is the He's polynomials, which can be obtained from (1.1), i.e., $H_0(v_0) = v_0^2$, $H_1(v_0, v_1) = 2v_0v_1$.

The approximate solution $\Phi_n(x)$ is defined by $\Phi_n(x) = \sum_{m=0}^{n-1} v_m(x)$. Now, we produce the absolute residual error $E_n^{\infty}(x)$ to show the effectiveness of our present method. These point-wise errors are defined as follows [24,26,27,30,31]:

$$E_n^{\infty}(x) = |\mathbf{A}(\Phi_n(x)) - P(x)| = \left|\mathbf{A}\left(\sum_{m=0}^{n-1} v_m(x)\right) - P(x)\right|, \tag{5.11}$$

where $\mathbf{A}$ is defined as

$$\mathbf{A}(\Phi_n)(x) \equiv D^{\frac{3}{4}}(\Phi_n(x)) - Q(x)(\Phi_n(x))^2 - \int_0^x K(x,t)(\Phi_n(t))^2 dt.$$

For Example 5.2, we took $n = 3$ and $n = 4$ respectively, to obtain the errors. They are given in Table 3.

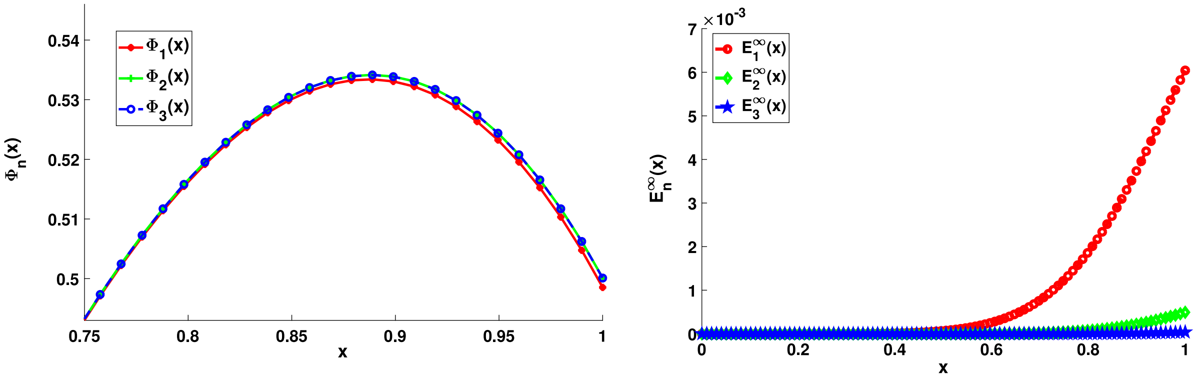
Now we present the errors with respect to L^2 norm from (5.6) at Table 4 to show the effectiveness of our present approach. This clearly shows that the errors are decreasing as n increases. Hence the semianalytical methods can be considered as an alternative approach compared to numerical discretizations, available in [26,27,30,31] (see Fig. 2).

Table 3
Absolute point-wise errors of Example 5.2.

x	$E_3^\infty(x)$	$E_4^\infty(x)$	x	$E_3^\infty(x)$	$E_4^\infty(x)$
0.1	7.74521E−14	2.53200E−17	0.6	4.00389E−06	5.90583E−08
0.2	5.32632E−11	2.71633E−14	0.7	1.98466E−05	4.86019E−07
0.3	2.95633E−09	4.93499E−12	0.8	7.49114E−05	2.84490E−06
0.4	5.65572E−08	2.27652E−10	0.9	2.19169E−04	1.21747E−05
0.5	5.86245E−07	4.78438E−09	1	4.92942E−04	3.80106E−05

Table 4
L^2 norm errors of Example 5.2.

x	$n=2$	$n=3$	$n=4$	$n=5$
1	1.89450E−02	1.26255E−03	8.48495E−05	5.72371E−06



(a) Comparison of approximate solutions (b) Absolute point-wise errors

Fig. 2. Computed solutions and error plots of Example 5.2 for different values of n.

Example 5.3. Now consider the following fractional order Volterra integro-differential equation:

$$\begin{cases} \dfrac{1}{20}\displaystyle\int_0^x t^2 e^{2x} u(t)\,dt + \int_0^x e^t D^{3/4} u(t)\,dt = xe^x, & x \in (0,1], \\ u(0) = 0. \end{cases} \tag{5.12}$$

The exact solution of Example 5.3 is unknown. Using Leibniz rule, we obtain

$$\begin{cases} D^{\frac{3}{4}} u(x) = P(x) + Q(x)u(x) + \displaystyle\int_0^x K(x,t)\, u(t)dt, \\ u(0) = 0, \end{cases} \tag{5.13}$$

where $P(x) = x+1$, $Q(x) = -\dfrac{x^2 e^x}{20}$ and $K(x,t) = -\dfrac{t^2 e^x}{10}$. For simplification, we approximate e^x by Chebyshev polynomial of order 3 which is given by $e^x \approx 0.999509 + 1.01563x + 0.424301x^2 + 0.27824x^3$. Based on this Chebyshev approximation, the errors accumulated due to this third order polynomial can be at most 0.0006. Details of the Chebyshev based approximations are given in Appendix.

Now, let us construct the homotopy as follows

$$H(v,p) \equiv D^{\frac{3}{4}} v(x) - P(x) + p\left(-Q(x)v(x) - \int_0^x K(x,t)\, v(t)dt\right) = 0\,. \tag{5.14}$$

Substituting (4.2) into the above equation and equating the identical powers of p, we obtain

$$D^{\frac{3}{4}} v_0 = P(x).$$

Now applying $J^{3/4}$ from (2.1) on both sides of the above equation and using the initial condition, we obtain

$$v_0(x) = \frac{x^{3/4}}{\Gamma(7/4)} + \frac{x^{7/4}}{\Gamma(11/4)}.$$

Table 5
Absolute point-wise errors of Example 5.3.

x	$E_2^\infty(x)$	$E_3^\infty(x)$	x	$E_2^\infty(x)$	$E_3^\infty(x)$
0.1	4.98859E−08	4.54542E−08	0.6	3.46644E−04	1.07404E−05
0.2	6.14506E−07	3.44489E−07	0.7	1.09558E−03	8.07762E−06
0.3	3.46540E−06	1.02970E−07	0.8	3.02440E−03	1.35582E−05
0.4	1.94060E−05	2.40845E−06	0.9	7.54262E−03	1.16908E−04
0.5	9.16005E−05	7.31310E−06	1	1.74502E−02	6.17948E−04

Now, we use the following relation to obtain the next terms of the series, for $i \geq 0$:

$$D^{\frac{3}{4}} v_{i+1}(x) = Q(x)v_i(x) + \int_0^x K(x,t)v_i(t)dt. \tag{5.15}$$

For Example 5.3, we took $n = 2$ and $n = 3$ respectively, to obtain the maximum errors from the formula (5.11), where **A** is defined as

$$\mathbf{A}(\Phi_n)(x) \equiv D^{\frac{3}{4}}(\Phi_n(x)) - Q(x)\Phi_n(x) - \int_0^x K(x,t)\Phi_n(t)dt.$$

These errors are given at Table 5 and clearly show that the residual errors are decreasing as n increases.

Example 5.4. Now consider the following fractional order Volterra integro-differential equation with a boundary condition:

$$\begin{cases} \dfrac{1}{12}\displaystyle\int_0^x (2t-1)e^{3x}u(t)\,dt + \int_0^x e^{3t}D^{1/2}u(t)\,dt = (x^2+3x-2x^3)e^{3x}, \quad x \in (0,1], \\ u(0) + 2u(1) = 3. \end{cases} \tag{5.16}$$

We formulate the above boundary value problem as

$$\begin{cases} D^{\frac{1}{2}}u(x) = P(x) + Q(x)u(x) + \displaystyle\int_0^x K(x,t)\, u(t)dt, \\ u(0) + 2u(1) = 3, \end{cases} \tag{5.17}$$

where $P(x) = 3 + 11x - 3x^2 - 6x^3$, $Q(x) = \dfrac{2x-1}{12}$ and $K(x,t) = \dfrac{2t-1}{4}$.

Finding the solution of Eq. (5.16) is equivalent to find the solution of the integral equation, (based on (3.9) of Theorem 3.2) which is given by

$$\begin{aligned} u(x) = h(x) &- \frac{2}{3\Gamma(1/2)}\int_0^1 (1-t)^{-1/2}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt \\ &+ \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2}\left[Q(t)F(u(t)) + \int_0^t K(t,s)F(u(s))ds\right]dt, \end{aligned} \tag{5.18}$$

where $h(x) = 1 - \dfrac{2}{3\Gamma(1/2)}\int_0^1 (1-t)^{-1/2}P(t)dt + \dfrac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2}P(t)dt$. Now, we solve the above equation by using homotopy perturbation method. By relation (4.3), we obtain

$$v_0(x) = h(x) = 1 - \frac{2516}{315\sqrt{\pi}} + \frac{6x^{1/2}}{\sqrt{\pi}} + \frac{44x^{3/2}}{3\sqrt{\pi}} - \frac{16x^{5/2}}{5\sqrt{\pi}} - \frac{192x^{7/2}}{35\sqrt{\pi}}.$$

Similarly, we obtain

$$v_1(x) = Q(x)v_0(x) + \int_0^x K(x,t)\, v_0(t)dt.$$

This implies

$$v_1(x) = \frac{x^{1/2}}{3\sqrt{\pi}}\left(\frac{1}{2} + \frac{x}{3} - \frac{4x^2}{5}\right) - \frac{1258x^{1/2}}{945\pi}\left(1 + \frac{2x}{3} - \frac{8x^2}{5}\right) + \frac{x}{4} + \frac{11x^2}{24} - \frac{55x^3}{72} - \frac{59x^4}{96} + \frac{19x^5}{80} + \frac{9x^6}{80}.$$

Now, we use the following relation to obtain the next terms of the series, for $i \geq 1$:

$$v_{i+1}(x) = Q(x)v_i(x) + \int_0^x K(x,t)\, v_i(t)dt. \tag{5.19}$$

Table 6
Absolute point-wise errors of Example 5.4.

x	$E_4^\infty(x)$	$E_5^\infty(x)$	x	$E_4^\infty(x)$	$E_5^\infty(x)$
0.1	7.10869E−06	1.83915E−07	0.6	3.99378E−06	9.91608E−08
0.2	6.39764E−06	1.44964E−07	0.7	3.50314E−06	9.64669E−08
0.3	6.13705E−06	1.17742E−07	0.8	3.32471E−06	9.36716E−08
0.4	5.58433E−06	1.04115E−07	0.9	4.65206E−06	1.12852E−07
0.5	4.75636E−06	1.00451E−07	1	1.37257E−05	3.24812E−07

For Example 5.4, we took $n = 4$ and $n = 5$ respectively, to obtain the errors from (5.11) where $\mathbf{A}$ is defined as

$$\mathbf{A}(\Phi_n)(x) \equiv D^{\frac{1}{2}}(\Phi_n(x)) - Q(x)\Phi_n(x) - \int_0^x K(x,t)\Phi_n(t)dt.$$

They are given at Table 6 which clearly show that the errors are decreasing as n increases.

6. Conclusions

In this work, the existence uniqueness and the approximation of the solution of a fractional order nonlinear Volterra integro-differential equations of first kind with an initial condition or a boundary condition, respectively, are considered. It is observed that we require a sufficient condition for the existence and uniqueness of the solution. We also provide a perturbation based computational algorithm for the approximate solutions of the IVP and BVP. Numerical experiments with error analysis show that the present approach is effective to obtain an efficient approximation of the solution.

Acknowledgements

The author, Pratibhamoy Das, wants to thank the Science and Engineering Research Board (SERB), Govt. of India, for providing the research support for the present work under the grant ECR/2017/00197 as Early Career Research Award Program. The author Subrata Rana acknowledges University Grant Commission (UGC), Govt. of India, for providing his financial assistance for present research.

Appendix

Let $f(x)$ be a real valued function on $[a, b]$ where $a, b \in \mathbb{R}$ such that $f \in C^{n+1}[a, b]$. On $[a, b]$, we can obtain a best approximation of f by using Chebyshev polynomials. Considering interpolating points as zeros of nth order Chebyshev polynomial (after transforming the domain $[a, b]$ to $[-1, 1]$)

$$x_k = \left(\frac{b-a}{2}\right)\cos\frac{(2k+1)\pi}{2(n+1)} + \left(\frac{b+a}{2}\right), \; k = 0, 1, 2, \dots, n,$$

we can obtain a nth order polynomial $P_n(x)$ by Lagrange interpolation. The interpolating error will be

$$\|f - P_n\| \le \frac{(b-a)^{(n+1)}}{2^{2n+1}(n+1)!} \max_{c\in[a,b]} |f^{n+1}(c)|.$$

For more details of the analysis, one can see [32].

For Example 5.3, we approximate e^x by a third order polynomial $P_3(x)$ on $[0, 1]$. The interpolating points (based on Chebyshev zeros) are $x_0 = \dfrac{1}{2}(\cos(\frac{\pi}{8}) + 1)$, $x_1 = \dfrac{1}{2}(\cos(\frac{3\pi}{8}) + 1)$, $x_2 = \dfrac{1}{2}(\cos(\frac{5\pi}{8}) + 1)$ and $x_3 = \dfrac{1}{2}(\cos(\frac{7\pi}{8}) + 1)$. Therefore, by Lagrange interpolation, we obtain $P_3(x) = 0.999509 + 1.01563x + 0.424301x^2 + 0.27824x^3$. The error is given by

$$\|e - P_3\| \le \frac{1}{2^7(4)!} \max_{c\in[0,1]} |e^c| \le 0.0006.$$

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B.V. Amsterdam, 2006.
[2] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[3] L.M. Delves, J.L. Mohamed, Computational Methods for Integral Equations, Cambridge University Press, 1985.
[4] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley-Interscience, New York, 1993.
[5] A.M. Yang, Y. Han, Y.Z. Mang, et al., On local fractional Volterra integro-differential equations in fractal steady heat transfer, Therm. Sci. 20 (2016) 789–793.
[6] S. Yuzbasi, A numerical approximation for Volterra's population growth model with fractional order, Appl. Math. Model. 37 (2013) 3216–3227.
[7] S. Momani, M.A. Noor, Numerical methods for fourth-order fractional integro-differential equations, Appl. Math. Comput. 182 (2006) 754–760.
[8] K. Sayevand, Analytical treatment of Volterra integro-differential equations of fractional order, Appl. Math. Model. 39 (2015) 4330–4336.

[9] J. Wei, T. Tian, Numerical solution of nonlinear Volterra integro-differential equations of fractional order by the reproducing kernel method, Appl. Math. Model. 39 (2015) 4871–4876.
[10] A. Arikoglu, I. Ozkol, Solution of fractional integro-differential equations by using fractional differential transform method, Chaos Solitons Fractals 34 (2007) 1473–1481.
[11] X.H. Ma, C.M. Huang, Numerical solution of fractional integro-differential equations by a hybrid collocation method, Appl. Math. Comput. 219 (2013) 6750–6760.
[12] L. Zhu, Q.B. Fan, Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 1203–1213.
[13] C.W. Groetsch, Integral equations of the first kind, inverse problems and regularization: a crash course, J. Phys.: Conf. Ser. 73 (2007) 012001.
[14] C. Constanda, Integral equation of the first kind in plane elasticity, J. Quart. Appl. Math. LIII (4) (1995) 783–793.
[15] P. Linz, Analytical and Numerical Methods for Volterra Equations, SIAM, Philadelphia, 1985.
[16] P. Linz, A simple approximation method for solving Volterra integro-differential equations of first kind, J. Inst. Math. Appl. 14 (1974) 211–215.
[17] E. Babolian, Z. Masouri, Direct method to solve Volterra integral equation of first kind using operational matrix with block-pulse functions, J. Comput. Appl. Math. 220 (2008) 51–57.
[18] P. Das, S. Rana, H. Ramos, A perturbation based approach for solving fractional order Volterra-Fredholm integro differential equations and its convergence analysis, Int. J. Comput. Math. (2019) http://dx.doi.org/10.1080/00207160.2019.1673892.
[19] P. Das, S. Rana, H. Ramos, Homotopy perturbation method for solving caputo type fractional order Volterra-Fredholm integro-differential equations, Comput. Math. Methods (2019) http://dx.doi.org/10.1002/cmm4.1047.
[20] A.M. Wazwaz, The combined Laplace transform-adomian decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Math. Comput. 216 (2010) 1304–1309.
[21] A.M. Wazwaz, Nonlinear Volterra Integro-Differential Equations: Linear and Nonlinear Integral Equation, Springer, Berlin, 2011.
[22] D.R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.
[23] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257–262.
[24] P. Das, An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh, Numer. Algorithms 81 (2019) 465–487.
[25] P. Das, Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems, J. Comput. Appl. Math. 290 (2015) 16–25.
[26] P. Das, J. Vigo-Aguiar, Parameter uniform numerical approximation of the solution of a system of reaction diffusion problems involving a small perturbation parameter, J. Comput. Appl. Math. 354 (2019) 533–544.
[27] P. Das, S. Rana, J. Vigo-Aguiar, Higher order accurate approximations on equidistributed meshes for boundary layer originated mixed type reaction diffusion systems with multiple scale nature, Appl. Numer. Math. 148 (2020) 79–97.
[28] P. Das, A higher order difference method for singularly perturbed parabolic partial differential equations, J. Difference Equ. Appl. 24 (3) (2018) 452–477.
[29] M. Chandru, P. Das, H. Ramos, Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non smooth data, Math. Methods Appl. Sci. 41 (14) (2018) 5359–5387.
[30] P. Das, V. Mehrmann, Numerical solution of singularly perturbed convection diffusion reaction problems with two small parameters, BIT 56 (2016) 51–76.
[31] M. Chandru, T. Prabha, P. Das, V. Shanthi, A numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms, Diff. Equ. Dyn. Syst. 27 (1–3) (2019) 91–112.
[32] E. Suli, D. Mayers, A Introduction to Numerical Analysis, Cambridge University Press, Cambridge, 2003.