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A fitted numerical method for a singularly perturbed Fredholm integro-differential equation with discontinuous source term

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In this article, a singularly perturbed second-order Fredholm integro-differential equation with a discontinuous source term is examined. An exponentially-fitted numerical method on a Shishkin mesh is applied to solve the problem. The method is shown to be uniformly convergent with respect to the singular perturbation parameter. Some numerical results are given, which validate the theoretical results.

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1. Introduction

Fredholm integro-differential equations (FIDEs) are widely used in scientific modeling, including economics, mechanics, fluid dynamics (see, e.g., [11,12]). Furthermore, due to the fact that many scientific applications include large gradients, singularly perturbed Fredholm integro-differential equations (SPFIDEs) are very common. It is common in these problems the presence of thin boundary layers. This is because the solution fluctuates quickly in some parts of the domain, while the regularity is maintained in the rest of the domain, where its variability has slowed down. For further information on these issues, one can see [1,10] and the works referenced therein. Because of these boundary layers, singularly perturbed problems are difficult to solve using standard numerical techniques.

When dealing with these issues, two types of methods are typically employed; namely, fitted operator techniques, which reproduce the boundary layers' solution [13], and layer-adaptive techniques [9]. A reliable numerical method is required since SPFIDEs cannot be solved analytically due to their complexity. Given this, many scientists have focused their efforts on developing numerical methods for solving SPFIDEs. An overview of the many different methods that can be used for solving SPFIDEs can be found in [4,6]. For singularly perturbed ordinary differential problems with discontinuous data, one can refer to [5,8,14]. In [16] the authors worked on a delay differential equation with exponentially fitted methods. In [15] the author proved second order convergence for singularly perturbed delay differential equations using fitted mesh method.

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Amiraliyev et al. [2] showed that first-order convergence on a uniform mesh was obtained globally for a first-order equation considering an exponential fitting approach. A linearly accurate approach for singularly perturbed Volterra delay-integro–differential equations (SPVDIDEs) was presented in [3].

Motivated by the works by Amiraliyev [7] and Farrell et al. [8], we consider the following problem in which a single jump discontinuity in the source term is assumed to occur at a point $d \in \Omega = (0, l)$:

$$Ly \equiv L_1 y + \lambda \int_0^l K(t,s)y(s)ds = F(t), \quad \forall t \in \Omega^- \cup \Omega^+, \tag{1.1}$$

$$\text{with} \quad L_1 y = -\epsilon y'' + b(t)y, \quad y(0) = A, \ y(l) = B, \tag{1.2}$$

$$F(d+) \neq F(d-), \quad b(t) \geq \beta > 0,$$

$$\Omega^- = (0,d), \ \Omega^+ = (d,l), \ \bar{\Omega} = [0,l],$$

where ϵ $(0 < \epsilon << 1)$ is a perturbation parameter that multiplies the higher order derivative, and λ is a real parameter. We assume that $K(t,s)$ for $(t,s) \in \bar{\Omega} \times \bar{\Omega}$ and $b(t)$ for $t \in \bar{\Omega}$ are sufficiently smooth functions. To denote the jump discontinuity at d for any function with $[y](d) = y(d+) - y(d-)$, $[y'](d) \leq 0$.

Furthermore, we assume that the source term $F(t)$ has a jump discontinuity, being

$$F(t) = \begin{cases} F_1(t), & t \in \Omega^-, \\ F_2(t), & t \in \Omega^+. \end{cases}$$

To provide an approximate solution for (1.1)–(1.2) we consider a finite difference method on a Shishkin mesh. The present difference scheme uses exponential basis functions and interpolating quadrature rules with integral forms for the weighted and remainder terms. The novelty of the article is applying Shishkin mesh, especially at the point $t = \mathcal{M}/2$, to obtain the error as $D^+ = D^-$. Only the exponentially fitted mesh method is not sufficient to find the error at $t = \mathcal{M}/2$. If we want to use exponentially fitted mesh only, we can use Boundary Value Technique hybrid methods. The composite trapezoidal rule, including a residual term, will be used to approximate the integral in (1.1). Throughout the paper, C is used to denote a generic constant, $\mathbb{C}^k(D)$ denotes the set of k times differentiable functions on the domain D and we use the supremum norm that is $||g||_D = \sup_{t \in D} |g(t)|$ (on any domain D), $||g||_{\infty, \bar{\Omega}_{\mathcal{M}}} = \max |g_i|$ (on a discrete domain $\bar{\Omega}_M$ with points t_i).

The rest of the article is as follows. Section 2 presents some analytical results of our problem. Section 3 explains how to create the mesh and the discretization of the problem that will be used to get an approximate solution. The theorems and lemmas in Section 4 demonstrate that the approach is accurate to roughly a $(\mathcal{M}^{-1} \ln \mathcal{M})$ order where $\mathcal{M}$ is an even number being $\mathcal{M}+1$ the number of mesh points. The final findings are summarized in Section 5, together with some examples to confirm the order of convergence and the pointwise errors. Finally, we compose a conclusion for the proposed approach.

2. Analytical results

The maximum principle and a stability result are presented in this section for the exact solution of problem (1.1)–(1.2). Additionally, some derivative bounds are also provided.

Lemma 2.1. *([7,8,10]): Let $y \in \mathbb{C}^0(\bar{\Omega}) \cap \mathbb{C}^1(\Omega) \cap \mathbb{C}^2(\Omega^- \cup \Omega^+)$ and*

$$|\lambda| < \frac{\beta}{\max\limits_{0 \leq t \leq l} \int_0^l |K(t,s)| ds},$$

with $y(0) \geq 0$, $y(l) \geq 0$, $Ly(t) \geq 0$, $\forall t \in \Omega^- \cup \Omega^+$, $[y](d) = 0$, $[y'](d) \leq 0$. Then, it holds that $y(t) \geq 0$, $\forall t \in \bar{\Omega}$.

Proof. Let q be a point at which y attains its minimum value in $\bar{\Omega}$. If $y(q) \geq 0$, then there is nothing to prove. If, $y(q) < 0$ to complete the proof consider

$$-\epsilon y'' + b(t)y = G(t) = \begin{cases} G_1(t), & t \in \Omega^-, \\ G_2(t), & t \in \Omega^+, \end{cases}$$

where $G_1(t) = F_1(t) - \lambda \int_0^l K(t,s)y(s)ds$, $G_2(t) = F_2(t) - \lambda \int_0^l K(t,s)y(s)ds$. We distinguish two cases.

Case1: If $q \in \Omega^- \cup \Omega^+$ then $y(q) < 0$, $y'(q) = 0$, $y''(q) > 0$. This implies that $L_1 y \leq 0$, which is a contradiction.

Case2: If $q = d$ then $y'(d+) > 0$ and $y'(d-) < 0$. Hence, we have $[y'(d)] = y'(d+) - y'(d-) > 0$, which is again a contradiction.

Therefore, it is $y(q) > 0$ always, and the proof is complete. □

Using the above Lemma 2.1 and the procedure adopted in [14] we can get the following result.

Lemma 2.2. *Let* $\frac{\partial^k K}{\partial t^k} \in \mathbb{C}(\bar{\Omega}\times\bar{\Omega})$, $(k=0,1,2)$ *and* $y(t)$ *the solution of the problem* (1.1)–(1.2). *It holds that*

$$\max_{0\le t\le l}|y(t)| \le C, \tag{2.1}$$

$$\left\| y^{(k)}(t)\right\| \le C \begin{cases} 1+\epsilon^{\frac{-k}{2}}\left(e^{\frac{-t\sqrt{\beta}}{\sqrt{\epsilon}}}+e^{-\frac{(d-t)\sqrt{\beta}}{\sqrt{\epsilon}}}\right), & t\in\Omega^-,\\ 1+\epsilon^{\frac{-k}{2}}\left(e^{\frac{-(t-d)\sqrt{\beta}}{\sqrt{\epsilon}}}+e^{-\frac{(1-t)\sqrt{\beta}}{\sqrt{\epsilon}}}\right), & t\in\Omega^+, \qquad k=1,2. \end{cases} \tag{2.2}$$

3. Discretization of the problem

Consider a non-uniform mesh $(0=t_0<t_1<\cdots<t_{\mathcal{M}}=l)$ with $\mathcal{M}$ a natural number multiple of eight (the reason will become clear later in the construction of the Shishkin mesh) and $h_i=t_i-t_{i-1}$, on $\Omega^{\mathcal{M}}=\left\{\Omega^{\mathcal{M}-}\cup\Omega^{\mathcal{M}+}\right\}$, where, $\Omega^{\mathcal{M}-}=\left(t_i: 1\le i\le \frac{\mathcal{M}}{2}-1\right)$, $\Omega^{\mathcal{M}+}=\left(t_i:\frac{\mathcal{M}}{2}+1\le i\le \mathcal{M}-1\right)$,

$$\bar{\Omega}^{\mathcal{M}}=\Omega^{\mathcal{M}}\cup\{t_0=0,t_{\mathcal{M}}=l\}\cup\{t_{\frac{\mathcal{M}}{2}}=d\}.$$

With respect to any mesh function $g(t)$ that is specified on $\bar{\Omega}^{\mathcal{M}}$ we denote

$$g_i=g(t_i),\ \bar{h}_0=\frac{h_1}{2},\ \bar{h}_i=\frac{h_i+h_{i+1}}{2},\ \bar{h}_{\mathcal{M}}=\frac{h_{\mathcal{M}}}{2},\ ||g||_\infty\equiv||g||_{\infty,\bar{\Omega}_{\mathcal{M}}}=\max_{0\le i\le\mathcal{M}}|g_i|,$$
$$D^+g_i=\frac{g_{i+1}-g_i}{h_{i+1}},\quad D^-g_i=\frac{g_i-g_{i-1}}{h_i}.$$

To discretize the problem (1.1)-(1.2), we proceed similarly as in [7]. Firstly, we multiply the equation in (1.1) by $\psi_i(t)$ and integrate it on $[t_{i-1},t_{i+1}]$. Then, multiplying by $\chi_i\bar{h}_i^{-1}$ we obtain

$$\left(\chi_i\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}}L_1 y\psi_i(t)dt+\chi_i\bar{h}_i^{-1}\lambda\int_{t_{i-1}}^{t_{i+1}}\psi_i(t)\left(\int_0^l K(t,s)y(s)ds\right)dt\right)$$
$$=\left(\chi_i\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}}F(t)\psi_i(t)dt\right), \tag{3.1}$$

where $\chi_i=\frac{\bar{h}_i}{\int_{t_{i-1}}^{t_{i+1}}\psi_i(t)dt}$, and the basis functions for $i\ne\frac{\mathcal{M}}{2}$ are given by

$$\psi_i(t)=\begin{cases}\psi_i^L(t)=\frac{\sinh(\Upsilon_i(t-t_{i-1}))}{\sinh(\Upsilon_i h_i)}, & t\in(t_{i-1},t_i),\\ \psi_i^R(t)=\frac{\sinh(\Upsilon_i(t_{i+1}-t))}{\sinh(\Upsilon_i h_{i+1})}, & t\in(t_i,t_{i+1}),\\ 0, & \text{otherwise},\end{cases} \tag{3.2}$$

being $\psi_i{}^L(t)$ and $\psi_i{}^R(t)$, respectively, the solutions of the following problems

$$\begin{cases}-\epsilon\psi''(t)+b_i\psi(t)=0, & t\in(t_{i-1},t_i),\\ \psi(t_{i-1})=0, \quad \psi(t_i)=1,\end{cases}$$

$$\begin{cases}-\epsilon\psi''(t)+b_i\psi(t)=0, & t\in(t_i,t_{i+1}),\\ \psi(t_i)=1, \quad \psi(t_{i+1})=0,\end{cases}$$

and $\Upsilon_i=\sqrt{\dfrac{b(t_i)}{\epsilon}}$.

For $i=\frac{\mathcal{M}}{2}$ we consider the following basis function for (1.1)–(1.2)

$$\psi_{\frac{\mathcal{M}}{2}}(t) = \begin{cases} \psi^L_{\frac{\mathcal{M}}{2}}(t) = \frac{\sinh(\Upsilon_i(t - t_{\frac{\mathcal{M}}{2}-1}))}{\sinh(\Upsilon_i(t_{\frac{\mathcal{M}}{2}} - t_{\frac{\mathcal{M}}{2}-1}))}, & t \in (t_{\frac{\mathcal{M}}{2}-1}, t_{\frac{\mathcal{M}}{2}}), \\ \psi^R_{\frac{\mathcal{M}}{2}}(t) = \frac{\sinh(\Upsilon_i(t_{\frac{\mathcal{M}}{2}+1} - t))}{\sinh(\Upsilon_i(t_{\frac{\mathcal{M}}{2}+1} - t_{\frac{\mathcal{M}}{2}}))}, & t \in (t_{\frac{\mathcal{M}}{2}}, t_{\frac{\mathcal{M}}{2}+1}), \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

where $\psi^L_{\frac{\mathcal{M}}{2}}(t)$ and $\psi^R_{\frac{\mathcal{M}}{2}}(t)$ are, respectively, the solutions to the following problems

$$\begin{cases} -\epsilon\psi''(t) + b_{\frac{\mathcal{M}}{2}}\psi(t) = 0, & t \in (t_{\frac{\mathcal{M}}{2}-1}, t_{\frac{\mathcal{M}}{2}}), \\ \psi(t_{\frac{\mathcal{M}}{2}-1}) = 0, \quad \psi(t_{\frac{\mathcal{M}}{2}}) = 1, \end{cases}$$

$$\begin{cases} -\epsilon\psi''(t) + b_{\frac{\mathcal{M}}{2}}\psi(t) = 0, & t \in (t_{\frac{\mathcal{M}}{2}}, t_{\frac{\mathcal{M}}{2}+1}), \\ \psi(t_{\frac{\mathcal{M}}{2}}) = 1, \quad \psi(t_{\frac{\mathcal{M}}{2}+1}) = 0. \end{cases}$$

Based on [7], we have that

$$\chi_i = \frac{1}{\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}} \psi_i(t)dt} = \frac{\Upsilon_i \bar{h}_i}{\tanh(\frac{\Upsilon_i h_i}{2}) + \tanh(\frac{\Upsilon_i h_{i+1}}{2})} \quad \text{and}$$

$$\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} L_1 y \psi_i(t)dt = -\epsilon\delta^2 y_i + b_i y_i + \chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} (b(t) - b(t_i)) y(t)\psi_i(t)dt, \tag{3.4}$$

where

$$\delta^2 y_i = \frac{\vartheta_i^{(2)} D^+ y_i - \vartheta_i^{(1)} D^- y_i}{\bar{h}_i} \tag{3.5}$$

with

$$\vartheta_i^{(1)} = \frac{b_i h_i \bar{h}_i}{\epsilon \sinh(\Upsilon_i h_i)\left[\tanh\left(\frac{\Upsilon_i h_i}{2}\right) + \tanh\left(\frac{\Upsilon_i h_{i+1}}{2}\right)\right]},$$

$$\vartheta_i^{(2)} = \frac{b_i h_{i+1} \bar{h}_i}{\epsilon \sinh(\Upsilon_i h_{i+1})\left[\tanh\left(\frac{\Upsilon_i h_i}{2}\right) + \tanh\left(\frac{\Upsilon_i h_{i+1}}{2}\right)\right]}.$$

In the case of constant step size, that is, $h_i = h$, it is

$$\vartheta_i^{(1)} = \vartheta_i^{(2)} = \vartheta_i = b\rho^2\{4\sinh^2\left(\sqrt{a_i}\frac{\rho}{2}\right)\}^{-1}, \quad \left(\rho = h(\sqrt{\epsilon})^{-1}\right).$$

Using Newton's interpolation method for the mesh points t_i, t_{i+1} we get

$$b(t) = b(t_i) + (t - t_i)D^+ b_i + \frac{1}{2}(t - t_i)(t - t_{i+1})b''(\zeta_i(t)).$$

Therefore, we get

$$\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} (b(t) - b(t_i)) y(t)\psi_i(t)dt = \chi_i \bar{h}_i^{-1} D^+ b_i \int_{t_{i-1}}^{t_{i+1}} (t - t_i) y(t)\psi_i(t)dt$$
$$+ \frac{1}{2}\chi_i \bar{h}_i^{-1} D^+ b_i \int_{t_{i-1}}^{t_{i+1}} (t - t_i)(t - t_{i+1}) y(t)\psi_i(t) b''(\zeta_i(t))dt. \tag{3.6}$$

That is,

$$\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} (b(t) - b(t_i)) y(t)\psi_i(t)dt = (D^+ b_i)\chi_i \mu_i y_i + \mathcal{R}_i^{(1)},$$

where

$$\mu_i = \int_{t_{i-1}}^{t_{i+1}} (t - t_i)\psi_i(t)dt,$$

$$\mathcal{R}_i^{(1)} = \frac{1}{2}\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} (t - t_i)(t - t_{i+1}) b''(\zeta_i(t))dt + \chi_i h_i^{-1}(D^+ b_i) \int_{t_{i-1}}^{t_{i+1}} (t - t_i)\psi_i(t) \left(\int_{t_i}^{t} y'(t)dt \right) dt. \tag{3.7}$$

A simple calculation gives

$$\mu_i = \bar{h}_i^{-1} \Upsilon_i \left(\frac{h_i}{\sinh(\Upsilon_i h_i)} - \frac{h_{i+1}}{\sinh(\Upsilon_i h_{i+1})} \right), \tag{3.8}$$

and for a constant stepsize mesh, $\mu_i = 0$. Now, the first term in (3.1) can be written as

$$\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} L_1 y \psi_i(t)dt = -\epsilon \delta^2 y_i + \bar{b}_i y_i + \mathcal{R}_i^{(1)}, \tag{3.9}$$

where

$$\bar{b}_i = b_i + (D^+ b_i)\chi_i \mu_i \tag{3.10}$$

and μ_i is given in (3.8). The residual calculation of the right hand side of (3.1) gives

$$\chi_i \bar{h}_i^{-1} \int_{t_{i-1}}^{t_{i+1}} F(t)\psi_i(t)dt = \bar{F}_i + \mathcal{R}_i^{(2)}. \tag{3.11}$$

Since F is discontinuous, we require the following extension of the source term to maintain the differentiability conditions as in [14]

$$(F^*)^{(k)}(t) = \begin{cases} (F_1^*)^{(k)}, & t \in [0, d], \\ (F_2^*)^{(k)}, & t \in [d, l], \end{cases}$$

where

$$(F_1^*)^{(k)}(t) = \begin{cases} (F_1)^{(k)}(t), & t \in [0, d), \\ (F_1)^{(k)}(d-), & t = d, \end{cases}$$

$$(F_2^*)^{(k)}(t) = \begin{cases} (F_2)^{(k)}(t), & t \in (d, l], \\ (F_2)^{(k)}(d+), & t = d. \end{cases}$$

Clearly, $F^*(t) \in \mathbb{C}^2[0, l]$. It follows the same integrability conditions [7] for any subdomain $[a, b]$ within the domain $[0, l]$ where

$$\bar{F}_i = F_i + (D^+ F_i)\chi_i \mu_i, \tag{3.12}$$

$$\mathcal{R}_i^{(2)} = \frac{1}{2}\chi_i \bar{h}_i^{-1} \left(\int_{t_{i-1}}^{t_{i+1}} (t - t_i)(t - t_{i+1}) F^{*\prime\prime}(\zeta_i(t))\psi_i(t)dt \right). \tag{3.13}$$

The second term on the left-hand side of (3.1) is approximated using the Taylor series expansion

$$\mathcal{K}(t, s) = \mathcal{K}(t_i, s) + (t - t_i)\frac{\partial}{\partial t}\mathcal{K}(t_i, s) + \frac{(t - t_i)^2}{2}\frac{\partial^2}{\partial t^2}\mathcal{K}(\zeta_i, s),$$

and thus, we have

$$\chi_i \bar{h}_i^{-1}\lambda \int_{t_{i-1}}^{t_{i+1}} \psi_i(t)\left(\int_0^l K(t,s)y(s)dsdt\right)dt = \lambda \int_0^l K(t_i,s)y(s)ds$$

$$+\mu_i\lambda \int_0^l \frac{\partial}{\partial t}K(t_i,s)y(s)ds + \mathcal{R}_i^{(3)} \equiv \lambda \int_0^l K(t_i,s)y(s)ds + \mathcal{R}_i^{(3)}, \tag{3.14}$$

where

$$\mathcal{K}(t_i,s) = K(t_i,s) + \mu_i \frac{\partial}{\partial t}K(t_i,s)y(s), \tag{3.15}$$

$$\mathcal{R}_i^{(3)} = \chi_i h_i^{-1} \int_{t_{i-1}}^{t_{i+1}} (t-t_i)^2 \psi_i(t)\left(\int_0^l \frac{\partial^2}{\partial t^2}K(\zeta_i(t),s)y(s)ds\right)dt. \tag{3.16}$$

Applying the composite trapezoidal rule with the integrand $\mathcal{K}(t_i,s)u(s)$, we get

$$\int_0^l \mathcal{K}(t_i,s)u(s)ds = \sum_{j=0}^{\mathcal{M}} \bar{h}_j \mathcal{K}_{ij} y_j + \mathcal{R}_i^{(4)}, \tag{3.17}$$

where

$$\mathcal{R}_i^{(4)} = \frac{1}{2}\sum_{j=1}^{\mathcal{M}} \int_{t_{j-1}}^{t_j} (t_j-\zeta)(t_{j-1}-\zeta)\frac{d^2}{d\zeta^2}\big(\mathcal{K}(t_i,\zeta)y(\zeta)\big)d\zeta. \tag{3.18}$$

Now, we get a discretization for problem (1.1) as

$$L^{\mathcal{M}} y_i = \begin{cases} -\epsilon\delta^2 y_i + \bar{b}_i y_i + \lambda \sum_{j=0}^{\mathcal{M}} \bar{h}_j \mathcal{K}_{ij} y_j = \bar{F}_i - \mathcal{R}_i, & i \neq \frac{\mathcal{M}}{2}, \\ \mathcal{L}^{\mathcal{M}} y_{\frac{\mathcal{M}}{2}}, \end{cases} \tag{3.19}$$

$$\text{with} \quad \mathcal{L}^{\mathcal{M}} y_{\mathcal{M}/2} = D^+ y_{\mathcal{M}/2} - D^- y_{\mathcal{M}/2}, \tag{3.20}$$

and the remainder term is

$$\mathcal{R}_i = \sum_{m=1}^{4} \mathcal{R}_i^{(m)} + \mathcal{R}_{\mathcal{M}/2}^{(5)}, \tag{3.21}$$

where $\mathcal{R}_i^{(m)} (1 \le m \le 4)$ are defined in (3.7), (3.13), (3.16) and (3.18), respectively, and $\mathcal{R}_{\mathcal{M}/2}^{(5)}$ is as follows

$$\mathcal{R}_{\mathcal{M}/2}^{(5)} = |y'(t_{\frac{\mathcal{M}}{2}})| \le \int_{t_{\frac{\mathcal{M}}{2}-1}}^{t_{\frac{\mathcal{M}}{2}+1}} y''(t)dt.$$

Based on (3.19), after removing the remainder term we propose the following discrete scheme for (1.1)-(1.2)

$$L^{\mathcal{M}} v_i = \begin{cases} -\epsilon\delta^2 v_i + \bar{b}_i v_i + \lambda \sum_{j=0}^{\mathcal{M}} \bar{h}_j \mathcal{K}_{ij} v_j = \bar{F}_i, & i \neq \frac{\mathcal{M}}{2}, \\ \mathcal{L}^{\mathcal{M}} v_{\frac{\mathcal{M}}{2}}, \end{cases} \quad \text{with } v_0 = y(0),\ v_{\mathcal{M}} = y(l). \tag{3.22}$$

Lemma 3.1. *(Discrete Maximum Principle) Let Y be the solution of (3.22). If*

$$Y(0) \ge 0, \quad Y(l) \ge 0, \quad L^{\mathcal{M}} Y(t_i) \ge 0 \,\forall t_i \in \Omega^{\mathcal{M}-} \cup \Omega^{\mathcal{M}+}$$

with

$$|\lambda| < \frac{\bar{\beta}}{\max\limits_{0\le t\le l} \sum_{m=0}^{\mathcal{M}} \bar{h}_m |\mathcal{K}_{i,j}|} \quad \text{and} \quad \mathcal{L}^{\mathcal{M}} Y_{\frac{\mathcal{M}}{2}} \le 0,$$

where $\bar{b}_i \ge \bar{\beta}$ then it is

$$Y(t_i) \ge 0,\ \forall t_i \in \bar{\Omega}^{\mathcal{M}}.$$

Proof. Using a similar argument as the one given in [8], and a suitable barrier function, the proof can be readily obtained. □

4. Construction of the Shishkin mesh and convergence analysis

This section presents a non-uniform mesh to address the approximate solution of the problem in (3.22). We construct a piecewise uniform Shishkin mesh, taking into account the point of discontinuity $t_{\mathcal{M}/2} = d$. To do that we divide the domain $\bar{\Omega} = [0, l]$ into 6 subintervals as

$$[0, \sigma_1], \quad [\sigma_1, d - \sigma_1], \quad [d - \sigma_1, d], \quad [d, d + \sigma_2], \quad [d + \sigma_2, l - \sigma_2], \quad [l - \sigma_2, l],$$

where

$$\sigma_1 = \min\left\{\frac{d}{4}, 2(\sqrt{\beta})^{-1}\sqrt{\epsilon}\ln\mathcal{M}\right\}, \quad \sigma_2 = \min\left\{\frac{l-d}{4}, 2(\sqrt{\beta})^{-1}\sqrt{\epsilon}\ln\mathcal{M}\right\}.$$

Later, each of the intervals $[\sigma_1, d - \sigma_1]$ and $[d + \sigma_2, l - \sigma_2]$ are divided into $\frac{\mathcal{M}}{4}$ subintervals while the others are divided into $\frac{\mathcal{M}}{8}$ subintervals. The corresponding step sizes are $h^{(1)} = \frac{8\sigma_1}{\mathcal{M}} = h^{(3)}$, $h^{(2)} = \frac{4(d-2\sigma_1)}{\mathcal{M}}$, $h^{(4)} = \frac{8\sigma_2}{\mathcal{M}} = h^{(6)}$, $h^{(5)} = \frac{4(l-d-2\sigma_2)}{\mathcal{M}}$, respectively. Note that if $d = \frac{l}{2}$ then $\sigma_1 = \sigma_2 = \sigma = \min\left\{\frac{d}{4}, 2(\sqrt{\beta})^{-1}\sqrt{\epsilon}\ln\mathcal{M}\right\}$.

Lemma 4.1. *[7] The remainder term $\mathcal{R}_i$ satisfies the bound*

$$||\mathcal{R}_i||_{\infty,\bar{\Omega}^{\mathcal{M}}} \le C\mathcal{M}^{-1}\ln\mathcal{M}. \tag{4.1}$$

Proof. We estimate each of the terms $\mathcal{R}_i^{(m)}$, $(1 \le m \le 4)$ separately, for the domain $\Omega^{\mathcal{M}-} \cup \Omega^{\mathcal{M}+}$ and $\mathcal{R}_{\mathcal{M}}^{(5)}$ at the point of discontinuity d.

Firstly, in the domain $\Omega^{\mathcal{M}-}$, we start with $\sigma_1 = 2(\sqrt{\beta})^{-1}\sqrt{\epsilon}\ln\mathcal{M}$ (note that if $\sigma_1 = \sigma_2 = \sigma = l/8$ we could use standard arguments). We begin by estimating the remainder $\mathcal{R}_i^{(2)}$ as given in (3.13). Since $|t - t_i| \le \max\{h_i, h_{i+1}\}$, $\quad |t - t_{i+1}| \le 2\bar{h}_i$ and $h^{(k)} \le C\mathcal{M}^{-1}$, $(k = 1, \ldots, 6)$, we have

$$|\mathcal{R}_i^{(2)}| \le C\chi_i\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}} |(t - t_i)(t - t_{i+1})|\psi_i(t)dt \le C\{\max(h_i, h_{i+1})\}^2 \le C\mathcal{M}^{-2}. \tag{4.2}$$

Then, using similar arguments, it can be concluded that $\mathcal{R}_i^{(3)}$ is bounded as follows

$$|\mathcal{R}_i^{(3)}| \le |\frac{1}{2}\chi_i\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}} (t - t_i)^2\left(\int_0^l \frac{\partial^2}{\partial t^2}K(\zeta_i(t), s)y(s)ds\right)\psi_i(t)dt| \le C\{\max(h_i, h_{i+1})\}^2 \le C\mathcal{M}^{-2}. \tag{4.3}$$

For $\mathcal{R}_i^{(1)}$, since $b \in C^2(\bar{\Omega})$ and $\|y(t)\| \le C$, we have

$$|\mathcal{R}_i^{(1)}| \le |\frac{1}{2}\chi_i\bar{h}_i^{-1}\int_{t_{i-1}}^{t_{i+1}} (t - t_i)(t - t_{i+1})b''(\zeta_i(t)y(t)\psi_{id})dt| + |\chi_i\bar{h}_i^{-1}(D^+b_i)\int_{t_{i-1}}^{t_{i+1}} (t - t_i)\chi_i\left(\int_{t_i}^{t} y'(t)dt\right)dt|.$$

In the domain $\bar{\Omega}^{\mathcal{M}-}$ we have

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \int_{t_{i-1}}^{t_{i+1}} \frac{1}{\sqrt{\epsilon}}\left(e^{-\frac{\sqrt{\beta}t}{\sqrt{\epsilon}}} + e^{-\frac{\sqrt{\beta}(d-t)}{\sqrt{\epsilon}}}\right)dt\right\}, \tag{4.4}$$

and for the domain $\bar{\Omega}^{\mathcal{M}+}$ we get,

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \int_{t_{i-1}}^{t_{i+1}} \frac{1}{\sqrt{\epsilon}}\left(e^{-\frac{\sqrt{\beta}(d-t)}{\sqrt{\epsilon}}} + e^{-\frac{\sqrt{\beta}(l-t)}{\sqrt{\epsilon}}}\right)dt\right\}. \tag{4.5}$$

The estimation $\mathcal{R}_i^{(1)}$ in the layer region $[0, \sigma_1]$ becomes

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-1}\left(\mathcal{M}^{-1} + \frac{h^{(1)}}{\sqrt{\epsilon}}\right) = C\mathcal{M}^{-1}\big(\mathcal{M}^{-1} + 16(\sqrt{\beta})^{-1}\mathcal{M}^{-1}\ln\mathcal{M}\big),$$

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-2}\ln\mathcal{M}, \qquad 1 \le i \le \frac{\mathcal{M}}{8} - 1. \tag{4.6}$$

Similar bounds can be obtained for the other layer regions $[d-\sigma_1, d]$, $[d, d+\sigma_1]$ and $[l-\sigma_1, l]$.

Now, let us consider the subinterval $[\sigma_1, d-\sigma_1]$

$$\begin{aligned}
|\mathcal{R}_i^{(1)}| &\le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}\left(e^{-\frac{\sqrt{\beta}t_{i-1}}{\sqrt{\epsilon}}} - e^{-\frac{\sqrt{\beta}t_{i+1}}{\sqrt{\epsilon}}}\right)\right\} + C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}\left(e^{-\frac{\sqrt{\beta}(d-t_{i-1})}{\sqrt{\epsilon}}} - e^{-\frac{\sqrt{\beta}(d-t_{i+1})}{\sqrt{\epsilon}}}\right)\right\} \\
&\le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}t_{i-1}}{\sqrt{\epsilon}}}\left(1 - e^{-\frac{2\sqrt{\beta}h^{(2)}}{\sqrt{\epsilon}}}\right)\right\} + C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}(d-t_{i+1})}{\sqrt{\epsilon}}}\left(1 - e^{-\frac{2\sqrt{\beta}h^{(2)}}{\sqrt{\epsilon}}}\right)\right\} \\
&\le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}t_{\frac{\mathcal{M}}{8}}}{\sqrt{\epsilon}}} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}\left(d-t_{\frac{3\mathcal{M}}{8}}\right)}{\sqrt{\epsilon}}}\right\}
\end{aligned}$$

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-2}, \qquad \frac{\mathcal{M}}{8} + 1 \le i \le \frac{3\mathcal{M}}{8} - 1. \tag{4.7}$$

Similarly, in $[d-\sigma_1, l-\sigma_1]$ we get the same result.

Now, at the transition points $i = \frac{\mathcal{M}}{8}, \frac{3\mathcal{M}}{8}, \frac{5\mathcal{M}}{8}$ and $\frac{7\mathcal{M}}{8}$ we evaluate the approximate value of $\mathcal{R}_i^{(1)}$. For $i = \frac{\mathcal{M}}{8}$, inequality (4.4) becomes

$$\begin{aligned}
|\mathcal{R}_{\frac{\mathcal{M}}{8}}^{(1)}| &\le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \int_{t_{\frac{\mathcal{M}}{8}-1}}^{t_{\frac{\mathcal{M}}{8}+1}} \frac{1}{\sqrt{\epsilon}}\left(e^{-\frac{t\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-t)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)dt\right\} \\
&= C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}\left(e^{-\frac{\sqrt{\beta}t_{\frac{\mathcal{M}}{8}-1}}{\sqrt{\epsilon}}} - e^{-\frac{\sqrt{\beta}t_{\frac{\mathcal{M}}{8}+1}}{\sqrt{\epsilon}}}\right)\right\} \\
&\quad + C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}\left(e^{-\frac{\sqrt{\beta}\left(d-t_{\frac{\mathcal{M}}{8}+1}\right)}{\sqrt{\epsilon}}} - e^{-\frac{\sqrt{\beta}\left(d-t_{\frac{M}{8}-1}\right)}{\sqrt{\epsilon}}}\right)\right\} \\
&= C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}t_{\frac{\mathcal{M}}{8}-1}}{\sqrt{\epsilon}}}\left(1 - e^{-\frac{2\sqrt{\beta}\left(t_{\frac{\mathcal{M}}{8}-1}-h_i^{(1)}-h_i^{(2)}\right)}{\sqrt{\epsilon}}}\right)\right\} \\
&\quad + C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}\left(d-t_{\frac{M}{8}+1}\right)}{\sqrt{\epsilon}}}\left(1 - e^{-\frac{2\sqrt{\beta}\left(d-t_{\frac{\mathcal{M}}{8}-1}-h_i^{(1)}-h_i^{(2)}\right)}{\sqrt{\epsilon}}}\right)\right\} \\
&\le C\mathcal{M}^{-1}\left\{\mathcal{M}^{-1} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}t_{\frac{M}{8}}}{\sqrt{\epsilon}}}e^{\frac{8\ln(\mathcal{M})}{\mathcal{M}}} + \frac{1}{\sqrt{\beta}}e^{-\frac{\sqrt{\beta}\left(d-t_{\frac{3\mathcal{M}}{8}}\right)}{\sqrt{\epsilon}}}e^{-\frac{\sqrt{\beta}(\sigma)}{\sqrt{\epsilon}}}\right\},
\end{aligned}$$

$$|\mathcal{R}_{\frac{\mathcal{M}}{8}}^{(1)}| \le C\mathcal{M}^{-2}. \tag{4.8}$$

The remaining estimation on the transition points is the same or can be done in a similar way.

So, the following inequality deduced from (4.4)-(4.7) holds,

$$|\mathcal{R}_i^{(1)}| \le C\mathcal{M}^{-2}\ln(\mathcal{M}).$$

For $\mathcal{R}_i^{(4)}$, in the domain $[0, \sigma_1]$

$$\begin{aligned}
|\mathcal{R}_i^{(4)}| &\le \frac{1}{2}\sum_{m=1}^{\mathcal{M}}\int_{t_{m-1}}^{t_m}(t_m-\zeta)(\zeta-t_{m-1})\frac{d^2}{d\zeta^2}(\mathcal{K}(t_m,\zeta)u(\zeta))d\zeta \\
&\le C\sum_{m=1}^{\mathcal{M}}\int_{t_{m-1}}^{t_m}(t_m-\zeta)(\zeta-t_{m-1})(1+|u'(\zeta)|+|u''(\zeta)|)d\zeta,
\end{aligned}$$

from which we have

$$|\mathcal{R}_i^{(4)}| \le C\left\{\sum_{m=1}^{\mathcal{M}} h_m^3 + \sum_{m=1}^{\mathcal{M}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta\right\}. \tag{4.9}$$

Hence,

$$\sum_{m=1}^{\mathcal{M}} h_m^3 = \frac{\mathcal{M}}{4}|h^{(1)}|^3 + \frac{\mathcal{M}}{4}|h^{(2)}|^3 \le C\mathcal{M}^{-2}. \tag{4.10}$$

The remaining term in (4.9) can be expanded as

$$\begin{aligned}
&\sum_{m=1}^{\mathcal{M}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad = \sum_{m=1}^{\frac{\mathcal{M}}{8}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad + \sum_{m=\frac{\mathcal{M}}{8}+1}^{\frac{3\mathcal{M}}{8}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad + \sum_{m=\frac{3\mathcal{M}}{8}+1}^{d} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta.
\end{aligned} \tag{4.11}$$

For the first term of the above inequality we have

$$\begin{aligned}
&\sum_{m=1}^{\frac{\mathcal{M}}{8}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\qquad \le |h^{(1)}|^2 \int_0^{\sigma} (\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\qquad \le |h^{(1)}|^2 4(\sqrt{\beta\epsilon})^{-1} \le C\mathcal{M}^{-2}\ln\mathcal{M}.
\end{aligned} \tag{4.12}$$

Now, for the second term in inequality (4.11)

$$\begin{aligned}
&\sum_{m=\frac{\mathcal{M}}{8}+1}^{\frac{3\mathcal{M}}{8}} \int_{t_{m-1}}^{t_m} (t_m-\zeta)(\zeta-t_{m-1})(\epsilon)^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad = 4\frac{1}{\sqrt{\beta}} \sum_{m=\frac{\mathcal{M}}{8}+1}^{\frac{3\mathcal{M}}{8}} \int_{t_{m-1}}^{t_m} (t_m-\zeta-\frac{h^{(2)}}{2})(\sqrt{\epsilon})^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad \le 4\frac{1}{\sqrt{\beta}} h^{(2)} \int_{\sigma}^{d-\sigma} (\sqrt{\epsilon})^{-1}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad = 8\frac{1}{(\sqrt{\beta})^2} h^{(2)}\left(e^{-\frac{\zeta\sqrt{\beta}}{\sqrt{\epsilon}}} + e^{-(d-\zeta)\frac{\sqrt{\beta}}{\sqrt{\epsilon}}}\right)d\zeta \\
&\quad \le 8\frac{1}{(\sqrt{\beta})^2} h^{(2)}\mathcal{M}^{-1} \le C\mathcal{M}^{-2}.
\end{aligned} \tag{4.13}$$

We can deduce from equations (4.11) to (4.13) that

$$|\mathcal{R}_i^{(4)}| \le C\mathcal{M}^{-2}\ln\mathcal{M}, \qquad 1 \le i \le \mathcal{M}-1.$$

Finally, we would determine $\mathcal{R}_i^{(5)}$ at $t_{\frac{\mathcal{M}}{2}} = d$ using the scheme $\mathcal{L}^{\mathcal{M}}$.

Consider,

$$
\begin{aligned}
|[y'] - \mathcal{L}^{\mathcal{M}}| &= |[y'(t_{\frac{\mathcal{M}}{2}})] - (D^- - D^+)(t_{\frac{\mathcal{M}}{2}})| \\
&= |y'(t_{\frac{\mathcal{M}}{2}})| \le \int_{t_{\frac{\mathcal{M}}{2}-1}}^{t_{\frac{\mathcal{M}}{2}+1}} y''(t)dt \\
&\le \int_{t_{\frac{\mathcal{M}}{2}-1}}^{t_{\frac{\mathcal{M}}{2}}} y''(t)dt + \int_{t_{\frac{\mathcal{M}}{2}}}^{t_{\frac{\mathcal{M}}{2}+1}} y''(t)dt.
\end{aligned}
$$

From Lemma (2.2) and the result from [8] we have

$$
\begin{aligned}
|[y'] - L^M| &\le \int_{t_{\frac{\mathcal{M}}{2}-1}}^{t_{\frac{\mathcal{M}}{2}}} \frac{1}{\sqrt{\epsilon}}\left(e^{-\frac{(d-t)\sqrt{\beta}}{\sqrt{\epsilon}}}\right)dt + \int_{t_{\frac{\mathcal{M}}{2}}}^{t_{\frac{\mathcal{M}}{2}+1}} \frac{1}{\sqrt{\epsilon}}\left(e^{-\frac{(t-d)\sqrt{\beta}}{\sqrt{\epsilon}}}\right)dt \\
&\le \frac{C}{\sqrt{\epsilon}} \max(h^{(3)}, h^{(4)}) \\
&\le \frac{8C(\sqrt{\beta})^{-1}\ln\mathcal{M}}{\mathcal{M}}
\end{aligned}
$$

$$
\mathcal{R}_i^{(5)} \le C\mathcal{M}^{-1}\ln\mathcal{M}. \tag{4.14}
$$

If we denote the error as $\mathcal{Z}_i = y_i - v_i$ from (3.19) and (3.22) we get

$$
L^{\mathcal{M}}\mathcal{Z}_i = \mathcal{R}_i, \qquad 1 \le i \le \mathcal{M}-1, \tag{4.15}
$$

$$
\mathcal{Z}_0 = 0, \quad \mathcal{Z}_{\mathcal{M}} = 0, \tag{4.16}
$$

where $\mathcal{R}_i$ are taken from (3.21) and (4.14). □

Theorem 4.2. *Let v_i be the solution of (3.22) and y be the solution of (1.1)-(1.2). Then for $\mathcal{M}$ sufficiently large it is*

$$
||v_i - y||_{\infty,\bar{\Omega}^{\mathcal{M}}} \le C\mathcal{M}^{-1}\ln\mathcal{M}.
$$

Proof. By applying Lemma 3.1 to (4.15) and (4.16) and also using Lemma 4.1 we can get the required result. □

5. Numerical outcome

We present here two examples to validate the proposed algorithm.

Example 5.1.

$$
-\epsilon y'' + (2 - e^{-t})y + \frac{1}{2}\int_0^1 (e^{(t\cos(\pi s))} - 1)y(s)ds = F(t),
$$

where

$$
F(t) = \begin{cases} -\frac{1}{(1+t)}, & 0 < t \le 0.5, \\ \frac{1}{(1+t)}, & 0.5 < t < 1, \end{cases}
$$

$$
y(0) = 1, \quad y(1) = 0.
$$

Since the exact solution is unknown we use to measure the efficiency of the proposed method the same procedure as in the double mesh approach, taking:

$$
E_\epsilon^{\mathcal{M}} = \|v_i^{\mathcal{M}} - v_i^{2\mathcal{M}}\|_\infty, \qquad E^{\mathcal{M}} = \max_\epsilon E_\epsilon^{\mathcal{M}}.
$$

Table 1
Maximum point-wise errors $E^{\mathcal{M}}$ and orders of convergence $Q^{\mathcal{M}}$ for Example 5.1.

$\epsilon \setminus \mathcal{M}$	32	64	128	256	512	1024
10^{-1}	2.612e-03	1.212e-03	5.839e-04	2.866e-04	1.419e-04	7.064e-05
10^{-2}	1.265e-02	6.096e-03	2.994e-03	1.483e-03	7.382e-04	3.683e-04
10^{-3}	2.162e-02	1.036e-02	5.072e-03	2.510e-03	1.248e-03	6.224e-04
10^{-4}	3.380e-02	1.445e-02	6.356e-03	2.884e-03	1.616e-03	8.071e-04
10^{-5}	4.965e-02	2.468e-02	1.193e-02	5.668e-03	2.720e-03	1.310e-03
10^{-6}	5.551e-02	2.836e-02	1.426e-02	7.103e-03	3.478e-03	1.720e-03
10^{-7}	5.747e-02	2.961e-02	1.500e-02	7.539e-03	3.773e-03	1.879e-03
10^{-8}	5.810e-02	3.001e-02	1.525e-02	7.680e-03	3.852e-03	1.928e-03
10^{-9}	5.831e-02	3.014e-02	1.532e-02	7.725e-03	3.878e-03	1.942e-03
10^{-10}	5.837e-02	3.018e-02	1.535e-02	7.739e-03	3.886e-03	1.947e-03
10^{-11}	5.839e-02	3.019e-02	1.536e-02	7.744e-03	3.888e-03	1.948e-03
10^{-12}	5.840e-02	3.020e-02	1.536e-02	7.745e-03	3.889e-03	1.949e-03
10^{-13}	5.840e-02	3.020e-02	1.536e-02	7.745e-03	3.889e-03	1.949e-03
10^{-14}	5.840e-02	3.020e-02	1.536e-02	7.746e-03	3.889e-03	1.949e-03
10^{-15}	5.840e-02	3.020e-02	1.536e-02	7.746e-03	3.889e-03	1.949e-03
$E^{\mathcal{M}}$	5.840e-02	3.020e-02	1.536e-02	7.746e-03	3.889e-03	1.949e-03
$Q^{\mathcal{M}}$	0.9522	0.9754	0.9877	0.9941	0.9967	–

Table 2
Maximum point-wise errors $E^{\mathcal{M}}$ and orders of convergence $Q^{\mathcal{M}}$ for Example 5.2.

$\epsilon \setminus \mathcal{M}$	32	64	128	256	512	1024
10^{-1}	7.441e-03	4.276e-03	2.276e-03	1.173e-03	5.950e-04	2.996e-04
10^{-2}	1.072e-01	5.518e-02	2.799e-02	1.410e-02	7.074e-03	3.543e-03
10^{-3}	2.429e-01	1.407e-01	7.555e-02	3.910e-02	1.989e-02	1.003e-02
10^{-4}	1.844e-01	7.982e-02	3.562e-02	1.588e-02	8.293e-03	6.205e-03
10^{-5}	2.591e-01	1.283e-01	6.123e-02	2.946e-02	1.431e-02	6.938e-03
10^{-6}	2.837e-01	1.438e-01	7.208e-02	3.581e-02	1.760e-02	8.699e-03
10^{-7}	2.917e-01	1.488e-01	7.507e-02	3.765e-02	1.882e-02	9.344e-03
10^{-8}	2.943e-01	1.505e-01	7.603e-02	3.820e-02	1.914e-02	9.576e-03
10^{-9}	2.951e-01	1.510e-01	7.634e-02	3.838e-02	1.924e-02	9.631e-03
10^{-10}	2.953e-01	1.511e-01	7.644e-02	3.843e-02	1.927e-02	9.649e-03
10^{-11}	2.954e-01	1.512e-01	7.647e-02	3.845e-02	1.928e-02	9.654e-03
10^{-12}	2.955e-01	1.512e-01	7.648e-02	3.846e-02	1.928e-02	9.656e-03
10^{-13}	2.955e-01	1.512e-01	7.648e-02	3.846e-02	1.929e-02	9.657e-03
10^{-14}	2.955e-01	1.512e-01	7.648e-02	3.846e-02	1.929e-02	9.657e-03
10^{-15}	2.955e-01	1.512e-01	7.648e-02	3.846e-02	1.929e-02	9.657e-03
$E^{\mathcal{M}}$	2.955e-01	1.512e-01	7.648e-02	3.910e-02	1.989e-02	1.003e-02
$Q^{\mathcal{M}}$	0.9667	0.9833	0.9679	0.9751	0.9877	–

The approximate order of convergence is obtained using

$$Q^{\mathcal{M}} = \log_2 \frac{E^{\mathcal{M}}}{E^{2\mathcal{M}}}$$

Example 5.2.

$$-\epsilon y'' + (1+\sin(\frac{\pi t}{2}))y + \frac{1}{2}\int_0^1 \left(e^{(1-ts)}y(s)\right)ds = F(t),$$

where

$$F(t) = \begin{cases} -8, & 0 < t \le 0.4, \\ 8, & 0.4 < t < 1, \end{cases}$$
$$y(0) = 1, \quad y(1) = 1.$$

Tables 1 and 2 show ϵ values ranging from 10^{-1} to 10^{-15} and $\mathcal{M}$ from 2^5 to 2^{10} for Examples 5.1 and 5.2 respectively for the maximum pointwise error and order of convergence. Furthermore, the table shows that our numerical method is always first-order convergence. The numerical solution to Example 5.1 is shown in Fig. 1 when $\epsilon = 10^{-3}$, $\mathcal{M} = 128$ and $d = 0.5$, whereas Fig. 2 shows the numerical solution to Example 5.2 when $\epsilon = 10^{-3}$, $\mathcal{M} = 128$ and $d = 0.4$.

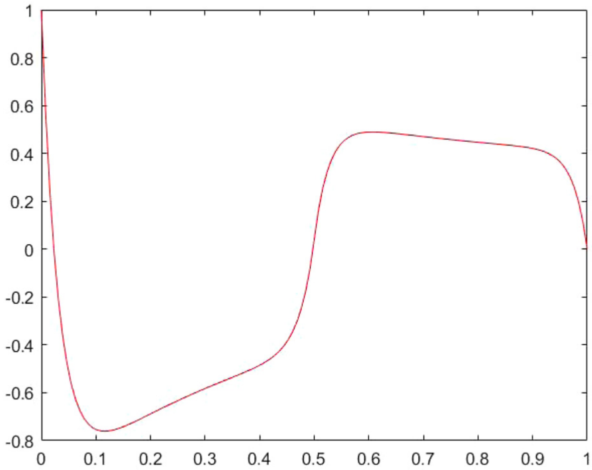

Fig. 1. The numerical solution for Example 5.1 when $\epsilon = 10^{-3}$, $\mathcal{M} = 128$ and $d = 0.5$.

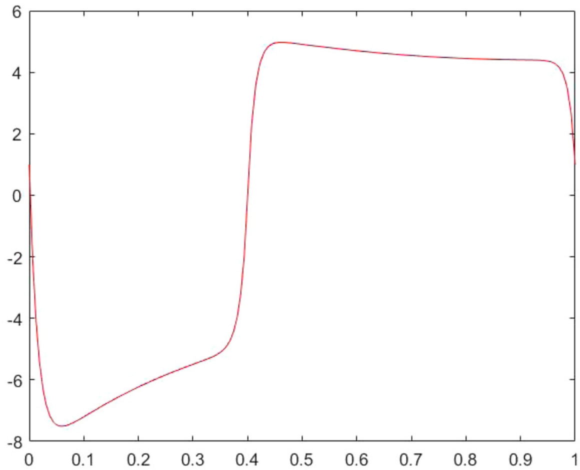

Fig. 2. The numerical solution for Example 5.2 when $\epsilon = 10^{-3}$, $\mathcal{M} = 128$ and $d = 0.4$.

6. Conclusions

This paper considers singularly perturbed Fredholm integro-differential equations with discontinuous source terms. An almost first order ϵ-uniformly convergent numerical method for solving this problem is presented, which comprises an exponentially fitted scheme on a Shishkin mesh. Using the integral representation, together with a quadrature rule, the weights and remainder terms in the integral form are used to develop a difference scheme to solve the problem. A theoretical analysis is conducted to prove the first-order convergence of the proposed method. Some examples are given to illustrate its performance.

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