On How the 2 Set-Up Routley-Meyer Semantics Are a Specific Case of the Reduced General Routley-Meyer Semantics in the Context of Some 4-Valued Logics

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Abstract

Routley-Meyer ternary relational semantics can be introduced for models in different ways depending on how the set of regular elements of the model is defined. Two of the most prominent ones are the Reduced General semantics and the 2 Set-up semantics. On the other hand, Lti-logics are 4-valued logics characterized by variations of the conditional of the matrices upon which Brady's logic BN4, and Robles and Méndez's E4 are built. When Lti-logics are endowed with the Reduced General semantics they conform Lti-models; when endowed with 2 Set-up semantics, they conform 2 Set-up Lti-models. Then, it is shown that 2 Set-up Lti-models are actually a specific case of the more general structure that are the Lti-models.

Keywords: Routley-Meyer Semantics; 4-valued Logics; General Reduced Routley-Meyer Semantics; 2 Set-up Routley-Meyer Semantics

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1 Introduction

Routley-Meyer semantics, also known as ternary relational semantics, was introduced in the early 70s [30, 31, 32] to solve some long term problems associated with relevance logics, such as completeness [1]. Since its inception, it has been expanded beyond its original motivation and has in fact been applied to a wide range of systems with very satisfactory results such as [11], [23], and [26]. Going into detail, it is possible to point out two elements as the main characteristics of the Routley-Meyer semantics. The first one is the Routley star, a set-theoretical approach to negation [22] whose usefulness it is out of doubt [21]. The second characteristic is the ternary accessibility relation akin to that of Kripke semantics. However, instead of enabling access among two possible worlds [12], Routley-Meyer semantics extends it up to three elements [3]. The development of this ternary relation is due to Kripke-style semantics being unable to prevent the apparition of implicational paradoxes such as $B \to (A \to A)$, a characteristic axiom of S4 [28]. It is also worth noting that ternary relational semantics require a set of designated worlds, as otherwise certain paradoxes such as the rule *Verum ex quodlibet* ($A \Rightarrow B \to A$) would persist.

When defining Routley-Meyer semantics we can find multiple ways to tackle this according to the kind of model that we are introducing [9, 33]. With this in mind, there are two options that stick out: the Reduced General models and the 2 Set-up models[1]. The first one is an offspring of the general model used for Routley-Meyer semantics. The main difference lies on the approach to the set of designated worlds: in the case of reduced models, this set is not itself a subset of the set of possible worlds but rather an element of said set [10]. On the other hand, the 2 Set-up model relies on a restriction of the set of possible worlds. The set is restricted to just two different elements, instead of having, virtually, infinite elements as it is the case of general models [27]. The first version of the reduced models can be traced almost to the very inception of Routley-Meyer semantics, although the main treatise on them can be found in Chapter IV of [33]. Furthermore, the impact that the reduced models have had in recent research is undeniable, as it can be seen in [10]. This interest has motivated the application of Routley-Meyer semantics to systems that are borderline with relevance logics, such as 3 and 4-valued logics [24], or modal logics [11]. On the other hand, the inception of 2 Set-up models is much more unclear. It is possible to find a precedent of these models in [18], although Brady's paper points

[1]Let it be understood that the term Reduced General models is used to generally refer to any model endowed with the reduced general version of the Routley-Meyer semantics. On the other hand, 2 Set-up models may refer to any model endowed with the 2 Set-up version of the Routley-Meyer semantics.

towards a detailed definition of the 2 Set-up models semantics in [33][2]. It is also of importance to note that, even though they do not introduce 2 Set-up models *per se*, the work of [16] is also seminal for their further development. Additionally, it is quite possible that the most important work on the inception of the 2 Set-up models and their very first published definition, [17], is now lost according to the author[3].

Despite all the above, it is necessary to remark that there has been, to the best of our knowledge, no previous research exploring the relationship between these two kinds of semantics. We can only point out the work done in [5], where a very specific proof was given. Aside from this example, there are no published and widely available records of how this two different ways of interpreting the notions of Routley-Meyer semantics intertwine. Furthermore, the research has usually been focused on the Reduced General semantics rather than the 2 Set-up semantics.

R. T. Brady introduced the logic BN4 in [7] as a system built upon the matrix MBN4. This matrix was defined as a modification of Smiley's 4-valued matrix, the characteristic matrix of First Degree Entailment (FDE). According to Brady, BN4 was meant to be a 4-valued extension of Routley and Meyer's basic logic B [7]. Furthermore, it was J. Slaney who pointed out that the system seemed to be the adequate extension of FDE if it was to be endowed with a relevant conditional akin to that of R [34]. On the other hand, E4 was introduced in [29] by G. Robles and J. M. Méndez. They proposed the system as a companion to BN4, where BN4 could be understood as a 4-valued version of R (the system of relevant conditional), E4 would be a 4-valued version of E (the system of –relevant– entailment). Additionally, there are six different 4-valued conditional variants of the characteristics matrices of BN4 and E4 that verify B [13]. To name the logics characterized by these matrices, the term used is Lti-logics, where i refers to a numerical value assigned to each one of the logics considered. This way there are up to 8 Lti-logics. In particular Lt1 is BN4 and Lt5 is E4, while the other 6 logics do not have specific names with the exception of Lt2, which is known as EF4 [6].

Nowadays there seems to be a rising interest in 4-valued logics, as it can be seen in some recent papers such as [2, 20, 35]. One of the reasons for this rising interest is that they are useful for addressing philosophical topics [19], as well as topics from computer science [4]. Furthermore, as it can be seen below, there is a trend of endowing 4-valued logics with Routley-Meyer semantics, thus offering us a bridge to connect both together.

All of the Lti-logics have been endowed with both of the previously mentioned versions of the Routley-Meyer semantics, the Reduced General models in [14] and

[2] Let us state that by the time [7] was published, [33] was not. This is the main reason why the author states something that never happened until [8] was out.

[3] This was stated by the author in private correspondence.

the 2 Set-up models in [15]. In these papers it was shown that all these systems are both sound and complete in the strong sense with respect to both corresponding semantics. Nevertheless, the relation between these different semantics is, to the best of our knowledge, still unexplored. Therefore, the main aim of this paper is to study the relationship between the Reduced General models and the 2 Set-up models in the context of the Lti-logics. For that matter we will first introduce the logics themselves with their corresponding characteristic matrices, and then we will endow them with the two different models. Afterwards, we will proceed to show how the 2 Set-up models are indeed a specific case of the Reduced General models. This was already shown in [5] for Lt2/EF4.

With all of the above, this article is structured as follows: In Section 2 we introduce the Lti-logics and their characteristic matrices. In Section 3 we define the Reduced General models for the Lti-logics. In Section 4 we display the 2 Set-up models for the different Lti-logics. In Section 5 we provide the proof in which we show that the 2 Set-up models are a specific case of a more general structure that is the Reduced General models in the context of Lti-logics. Finally in Section 6 we recap all the work done in the article and sum up the conclusions to the paper.

2 Lti-logics

We begin this section by introducing the characteristic matrices of all the Lti-logics. Firstly we define the structure on which they are based on and all the common functions. Afterwards we introduce the notions that make each Lti-logic their own. Beforehand we define what is a language and what is a logic.

Definition 1 (Propositional Languages)**.** *A propositional language $\mathfrak{L}$ is a denumerable set of propositional variables $p_1, p_2, ..., p_n, ...$ and all or some of the connectives $\wedge$ (Conjunction), $\vee$ (Disjunction), $\neg$ (Negation) and $\rightarrow$ (Conditional)[4]. The set of well-formed formulas (wff) is defined as usual. Finally $A, B, ...$ are used to represent metalinguistic variables.*

Definition 2 (Logics)**.** *A logic S is defined as a structure $\langle \mathfrak{L}, \vdash_S \rangle$ where $\mathfrak{L}$ is a propositional language from Definition 1 and $\vdash_S$ is a (proof-theoretical) consequence relation defined on $\mathfrak{L}$ by a set of axioms and rules of inference. The notions of proof and theorem are the usual ones of Hilbert-style axiomatic systems[5].*

[4] We define $\leftrightarrow$ as is customary: $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$.

[5] $\Gamma \vdash_S A$ means that A is derivable from the set of wff Γ in S; $\vdash_S A$ means that A is a theorem of S.

Now we proceed unto defining the characteristic matrices of the Lti-logics. Let it be understood that these matrices are based on the notions of Definition 1.

Definition 3 (Characteristic Matrices of Lti-logics)**.** *With the propositional language $\mathfrak{L}$ consisting on the connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$, the matrices of the Lti-logics are structures $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}\rangle$, where $\mathcal{V} = \{0, 1, 2, 3\}$ and its partially ordered according to the following lattice:*

$$\begin{tikzcd}
 & 3 & \\
1 \arrow[ur] & & 2 \arrow[ul] \\
 & 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Also, $\mathcal{D} = \{2, 3\}$, and $\mathcal{F} = \{f_\wedge, f_\vee, f_\neg, f_\rightarrow\}$, where $f_\wedge$ and $f_\vee$ are defined as the greatest lower bound (or lattice meet) and the lowest upper bound (or lattice join) respectively. $f_\neg$ is defined as an involutionary operation such that $f_\neg(0) = 3$, $f_\neg(1) = 1$, $f_\neg(2) = 2$, $f_\neg(3) = 0$. Finally, for $f_\rightarrow$ is defined for each matrix of the Lti-logics according to the following tables:

MLt1	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	1	2	3
3	0	1	0	3

MLt2	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	2	3
3	0	0	0	3

MLt3	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	0	2	3
3	0	0	0	3

MLt4	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	1	2	3
3	0	1	0	3

MLt5	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

MLt6	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	1	2	3
3	0	0	0	3

MLt7	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	0	2	3
3	0	0	0	3

MLt8	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	1	2	3
3	0	0	0	3

Now that we have defined the matrices of the Lti-logics, it is time for us to present the Lti-logics themselves. For that matter we provide a Hilbert-style axiomatization of these logics based on the notions from Definitions 1 and 2.

Definition 4 (The Lti-logics)**.** *The logics considered in this paper are defined by means of a subset of the axioms as well as all the rules of inference displayed below:*

Axioms

A1. $A \to A$
A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A7. $\neg\neg A \to A$
A8. $A \to \neg\neg A$
A9. $\neg A \to [A \vee (A \to B)]$
A10. $B \to [\neg B \vee (A \to B)]$
A11. $(A \vee \neg B) \vee (A \to B)$
A12. $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$
A13. $A \to [B \to [[(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)]]$
A14. $(A \wedge \neg B) \to [(A \wedge \neg B) \to \neg(A \to B)]$
A15. $A \vee [\neg(A \to B) \to A]$
A16. $\neg B \vee [\neg(A \to B) \to \neg B]$
A17. $[A \wedge (A \to B)] \to B$
A18. $[(A \to B) \wedge \neg B] \to \neg A$
A19. $A \to [B \vee \neg(A \to B)]$
A20. $\neg B \to [\neg A \vee \neg(A \to B)]$
A21. $[\neg(A \to B) \wedge \neg A] \to A$
A22. $\neg(A \to B) \to (A \vee \neg B)$
A23. $[\neg(A \to B) \wedge B] \to \neg B$
A24. $B \to \{[B \wedge \neg(A \to B)] \to A]\}$
A25. $(A \to B) \vee \neg(A \to B)$
A26. $(\neg A \vee B) \vee \neg(A \to B)$
A27. $[(A \to B) \wedge (A \wedge \neg B)] \to \neg(A \to B)$
A28. $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)]$
A29. $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$
A30. $\{[\neg(A \to B) \wedge B] \to A\} \vee A$

Rules of inference

R1. A, $B \Rightarrow A \wedge B$
R2. A, $A \rightarrow B \Rightarrow B$
R3. $C \vee (A \rightarrow B) \Rightarrow C \vee [(B \rightarrow D) \rightarrow (A \rightarrow D)]$
R4. $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \rightarrow B)$
R5. $C \vee A$, $C \vee (A \rightarrow B) \Rightarrow C \vee B$
R6. $C \vee (A \rightarrow B) \Rightarrow C \vee [(D \rightarrow A) \rightarrow (D \rightarrow B)]$
R7. $C \vee (A \rightarrow B) \Rightarrow C \vee (\neg B \rightarrow \neg A)$

Where R1 is Adjunction, R2 is Modus Ponens, R3 is Disjunctive Suffixing, R4 is Disjunctive Counterexample, R5 is Disjunctive Modus Ponens, R6 is Disjunctive prefixing, and R7 is Disjunctive Contraposition.

In particular, each one of the Lti-logics are axiomatized by the subset A1-A13 plus the axioms of the following list and the rules of inference R1-R7:

Lt1: A14-A16
Lt2: A17-A23
Lt3: A14, A15, A18, A19, A22-A24
Lt4: A16, A17, A20-A22
Lt5: A17-A21, A23, A25-A27
Lt6: A17, A20, A21, A23, A26, A28, A29
Lt7: A14, A18, A19, A21, A23, A26, A30
Lt8: A14, A21, A23, A26, A29, A30

Thus we have presented all the Lti-logics. Furtheremore, we would like to point out that Lt1 is BN4, Lt2 is EF4, and Lt5 is E4, as we specified in Section 1. Moreover, it is obvious that each of the Lti-logics has a characteristic matrix from Defintion 3, and said matrix is the one whose name they bear.

3 Reduced General Routley-Meyer Semantics for Lti-logics

Now we proceed with the definition of the Reduced General models for the Lti-logics. For that matter we define the model generally with the whole set of semantic postulates and afterwards we show how each of the Lti-logics relates to a subset of said semantic postulates.

Definition 5 (Lti-models)**.** *An Lti-model* M *is a structure* $< T, K, R, *, \models>$ *where* K *is a non-empty set,* $T \in K$*,* R *is a ternary relation on* K *and* $*$ *is an involutive unary operator on* K *subject to a subset of the following definitions and postulates for all* a*,* b*,* $c \in K$*:*

d1. $a \leq b =_{df} RTab$

d2. $a = b =_{df} a \leq b$ *&* $b \leq a$

d3. $R^2abcd =_{df} (\exists x \in K)\ (Rabx$ *&* $Rxcd)$

p1. $a \leq a$

p2. $(a \leq b$ *&* $Rbcd) \Rightarrow Racd$

p3. $R^2Tabc \Rightarrow (\exists x \in K)\ (RTbx$ *&* $Raxc)$

p4. $R^2Tabc \Rightarrow (\exists x \in K)\ (Rabx$ *&* $RTxc)$

p5. $a^{**} \leq a$

p6. $a \leq a^{**}$

p7. $a \leq b \Rightarrow b^* \leq a^*$

p8. RT^*TT^*

p9. $Rabc \Rightarrow (b \leq a^*$ *or* $b \leq a)$

p10. $Rabc \Rightarrow (a \leq c$ *or* $a^* \leq c)$

p11. $RTab \Rightarrow (T^* \leq b$ *or* $a \leq T)$

p12. $(RTab$ *&* $R^2Tcde) \Rightarrow (a \leq c^*$ *or* $d \leq c^*$ *or* $c \leq b$ *or* $c \leq e)$

p13. $(Rabc$ *&* $Rcde) \Rightarrow (a \leq c$ *or* $b \leq c$ *or* $c^* \leq c$ *or* $d \leq c$ *or* $b \leq e)$

p14. $Rabc \Rightarrow (Rc^*ab^*$ *or* Rc^*ba^* *or* Rc^*aa^* *or* $Rc^*bb^*)$

p15. $(RTab$ *&* $Ra^*cd) \Rightarrow (c \leq T$ *or* $c \leq b)$

p16. $(RTab$ *&* $Ra^*cd) \Rightarrow (T^* \leq d$ *or* $b^* \leq d)$

p17. $Raaa$

p18. Raa^*a^*

p19. Ra^*aa

p20. $Ra^*a^*a^*$

p21. $Ra^*bc \Rightarrow (b \leq a$ *or* $b \leq a^*)$

p22. $Ra^*bc \Rightarrow (a^* \leq c$ *or* $b \leq a)$

p23. $Ra^*bc \Rightarrow (a \leq c$ *or* $a^* \leq c)$

p24. $(Rabc$ *&* $Rb^*de) \Rightarrow (a \leq e$ *or* $b \leq e$ *or* $d \leq c)$

p25. $RTab \Rightarrow RT^*ab$

p26. RT^*T^*T

p27. $Raaa^*$ *or* Ra^*aa^*

p28. $RTab \Rightarrow (RT^*aa^*$ *or* $Rb^*aa^*)$

p29. $(RTab$ *&* $Ra^*cd) \Rightarrow (T^* \leq d$ *or* $b^* \leq d$ *or* $c \leq a^*)$

p30. $(RTab$ *&* $Ra^*cd) \Rightarrow (c \leq T$ *or* $c \leq b$ *or* $a \leq d)$

Finally, $\models$ is a valuation relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(a \leq b \ \&\ a \models p) \Rightarrow b \models p$

(ii) $a \models A \wedge B$ *iff* $a \models A \ \&\ a \models B$

(iii) $a \models A \vee B$ *iff* $a \models A$ *or* $a \models B$

(iv) $a \models A \rightarrow B$ *iff for all* $b, c \in K$, $(Rabc \ \&\ b \models A) \Rightarrow c \models B$

(v) $a \models \neg A$ *iff* $a^* \not\models A$

Every Lti-logic is subject to d1-d3, p1-p13 and they differ with each other in the additional characteristic subset of corresponding postulates listed above as p14-p30. In particular, for any axiom A_j (where $14 \leq j \leq 30$) belonging to any of them, there is a corresponding postulate p_j from the list above.

The postulates of the Lti-models are summarized as follows:

Remark 1 (Postulates for Lti-models)**.** *Each Lti-model relation R is characterized by d1-d3, p1-p13 plus:*

Lt1: p14-p16

Lt2: p17-p23

Lt3: p14, p15, p18, p19, p22-p24

Lt4: p16, p17, p20-p22

Lt5: p17-p21, p23, p25-p27

Lt6: p17, p20, p21, p23, p26, p28, p29

Lt7: p14, p18, p19, p21, p23, p26, p30

Lt8: p14, p21, p23, p26, p29, p30

To conclude this section, let us point out some of the most important results of these logics w.r.t. the model that we have just defined.

Remark 2 (Results for Lti-logics)**.** *All the logics from Definition 4 are sound and complete in the strong sense w.r.t. the Reduced General Routley-Meyer semantics and their corresponding model from 5 as shown in [5, 14, 25].*

4 2 Set-up Routley-Meyer Semantics for Lti-logics

We proceed unto defining the 2 Set-up models for the Lti-logics. We define the model the same way we did for the Lti-models of Definition 5; the main difference resides that in this case, instead of dealing with a set of semantic postulates, we deal with a set of accessibility relationships. This set of accessibility relationships will be tailored to fit each logic as it shown below.

Definition 6 (2 Set-up Lti-models)**.** *A 2 Set-up Lti-model* $\mathfrak{M}$ *is a structure* $< \mathfrak{K}$, $\mathfrak{R}$, $*_2$, $\models_2>$ *where* $\mathfrak{K}$ *is a set which contains two elements –labelled* O *and* O^{*_2} *– and no other elements.* $*_2$ *is an involutive unary operator defined on* $\mathfrak{K}$ *such that for any* $x \in \mathfrak{K}$, $x = x^{*_2*_2}$. $\mathfrak{R}$ *is a ternary relation on* $\mathfrak{K}$ *defined as follows for each Lti-model class considered in this article: if* a, b, $c \in \mathfrak{K}$, *then* $\mathfrak{R}abc$ *iff:*

Lt1-models: $(a = O$ *&* $b = c)$ *or* $(a \neq b$ *&* $c = O^{*_2})$[6]

Lt2-models: $b = c$ *or* $(a = c = O^{*_2}$ *&* $b = O)$.

Lt3-models: $(a = O$ *&* $b = c)$ *or* $(a = O^{*_2}$ *&* $b = O)$.

Lt4-models: $a = b = c$ *or* $(c = O^{*_2}$ *&* $a \neq b)$.

Lt5-models: $a = O^{*_2}$ *or* $b = c$.

Lt6-models: $(a = O$ *&* $b = c)$ *or* $a = b = c$ *or* $(a = O^{*_2}$ *&* $b \neq c)$.

Lt7-models: $(a = O$ *&* $b = c)$ *or* $(b = c = O)$ *or* $(a = O^{*_2}$ *&* $b \neq c)$.

Lt8-models: $(a = O$ *&* $b = c)$ *or* $(a \neq O$ *&* $b \neq c)$.

$\models_2$ *is a (valuation) relation from* $\mathfrak{K}$ *to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable* p, *wffs* A, B *and* $a \in \mathfrak{K}$*:*

(i) $a \models_2 p$ *or* $a \not\models_2 p$

(ii) $a \models_2 A \wedge B$ *iff* $a \models_2 A$ *&* $a \models_2 B$

(iii) $a \models_2 A \vee B$ *iff* $a \models_2 A$ *or* $a \models_2 B$

(iv) $a \models_2 A \rightarrow B$ *iff for all* b, $c \in \mathfrak{K}$, $(\mathfrak{R}abc$ *&* $b \models_2 A) \Rightarrow c \models_2 B$

(v) $a \models_2 \neg A$ *iff* $a^{*_2} \not\models_2 A$

[6] This clause is equivalent to Brady's clause for BN4-models (i.e., our Lt1-models): $(a \neq O$ or $b = c)$ & $[a \neq O^*$ or $(b = O$ & $c = O^*)]$. Cf. [7, 15].

Let it be noted that we are writing O^* instead of O^{*_2} for the sake of the clarity of the text. We will also accept that $O^{**} = O$, as it is a common use that has been seen multiple times in references such as [5, 15, 27].

Additionally, we explicit the set of accessibility relations that each model associated to a Lti-logic has.

Remark 3 (Ternary relations in $\mathfrak{K}$)**.** *Suppose $O \neq O^*$. Now, given the definition of $\mathfrak{R}$ (cf. Definition 6), the following ternary relations are the only ones holding for each 2 Set-up Lti-model considered:*

Lt1 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*\}$.

Lt2 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O^*, \mathfrak{R}O^*OO\}$.

Lt3 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*OO\}$.

Lt4 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O^*\}$.

Lt5 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O^*, \mathfrak{R}O^*O^*O, \mathfrak{R}O^*OO\}$.

Lt6 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O^*, \mathfrak{R}O^*O^*O\}$.

Lt7 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O, \mathfrak{R}O^*OO\}$.

Lt8 $\mathfrak{R} = \{\mathfrak{R}OOO, \mathfrak{R}OO^*O^*, \mathfrak{R}O^*OO^*, \mathfrak{R}O^*O^*O\}$.

And in order to conclude the section, we do as we did before and point out some of the most interesting results of the Lti-logics w.r.t. this kind of models.

Remark 4 (Results for Lti-logics in 2 Set-up Lti-models)**.** *All the logics from Definition 4 are sound and complete in the strong sense w.r.t. the 2 Set-up Routley-Meyer semantics as shown in [5, 15].*

5 The 2 Set-up Lt*i*-models are a specific case of the Lt*i*-models

After we have introduced both models, Lti-models and 2 Set-up Lti-models, Definitions 5 and 6 respectively, we proceed to show how the latter is actually a specific case of the former. We prove this in the following theorem.

Theorem 1 (The 2 Set-up Lti-models are a specific case of the Lti-models)**.** *The 2 Set-up Lti-models of Definition 6 are a specific case of the Lti-models of Definition 5.*

Proof. We have to prove: (a) that whenever a postulate is included in a Lti-model, that postulate is verified in the corresponding 2 Set-up Lti-model according to the definition of the ternary relation in that Lti-model; (b) that there is no equivalence between Lti-models and 2 Set-up Lti-models; (c) that both clauses (i) of Definitions 5 and 6, are equivalent.

The fact (b) is easy to verify. It suffices to show that there are some relations in Lti-models which cannot be considered in 2 Set-up Lti-models. For any of the Lti-models in Definition 5, for a, b, $c \in K$, we could have $Rabc$, $a \neq b$, $a \neq c$ and $b \neq c$. This situation cannot be in any 2 Set-up Lti-model.

In the case of (c), let us remember that the corresponding clause, in Definition 5 (i) reads as $(a \leq b \;\&\; a \models p) \Rightarrow b \models p$, while in Definition 6 (i) reads as $a \models_2 p$ or $a \not\models_2 p$. To show that both clauses are equivalent we need to show that whenever $\mathfrak{R}Oab$ and $a \models_2 p$, then $b \models_2 p$. This follows automatically as whenever $\mathfrak{R}Oab$, then $a = b$ as it can be seen in Remark 3. And in that case, if $a \models_2 p$, necessarily $b \models_2 p$, as $a = b$.

Then, it remains to prove (a), this is, that postulates of Definition 5 are also verified in the 2 Set-up counterpart Lti-models. A few instances will suffice to illustrate (a).

p1, $a \leq a$, p5, $a^{**} \leq a$, p6, $a \leq a^{**}$ and p7, $a \leq b \Rightarrow b^{*} \leq a^{*}$, clearly hold in any 2 Set-up Lti-model given d1 and the fact that $\mathfrak{R}OOO$ and $\mathfrak{R}OO^{*}O^{*}$ are valid relations in any of those models.

p2, $(a \leq b \;\&\; Rbcd) \Rightarrow Racd$, holds in any 2 Set-up Lti-model. By d1, we have $(RTab \;\&\; Rbcd) \Rightarrow Racd$. Thus, we only need to consider cases $\mathfrak{R}OOO$ and $\mathfrak{R}OO^{*}O^{*}$ given Remark 3 and the fact that T (i.e., O in the 2 Set-up Lti-models) is the first element in the ternary relation. Consequently, we have $a = b$. Therefore, $(RTaa \;\&\; Racd) \Rightarrow Racd$, which is trivial.

p3, $R^{2}Tabc \Rightarrow (\exists x \in K)\,(RTbx \;\&\; Raxc)$, holds in any 2 Set-up Lti-model. Given that the first element in this (double) ternary relation is T, we just have to consider eight different cases, i.e., the cases when the first element in the 2 Set-up Lti-models is O. For each one of the following cases, we have to prove that whenever the antecedent of the postulate holds in a 2 Set-up Lti-model, the consequence also holds in the same model. The eight cases we initially have to consider are:

(1) $\mathfrak{R}^{2}OOOO \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}OxO)$

(2) $\mathfrak{R}^2OO^*OO \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}O^*xO)$
(3) $\mathfrak{R}^2OO^*O^*O \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}O^*xO)$
(4) $\mathfrak{R}^2OO^*OO^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}O^*xO^*)$
(5) $\mathfrak{R}^2OO^*O^*O^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}O^*xO^*)$
(6) $\mathfrak{R}^2OOO^*O \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}OxO)$
(7) $\mathfrak{R}^2OOO^*O^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}OxO^*)$
(8) $\mathfrak{R}^2OOOO^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}OxO^*)$

Now, we note that double ternary relations are understood according to d3 in Definition 5. Then, there is no possibility for the antecedent of cases (6) and (8) to hold in any of the 2 Set-up Lti-models (see Remark 3). As for the rest of the cases, they will hold in some or all the 2 Set-up Lti-models. Let us first consider the situation for Lt5-models, where the antecedent of cases (1)-(5) and (7) does hold. For any of those cases, it is easy to see that the consequent also holds in 2 Set-up Lt5-models for some x: in particular, when (1) $x = O$; (2) $x = O$; (3) $x = O^*$; (4) $x = O$; (5) $x = O^*$; (7) $x = O^*$. Finally, the proof for the rest of the 2 Set-up Lti-models is similar. However, given d3 in Definition 5, Remark 3 and the specific antecedent of each case, a different subset of the six previously considered cases has to be contemplated for each 2 Set-up Lti-model. In particular, cases (1), (4) and (7) have to be considered in any 2 Set-up Lti-model. Case (2) must not be considered in 2 Set-up Lti-models where $i = \{1, 4, 6, 8\}$. Similarly, case (3) must not be considered in 2 Set-up Lti-models where $i = \{1, 2, 3, 4\}$. Lastly, case (5) must not be considered in 2 Set-up Lti-models where $i = \{1, 3, 7, 8\}$.

p10, $Rabc \Rightarrow (a \leq c$ or $a^* \leq c)$, holds in any of the 2 Set-up Lti-models. We note that, by d1 ($a \leq b =_{df} RTab$), p10 can also be read as $Rabc \Rightarrow (RTac$ or $RTa^*c)$. Let us consider the case of Lt5 since its ternary relation in the 2 Set-up Lti-models is the most complex among the Lti-logics. Then, we can simply obtain the proof for the rest of them by eliminating some considered cases. Given the definition of $\mathfrak{R}$ in Lt5 and assuming $\mathfrak{R}abc$, six different cases have to be considered: (1) $\mathfrak{R}OOO$, (2) $\mathfrak{R}OO^*O^*$, (3) $\mathfrak{R}O^*OO^*$, (4) $\mathfrak{R}O^*O^*O^*$, (5) $\mathfrak{R}O^*O^*O$ and (6) $\mathfrak{R}O^*OO$. By assuming each one of these, we obtain at least another valid relation in each case, $\mathfrak{R}Oac$ or $\mathfrak{R}Oa^*c$. Let us take case (2), this is, $a = O$ and $b = c = O^*$. Then, we have either $\mathfrak{R}OO^*O^*$ or $\mathfrak{R}OOO^*$ where $\mathfrak{R}OO^*O^*$ is a relation appearing in Lt5 –actually, in all the Lti-logics. The reader can easily check that results in the other five cases are similar.

p12, $(RTab \;\&\; R^2Tcde) \Rightarrow (a \leq c^*$ or $d \leq c^*$ or $c \leq b$ or $c \leq e)$, holds in any of the 2 Set-up Lti-models. Firstly, given d1 and d3, p12 can be more easily read as follows: $(RTab \;\&\; RTcx \;\&\; Rxde) \Rightarrow (RTac^*$ or $RTdc^*$ or $RTcb$ or $RTce)$. Let us show the case of Lt5. The proof for the rest of the Lti-logics is similar. In the case of the system Lt5, twelve different cases have to be considered. For the first six cases, we have $a = b = O$; for the other six: $a = b = O^*$. Let us consider now the first six, where $a = b = O$. (1) $\mathfrak{R}OOO$ & $\mathfrak{R}OOO$; (2) $\mathfrak{R}OOO$ & $\mathfrak{R}OO^*O^*$; (3) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*OO^*$; (4) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*O^*O^*$; (5) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*O^*O$; (6) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*OO$. It is easy to see that ($\mathfrak{R}Oac^*$ or $\mathfrak{R}Odc^*$ or $\mathfrak{R}Ocb$ or $\mathfrak{R}Oce$) actually holds in any of them. On the one hand, we have $c = x = O$ for cases (1) and (2). Thus, we obtain at least one of those ternary relations (i.e., $\mathfrak{R}Ocb$) for both of these cases. On the other hand, we get $c = x = O^*$ for cases (3)-(6). Therefore, we obtain at least the relation $\mathfrak{R}Oac^*$. When we study the remaining six cases –i.e., the cases (1)-(6) written above plus a third ternary relation where $a = b = O^*$–, similar results are obtained. In particular, at least the relations $RTac^*$ and $RTcb$ are obtained for cases (1)-(2) and (3)-(6), respectively.

p14, $Rabc \Rightarrow (Rc^*ab^*$ or Rc^*ba^* or Rc^*aa^* or $Rc^*bb^*)$, holds in 2 Set-up Lti-models where $i = \{1, 3, 7, 8\}$. Let us prove the case where $i = 1$ –i.e., the case of Lt1-models– the rest of them are proved in a similar way. Given $i = 1$, we have to consider three different cases: (1) $\mathfrak{R}OOO$, (2) $\mathfrak{R}OO^*O^*$, (3) $\mathfrak{R}O^*OO^*$. For each one of these, the reader can easily see that at least another one of the valid ternary relations in Lt1-models is gotten: $\mathfrak{R}O^*OO^*$ in the first case and $\mathfrak{R}OO^*O^*$ in the other two.

p19, Ra^*aa, holds in Lti-models where $i = \{2, 3, 5, 7\}$, as the relation $\mathfrak{R}O^*OO$ also holds in the 2 Set-up Lti-models for these logics (see Definition 5 and Remark 3).

p25, $RTab \Rightarrow RT^*ab$, holds in the 2 Set-up Lt5-models. Given the fact that the first element in the ternary relation is T, only two cases need to be considered in the 2 Set-up Lt5-models: (1) $\mathfrak{R}OOO$ and (2) $\mathfrak{R}OO^*O^*$. Then, we get $\mathfrak{R}O^*OO$ and $\mathfrak{R}O^*O^*O^*$, respectively for each case. Both relations are included in 2 Set-up Lt5-models, therefore p25 holds in the 2 Set-up Lt5-models.

p29, $(RTab$ & $Ra^*cd) \Rightarrow (T^* \leq d$ or $b^* \leq d$ or $c \leq a^*)$, holds in the 2 Set-up Lti-models such that $i = \{6, 8\}$. Given d2 in Definition 5, p29 can also be read as follows: $(RTab$ & $Ra^*cd) \Rightarrow (RTT^*d$ or RTb^*d or $RTca^*)$. Thus, assuming $\mathfrak{R}Oab$ & $\mathfrak{R}a^*cd$ and Definition 6, only five different cases need to be considered in the 2 Set-up Lti-models: (1)$\mathfrak{R}OOO$ & $\mathfrak{R}O^*OO^*$; (2) $\mathfrak{R}OOO$ & $\mathfrak{R}O^*O^*O^*$; (3) $\mathfrak{R}OOO$ & $\mathfrak{R}O^*O^*O$; (4) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}OOO$; (5) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}OO^*O^*$. Now, each one of these cases should result in ($\mathfrak{R}OO^*d$ or $\mathfrak{R}Ob^*d$ or $\mathfrak{R}Oca^*$). For instance, let us take case (1). Then, we have $T = a = b = c = O$ and $d = O^*$. Then, $\mathfrak{R}OO^*O^*$

or $\mathfrak{R}OO^*O^*$ or $\mathfrak{R}OOO^*$. Thus, p29 can be correctly read in terms of 2 Set-up Lti-models because at least one of these relations ($\mathfrak{R}OO^*O^*$) holds in the models of the considered Lti-logics. Same could be said of the other four cases. In particular, by applying the same method to any of those cases we get at least one valid ternary relation in the considered Lti-model. For cases (2), (3) and (5), we also get $\mathfrak{R}OO^*O^*$. For case (4), we get $\mathfrak{R}OOO$.

Finally, let us note that p4 can be proved as p3. Also, proofs of p9, p11, p21, p22 and p23 are similar to that of p10 displayed above. Proofs of p8, p17, p18, p20, p26 and p27 are trivial –see the case of p19 showed above. Proof of p28 follows similar lines to that of p14. Lastly, proofs of p13, p15, p16, p24 and p30 are similar to the proof of p29. □

Thus we have shown that the 2 Set-up Lti-models are, indeed, a specific case of the Lti-models. Let us remind the reader that the former models have been introduced using a 2 Set-up Routley-Meyer semantics, while the latter were defined using the Reduced General Routley-Meyer semantics.

6 Conclusion

The main goal of the paper was to show that the 2 Set-up Routley Meyer models for the Lti-logics were a specific case of the corresponding Reduced General Routley-Meyer models for the same logics. And, as it has been shown in Theorem 1, we can easily conclude that, indeed, 2 Set-up Lti-models are instances or specific cases of the more general ones with an unrestricted number of set-ups.

The fact that the ternary accessibility relation needs three elements to operate over the Routley-Meyer semantics makes obvious the observation that, in the case of being evaluated in a 2 Set-up model at least two of the three members needed are equal to each other (being the third the same or its set-theoretical negation counterpart). Given that, it is necessary to remind that the requirement for a special set-up T (such that $T \in K$ and $a \leq b =_{df} RTab$) in Reduced General Routley-Meyer models is crucial for the development and feasibility of this proof. It is absolutely required that one of the set-ups in 2 Set-up Routley-Meyer models is equivalent to T and the other to the set-theoretical negation counterpart.

Under the Lti-logics –with these requirements well understood– and using 2 Set-up Routley-Meyer semantics, there are a limited amount of possible ternary accessibility relations [7]. Then it is concluded, following the proof of Theorem 1 here

[7] 8 possible relationships with $\mathfrak{K}$ cardinality restricted to 2.

presented, that any 2 Set-up Lti-model will be an instance or a special case of a more general one, such as the Reduced General Routley-Meyer models for the Lti-logics.

Thanks to the present work we would be able to study, in the future, a more abstract and general feature that would extrapolate the relationship between 2 Set-up Routley-Meyer models and Reduced General Routley-Meyer models. Proving this, not only for a given group of logics (like in this case), but for all the ones that can be modelled with Routley-Meyer semantics.

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