

THE CLASS OF ALL 3-VALUED NATURAL CONDITIONAL VARIANTS OF RM3 THAT ARE PLUMWOOD ALGEBRAS

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Abstract

Valerie Plumwood introduced in *Some false laws of logic* [15] a series of arguments on how the rules *Exported Syllogism*, *Disjunctive Syllogism*, *Commutation*, and *Exportation* are not acceptable. Based on this we define the class of Plumwood algebras, logical matrices that do not verify any of these theses. Afterwards we provide conditional variants of the characteristic matrix of the logic RM3 that are also Plumwood algebras. These matrices are given an axiomatization based on First Degree Entailment and are endowed with Belnap-Dunn Semantics. Finally we provide results of Soundness and Completeness in the strong sense for each of the defined variants.

Keywords: Plumwood Algebras; Belnap-Dunn Semantics; 3-valued Logics.

1 Introduction

V. Plumwood determined that there are four rules of proof, namely Exported Syllogism, Disjunctive Syllogism, Commutation, and Exportation, that are unacceptable as they *fail to preserve the property of sufficiency of premiss set for conclusion* [15]. She went on to argue that this failure is independent from their role of responsibility on paradoxes and their interpretations are outright false. These aforementioned laws, are formalized as follows:

TP1. $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$ – *Exported Syllogism*

TP2. $A, (\neg A \vee B) \Rightarrow B$ – *Disjunctive Syllogism*

TP3. $A \rightarrow (B \rightarrow C) \Rightarrow B \rightarrow (A \rightarrow C)$ – *Commutation*

TP4. $(A \wedge B) \rightarrow C \Rightarrow A \rightarrow (B \rightarrow C)$ – *Exportation*

These rules are more than often regarded as key items in many different systems. For example, it is well known the role of TP1 in the relevance systems such as B [14] only that it is presented under the name of Sufficing. TP2 is equivalent to Modus Ponens in any logic that validates the interdefinition of the conditional with respect to negation and disjunction, i.e., $A \rightarrow B =_{df} \neg A \vee B$. With TP3 and TP4 being classically valid theses that play a minor role in some modal systems such as S5 or T [12]. The arguments of Plumwood highlighting the reasons for discarding these rules, while never published until recently [15], were influential enough that their imprint can be felt in many reference texts such as [19] where she is one of the authors.

Despite the importance of Plumwood's contributions, we fail to see the explicit footprint of her work, and are rather shown the implicit outcome in the further development of relevance logics in work such as [18]. With that in mind, we set ourselves to explore the class of algebras that do not validate the rules that Plumwood shown to be false. Thus, we can define the notion of Plumwood algebras as follows:

Definition 1.1 (Plumwood algebras). A class of algebra \mathcal{A} is a Plumwood algebra if and only if (iff) no Plumwood thesis TPn is valid in said class of algebra. That is to say that $\not\models_{\mathcal{A}} TPn$ where, of course, $1 \leq n \leq 4$.

Nevertheless, setting ourselves with the task of exploring all the Plumwood algebras would be a daunting task, if not impossible to carry on. Thus,

we need to provide further context on which this exploration can be performed. For that matter we will consider the framework that the logic RM3 provides. Axiomatizations for the system RM3 can be found in [2] but probably the most recognizable work related to it was written by R. T. Brady [9], which was seminal for multiple works afterwards [5]. RM3, which is part of the family of relevance logics, was born as a three-valued extension of the logic R-Mingle (RM), where RM is the result of adding the Mingle axiom, $A \rightarrow (A \rightarrow A)$, to the system of relevant conditional R [10]. Thus, RM3 is a logic in which propositions could be assigned an interpretation in terms of three truth-values: True, False or Both. RM3 also lacks the more jarring implication paradoxes [16], such as Verum Ex Quodlibet, $A \rightarrow (B \rightarrow A)$.

It should be easy to see the connection between RM3 and Plumwood's work if we take into account the development of the first axiomatization of RM3 by J. M. Dunn (cf. [2, 9]), and how he relates RM3 to the work of A. R. Anderson and N. Belnap. In particular, Plumwood explicitly states that *only Anderson and Belnap have clearly committed themselves to the interpretation of $p \Rightarrow q$ as p is sufficient for the deduction of q* [15]; this is especially clear in [1, 4]. In this line of research, it is of common knowledge the work that Belnap and Dunn did with regards to inconsistent data. It is the inception of what nowadays is known as Belnap-Dunn semantics, that is extensively used to explore different algebras whose interpretations are based upon the four characteristic values of Belnap's work [6, 8, 13] all the way up to colliding data in networks [20].

Extending our scope beyond the work of Dunn, another topic that focuses on the nature of implication is the one of N. Tomova. In [21] Tomova introduces the notion of natural conditional. In particular, Tomova defends that a conditional is said to be natural if and only if it satisfies three different conditions: (I) If the truth-values are restricted to T and F , i.e.: the classical truth-values, the conditional is the one of classical logic; (II) the conditional verifies Modus Ponens; and (III) whenever the antecedent of the conditional is assigned a lower value than the consequent, the interpretation of the conditional is a designated truth-value; that is to say that for any propositional variables p and q , if $p \leq q$, then $I(p \rightarrow q) \in \mathcal{D}^{12}$. While there are certain weakenings of Tomova's definition [17] we prefer to focus on the

¹For the technical definitions of the notions of interpretation and designated values, we refer the reader to Definitions 2.3 and 2.5 below.

²The definition of natural conditional can be found in this paper under Definition 7.3.

original iteration of the idea.

Thus, the main aim of this paper is to explore all of the conditional variants of RM3 characteristic matrix that are Plumwood algebras. That is, variants that do not verify the rules Exported Syllogism, Disjunctive Syllogism, Commutation, and Exportation. Additionally, to provide a more specific context, in particular, we will focus only on those variants that happen to have a natural conditional according to Tomova's definition. Given RM3 conditional:

\rightarrow_{RM3}	0	1	2
0	2	2	2
1*	0	1	2
2*	0	0	2

We have to take into consideration variants in which the interpretation of the cases $f_{\rightarrow_{RM3}}\{(0, 1), (1, 1), (1, 2)\}$ varies because these are the only cases with designated values whose possible variations will still result in other natural conditionals. This means that we have to consider eight (8) different variants including the characteristic case of RM3. Out of this variants, only four (4) are Plumwood algebras and will be explored in this paper under the name of M_{P_i} matrices. Let us note that the characteristic matrix of RM3 is not a Plumwood algebra since it validates both the aforementioned rules TP1 and TP3. We will provide a definition of these logical matrices as well as the inherent algebraic semantics. Afterwards we will axiomatize the M_{P_i} matrices as P_i -logics, which we will endow with Belnap-Dunn semantics. Finally, we will show how the P_i -logics are both sound and complete in the strong sense w.r.t. both semantics.

Finally, the paper is structured as follows: In Section 2 we introduce the P_i -logics as well as the algebraic semantics that characterize them. In Section 3 we define the Belnap-Dunn semantics and endow the P_i -logics with it. In Section 4 we provide a proof of the coextensiveness of the two semantics that we have previously introduced and give a proof of soundness in the strong sense for the P_i -logics. In Section 5 we prove the extension and primeness lemmas that are used for the completeness proof. In Section 6 we show how the P_i -logics are complete in the strong sense w.r.t. both semantics that we have defined. In Section 7 we show some of the properties that the P_i -logics have. Finally, in Section 8 we sum up the work done and present the conclusions of the paper.

2 P_i -logics and Algebraic Semantics

We begin by defining what a propositional language is.

Definition 2.1 (Propositional Languages). A propositional language \mathfrak{L} is a denumerable set of propositional variables $p_1, p_2, \dots, p_n, \dots$ and all or some of the connectives \wedge (Conjunction), \vee (Disjunction), \neg (Negation) and \rightarrow (Conditional). We define \leftrightarrow as is customary: $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$. The set of well-formed formulae (wff) is defined as usual. Finally A, B, \dots are used to represent metalinguistic variables.

Let us note that we use the term formula for singular and formulae for plural. Thus, the abbreviation wff covers both. Additionally, we use \Rightarrow and $\&$ as metaconnectives in their customary sense, and similarly, parentheses are omitted around conjunctions and disjunctions when convenient. Next we define what is a logic.

Definition 2.2 (Logics). A logic L is defined as a structure $\langle \mathfrak{L}, \vdash_L \rangle$ where \mathfrak{L} is a propositional language from Definition 2.1 and \vdash_L is a (proof-theoretical) consequence relation defined on \mathfrak{L} by a set of axioms and rules of inference. The notions of proof and theorem are the usual ones of Hilbert-style axiomatic systems, that is to say that $\Gamma \vdash_S A$ means that A is derivable from the set of wff Γ in S , and $\vdash_S A$ means that A is a theorem of S .

With the previous definitions, now we can proceed unto defining what a logical matrix is and, afterwards, we define the M_{P_i} matrices that are characteristics of the P_i -logics.

Definition 2.3 (Logical Matrix). A logical matrix M is a structure $\langle K, \mathcal{D}, \mathcal{F}, f \rangle$ where K is a set, \mathcal{D} and \mathcal{F} are non-empty subsets of K such that $\mathcal{D} \cup \mathcal{F} = K$ and $\mathcal{D} \cap \mathcal{F} = \emptyset$, and f is the set of functions defined over K . Thus, K is the set of elements of M ; \mathcal{D} is the set of designated values, while \mathcal{F} is the set of non-designated values. Finally, the functions of f provide the different interpretations of the connectives over M .

Definition 2.4 (M_{P_i} Matrices). The M_{P_i} matrices are defined as a logical matrix (cf. Definition 2.3) such that $M_{P_i} = \langle K_{P_i}, \mathcal{D}_{P_i}, \mathcal{F}_{P_i}, f_{P_i} \rangle$, where $K_{P_i} = \{0, 1, 2\}$, $\mathcal{D}_{P_i} = \{1, 2\}$, $\mathcal{F}_{P_i} = \{0\}$, and $f_{P_i} = \{f_\wedge, f_\vee, f_\neg, f_\rightarrow\}$, where $\forall a, b \in K_{P_i}$, $f_\wedge(a, b)$ is defined as $\min.(a, b)$ and $f_\vee(a, b)$ is defined as $\max.(a, b)$. $f_\neg(a)$ is defined as an operation such that $f_\neg(0) = 2$, $f_\neg(1) = 1$, $f_\neg(2) = 0$.

These connectives, in order to facilitate the reading, are explicitated in the following truth-tables³:

\vee	0	1	2	\wedge	0	1	2	\neg
0	0	1	2	0	0	0	0	0
1*	1	1	2	1*	0	1	1	1*
2*	2	2	2	2*	0	1	2	2*

Finally, $f_{\rightarrow}(a, b)$ is defined for each matrix of the P_i -logics according to the following tables:

P_1	0	1	2	P_2	0	1	2
0	2	2	2	0	2	1	2
1*	0	1	1	1*	0	1	1
2*	0	0	2	2*	0	0	2

P_3	0	1	2	P_4	0	1	2
0	2	1	2	0	2	1	2
1*	0	2	1	1*	0	2	2
2*	0	0	2	2*	0	0	2

Once we have defined the matrices, we define the interpretations and the notions of validity and consequence that derive from them.

Definition 2.5 (M_{P_i} -interpretations). A M_{P_i} -interpretation $I_{M_{P_i}}$ is a function from the set of wff \mathfrak{F} to K and adjusted according to the connectives defined in Definition 2.4. Same follows for any subset of wffs Γ .

Definition 2.6 (M_{P_i} -consequence and M_{P_i} -validity). For any set of wff Γ and wff A , $\Gamma \models_{M_{P_i}} A$, A is consequence of Γ in the M_{P_i} matrices iff $I_{M_{P_i}}(A) \in \mathcal{D}$ as long as $I_{M_{P_i}}(\Gamma) \in \mathcal{D}$ for every M_{P_i} -interpretation $I_{M_{P_i}}$. Furthermore, A is M_{P_i} -valid, $\models_{M_{P_i}} A$, iff $I_{M_{P_i}}(A) \in \mathcal{D}$ for every M_{P_i} -interpretation.

Given these definitions, we now provide the following proposition so the characterization of the M_{P_i} matrices is fully fleshed out.

³Let us note that the truth-tables for these connectives are identical to those of RM3.

Proposition 2.7 (The M_{P_i} matrices are Plumwood algebras). *All the M_{P_i} matrices from Definition 2.4 are Plumwood algebras as seen in Definition 1.1.*

Proof. The proof is straight forward and is left to the reader, who might feel compelled to use [11]. \square

With all this, we now proceed to introduce the axiomatizations of the P_i -logics. First of all, let us note that the axiomatizations here presented have substituted most of the usual derivation rules by their disjunctive counterparts. This is due to the extension lemma that we will be using to show the completeness of the logics. To further expand on disjunctive rules, the reader can refer to [9] ⁴. With that in mind, the following is the axiomatization of Disjunctive FDE, i.e., the result of adding the disjunctive version of FDE derivation rules to its characteristic axioms ⁵. Let it be noted that when convenient, we use the dot (.) notation in such a way that, whenever the main connective is a conditional, it is denoted by the dot and parenthesis are omitted.

⁴A necessary condition to apply the extension lemma that we will use later in this work to a given logic L is that it is possible to build up prime regular theories closed under the primitive rules of inference of L. But this necessary condition is not generally met by weak logics. Thus, suppose that L is a logic closed by a rule R but lacking the corresponding axiom. Then, following the aforementioned method, it is not possible to build prime L-theories closed under that rule R , in general. Nevertheless, Brady himself showed that, despite the absence of the axiom corresponding to the rule R , prime L-theories are available if in addition to being closed by R , L is also closed under the disjunctive version of R .

⁵Regarding axioms A1-A6, we want to note that Anderson and Belnap's original axiomatization of FDE: (1) does not contain A1 as an axiom but, as they stated on page 160 [2], it can be easily derived from A5, A6 and transitivity rule; (2) the metalinguistic variables are restricted to wff composed only by truth-functional connectives –thus, excluding any possible non first-degree formula.

- A1. $A \rightarrow A$
 A2. $A \wedge B \rightarrow .A / A \wedge B \rightarrow .B$
 A3. $A \rightarrow .A \vee B / B \rightarrow .A \vee B$
 A4. $A \wedge (B \vee C) \rightarrow .(A \wedge B) \vee (A \wedge C)$
 A5. $\neg\neg A \rightarrow A$
 A6. $A \rightarrow \neg\neg A$

- R1. $A, B \Rightarrow A \wedge B$
 R2. $A \rightarrow B, A \Rightarrow B$
 R3. $C \vee (A \rightarrow B), C \vee A \Rightarrow C \vee B$
 R4. $D \vee (A \rightarrow B), D \vee (B \rightarrow C) \Rightarrow D \vee (A \rightarrow C)$
 R5. $D \vee (A \rightarrow B), D \vee (A \rightarrow C) \Rightarrow D \vee (A \rightarrow (B \wedge C))$
 R6. $D \vee (A \rightarrow C), D \vee (B \rightarrow C) \Rightarrow D \vee ((A \vee B) \rightarrow C)$
 R7. $C \vee (A \rightarrow \neg B) \Rightarrow C \vee (B \rightarrow \neg A)$

FDE can be extended into $\text{FDE}_{P_i}+$, FDE plus the common part of the conditional of P_i -logics, with the addition of the following axioms:

- A7. $(A \rightarrow B) \wedge A \wedge \neg B \rightarrow .\neg A$ A10. $(A \rightarrow B) \vee \neg B$
 A8. $(A \rightarrow B) \wedge A \wedge \neg B \rightarrow .B$ A11. $\neg A \wedge B \rightarrow .A \rightarrow B$
 A9. $(A \rightarrow B) \vee A$

Thus, P_1 is $\text{FDE}_{P_i}+$ plus the following axioms and rules:

- A12. $\neg(A \rightarrow B) \rightarrow .A \vee \neg B$ R8. $C \vee \neg(A \rightarrow B) \Rightarrow C \vee A$
 A13. $A \wedge \neg B \rightarrow .\neg(A \rightarrow B)$ R9. $C \vee \neg(A \rightarrow B) \Rightarrow C \vee (A \vee \neg A)$
 A14. $A \wedge \neg A \rightarrow .\neg(A \rightarrow B)$ R10. $C \vee \neg(A \rightarrow B) \Rightarrow C \vee (\neg A \vee \neg B)$

P_2 is $\text{FDE}_{P_i}+$, A12-A14, and R9 plus the following axioms and rules:

- A15. $B \wedge \neg B \rightarrow .\neg(A \rightarrow B)$ R11. $C \vee \neg(A \rightarrow B) \Rightarrow C \vee (A \vee B)$
 R12. $C \vee \neg(A \rightarrow B) \Rightarrow C \vee (\neg A \vee B \vee \neg B)$

P_3 is $\text{FDE}_{P_i}+$ and A12 plus the following axioms:

- A16. $\neg(A \rightarrow B) \wedge \neg A \rightarrow .A \vee B$
 A17. $\neg(A \rightarrow B) \wedge \neg A \rightarrow .\neg A \vee \neg B$
 A18. $\neg(A \rightarrow B) \wedge A \wedge \neg A \wedge B \wedge \neg B \rightarrow .C$
 A19. $\neg(A \rightarrow B) \wedge \neg A \wedge B \wedge \neg B \rightarrow .B$
 A20. $A \rightarrow .\neg(A \rightarrow B) \vee B$
 A21. $A \wedge \neg A \rightarrow .\neg(A \rightarrow B) \vee \neg B$
 A22. $\neg B \rightarrow .\neg(A \rightarrow B) \vee \neg A$
 A23. $B \wedge \neg B \rightarrow .\neg(A \rightarrow B) \vee A$

P_4 if $\text{FDE}_{P_i}+$, A16, A18, A20, A22, and A23 plus the following axiom:

- A24. $(\neg(A \rightarrow B) \rightarrow \neg B) \vee \neg B$

Also, we add a remark about disjunctive rules:

Remark (Regular rules follow from their disjunctive counterpart). For any of the disjunctive rules of the P_i -logics, the regular version follows from the disjunctive one, except for the case of R3, as R2 is included as a primitive rule. In particular, the proofs follow by A3, R2, and the corresponding disjunctive rule.

Finally, to conclude the section, we mention some derived rules and theorems that are common to all the P_i -logics. These are Modus Tollens Rule, $A \rightarrow B, \neg B \Rightarrow \neg A$, that follows from R2 and R7; Summation Rule, $A \rightarrow B \Rightarrow C \vee A \rightarrow .C \vee B$, that follows by A3, R6 and R7; De Morgan (I), $\neg A \vee \neg B \rightarrow .\neg(A \wedge B)$, that follows from A2, R6 and R7; De Morgan (II), $\neg(A \wedge B) \rightarrow .\neg A \vee \neg B$, that follows from A3, A5, R5 and R6; De Morgan (III), $\neg A \wedge \neg B \rightarrow .\neg(A \vee B)$, that follows from A2, A6, R6 and R7; De Morgan (IV), $\neg(A \vee B) \rightarrow .\neg A \wedge \neg B$, that follows from A3, R5 and R7; Distribution, $[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (A \vee C)]$, that follows by A2, A3, R4, R5 and R6 from left to right, and by A4, A5, R4, R5, and De Morgan (II) and (IV) from right to left; Associativity of Disjunction, $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$, that follows by A3, R4 and R6 in both directions; and Idempotence of Disjunction, $A \leftrightarrow (A \vee A)$, that follows by A3 from left to right, and by A1 and R6 from right to left.

3 Belnap-Dunn Semantics

In this section we introduce the Belnap-Dunn semantics for the P_i -logics. We begin by defining the models and the notions of validity and consequence.

Definition 3.1 (P_i -models). A P_i -model \mathcal{M}_{P_i} is a structure such that $\langle \mathcal{K}_{P_i}, I_{P_i} \rangle$ where $\mathcal{K}_{P_i} = \{\{T\}, \{F\}, \{T, F\}\}$, and I_{P_i} is a P_i -interpretation, a function from \mathfrak{F} over \mathcal{K}_{P_i} where the following clauses apply for a propositional variable p and wff A and B :

- (I) $I_{P_i}(p) \in \mathcal{K}_{P_i}$
- (II) $T \in I_{P_i}(\neg A)$ iff $F \in I_{P_i}(A)$
- (III) $F \in I_{P_i}(\neg A)$ iff $T \in I_{P_i}(A)$
- (IV) $T \in I_{P_i}(A \wedge B)$ iff $T \in I_{P_i}(A)$ and $T \in I_{P_i}(B)$
- (V) $F \in I_{P_i}(A \wedge B)$ iff $F \in I_{P_i}(A)$ or $F \in I_{P_i}(B)$
- (VI) $T \in I_{P_i}(A \vee B)$ iff $T \in I_{P_i}(A)$ or $T \in I_{P_i}(B)$
- (VII) $F \in I_{P_i}(A \vee B)$ iff $F \in I_{P_i}(A)$ and $F \in I_{P_i}(B)$
- (VIII) $T \in I_{P_i}(A \rightarrow B)$ iff $T \notin I_{P_i}(A)$ or $F \notin I_{P_i}(B)$ or ($F \in I_{P_i}(A)$ and $T \in I_{P_i}(B)$)

We distinguish four different instances of the clause (IX), one for each model for the corresponding logic. These clauses are as follow:

- (IX₁) $F \in I_{P_1}(A \rightarrow B)$ iff ($T \in I_{P_1}(A)$ and $F \in I_{P_1}(B)$) or ($T \in I_{P_1}(A)$ and $F \in I_{P_1}(A)$)
- (IX₂) $F \in I_{P_2}(A \rightarrow B)$ iff [$T \in I_{P_2}(A)$ and ($F \in I_{P_2}(A)$ or $F \in I_{P_2}(B)$)] or ($T \in I_{P_2}(B)$ and $F \in I_{P_2}(B)$)
- (IX₃) $F \in I_{P_3}(A \rightarrow B)$ iff $\{T \in I_{P_3}(A)$ and [$T \notin I_{P_3}(B)$ or ($F \in I_{P_3}(A)$ and $F \notin I_{P_3}(B)$)]\} or $\{F \in I_{P_3}(B)$ and [$F \notin I_{P_3}(A)$ or ($T \notin I_{P_3}(A)$ and $T \in I_{P_3}(B)$)]\}
- (IX₄) $F \in I_{P_4}(A \rightarrow B)$ iff $F \in I_{P_4}(B)$ and [$F \notin I_{P_4}(A)$ or ($T \in I_{P_4}(A)$ and $T \notin I_{P_4}(B)$)] or ($T \notin I_{P_4}(A)$ and $T \in I_{P_4}(B)$)

Definition 3.2 (P_i -consequence and P_i -validity). For any set of wff Γ and wff A , $\Gamma \models_{\mathcal{M}_{P_i}} A$, A is consequence of Γ in the P_i -model \mathcal{M}_{P_i} iff $T \in I_{P_i}(A)$ if $T \in I_{P_i}(\Gamma)$ ($T \in I_{P_i}(\Gamma)$ iff $\forall B \in \Gamma \mid T \in I_{P_i}(B)$; $F \in I_{P_i}(\Gamma)$ iff $\exists B \in \Gamma \mid F \in I_{P_i}(B)$). Particularly, $\models_{\mathcal{M}_{P_i}} A$, A is true in \mathcal{M}_{P_i} iff $T \in I_{P_i}(A)$. Then, $\Gamma \models_{P_i} A$, A is semantic consequence of Γ in the semantics for the P_i -logics, iff $\Gamma \models_{\mathcal{M}_{P_i}} A$ for every P_i -model. Particularly, $\models_{P_i} A$, A is valid in the semantics for the P_i -logics, iff $\models_{\mathcal{M}_{P_i}} A$ for every P_i -model \mathcal{M}_{P_i} .

4 Soundness of P_i -logics

After introducing the P_i -logics as well as the Algebraic and Belnap-Dunn semantics for them, we proceed to show the soundness of these, the P_i -logics, w.r.t. both semantics defined above. Before that, we prove that both semantics are equivalent so the proof can be shown in a more straight forward way. We begin by giving a establishing a correspondence between the interpretations of the semantics; first it is defined for propositional variables and then is extended to wff and sets of wff.

Definition 4.1 (Corresponding Interpretations). Given any M_{P_i} -interpretation, a corresponding P_i -interpretation can be defined. Conversely, the same thing happens. In particular, given a M_{P_i} -interpretation $I_{M_{P_i}}$ for a propositional variable p , the corresponding P_i -interpretation is defined as follows:

- (1) $I_{M_{P_i}}(p) = 0$ iff $I_{P_i}(p) = \{F\}$
- (2) $I_{M_{P_i}}(p) = 1$ iff $I_{P_i}(p) = \{T, F\}$
- (3) $I_{M_{P_i}}(p) = 2$ iff $I_{P_i}(p) = \{T\}$

Proposition 4.2 (Extension of the corresponding interpretations to any wff). *Given the corresponding interpretations of Definition 4.1, we extend them to any wff as follows:*

- (1) $I_{M_{P_i}}(A) = 0$ iff $I_{P_i}(A) = \{F\}$
- (2) $I_{M_{P_i}}(A) = 1$ iff $I_{P_i}(A) = \{T, F\}$
- (3) $I_{M_{P_i}}(A) = 2$ iff $I_{P_i}(A) = \{T\}$

Proof. I_{P_i} integrates the wff according to the clauses of Definition 3.1. With that in mind, the proof proceeds by induction and is left to the reader. Let us state that there are two different cases to be considered. The case in which A is a propositional variable, which is trivial as it corresponds to Definition 4.1. And the case in which A is, indeed, a wff. There are four different cases, one for each connective; in particular, the case of the conditional is specific to each of the P_i -logics, as the clause for falsehood varies. It is obvious that the proof needs to be addressed in both directions: once assuming the $I_{M_{P_i}}$ as the hypothesis, and another one where the assumed hypothesis is the I_{P_i} . As stated above, the proof is left to the reader, who might be interested in similar proofs to this like the ones appearing in [7, 13]. \square

Proposition 4.3 (Extension of the corresponding interpretation to a set of wff). *Given a I_{P_i} -interpretation and a $I_{M_{P_i}}$ -interpretation, for any set of wff Γ , it follows that:*

- (1) $I_{M_{P_i}}(\Gamma) = 0$ iff $I_{P_i}(\Gamma) = \{F\}$
- (2) $I_{M_{P_i}}(\Gamma) = 1$ iff $I_{P_i}(\Gamma) = \{T, F\}$
- (3) $I_{M_{P_i}}(\Gamma) = 2$ iff $I_{P_i}(\Gamma) = \{T\}$

Proof. This proof, as the case of Proposition 4.2, proceeds by induction. As above, it is also left to the reader. \square

Now, we are able to show the coextensiveness of both semantics, the algebraic one and the Belnap-Dunn semantics.

Theorem 4.4 (Coextensiveness of $\models_{M_{P_i}}$ and \models_{P_i}). *For any set of wff Γ and wff A , $\Gamma \models_{M_{P_i}} A$ iff $\Gamma \models_{P_i} A$.*

Proof. There are two different cases:

- (a) $\Gamma = \emptyset$
- (b) $\Gamma \neq \emptyset$

For (a) we have to show that $\models_{M_{P_i}} A$ iff $\models_{P_i} A$. This means that for any $I_{M_{P_i}}$ and I_{P_i} , $I_{M_{P_i}}(A) = 1$ or 2 iff $T \in I_{P_i}$, which was already shown in Proposition 4.2.

For (b) we need to show that $\Gamma \models_{M_{P_i}} A$ iff $\Gamma \models_{P_i} A$. This is divided in two different subcases: (i) from left to right, if $\Gamma \models_{M_{P_i}} A$, then $\Gamma \models_{P_i} A$;

(ii) and from right to left, if $\Gamma \models_{P_i} A$ then $\Gamma \models_{M_{P_i}} A$. We proceed with (i). We assume (1). $\Gamma \models_{M_{P_i}} A$ and need to show that $T \in I_{P_i}(A)$. We also assume a P_i -interpretation such that (2). $T \in I_{P_i}(\Gamma)$. By Proposition 4.3, we have a corresponding M_{P_i} -interpretation such that (3). $I_{M_{P_i}}(\Gamma) = 1$ or 2 . Given (1) and (3), it follows (4). $I_{M_{P_i}}(A) = 1$ or 2 . Given Proposition 4.2 and (4) we have $T \in I_{P_i}$ as we needed to show. For subcase (ii) we assume (1). $\Gamma \models_{P_i} A$ and need to show that $I_{M_{P_i}}(A) = 1$ or 2 . Additionally we also assume (2). $I_{M_{P_i}}(\Gamma) = 1$ or 2 by the corresponding interpretation defined above. Given (2) and Definition 3.2 it follows (3). $T \in I_{P_i}(\Gamma)$. From (1), (3) and Definition 3.2 it follows (4). $T \in I_{P_i}(A)$. Finally, using Proposition 4.2 we have $I_{M_{P_i}} = 1$ or 2 just as we need.

Thus, we have shown the coextensiveness of $\models_{M_{P_i}}$ and \models_{P_i} . \square

With the result of Theorem 4.4 we can proceed to the soundness proof.

Theorem 4.5 (Soundness of P_i -logics). *For any set of wff Γ and wff A , if $\Gamma \vdash_{P_i} A$ then $\Gamma \models_{P_i} A$ and $\Gamma \models_{M_{P_i}} A$.*

Proof. We firstly show that if $\Gamma \vdash_{P_i} A$ then $\Gamma \models_{M_{P_i}} A$. We proceed by induction on the length of proofs. For that matter we have to distinguish three different cases:

- (1) $A \in \Gamma$
- (2) A is an axiom of P_i
- (3) A is derived using a P_i rule

(1) follows automatically and it is trivial. (2) requires to show the validity of the axioms in their corresponding matrices. This is left to the reader, who is advised to use [11] to simplify the process. Finally, (3) requires us to show the validity of the rules. Although it could be done similarly to (2), we provide one example and the rest of the rules are left to the reader. We show the case of non-disjunctive R10.

$$\begin{array}{ll}
 \text{R10. } \neg(B \rightarrow C) \Rightarrow \neg C \vee \neg B & \\
 1. \Gamma \vdash_{P_i} \neg(B \rightarrow C) & \text{Hypothesis} \\
 2. \Gamma \models_{M_{P_i}} \neg(B \rightarrow C) & \text{Induction Hypothesis, 1}
 \end{array}$$

We need to demonstrate $\Gamma \models_{M_{P_i}} \neg C \vee \neg B$. Thus, we assume $I_{M_{P_i}}(\Gamma) = 1$ or 2 , and prove $I_{M_{P_i}}(\neg C \vee \neg B) = 1$ or 2 .

- | | |
|--|----------------------|
| 3. $I_{M_{P_i}}(\Gamma) = 1$ or 2 | Hypothesis |
| 4. $I_{M_{P_i}}(\neg(B \rightarrow C)) = 1$ or 2 | Definition 2.6, 2, 3 |
| 5. $I_{M_{P_i}}(\neg C \vee \neg B) = 1$ or 2 | M_{P_i} , 4 |

Now, to show that if $\Gamma \vdash_{P_i} A$ then $\Gamma \models_{P_i} A$, follows automatically by the coextensiveness of \models_{P_i} and $\models_{M_{P_i}}$ shown in Theorem 4.4 and what we have shown above. \square

5 Extension and Primeness Lemmas

In this section we introduce the extension and primeness lemmas that are required for showing that the P_i -logics are complete. We begin by defining the notion of P_i -theories and their classes. Afterwards we proceed to introduce the notion of disjunctive derivability associated with P_i -logics and what is a maximal set in said logics.

Definition 5.1 (P_i -theories and their Classes). A P_i -theory \mathcal{T} is a set of wff closed under Adjunction and P_i -entailment. In particular, for wff A and B ,

- (1) \mathcal{T} is closed under Adjunction iff if $A \in \mathcal{T}$ and $B \in \mathcal{T}$, then $A \wedge B \in \mathcal{T}$;
- (2) \mathcal{T} is closed under P_i -entailment iff if $\vdash_{P_i} A \rightarrow B$ and $A \in \mathcal{T}$, then $B \in \mathcal{T}$.

Additionally, the following classes of P_i -theories are defined for a P_i -theory \mathcal{T} and wff A and B :

- (I) Prime P_i -theories: \mathcal{T} is prime iff if $A \vee B \in \mathcal{T}$, then $A \in \mathcal{T}$ or $B \in \mathcal{T}$.
- (II) Regular P_i -theories: \mathcal{T} is a regular P_i -theory iff if $\vdash_{P_i} A$, then $A \in \mathcal{T}$.
- (III) A-consistent P_i -theories: \mathcal{T} is an a-consistent P_i -theory iff \mathcal{T} is not trivial, that is, \mathcal{T} does not contain all the wff.
- (IV) Empty P_i -theories: \mathcal{T} is an empty P_i -theory iff \mathcal{T} does not contain any wff.
- (V) Appropriately closed P_i -theories: \mathcal{T} is an appropriately closed P_i -theory iff for all the P_i -logic derivation rules of the form $\Gamma \Rightarrow B$, it follows that if $\Gamma \subseteq \mathcal{T}$, then $B \in \mathcal{T}$, for a set of wff Γ .

In the following sections we will show that the theories for P_1 and P_2 need to be appropriately closed, while the theories for P_3 and P_4 need to be a-consistent but not appropriately closed.

Remark (Closure under derived rules). For any \mathcal{T} P_i -theory that is appropriately closed, it follows automatically that it also closed under any derived rules as long as \mathcal{T} is also a regular theory. As an example, the closure under Modus Tollens Rule can be shown thanks to the closure of \mathcal{T} under R2 and R7.

Definition 5.2 (Disjunctive P_i -derivability). For any set of wff Γ , Θ , and wff A_1, \dots, A_m and B_1, \dots, B_n such that $A_1, \dots, A_m \in \Gamma$ and $B_1, \dots, B_n \in \Theta$, Γ disjunctively implies Θ in a P_i -logic L , in symbols $\Gamma \vdash_L^d \Theta$, iff $\vdash_L (A_1 \wedge \dots \wedge A_m) \rightarrow (B_1 \vee \dots \vee B_n)$.

Definition 5.3 (P_i -maximal set). For any wff set Γ and its complement, $\bar{\Gamma}$, Γ is a P_i -maximal set iff $\Gamma \not\vdash_L^d \bar{\Gamma}$ for any P_i -logic L .

Now we can introduce the auxiliary lemma for the extension lemma and afterwards we will prove this extension lemma for the P_i -logics. Then we introduce the primeness lemma:

Lemma 5.4 (Auxiliary lemma to the extension lemma). *For an appropriately closed P_i -logic L , and wff A, B_1, \dots, B_n , if $\{B_1, \dots, B_n\} \vdash_L A$, then for any wff C , it follows that $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_L C \vee A$.*

Proof. The proof proceeds by induction over the length of the proof of A from $\{B_1, \dots, B_n\}$. With this in mind there are three main cases:

- (a) $A \in \{B_1, \dots, B_n\}$: It follows by Summation Rule and the basic properties of conjunction.
- (b) A is an axiom: It follows by A3 and R2.
- (c) A is the result of the application of a rule: This case has multiple subcases, one for each derivation rule that is part of L . In particular, the proof is different for the cases of P_1 and P_2 as both have more rules in the axiomatization than P_3 and P_4 ; thus, the proof has a common ground, in which we show the cases in which A follows by R1-R7, and then we show the cases of P_1 , in which it might follow by R8-R10, and P_2 , in which it might follow by R11 or R12. The common part

of the proof is as follows: if A is derived by R1, it follows by R1 and Distribution. All the other cases, R2-R7, follow by the corresponding rule and Associativity of Disjunction, except for R2 that requires R3. For the cases of P_1 and P_2 , in which A follows by one of the rules that are part of their axiomatization but not of the common one, it follows by the rule of the case and Associativity of Disjunction. That is, in P_1 , if A follows by R8-R10, it would suffice to use the corresponding rule of the case and the aforementioned theorem. A similar method can be used when A is derived by R11 and R12 in P_2 .

□

Lemma 5.5 (Extension to P_i -maximal sets). *For a P_i -theory L , let Γ, Θ be sets of wff, such that $\Gamma \not\vdash_L^d \Theta$. Then, there are sets of wff Γ', Θ' , such that $\Gamma \subseteq \Gamma', \Theta \subseteq \Theta', \Theta' = \overline{\Gamma'}$ and $\Gamma' \not\vdash_L^d \Theta'$. That is, Γ' is P_i -maximal set such that it does not disjunctively P_i -derive its complement.*

Proof. Let A_1, \dots, A_n, \dots be an enumeration of wff. For $k \in \mathbb{N}$, the sets Γ' and Θ' are defined as follows: $\Gamma' = \bigcup_k \Gamma_k$ and $\Theta' = \bigcup_k \Theta_k$, where $\Gamma_0 = \Gamma$ and $\Theta_0 = \Theta$. For each $k \in \mathbb{N}$, Γ_{k+1} and Θ_{k+1} are defined as follows:

- (i) If $\Gamma_{k+1} \cup \{A_{k+1}\} \vdash_L^d \Theta$, then $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k \cup \{A_{k+1}\}$.
- (ii) If $\Gamma_{k+1} \cup \{A_{k+1}\} \not\vdash_L^d \Theta_k$, then $\Gamma_{k+1} = \Gamma_k \cup \{A_{k+1}\}$ and $\Theta_{k+1} = \Theta_k$.

It is obvious, but should be noted nonetheless, that $\Gamma \subseteq \Gamma', \Theta \subseteq \Theta'$ and $\Gamma' \cup \Theta' = \mathfrak{F}$. We prove: (I) $\Gamma_k \not\vdash_L^d \Theta_k$ for all $k \in \mathbb{N}$. We proceed by reductio, so we suppose that for some $i \in \mathbb{N}$, (II) $\Gamma_i \not\vdash_L^d \Theta_i$ but $\Gamma_{i+1} \vdash_L^d \Theta_{i+1}$. Then we have to take into consideration (i) and (ii), how the sets Γ_{k+1} and Θ_{k+1} are built. We begin with (ii): (a) $\Gamma_{i+1} \cup \{A_{i+1}\} \not\vdash_L^d \Theta_i$. By (ii) we have $\Gamma_{i+1} = \Gamma_i \cup \{A_{i+1}\}$ and $\Theta_{i+1} = \Theta_i$. By the reductio hypothesis, (II), we get $\Gamma_i \cup \{A_{i+1}\} \vdash_L^d \Theta_i$, which leads to a contradiction. We continue with the case (i): (b) $\Gamma_{i+1} \cup \{A_{i+1}\} \vdash_L^d \Theta_i$. By (i) we have $\Gamma_{i+1} = \Gamma_i$ and $\Theta_{i+1} = \Theta_i \cup \{A_{i+1}\}$. By the reductio hypothesis, (II), we get (1) $\Gamma_i \vdash_L^d \Theta_i \cup \{A_{i+1}\}$. Now, we assume the wff of this derivation to be B_1, \dots, B_m and C_1, \dots, C_n respectively; also we will refer to B_1, \dots, B_m by B , and by C to C_1, \dots, C_n . Therefore, (1) is as follows: (2) $\vdash_L B \rightarrow .C \vee A_{i+1}$. On the other hand, given (b), there is a conjunction of elements of Γ_i that we name B' ; and also a disjunction of elements of Θ_i that we name C' . This leads to: (3) $\vdash_L B' \wedge A_{i+1} \rightarrow .C'$. Now, by B'' and C'' we refer to $B \wedge B'$ and $C \vee C'$ respectively. We will show

(III) $\vdash_L B'' \rightarrow C''$, and hence $\Gamma_i \vdash_L^d \Theta_i$, contradicting the reductio hypothesis and proving (I). From (2) and some elementary properties of conjunction and disjunction, we have: (4) $\vdash_L (B \wedge B') \rightarrow [(C \vee C') \vee A_{i+1}]$. From (3) and by R2 we have: (5) $\vdash_L [(B \wedge B') \wedge A_{i+1}] \rightarrow (C \vee C')$. Now, by applying R4, Idempotence of Disjunction, and Lemma 5.4 in (4) and (5), we have: (6) $\vdash_L (B \wedge B') \rightarrow (C \vee C')$. But (6) is actually (III), $\vdash_L B'' \rightarrow C''$, and hence, as we pointed out above, $\Gamma_i \vdash_L^d \Theta_i$, contradicting the reductio hypothesis. Consequently, (I), $\Gamma_k \not\vdash_L^d \Theta_k$, for all $k \in \mathbb{N}$, follows. Thus, we have two sets of wff, Γ' and Θ' , such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Gamma' \not\vdash_L^d \Theta'$, and $\Theta' = \overline{\Gamma'}$ as it was required all along. Also, notice that Γ' is a P_i -maximal set, since $\Gamma' \not\vdash_L^d \overline{\Gamma'}$. \square

Lemma 5.6 (P_i -maximal sets are prime P_i -theories). *If Γ is a P_i -maximal set, then Γ is a prime P_i -theory.*

Proof. We need to prove that Γ is closed under Adjunction, P_i -entailment and that is prime. Let A, B be wff and Γ a set of wff. For Adjunction we assume that $A, B \in \Gamma$ but $A \wedge B \notin \Gamma$. By A1 we have $\vdash_L (A \wedge B) \rightarrow (A \wedge B)$, contradicting Γ maximality. For P_i -entailment, we assume $\vdash_L A \rightarrow B$ and $A \in \Gamma$. If $B \notin \Gamma$ happens, then there is a contradiction with Γ maximality. The primeness of Γ is proven just like the closure under Adjunction but using A1 in the form $\vdash_L (A \vee B) \rightarrow (A \vee B)$. \square

6 Completeness of P_i -logics

To tackle the completeness proof we need to prove some preliminary lemmas that will help us after defining the canonical model. We begin by defining the notion of \mathcal{T} -interpretation.

Definition 6.1 (\mathcal{T} -interpretations). Let \mathcal{K} be the set from Definition 3.1, L a P_i -logic, and \mathcal{T} a prime, regular and a-consistent theory. Then, a function $I_{\mathcal{T}}$ is defined in a way such that for each propositional variable p it follows that:

- (a) $T \in I_{\mathcal{T}}(p)$ iff $p \in \mathcal{T}$
- (b) $F \in I_{\mathcal{T}}(p)$ iff $\neg p \in \mathcal{T}$

Additionally, each wff is assigned an element from \mathcal{K} according to the clauses (IV)-(IX) from Definition 3.1. Thus, $I_{\mathcal{T}}$ is a \mathcal{T} -interpretation, and following Definition 3.2 it follows that $T \in I_{\mathcal{T}}(\Gamma)$ iff $\forall B \in \Gamma | T \in I_{\mathcal{T}}(B)$; $F \in I_{\mathcal{T}}(\Gamma)$ iff $\exists B \in \Gamma | F \in I_{\mathcal{T}}(B)$.

With the above definition we can now introduce the canonical model, its consequence relation and show that it is, indeed, a model.

Definition 6.2 (Canonical P_i -models). A canonical P_i -model is a structure $\langle \mathcal{K}_{P_i}, I_{\mathcal{T}} \rangle$, where \mathcal{K}_{P_i} is the set from Definition 3.1, and $I_{\mathcal{T}}$ is a \mathcal{T} -interpretation from Definition 6.1 above.

Definition 6.3 (Canonical Relation $\models_{\mathcal{T}}$). The canonical relation $\models_{\mathcal{T}}$ is defined as an evaluation function such that for any set of wff Γ and wff A , $\Gamma \models_{\mathcal{T}} A$ iff $T \in I_{\mathcal{T}}(A)$ if $T \in I_{\mathcal{T}}(\Gamma)$. Particularly, $\models_{\mathcal{T}} A$ iff $T \in I_{\mathcal{T}}(A)$.

Proposition 6.4 (The canonical P_i -model is a P_i -model). *For a P_i logic L , a P_i -theory \mathcal{T} , and canonical P_i -model \mathcal{M} , it follows that each \mathcal{M} is, indeed, a P_i -model, for $(1 \leq i \leq 4)$.*

Proof. It follows automatically by Definitions 3.1 and 6.2. Of course, there are four different cases, regarding each of the different falsity clauses. Let us additionally note that each propositional variable and wff is assigned $\{T\}$, $\{F\}$, or $\{T, F\}$, as \mathcal{T} is required to be a-consistent, but does not require to be complete or consistent in the classical sense. \square

At this point we introduce a number of lemmas that will help greatly with the completeness theorem. All these lemmas are directed towards creating a solid interpretation of the theory upon which the canonical model is created.

Lemma 6.5 (Theories and Double Negation). *For a P_i -logic L , a P_i -theory \mathcal{T} , and wff A it follows that $A \in \mathcal{T}$ iff $\neg\neg A \in \mathcal{T}$.*

Proof. From left to right it follows by A6 and the closure of \mathcal{T} by P_i -entailment. The proof from right to left uses A5 and the same closure. \square

Lemma 6.6 (Conjunction and Disjunction in Prime Theories). *For a P_i -logic L , a prime P_i -theory \mathcal{T} , and wff A and B it follows that (1) $A \wedge B \in \mathcal{T}$ iff $A \in \mathcal{T}$ and $B \in \mathcal{T}$; (2) $\neg(A \wedge B) \in \mathcal{T}$ iff $\neg A \in \mathcal{T}$ or $\neg B \in \mathcal{T}$; (3) $A \vee B \in \mathcal{T}$ iff $A \in \mathcal{T}$ or $B \in \mathcal{T}$; (4) $\neg(A \vee B) \in \mathcal{T}$ iff $\neg A \in \mathcal{T}$ and $\neg B \in \mathcal{T}$.*

Proof. (1), from left to right, follows by A2 and the closure of \mathcal{T} under P_i -entailment. From right to left follows by A2 and the closure \mathcal{T} under adjunction. (2), from left to right, follows by De Morgan (II) and the primeness and closure under P_i -entailment of \mathcal{T} . From right to left it follows by A3, De Morgan (I), and the closure of \mathcal{T} under P_i -entailment. (3), from left to right, follows by the primeness of \mathcal{T} . From right to left it follows by A3, and the closure of \mathcal{T} under P_i -entailment. (4), from left to right, follows by De Morgan (IV), A2, and the closure under P_i -entailment of \mathcal{T} . From right to left it follows by De Morgan (III) and the closure of \mathcal{T} under P_i -entailment. \square

Lemma 6.7 (Conditional in regular, prime and appropriately closed theories). *For any P_i -logic L , wff A and B , and a regular, prime and appropriately closed P_i -theory \mathcal{T} , it follows that $A \rightarrow B \in \mathcal{T}$ iff $A \notin \mathcal{T}$ or $\neg B \notin \mathcal{T}$ or $(\neg A \in \mathcal{T}$ and $B \in \mathcal{T})$*

Proof. From left to right there are two subcases. The first follows by the closure of the theories under R2 and the second by the closure under the Modus Tollens Rule. Both subcases use the closure of \mathcal{T} under P_i -entailment and its regularity. From right to left there are three subcases. The first follows by A9 and the primeness of \mathcal{T} ; the second follows by A10 and, again, the primeness of \mathcal{T} . The final third subcase follows by A11, R2 and the regularity of \mathcal{T} . \square

Lemma 6.8 (Negated conditional in regular, prime, a-consistent, and appropriately closed theories). *For the corresponding P_i -logic L , wff A and B , and a regular, prime, a-consistent, and appropriately closed P_i -theory \mathcal{T} , it follows that:*

$$(P_1) \neg(A \rightarrow B) \in \mathcal{T} \text{ iff } (A \in \mathcal{T} \text{ and } \neg B \in \mathcal{T}) \text{ or } (A \in \mathcal{T} \text{ and } \neg A \in \mathcal{T})$$

$$(P_2) \neg(A \rightarrow B) \in \mathcal{T} \text{ iff } [A \in \mathcal{T} \text{ and } (\neg A \in \mathcal{T} \text{ or } \neg B \in \mathcal{T})] \text{ or } (B \in \mathcal{T} \text{ and } \neg B \in \mathcal{T})$$

$$(P_3) \neg(A \rightarrow B) \in \mathcal{T} \text{ iff } \{A \in \mathcal{T} \text{ and } [B \notin \mathcal{T} \text{ or } (\neg A \in \mathcal{T} \text{ and } \neg B \in \mathcal{T})]\} \text{ or } \{ \neg B \in \mathcal{T} \text{ and } [\neg A \notin \mathcal{T} \text{ or } (A \notin \mathcal{T} \text{ and } B \in \mathcal{T})] \}$$

$$(P_4) \neg(A \rightarrow B) \in \mathcal{T} \text{ iff } \neg B \in \mathcal{T} \text{ and } [\neg A \notin \mathcal{T} \text{ or } (A \in \mathcal{T} \text{ and } B \notin \mathcal{T}) \text{ or } (A \notin \mathcal{T} \text{ and } B \in \mathcal{T})]$$

Proof. We show the proof of P_1 in detail and then propose a draft of the other proofs that the reader might follow if interested. To prove P_1 , from left to right, we proceed by reductio. Thus, we have (I) $\neg(A \rightarrow B) \in \mathcal{T}$ as hypothesis, and (II) $(A \notin \mathcal{T} \text{ or } \neg B \notin \mathcal{T})$ and $(A \notin \mathcal{T} \text{ or } \neg A \notin \mathcal{T})$ as reductio hypothesis. From (II), we have four (4) different cases: (III) $A \notin \mathcal{T}$; (IV) $A \notin \mathcal{T}$ and $\neg A \notin \mathcal{T}$; (V) $\neg B \notin \mathcal{T}$ and $A \notin \mathcal{T}$; (VI) $\neg B \notin \mathcal{T}$ and $\neg A \notin \mathcal{T}$. Let us start with case (III), we apply R8 and closure of \mathcal{T} on (I) to get $A \in \mathcal{T}$, which contradicts the reductio hypothesis. Regarding case (IV), we get $A \vee \neg A \in \mathcal{T}$ given that \mathcal{T} is also closed under R9 and, by the primeness of \mathcal{T} , we get $A \in \mathcal{T}$ or $\neg A \in \mathcal{T}$, which contradicts the reductio hypothesis. The case (V) follows by the fact that \mathcal{T} is closed under P_i -entailment; thus, using A12, we get $A \vee \neg B \in \mathcal{T}$. Then, by the primeness of \mathcal{T} , we get $A \in \mathcal{T}$ or $\neg B \in \mathcal{T}$, whence contradicting the reductio hypothesis. The last case (VI) is solved by the fact that \mathcal{T} is closed under R10, i.e., we get $\neg A \vee \neg B \in \mathcal{T}$ and, again by primeness of \mathcal{T} , $\neg A \in \mathcal{T}$ or $\neg B \in \mathcal{T}$, contradicting the reductio hypothesis. Now, to prove it from right to left, we have two different cases. The first one has (VII) $A \in \mathcal{T}$ and $\neg B \in \mathcal{T}$ as main hypothesis, while the second one has (VIII) $A \in \mathcal{T}$ and $\neg A \in \mathcal{T}$ as main hypothesis. Let us also suppose for both cases (IX) $\neg(A \rightarrow B) \notin \mathcal{T}$ as reductio hypothesis. Starting with case (VII), by A13 and the closure of \mathcal{T} under P_i -entailment, we have $\neg(A \rightarrow B) \in \mathcal{T}$, contradicting the reductio hypothesis (IX). Finally, case (VIII) is solved using A14 and the closure of \mathcal{T} under P_i -entailment, this is, we get $\neg(A \rightarrow B) \in \mathcal{T}$, again contradicting the reductio hypothesis (IX).

P_2 , from left to right, proceeding by reductio and after distributing the reductio hypothesis, has four (4) cases. The first case is proven using the closure of \mathcal{T} under R11 as well as primeness. The second one can be proven using primeness too, and also A12 and the closure of \mathcal{T} under P_i -entailment. The third case is based on the closure under R12 of \mathcal{T} and primeness. Finally, in the fourth case the primeness of \mathcal{T} as well as the closure of the theory under R10 are applied. On the other hand, we get (3) different cases from right to left –after proceeding by reductio and distributing the reductio hypothesis. For all three cases, the closure of \mathcal{T} under adjunction and P_i -entailment is used in addition to A14, A13 and A15, respectively for each case.

P_3 , from left to right. By proceeding as in P_2 , we distinguish nine (9) cases. The second, fifth, sixth, and ninth lead to a contradiction right on the reductio hypothesis. The first one is solved using A12, and the primeness and closure under P_i -entailment of \mathcal{T} . The third one uses A16 as well as the primeness and closure under P_i -entailment of \mathcal{T} . The fourth one follows

by A17, the closure of \mathcal{T} under P_i -entailment, and its primeness. The seventh uses A19, and the closure of \mathcal{T} under P_i -entailment. The eighth one utilizes A18, the closure under P_i -entailment of \mathcal{T} , and the fact that \mathcal{T} is a-consistent. From right to left, proceeding by reductio and after distributing the reductio hypothesis, there are four cases. These cases can be proven by the aforementioned properties of \mathcal{T} and A20, A21, A22 and A23, respectively for each case.

P_4 , from left to right. By following the previous method, we get five different cases. The second and fifth cases lead to a contradiction right away. The first and third case can be solved by applying the closure of \mathcal{T} under P_i -entailment, primeness and A24 for the former, A16 for the latter. The fourth case also needs the closure of \mathcal{T} under P_i -entailment, but also the a-consistency of \mathcal{T} and A18. Lastly, from right to left and proceeding by reductio, there are three (3) cases (after distributing the reductio hypothesis), each of them can be proven by using respectively A22, A20, A23, and the properties of \mathcal{T} . \square

Lemma 6.9 (\mathcal{T} -interpretation of the set of wff). *For a P_i -logic L , a regular, prime, a-consistent and appropriately closed P_i -theory \mathcal{T} , and a \mathcal{T} -interpretation $I_{\mathcal{T}}$, it follows that for each wff A :*

$$(I) T \in I_{\mathcal{T}} \text{ iff } A \in \mathcal{T}$$

$$(II) F \in I_{\mathcal{T}} \text{ iff } \neg A \in \mathcal{T}$$

Proof. The proof proceeds by induction over the complexity of the wff A . In particular, for wff B and C , we have the following cases: (a) A is a propositional variable; (b) A is of the type $B \vee C$; (c) A is of the type $B \wedge C$; (d) A is of the type $\neg A$; (e) A is of the type $B \rightarrow C$. Given the nature of the P_i -logics, case (e) will be divided in multiple subcases, in particular, one in which the conditional is assigned the value T , and one for each of the cases in which the conditional is assigned the value F ; this way we will have five (5) different subcases for case (e).

- (a) A is a propositional variable: It follows automatically by clauses (a) and (b) of Definition 6.1.
- (b) A is of the type $B \vee C$: (I) $T \in I_{\mathcal{T}}(B \vee C)$ iff $T \in I_{\mathcal{T}}(B)$ or $T \in I_{\mathcal{T}}(C)$, by Lemma 6.6 is $B \in \mathcal{T}$ or $C \in \mathcal{T}$ iff $B \vee C \in \mathcal{T}$. (II) $F \in I_{\mathcal{T}}(B \vee C)$ iff $F \in I_{\mathcal{T}}(B)$ and $F \in I_{\mathcal{T}}(C)$, by Lemma 6.6 is $\neg B \in \mathcal{T}$ and $\neg C \in \mathcal{T}$ iff $\neg(B \vee C) \in \mathcal{T}$.

- (c) A is of the type $B \wedge C$: (I) $T \in I_{\mathcal{T}}(B \wedge C)$ iff $T \in I_{\mathcal{T}}(B)$ and $T \in I_{\mathcal{T}}(C)$, by Lemma 6.6 is $B \in \mathcal{T}$ and $C \in \mathcal{T}$ iff $B \wedge C \in \mathcal{T}$. (II) $F \in I_{\mathcal{T}}(B \wedge C)$ iff $F \in I_{\mathcal{T}}(B)$ or $F \in I_{\mathcal{T}}(C)$, by Lemma 6.6 is $\neg B \in \mathcal{T}$ or $\neg C \in \mathcal{T}$ iff $\neg(B \wedge C) \in \mathcal{T}$.
- (d) A is of the type $\neg B$: (I) $T \in I_{\mathcal{T}}(\neg B)$ iff $F \in I_{\mathcal{T}}(B)$, by Lemma 6.5 is $\neg B \in \mathcal{T}$. (II) $F \in I_{\mathcal{T}}(\neg B)$ iff $T \in I_{\mathcal{T}}(B)$, by Lemma 6.5 is $B \in \mathcal{T}$ iff $\neg\neg B \in \mathcal{T}$.
- (e₁) A is of the type $B \rightarrow C$ and is assigned the value T : $T \in I_{\mathcal{T}}(A \rightarrow B)$ iff $T \notin I_{\mathcal{T}}(A)$ or $F \notin I_{\mathcal{T}}(B)$ or $(F \in I_{\mathcal{T}}(A) \ \& \ T \in I_{\mathcal{T}}(B))$, by Lemma 6.7 is $A \notin \mathcal{T}$ or $\neg B \notin \mathcal{T}$ or $(\neg A \in \mathcal{T} \ \& \ B \in \mathcal{T})$ iff $A \rightarrow B \in \mathcal{T}$.
- (e₂) A is of the type $B \rightarrow C$ and is assigned the value F in P_1 : $F \in I_{\mathcal{T}}(A \rightarrow B)$ iff $(T \in I_{\mathcal{T}}(A) \ \& \ F \in I_{\mathcal{T}}(B))$ or $(T \in I_{\mathcal{T}}(A) \ \& \ F \in I_{\mathcal{T}}(A))$, by Lemma 6.8 is $(A \in \mathcal{T} \ \& \ \neg B \in \mathcal{T})$ or $(A \in \mathcal{T} \ \& \ \neg A \in \mathcal{T})$ iff $\neg(A \rightarrow B) \in \mathcal{T}$.
- (e₃) A is of the type $B \rightarrow C$ and is assigned the value F in P_2 : $F \in I_{\mathcal{T}}(A \rightarrow B)$ iff $[T \in I_{\mathcal{T}}(A) \ \& \ (F \in I_{\mathcal{T}}(A) \ \text{or} \ F \in I_{\mathcal{T}}(B))]$ or $(T \in I_{\mathcal{T}}(B) \ \& \ F \in I_{\mathcal{T}}(B))$, by Lemma 6.8 is $[A \in \mathcal{T} \ \& \ (\neg A \in \mathcal{T} \ \text{or} \ \neg B \in \mathcal{T})]$ or $(B \in \mathcal{T} \ \& \ \neg B \in \mathcal{T})$ iff $\neg(A \rightarrow B) \in \mathcal{T}$.
- (e₄) A is of the type $B \rightarrow C$ and is assigned the value F in P_3 : $F \in I_{\mathcal{T}}(A \rightarrow B)$ iff $\{T \in I_{\mathcal{T}}(A) \ \& \ [T \notin I_{\mathcal{T}}(B) \ \text{or} \ (F \in I_{\mathcal{T}}(A) \ \& \ F \notin I_{\mathcal{T}}(B))]\}$ or $\{F \in I_{\mathcal{T}}(B) \ \& \ [F \notin I_{\mathcal{T}}(A) \ \text{or} \ (T \notin I_{\mathcal{T}}(A) \ \& \ T \in I_{\mathcal{T}}(B))]\}$, by Lemma 6.8 is $\{A \in \mathcal{T} \ \& \ [B \notin \mathcal{T} \ \text{or} \ (\neg A \in \mathcal{T} \ \& \ \neg B \in \mathcal{T})]\}$ or $\{\neg B \in \mathcal{T} \ \& \ [\neg A \notin \mathcal{T} \ \text{or} \ (A \notin \mathcal{T} \ \& \ B \in \mathcal{T})]\}$ iff $\neg(A \rightarrow B) \in \mathcal{T}$.
- (e₅) A is of the type $B \rightarrow C$ and is assigned the value F in P_4 : $F \in I_{\mathcal{T}}(A \rightarrow B)$ iff $F \in I_{\mathcal{T}}(B) \ \& \ [F \notin I_{\mathcal{T}}(A) \ \text{or} \ (T \in I_{\mathcal{T}}(A) \ \& \ T \notin I_{\mathcal{T}}(B))]$ or $(T \notin I_{\mathcal{T}}(A) \ \& \ T \in I_{\mathcal{T}}(B))$, by Lemma 6.8 is $\neg B \in \mathcal{T} \ \& \ [\neg A \notin \mathcal{T} \ \text{or} \ (A \in \mathcal{T} \ \& \ B \notin \mathcal{T})]$ or $(A \notin \mathcal{T} \ \& \ B \in \mathcal{T})$ iff $\neg(A \rightarrow B) \in \mathcal{T}$.

□

Definition 6.10 (The set $Cn\Gamma(L)$). The set of consequences in a logic L of a set of wff Γ , in symbols $Cn\Gamma(B_{E4})$, is defined as follows: $Cn\Gamma(L) := \{A \mid \Gamma \vdash_L A\}$.

Remark (The set of consequences is a regular theory). For a P_i -logic L , and a set of wff Γ , the set of consequences of Γ in L , $Cn\Gamma[L]$, is regular theory. On the one hand $Cn\Gamma[L]$ is closed under all the rules of L , thus being also appropriately closed, and contains all its theorems and, therefore, is closed under P_i -implication. As L is also closed under adjunction, it is obvious that $Cn\Gamma[L]$ is a regular theory.

With all of the above, we proceed to show the completeness of the P_i -logics w.r.t. the Belnap-Dunn and algebraic semantics that we have defined above.

Theorem 6.11 (Completeness of P_i -logics). *For any P_i -logic, set of wff Γ and wff A , it follows that:*

(I) *If $\Gamma \models_{P_i} A$, then $\Gamma \vdash_{P_i} A$*

(II) *If $\Gamma \models_{MP_i} A$, then $\Gamma \vdash_{P_i} A$*

Proof. For (I) we assume a set of wff Γ and wff A such that $\Gamma \not\vdash_{P_i} A$, then we will show that $\Gamma \not\models_{P_i} A$. Given the hypothesis we know that $A \notin Cn\Gamma[P_i]$, which is equivalent to $Cn\Gamma[P_i] \not\vdash_{P_i}^d A$; otherwise it would follow that $B_1 \wedge \dots \wedge B_n \vdash_{P_i} A$ for some wff $B_1, \dots, B_n \in Cn\Gamma[P_i]$ and thus leading to $A \in Cn\Gamma[P_i]$ and a contradiction. By Lemma 5.5 there is a maximal set \mathcal{T} such that $Cn\Gamma[P_i] \subseteq \mathcal{T}$ and, consequently, $\Gamma \in \mathcal{T}$ and $A \notin \mathcal{T}$. By Lemma 5.6 we know that \mathcal{T} is a prime theory and, furthermore, thanks to Remark 6 we know that \mathcal{T} is also a regular and an appropriately closed theory. Additionally, \mathcal{T} is also a-consistent as $A \notin \mathcal{T}$. Finally, we have a \mathcal{T} -interpretation $I_{\mathcal{T}}$ such that by Lemma 6.9 $T \in I_{\mathcal{T}}(\mathcal{T})$ but $T \notin I_{\mathcal{T}}(A)$. With all this, by Definition 6.3 and Proposition 6.4, we have $\Gamma \not\models_{I_{\mathcal{T}}} A$ and, subsequently, $\Gamma \not\models_{P_i} A$ by Definition 3.2 as we wanted to show. (II) follows automatically by part (I) and the coextensiveness of $\models_{P_i} A$ and $\Gamma \models_{MP_i} A$ shown in Theorem 4.4. \square

7 Properties of the P_i -logics

In this section we introduce a series of concepts that are of interest when characterizing logics related to RM3. In particular we will provide an overview of what paraconsistent logics, natural conditionals, variable sharing property, and quasi-relevance property are. Of course, we will also show how these concepts relate to the P_i -logics. First of all, we take a look at the idea of paraconsistent logics.

Definition 7.1 (Paraconsistent logics). Let \Vdash represent a consequence relation (may it be defined either semantically or proof-theoretically). Then, a logic L is paraconsistent if, for any wffs A, B , the rule ECQ (*E Contradictione Quodlibet*) $A, \neg A \Vdash B$ does not hold in L .

In other words, a logic is paraconsistent if theories built upon L are not necessarily trivial when a contradiction arises.

Proposition 7.2 (*Pi-logics are paraconsistent*). *Let L be a Pi-logic ($1 \leq i \leq 4$). Then, L is paraconsistent.*

Proof. Let M be the matrix determining the logic L and let p and q be distinct propositional variables. There is an M -interpretation I such that $I(p) = 1$ and $I(q) = 0$. Therefore, $\{p, \neg p\} \not\equiv_M q$, this is, ECQ does not hold in any Pi-logic. \square

Now, we address natural conditionals that we already mentioned in Section 1.

Definition 7.3 (Natural conditionals). Let \mathcal{L} be a propositional language with \rightarrow among its connectives and M be a matrix for \mathcal{L} where the values x and y represent the maximum and the infimum in K . Then, an f_{\rightarrow} -function on K defines a natural conditional if the following conditions are satisfied:

1. f_{\rightarrow} coincides with the classical conditional when restricted to the subset $\{x, y\}$ of K ;
2. f_{\rightarrow} satisfies Modus Ponens, that is, for any $a, b \in K$, if $a \rightarrow b \in \mathcal{D}$ and $a \in \mathcal{D}$, then $b \in \mathcal{D}$;
3. For any $a, b \in K$, $a \rightarrow b \in \mathcal{D}$ if $a \leq b$.

Proposition 7.4 (Natural conditionals in 3-valued matrices). *Let \mathcal{L} be a propositional language and M a 3-valued matrix for \mathcal{L} where K y \mathcal{D} are defined exactly as in Definition 2.4. Consider the 24 f_{\rightarrow} -functions defined in the following general table:*

\rightarrow	0	1	2
0	2	a_1	2
1	b	a_2	a_3
2	0	c	2

Where $a_i (1 \leq i \leq 3) \in \mathcal{D}$, $b \notin \mathcal{D}$ and $c \in K$. The set of functions of the previous truth-table is the set of all natural conditionals definable in M .

Proof. (1) The following cases are needed in order to fulfill clause 1 in the previous definition: $f_{\rightarrow}(0, 0) = 2$, $f_{\rightarrow}(0, 2) = 2$, $f_{\rightarrow}(2, 2) = 2$ and $f_{\rightarrow}(2, 0) = 0$. (2) Regarding clause 2 in the same definition, a non-designated value (i.e., 0) needs to be assigned to $f_{\rightarrow}(1, 0)$. (3) Finally, we also need $f_{\rightarrow}(0, 1) \in \mathcal{D}$, $f_{\rightarrow}(1, 1) \in \mathcal{D}$, $f_{\rightarrow}(1, 2) \in \mathcal{D}$ for the last condition to be guaranteed. \square

Corollary 7.5 (All M_{P_i} possess a natural conditional). *Each M_{P_i} possess one of the 24 natural conditionals reflected in Proposition 7.4.*

Proof. It is obvious given that each M_{P_i} is one of the 24 f_{\rightarrow} -functions defined in Proposition 7.4 and therefore all of them fulfill the requirements mentioned in Definition 7.3. \square

To finish our look at the properties of the P_i -logics we will investigate the cases of the variable sharing property and the quasi-relevance property in relation to them.

Definition 7.6 (Variable-sharing property). A logic L has the “variable-sharing property” (vsp) if for every theorem of L of the form $A \rightarrow B$, A and B share at least a propositional variable.

Proposition 7.7 (All P_i -logics lack the vsp). *Let L be a P_i -logic. Then, L lacks the vsp.*

Proof. Let M be the matrix determining the logic L . The proof is immediate since, for any distinct propositional variables p and q , the wff $\neg(p \rightarrow p) \rightarrow (q \rightarrow q)$ is M -valid, this is, the wff is valid in any P_i -logic (cf. [11]). \square

Definition 7.8 (Quasi-relevance property). A logic L has the “quasi-relevance property” (qrp) if for every theorem of L of the form $A \rightarrow B$, either A and B share at least a propositional variable or both $\neg A$ and B are also theorems of L .

Proposition 7.9 (Logics P_1 and P_2 possess the qrp). *Let L be either the logic P_1 or P_2 . Then, L possess the qrp.*

Proof. Let M be the matrix determining the logic L . By reductio, suppose that there are wffs A and B which have no propositional variable in common and such that $A \rightarrow B$ is M -valid but either $\neg A$ or B is not.

(i) Let us suppose that $\neg A$ is not M -valid. Then, there is an M -interpretation I such that $I(\neg A) = 0$ (i.e., $I(A) = 2$). Now, let I' be exactly as I except

that for each propositional variable p in B , $I'(p) = 1$. Then, clearly $I'(B) = 1$ since $\{1\}$ is closed under \rightarrow , \wedge , \vee and \neg , and $I'(A) = 2$, since A and B do not share propositional variables. Consequently, we get $I'(A \rightarrow B) = 0$, contradicting the M-validity of the wff $A \rightarrow B$.

(ii) Let us suppose now that B is not M-valid. Then, there is an M-interpretation I such that $I(B) = 0$. Let I' be exactly as I except that for each propositional variable p in A , $I'(p) = 1$. Similarly, as in case (i), we get $I'(A) = 1$ and $I'(B) = 0$. Then, $I'(A \rightarrow B) = 0$, contradicting the M-validity of $A \rightarrow B$. \square

Proposition 7.10 (Logics P_3 and P_4 lack the qrp). *Let L be either the logic P_3 or P_4 . Then, L lacks the qrp.*

Proof. Let M be the matrix determining the logic L . The proof is immediate since, for any distinct propositional variables p and q , the wff $\neg(p \rightarrow p) \rightarrow q$ is M-valid, that is, the wff is valid in both P_3 and P_4 (cf. [11]). \square

To conclude this section we will take a look at certain theses and their provability and validity in the different P_i -logics. This will help to give a more broad overview of what these Plumwood algebras can offer.

Proposition 7.11 (Some theses provable in the P_i -logics). *The following are provable in the P_i -logics: $A \rightarrow (A \rightarrow A)$; $\neg(A \wedge \neg A)$; $A \vee \neg A$; $(A \rightarrow B) \vee (B \rightarrow A)$; $A \vee (A \rightarrow B)$; $\neg(A \rightarrow A) \rightarrow (B \rightarrow B)$; $[(A \rightarrow A) \rightarrow B] \rightarrow B$; $[(A \rightarrow A) \wedge (B \rightarrow B)] \rightarrow C \rightarrow C$.*

Proof. All these theses are verified by any P_i -interpretation. Then, they are provable by the completeness theorem (cf. Theorem 6.11). \square

Thus, among the theses verified by any P_i -logics, we find *Mingle axiom*, *non-contradiction principle*, *excluded middle principle*, *Dummet axiom*, *specialized assertion axiom* and the characteristic axiom of the logic of entailment E .

Proposition 7.12 (Some theses provable in P_1 and P_2). *The following are provable in P_1 and P_2 : $(A \rightarrow \neg A) \rightarrow \neg A$; $(A \rightarrow B) \rightarrow (\neg A \vee B)$; $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$.*

Also, contraction axiom is provable in P_1 : $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$.

Proof. All these theses are verified by any P_1 -interpretation and/or P_2 -interpretation. Then, they are provable by the corresponding completeness theorems. \square

Consequently, P_1 and P_2 are the only ones that verify *reductio axiom*, and the classical interdefinitions between the conditional and the disjunction/conjunction. Moreover, P_1 verifies the contraction axiom.

Proposition 7.13 (Some wff not provable in the P_i -logics). *The following are not provable in the P_i -logics: $\neg A \rightarrow (B \rightarrow \neg A)$; $\neg A \rightarrow (A \rightarrow B)$; $A \rightarrow (\neg A \rightarrow B)$; $A \rightarrow (B \rightarrow A)$; $(A \rightarrow B) \rightarrow [C \rightarrow (A \rightarrow B)]$.*

Proof. All these wff are falsified in the matrices M_{P_i} . Then, they are not provable by the soundness theorem (cf. Theorem 4.5). \square

Interestingly, any of the P_i -logics falsified conditional paradoxes such as the various forms of *ex falso quodlibet* and *verum et quodlibet* showed in the previous proposition.

Proposition 7.14 (Some wff not provable in P_1 and P_2). *The following are not provable in P_1 and P_2 : $B \rightarrow (A \rightarrow A)$; $\neg(A \rightarrow A) \rightarrow B$.*

Proof. The previous wff are falsified in the matrices M_{P_1} and M_{P_2} . Then, they are not provable by the corresponding soundness theorems. \square

8 Conclusion

Throughout this article, we have coined the notion of Plumwood algebra to refer to a class of matrices that do not verify any of the theses Plumwood criticised in [15]. Then, in order to narrow the spectrum of logical systems, we focused on three-valued matrices. In particular, we put the spotlight on the variants of the well-known logic RM3, which is considered of great interest among weak relevance logics. More specifically, we have studied the conditional variants of RM3 and determined that there are only four systems among them that are indeed Plumwood algebras and, at the same time, possess what Tomova called 'natural conditionals'. We have axiomatized the logics characterized by the four resulting matrices and named them P_i -logics. Finally, we have provided a Belnap-Dunn semantics for all the four logics and proved that they are strongly sound and complete with respect to this semantics. It is of interest to note that, given the results shown in the paper, not all the P_i -logics behave in the same way: the theories for P_1 and P_2 need to be appropriately closed, while the theories for P_3 and P_4 need to be a-consistent but not appropriately closed (cf. Definition 5.1). Coincidentally,

P_1 and P_2 possess the quasi-relevance property, while P_3 and P_4 do not possess it. To which extent this coincidence is something more than pure fortuity is a question that remains open.

It is worth recalling that Plumwood argued against these four rules of proof regarding their unacceptable character as rules of relevance logics. In particular, they failed to preserve the property of ‘sufficiency of the premiss set for conclusion’, according to her. However, we have shown that the lack of these four rules is not enough for the resulting systems to have some other properties related to relevance, such as the vsp. In any case, the lack of the vsp in the case of the P_i -logics is not surprising since they are variants of RM3, a logic which lacks itself said property. Nevertheless, the fact that RM3 lacks the vsp has never been reason enough to claim that RM3 lacks interest within the relevance family. Thus, we believe that the same could be said about some of the logics that have been presented here. In particular, as it has been proved in the previous section of the article, logics P_1 and P_2 have the quasi-relevance property and therefore lack the most controversial paradoxes of implication. Furthermore, not only the latter couple of systems but the four of them are paraconsistent logics in the sense that they do not verify the explosion principle, as it has been shown in Proposition 5.

To end this article, we would like to note a couple of final remarks. On the one hand, since we decided to narrow the spectrum of considered matrices to the variants of RM3, one could naturally wonder which would be the case regarding other families of many-valued logics related in some way to relevance⁶. Thus, we believe that there is some room for further research on the topic of Plumwood algebras. On the other hand, some research relating Plumwood algebras to the Australian Plan may be of interest too. In particular, providing a ternary relational semantics to the P_i -logics. In this respect, we note that some workaround would provably be needed in order to find a replacement for Sufficing rule, the one Plumwood called Exported Syllogism in [15] (named TP1 at the beginning of this article).

⁶We are thinking, for example, in possible variants of four-valued logics of historical importance in the relevance project, such as Smiley’s expansion of B4 [3] or Brady’s BN4 [9]

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References

- [1] Alan Ross Anderson and Nuel D. Belnap. Enthymemes. The Journal of Philosophy, 58(23):713–723, 1961.
- [2] Alan Ross Anderson and Nuel D. Belnap. Entailment: The Logic of Relevance and Necessity. Vol. I. Princeton University Press, Princeton, N.J, 1976.
- [3] Alan Ross Anderson, Nuel D. Belnap Jr, and J. Michael Dunn. Entailment, Vol. II: The Logic of Relevance and Necessity. Princeton University Press, March 2017.
- [4] Alan Ross Anderson and Jr. Nuel D. Belnap. The Pure Calculus of Entailment. The Journal of Symbolic Logic, 27(1):19–52, 1962. Publisher: Association for Symbolic Logic.
- [5] Arnon Avron. Natural 3-valued logics—characterization and proof theory. The Journal of Symbolic Logic, 56(1):276–294, March 1991. Publisher: Cambridge University Press.
- [6] Nuel D. Belnap. How a Computer Should Think, pages 35–53. Springer International Publishing, Cham, 2019.
- [7] José Miguel Blanco. An implicative expansion of belnap’s four-valued matrix: A modal four-valued logic without strong modal lukasiewicz-type paradoxes. The Bulletin of Symbolic Logic, 26(3-4):297–298, 2020.
- [8] José Miguel Blanco. EF4, EF4-M and EF4-L: A companion to BN4 and two modal four-valued systems without strong Łukasiewicz-type modal paradoxes. Logic and Logical Philosophy, 31(1):75–104, 2022. Number: 1.

- [9] Ross T. Brady. Completeness Proofs for the Systems RM3 and BN4. Logique et Analyse, 25(97):9–32, 1982. Publisher: Peeters Publishers.
- [10] J. Michael Dunn. Algebraic completeness results for R-mingle and its extensions. The Journal of Symbolic Logic, 35(1):1–13, March 1970. Publisher: Cambridge University Press.
- [11] César González. Matest, February 2020. original-date: 2014-07-17T18:39:02Z.
- [12] George Edward Hughes and M. J. Cresswell. A New Introduction to Modal Logic. Psychology Press, 1996. Google-Books-ID: Dsn1xWNB4MEC.
- [13] Sandra M. López. Belnap-Dunn Semantics for the Variants of BN4 and E4 which Contain Routley and Meyer’s Logic B. Logic and Logical Philosophy, 31(1):29–56, 2022. Number: 1.
- [14] Edwin D. Mares. Relevant Logic: A Philosophical Interpretation. Cambridge University Press, Cambridge, 2008.
- [15] Valerie Plumwood. Some False Laws of Logic. Australasian Journal of Logic.
- [16] Gemma Robles. The quasi-relevant 3-valued logic RM3 and some of its sublogics lacking the variable-sharing property. Reports on Mathematical Logic, 2016(Number 51):105–131, September 2016. Number: Number 51 Publisher: Portal Czasopism Naukowych Ejournal.eu.
- [17] Gemma Robles and José M. Méndez. The Class of All Natural Implicative Expansions of Kleene’s Strong Logic Functionally Equivalent to Łukasiewicz’s 3-Valued Logic Ł3. Journal of Logic, Language and Information, 29(3):349–374, September 2020.
- [18] R. Routley and V. Routley. The Semantics of First Degree Entailment. Noûs, 6(4):335–359, 1972. Publisher: Wiley.
- [19] Richard Routley, Robert K. Meyer, Valerie Plumwood, and Ross T. Brady. Relevant Logics and Their Rivals, Vol. 1. Ridgeview, Atascadero, CA, 1982.

- [20] Yaroslav Shramko and Heinrich Wansing. Some Useful 16-Valued Logics: How a Computer Network Should Think. Journal of Philosophical Logic, 34(2):121–153, April 2005.
- [21] Natalya Tomova. A Lattice of Implicative Extensions of Regular Kleene’s Logics. Reports on Mathematical Logic, 2012(Number 47):173–182, August 2012. Number: Number 47 Publisher: Portal Czasopism Naukowych Ejournals.eu.