Mediterr. J. Math. (2023) 20:166
https://doi.org/10.1007/s00009-023-02373-7

Mediterranean Journal of Mathematics

An Improved Oscillation Result for a Class of Higher Order Non-canonical Delay Differential Equations

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Abstract. In this work, by obtaining a new condition that excludes a class of positive solutions of a type of higher order delay differential equations, we were able to construct an oscillation criterion that simplifies, improves and complements the previous results in the literature. The adopted approach extends those commonly used in the study of second-order equations. The simplification lies in obtaining an oscillation criterion with two conditions, unlike the previous results, which required at least three conditions. In addition, we illustrate the improvement with the new criterion, applying it to some examples and comparing the results obtained with previous results in the literature.

Mathematics Subject Classification. Primary 34C10; Secondary 34K11.

Keywords. Non-canonical differential equations, higher order, delay argument, oscillation.

1. Introduction

The objective of this work is to study the asymptotic behavior of solutions of a class of higher order delay differential equations (DDEs) of the form

$$\left(a \cdot v^{(n-1)}\right)' (t) + (h \cdot (F \circ v \circ g)) (t) = 0, \tag{1.1}$$

where $t \geq t_0$, $n \in \mathbb{Z}^+$ is even, and $n \geq 4$. We also assume the following:

(H1) $a, h \in C([t_0, \infty), [0, \infty))$, $a(t) > 0$ and $A_{n-2}(t_0) < \infty$, where

$$A_0(t) := \int_t^\infty a^{-1}(\varrho)\, \mathrm{d}\varrho, \text{ and } A_k(t) := \int_t^\infty A_{k-1}(\varrho)\, \mathrm{d}\varrho,$$

for $k = 1, 2, \ldots, n-2$.

(H2) $g \in C([t_0, \infty), \mathbb{R}^+)$, $g(t) \leq t$, $g'(t) > 0$ and $\lim_{t\to\infty} g(t) = \infty$.

(H3) $F \in C(\mathbb{R}, \mathbb{R})$, $F'(u) \geq 0$, $uF(u) > 0$ for $u \neq 0$, and $F(uv) \geq F(u) F(v)$ for $uv > 0$.

Published online: 21 March 2023

By a solution of (1.1), we mean a function $\upsilon \in C^{n-1}([t_\upsilon, \infty), \mathbb{R})$, for $t_\upsilon \geq t_0$, with $a\upsilon^{(n-1)} \in C^1([t_\upsilon, \infty), \mathbb{R})$, that satisfies (1.1) for all $t \geq t_\upsilon$. We consider only those solutions of (1.1) that do not eventually vanish. A solution υ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be non-oscillatory [1].

The many applications of DDEs in different sciences were and continue to be the motivation behind the growing interest in studying the qualitative behavior of the solutions of these equations.

In the non-canonical case, it is easy to see how much research has progressed on the oscillatory behavior of the solutions of second-order DDEs. This progress can be traced through the recent results of Baculí ková [2,3] and Džurina and Jadlovská [4,5]. They provided improved techniques and sharper criteria for the oscillation of second-order DDE solutions.

In the study of oscillatory behavior, there are two common techniques: Riccati substitution and comparison with first-order equations. In the non-canonical case, Baculíková et al. [6] used the comparison technique to establish the oscillation conditions for the solutions of the DDE

$$\left(a \cdot \left(\upsilon^{(n-1)}\right)^{\alpha}\right)'(t) + (h \cdot (F \circ \upsilon \circ g))(t) = 0. \tag{1.2}$$

On the other hand, Zhang et al. [7] used the Riccati substitution to establish criteria for deciding that all the solutions of the DDE

$$\left(a \cdot \left(\upsilon^{(n-1)}\right)^{\alpha}\right)'(t) + \left(h \cdot \left(\upsilon^{\beta} \circ g\right)\right)(t) = 0 \tag{1.3}$$

are oscillatory, where α and β are ratios of odd positive integers. Below, we present two results obtained from the literature, to which we will refer to later.

Theorem 1.1. [6, Theorem 4 with $F(\upsilon) = \upsilon^{\alpha}$] *All solutions of (1.3) are oscillatory if the first-order DDEs*

$$w'(t) + \left(h \cdot \left(\frac{\epsilon_1 g^{n-1}}{(n-1)!\left(a^{1/\alpha} \circ g\right)}\right)^{\alpha} \cdot (w \circ g)\right)(t) = 0, \tag{1.4}$$

$$w'(t) + \frac{1}{a^{1/\alpha}(t)} \cdot \left(\int_{t_0}^{t} h(\varrho)\left(\frac{\epsilon_2 g^{n-2}(\varrho)}{(n-2)!}\right)^{\alpha} \mathrm{d}\varrho\right)^{1/\alpha} \cdot (w \circ g)(t) = 0$$

are oscillatory, and there is a $\eta \in C^1([t_0, \infty))$ *with* $\eta(t) > t$, $\eta'(t) \geq 0$ *and* $(\eta_{n-2} \circ g)(t) < t$, *such that*

$$w'(t) + \frac{1}{a^{1/\alpha}(t)} \cdot \left(\int_{t_0}^{t} h(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha} \cdot (J_{n-2} \circ g)(t) \cdot (w \circ \eta_{n-2} \circ g)(t) = 0$$

is oscillatory for some $\epsilon_1, \epsilon_2 \in (0, 1)$, *where*

$$\eta_1 = \eta,\ \eta_{i+1} = \eta_i \circ \eta,\ J_1(t) = \eta - t \text{ and } J_{i+1}(t) = \int_t^{\eta} J_i(\varrho)\,\mathrm{d}\varrho,$$

for $i = 1, 2, \ldots, n-3$.

Theorem 1.2. [7, Theorem 2.1] *All solutions of (1.3) are oscillatory if the first-order DDE (1.4) is oscillatory and the following conditions hold:*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\left(\frac{\epsilon_0 g^{n-2}(\varrho)}{(n-2)!}\right)^{\alpha} h(\varrho)\,A_0^{\alpha}(\varrho)-\frac{\alpha^*}{a^{1/\alpha}(\varrho)\,A_0(\varrho)}\right)\mathrm{d}\varrho=\infty,$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(h(\varrho)\,B^{\alpha}(\varrho)-\frac{\alpha^*\left(B'(\varrho)\right)^{\alpha+1}}{B(\varrho)\,B_1^{\alpha}(\varrho)}\right)\mathrm{d}\varrho=\infty,$$

where $\epsilon_0\in(0,1)$, $\alpha^*=(\alpha/(\alpha+1))^{\alpha+1}$, $A_0(t)=\int_t^{\infty}a^{-1/\alpha}(\varrho)\,\mathrm{d}\varrho$,

$$B(t)=\frac{1}{(n-3)!}\int_t^{\infty}(\varrho-t)^{n-3}A_0(\varrho)\,\mathrm{d}\varrho$$

and

$$B_1(t)=\frac{1}{(n-4)!}\int_t^{\infty}(\varrho-t)^{n-4}A_0(\varrho)\,\mathrm{d}\varrho.$$

We note here that the linear delay equation

$$\upsilon^{(n)}(t)+(h\cdot(\upsilon\circ g))(t)=0$$

has been studied by Koplatadze et al. [8]. They took into account the odd- and even-order cases of this equation.

Very recently, Moaaz et al. [9] extended the results about the second-order equations to even-order equations in the non-canonical case. They adopted a strategy that involved new monotonic properties for positive decreasing solutions and used those properties to iteratively develop new oscillation criteria.

This study aims to establish a new criterion to determine the oscillation of all solutions of Eq. (1.1) in the non-canonical case. The approach followed is an extension of the approach used by Koplatadze et al. [8] and later by Baculíková [2] to obtain an effective oscillation criterion for second-order equations. The new criterion ensures that Eq. (1.1) is oscillatory without the need to check the additional condition (1.4), which has traditionally been imposed on all previous related results. In addition, the new criterion also introduces a measure of oscillation that is sharper than previous results in the literature.

2. Main Results

Using Lemma 2.2.1 in [10], we can classify the positive solutions of (1.1) as follows:

Lemma 2.1. [11, Lemma 3] *Suppose that υ is an eventually positive solution of (1.1). Then, $\left(a\cdot\upsilon^{(n-1)}\right)'(t)\le 0$, and there are eventually the following three cases:*

(C_1) *υ' and $\upsilon^{(n-1)}$ are positive, and $\upsilon^{(n)}$ is negative;*
(C_2) *υ' and $\upsilon^{(n-2)}$ are positive, and $\upsilon^{(n-1)}$ is negative;*

(C_3) $(-1)^m \upsilon^{(m)}$ *are positive, for* $m = 1, \ldots, n-1$.

Lemma 2.2. *Suppose that* υ *is an eventually positive solution of (1.1) and satisfies* (C_3). *Then, there is a* $t_1 \geq t_0$ *such that*

$$\frac{\upsilon\left(g\left(t\right)\right)}{F\left(\upsilon\left(g\left(t\right)\right)\right)} \geq A_{n-2}\left(g\left(t\right)\right)\int_{t_1}^{g(t)} h\left(\varrho\right)\mathrm{d}\varrho + \int_{g(t)}^{t} A_{n-2}\left(\varrho\right)h\left(\varrho\right)\mathrm{d}\varrho$$
$$+F\left(A_{n-2}^{-1}\left(g\left(t\right)\right)\right)\int_{t}^{\infty} A_{n-2}\left(\varrho\right)h\left(\varrho\right)F\left(A_{n-2}\left(g\left(\varrho\right)\right)\right)\mathrm{d}\varrho. \tag{2.1}$$

Proof. Since υ is an eventually positive solution of (1.1) and satisfies (C_3), there is a $t_1 \geq t_0$ such that $\upsilon(t) > 0$ and $\upsilon(g(t)) > 0$ for all $t \geq t_1$. We also have

$$\int_t^\infty \upsilon^{(n-1)}\left(\varrho\right)\mathrm{d}\varrho = \lim_{s\to\infty}\upsilon^{(n-2)}\left(s\right) - \upsilon^{(n-2)}\left(t\right) \geq -\upsilon^{(n-2)}\left(t\right).$$

Furthermore, according to Lemma 2.1 it is $\left(a\cdot\upsilon^{(n-1)}\right)'(t) \leq 0$, and thus we have

$$-\upsilon^{(n-2)}\left(t\right) \leq \int_t^\infty \frac{1}{a\left(\varrho\right)}a\left(\varrho\right)\upsilon^{(n-1)}\left(\varrho\right)\mathrm{d}\varrho \leq a\left(t\right)\upsilon^{(n-1)}\left(t\right)\int_t^\infty \frac{1}{a\left(\varrho\right)}\mathrm{d}\varrho$$
$$= a\left(t\right)\upsilon^{(n-1)}\left(t\right)A_0\left(t\right). \tag{2.2}$$

Integrating this inequality over $[t,\infty)$, we obtain

$$\begin{aligned}\upsilon^{(n-3)}\left(t\right) &\leq \int_t^\infty a\left(\varrho\right)\upsilon^{(n-1)}\left(\varrho\right)A_0\left(\varrho\right)\mathrm{d}\varrho \\ &\leq a\left(t\right)\upsilon^{(n-1)}\left(t\right)\int_t^\infty A_0\left(\varrho\right)\mathrm{d}\varrho \\ &= \left(a\cdot\upsilon^{(n-1)}\right)\left(t\right)\cdot A_1\left(t\right).\end{aligned}$$

Integrating this inequality $n-3$ times over $[t,\infty)$, and taking into account the behavior of the derivatives of υ in (C_3), we conclude that

$$\upsilon'\left(t\right) \leq \left(a\cdot\upsilon^{(n-1)}\right)\left(t\right)\cdot A_{n-3}\left(t\right) \tag{2.3}$$

and

$$\upsilon\left(t\right) \geq -\left(a\cdot\upsilon^{(n-1)}\right)\left(t\right)\cdot A_{n-2}\left(t\right). \tag{2.4}$$

Next, we define

$$H\left(t\right) := \left(a\cdot\upsilon^{(n-1)}\right)\left(t\right)\cdot A_{n-2}\left(t\right) + \upsilon\left(t\right).$$

From (2.4), we have that $H(t) \geq 0$ for $t \geq t_1$. Using (2.3) and (1.1), we have

$$\begin{aligned}H'\left(t\right) &= \left(a\cdot\upsilon^{(n-1)}\right)'\left(t\right)\cdot A_{n-2}\left(t\right) - a\left(t\right)\upsilon^{(n-1)}\left(t\right)A_{n-3}\left(t\right) + \upsilon'\left(t\right) \\ &\leq \left(a\cdot\upsilon^{(n-1)}\right)'\left(t\right)\cdot A_{n-2}\left(t\right) \\ &= -A_{n-2}\left(t\right)\cdot h\left(t\right)\cdot\left(F\circ\upsilon\circ g\right)\left(t\right) \leq 0.\end{aligned}$$

Integrating this inequality over $[t, \infty)$, we arrive at

$$H(t) \geq \int_t^\infty A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho. \tag{2.5}$$

Integrating (1.1) from t_1 to t, we find that

$$\begin{aligned} a(t)\, \upsilon^{(n-1)}(t) &= a(t_1)\, \upsilon^{(n-1)}(t_1) - \int_{t_1}^t h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &< -\int_{t_1}^t h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\begin{aligned} \upsilon(t) &\geq -a(t)\, \upsilon^{(n-1)}(t)\, A_{n-2}(t) + \int_t^\infty A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &> A_{n-2}(t) \int_{t_1}^t h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &\quad + \int_t^\infty A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho. \end{aligned}$$

Thus, we have

$$\begin{aligned} \upsilon(g(t)) &> A_{n-2}(g(t)) \int_{t_1}^{g(t)} h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &\quad + \int_{g(t)}^\infty A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &= A_{n-2}(g(t)) \int_{t_1}^{g(t)} h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &\quad + \int_{g(t)}^t A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho \\ &\quad + \int_t^\infty A_{n-2}(\varrho)\, h(\varrho)\, F(\upsilon(g(\varrho)))\, \mathrm{d}\varrho. \end{aligned} \tag{2.7}$$

On the other hand, from (2.2), we get

$$\left(\frac{\upsilon^{(n-2)}}{A_0}\right)' = \frac{1}{A_0^2(t)} \left(\left(A_0 \cdot \upsilon^{(n-1)}\right)(t) + \left(a^{-1} \cdot \upsilon^{(n-2)}\right)(t) \right) \geq 0, \tag{2.8}$$

which leads to

$$-\upsilon^{(n-3)}(t) \geq \int_t^\infty A_0(\varrho)\, \frac{\upsilon^{(n-2)}(\varrho)}{A_0(\varrho)} \mathrm{d}\varrho \geq \frac{\upsilon^{(n-2)}(t)}{A_0(t)} A_1(t).$$

This implies

$$\left(\frac{\upsilon^{(n-3)}}{A_1}\right)'(t) = \frac{1}{A_1^2(t)} \left(\left(A_1 \cdot \upsilon^{(n-2)}\right)(t) + \left(A_0 \cdot \upsilon^{(n-3)}\right)(t) \right) \leq 0.$$

Repeating the same procedure $n-3$ times, we obtain that $\left(\frac{\upsilon}{A_{n-2}}\right)'(t) \geq 0$. Hence, $\upsilon(g(\varrho)) \geq \frac{A_{n-2}(g(\varrho))}{A_{n-2}(g(t))}\upsilon(g(t))$, for $t \leq \varrho$. Then, from (H3), we have

$$F(\upsilon(g(\varrho))) \geq F\left(A_{n-2}^{-1}(g(t))\right) F(A_{n-2}(g(\varrho))) F(\upsilon(g(t))), \text{ for } t \leq \varrho. \tag{2.9}$$

Using the fact that $\upsilon'(t) < 0$ and (2.9), the inequality (2.7) becomes

$$\begin{aligned}\frac{\upsilon(g(t))}{F(\upsilon(g(t)))} \geq{}& A_{n-2}(g(t)) \int_{t_1}^{g(t)} h(\varrho)\,\mathrm{d}\varrho + \int_{g(t)}^{t} A_{n-2}(\varrho)\,h(\varrho)\,\mathrm{d}\varrho \\ &+F\left(A_{n-2}^{-1}(g(t))\right)\int_t^\infty A_{n-2}(\varrho)\,h(\varrho)\,F(A_{n-2}(g(\varrho)))\,\mathrm{d}\varrho,\end{aligned}$$

and the proof is complete. □

Theorem 2.3. *Suppose that* $\lim_{\omega\to 0}\frac{\omega}{F(\omega)} = L < \infty$ *and*

$$\begin{aligned}\limsup_{t\to\infty}\Bigg[&A_{n-2}(g(t))\int_{t_0}^{g(t)} h(\varrho)\,\mathrm{d}\varrho + \int_{g(t)}^{t} A_{n-2}(\varrho)\,h(\varrho)\,\mathrm{d}\varrho \\ &+F\left(A_{n-2}^{-1}(g(t))\right)\int_t^\infty A_{n-2}(\varrho)\,h(\varrho)\,F(A_{n-2}(g(\varrho)))\,\mathrm{d}\varrho\Bigg] > L\,. \end{aligned}\tag{2.10}$$

If for some $\epsilon_0 \in (0,1)$, *the DDE*

$$\begin{aligned}&\left(a(t)\,A_0^2(t)\,w'(t)\right)' \\ &\quad+A_0(t)\,h(t)\,F\left(\frac{\epsilon_0}{(n-2)!}g^{n-2}(t)\right)F(A_0(g(t))\,w(g(t))) = 0\end{aligned}\tag{2.11}$$

is oscillatory, then all solutions of (1.1) are oscillatory.

Proof. We proceed by contradiction. Let us assume that υ is an eventually positive solution of (1.1). Then, there is a $t_1 \geq t_0$, such that $\upsilon(t) > 0$ and $\upsilon(g(t)) > 0$ for all $t \geq t_1$. It follows from Lemma 2.1 that υ satisfies one of the cases $(C_1)-(C_3)$.

Assume that case (C_1) holds. Proceeding similarly as in the proof of Theorem 1 in [2], we can prove that (2.10) implies that

$$\int_{t_1}^\infty h(\varrho)\,A_2(\varrho)\,\mathrm{d}\varrho = \infty. \tag{2.12}$$

Integrating (1.1) from t_1 to ∞ and using the fact that $\upsilon^{(n-1)}$ is a decreasing positive function, we find that

$$a(t_1)\,\upsilon^{(n-1)}(t_1) \geq \int_{t_1}^\infty h(\varrho)\,F(\upsilon(g(\varrho)))\,\mathrm{d}\varrho.$$

Since $\upsilon(t) > 0$ and $\upsilon'(t) > 0$, there is a $t_2 \geq t_1$, such that $\upsilon(g(t)) > l$ for $t \geq t_1$. Then, from (H3), we arrive at

$$a(t_1)\,\upsilon^{(n-1)}(t_1) \geq F(l)\int_{t_1}^\infty h(\varrho)\,\mathrm{d}\varrho \geq F(l)\int_{t_1}^\infty h(\varrho)\,A_2(\varrho)\,\mathrm{d}\varrho,$$

which contradicts (2.12).

Assume that case (C_2) holds. We have

$$\begin{aligned}
&\left(a(t)\cdot A_0^2(t)\cdot\left(\frac{\upsilon^{(n-2)}}{A_0}\right)'(t)\right)' \\
&\quad=\left(a(t)\cdot A_0^2(t)\cdot\left(\frac{\left(A_0\cdot\upsilon^{(n-1)}\right)(t)+\left(a^{-1}\cdot\upsilon^{(n-2)}\right)(t)}{A_0^2(t)}\right)\right)' \\
&\quad=\left(A_0(t)\cdot\left(a\cdot\upsilon^{(n-1)}\right)(t)+\upsilon^{(n-2)}(t)\right)' \\
&\quad=A_0(t)\cdot\left(a\cdot\upsilon^{(n-1)}\right)'(t),
\end{aligned}\tag{2.13}$$

which, by virtue of (1.1), yields that

$$\left(a(t)\cdot A_0^2(t)\cdot\left(\frac{\upsilon^{(n-2)}}{A_0}\right)'(t)\right)'+A_0(t)\cdot h(t)\cdot(F\circ\upsilon\circ g)(t)=0.\tag{2.14}$$

From Lemma [10, Lemma 2.2.3], we get

$$(\upsilon\circ g)(t)\geq\frac{\epsilon_0}{(n-2)!}g^{n-2}(t)\cdot\left(\upsilon^{(n-2)}\circ g\right)(t),\tag{2.15}$$

for all $\epsilon_0\in(0,1)$. Combining (2.15) and (2.14) and using the transformation $\upsilon^{(n-2)}(t)=A_0(t)\,w(t)$, we find that

$$\begin{aligned}
&\left(a(t)\,A_0^2(t)\,w'(t)\right)' \\
&\quad+A_0(t)\,h(t)\,F\left(\frac{\epsilon_0}{(n-2)!}g^{n-2}(t)\right)F\left(A_0(g(t))\,w(g(t))\right)\leq 0.
\end{aligned}$$

It follows from Corollary 1 in [12] that the DDE (2.11) also has a positive solution, which is a contradiction.

Assume that case (C_3) holds. From Lemma 2.2, we have that (2.1) holds, which contradicts the hypothesis (2.10). The proof is complete. □

Corollary 2.4. *Suppose that $F(u)=u$. Then, conditions*

$$\begin{aligned}
\limsup_{t\to\infty}&\left[A_{n-2}(g(t))\int_{t_0}^{g(t)}h(\varrho)\,\mathrm{d}\varrho+\int_{g(t)}^{t}A_{n-2}(\varrho)\,h(\varrho)\,\mathrm{d}\varrho\right. \\
&\left.+A_{n-2}^{-1}(g(t))\int_t^{\infty}A_{n-2}(\varrho)\,h(\varrho)\,A_{n-2}(g(\varrho))\,\mathrm{d}\varrho\right]>1\,,
\end{aligned}\tag{2.16}$$

and for some $\epsilon_0\in(0,1)$

$$\begin{aligned}
&\liminf_{t\to\infty}\int_{g(t)}^{t}g^{n-2}(s)\,h(s)\,A_0(s)\,A_0(g(s))\left(\int_{t_0}^{g(s)}\frac{\mathrm{d}\varrho}{a(\varrho)\,A_0^2(\varrho)}\right)\mathrm{d}s \\
&\quad>\frac{(n-2)!}{\epsilon_0\mathrm{e}},
\end{aligned}\tag{2.17}$$

guarantee that all solutions of (1.1) are oscillatory.

Proof. The proof is the same as the proof of Theorem 2.3. It is enough just to know from Theorem 4 in [13] that the criterion (2.17) guarantees that (2.11) is oscillatory. □

3. Examples

We present two examples taken from the literature to compare different oscillation criteria, showing that Theorem 2.1 provides the sharpest results.

Example 3.1. Consider the DDE

$$\left(\mathrm{e}^{t}\upsilon'''(t)\right)' + h_0\mathrm{e}^{t}\upsilon(t-\lambda) = 0, \tag{3.1}$$

where $\lambda > 0$ and $h_0 > 0$. It is easy to check that $A_i(t) = \mathrm{e}^{-t}$, for $i = 0, 1, 2$. Furthermore, conditions (2.17) and (2.16) reduce to

$$\liminf_{t\to\infty}\int_{t-\lambda}^{t} h_0(s-\lambda)^2\left(1 - e^{t_0}\mathrm{e}^{-s+\lambda}\right)\mathrm{d}s = \infty$$

and

$$h_0 > \frac{1}{2+\lambda}, \tag{3.2}$$

respectively. Then, according to Corollary 2.4, every solution of (3.1) is oscillatory if (3.2) holds.

Remark 3.1. From Example 3 in [6], Eq. (3.1), when $\lambda = 1$, is oscillatory if $h_0 > 2^5/\mathrm{e}$. However, from Example 3.1, the finer bound $h_0 > 1/3$ guarantees that for $\lambda = 1$, Eq. (3.1) is oscillatory. Moreover, for all $\lambda < 2$, the results in [7] give the most efficient condition as $h_0 < 0.25$, while our results support the most efficient condition for all $\lambda > 2$. On the other hand, the results in [6,7] do not take into account the impact of λ.

Example 3.2. Consider the DDE of the Euler type

$$\left(t^4\upsilon'''(t)\right)' + h_0\upsilon(\lambda t) = 0, \tag{3.3}$$

where $\lambda \in (0,1)$ and $h_0 > 0$. It is easy to check that $A_0(t) = \frac{1}{3}t^{-3}$, $A_1(t) = \frac{1}{6}t^{-2}$, $A_2(t) = \frac{1}{6}t^{-1}$. Conditions (2.16) and (2.17) reduce to $\frac{1}{6}h_0\left(2 + \ln\frac{1}{\lambda}\right) > 1$, and $h_0\lambda^2\ln\frac{1}{\lambda} > \frac{6}{\mathrm{e}}$, respectively. Then, from Corollary 2.4, every solution of (3.3) is oscillatory if

$$h_0 > \frac{6}{\mathrm{e}\lambda^2\ln\frac{1}{\lambda}}. \tag{3.4}$$

Remark 3.2. By choosing $\eta(t) = ct$, where $c = \left(1 + \lambda^{-1/2}\right)/2$, we can apply Theorem 1.1 to Example 3.2. Then, Eq. (3.3) is oscillatory if

$$h_0 > \max\left\{\frac{6\lambda}{\mathrm{e}\ln\frac{1}{\lambda}}, \frac{6}{\mathrm{e}\lambda^2\ln\frac{1}{\lambda}}, \frac{2}{\mathrm{e}\lambda^2(c-1)(c^2-1)\ln\frac{1}{c^2\lambda}}\right\}. \tag{3.5}$$

We note the difficulty of obtaining an unknown function η that satisfies the conditions of Theorem 1.1. On the other hand, from Theorem 1.2, Eq. (3.3) is oscillatory if

$$h_0 > \max\left\{\frac{6\lambda}{\mathrm{e}\ln\frac{1}{\lambda}}, \frac{9}{2\lambda^2}, \frac{3}{2}\right\}. \tag{3.6}$$

If we set $\lambda = 1/2$, then conditions (3.5) and (3.6) reduce to $h_0 > 98.162$ and $h_0 > 18$, respectively, while condition (3.4) gives $h_0 > 12.738$.

4. Conclusion

The study of the oscillatory behavior of solutions of higher order differential equations depends on finding conditions that exclude positive solutions. By extending the results in [8] to higher order equations, we have established a new condition that excludes positive decreasing solutions. Our results involve two conditions to ensure that all solutions of (1.1) are oscillatory, while all previous results need three assumptions. Through some examples, we show that our results improve other results in the literature. As future work, it would be interesting to extend the obtained result to Eq. (1.3), as well as to the neutral case. In addition, as an open problem, it is proposed to extend our results to equations of odd order in the non-canonical or non-linear cases.

Acknowledgements

The authors present their sincere thanks to the editors and reviewers.

Author contributions OM conceptualized the issue, investigated, and written original draft preparation; HR was a major contributor in methodology, analysis, review, and editing. All authors read and approved the final manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There are no competing interests.

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Received: August 3, 2022.
Revised: December 21, 2022.
Accepted: February 27, 2023.