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Advanced numerical scheme and its convergence analysis for a class of two-point singular boundary value problems

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Abstract

In the past decades, many applications related to applied physics, physiology and astrophysics have been modelled using a class of two-point singular boundary value problems (SBVPs). In this article, a novel approach based on the shooting projection method and the Legendre wavelet operational matrix formulation for approximating a class of two-point SBVPs with Dirichlet and Neumann–Robin boundary conditions is proposed. For the new approach, an initial guess is postulated in contrast to the boundary conditions in the first step. The second step deals with the usage of the Legendre wavelet operational matrix method to solve the initial value problem (IVP). Further, the resulting solution of the IVP is utilized at the second endpoint of the domain of a differential equation in a shooting projection method to improve the initial condition. These two steps are repeated until the desired accuracy of the solution is achieved. To support the mathematical formulation, a detailed convergence analysis of the new approach is conducted. The new approach is tested against some existing methods such as various types of the variational iteration method, considering several numerical examples to which it provides high-quality solutions.

Keywords: Two-point singular boundary value problem; Shooting-projection method; Legendre wavelets; Operational matrix of integration; Convergence analysis

1. Introduction

Boundary value problems have been the subject of many studies in the areas of elastic beams [1], astrophysics [2–6] and chemistry [7] in the last few decades. Other relevant applications of these equations can be found in the thermal behavior of spherical gas clouds, the theory of thermionic current, Thomas–Fermi type equations, the flow in a circular cylindrical conduit, a nonlinear heat conduction model in a human head, and the mathematical model

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of spherical biocatalyst equation (see [8] and references therein). In this paper, the following class of two-point singular boundary value problems (SBVPs) is considered

$$-(x^\alpha y')' = x^\alpha f(x, y), x \in (0, 1), \tag{1}$$

subject to the boundary conditions

$$y(0) = c_1, \quad y(1) = c_2, \tag{2}$$

or to

$$y'(0) = c_3, \quad y(1) = c_4, \tag{3}$$

where $\alpha > 0, c_1, c_2, c_3$ and c_4 are real constants, and $f(x, y) \in C([0, 1] \times [0, \infty), [0, \infty))$. The singular behavior of the differential equation (1) at $x = 0$, poses difficulties for obtaining the solution of problems (1)–(2) or (1)–(3). Eq. (1) is known as the Emden–Fowler equation in the literature. Although the exact solutions are known in few cases, finding the exact solution of this problem is very difficult due to its complex nature. This leads researchers to develop numerical approaches for computing the solution.

There are several numerical techniques available to deal with SBVPs such as block methods [9–11], a spline method [12], a B-spline method [13], the variational iteration method [14], the homotopy analysis method (HAM) [15–17], the homotopy perturbation method (HPM) [16,18,19], collocation methods [20,21], a spectral method [22], the localized collocation method [23], singular boundary methods [24,25] or the Adomian decomposition method [26,27]. Chawla and Subramanian [12] have developed a new spline method to find a numerical solution of the differential equation (DE) (1) with the boundary conditions $y'(0) = 0, y(1) = A$. Çağlar et al. [13] have developed a method based on B-splines to deal with the differential equation (1) with the boundary conditions $y'(0) = 0, \alpha y(1) + \beta y'(1) = \gamma$. Singh et al. [28] have depicted a Quasi-Newton Method (in fact a variational iteration method) to solve problem (1) with $\alpha \geq 1$ and boundary conditions $y'(0) = A, a_1 y(1) + b_1 y'(1) = c_1$ or $y'(0) = 0, y(1) = \delta y(\eta), \ 0 < \eta < 1$. Roul and Biswal [17] used a recursive scheme based on a combination of an integral equation formalism and the homotopy analysis method for the numerical solution of (1) with the boundary conditions $y'(0) = 0, \mu y(1) + \delta y'(1) = B$.

All the merits and demerits of the aforementioned numerical methods are discussed below. The Adomian decomposition method is a demonstrable, sustainable and rapidly convergent method, but the major issue with this method is to construct the complicated Adomian polynomials. Turkyilmazoglu [29] proposed an analytical approach to determine the convergence of the homotopy series and found that for particular initial guesses the convergence-control parameter does not satisfy the HPM. The HAM does not guarantee the convergence of the approximate solution to the desired solution [30]. The spline and B-spline methods provide the approximate solution accurately, but at a high computational expense. In recent decades, wavelets have become a powerful tool for the numerical solution of BVPs. Wavelets are useful numerical techniques due to their compact support and orthogonality. Wavelets are called numerical microscopes due to their ability to represent functions at different levels of resolution. Wavelets based numerical techniques are available for (1) with the Dirichlet boundary conditions in (2), or the Neumann–Robin conditions in (3) [8,31,32].

Analytical results are also discussed in numerous research articles on two-point SBVPs (see [33–35] and the references therein). Chawla and Shivakumar [33] have stated that the SBVP (1) with $y'(0^+) = 0, y(1) = A$, has a unique solution if $u^* < k_1$ where $u^* = \sup \frac{\partial f}{\partial y}$ and k_1 is the first positive zero of $J_{(\alpha-1)/2}(\sqrt{k})$, with $J_\nu(z)$ the Bessel's function of first kind and order ν. Pandey [34] has discussed an existence theorem about the uniqueness of the SBVPs $-(p(x)y')' = p(x)f(x, y), 0 < x \leq b, \lim_{x\to 0^+} y'(0) = 0, y(b) = B$.

This article aims at introducing an advanced numerical approach based on a combination of a shooting projection method and the Legendre wavelet operational matrix method for approximating two-point SBVPs. The basic idea of any shooting method for solving SBVPs is to replace a boundary condition (BC) with an initial condition (IC), hence transforming it into an initial value problem (IVP). In the literature, several numerical techniques are available to find numerical solutions of IVPs (see [20,36–38] and references therein).

In this work, the Legendre wavelet operational matrix approach is adapted and implemented to solve the IVP due to its orthogonality and mutual spectral accuracy. The value provided by this approach is used at the right end of the domain in (1) through an iterative formula, namely the shooting projection method in [39], to get improved initial conditions. To find the numerical solution of these IVPs, the Legendre wavelet operational matrix of integration method is implemented (details can be found in [32,40]). The product operation of the Legendre wavelets, the operational matrix of integration, and the integer power of a function are computed on the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ which decreases the computational complexity, hence increasing the efficiency of the method. The Legendre wavelet operational matrix method converts the IVP into an equivalent system of a nonlinear algebraic equations that speeds up the process of obtaining an approximate solution of the IVP.

The rest of the work is organized as follows. A description of the formulations of the shooting projection method and that of the Legendre wavelet operational matrix method are provided in Sections 2.1 and 2.2, respectively. Section 3 is devoted to provide a new methodology to tackle SBVPs (1)–(3) and the convergence analysis is discussed in Section 4. In Section 5, a number of examples are performed to assess the applicability and efficiency of the proposed technique. Finally, Section 6 is used to present the conclusions of the current work.

2. Numerical approaches

The Legendre wavelet operational matrix method yields very accurate approximations for IVPs. So, to handle a two-point singular BVP, we propose to combine two methods: (a) the shooting projection method, and (b) the Legendre wavelet operational matrix method. In the following lines it is outlined how to apply those techniques to the considered problem.

2.1. The shooting-projection method

Consider the two-point SBVP (1)–(2) or (1)–(3). The rationale behind the shooting method is to transform a SBVP into an IVP. So after changing the boundary conditions (2) and (3) by the initial conditions

$$y(0) = c_1, \quad y'(0) = \mu_0, \tag{4}$$

or

$$y(0) = \eta_0, \quad y'(0) = c_3, \tag{5}$$

respectively, we assume that the transformed problems have the same solution as the SBVP (1)–(2) or (1)–(3). Here, μ_0 and η_0 are two unknowns.

Thus, the main objective is to find the values of the unknowns μ_0 and η_0, such that the transformed IVP has the same solution as the corresponding SBVP. So, to find the values of μ_0 and η_0, we have developed two iterative formulas in the next two sub-subsections, corresponding to the boundary conditions (2) and (3), respectively.

2.1.1. Iterative formula (I)

Let γ_0 be the initial guess of $y'(0)$ corresponding to the SBVP (1)–(2), and $y(x, \gamma_0)$ be the obtained solution of the IVP, satisfying $y(0, \gamma_0) = c_1$ and $y'(0, \gamma_0) = \gamma_0$. Usually, the solution $y(x, \gamma_0)$ does not satisfy the second BC, that is, there is a deviation $e(\gamma_0)$ given by

$$e(\gamma_0) = y(1, \gamma_0) - c_2.$$

It is clear that the solution of the IVP will also be the solution of the SBVP (1)–(2) if $e(\gamma_0) = 0$, that is, $\gamma_0 = \mu_0$. This means that μ_0 should be a root of $e(\gamma_0)$. So, we require a root finding procedure to solve $e(\gamma_0) = 0$.

Let us consider an auxiliary function y^*, a non-classical H^1-projection of the solution $y(x, \gamma_0)$ of the IVP, which satisfies (2), and minimizes the H^1 semi-norm of the difference between $y(x)$ and $y^*(x)$, that is, it minimizes the functional

$$F(y^*) = \int_0^1 (y^{*\prime} - y')^2 dx. \tag{6}$$

Therefore, it holds the following Euler–Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial (y^{*\prime} - y')^2}{\partial y^{*\prime}}\right) - \frac{\partial (y^{*\prime} - y')^2}{\partial y^*} = 0, \quad x \in (0, 1). \tag{7}$$

Differentiating (7), we get

$$y^{*\prime\prime} = y''. \tag{8}$$

Integrating (8), from 0 to x leads to

$$y^{*\prime}(x) - y'(x) = \gamma_0^* - \gamma_0. \tag{9}$$

Here $y'(0) = \gamma_0$ and $y^{*\prime}(0) = \gamma_0^*$. Multiplying (9) by x^α, we get

$$x^\alpha(y^{*\prime}(x) - y'(x)) = x^\alpha(\gamma_0^* - \gamma_0). \tag{10}$$

$\gamma_0^* - \gamma_0 \to 0$ as γ_0^* and γ_0 approach μ_0. Thus, from Eq. (10), the following expression is obtained:

$$x^\alpha y^{*\prime}(x) = x^\alpha y'(x), \tag{11}$$

which implies that

$$(x^\alpha y^{*\prime}(x))' = (x^\alpha y'(x))'. \tag{12}$$

Since the solution y of the IVP must satisfy the differential equation (1), then $(x^\alpha y'(x))'$ can be replaced with $-x^\alpha f(x, y)$ in Eq. (12). Further, the application of the Taylor series expansion to the function $f(x, y)$ around 'y^*' provides

$$-(x^\alpha y^{*\prime})' = x^\alpha \left(f(x, y^*) + f_y(x, y)|_{y=y^*}(y - y^*) + \frac{1}{2} f_{yy}(x, y)|_{y=y^*}(y - y^*)^2 + \cdots \right). \tag{13}$$

Since y^* is close to y as γ_0 approaches μ_0, y^* satisfies the differential equation (1) approximately and, therefore y^* is an approximate solution of the SBVP (1)–(2).

To get an improved initial condition corresponding to the SBVP (1)–(2), integrating (9) from 0 to 1, we have

$$\gamma_0^* = \gamma_0 - e(\gamma_0). \tag{14}$$

This is a shooting-projection iterative formula to improve the remaining initial guess of the IVP corresponding to the SBVP (1)–(2).

2.1.2. Iterative formula (II)

In a similar fashion, let λ_0 be the initial guess of $y(0)$, corresponding to the SBVP (1)–(3), and $y(x, \lambda_0)$ be the obtained solution of the IVP, satisfying $y(0, \lambda_0) = \lambda_0$, $y'(0) = c_3$. In most cases, the solution $y(x, \lambda_0)$ does not satisfy the second boundary condition, that is, there is a deviation given by

$$\bar{e}(\lambda_0) = y(1, \lambda_0) - c_4.$$

Clearly, the solution of the IVP will also be the solution of the SBVP (1)–(3), if $\bar{e}(\lambda_0) = 0$, that is, $\lambda_0 = \eta_0$. This means that η_0 should be a root of $\bar{e}(\lambda_0) = 0$. So, we require a root finding procedure to solve $\bar{e}(\lambda_0) = 0$.

Let us consider an auxiliary function y^*, a non-classical H^1-projection of the solution $y(x, \lambda_0)$ of the IVP, which satisfies the first BC in (3), and minimizes the H^1 semi-norm of the difference between $y(x)$ and $y^*(x)$, that is, it minimizes the functional

$$F(y^*) = \int_0^1 (y^{*\prime} - y')^2 dx. \tag{15}$$

Therefore, it holds the following Euler–Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial(y^{*\prime} - y')^2}{\partial y^{*\prime}}\right) - \frac{\partial(y^{*\prime} - y')^2}{\partial y^*} = 0, x \in (0, 1). \tag{16}$$

Differentiating (16), we get

$$y^{*\prime\prime} = y''. \tag{17}$$

Integrating (17) from 0 to x leads to

$$y^{*\prime}(x) - c_3 = y'(x) - c_3, \tag{18}$$

which implies

$$y^{*\prime}(x) = y'(x). \tag{19}$$

Multiplying (19) by x^α, we get

$$x^\alpha {y^*}'(x) = x^\alpha y'(x), \tag{20}$$

and differentiating this we arrive at

$$(x^\alpha {y^*}'(x))' = (x^\alpha y'(x))'. \tag{21}$$

Since the solution y of the IVP must satisfy the differential equation (1), then $(x^\alpha y'(x))'$ can be replaced with $-x^\alpha f(x, y)$ in Eq. (21). Further, the application of the Taylor series expansion to the function $f(x, y)$ around 'y^*' provides

$$-(x^\alpha {y^*}')' = x^\alpha \left(f(x, y^*) + f_y(x, y)|_{y=y^*}(y - y^*) + \frac{1}{2} f_{yy}(x, y)|_{y=y^*}(y - y^*)^2 + \cdots \right). \tag{22}$$

Since y^* is close to y as λ_0 approaches η_0, then y^* satisfies the differential equation (1) approximately and y^* is an approximate solution of the SBVP (1)–(3). To get an improved initial condition corresponding to the SBVP (1)–(3), integrating (19) from 0 to 1, we have

$$\lambda_0^* = \lambda_0 - \bar{e}(\lambda_0). \tag{23}$$

This is a shooting-projection iterative formula to improve the remaining initial guess of the IVP corresponding to the SBVP (1)–(3).

2.2. Legendre wavelet operational matrix method (LWOMM)

The Legendre wavelet approach and its comprehensive properties are given in [32,40]. Here we briefly introduce the Legendre wavelet method and its operational matrix of integration. Yousefi [32] has defined a family of Legendre wavelets in terms of Legendre polynomials (P_m) for any positive integers M and k, as

$$\psi_{nm}(x) = \begin{cases} \sqrt{2^k(m + \frac{1}{2})} P_m(2^k x - \hat{n}), & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}} \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Here, $\hat{n} = 2n - 1$, $n = 1, 2, 3, \ldots, 2^{(k-1)}$ and $m = 0, 1, 2, 3, \ldots, M - 1$, M is the maximum possible order of the Legendre polynomial. Particularly, for $M = 3$ and $k = 2$, we obtain six wavelet basis functions on the interval $[0, 1]$, defined by

$$\begin{aligned} \psi_{10}(x) &= \sqrt{2}, \\ \psi_{11}(x) &= \sqrt{6}(4x - 1), \quad 0 \le x \le \frac{1}{2} \\ \psi_{12}(x) &= \frac{\sqrt{10}}{2} \left(3(4x - 1)^2 - 1 \right), \end{aligned} \tag{25}$$

$$\begin{aligned} \psi_{20}(x) &= \sqrt{2}, \\ \psi_{21}(x) &= \sqrt{6}(4x - 3), \quad \frac{1}{2} \le x \le 1 \\ \psi_{22}(x) &= \frac{\sqrt{10}}{2} \left(3(4x - 3)^2 - 1 \right). \end{aligned} \tag{26}$$

Every function $f \in L_2[0, 1]$ can be written in the form of an infinite series of Legendre wavelets using the theory of multi-resolution [40] as follows

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{27}$$

where

$$c_{nm} = \langle f(x), \psi_{nm}(x) \rangle. \tag{28}$$

Here, $\langle .,. \rangle$ represents the inner product on $L_2[0,1]$. A suitable truncation of (27) yields

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\psi(x), \tag{29}$$

where C and $\psi(x)$ are vectors given by

$$\begin{aligned} C &= [c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}]^T, \\ \psi(x) &= [\psi_{10}(x), \psi_{11}(x), \dots, \psi_{1M-1}(x), \psi_{20}(x), \dots, \psi_{2M-1}(x), \dots, \psi_{2^{k-1}0}(x), \dots, \psi_{2^{k-1}M-1}(x)]^T. \end{aligned}$$

For our convenience, we rewrite the right hand side in (29) as

$$C^T\psi(x) = \sum_{n=1}^{2^{k-1}} C_n^T\psi_n(x), \tag{30}$$

where

$$\begin{aligned} C_n &= [c_{n0}, c_{n1}, \dots, c_{nM-1}]^T, \\ \psi_n(x) &= [\psi_{n0}(x), \psi_{n1}(x), \dots, \psi_{nM-1}(x)]^T, \end{aligned}$$

in order to facilitate the computations with the Lagrange operational matrix on the sub-intervals $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$.

Now, the Legendre wavelet operational matrix of integration [40] on $[0,1)$ is described. In order to get the matrix of integration, the interval $[0,1)$ is divided into subintervals $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ of equal length. The matrix of integration is the same on each subinterval. This property of the Legendre wavelet operational matrix of integration makes the method highly efficient. For $M=3$ and $k=2$, the operational matrix by integrating the wavelet vector $\psi_1(x) = [\psi_{10}(x), \psi_{11}(x), \psi_{12}(x)]^T$ from 0 to x with $x \in \left[0, \frac{1}{2}\right)$ is given by

$$\int_0^x \psi_1(x)dx \approx B\psi_1(x), \tag{31}$$

and on integrating the wavelet vector $\psi_2(x) = [\psi_{20}(x), \psi_{21}(x), \psi_{22}(x)]^T$ from $\frac{1}{2}$ to x with $x \in \left[\frac{1}{2}, 1\right)$, we have

$$\int_{1/2}^x \psi_2(x)dx \approx B\psi_2(x), \tag{32}$$

where

$$B = \frac{1}{4}\begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}\sqrt{5}}{35} \\ 0 & -\frac{\sqrt{3}\sqrt{5}}{35} & 0 \end{bmatrix}. \tag{33}$$

Razzaghi and Yousefi [41] have constructed the Legendre wavelet operational matrix of integration on the interval $[0,1]$. Hence for $x \in [0,1)$, we have

$$\int_0^x \psi(x)dx \approx P\psi(x), \tag{34}$$

where

$$\psi(x) = [\psi_{10}(x), \psi_{11}(x), \psi_{12}(x), \psi_{20}(x), \psi_{21}(x), \psi_{22}(x)]^T, \tag{35}$$

and

$$P=\frac{1}{2^2}\begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & 2 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}\,\sqrt{5}}{3\,5} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}\,\sqrt{5}}{3\,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}\,\sqrt{5}}{3\,5} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}\,\sqrt{5}}{3\,5} & 0 \end{bmatrix}. \tag{36}$$

Note that P can be written as

$$P=\begin{bmatrix} B & F \\ O & B \end{bmatrix},$$

where B is defined in (33), O is a null matrix of order three, and

$$F=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general form of the matrix P adopts a $2^{k-1}\times 2^{k-1}$ diagonal block structure given by

$$P=\frac{1}{2^k}\begin{bmatrix} B & F & F & \cdots & F \\ O & B & F & \cdots & F \\ \vdots & O & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & F \\ O & O & \cdots & O & B \end{bmatrix},$$

where B is a $M\times M$ tridiagonal matrix given by

$$B=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}\,\sqrt{5}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}\,\sqrt{5}} & 0 & \frac{1}{\sqrt{5}\,\sqrt{7}} & \ddots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}\,\sqrt{7}} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{\sqrt{(2M-3)}\,\sqrt{(2M-1)}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{\sqrt{(2M-3)}\,\sqrt{(2M-1)}} & 0 \end{bmatrix},$$

$$F=\begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \text{and} \quad O=\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Here F and O are square matrices of order M. The matrices B and P are operational matrices of integration on the intervals $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ and $[0, 1)$, respectively.

3. Methodology

To get the solution of the SBVPs (1)–(3), we first guess the corresponding initial condition instead of the boundary conditions (2) or (3), respectively. Consequently, a series of IVPs corresponding to each SBVP is obtained. Further, the Legendre wavelet operational matrix method is used for finding the solution of these IVPs. To implement this method, first divide the interval $[0, 1)$ into subintervals $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ for any positive integer k, where $n = 1, \ldots, 2^{k-1}$. Let us assume that $y_1(x)$ is the solution of the IVP on $[0, \frac{1}{2^{k-1}})$, such that

$$y_1''(x) \approx C_1^T \psi_1(x). \tag{37}$$

On integrating Eq. (37) twice from 0 to x, we have

$$y_1'(x) \approx y_1'(0) + C_1^T B\psi_1(x) = D^T \psi_1(x)$$

and

$$y_1(x) \approx y_1(0) + y_1'(0)x + C_1^T B^2\psi_1(x). \tag{38}$$

The differential equation (1) can be written as

$$xy'' + \alpha y' + xf(x, y) = 0. \tag{39}$$

The functions x, α and $f(x, y)$ can be approximated in terms of the Legendre wavelet vector $\psi_1(x)$ on the sub-interval $[0, \frac{1}{2^{k-1}})$, as

$$\begin{aligned} x &\approx g^T \psi_1(x), \\ \alpha &\approx r^T \psi_1(x), \\ f(x, y) &\approx F^T \psi_1(x), \end{aligned}$$

where g, r and F are $(M \times 1)$-matrices. Thus, the product of functions can be approximated using the product operation of Legendre wavelet vector functions [40], as follows

$$\begin{aligned} xy'' &\approx g^T \psi_1(x)\psi_1(x)^T C_1 = \psi_1(x)^T GC_1, \\ \alpha y' &\approx r^T \psi_1(x)\psi_1(x)^T D = \psi_1(x)^T RD, \\ xf(x, y) &\approx g^T \psi_1(x)\psi_1(x)^T F = \psi_1(x)^T GF, \end{aligned} \tag{40}$$

where G and R are $(M \times M)$-matrices. Consequently, the matrix form of Eq. (39) using (40) is given by

$$GC_1 + RD + GF = 0. \tag{41}$$

Here, Eq. (41) is a system of algebraic equations in the variables $C_1 = [c_{10}, c_{11}, \ldots, c_{1M-1}]^T$. The solution of the algebraic Eq. (41) for C_1 provides the approximate solution $y_1(x)$ of the IVP (1)–(3) on the subinterval $\left[0, \frac{1}{2^{k-1}}\right)$. Thus, we have the approximate solution on $\left[0, \frac{1}{2^{k-1}}\right)$

$$y_1(x) = y_1(0) + y_1'(0)x + \hat{C}_1^T B\psi_0(x), \tag{42}$$

where $\hat{C}_1$ is the approximate value of C_1 after solving (41).

Similarly, the solution $y_n(x)$ on the sub-interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ using the initial conditions $y_n\left(\frac{n-1}{2^{k-1}}\right) = y_{n-1}\left(\frac{n-1}{2^{k-1}} - \Delta x\right)$, $y_n'\left(\frac{n-1}{2^{k-1}}\right) = \left(\frac{y_{n-1}\left(\frac{n-1}{2^{k-1}} - \Delta x\right) - y_{n-1}\left(\frac{n-1}{2^{k-1}} - 2\Delta x\right)}{\Delta x}\right)$, $n = 2, 3, \ldots, 2^{k-1}$, is obtained. Here Δx is arbitrary, mostly we have used $\Delta x = 1.0 \times 10^{-6}$. Consequently, combining the solutions $y_1(x), y_2(x), \ldots, y_{2^{k-1}}(x)$, yields a piecewise approximate solution $y(M, k)(x)$ of the corresponding IVP on $[0, 1]$.

The value of the obtained solution at $x = 1$ is used in the iterative formula of the shooting projection method (14) or (23) to get the improved initial condition. At each iteration of the shooting projection method, there is an improved IVP. We solve this series of IVPs using the Legendre wavelet operational matrix method. The schematic representation of the newly proposed algorithm is depicted in Fig. 1.

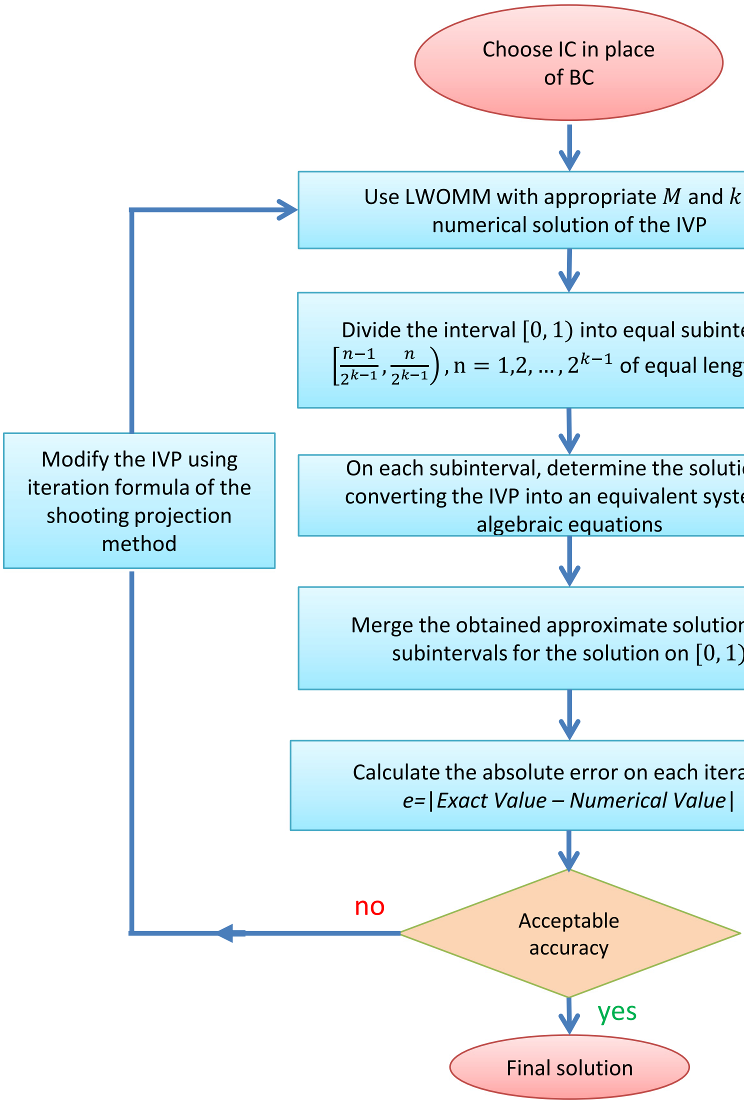


Fig. 1. Schematic representation of the new approach.

4. Convergence analysis

In this section, the convergence analysis of the proposed approach is conducted in order to support the theoretical formulation.

Theorem 4.1. *Suppose that $y(x) \in L_2[0, 1]$ has a bounded second order derivative, that is, $\exists\ K \in R$ such that $|y''(x)| \le K$. Then, the following bound for the error norm holds*

$$\begin{aligned} \|e(M,k)(x)\| &= \sup_{x\in[0,1]} |y(x) - y(M,k)(x)| \\ &\le \frac{\sqrt{10}K}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}. \end{aligned}$$

Proof. The error is defined as

$$
\begin{aligned}
|e(M,k)(x)| &= |y(x) - y(M,k)(x)| \\
&= \left| \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}\,\Psi_{nm}(x) \right|,
\end{aligned}
$$

where

$$
y(M,k)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\,\Psi_{nm}(x).
$$

Hence, it is

$$
\begin{aligned}
\|e(M,k)\|^2 &= \int_0^1 \left\langle \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}\,\Psi_{nm}(x), \sum_{p=2^{k-1}+1}^{\infty} \sum_{q=M}^{\infty} c_{pq}\,\Psi_{pq}(x) \right\rangle dx \\
&= \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \sum_{p=2^{k-1}+1}^{\infty} \sum_{q=M}^{\infty} c_{nm} c_{pq} \int_0^1 \Psi_{nm}(x)\,\Psi_{pq}(x)dx \qquad (43) \\
&\le \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2, \qquad (44)
\end{aligned}
$$

where

$$
\begin{aligned}
c_{nm} &= \int_0^1 y(x)\,\Psi_{nm}(x)dx \\
&= \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} y(x)\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} L_m(2^k x - \hat{n})dx \\
&= \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} \int_{(n-1)/2^{k-1}}^{n/2^{k-1}} y(x)L_m(2^k x - \hat{n})dx.
\end{aligned}
$$

Making the substitution $2^k x - \hat{n} = t$ in the above integral, we get

$$
c_{nm} = \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} \int_{-1}^{1} y\left(\frac{t+\hat{n}}{2^k}\right) L_m(t)\frac{dt}{2^k} = \frac{\sqrt{m+\frac{1}{2}}}{2^{\frac{k}{2}}} \int_{-1}^{1} y\left(\frac{t+\hat{n}}{2^k}\right) L_m(t)dt.
$$

Since

$$
(2\,m+1)L_m(t) = \left(L'_{m+1}(t) - L'_{m-1}(t)\right), \qquad (45)
$$

it provides

$$
\begin{aligned}
c_{nm} &= \frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}} \int_{-1}^{1} y\left(\frac{t+\hat{n}}{2^k}\right) d(L_{m+1}(t) - L_{m-1}(t)) \\
&= \frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}} \left(y\left(\frac{t+\hat{n}}{2^k}\right)(L_{m+1}(t) - L_{m-1}(t))\Big|_{-1}^{1} \right) \\
&\quad - \frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}} \left(\int_{-1}^{1} y'\left(\frac{t+\hat{n}}{2^k}\right) \frac{1}{2^k}(L_{m+1}(t) - L_{m-1}(t))dt \right) \\
&= -\frac{1}{2^{\frac{3k}{2}+1}\sqrt{m+\frac{1}{2}}} \left(\int_{-1}^{1} y'\left(\frac{t+\hat{n}}{2^k}\right)(L_{m+1}(t) - L_{m-1}(t))dt \right).
\end{aligned}
$$

Using Eq. (45), we have

$$c_{nm} = -\frac{1}{2^{\frac{3k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} y'\left(\frac{t+\hat{n}}{2^k}\right) d\left(\frac{L_{m+2}(t)-L_m(t)}{2\ m+3} - \frac{L_m(t)-L_{m-2}(t)}{2\ m-1}\right)\right).$$

Proceeding similarly as before, we have

$$\begin{aligned} c_{nm} &= \frac{1}{2^{\frac{5k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} y''\left(\frac{t+\hat{n}}{2^k}\right)\left(\frac{L_{m+2}(t)-L_m(t)}{2\ m+3} - \frac{L_m(t)-L_{m-2}(t)}{2\ m-1}\right)dt\right) \\ &= \frac{1}{2^{\frac{5k}{2}+1}(2\ m+3)(2m-1)\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} y''\left(\frac{t+\hat{n}}{2^k}\right) T_m(t)dt\right), \end{aligned}$$

where $T_m(t) = (2\ m-1)L_{m+2}(t) - 2(2\ m+1)L_m(t) + (2\ m+3)L_{m-2}(t)$ and hence,

$$\begin{aligned} |c_{nm}| &\leq \frac{1}{2^{\frac{5k}{2}+1}(2\ m+3)(2\ m-1)\sqrt{m+\frac{1}{2}}}\int_{-1}^{1}\left|y''\left(\frac{t+\hat{n}}{2^k}\right)\right||T_m(t)|dt \\ &\leq \frac{4\sqrt{6}K}{2^{\frac{5k}{2}}(2\ m-3)^2}, \end{aligned} \tag{46}$$

where we have used the inequality shown in [42], namely

$$|T_m(t)| \leq \sqrt{24}\frac{(2\ m+3)}{\sqrt{(2\ m-3)}}.$$

Since $n \leq 2^k$, from the inequality (46) we obtain that

$$|c_{nm}| \leq \frac{4\sqrt{6}K}{n^{\frac{5}{2}}(2\ m-3)^2}. \tag{47}$$

Thus, we arrive at

$$\begin{aligned} \|e(M,k)\|^2 &\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{96K^2}{n^5(2\ m-3)^4} \\ &= 96K^2\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(2\ m-3)^4} \qquad (48) \\ &\leq \frac{10K^2}{(2^{k-1}+1)^4(M-\frac{3}{2})^3}, \qquad (49) \end{aligned}$$

and hence

$$\|e(M,k)\| \leq \frac{\sqrt{10}K}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}. \quad \square \tag{50}$$

Remark. From Theorem 4.1 it is clear that the approximate solution converges to the exact solution of the IVP when M and k tend to ∞.

The iteration formulae represented by Eqs. (14) and (23) are also known as fixed point iteration formulae. The iteration formula in (14) will converge [39] to the desired value when $|d\gamma_0^*/d\gamma_0| < 1$ in some neighborhood of the root of $\bar{e}(\gamma_0) = 0$, that is, when

$$|1-m| < 1, \tag{51}$$

which implies

$$0 < m < 2, \tag{52}$$

where m is the slope of $\bar{e}(\gamma_0)$ at the root μ_0. For the fixed point iteration corresponding to (23) we obtain a similar result, now being m the slope of $\bar{e}(\lambda_0)$ at η_0.

Table 1
Comparison of proposed approach (Num) with exact solution (Exact) of Example 5.1.

t	Exact	Num(4)	δ	Num(8)	δ	Method [28]	δ [28]	Method [43]	δ [43]
0.0	0.3166	0.3296	0.0409	0.3166	1.2×10^{-4}	0.3317	0.0474	0.3181	0.0044
0.1	0.3132	0.3261	0.0412	0.3132	1.1×10^{-4}	0.3282	0.0478	0.3146	0.0044
0.2	0.3030	0.3158	0.0421	0.3029	9.7×10^{-5}	0.3178	0.0488	0.3044	0.0045
0.3	0.2860	0.2986	0.0439	0.2860	7.5×10^{-5}	0.3005	0.0505	0.2873	0.0046
0.4	0.2625	0.2748	0.0467	0.2625	6.2×10^{-5}	0.2764	0.0529	0.2637	0.0048
0.5	0.2326	0.2445	0.0511	0.2326	8.0×10^{-5}	0.2456	0.0558	0.2338	0.0049
0.6	0.1968	0.2082	0.0581	0.1968	8.9×10^{-5}	0.2085	0.0593	0.1977	0.0048
0.7	0.1552	0.1662	0.0703	0.1552	4.8×10^{-5}	0.1650	0.0632	0.1559	0.0046
0.8	0.1083	0.1187	0.0955	0.1083	3.6×10^{-5}	0.1156	0.0675	0.1087	0.0041
0.9	0.0564	0.0662	0.1732	0.0564	2.3×10^{-4}	0.0605	0.0722	0.0566	0.0032

As we have discussed in Section 2 if $\gamma_0 \to \mu_0$ then $y(x, \gamma_0)$ converges to $y(x)$, and if $\lambda_0 \to \eta_0$ then $y(x, \gamma_0)$ converges to $y(x)$, corresponding the SBVPs (1)–(2) and (1)–(3), respectively, under the condition (52). Hence, the use of the shooting projection method in combination with the Legendre wavelet operational matrix approach ensures the convergence of the numerical approximation to the corresponding exact solutions of the SBVPs (1)–(2) and (1)–(3).

5. Numerical testing and discussion

In order to show the accuracy and efficiency of the new approach for solving the considered SBVPs, five numerical examples have been tested and compared against existing methods. The qualitative comparison of the numerical solutions is done against the exact solution using 2D graphs. In order to show the deviation of the numerical results from the exact solution quantitatively, the absolute error norm $e = |Exact - Numerical|$, the relative error norm $\delta = \left|\frac{Exact-Numerical}{Exact}\right|$, and $L_\infty = \max_{x\in[0,1]} |Exact - Numerical|$ have been calculated for all tests.

Example 5.1. Consider a nonlinear second order SBVP arising in physiology [44]

$$-(xy')' = xe^y, \tag{53}$$

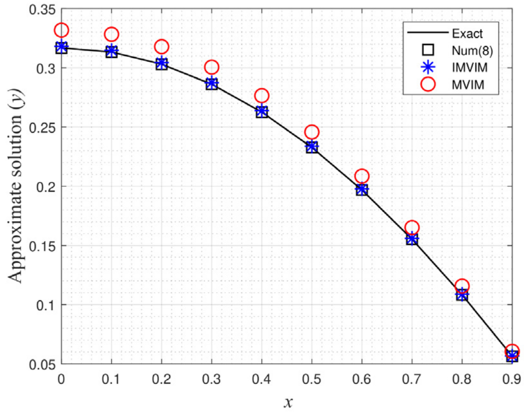
subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \tag{54}$$

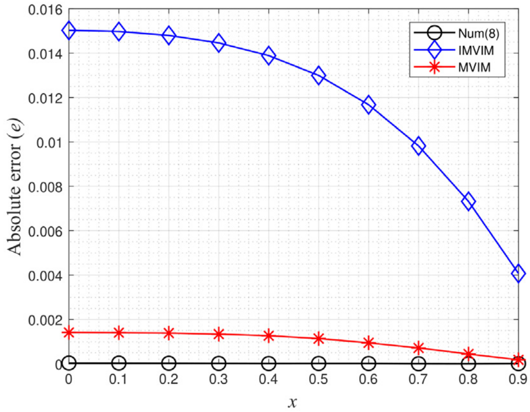
The exact solution of the SBVP (53)–(54) is $y(x) = 2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1}\right)$.

Here we have started the iteration with the initial guess $\lambda_0 = 5$. The slope of $\bar{e}(\lambda_0)$ is found to be negative for some successive iterations i.e. this initial guess does not satisfy the condition (52). The slope m of $\bar{e}(\lambda_0)$ is positive and satisfies the condition (52) for $\lambda_0 = 2$. Thus, the successive iterations will converge for $\lambda_0 = 2$ but not for $\lambda_0 = 5$. Another initial guess for λ_0 considered here is $\lambda_0 = 0$, for which it is found that the slopes of $\bar{e}(\lambda_0)$ are positive and satisfy condition (52). Thus successive iterations will also converge for $\lambda_0 = 0$.

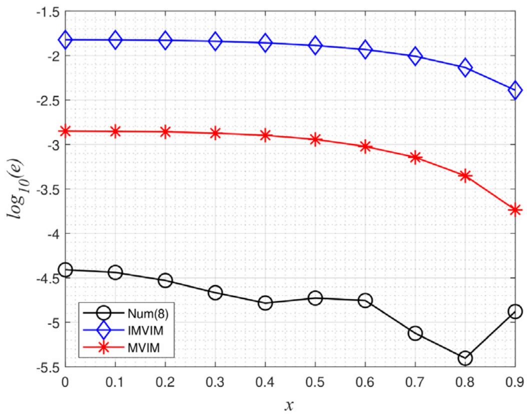
Now, we use the methodology developed in Section 3 with starting initial guess $\lambda_0 = 0$. The solutions obtained using the new approach after four iterations (Num(4)) and eight iterations (Num(8)) are compared to the exact solution and the solutions obtained using some existing numerical techniques (Modified variational iteration method (MVIM) [28], Improved modified variational iteration method (IMVIM) [43]) in Table 1. The numerical results of IVPs for different numbers of iterations are provided for $M = 6$ and $k = 2$. Fig. 2 compares the solution of the proposed numerical technique to IMVIM and MVIM. The numerical result of MVIM is given at first iteration for parameter $\omega = 0.78$, for which an improved result is given in [28]. The absolute errors are given in Fig. 2 and Table 1. The log of absolute errors is plotted in Fig. 2(c) to depict the errors precisely. One can observe from Table 1 that as the number of iterations increases the quality of the obtained solution increases significantly and matches well with the exact solution. A similar procedure has also been conducted for the starting initial guess $\lambda_0 = 2$, with $M = 6$ and $k = 2$ and found that $L_\infty = 0.047093$ after five iterations, and $L_\infty = 7.0 \times 10^{-5}$ after ten iterations.



(a) Numerical approximate and exact solutions



(b) Absolute errors (e)



(c) Log of absolute errors ($log_{10}(e)$)

Fig. 2. Comparison of results for Example 5.1.

Example 5.2. Consider a nonlinear second order SBVPs describing the isothermal gas sphere in astrophysics

$$-(x^2y')' = x^2y^5, \tag{55}$$

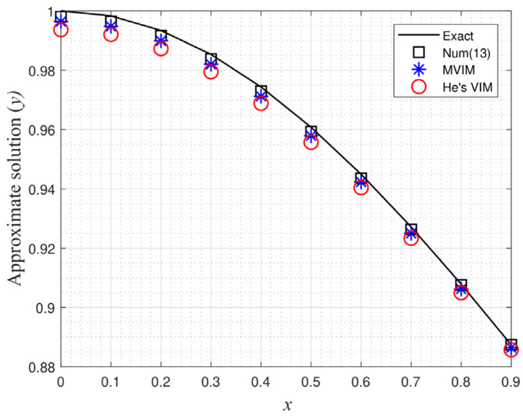
subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = \sqrt{\frac{3}{4}}. \tag{56}$$

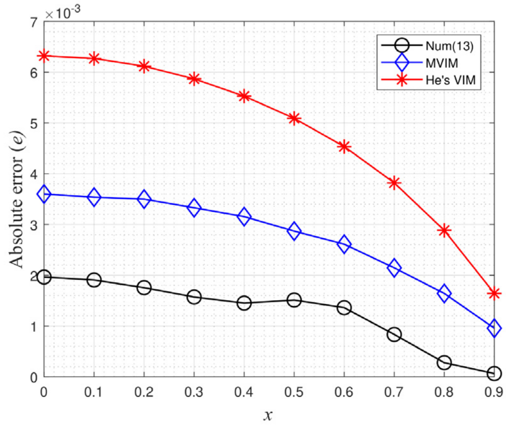
The exact solution is $y(x) = \sqrt{\frac{3}{3+x^2}}$. As in the previous example, the approximate solutions using the proposed method after six iterations (Num(6)) and thirteen iterations (Num(13)) are compared with the exact solution, MVIM [28] at $\omega = 2.3$ and He's VIM [14] in terms of absolute errors (refer to Table 2). The numerical results of IVPs after different number of iterations are provided for $M = 8$ and $k = 2$. From Fig. 3 and Table 2, it can be concluded that the proposed method requires fewer iterations to match the exact solution, and provides better results than MVIM and He's VIM. The logplot of the absolute errors is shown in Fig. 3(c) to depict the errors precisely.

Example 5.3. Consider a nonlinear SBVP given by

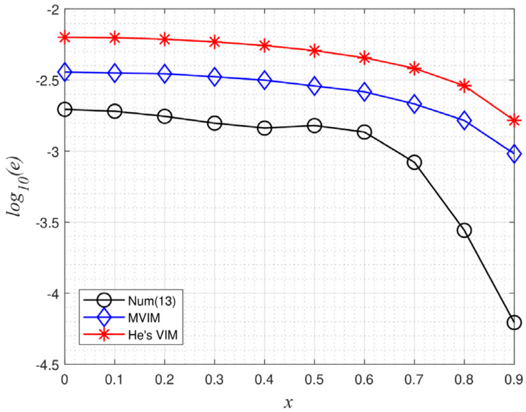
$$-(x^2y')' = x^2\left(6 - 12x + \cos(x^2 - x^3) - \cos y\right), \tag{57}$$



(a) Numerical approximate and exact solutions



(b) Absolute errors (e)



(c) Log of absolute errors ($log_{10}(e)$)

Fig. 3. Comparison of results for Example 5.2.

Table 2
Comparison of proposed approach (Num) with exact solution (Exact) of Example 5.2.

t	Exact	Num(6)	e	Num(13)	e	Method [28]	e [28]	Method [14]	e [14]
0.0	1.0000	0.9965	0.0034	0.9980	0.0019	0.9964	0.0035	0.9936	0.0063
0.1	0.9983	0.9949	0.0033	0.9964	0.0019	0.9948	0.0035	0.9920	0.0062
0.2	0.9933	0.9902	0.0031	0.9916	0.0017	0.9899	0.0034	0.9872	0.0061
0.3	0.9853	0.9823	0.0029	0.9837	0.0015	0.9820	0.0032	0.9794	0.0058
0.4	0.9743	0.9716	0.0027	0.9729	0.0014	0.9712	0.0030	0.9688	0.0055
0.5	0.9607	0.9580	0.0027	0.9592	0.0015	0.9579	0.0028	0.9556	0.0050
0.6	0.9449	0.9424	0.0024	0.9435	0.0013	0.9423	0.0025	0.9403	0.0045
0.7	0.9271	0.9253	0.0018	0.9263	0.0008	0.9250	0.0021	0.9233	0.0038
0.8	0.9078	0.9066	0.0011	0.9075	0.0002	0.9062	0.0015	0.9049	0.0028
0.9	0.8873	0.8866	0.0007	0.8874	6.2×10^{-5}	0.8864	0.0008	0.8857	0.0016

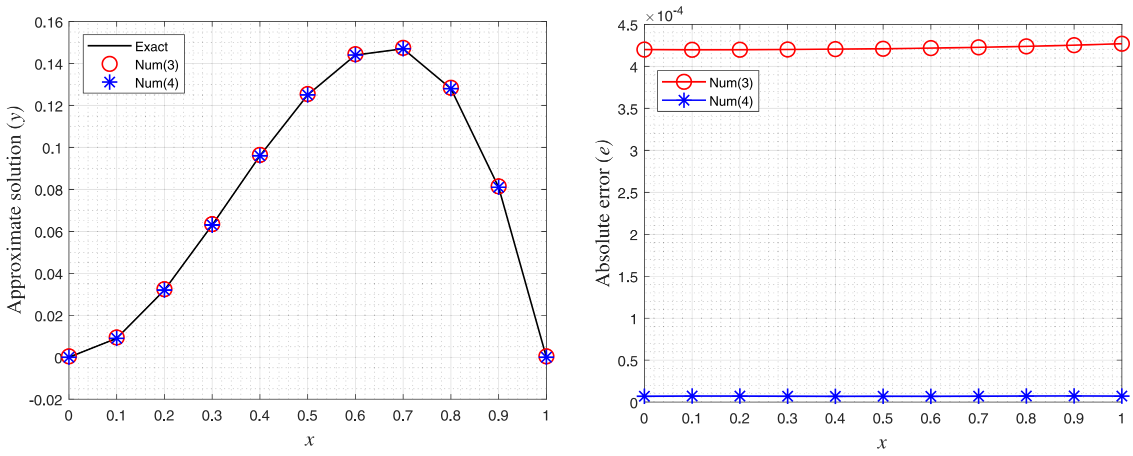
subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \tag{58}$$

The exact solution of SBVP (57)–(58) is $y(x) = x^2 - x^3$. Now, we use the methodology developed in Section 3 with initial guess $\lambda_0 = 0.5$. We use the Legendre wavelet operational matrix of integration with $M = 4$ and $k = 2$ at

Table 3
Comparison of proposed approach (Num) with exact solution (Exact) of Example 5.3.

t	Exact	Num(3)	e	Num(4)	e
0.0	0.0000	0.00042	0.00042	-6.9×10^{-6}	1.3×10^{-6}
0.1	0.0090	0.00942	0.00042	0.008992750	7.2×10^{-6}
0.2	0.0320	0.03242	0.00042	0.031992798	7.2×10^{-6}
0.3	0.0630	0.06342	0.00042	0.062992983	7.0×10^{-6}
0.4	0.0960	0.09642	0.00042	0.095993124	6.8×10^{-6}
0.5	0.1250	0.12542	0.00042	0.124993029	6.9×10^{-6}
0.6	0.1440	0.14442	0.00042	0.143993098	6.9×10^{-6}
0.7	0.1470	0.14742	0.00042	0.146992937	7.0×10^{-6}
0.8	0.1280	0.12842	0.00042	0.127992721	7.2×10^{-7}
0.9	0.0810	0.08142	0.00042	0.080992624	7.3×10^{-7}
1.0	0.0000	0.00042	0.00042	-7.1×10^{-6}	7.1×10^{-6}



(a) Numerical approximate and exact solutions (b) Absolute errors (e)

Fig. 4. Comparison of results for Example 5.3.

each iteration to find the numerical solution of the IVP. Table 3 and Fig. 4 show the comparison of the approximate solutions using the present method after three iterations (Num(3)) and four iterations (Num(4)) against the exact solution (Exact). The absolute errors for different numbers of iterations have been presented in Table 3 and Fig. 4. For small number of iterations brings the numerical solution approach extremely close to the exact solution for such an extreme nonlinear SBVP.

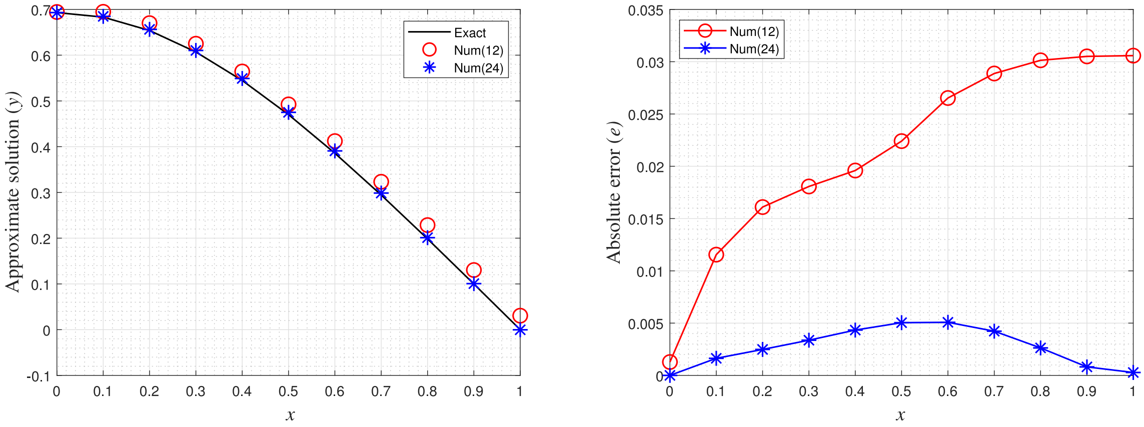
Example 5.4. Consider the following Lane–Emden equation arising in astrophysics [8]

$$-\left(x^{1/2}y'\right)' = x^{1/2}\left(e^{2y} - \frac{1}{2}e^{y}\right), \tag{59}$$

subject to the boundary conditions

$$y(0) = \ln(2), \quad y(1) = 0. \tag{60}$$

The exact solution is $y(x) = \ln\left(\frac{2}{x^2+1}\right)$. In Table 4, we have presented the quantitative comparison of approximate solutions after five iterations (Num(5)), ten iterations (Num(10)) and thirty two iterations (Num(32)) and the absolute errors. Fig. 5 is used to compare the numerical solution for different iterations with the exact solution, qualitatively. The numerical results of IVPs at different iterations are provided for $M = 4$ and $k = 2$. In a recent study, Singh et al. [8] estimated the maximum error norm as $L_\infty = max_{x\in[0,1]}|Exact - Numerical| = 10^{-4}$ for $(J = 2, 2M = 8)$ and $(J = 3, 2M = 16)$.



(a) Numerical approximate and exact solutions

(b) Absolute errors (e)

Fig. 5. Comparison of results for Example 5.4.

Table 4
Comparison of proposed approach (Num) with exact solution (Exact) of Example 5.4.

t	Exact	Num(5)	e	Num(10)	e	Num(32)	e
0.0	0.6931	0.6946	0.0014	0.6931	8.9×10^{-8}	0.6929	1.7×10^{-4}
0.1	0.6831	0.6934	0.0103	0.6845	1.3×10^{-3}	0.6835	3.7×10^{-4}
0.2	0.6539	0.6692	0.0152	0.6559	1.9×10^{-3}	0.6543	4.5×10^{-4}
0.3	0.6069	0.6266	0.0196	0.6096	2.6×10^{-3}	0.6077	7.5×10^{-4}
0.4	0.5447	0.5705	0.0258	0.5484	3.7×10^{-3}	0.5455	8.2×10^{-4}
0.5	0.4700	0.5060	0.0360	0.4750	5.0×10^{-3}	0.4709	9.2×10^{-4}
0.6	0.3856	0.4327	0.0471	0.3914	5.7×10^{-3}	0.3866	8.6×10^{-4}
0.7	0.2943	0.3502	0.0559	0.3008	6.5×10^{-3}	0.2951	7.8×10^{-4}
0.8	0.1984	0.2607	0.0623	0.2053	6.9×10^{-3}	0.1989	4.9×10^{-4}
0.9	0.0998	0.1665	0.0667	0.1069	7.1×10^{-3}	0.0999	1.2×10^{-4}
1.0	0.0000	0.0698	0.0698	0.0074	7.4×10^{-4}	7.2×10^{-5}	7.5×10^{-5}

Example 5.5. Consider the following nonlinear SBVP

$$-(x^{1/2}y')' = x^{1/2}\left(-3 + \frac{45}{7}x - x^4 + \frac{12}{7}x^5 - \frac{36}{49}x^6 + y^2\right), \tag{61}$$

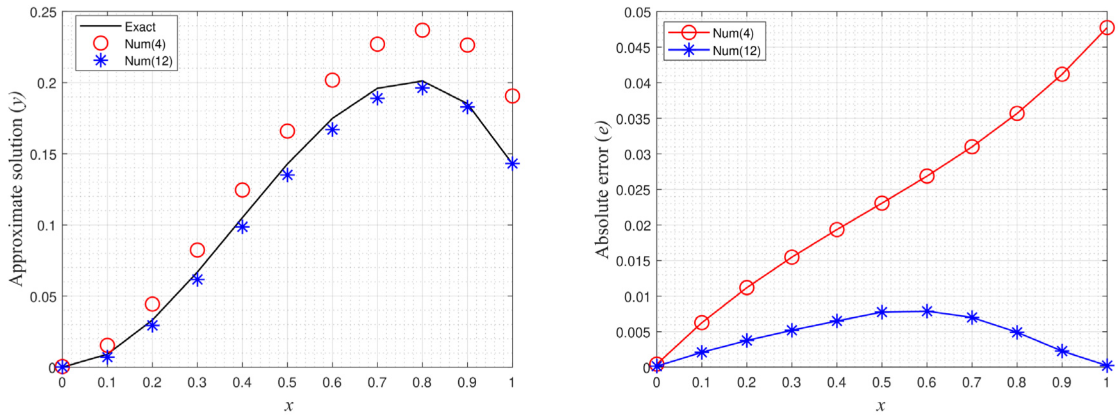
subject to the boundary conditions

$$y(0) = 0, \quad y(1) = \frac{1}{7}. \tag{62}$$

The exact solution is $y(x) = -\frac{6}{7}x^3 + x^2$. We have used the proposed method and compared it with the exact solution in Table 5 and Fig. 6. It can be observed that the new method is highly accurate and efficient for approximating the solution of this example.

6. Conclusions and future scope

This research work presents a highly efficient numerical strategy based on a combination of the shooting projection method and the Legendre wavelet operational matrix of integration method to approximate a class of two point singular boundary value problems. For initial value problems, the Legendre wavelet operational matrix approach delivers very accurate and efficient results, but it is difficult to solve two point singular boundary value problems with this method. A shooting projection approach for transforming boundary value problems into a series of initial value problems has been adapted to overcome the limitations of the Legendre wavelet operational matrix method for solving boundary value problems. The IVPs are then solved using the Legendre wavelet operational matrix approach. The assessment of the new approach has been done by considering some existing



(a) Numerical vs Exact

(b) Absolute error (e)

Fig. 6. Comparison of results for Example 5.5.

Table 5
Comparison of proposed approach (Num) with exact solution (Exact) of Example 5.5.

t	Exact	Num(1)	e	Num(4)	e	Num(12)	e
0.0	0.0000	0.0063	0.0063	0.0004	0.0004	0.0001	0.0001
0.1	0.0091	0.1004	0.0913	0.0154	0.0062	0.0070	0.0021
0.2	0.0331	0.1963	0.1632	0.0443	0.0111	0.0293	0.0038
0.3	0.0668	0.2923	0.2254	0.0823	0.0154	0.0616	0.0051
0.4	0.1051	0.3865	0.2813	0.1244	0.0193	0.0986	0.0065
0.5	0.1428	0.4772	0.3343	0.1659	0.0230	0.1351	0.0077
0.6	0.1748	0.5611	0.3862	0.2017	0.0268	0.1669	0.0078
0.7	0.1960	0.6301	0.4341	0.2269	0.0309	0.1890	0.0069
0.8	0.2011	0.6800	0.4788	0.2368	0.0356	0.1962	0.0049
0.9	0.1851	0.7039	0.5188	0.2263	0.0411	0.1828	0.0022
1.0	0.1428	0.6951	0.5522	0.1906	0.0477	0.1430	0.0002

approaches [8,28,43] to solve five standard highly nonlinear examples (most of the test examples having a non-polynomial solution). We note that the proposed approach can be implemented for approximating the BVP (1)–(2) or (1)–(3) for any positive α. Nevertheless, the application of the variational iteration method is different for each α. For each α there are different Lagrange multipliers or different relaxation parameters, and the process to find these parameters is very expensive computationally. The results show that the new approach produces high quality solutions for a class of two point singular boundary value problems within few iterations. The convergence analysis guarantees that the provided numerical solution converges to the exact solution.

Note that study made in this article can be generalized to a wide range of higher order singular boundary value problem of the form [45,46]

$$-\frac{1}{x^k}(x^k y')'' = f(x, y), \text{ or } -\frac{1}{x^k}(x^k y'')' = f(x, y), \; x \in (0, 1) \tag{63}$$

subject to the boundary conditions

$$y'(0) = c_1, \; y''(0) = c_2, \text{ and } y(1) = c_3, \tag{64}$$

or

$$y(0) = c_1, \; y'(0) = c_2, \text{ and } y(1) = c_3. \tag{65}$$

Data availability

Our manuscript has no associate data.

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