Differential Equations and Dynamical Systems
https://doi.org/10.1007/s12591-024-00676-z

ORIGINAL RESEARCH

Numerical Treatment of a Two-Parameter Singularly Perturbed Elliptic Problem with Discontinuous Convection and Source Terms

Ram Shiromani[1,3] · Vembu Shanthi[1] · Higinio Ramos[2,4]

Accepted: 31 December 2023

Abstract

In this paper, we address a two-parameter singularly perturbed convection-reaction-diffusion 2-D problem. We also consider that the convection and source terms are discontinuous in space. Due to these discontinuities and the presence of perturbation parameters, solutions to such problems show boundary and interior layers. In this study, we have carried out a numerical approach using a finite-difference technique with an appropriate layer-adapted piecewise uniform Shishkin mesh. Some examples are presented which show the best performance of the proposed method and its agreement with the theoretical analysis.

Keywords Discontinuous convection and source terms · Finite difference scheme · Shishkin mesh · Elliptic equation · Two-parameter singularly perturbed problem · Two-dimensional space

Mathematics Subject Classification 35J25 · 35J40 · 35B25 · 65N06 · 65N12 · 65N15 · 65N50

Introduction

Several partial differential equations found in practice are parameter-dependent, with a singularly perturbed nature for small values of this parameters. The solutions to these problems have boundary layers (and/or interior layers due to the discontinuities in the convection and source terms), which are near the boundary of the domain where the solution has an extremely high gradient. The layers might be either regular (exponential) or of parabolic

✉ Higinio Ramos
higra@usal.es

Ram Shiromani
ram.panday.786@gmail.com

[1] Department of Mathematics, National Institute of Technology, Tiruchirappalli, Tamilnadu, India

[2] Scientific Computing Group, Universidad de Salamanca, Salamanca, Spain

[3] School of Sciences, Woxsen University, Hyderabad, India

[4] Escuela Politécnica Superior de Zamora, Campus Viriato, 49029 Zamora, Spain

Published online: 08 February 2024

type (characteristic). The problem considered is of the convection-reaction-diffusion type, which is one of the most common singular perturbation problems in the literature.

Singularly perturbed elliptic problems are widespread in mathematical modeling, ranging from simulation of oil and gas reservoirs, as well as magnetohydrodynamic flow, to chemical flow reactor theory [1], to boundary layers influenced by suction (or blowing) of some fluid and semiconductor model [2]. In most cases the equations are too complicated to be solved exactly, so the use of numerical techniques is required. On the other hand, classical techniques can utterly fail in the presence of layers (see, e.g. [3, 4]), thus this is a highly active and interesting topic of research for numerical analysts.

There is a lot of literature on parameter-uniform numerical methods (see, for example, [3, 5]).

Butuzov [6] studied the asymptotic structure of a solution to a problem similar to the one under consideration here, and stablished a substantial relationship between the ordering of small parameters. When $\mu = O(\epsilon^{1/2})$, we are approaching the reaction-diffusion type with layer structure. The solutions to 2-D reaction-diffusion problems contain characteristic (parabolic) boundary layers along all edges of the unit square. Clavero considered a class of 2-D reaction-diffusion singularly perturbed problems in [7] and used a finite difference method with a piecewise uniform Shishkin mesh to find a more accurate estimate. When $\mu = 1$, we are approaching the convection-diffusion type with a regular (exponential) layer near the outflow of the boundary, and a corner layer near the boundary layers junction. In a 2-D singularly perturbed convection-diffusion problem, the solution has boundary and interior layers in the domain. Therefore, the derivative's norms of z in [8] show that the solution z contains regular (exponential) boundary layers along all of the unit square's edges, while certain bounds in [9] demonstrate both exponential and parabolic boundary layers in z. In [10], they introduced the asymptotic character of singularly perturbed convection-diffusion problems to better understand their solution and related challenges with numerical techniques. Lin et al. [11] analyzed the weak form of 2-D singularly perturbed convection-diffusion problems using an effective approach based on local discontinuous Galerkin (LDG) discretization. Nhan and Vulanovic [12] explain a class of 2-D convection-diffusion singularly perturbed problems using a complete finite-difference scheme with a Bakhvalov mesh to find more accurate estimates. When μ moves away from $\epsilon^{1/2}$ and remains small when compared to $\mu = 1$, we have an entirely different layer structure.

In a 2-D singularly perturbed convection-diffusion problem with two parameters, Zhang and Lv [13] consider an efficient approach based on the finite-element method using a Bakhvalov mesh. In O'Riordan et al. [14, 15], they consider the finite-difference method with a piecewise uniform Shishkin mesh for a two-parameter convection-diffusion singularly perturbed 2-D problems. Teofanov and Roos [16, 17] handle an efficient approach based on the finite-element method using linear or bilinear elements in a piecewise uniform Shishkin mesh for a 2-D singularly perturbed convection-diffusion problem with two parameters. In that paper, they discuss the decomposition of the solution and derivative's bounds on the components. In this study, we develop a finite-difference method (FDM) for solving two-parameter singularly perturbed convection-diffusion 2-D problems, where a, b and f are discontinuous along two lines, namely $x = d$ and $y = d$.

Let us consider a two-parameter singularly perturbed 2-D steady-state convection-reaction-diffusion equation

$$L_{\epsilon,\mu} z(x,y) = f(x,y), \quad \forall (x,y) \in \Omega, \qquad z(x,y) = g(x,y), \quad \forall (x,y) \in \partial\Omega, \tag{1.1a}$$

where the differential operator is represented by

$$L_{\epsilon,\mu}z(x,y) \equiv \epsilon^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)+\mu^2\left(a(x,y)\frac{\partial z}{\partial x}+b(x,y)\frac{\partial z}{\partial y}\right)-c(x,y)z. \tag{1.1b}$$

The two small perturbation parameters satisfy $0<\epsilon,\mu<<1$. The problem's domain is $\Omega=\bigcup_{k=1}^{4}\Omega_k$, being $\Omega_1=(0,d)\times(0,d), \Omega_2=(d,1)\times(0,d), \Omega_3=(0,d)\times(d,1)$ and $\Omega_4=(d,1)\times(d,1)$. Let $\bar{\Omega}=[0,1]\times[0,1]$ and $\Gamma_1=\{(d,y):0\le y\le 1\}, \Gamma_2=\{(x,d):0\le x\le 1\}$, with d any point in (0, 1).

We assume that the convection terms are bounded as follows with

$$\begin{aligned}
&\alpha_1^*>a(x,y)\ge\alpha_1>0,\ \text{for } x<d,\ -\alpha_2^*<a(x,y)\le-\alpha_2<0,\ \text{for } x>d,\\
&\beta_1^*>b(x,y)\ge\beta_1>0,\ \text{for } y<d,\ -\beta_2^*<b(x,y)\le-\beta_2<0,\ \text{for } y>d,\\
&\alpha=\min(\alpha_1,\alpha_2),\ \alpha^*=\max(\alpha_1^*,\alpha_2^*),\ \lambda=\min_{\bar{\Omega}}\left\{\frac{c}{2a},\frac{c}{2b}\right\},\\
&\beta=\min(\beta_1,\beta_2),\ \beta^*=\max(\beta_1^*,\beta_2^*),
\end{aligned}$$

for some constants α_1, α_2 and β, while the reaction coefficient verifies $c(x,y)>0$. We will also assume that $a|_{\Omega_k}, b|_{\Omega_k}, f|_{\Omega_k}\in C^{3,\gamma}(\Omega_k)$, $c\in C^{3,\gamma}(\bar{\Omega})$, and $g\in C^{4,\gamma}(\Omega)$, for some $\gamma\in(0,1]$, $k=1,2,3,4$. Further, we assume that the exact solution is such that $z\in C^{4,\gamma}(\Omega_k)$.

In our problem (1.1), the source term $f(x,y)$ has a jump discontinuity at both lines $x=d$ and $y=d$, and the convection terms $a(x,y)$, $b(x,y)$, have jump discontinuities at the line $x=d$ or $y=d$, respectively. So, it is congruent to denote the jump discontinuity in any function κ at a point $(x,y)\in\Omega$ along the lines parallel to the x- and y-axe as $[\kappa](d,y)=\kappa(d^+,y)-\kappa(d^-,y)$ and $[\kappa](x,d)=\kappa(x,d^+)-\kappa(x,d^-)$, respectively. The following symbols are used to specify the boundaries:

$$\begin{aligned}
\Lambda_1&=\{(0,y)\,|\,(0\le y<d)\cup(d<y\le 1\}, \quad \Lambda_2=\{(x,0)\,|\,(0\le x<d)\cup(d<x\le 1)\},\\
\Lambda_3&=\{(1,y)\,|\,(0\le y<d)\cup(d<y\le 1)\}, \quad \Lambda_4=\{(x,1)\,|\,(0\le x<d)\cup(d<x\le 1)\},
\end{aligned}$$

and $\Lambda=\Lambda_1\cup\Lambda_2\cup\Lambda_3\cup\Lambda_4$. Recalling from (1.1) that $z=g$ on the boundary, we denote by g_i the restriction of g onto Λ_i, $i=1,2,3,4$.

We further denote the continuous subsets of the boundaries, and the interior line segments on both sides of the discontinuities as $\Lambda_{k,j}$, $c_{k,j}$ where $j=1,2,3,4$ (which indicate the edges and corners of Ω_k respectively) (see Fig. 1).

The article is structured as follows. In Section 2, we derived the minimum principle, a stability result and bounds of the exact solution and its derivatives exhibiting their dependence on the singular perturbation parameters. Section 3 explores the numerical approach of the standard 5-point finite difference scheme built on a Shishkin mesh. In Section 4, we derived the error estimation, which results in an almost first-order convergence. At the end of Section 5, some test problems are provided to verify the theoretical results.

A Priori Bounds of the Solution and its Derivatives

The present section contains the minimum principle, a stability result, and some useful bounds for the derivatives of the true solution. In addition, we obtain some bounds of the regular, singular, and corner components of the solution.

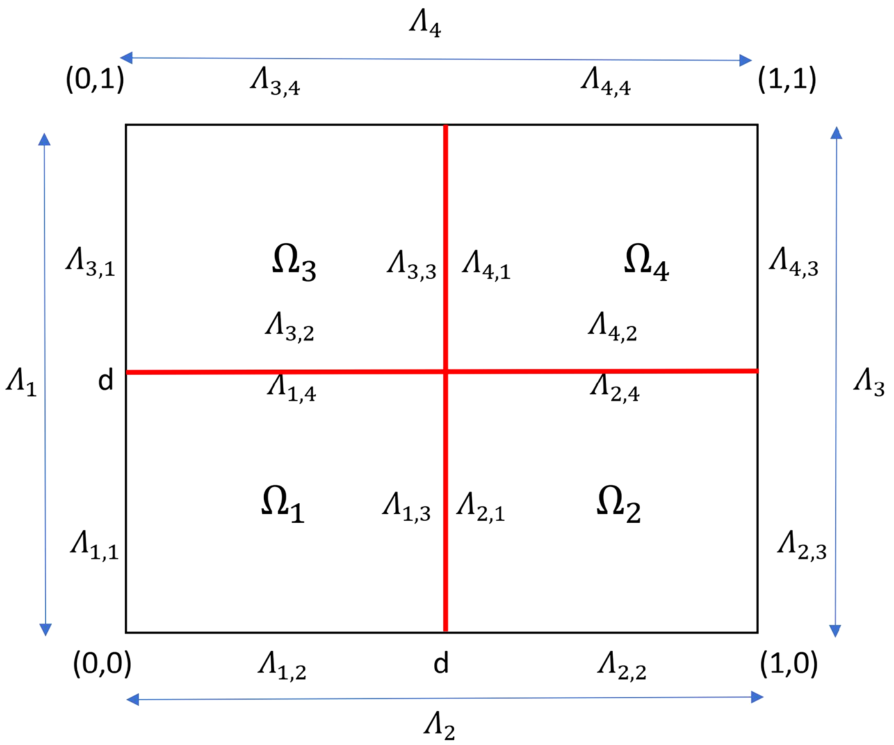


Fig. 1 Notation for Subregions and Domain Boundaries

Lemma 2.1 (Minimum principle) *Let* $L_{\epsilon,\mu}$ *be the differential operator given in* (1.1). *If* $\phi(x,y) \geq 0$ *on* $\partial\Omega$, $L_{\epsilon,\mu}\phi(x,y) \leq 0$ *for all* $(x,y) \in \Omega$, $\left[\frac{\partial \phi}{\partial x}\right](d,y) \leq 0$ *on* Γ_1 *and* $\left[\frac{\partial \phi}{\partial y}\right](x,d) \leq 0$ *on* Γ_2 *then it is* $\phi(x,y) \geq 0$ *for all* $(x,y) \in \bar{\Omega}$.

Proof Consider the function ω on $\bar{\Omega}$ defined through $\phi(x,y) = \omega(x,y)\psi(x,y)$, with the function

$$\psi(x,y) = \exp\left(\frac{\mu^2(d-x)\lambda_1(x)}{2\epsilon^2} + \frac{\mu^2(d-y)\lambda_2(y)}{2\epsilon^2}\right), \quad (x,y) \in \bar{\Omega}.$$

The functions $\lambda_1(x)$, $\lambda_2(y)$ on $[0,1]$ are defined as

$$\lambda_1(x) = \begin{cases} \lambda_{11}, & \text{if } x \leq d, \\ -\lambda_{12}, & \text{if } x > d, \end{cases} \qquad \lambda_2(y) = \begin{cases} \lambda_{21}, & \text{if } y \leq d, \\ -\lambda_{22}, & \text{if } y > d, \end{cases}$$

where $\alpha \geq \lambda_{11} > \lambda_{12} > 0$, and $\beta \geq \lambda_{21} > \lambda_{22} > 0$ are some constants. Let be $\omega(x^*,y^*) = \min_{(x,y)\in\bar{\Omega}} \{\omega(x,y)\}$. If $\omega(x^*,y^*) \geq 0$, there is nothing to prove. Suppose $\omega(x^*,y^*) < 0$. Then, by the assumption on the boundary values, either the point $(x^*,y^*) \in \Omega$ or $(x^*,y^*) \in \Gamma_1 \cup \Gamma_2$. Let us consider both cases.

Case(i): Firstly, assume that $(x^*, y^*) \in \Omega$.

Then, at the point (x^*, y^*) it is $\frac{\partial \omega}{\partial x}(x^*, y^*) = \frac{\partial \omega}{\partial y}(x^*, y^*) = 0$ and $\frac{\partial^2 \omega}{\partial x^2}(x^*, y^*) \geq 0$, $\frac{\partial^2 \omega}{\partial y^2}(x^*, y^*) \geq 0$, we have

$$\begin{aligned} L_{\epsilon,\mu}\phi(x^*, y^*) = \psi(x^*, y^*)\Bigg(\epsilon^2 \Delta\omega + \Bigg(\frac{\mu^4 \lambda_1(x^*)}{2\epsilon^2}\Big(\frac{\lambda_1(x^*)}{2} - a(x^*, y^*)\Big) \\ + \frac{\mu^4 \lambda_2(y^*)}{2\epsilon^2}\Big(\frac{\lambda_2(y^*)}{2} - b(x^*, y^*)\Big)\Bigg)\omega(x^*, y^*) \\ - c\omega(x^*, y^*)\Bigg) > 0, \end{aligned}$$

which contradicts the hypothesis.

Case(ii): Now suppose that $(x^*, y^*) \in \Gamma_1 \cup \Gamma_2$.

Here, either $(x^*, y^*) = (d, y^*)$, or $(x^*, y^*) = (x^*, d)$. Let us assume $(x^*, y^*) = (d, y^*)$. Since ω takes its minimum value at (x^*, y^*), this implies that $\frac{\partial \omega}{\partial x}(d^+, y^*) \geq 0$ and $\frac{\partial \omega}{\partial x}(d^-, y^*) \leq 0$. Then, it is evident that $[\frac{\partial \omega}{\partial x}](d, y^*) \geq 0$. Then, since $\omega(d, y^*) < 0$, it follows that

$$\left[\frac{\partial \phi}{\partial x}\right](d, y^*) = \exp\left(\frac{\mu^2 \lambda_2(d - y^*)}{2\epsilon^2}\right)\left(\left[\frac{\partial \omega}{\partial x}\right](d, y^*) + \mu^2 \frac{\lambda_{12} - \lambda_{11}}{2\epsilon^2}\omega(d, y^*)\right) > 0,$$

which contradicts the hypothesis $[\frac{\partial \phi}{\partial x}](x, y) \leq 0$, $\forall (x, y) \in \Gamma_1$. The case when $(x^*, y^*) = (x^*, d)$ can be proved similarly. This completes the proof. □

A consequence of this minimum principle is the parameter uniform boundedness of the solution of (1.1) given below.

Lemma 2.2 (Stability result) *Let $z(x, y)$ be the solution of* (1.1). *Then, it holds*

$$||z(x, y)||_{\bar{\Omega}} \leq \frac{1}{\nu}||L_{\epsilon,\mu}z||_{\bar{\Omega}} + \max\left\{||z||_{\Lambda_1}, ||z||_{\Lambda_2}, ||z||_{\Lambda_3}, ||z||_{\Lambda_4}\right\},$$

where $||.||$ *represents the pointwise maximum norm and* $\nu = \min\{\alpha, \beta\}$.

Proof We define the barrier functions

$$\phi^{\pm}(x, y) = \begin{cases} M + \frac{||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{\nu}\left(\frac{1}{2} + \frac{x}{8} + \frac{y}{8} - \frac{d}{4}\right) \pm z(x, y), & (x, y) \in [0, d] \times [0, d], \\ M + \frac{||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{\nu}\left(\frac{1}{2} - \frac{x}{4} + \frac{y}{8} + \frac{d}{8}\right) \pm z(x, y), & (x, y) \in (d, 1] \times [0, d], \\ M + \frac{||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{\nu}\left(\frac{1}{2} + \frac{x}{8} - \frac{y}{4} + \frac{d}{8}\right) \pm z(x, y), & (x, y) \in [0, d] \times (d, 1], \\ M + \frac{||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{\nu}\left(\frac{1}{2} - \frac{x}{4} - \frac{y}{4} + \frac{d}{2}\right) \pm z(x, y), & (x, y) \in (d, 1] \times (d, 1], \end{cases}$$

where $M = \max\{||z||_{\Lambda_1}, ||z||_{\Lambda_2}, ||z||_{\Lambda_3}, ||z||_{\Lambda_4}\}$.

Then, clearly $\phi^{\pm}(x, 0) \geq 0$, $\phi^{\pm}(0, y) \geq 0$, $\phi^{\pm}(x, 1) \geq 0$, $\phi^{\pm}(1, y) \geq 0$. For each $(x, y) \in \Omega$, we have

$$L_{\epsilon,\mu}\phi^{\pm}(x,y) \leq 0.$$

Since $z(x,y) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, we have

$$\left[\frac{\partial \phi^{\pm}}{\partial x}\right](d,y) = \frac{-3||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{8\nu} \pm \left[\frac{\partial z^{\pm}}{\partial x}\right](d,y) \leq 0,$$
$$\left[\frac{\partial \phi^{\pm}}{\partial y}\right](x,d) = \frac{-3||L_{\epsilon,\mu}z||_{\bar{\Omega}}}{8\nu} \pm \left[\frac{\partial z^{\pm}}{\partial y}\right](x,d) \leq 0.$$

It follows from Lemma 2.1 that $\phi^{\pm}(x,y) \geq 0, \quad \forall (x,y) \in \bar{\Omega}$, which allows to get the bound on $||z(x,y)||_{\bar{\Omega}}$. □

The derivatives of the solution satisfy the parameter-explicit bounds shown as follows.

Lemma 2.3 *Let z be the solution of (1.1). Then, for $1 \leq i+j \leq 4$, it holds*

$$\left|\left|\frac{\partial^{i+j} z}{\partial x^i \partial y^j}\right|\right|_{\Omega_k} \leq C\epsilon^{-(i+j)}, \text{ if } \alpha\mu^2 \leq \lambda\epsilon, \tag{2.1}$$

$$\left|\left|\frac{\partial^{i+j} z}{\partial x^i \partial y^j}\right|\right|_{\Omega_k} \leq C\left(\frac{\epsilon}{\mu}\right)^{-(2i+2j)}, \text{ if } \alpha\mu^2 > \lambda\epsilon, \tag{2.2}$$

where C is independent of ϵ and μ.

Proof It can be easily obtained using standard procedures, as in [18, 19]. □

Now, we decompose the solution $z(x, y)$ into the regular and singular components. The regular components $r_k(x,y)$, $k = 1, 2, 3, 4$, are obtained, respectively, as the solution of the following problems

$$\begin{cases} L_{\epsilon,\mu} r_k(x,y) = f(x,y), \ \forall (x,y) \in \Omega_k, \ k = 1,2,3,4 \\ r_k(x,y) = g_1(y), \quad \forall (x,y) \in \Lambda_{k,1}, \ k = 1,3, \quad r_k(x,y) = g_2(x), \quad \forall (x,y) \in \Lambda_{k,2}, \ k = 1,2, \\ r_k(x,y) = g_3(y), \quad \forall (x,y) \in \Lambda_{k,3}, \ k = 2,4, \quad r_k(x,y) = g_4(x), \quad \forall (x,y) \in \Lambda_{k,4}, \ k = 3,4, \\ [r_k](x,y) = 0, \quad \forall (x,y) \in \Gamma_1 \cup \Gamma_2, \ k = 1,2,3,4, \\ [(r_k)_x](x,y) = 0, \quad \forall (x,y) \in \Gamma_1, \quad [(r_k)_y](x,y) = 0, \quad \forall (x,y) \in \Gamma_2, \ k = 1,2,3,4. \end{cases} \tag{2.3}$$

Lemma 2.4 *The regular components $r_k(x,y)$ at (2.3) and their derivatives satisfy the bounds*:

$$\left|\left|\frac{\partial^{i+j} r_k}{\partial x^i \partial y^j}\right|\right|_{\Omega} \leq C\left(1 + \epsilon^{2-(i+j)}\right), \quad \text{for} \quad 1 \leq i+j \leq 4, \quad \text{if} \quad \alpha\mu^2 \leq \lambda\epsilon,$$
$$\left|\left|\frac{\partial^{i+j} r_k}{\partial x^i \partial y^j}\right|\right|_{\Omega} \leq C\left(1 + \left(\frac{\epsilon}{\mu}\right)^{4-(2i+2j)}\right), \quad \text{for} \quad 1 \leq i+j \leq 4, \quad \text{if} \quad \alpha\mu^2 > \lambda\epsilon.$$

Proof Let us consider the two cases.

Case(i): Firstly, assume that $\alpha\mu^2 \leq \lambda\epsilon$.

Suppose the regular component $r_k(x,y)$, $k = 1, 2, 3, 4$ can be decomposed as

$$r_k(x,y) = r_{k,0}(x,y,\epsilon,\mu) + \epsilon r_{k,1}(x,y,\epsilon,\mu) + \epsilon^2 r_{k,2}(x,y,\epsilon,\mu), \tag{2.4}$$

where $r_{k,0}$, $r_{k,1}$ and $r_{k,2}$ are the solutions to the following problems without any compatibility inequalities:

$$-r_{k,0} = f, \quad r_{k,1} = \epsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r_{k,0} + \frac{\mu^2}{\epsilon}\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) r_{k,0},$$
$$L_{\epsilon,\mu} r_{k,2} = -\epsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r_{k,1} - \frac{\mu^2}{\epsilon}\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) r_{k,1},$$
$$r_{k,2} = 0, \ \forall (x,y) \in \Lambda_{k,j}, \ k,j = 1,2,3,4.$$

Since, $r_{k,0} \in C^{4,\gamma}(\bar{\Omega}_k)$, $k = 1,2,3,4$, we get $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r_{k,0} \in C^{2,\gamma}(\bar{\Omega}_k)$, $k = 1,2,3,4$.

Applying Lemma 2.2 and Lemma 2.3 to the problem (2.3), it results that $r_k \in C^{4,\gamma}(\Omega_k)$ and

$$\left\| \frac{\partial^{i+j} r_k}{\partial x^i \partial y^j} \right\| \le C(1 + \epsilon^{2-(i+j)}), \quad 1 \le i+j \le 4, \ k = 1,2,3,4. \tag{2.5}$$

Case(ii): $\alpha\mu^2 > \lambda\epsilon$.

Suppose the regular component $r_k(x,y;\epsilon,\mu)$, $k = 1,2,3,4$ can be decomposed as

$$r_k(x,y;\epsilon,\mu) = r_{k,0}(x,y;\mu) + \epsilon^2 r_{k,1}(x,y;\mu) + \epsilon^4 r_{k,2}(x,y;\epsilon,\mu),$$

where

$$L_\mu r_{k,0} = f, \ r_{k,0}(x,y) = z(x,y), \ \forall (x,y) \in \Lambda_3 \cup \Lambda_4,$$
$$L_\mu r_{k,1} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r_{k,0}, \ r_{k,1}(x,y) = 0, \ \forall (x,y) \in \Lambda_3 \cup \Lambda_4,$$
$$L_{\epsilon,\mu} r_{k,2} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r_{k,1}, \quad r_{k,2}(x,y) = 0, \ \forall (x,y) \in \Lambda_{k,j}, \ k,j = 1,2,3,4.$$

Applying Lemma 2.6 and Lemma 2.3 to the problem (2.3) we get

$$\left\| \frac{\partial^{i+j} r_k}{\partial x^i \partial y^j} \right\| \le C\left(1 + \left(\frac{\epsilon}{\mu}\right)^{4-(2i+2j)}\right), \quad 1 \le i+j \le 4, \ k = 1,2,3,4. \tag{2.6}$$

□

Now, let us consider the first order IBVP:

$$L_\mu z(x,y) = \mu^2\left(a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y}\right) - c(x,y)z = f(x,y), \ \forall (x,y) \in \Omega, \tag{2.7a}$$

$$z(x,y) = g_i, \ (x,y) \in \Lambda_i, \ i = 3,4. \tag{2.7b}$$

Note that L_μ satisfies the following comparison principle:

Lemma 2.5 *Let L_μ be the differential operator given in (2.7). If $\phi(x,y) \geq 0$ on Λ_i, $i = 3,4$, $L_\mu \phi(x,y) \leq 0$ for all $(x,y) \in \Omega$, $\left[\frac{\partial \phi}{\partial x}\right](d,y) \leq 0$ on Γ_1, and $\left[\frac{\partial \phi}{\partial y}\right](x,d) \leq 0$ on Γ_2 then it is $\phi(x,y) \geq 0$ for all $(x,y) \in \bar{\Omega}$.*

Proof The proof is similar to the one of Lemma 2.1. □

Lemma 2.6 *Let $z(x,y)$ be the solution of problem (2.7). Then, it holds the stability estimate*

$$||z(x,y)||_{\bar{\Omega}} \leq \frac{1}{\nu}||L_\mu z||_\Omega + \max\left\{||z||_{\Lambda_3}, ||z||_{\Lambda_4}\right\}.$$

Proof This lemma can be proved similarly to Lemma 2.2. □

Corresponding to the edge $x = 0$ in Ω_1 (see, Fig. 2), a layer function s_1 exists that is determined by:

$$L_{\epsilon,\mu} s_1(x,y) = 0, \quad \forall (x,y) \in \Omega, \tag{2.8a}$$

$$s_1(x,y) = (z - r_1)(x,y), \quad \forall (x,y) \in \Lambda_{1,1}, \tag{2.8b}$$

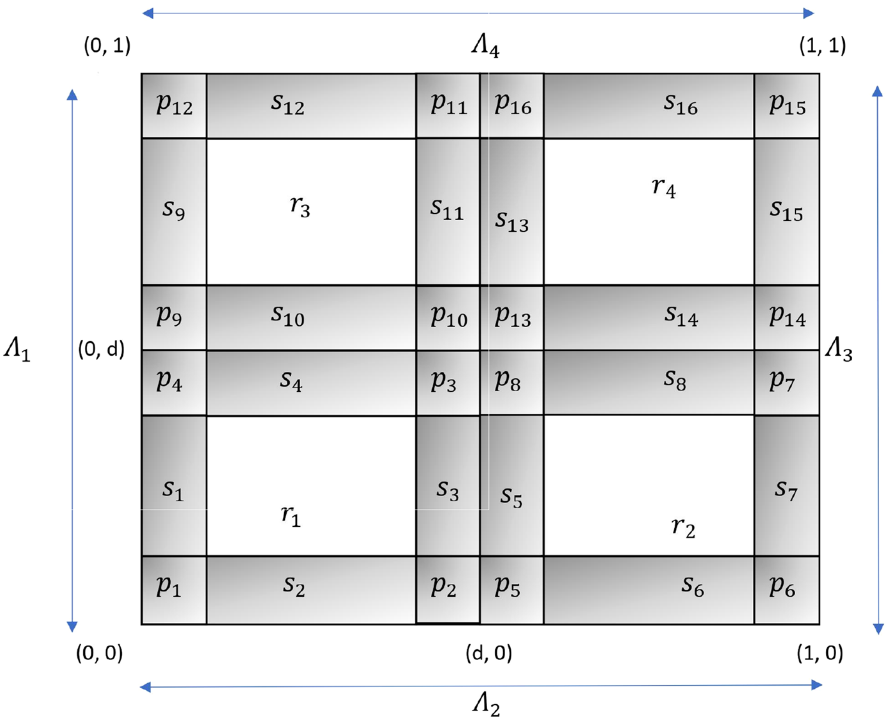


Fig. 2 Situation of the layer functions in the domain Ω, when $\alpha\mu^2 \leq \lambda\epsilon$ and $\alpha\mu^2 > \lambda\epsilon$

$$s_1(x,y) = 0, \quad \forall (x,y) \in \Lambda_{3,1} \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4, \tag{2.8c}$$

$$s_1(x,y) = 0, \quad \forall (x,y) \in \Gamma_1 \cup \Gamma_2, \tag{2.8d}$$

$$\left[(s_1)_x\right](x,y) = 0, \quad \forall (x,y) \in \Gamma_1,\ [(s_1)_y](x,y) = 0, \quad \forall (x,y) \in \Gamma_2. \tag{2.8e}$$

The following lemmas give some bounds on the derivatives of the layer components, which are necessary for the convergence analysis.

Lemma 2.7 *Let s_1 be the boundary layer component satisfying the equations in (2.8). If $\alpha\mu^2 \le \lambda\epsilon$, then it holds*

$$\begin{aligned} |s_1(x,y)| &\le C \exp\left(- \frac{\alpha\lambda}{\epsilon} x \right), \\ \left\| \frac{\partial^j s_1}{\partial y^j} \right\| &\le C(1 + \epsilon^{1-j}), \quad j = 1,2,3,4. \end{aligned}$$

If $\alpha\mu^2 > \lambda\epsilon$, then it holds

$$\begin{aligned} |s_1(x,y)| &\le C \exp\left(- \frac{\alpha\mu^2}{\epsilon^2} x \right), \\ \left\| \frac{\partial^j s_1}{\partial y^j} \right\| &\le C\left(1 + \left(\frac{\epsilon}{\mu} \right)^{-2j} \right), \quad j = 1,2,3,4. \end{aligned}$$

Proof It can be referred from the works by O'Riordan et al. [14, 15]. □

Similarly, as has been done for s_1, for the different edges of Ω_k, $k = 1,2,3,4$, (see Fig. 1) we can consider the corresponding boundary layer components s_i, $i = 2,3,\dots,16$, (see Fig. 2) for which we can obtain similar bounds as in Lemma 2.7.

Related to the corner at $c_{1,1} = (0,0)$ in Ω_1, we consider the corner layer component p_1, which is determined by

$$L_{\epsilon,\mu} p_1(x,y) = 0, \quad \forall (x,y) \in \Omega, \tag{2.9a}$$

$$p_1(x,y) = -s_1(x,y),\ \forall (x,y) \in \Lambda_{1,1}, \quad p_1(x,y) = -s_2(x,y),\ \forall (x,y) \in \Lambda_{1,2}, \tag{2.9b}$$

$$p_1(x,y) = 0,\ \forall (x,y) \in \Lambda_{2,2} \cup \Lambda_{3,1} \cup \Lambda_3 \cup \Lambda_4, \tag{2.9c}$$

$$p_1(x,y) = 0, \quad \forall (x,y) \in \Gamma_1 \cup \Gamma_2, \tag{2.9d}$$

$$\left[(p_1)_x\right](x,y) = 0, \quad \forall (x,y) \in \Gamma_1,\ [(p_1)_y](x,y) = 0, \quad \forall (x,y) \in \Gamma_2. \tag{2.9e}$$

Lemma 2.8 *Let p_1 be the corner layer component satisfying the equations in (2.9). If $\alpha\mu^2 \le \lambda\epsilon$, then it holds*

$$|p_1(x,y)| \leq C\exp\left(-\frac{\alpha\lambda}{\epsilon}x\right)\exp\left(-\frac{\alpha\lambda}{\epsilon}y\right), \quad \left\|\frac{\partial^{i+j}p_1}{\partial x^i \partial y^j}\right\| \leq C(1+\epsilon^{-(i+j)}), \quad 1 \leq i+j \leq 4.$$

If $\alpha\mu^2 > \lambda\epsilon$, *then it holds*

$$|p_1(x,y)| \leq C\exp\left(-\frac{\alpha\mu^2}{\epsilon^2}x\right)\exp\left(-\frac{\alpha\mu^2}{\epsilon^2}y\right), \quad \left\|\frac{\partial^{i+j}p_1}{\partial x^i \partial y^j}\right\| \leq C\left(\frac{\epsilon}{\mu}\right)^{-(2i+2j)}, \quad 1 \leq i+j \leq 4.$$

Proof It can be referred from the works by O'Riordan et al. [14, 15]. □

Similarly, we can describe other corner layer components p_k, $k = 2, 3, \ldots, 16$, corresponding to the different corners of Ω_k, $k = 1, 2, 3, 4$, which verify similar bounds as the ones in Lemma 2.8.

Finally, from the above lemmas we can establish the following theorem.

Theorem 2.9 *The solution* z *of* (1.1) *may be written as*

$$z = \sum_{k=1}^{4} r_k + \sum_{j=1}^{16} s_j + \sum_{j=1}^{16} p_j,$$

where

$$L_{\epsilon,\mu} r_k = f, \quad L_{\epsilon,\mu} s_j = 0, \quad L_{\epsilon,\mu} p_j = 0, \ k = 1, 2, 3, 4, \ j = 1, 2, 3, \ldots, 16.$$

Furthermore, the regular and singular components and their derivatives satisfy the following bounds

$$\begin{cases} \left\|\dfrac{\partial^{i+j} r_k}{\partial x^i \partial y^j}\right\| \leq C(1+\epsilon^{2-(i+j)}), & 1 \leq i+j \leq 4, \ k = 1,2,3,4, \ \text{ if } \alpha\mu^2 \leq \lambda\epsilon, \\ \left\|\dfrac{\partial^{i+j} r_k}{\partial x^i \partial y^j}\right\| \leq C(1+\left(\frac{\epsilon}{\mu}\right)^{4-(2i+2j)}), & 1 \leq i+j \leq 4, \ k = 1,2,3,4, \ \text{ if } \alpha\mu^2 > \lambda\epsilon, \end{cases}$$

$$\begin{cases} |s_1(x,y)| & \leq Ce^{-\alpha\theta_1 x}; \\ |s_2(x,y)| & \leq Ce^{-\beta\theta_1 y}; \\ |s_3(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}; \\ |s_4(x,y)| & \leq Ce^{-\beta\theta_2 (d-y)}; \end{cases}$$

$$\begin{cases} |s_5(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}; \\ |s_6(x,y)| & \leq Ce^{-\beta\theta_1 y}; \\ |s_7(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}; \\ |s_8(x,y)| & \leq Ce^{-\beta\theta_2 (d-y)}; \end{cases}$$

$$\begin{cases} |s_9(x,y)| & \leq Ce^{-\alpha\theta_1 x}; \\ |s_{10}(x,y)| & \leq Ce^{-\beta\theta_2 (y-d)}; \\ |s_{11}(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}; \\ |s_{12}(x,y)| & \leq Ce^{-\beta\theta_1 (1-y)}; \end{cases}$$

$$\begin{cases} |s_{13}(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}; \\ |s_{14}(x,y)| & \leq Ce^{-\beta\theta_2 (y-d)}; \\ |s_{15}(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}; \\ |s_{16}(x,y)| & \leq Ce^{-\beta\theta_1 (1-y)}; \end{cases}$$

$$\begin{cases} |p_1(x,y)| & \leq Ce^{-\alpha\theta_1 x}e^{-\beta\theta_1 y}, \\ |p_2(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}e^{-\beta\theta_1 y}, \\ |p_3(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}e^{-\beta\theta_2 (d-y)}, \\ |p_4(x,y)| & \leq Ce^{-\alpha\theta_1 x}e^{-\beta\theta_2 (d-y)}, \end{cases}$$

$$\begin{cases} |p_5(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}e^{-\beta\theta_1 y}, \\ |p_6(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}e^{-\beta\theta_1 y}, \\ |p_7(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}e^{-\beta\theta_2 (d-y)}, \\ |p_8(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}e^{-\beta\theta_2 (d-y)}, \end{cases}$$

$$\begin{cases} |p_9(x,y)| & \leq Ce^{-\alpha\theta_1 x}e^{-\beta\theta_2 (y-d)}, \\ |p_{10}(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}e^{-\beta\theta_2 (y-d)}, \\ |p_{11}(x,y)| & \leq Ce^{-\alpha\theta_2 (d-x)}e^{-\beta\theta_1 (1-y)}, \\ |p_{12}(x,y)| & \leq Ce^{-\alpha\theta_1 x}e^{-\beta\theta_1 (1-y)}, \end{cases}$$

$$\begin{cases} |p_{13}(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}e^{-\beta\theta_2 (y-d)}, \\ |p_{14}(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}e^{-\beta\theta_2 (y-d)}, \\ |p_{15}(x,y)| & \leq Ce^{-\alpha\theta_1 (1-x)}e^{-\beta\theta_1 (1-y)}, \\ |p_{16}(x,y)| & \leq Ce^{-\alpha\theta_2 (x-d)}e^{-\beta\theta_1 (1-y)}, \end{cases}$$

$$\text{where } \theta_1 = \begin{cases} \frac{\lambda}{\epsilon}, & \alpha\mu^2 \leq \lambda\epsilon, \\ \frac{\mu^2}{\epsilon^2}, & \mu^2 > \lambda\epsilon, \end{cases} \qquad \theta_2 = \begin{cases} \frac{\lambda}{2\epsilon}, & \alpha\mu^2 \leq \lambda\epsilon, \\ \frac{\lambda}{2\mu^2}, & \alpha\mu^2 > \lambda\epsilon, \end{cases} \tag{2.10}$$

$$\begin{cases} \left|\left|\frac{\partial^i s_k}{\partial x^i}\right|\right| \leq C(1+\epsilon^{(1-i)}), & \text{where } k = 2,4,6,8,10,12,14,16 \text{ and } 1 \leq i \leq 4, \text{ if } \alpha\mu^2 \leq \lambda\epsilon, \\ \left|\left|\frac{\partial^i s_k}{\partial x^i}\right|\right| \leq C\left(1+\left(\frac{\epsilon}{\mu}\right)^{2-2i}\right), & \text{where } k = 2,6,12,16 \text{ and } 1 \leq i \leq 4, \text{ if } \alpha\mu^2 > \lambda\epsilon, \\ \left|\left|\frac{\partial^i s_k}{\partial x^i}\right|\right| \leq C\left(1+\left(\frac{1}{\mu}\right)^{2i-2}\right), & \text{where } k = 4,8,10,14 \text{ and } 1 \leq i \leq 4, \text{ if } \alpha\mu^2 > \lambda\epsilon, \end{cases}$$

$$\begin{cases} \left|\left|\frac{\partial^j s_k}{\partial y^j}\right|\right| \leq C(1+\epsilon^{(1-j)}), & \text{where } k = 1,3,5,7,9,11,13,15 \text{ and } 1 \leq j \leq 4, \text{ if } \alpha\mu^2 \leq \lambda\epsilon, \\ \left|\left|\frac{\partial^j s_k}{\partial y^j}\right|\right| \leq C\left(1+\left(\frac{\epsilon}{\mu}\right)^{2-2j}\right), & \text{where } k = 1,7,9,15 \text{ and } 1 \leq j \leq 4, \text{ if } \alpha\mu^2 > \lambda\epsilon, \\ \left|\left|\frac{\partial^j s_k}{\partial y^j}\right|\right| \leq C\left(1+\left(\frac{1}{\mu}\right)^{2j-2}\right), & \text{where } k = 3,5,11,13 \text{ and } 1 \leq j \leq 4, \text{ if } \alpha\mu^2 > \lambda\epsilon, \end{cases}$$

$$\begin{cases} \max\left\{\left\|\frac{\partial^{i+j}s_k}{\partial x^i\partial y^j}\right\|,\left\|\frac{\partial^{i+j}p_k}{\partial x^i\partial y^j}\right\|\right\} \le C\epsilon^{-(i+j)}, & 1\le i+j\le 4,\ \text{ if } \alpha\mu^2\le\lambda\epsilon, \\ \left\|\frac{\partial^{i+j}p_k}{\partial x^i\partial y^j}\right\| \le C\left(\frac{\epsilon}{\mu}\right)^{-(2i+2j)}, & 1\le i+j\le 4,\ \text{ if } \alpha\mu^2>\lambda\epsilon,\ k=1,6,12,15, \\ \left\|\frac{\partial^{i+j}p_k}{\partial x^i\partial y^j}\right\| \le C\left(\frac{1}{\mu}\right)^{2i}\left(\frac{\epsilon}{\mu}\right)^{-2j}, & 1\le i+j\le 4,\ \text{ if } \alpha\mu^2>\lambda\epsilon,\ k=2,5,11,16, \\ \left\|\frac{\partial^{i+j}p_k}{\partial x^i\partial y^j}\right\| \le C\left(\frac{\epsilon}{\mu}\right)^{-2i}\left(\frac{1}{\mu}\right)^{2j}, & 1\le i+j\le 4,\ \text{ if } \alpha\mu^2>\lambda\epsilon,\ k=4,7,9,14, \\ \left\|\frac{\partial^{i+j}p_k}{\partial x^i\partial y^j}\right\| \le C\left(\frac{1}{\mu}\right)^{2i+2j}, & 1\le i\le 4,\ \text{ if } \alpha\mu^2>\lambda\epsilon,\ k=3,8,10,13. \end{cases}$$ □

The solution $z(x,y)$ of (1.1) can be determined as follows

$$z(x,y)=\begin{cases} (r_1+s_1+s_2+s_3+s_4+p_1+p_2+p_3+p_4)(x,y), & \forall(x,y)\in\Omega_1, \\ (r_2+s_5+s_6+s_7+s_8+p_5+p_6+p_7+p_8)(x,y), & \forall(x,y)\in\Omega_2, \\ (r_3+s_9+s_{10}+s_{11}+s_{12}+p_9+p_{10}+p_{11}+p_{12})(x,y), & \forall(x,y)\in\Omega_3, \\ (r_4+s_{13}+s_{14}+s_{15}+s_{16}+p_{13}+p_{14}+p_{15}+p_{16})(x,y), & \forall(x,y)\in\Omega_4, \\ [(r+s+p)](x,y)=0,\ [(r+s+p)_x](x,y)=0, & \forall(x,y)\in\Gamma_1, \\ [(r+s+p)](x,y)=0,\ [(r+s+p)_y](x,y)=0, & \forall(x,y)\in\Gamma_2. \end{cases}$$

where, $r=\sum_{i=1}^{4} r_i,\ s=\sum_{k=1}^{16} s_k,\ p=\sum_{k=1}^{16} p_k.$

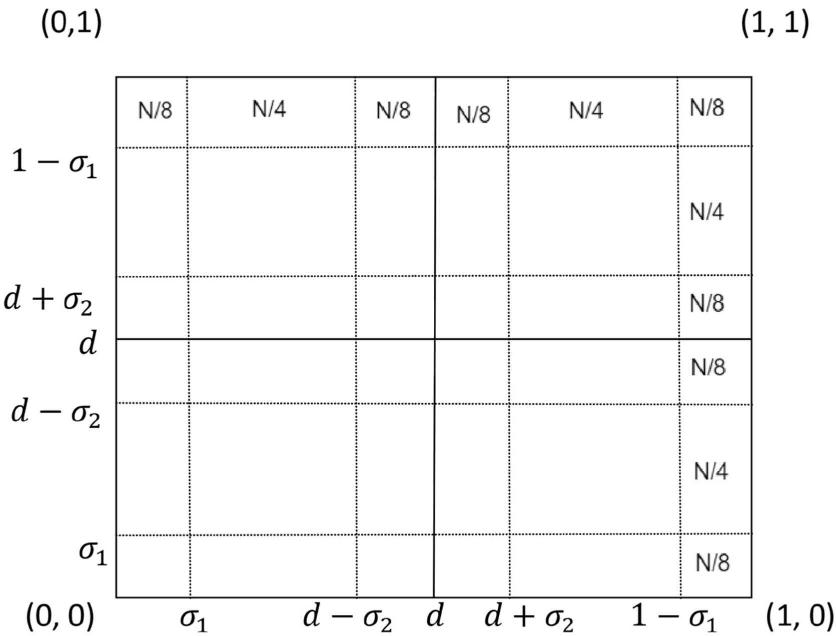


Fig. 3 Description of the Shishkin Mesh

Discretization of the Problem

In this section, we introduce a piecewise uniform Shishkin mesh for problem (1.1) and use the finite-difference technique on this mesh to get the numerical solution. To construct a suitable fitted piecewise uniform mesh, we first subdivided the unit interval in both x and y directions into six subintervals as

$$[0,1] = [0,\sigma_1] \cup [\sigma_1, d-\sigma_2] \cup [d-\sigma_2, d] \cup [d, d+\sigma_2] \cup [d+\sigma_2, 1-\sigma_1] \cup [1-\sigma_1, 1]. \tag{3.1}$$

On the subintervals $[\sigma_1, d-\sigma_2]$ and $[d+\sigma_2, 1-\sigma_1]$ we take $N/4$ subdivisions, while in the remaining subintervals we take $N/8$ subdivisions on each (see, Fig. 3). The transition points, σ_1 and σ_2, are defined as

$$\sigma_1 = \min\left\{\frac{d}{4}, \frac{2}{\theta_1}\ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{d}{4}, \frac{2}{\theta_2}\ln N\right\}, \tag{3.2}$$

where θ_1 and θ_2 are defined in (2.10).

In view of the above, the corresponding step sizes on each direction are given by

$$\begin{aligned} h_1 =& k_1 = \frac{8\sigma_1}{N}, \quad H_1 = K_1 = \frac{4(d-\sigma_1-\sigma_2)}{N}, \\ h_2 =& k_2 = \frac{8\sigma_2}{N}, \quad H_2 = K_2 = \frac{4(1-\sigma_1-\sigma_2-d)}{N}. \end{aligned}$$

The interior points of the mesh are denoted by $\Omega^{N,N} = \bigcup_{k=1}^{4} \Omega_k^{N,N}$, where

$$\begin{aligned} \Omega_1{}^{N,N} =& \left\{ (x_i, y_j) \,:\, 1 \le i \le \frac{N}{2} - 1, 1 \le j \le \frac{N}{2} - 1 \right\}; \\ \Omega_2{}^{N,N} =& \left\{ (x_i, y_j) \,:\, \frac{N}{2} + 1 \le i \le N - 1, 1 \le j \le \frac{N}{2} - 1 \right\}; \\ \Omega_3^{N,N} =& \left\{ (x_i, y_j) \,:\, 1 \le i \le \frac{N}{2} - 1, \frac{N}{2} + 1 \le j \le N - 1 \right\}; \\ \Omega_4^{N,N} =& \left\{ (x_i, y_j) \,:\, \frac{N}{2} + 1 \le i \le N - 1, \frac{N}{2} + 1 \le j \le N - 1 \right\}; \\ \Gamma_1^{N,N} =& \left\{ (x_{N/2}, y_j) \,:\, 1 \le j \le N \right\}; \quad \Gamma_2^{N,N} = \left\{ (x_i, y_{N/2}) \,:\, 1 \le i \le N \right\}. \end{aligned}$$

The boundaries of these subdomains are denoted as

$$\Lambda_1^{N,N} = \left\{ (0, y_j) \,|\, (0 \le j < N/2) \cup (N/2 < j \le N) \right\},$$
$$\Lambda_2^{N,N} = \left\{ (x_i, 0) \,|\, (0 \le i < N/2) \cup (N/2 < i \le N) \right\},$$
$$\Lambda_3^{N,N} = \left\{ (1, y_j) \,|\, (0 \le j < N/2) \cup (N/2 < j \le N) \right\},$$
$$\Lambda_4^{N,N} = \left\{ (x_i, 1) \,|\, (0 \le i < N/2) \cup (N/2 < i \le N) \right\},$$

and $\Lambda^{N,N} = \Lambda_1^{N,N} \cup \Lambda_2^{N,N} \cup \Lambda_3^{N,N} \cup \Lambda_4^{N,N}$. Note that $\bar{\Omega}^{N,N} = \left\{ (x_i, y_j) \,:\, 0 \le i \le N, 0 \le j \le N \right\}$.

On an arbitrary mesh, $\bar{\Omega}^{N,N}$, in order to discretized the problem (1.1) we define the standard upwind finite difference operator

$$\begin{cases} L_{\epsilon,\mu}^{N,N} Z(x_i, y_j) \equiv \epsilon^2(\delta_{xx}^2 + \delta_{yy}^2) Z(x_i, y_j) + \mu^2(a_{ij} D_x^* + b_{ij} D_y^*) Z(x_i, y_j) - c_{ij} Z(x_i, y_j) = f_{ij}, \ \forall (x_i, y_j) \in \Omega^{N,N}, \\ Z(x_i, y_j) = g_1(y_j), \ (x_i, y_j) \in \Lambda_1^{N,N}, \quad Z(x_i, y_j) = g_2(x_i), \ (x_i, y_j) \in \Lambda_2^{N,N}, \\ Z(x_i, y_j) = g_3(y_j), \ (x_i, y_j) \in \Lambda_3^{N,N}, \quad Z(x_i, y_j) = g_4(x_i), \ (x_i, y_j) \in \Lambda_4^{N,N}, \\ D_x^- Z(x_i, y_j) = D_x^+ Z(x_i, y_j), \ (x_i, y_j) \in \Gamma_1^{N,N}, \\ D_y^- Z(x_i, y_j) = D_y^+ Z(x_i, y_j), \ (x_i, y_j) \in \Gamma_2^{N,N}. \end{cases} \tag{3.3}$$

Further, the discrete differential operators D_x^*, D_y^*, δ_{xx}^2, and δ_{yy}^2 are considered as follows:

$$D_x^* Z(x_i, y_j) = \begin{cases} D_x^+ Z(x_i, y_j), \ i < N/2, \\ D_x^- Z(x_i, y_j), \ i > N/2, \end{cases} \qquad D_y^* Z(x_i, y_j) = \begin{cases} D_y^+ Z(x_i, y_j), \ j < N/2, \\ D_y^- Z(x_i, y_j), \ j > N/2, \end{cases}$$

$$\delta_{xx}^2 Z(x_i, y_j) = \frac{1}{\bar{h}_i}(D_x^+ Z(x_i, y_j) - D_x^- Z(x_i, y_j)), \quad \delta_{yy}^2 Z(x_i, y_j) = \frac{1}{\bar{k}_j}(D_y^+ Z(x_i, y_j) - D_y^- Z(x_i, y_j)), \quad \text{with}$$

$$D_x^+ Z(x_i, y_j) = \frac{Z(x_{i+1}, y_j) - Z(x_i, y_j)}{h_{i+1}},$$

$$D_x^- Z(x_i, y_j) = \frac{Z(x_i, y_j) - Z(x_{i-1}, y_j)}{h_i}.$$

Lemma 3.1 (Discrete minimum principle): *Let* $L_{\epsilon,\mu}^{N,N}$ *be the discrete operator given in* (3.3), *If* $\phi(x_i, y_j) \ge 0$ *on* $\Lambda^{N,N}$, $L_{\epsilon,\mu}^{N,N} \phi(x_i, y_j) \le 0$, $\forall (x_i, y_j) \in \Omega^{N,N}$, $(D_x^+ \phi(x_i, y_j) - D_x^- \phi(x_i, y_j)) \le 0$, $\forall (x_i, y_j) \in \Gamma_1^{N,N}$ *and* $(D_y^+ \phi(x_i, y_j) - D_y^- \phi(x_i, y_j)) \le 0$, $\forall (x_i, y_j) \in \Gamma_2^{N,N}$ *then* $\phi(x_i, y_j) \ge 0$, $\forall (x_i, y_j) \in \bar{\Omega}^{N,N}$.

Proof We can prove the present Lemma using [15, 20]. □

Lemma 3.2 (Discrete stability result) *Let* $Z(x_i, y_j)$ *be the solution of* (3.3). *Then it holds*

$$||Z(x_i, y_j)||_{\bar{\Omega}^{N,N}} \leq \frac{1}{\nu}||L_{\epsilon,\mu}Z||_{\Omega^{N,N}} + \max\left\{||Z||_{\Lambda_1^{N,N}}, ||Z||_{\Lambda_2^{N,N}}, ||Z||_{\Lambda_3^{N,N}}, ||Z||_{\Lambda_4^{N,N}}\right\},$$

where ||.|| *denotes the pointwise maximum norm.*

Proof It can be proved using Lemma 3.1. □

Error Analysis

Lemma (3.2), will be used to proof the uniform convergence. Using standard techniques, the local truncation error may be readily bounded as

$$|L_{\epsilon,\mu}^{N,N}(Z-z)(x_i,y_j)| \leq \begin{cases} C\epsilon^2\left(\bar{h}_i\left|\left|\frac{\partial^3 z}{\partial x^3}\right|\right| + \bar{k}_j\left|\left|\frac{\partial^3 z}{\partial y^3}\right|\right|\right) + C\mu^2\left(h_{i+1}\left|\left|\frac{\partial^2 z}{\partial x^2}\right|\right| + k_{j+1}\left|\left|\frac{\partial^2 z}{\partial y^2}\right|\right|\right), \\ \quad \text{if } x_i = \sigma_1, d-\sigma_2, d+\sigma_2, 1-\sigma_1, \\ \quad \text{or } y_j = \sigma_1, d-\sigma_2, d+\sigma_2, 1-\sigma_1, \\ C\epsilon^2\left({h_i}^2\left|\left|\frac{\partial^4 z}{\partial x^4}\right|\right| + {k_j}^2\left|\left|\frac{\partial^4 z}{\partial y^4}\right|\right|\right) + C\mu^2\left(h_i\left|\left|\frac{\partial^2 z}{\partial x^2}\right|\right| + k_j\left|\left|\frac{\partial^2 z}{\partial y^2}\right|\right|\right), \\ \quad \text{otherwise.} \end{cases} \tag{4.1}$$

To get suitable bounds of this error, we decompose the discrete solution as

$$Z = \sum_{k=1}^{4} R_k + \sum_{l=1}^{16} S_l + \sum_{m=1}^{16} P_m,$$

where R_k are the discrete regular components, S_l the discrete singular components, and P_m the discrete corner components. The three components are, respectively, solutions of the following problems

$$L_{\epsilon,\mu}^{N,N} R_k(x_i, y_j) = f(x_i, y_j), \quad \forall (x_i, y_j) \in \Omega^{N,N}, \quad k = 1, 2, 3, 4, \tag{4.2a}$$

$$R_k(x_i, y_j) = r_k(x_i, y_j), \quad \forall (x_i, y_j) \in \Lambda^{N,N}, \tag{4.2b}$$

$$\left[R_k\right](x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Gamma_1^{N,N} \cup \Gamma_2^{N,N}, \tag{4.2c}$$

$$\left[(R_k)_x\right](x_i, y_j) = 0, \ \forall (x_i, y_j) \in \Gamma_1^{N,N}, \ [(R_k)_y](x_i, y_j) = 0, \ \forall (x_i, y_j) \in \Gamma_2^{N,N}. \tag{4.2d}$$

$$L_{\epsilon,\mu}^{N,N} S_l(x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Omega^{N,N}, \quad l = 1, ..., 16, \tag{4.3a}$$

$$S_l(x_i, y_j) = s_l(x_i, y_j), \quad \forall (x_i, y_j) \in \Lambda^{N,N}, \tag{4.3b}$$

$$\left[S_l\right](x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Gamma_1^{N,N} \cup \Gamma_2^{N,N}, \tag{4.3c}$$

$$\left[(S_l)_x\right](x_i, y_j) = 0,\ \forall (x_i, y_j) \in \Gamma_1^{N,N},\ [(S_l)_y](x_i, y_j) = 0,\ \forall (x_i, y_j) \in \Gamma_2^{N,N}. \tag{4.3d}$$

$$L_{\epsilon,\mu}^{N,N} P_m(x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Omega^{N,N}, \quad m = 1, ..., 16, \tag{4.4a}$$

$$P_m(x_i, y_j) = p_m(x_i, y_j), \quad \forall (x_i, y_j) \in \Lambda^{N,N}, \tag{4.4b}$$

$$\left[P_m\right](x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Gamma_1^{N,N} \cup \Gamma_2^{N,N}, \tag{4.4c}$$

$$\left[(P_m)_x\right](x_i, y_j) = 0,\ \forall (x_i, y_j) \in \Gamma_1^{N,N},\ [(P_m)_y](x_i, y_j) = 0,\ \forall (x_i, y_j) \in \Gamma_2^{N,N}. \tag{4.4d}$$

Using the result (2.5), and those from (2.3) and (4.2) we get the following straightforward estimate

$$|L_{\epsilon,\mu}^{N,N}(R_k - r_k)(x_i, y_j)| \le \begin{cases} CN^{-1}, & \text{if } x_i = \sigma_1, d - \sigma_2, d + \sigma_2, 1 - \sigma_1, \text{ or } y_j = \sigma_1, d - \sigma_2, d + \sigma_2, 1 - \sigma_1, \\ C(N^{-2} + \epsilon N^{-1}), & \text{otherwise.} \end{cases}$$

Following [7, 19], we consider the barrier function

$$\Psi(x_i, y_j) = CN^{-2}(\Phi(x_i) + \Phi(y_j)) + CN^{-1},$$

where $\Phi(\phi_i)$ is the piecewise-linear polynomial

$$\Phi(\phi_i) = \begin{cases} 1, & 0 \le \phi_i \le \sigma_1, \\ 1 - \frac{\phi_i - \sigma_1}{2(d - \sigma_1 - \sigma_2)}, & \sigma_1 \le \phi_i \le d - \sigma_2, \\ \frac{d - \phi_i}{2\sigma_2}, & d - \sigma_2 \le \phi_i \le d, \\ \frac{\phi_i - d}{2\sigma_2}, & d \le \phi_i \le d + \sigma_2, \\ 1 - \frac{\phi_i - d - \sigma_2}{2(1 - d - \sigma_1 - \sigma_2)}, & d + \sigma_2 \le \phi_i \le 1 - \sigma_1, \\ \frac{1 - \phi_i}{2\sigma_1}, & 1 - \sigma_1 \le \phi_i \le 1. \end{cases}$$

Noting that $1/\sigma_2 \ge 4$, we have that

$$\epsilon^2 \delta_x^2 \Psi(x_i, y_j) = \begin{cases} O(-N^{-1}\epsilon^2), & x_i = \sigma_1, d - \sigma_2, d + \sigma_2, 1 - \sigma_1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\epsilon^2 \delta_y^2 \Psi(x_i, y_j) = \begin{cases} O(-N^{-1}\epsilon^2), & y_j = \sigma_1, d - \sigma_2, d + \sigma_2, 1 - \sigma_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$D_x^+ \Psi(x_i, y_j) \le 0,\ D_y^+ \Psi(x_i, y_j) \le 0,$$
$$D_x^- \Psi(x_i, y_j) \ge 0,\ D_y^- \Psi(x_i, y_j) \ge 0.$$

Combining this with Lemma 3.2 we get

$$||R_k - r_k|| \le CN^{-1}, \quad k = 1, 2, 3, 4, \tag{4.5}$$

which shows a suitable bound for the error of the regular components.

We utilize evidence-based on suitable barrier functions to show ϵ-uniform bounds of the errors related to the corner and edge components. We consider the barrier functions as shown below:

$$\begin{cases} G_{s_1;i} = \begin{cases} \prod_{a=1}^{i}(1+h_a\alpha\theta_1)^{-1} & , \quad i \neq 0, \quad 1 \leq i < N/2, \\ 1 & , \quad i = 0, \end{cases} \\ G_{s_2;j} = \begin{cases} \prod_{a=1}^{j}(1+k_a\beta\theta_1)^{-1} & , \quad j \neq 0, \quad 1 \leq j < N/2, \\ 1 & , \quad j = 0, \end{cases} \\ G_{s_3;i} = \begin{cases} \prod_{a=i+1}^{N/2}(1+h_a\alpha\theta_2)^{-1} & , \quad i \neq N/2, \quad 0 \leq i < N/2, \\ 1 & , \quad i = N/2, \end{cases} \\ G_{s_4;j} = \begin{cases} \prod_{a=j+1}^{N/2}(1+k_a\beta\theta_2)^{-1} & , \quad j \neq N/2, \quad 0 \leq j < N/2, \\ 1 & , \quad j = N/2, \end{cases} \end{cases}$$

$$\begin{cases} G_{s_5;i} = \begin{cases} \prod_{a=N/2+1}^{i}(1+h_a\alpha\theta_2)^{-1} & , \quad i \neq N, \quad N/2+1 \leq i < N, \\ 1 & , \quad i = N, \end{cases} \\ G_{s_6;j} = \begin{cases} \prod_{a=1}^{j}(1+k_a\beta\theta_1)^{-1} & , \quad j \neq N/2, \quad 1 \leq j < N/2, \\ 1 & , \quad j = N/2, \end{cases} \\ G_{s_7;i} = \begin{cases} \prod_{a=i+1}^{N}(1+h_a\alpha\theta_1)^{-1} & , \quad i \neq N, \quad N/2+1 \leq i < N, \\ 1 & , \quad i = N, \end{cases} \\ G_{s_8;j} = \begin{cases} \prod_{a=j+1}^{N/2}(1+k_a\beta\theta_2)^{-1} & , \quad j \neq N/2, \quad 0 \leq j < N/2, \\ 1 & , \quad j = N/2, \end{cases} \end{cases}$$

$$\begin{cases} G_{s_9;i} = \begin{cases} \prod_{a=1}^{i}(1+h_a\alpha\theta_1)^{-1} & , \quad i \neq 0, \quad 1 \leq i < N/2, \\ 1 & , \quad i = 0, \end{cases} \\ G_{s_{10};j} = \begin{cases} \prod_{a=N/2+1}^{j}(1+k_a\beta\theta_2)^{-1} & , \quad j \neq N, \quad N/2+1 \leq j < N, \\ 1 & , \quad j = N, \end{cases} \\ G_{s_{11};i} = \begin{cases} \prod_{a=i+1}^{N/2}(1+h_a\alpha\theta_2)^{-1} & , \quad i \neq N/2, \quad 1 \leq i < N/2, \\ 1 & , \quad i = N/2, \end{cases} \\ G_{s_{12};j} = \begin{cases} \prod_{a=N/2+1}^{j}(1+k_a\beta\theta_1)^{-1} & , \quad j \neq N, \quad N/2+1 \leq j < N, \\ 1 & , \quad j = N, \end{cases} \end{cases}$$

$$
\begin{cases}
G_{s_{13};i} = \begin{cases} \prod_{a=N/2+1}^{i} (1 + h_a \alpha \theta_2)^{-1}, & i \neq N, \quad N/2+1 \leq i < N, \\ 1 & , \quad i = N, \end{cases} \\
G_{s_{14};j} = \begin{cases} \prod_{a=N/2+1}^{j} (1 + k_a \beta \theta_2)^{-1}, & j \neq N, \quad N/2+1 \leq j < N, \\ 1 & , \quad j = N, \end{cases} \\
G_{s_{15};i} = \begin{cases} \prod_{a=i+1}^{N} (1 + h_a \alpha \theta_1)^{-1} & , \quad i \neq N, \quad N/2+1 \leq i < N, \\ 1 & , \quad i = N, \end{cases} \\
G_{s_{16};j} = \begin{cases} \prod_{a=N/2+1}^{j} (1 + k_a \beta \theta_1)^{-1}, & j \neq N, \quad N/2+1 \leq j < N, \\ 1 & , \quad j = N. \end{cases}
\end{cases}
$$

The above functions depict first-order Taylor estimates of the exponential functions associated with the singular components of problem (1.1). For all $1 \leq i < N/2$, we have

$\exp(-\alpha\theta_1 x_i) = \prod_{a=1}^{i} \exp(-\alpha\theta_1 h_a) \leq G_{s_{1,i}}$,

and for $\sigma_1 < \frac{d}{8}$ and $N/8 \leq i \leq N/2$ we have

$$G_{s_{1,i}} \leq G_{s_{1,N/8}} = \left(1 + \frac{16 \ln N}{N}\right)^{-N/8} \leq CN^{-1}, \tag{4.6}$$

$$L^N_{\epsilon,\mu} G_{s_{1,i}} \leq (\epsilon^2\alpha^2\theta_1^2 - \mu^2 a(x_i, y_j)\alpha\theta_1 - c(x_i, y_j))G_{s_{1,i}}. \tag{4.7}$$

Similar bounds can be obtained for the remaining edge functions.

Lemma 4.1 *If s_l and S_l are the solutions of (2.8) and (4.3), respectively, then, for $l = 1, 2, ..., 16$,*

$$|s_l(x_i, y_j) - S_l(x_i, y_j)| \leq C(N^{-1} \ln N), \quad \text{if } \alpha\mu^2 \leq \lambda\epsilon.$$

Proof If $\sigma_1 = d/4$ the proof can be obtained using standard techniques by taking into account that $\epsilon^{-2} \leq C(\ln N)^2$. Thus, we will asume that $\sigma_1 < d/4$. Here we merely provide the specifications pertaining to the edge layer function s_1. Similar results are valid for the remaining boundary layer functions. □

From (4.3) and Theorem 2.9, it follows that

$$|S_1(x_i, y_j)| = |s_1(x_i, y_j)| \leq Ce^{-\alpha\theta_1 x_i} \leq CG_{s_{1,i}}, \quad (x_i, y_j) \in \Lambda^{N,N}. \tag{4.8}$$

Further, for all internal grid points $(x_i, y_j) \in \Omega_1^{N,N}$, from (4.3), (4.7), and the discrete minimum principle, we have

$$|S_1(x_i, y_j)| \leq G_{s_{1,i}}. \tag{4.9}$$

After applying Theorem 2.9 and (4.9), we conclude that

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le |s_1(x_i, y_j)| + |S_1(x_i, y_j)| \le CG_{s_{1,i}}.$$

Finally, from (4.6), we have

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le CN^{-1}, \qquad N/8 \le i \le N/2,\ 0 \le j \le N/2. \tag{4.10}$$

To get similar bounds of the error in the region $\Omega_{1,1}^{N,N} = \left\{(x_i, y_j) \,|\, 0 < i < N/8,\ 0 < j < N/2\right\}$, we proceed as follows. Applying Taylor expansions, we get

$$|L_{\epsilon,\mu}^{N,N}[S_1 - s_1](x_i, y_j)| \le \begin{cases} C\epsilon^2\left({h_i}^2\left\|\frac{\partial^4 s_1}{\partial x^4}\right\| + \bar{k}_j\left\|\frac{\partial^3 s_1}{\partial y^3}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 s_1}{\partial x^2}\right\| + k_j\left\|\frac{\partial^2 s_1}{\partial y^2}\right\|\right), & \text{if } j = N/8,\ 3N/8, \\ C\epsilon^2\left({h_i}^2\left\|\frac{\partial^4 s_1}{\partial x^4}\right\| + {k_j}^2\left\|\frac{\partial^4 s_1}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 s_1}{\partial x^2}\right\| + k_j\left\|\frac{\partial^2 s_1}{\partial y^2}\right\|\right), & \text{otherwise.} \end{cases} \tag{4.11}$$

If $\alpha\mu^2 \le \lambda\epsilon$, from Theorem 2.9, we have

$$|L_{\epsilon,\mu}^{N,N}[S_1(x_i, y_j) - s_1(x_i, y_j)]| \le \begin{cases} CN^{-1}\ln N + CN^{-1}, & \text{if } j = N/8,\ 3N/8, \\ CN^{-1}\ln N, & \text{otherwise,} \end{cases}$$

and using the discrete minimum principle and a suitable barrier function on $\bar{\Omega}_{1,1}^{N,N}$, we obtain

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le CN^{-1}\ln N, \quad (x_i, y_j) \in \bar{\Omega}_{1,1}^{N,N}. \tag{4.12}$$

The result follows easily from (4.10) and (4.12).

Similar results can be obtained for the remaining boundary and interior layer functions s_l, $l = 2, 3, ..., 16$.

Lemma 4.2 *If s_l and S_l are the solutions of* (2.8) *and* (4.3), *respectively*, *then*, *for $\alpha\mu^2 > \lambda\epsilon$ it holds*

$$|s_l(x_i, y_j) - S_l(x_i, y_j)| \le \begin{cases} CN^{-1}\ln^2 N, & \text{if } k = 1, 2, 6, 7, 9, 12, 15, 16, \\ CN^{-1}\ln N, & \text{if } l = 3, 4, 5, 8, 10, 11, 13, 14. \end{cases}$$

Proof If $\sigma_2 = d/4$ the proof can be obtained using standard techniques by taking into account that $(\frac{1}{\mu})^2 \le C\ln N$ and $\left(\frac{\mu}{\epsilon}\right)^2 \le C\ln N$. Thus, we will asume that $\sigma_2 < d/4$. Here we merely provide the specifications pertaining to the edge layer function s_1. Similar results can be obtained for the remaining boundary layer functions. □

From (4.3) and Theorem 2.9, we have

$$|s_1(x_i, y_j)| = |s_1(x_i, y_j)| \le Ce^{-\theta_1 x_i} \le CG_{s_{1,i}}, \quad (x_i, y_j) \in \Lambda^{N,N}. \tag{4.13}$$

Further, for all internal grid points $(x_i, y_j) \in \Omega_1^{N,N}$, from (4.3), (4.7), and the discrete minimum principle, we have

$$|S_1(x_i, y_j)| \le G_{s_{1,i}}. \tag{4.14}$$

Therefore, applying Theorem 2.9 and (4.14), we conclude that

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le |s_1(x_, y_j)| + |S_1(x_, y_j)| \le CG_{s_{1,i}}.$$

Therefore, from the corresponding result as in (4.6), we have

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le CN^{-1}, \qquad N/8 \le i \le N/2,\ 0 \le j \le N/2. \tag{4.15}$$

To get appropriate bounds for the error in the region $\Omega^N_{1,1}$, we proceed as follows. Applying Taylor series, we get

$$|L^{N,N}_{\epsilon,\mu}[S_1 - s_1](x_i, y_j)| \le \begin{cases} C\epsilon^2\left(h_i{}^2\left\|\frac{\partial^4 s_1}{\partial x^4}\right\| + \bar{k}_j\left\|\frac{\partial^3 s_1}{\partial y^3}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 s_1}{\partial x^2}\right\| + \bar{k}_j\left\|\frac{\partial^2 s_1}{\partial y^2}\right\|\right), & \text{if } j = N/8,\ 3N/8, \\ C\epsilon^2\left(h_i{}^2\left\|\frac{\partial^4 s_1}{\partial x^4}\right\| + k_j{}^2\left\|\frac{\partial^4 s_1}{\partial y^4}\right\|\right) + C\mu^2\left(h_i\left\|\frac{\partial^2 s_1}{\partial x^2}\right\| + k_j\left\|\frac{\partial^2 s_1}{\partial y^2}\right\|\right), & \text{otherwise.} \end{cases}$$

If $\alpha\mu^2 > \lambda\epsilon$, from Theorem 2.9, it follows that

$$|L^{N,N}_{\epsilon,\mu}[S_1(x_i, y_j) - s_1(x_i, y_j)]| \le \begin{cases} C\mu^4\epsilon^{-2}(N^{-1}\ln N + N^{-1}), & \text{if } j = N/8,\ 3N/8, \\ C\mu^4\epsilon^{-2}(N^{-1}\ln N), & \text{otherwise.} \end{cases}$$

After using the barrier function $\mu^2\epsilon^{-2}(\sigma_1 - x_i)$ to get a feasible bound on the error in the layer region $\Omega^{N,N}_{1,1}$, the application of the discrete minimum principle on $\Omega^{N,N}_{1,1}$, gives

$$|s_1(x_i, y_j) - S_1(x_i, y_j)| \le CN^{-1}\ln^2 N, \quad (x_i, y_j) \in \bar{\Omega}^{N,N}_{1,1}. \tag{4.16}$$

The result follows easily from (4.15) and (4.16).

Similarly, we can derive the corresponding bounds for the remaining boundary and interior layer functions s_l, $l = 2, 6, 7, 9, 12, 15, 16$.

If $\alpha\mu^2 > \lambda\epsilon$, we examine the boundary layer function s_3. From (4.6-4.11) and Theorem 2.9, it follows that

$$|L^{N,N}_{\epsilon,\mu}[S_3(x_i, y_j) - s_3(x_i, y_j)]| \le \begin{cases} C(N^{-1}\ln N + N^{-1}), & \text{if } j = N/8,\ 3N/8, \\ C(N^{-1}\ln N), & \text{otherwise.} \end{cases}$$

Therefore, the discrete minimum priciple, only on $\bar{\Omega}^{N,N}_{2,1} = \{(x_i, y_j) \mid 3N/8 \le i \le N/2,\ 0 \le j \le N/2\}$, leads to

$$|s_3(x_i, y_j) - S_3(x_i, y_j)| \le CN^{-1}\ln N, \quad (x_i, y_j) \in \bar{\Omega}^{N,N}_{2,1}. \tag{4.17}$$

We can proceed similarly to get appropriate bounds for the remaining boundary and interior layer functions s_l, $l = 4, 5, 8, 10, 11, 13, 14$.

Lemma 4.3 *If p_m and P_m are the solutions of* (2.9) *and* (4.4), *respectively, then for $m = 1, 2, 3, ..., 16$, it holds*

$$|p_m(x_i, y_j) - P_m(x_i, y_j)| \le \begin{cases} C(N^{-1}\ln N), & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ C(N^{-1}\ln^2 N), & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases} \tag{4.18}$$

Proof Again, we merely provide the proof of (4.18) for the corner layer component p_1 and in case of $\sigma_1 < \frac{d}{4}$. Proceeding similarly as in Lemma 4.1, we get

$$|P_1(x_i, y_j)| \leq C \min\{G_{s_{2,j}}, G_{s_{1,i}}\}, \quad \text{if } (x_i, y_j) \in \Lambda^{N,N},$$
$$|p_1(x_i, y_j) - P_1(x_i, y_j)| \leq C \min\{G_{s_{2,j}}, G_{s_{1,i}}\}, \quad \text{if } (x_i, y_j) \in \Omega^{N,N}, \quad 0 < i, j < N/2.$$

Then, applying (4.6) we conclude that

$$|p_1(x_i, y_j) - P_1(x_i, y_j)| \leq CN^{-1}, \qquad (x_i, y_j) \in \Omega^{N,N} \backslash \Omega_{1,2}^{N,N}, \tag{4.19}$$

where, $\Omega_{1,2}^{N,N} = \{(x_i, y_j) \mid 0 < i, j < N/8\}$. Ultimately, in $\Omega_{1,2}^{N,N}$ the truncation error satisfies

$$|L_{\epsilon,\mu}^{N,N}[P_1(x_i, y_j) - p_1(x_i, y_j)]| \leq C\epsilon^2 \left({h_i}^2 \left\| \frac{\partial^4 p_1}{\partial x^4} \right\| + {k_j}^2 \left\| \frac{\partial^4 p_1}{\partial y^4} \right\| \right) + C\mu^2 \left(h_i \left\| \frac{\partial^2 p_1}{\partial x^2} \right\| + k_j \left\| \frac{\partial^2 p_1}{\partial y^2} \right\| \right).$$

If $\alpha\mu^2 \leq \lambda\epsilon$, from Theorem 2.9, it follows that

$$|L_{\epsilon,\mu}^{N,N}[P_1(x_i, y_j) - p_1(x_i, y_j)]| \leq C(N^{-1} \ln N),$$

and using the discrete minimum principle on $\bar{\Omega}_{1,2}^{N,N}$, we get

$$|p_1(x_i, y_j) - P_1(x_i, y_j)| \leq CN^{-1} \ln N, \quad (x_i, y_j) \in \bar{\Omega}_{1,2}^{N,N}. \tag{4.20}$$

The result follows from (4.19) and (4.20).

If $\alpha\mu^2 > \lambda\epsilon$, from Theorem 2.9, it follows that

$$|L_{\epsilon,\mu}^{N,N}[P_1(x_i, y_j) - p_1(x_i, y_j)]| \leq C\mu^4\epsilon^{-2}(N^{-1} \ln N),$$

Using the barrier function $\mu^2\epsilon^{-2}(\sigma_1 - x_i)$ to attain a feasible bound on the error in the layer region $\Omega_{1,2}^{N,N}$,

and the discrete minimum principle on $\bar{\Omega}_{1,2}^{N,N}$, we obtain

$$|p_1(x_i, y_j) - P_1(x_i, y_j)| \leq CN^{-1} \ln^2 N, \quad (x_i, y_j) \in \bar{\Omega}_{1,2}^{N,N}. \tag{4.21}$$

The result follows from (4.19) and (4.21).

The discrete solution $Z(x_i, y_j)$ of (3.3) can be written as,

$$Z(x,y) = \begin{cases} (R_1 + S_1 + S_2 + S_3 + S_4 + P_1 + P_2 + P_3 + P_4)(x,y), & \forall (x,y) \in {\Omega_1}^{N,N}, \\ (R_2 + S_5 + S_6 + S_7 + S_8 + P_5 + P_6 + P_7 + P_8)(x,y), & \forall (x,y) \in {\Omega_2}^{N,N}, \\ (R_3 + S_9 + S_{10} + S_{11} + S_{12} + P_9 + P_{10} + P_{11} + P_{12})(x,y), & \forall (x,y) \in {\Omega_3}^{N,N}, \\ (R_4 + S_{13} + S_{14} + S_{15} + S_{16} + P_{13} + P_{14} + P_{15} + P_{16})(x,y), & \forall (x,y) \in \Omega_4^{N,N}, \\ [(R + S + P)](x,y) = 0, & (x,y) \in \Gamma_1^{N,N}, \\ [(R + S + P)](x,y) = 0, & (x,y) \in \Gamma_2^{N,N}. \end{cases}$$

where, $R = \sum_{k=1}^{4} R_k,\ S = \sum_{l=1}^{16} S_l,\ P = \sum_{m=1}^{16} P_m.$

Lemma 4.4 *Let z be the solution of problem (1.1) and Z the numerical solution of (3.3) on the constructed piecewise-uniform Shishkin mesh. Then the error at the mesh points $(x_i, y_j) \in \bar{\Omega}^{N,N}$ satisfies*

$$|Z(x_i, y_j) - z(x_i, y_j)| \le \begin{cases} CN^{-1}\ln N, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ CN^{-1}\ln^2 N, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases}$$

Proof Combining Lemmas 4.1, 4.2 and 4.3, we obtain the following error bound for $\Omega^{N,N}$

$$|Z(x_i, y_j) - z(x_i, y_j)| \le \begin{cases} CN^{-1}\ln N, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ CN^{-1}\ln^2 N, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases}$$

At the point $(x_i, y_j) = (d, y_j)$, we have $(D_x^+ - D_x^-)Z(d, y_j)$. Therefore,

$$\begin{aligned} |(D_x^+ - D_x^-)(Z - z)(d, y_j)| &= |(D_x^+ - D_x^-)Z(d, y_j) - (D_x^+ - D_x^-)z(d, y_j)| \\ &\le |(D_x^+ - D_x^-)z(d, y_j)|. \end{aligned}$$

Now, note that $h_2 = \frac{8\sigma_2}{N}$ on either side of (d, y_j). Therefore,

$$\begin{aligned} |(D_x^+ - D_x^-)(Z - z)(d, y_j)| &\le |(D_x^+ - D_x^-)z(d, y_j)| \\ &\le \left|(D_x^+ - \frac{d}{dx})z(d^+, y_j) + (D_x^- - \frac{d}{dx})z(d^-, y_j)\right| \\ &\le C\frac{h_2}{2}\left|\frac{\partial^2 z}{\partial x^2}\right|_{\Omega^{N,N}} \\ |(D_x^+ - D_x^-)(Z - z)(d, y_j)| &\le C\begin{cases} \frac{h_2}{2}(\frac{1}{\epsilon})^2, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ \frac{h_2}{2}(\frac{1}{\mu})^2, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases} \end{aligned}$$

At the point $(x_i, y_j) = (x_i, d)$, we have $(D_y^+ - D_y^-)Z(x_i, d)$. Therefore,

$$\begin{aligned} |(D_y^+ - D_y^-)(Z - z)(x_i, d)| &= |(D_y^+ - D_y^-)Z(x_i, d) - (D_y^+ - D_y^-)z(x_i, d)| \\ &\le |(D_y^+ - D_y^-)z(x_i, d)|. \end{aligned}$$

Now, note that $k_2 = \frac{8\sigma_2}{N}$ on either side of (x_i, d). Therefore,

$$\begin{aligned} |(D_y^+ - D_y^-)(Z - z)(x_i, d)| &\le |(D_y^+ - D_y^-)z(x_i, d)| \\ &\le \left|(D_y^+ - \frac{d}{dy})z(x_i, d^+) + (D_y^- - \frac{d}{dy})z(x_i, d^-)\right| \\ &\le C\frac{k_2}{2}\left|\frac{\partial^2 z}{\partial y^2}\right|_{\Omega^{N,N}}, \\ |(D_y^+ - D_y^-)(Z - z)(x_i, d)| &\le C\begin{cases} \frac{k_2}{2}(\frac{1}{\epsilon})^2, & \text{if } \alpha\mu^2 \le \lambda\epsilon, \\ \frac{k_2}{2}(\frac{1}{\mu})^2, & \text{if } \alpha\mu^2 > \lambda\epsilon. \end{cases} \end{aligned}$$

□

Using the techniques given in [21, 22], we can also obtain the error for $(x_i, y_j) = (d, y_j) \cup (x_i, d)$. First, we obtain the result for the case $\alpha\mu^2 \le \lambda\epsilon$. Consider the following discrete barrier function:

$$\xi_1^{\pm}(x_i, y_j) = \begin{cases} C_1(N^{-1}\ln N) + C_2\frac{h_2}{\epsilon}\sigma_2^2(x_i - (d-\sigma_2)) \pm e(x_i, y_j), & \text{for } x_i \in (d-\sigma_2, d), \\ C_3(N^{-1}\ln N) + C_4\frac{h_2}{\epsilon}\sigma_2^2((d+\sigma_2) - x_i) \pm e(x_i, y_j), & \text{for } x_i \in (d, d+\sigma_2), \\ C_5(N^{-1}\ln N) + C_6\frac{k_2}{\epsilon}\sigma_2^2(y_j - (d-\sigma_2)) \pm e(x_i, y_j), & \text{for } y_j \in (d-\sigma_2, d), \\ C_7(N^{-1}\ln N) + C_8\frac{k_2}{\epsilon}\sigma_2^2((d+\sigma_2) - y_j) \pm e(x_i, y_j), & \text{for } y_j \in (d, d+\sigma_2), \end{cases}$$

where $h_2 = \frac{8\sigma_2}{N} = k_2$. Then it is easy to verify that

$$\xi_1^{\pm}(x_i, y_j) \ge 0,\ \forall (x_i, y_j) \in \bar{\Omega}^{N,N} \cap \Lambda^{N,N},$$

for a suitable large C_1. We also have

$$L_{\epsilon,\mu}^{N,N}\xi_1^{\pm}(x_i, y_j) \le 0,\ \forall (x_i, y_j) \in \Omega^{N,N}$$

and

$$(D_x^+ - D_x^-)\xi_1^{\pm}(d, y_j) \le 0,\ \ (D_y^+ - D_y^-)\xi_1^{\pm}(x_i, d) \le 0,$$

for a suitable large C_2. Thus, from the discrete comparison principle, we get

$$\xi_1^{\pm}(x_i, y_j) \ge 0,\ \forall (x_i, y_j) \in \bar{\Omega}^{N,N}.$$

Therefore, for a suitable large N, we obtain the following estimate

$$|Z(x_i, y_j) - z(x_i, y_j)| \le C(N^{-1}\ln N),\ \text{if}\ \alpha\mu^2 \le \lambda\epsilon.$$

In the second case, $\alpha\mu^2 > \lambda\epsilon$, using a suitable barrier function and a similar procedure from the above techniques based on the discrete comparison principle, we also obtain the error estimate

$$|Z(x_i, y_j) - z(x_i, y_j)| \le C(N^{-1}\ln^2 N),\ \text{if}\ \alpha\mu^2 > \lambda\epsilon.$$

Hence, we have the required result.

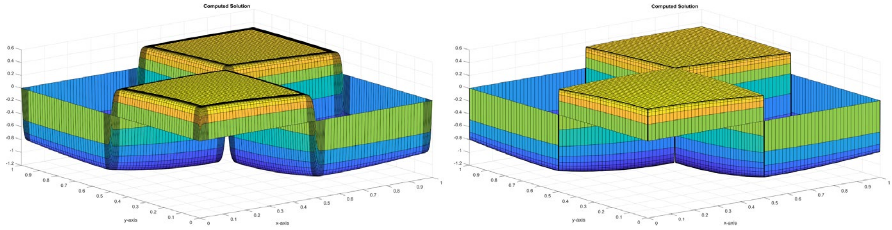
Numerical Experiments

To illustrate the performance of the method developed in the preceding sections, we focused on two examples of the type in (1.1).

Example 5.1 $\epsilon^2(z_{xx}(x,y) + z_{yy}(x,y)) + \mu^2(a(x,y)z_x(x,y) + b(x,y)z_y(x,y)) - c(x,y)z(x,y) = f(x,y)$, $\forall (x,y) \in \Omega$,

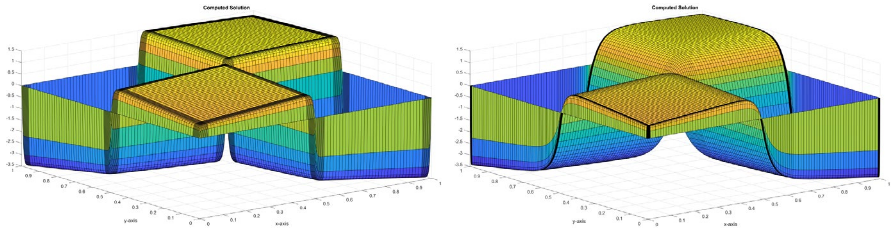
$f_1(x,y) = -(0.5 + xy/2)$; $f_2(x,y) = (0.6 + x + y)$; $f_3(x,y)$ $= (0.6 + x + y)$; $f_4(x,y) = -(0.5 + xy/2)$; $d = 0.5$,

where each f_k is defined over Ω_k, $k = 1, 2, 3, 4$, with boundary conditions as well as convection and reaction coefficients



(a) When$\epsilon = 2^{-12}, \mu = 2^{-8}, N = 128;$ (b) When$\epsilon = 2^{-12}, \mu = 2^{-4}, N = 128;$

Fig. 4 Surface graph of the numerical solution Z for Example 5.1



(a) When$\epsilon = 2^{-8}, \mu = 2^{-5}, N = 128;$ (b) When$\epsilon = 2^{-8}, \mu = 2^{-3}, N = 128;$

Fig. 5 Surface graph of the numerical solution Z for Example 5.2

$$z(x,0) = z(x,1) = z(0,y) = z(1,y) = 0,$$
$$a(x,y) = \begin{cases} 1+\exp(xy), & x<d, \\ -1-\exp(xy), & x>d, \end{cases} \quad b(x,y) = \begin{cases} 1+\exp(xy), & y<d, \\ -1-\exp(xy), & y>d, \end{cases} \quad c(x,y) = 1+x^2+y^2.$$

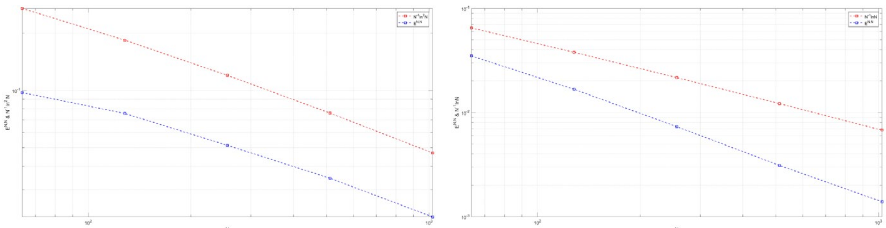
Example 5.2 $\epsilon^2(z_{xx}(x,y)+z_{yy}(x,y)) + \mu^2(a(x,y)z_x(x,y)+b(x,y)z_y(x,y)) - c(x,y)z(x,y) = f(x,y)$, $\forall (x,y) \in \Omega$

$f_1(x,y) = -(1+2x+2y) = f_4(x,y)$; $f_2(x,y) = -(4+3x+3y) = f_3(x,y)$; $d = 0.5$,

where each f_k is defined over Ω_k, $k = 1,2,3,4$, with boundary conditions as well as convection and reaction coefficients

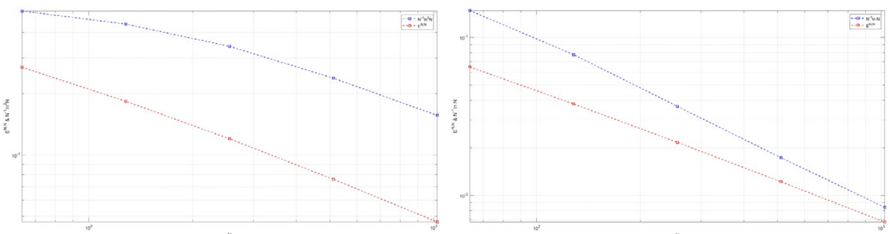
$$z(x,0) = z(x,1) = z(0,y) = z(1,y) = 0,$$
$$a(x,y) = \begin{cases} 2+x+y, & x<d, \\ -(3+2x-2y), & x>d, \end{cases} \quad b(x,y) = \begin{cases} 2+x+y, & y<d, \\ -(3+2x-2y), & y>d, \end{cases} \quad c(x,y) = 1+\exp(xy).$$

The exact solutions of these problems are not known. Therefore, we use the double mesh principle explained in [23] to estimate the maximum point-wise error. It is given by



(a) When $\epsilon = 2^{-12}, \mu = 2^{-4}$; (b) When $\epsilon = 2^{-12}, \mu = 2^{-8}$;

Fig. 6 Loglog plot of the numerical solution Z for Example 5.1



(a) When $\epsilon = 2^{-12}, \mu = 2^{-4}$; (b) When $\epsilon = 2^{-12}, \mu = 2^{-8}$;

Fig. 7 Loglog plot of the numerical solution Z for Example 5.2

$$E_{\epsilon,\mu}^{N,N} = \max_{(x_i,y_j)\in\bar{\Omega}^{N,N}} |Z^{2N,2N}(x_{2i}, y_{2j}) - Z^{N,N}(x_i, y_j)|$$

where $Z^{2N,2N}(x_{2i}, y_{2j})$ represents the numerical solution on a mesh with $2N$ subintervals on each direction. The parameter uniform maximum point-wise errors are determined applying the formula

$$E^{N,N} = \max_{\epsilon,\mu} E_{\epsilon,\mu}^{N,N}.$$

The numerical order of convergence is given by

$$Q^{N,N} = log_2\left(\frac{E^{N,N}}{E^{2N,2N}}\right).$$

In Tables 1 and 2 we have choosen $\epsilon = 2^{-12}$ and $\mu = 2^{-(2k+1)}$, $k = 0, 1, 2, 3...12$. These tables show the maximum point-wise errors and orders of convergence corresponding to Examples 5.1 and 5.2. Further, from this tables it is clear that our numerical scheme is almost first-order convergent as $\mu \to 0$, which is the rate required (see, e.g., [24]) when dealing with reaction-diffusion problems. Figure 4 shows the numerical solution for $\epsilon = 2^{-12}$, $\mu = 2^{-8}$, $N = 128$ and $\epsilon = 2^{-12}$, $\mu = 2^{-4}$, $N = 128$ respectively corresponding to the test Example 5.1. Figure 5 shows the numerical solution for $\epsilon = 2^{-8}$, $\mu = 2^{-5}$, $N = 128$

Table 1 Maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ for Example 5.1

$\epsilon = 2^{-12}$						
μ	N	64	128	256	512	1024
2.00e−01	$E^{N,N}$	9.918e−02	7.407e−02	5.194e−02	3.408e−02	2.093e−02
	$Q^{N,N}$	0.42116	0.51204	0.60792	0.70335	–
2.00e−03	$E^{N,N}$	9.902e−02	7.378e−02	5.090e−02	3.353e−02	2.065e−02
	$Q^{N,N}$	0.42449	0.53556	0.60221	0.69931	–
2.00e−05	$E^{N,N}$	9.341e−02	7.352e−02	4.965e−02	3.394e−02	2.120e−02
	$Q^{N,N}$	0.34544	0.56634	0.54881	0.67892	–
2.00e−07	$E^{N,N}$	4.643e−02	2.753e−02	1.479e−02	7.901e−03	4.236e−03
	$Q^{N,N}$	0.75405	0.89638	0.90451	0.89933	–
2.00e−09	$E^{N,N}$	3.079e−02	1.390e−02	5.365e−03	1.938e−03	7.075e−04
	$Q^{N,N}$	1.1474	1.3734	1.4690	1.4538	–
2.00e−11	$E^{N,N}$	2.936e−02	1.298e−02	4.734e−03	1.587e−03	5.040e−04
	$Q^{N,N}$	1.1776	1.4552	1.5768	1.6548	–
2.00e−13	$E^{N,N}$	2.926e−02	1.292e−02	4.695e−03	1.565e−03	4.920e−04
	$Q^{N,N}$	1.1793	1.4604	1.5850	1.6694	–
2.00e−15	$E^{N,N}$	2.926e−02	1.291e−02	4.692e−03	1.564e−03	4.913e−04
	$Q^{N,N}$	1.1804	1.4602	1.5850	1.6706	–
2.00e−17	$E^{N,N}$	2.926e−02	1.291e−02	4.692e−03	1.564e−03	4.912e−04
	$Q^{N,N}$	1.1804	1.4602	1.5850	1.6709	–
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2.00e−25	$E^{N,N}$	2.926e−02	1.291e−02	4.692e−03	1.564e−03	4.912e−04
	$Q^{N,N}$	1.1804	1.4602	1.5850	1.6709	–

and $\epsilon = 2^{-8}$, $\mu = 2^{-3}$, $N = 128$, respectively, corresponding to the test Example 5.2. Figures 6 and 7 show the Loglog plot for Examples 5.1 and 5.2, respectively, in which the second line from the below is represented the theoretical ϵ-uniform error estimations, whereas the ϵ-uniform error estimations $E^{N,N}$ are represented by the first almost linear curve.

Table 2 Maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ for Example 5.2

$\epsilon = 2^{-12}$						
μ	N	64	128	256	512	1024
2.00e−01	$E^{N,N}$	3.133e−01	2.677e−01	2.079e−01	1.451e−01	9.514e−02
	$Q^{N,N}$	0.22693	0.36473	0.51884	0.60892	–
2.00e−03	$E^{N,N}$	5.057e−01	4.353e−01	3.383e−01	2.361e−01	1.548e−01
	$Q^{N,N}$	0.21627	0.36371	0.51891	0.60899	–
2.00e−05	$E^{N,N}$	4.986e−01	4.359e−01	3.399e−01	2.371e−01	1.555e−01
	$Q^{N,N}$	0.19389	0.35889	0.51961	0.60858	–
2.00e−07	$E^{N,N}$	2.288e−01	1.519e−01	9.275e−02	5.526e−02	3.231e−02
	$Q^{N,N}$	0.59097	0.71170	0.74711	7.7425	–
2.00e−09	$E^{N,N}$	1.259e−01	5.931e−02	2.363e−02	9.035e−03	3.509e−03
	$Q^{N,N}$	1.0859	1.3277	1.3870	1.3645	–
2.00e−11	$E^{N,N}$	1.190e−01	5.320e−02	1.947e−02	6.645e−03	2.136e−03
	$Q^{N,N}$	1.1615	1.4502	1.5509	1.6374	–
2.00e−13	$E^{N,N}$	1.185e−01	5.282e−02	1.920e−02	6.497e−03	2.054e−03
	$Q^{N,N}$	1.1657	1.4600	1.5633	1.6613	–
2.00e−15	$E^{N,N}$	1.185e−01	5.279e−02	1.919e−02	6.487e−03	2.049e−03
	$Q^{N,N}$	1.1666	1.4599	1.5647	1.6626	–
2.00e−17	$E^{N,N}$	1.185e−01	5.279e−02	1.919e−02	6.487e−03	2.049e−03
	$Q^{N,N}$	1.1666	1.4599	1.5647	1.6626	–
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2.00e−25	$E^{N,N}$	1.185e−01	5.279e−02	1.919e−02	6.487e−03	2.049e−03
	$Q^{N,N}$	1.1666	1.4599	1.5647	1.6626	–

Conclusion

This study is concerned with two-parameter singularly perturbed steady-state 2-D convection-reaction-diffusion problems with interior layers. A finite-difference approach that yields almost first-order convergence is used to generate a parameter-uniform discrete solution. The analytical and discrete solutions are split into a sum of regular, singular, and corner components to address the convergence analysis and to obtain appropriate bounds. The numerical experiments show that the theoretical analysis agrees with the obtained results.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. The first author wishes to thank Ministry of Education (MoE), Govt. of India for their financial support during his PhD program.

Data availability Not applicable.

Declarations

Conflict of Interest The authors have no competing interests to declare that are relevant to the content of this article.

References

1. O'Malley Jr, R. E.: Introduction to singular perturbations. Appl. Math. Mech. Tech. rep. New York Univ. NY Courant Inst. Math. Sci.(1974)
2. Schlichting, H., Kestin, J.: Boundary layer theory, Vol. 121, Springer (1961)
3. Roos, H.-G., Stynes, M., Tobiska, L.: Robust numerical methods for singularly perturbed differential equations. Springer Series in Computational Mathematics, vol. 24, 2nd edn. Springer-Verlag, Berlin (2008)
4. Stynes, M., Stynes, D.: Convection-diffusion problems. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), vol. 196. Halifax, NS (2018)
5. Pandey, P., Jaboob, S.: A finite difference method for a numerical solution of elliptic boundary value problems. Appl. Math. Nonlinear Sci. **3**(1), 311–320 (2018)
6. Butuzov, V.: A singularly perturbed elliptic equation with two small parameters. Differ. Equ. **12**(10), 1261 (1976)
7. Clavero, C., Gracia, J., O'Riordan, E.: A parameter robust numerical method for a two dimensional reaction-diffusion problem. Math. Comput. **74**(252), 1743–1758 (2005)
8. Linß, T., Stynes, M.: Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem. J. Math. Anal. Appl. **261**, 6040 (2001)
9. Roos, H.-G.: Optimal uniform convergence of basic schemes for elliptic problems with strong parabolic boundary layers. J. Math. Anal. Applic **267**, 194–208 (2002)
10. Stynes, M.: Steady-state convection-diffusion problems. Acta Num. **14**(82), 445–508 (2005)
11. Lin, R., Ye, X., Zhang, S., Zhu, P.: A weak galerkin finite element method for singularly perturbed convection-diffusion-reaction problems. SIAM J. Num. Anal. **56**(3), 1482–1497 (2018)
12. Nhan, T.A., Vulanović, R.: The Bsakhvalov mesh: a complete finite-difference analysis of two-dimensional singularly perturbed convection-diffusion problems. Num. Algorithm. **87**(1), 203–221 (2021)
13. Zhang, J., Lv, Y.: Finite element method for singularly perturbed problems with two parameters on a Bakhvalov-type mesh in 2d, Num. Algorithm. 1–29 (2021)
14. O'Riordan, E., Pickett, M., Shishkin, G.: Numerical methods for singularly perturbed elliptic problems containing two perturbation parameters. Math. Model. Anal. **11**(2), 199–212 (2006)
15. O'Riordan, E., Pickett, M.: A parameter-uniform numerical method for a singularly perturbed two parameter elliptic problem. Adv. Comput. Math. **35**(1), 57–82 (2011)
16. Teofanov, L., Roos, H.-G.: An elliptic singularly perturbed problem with two parameters i: Solution decomposition. J. Comput. Appl.Math. **206**(2), 1082–1097 (2007)
17. Teofanov, L., Roos, H.-G.: An elliptic singularly perturbed problem with two parameters ii: Robust finite element solution. J. Comput. Sppl. Math. **212**(2), 374–389 (2008)
18. Volkov, E.A.: Differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle. Trudy Matematicheskogo Instit. imeni VA Stekl. **77**, 89–112 (1965)
19. Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Shishkina, L. P.: Fitted mesh methods for problems with parabolic boundary layers, Math. Proc. R. Ir. Acad. 98A (1998)
20. Aarthika, K., Shiromani, R., Shanthi, V.: A higher-order finite difference method for two-dimensional singularly perturbed reaction-diffusion with source-term-discontinuous problem. Comput. Math. Appl. **118**, 56–73 (2022)

21. Chandru, M., Prabha, T., Shanthi, V.: A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data. J. Comput. Appl. Math. **309**, 11–27 (2017)
22. Clavero, C., Gracia, J.L., Shishkin, G.I., Shishkina, L.P.: An efficient numerical scheme for 1d parabolic singularly perturbed problems with an interior and boundary layers. J. Comput. Appl. Math. **318**, 634–645 (2017)
23. Farrell, P., Hegarty, A., Miller, J. M., O'Riordan, E., Shishkin, G. I.: Robust computational techniques for boundary layers, CRC Press (2000)
24. Andreev, V.B.: On the accuracy of grid approximations to nonsmooth solutions of a singularly perturbed reaction-diffusion equation in the square. Diff. Equ. **42**(7), 954–966 (2006)