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RESEARCH ARTICLE

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A trigonometrically adapted 6(4) explicit Runge–Kutta–Nyström pair to solve oscillating systems

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In this study, a trigonometrically adapted 6(4) explicit Runge–Kutta–Nyström (RKN) pair with six stages is formulated, considering a previous method developed by El-Mikkawy and Rahmo. The obtained adapted pair integrates exactly the usual test equation: $y'' = -w^2 y$. The local truncation error of the new method is presented, showing that the algebraic order of the original method is maintained. The periodicity interval of the new method is computed, showing that the developed method is "almost" P-stable. The numerical examples considered clearly show the superiority of the new developed embedded pair over other RKN methods of algebraic orders 6(4) with six stages appeared in the literature.

KEYWORDS

initial value problems, oscillatory problems, Runge–Kutta–Nyström pair, trigonometrically fitted approach

MSC CLASSIFICATION

65L05, 65L06

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1 | INTRODUCTION

Various methods have been formulated to solve numerically the initial value problem (IVP) of the special second-order ordinary differential equation given by

$$y'' = f(x, y(x)), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0, \tag{1}$$

whose solutions exhibit an oscillatory behavior, where $y(x) \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ are assumed to be sufficiently differentiable. Problem (1) is usually met in the areas of fluid mechanics, quantum and physical chemistry, astronomy, and many more. The family of Runge–Kutta–Nyström (RKN) codes has been usually considered for solving (1) numerically. Related to such use, the embedding strategy was first used by Fehlberg[1] and has been used extensively since then to give an estimate of the error made at each step when applying Runge–Kutta type methods. Dormand et al[2] presented a 6(4) pair with six stages that uses the first same as last (FSAL) property for solving (1). Similarly, El-Mikkawy and Rahmo developed a 6(4) embedded pair having six stages in El-Mikkawy and Rahmo,[3] without the use of the FSAL property, which appeared to behave more efficiently than the one with the FSAL property by Dormand et al. Many RKN-adapted methods have been obtained by different authors, among them we mention those by Simos,[4] Kalogiratou and Simos,[5] Van de Vyver,[6] and Liu.[7] Senu et al[8] developed an explicit pair of embedded RKN methods for oscillatory problems, Franco et al[9] developed two pairs of explicit RKN embedded methods for solving problems with oscillating solutions, and Anastassi and Kosti[10] constructed an optimized embedded 6(4) RKN pair for solving periodic problems. Tsitouras[11] presented a study on fitted modifications of RKN pairs, putting an effort to avoid the order reduction phenomenon and showing that the modification of only four coefficients results in an adapted RKN pair with same orders as the standard RKN pair, which integrates precisely the harmonic oscillator. Demba et al[12,13] presented two pairs of explicit trigonometrically adapted embedded RKN methods for approximating the solution of the problem in (1). Recently, Demba et al[14] constructed a pair of explicit exponentially adapted RKN methods for solving the problem in (1). Later, Demba et al. developed a new phase- and amplification-fitted explicit RKN pair for solving the problem in (1).[15] In this paper, we obtain a new trigonometrically adapted embedded pair of explicit RKN methods based on the 6(4) pair of explicit type presented by El-Mikkawy and Rahmo[3] for solving (1). The obtained pair can solve exactly the common test oscillator: $y'' = -w^2 y$. The numerical examples highlight the performance of the new method compared to some embedded RKN codes with orders 6(4) and six stages, appeared in the literature. The outline of the paper is as follows: We give a detailed explanation on the explicit RKN pair and the concept of a RKN method trigonometrically adapted in Section 2. The next section focuses on the construction of the new RKN pair. Section 4 presents the order of the new pair and some details concerning its linear stability analysis. Some numerical experiments are shown in Section 5. A detailed explanation on the obtained figures is given in Section 6, and finally, in Section 7, we give some conclusions.

2 | BASIC CONCEPTS

An explicit r-stage RKN method is usually formulated as

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{l=1}^{r} b_l f(x_n + c_l h, Y_l), \tag{2}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l f(x_n + c_l h, Y_l), \tag{3}$$

$$Y_l = y_n + c_l h y'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), l = 1, 2, \dots, r, \tag{4}$$

where y_{n+1} and y'_{n+1} denote approximate values of $y(x_{n+1})$ and $y'(x_{n+1})$, respectively, and $x_{n+1} = x_n + h$, $n = 0, 1, \dots,$. This explicit RKN method can be expressed in an abbreviated form using the Butcher array

$$\begin{array}{c|c} c & A \\ \hline & b \\ & d \end{array}$$

being A a lower triangular matrix of dimension r containing the coefficients a_{ij}, the vector $c = (c_1, c_2, \dots, c_r)^T$ contains the intermediate stages, while $b = (b_1, b_2, \dots, b_r)$ and $d = (d_1, d_2, \dots, d_r)$ contain the remaining coefficients. This is usually expressed by (c, A, b, d).

A $p(q)$ pair of embedded RKN methods consists of two RKN methods, the first one with order p denoted by (c, A, b, d) and another with order $q < p$, denoted by $(c, A, \hat{b}, \hat{d})$, where both share the coefficients in c and A. The method with a higher order is used to give the approximate values y_{n+1}, y'_{n+1}, while the lower order method provides a second set of approximate values $\hat{y}_{n+1}$, $\hat{y}'_{n+1}$. These values are obtained for the sole purpose of providing a reliable estimate of the local error. An embedded-type pair of RKN methods may be formulated by the following Butcher tableau:

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \\ \hline & \hat{b}^T \\ & \hat{d}^T \end{array}$$

In this paper, we consider a variable step-size formulation based on the local error estimate obtained through the embedding strategy. The estimate of the local error at x_{n+1} is obtained by means of the differences $\eta_{n+1} = \hat{y}_{n+1} - y_{n+1}$ and $\eta'_{n+1} = \hat{y}'_{n+1} - y'_{n+1}$.

Let $\text{Est}_{n+1} = \max(\|\eta_{n+1}\|_\infty, \|\eta'_{n+1}\|_\infty)$ be the estimate of the local error on each iteration step, and *Tol* the tolerance selected by the user. To advance the integration from x_n to x_{n+1}, we consider the strategy for changing the step-length given in Liu et al.[7]

- If $Est_{n+1} < 0.01 \times Tol$, then we take $h_{n+1} = 2h_n$,
- If $0.01 \times Tol \le Est_{n+1} < Tol$, then we take $h_{n+1} = h_n$,
- If $Est_{n+1} \ge Tol$, then we reject the obtained values and repeat the calculations taking as step size $h_n/2$.

Definition 1 (Anastassi and Kosti[10]). An explicit RKN method as given in Equations (2)–(4) is said to have algebraic order k if it holds

$$\begin{cases} y_{n+1} - y(x_n + h) = O(h^{k+1}), \\ y'_{n+1} - y'(x_n + h) = O(h^{k+1}). \end{cases} \tag{5}$$

Definition 2. An explicit RKN method described by the formulas in (2)–(4) is said to be trigonometrically fitted if it can integrate exactly (except round-off errors) any problem $y'' = f(x, y)$, for which the solution is a linear combination of $\{\sin(wx), \cos(wx)\}$, where $w > 0$ is a parameter known as the principal frequency of the solution.

If we use the RKN method given in (2)–(4) to solve $y'' = -w^2 y$, we get

$$y_{n+1} = y_n + h y'_n + h^2 \sum_{l=1}^{r} b_l(-w^2 Y_l), \tag{6}$$

$$y'_{n+1} = y'_n + h\sum_{l=1}^{r} d_l(-w^2 Y_l), \tag{7}$$

where

$$Y_l = y_n + c_l h y'_n + h^2 \sum_{j=1}^{l-1} a_{lj}(-w^2 Y_j),\ l = 1, 2, 3, \ldots, r. \tag{8}$$

Let us assume that the exact solution of the test equation is given by $y(x) = e^{iwx}$. We compute the values of y_n, y_{n+1}, y'_n, and y'_{n+1} and introduce them in the equations of the method (5)–(7). If we use Euler's identity, $e^{iv} = \cos(\upsilon) + i\sin(\upsilon)$, and compare the real and imaginary parts of y_{n+1}, y'_{n+1} and $y(x_{n+1}), y'(x_{n+1})$, respectively, we get the following system (see Tsitouras[11]):

$$\cos(\upsilon) = 1 - \upsilon^2 \sum_{l=1}^{r} b_l \left(1 - \upsilon^2 \sum_{j=1}^{l-1} a_{lj} \Re[Y_j e^{-iwx}]\right), \tag{9}$$

$$\sin(\upsilon) = \upsilon - \upsilon^2 \sum_{l=1}^{r} b_l \left(c_l \upsilon - \upsilon^2 \sum_{j=1}^{l-1} a_{lj} \Im[Y_j e^{-iwx}]\right), \tag{10}$$

$$\sin(\upsilon) = \upsilon \sum_{l=1}^{r} d_l \left(1 - \upsilon^2 \sum_{j=1}^{l-1} a_{lj} \Re[Y_j e^{-iwx}]\right), \tag{11}$$

$$\cos(\upsilon) = 1 - \upsilon \sum_{l=1}^{r} d_l \left(c_l \upsilon - \upsilon^2 \sum_{j=1}^{l-1} a_{lj} \Im[Y_j e^{-iwx}]\right), \tag{12}$$

where $\upsilon = wh$ and $i = \sqrt{-1}$ is the imaginary unit ($\Re$ and $\Im$ denote the real and imaginary parts, respectively).

3 | DEVELOPMENT OF THE NEW EMBEDDED PAIR

This section is devoted to the development of a new 6(4) explicit pair of trigonometrically adapted embedded RKN methods.

The RKN6(4)6 ER embedded pair derived by El-Mikkawy and Rahmo[3] is used to derive the adapted embedded RKN method. The coefficients in Table 1 are those of the method in El-Mikkawy and Rahmo,[3] with the correct value of a_{54} as given in Anastassi and Kosti.[10]

TABLE 1 The RKN6(4)6 ER method in El-Mikkawy and Rahmo[3]

0						
$\frac{1}{77}$	$\frac{1}{11858}$					
$\frac{1}{3}$	$-\frac{7189}{17118}$	$\frac{4070}{8559}$				
$\frac{2}{3}$	$\frac{4007}{2403}$	$-\frac{589655}{355644}$	$\frac{25217}{118548}$			
$\frac{13}{15}$	$-\frac{4477057}{843750}$	$\frac{13331783894}{2357015625}$	$-\frac{281996}{5203125}$	$\frac{563992}{7078125}$		
1	$\frac{17265}{2002}$	$-\frac{1886451746}{212088107}$	$\frac{22401}{31339}$	$\frac{2964}{127897}$	$\frac{178125}{5428423}$	
	$-\frac{341}{780}$	$\frac{386683451}{661053840}$	$\frac{2853}{11840}$	$\frac{267}{3020}$	$\frac{9375}{410176}$	0
	$-\frac{341}{780}$	$\frac{29774625727}{50240091840}$	$\frac{8559}{23680}$	$\frac{801}{3020}$	$\frac{140625}{820352}$	$\frac{847}{18240}$
	$-\frac{95}{39}$	$\frac{89332243}{33052692}$	$\frac{317}{3552}$	$\frac{623}{5436}$	$\frac{54125}{1845792}$	0
	$-\frac{95}{39}$	$\frac{362030669}{132210768}$	$\frac{317}{2368}$	$\frac{623}{1812}$	$\frac{270625}{1230528}$	0

To obtain the adapted RKN pair, we take the coefficients of the lower order method in the RKN6(4)6 ER pair. Now we have to solve the system in (8)–(11) taking four of the coefficients as unknowns, specifically we take $\hat{b}_1, \hat{b}_2, \hat{d}_1, \hat{d}_2$ as unknowns and obtain the following values:

$$\begin{aligned}
\hat{b}_1 &= -\frac{1}{370581196986000\upsilon^3}\big(370581196986000\cos(\upsilon)\upsilon + 28164170970936000\upsilon + 77750016487\upsilon^9 + 252945285032700\upsilon^5 \\
&\quad -6668142131313\upsilon^7 - 3670210013373000\upsilon^3 - 28534752167922000\sin(\upsilon) + 2406371409000\sin(\upsilon)\upsilon^2\big), \\
\hat{b}_2 &= -\frac{11}{509852903969763000\upsilon^3}\big(3568970327788341000\sin(\upsilon) - 3568970327788341000\upsilon - 33518107575789975\upsilon^5 \\
&\quad + 469556502500247750\upsilon^3 - 10954502218055\upsilon^9 + 914008554301554\upsilon^7\big), \\
\hat{d}_1 &= -\frac{1}{49410826264800\upsilon^2}\big(-49410826264800\sin(\upsilon)\upsilon - 1732867129447200\upsilon^2 - 4379960615238\upsilon^6 \\
&\quad + 77750016487\upsilon^8 + 150451687982700\upsilon^4 + 3804633622389600 - 3804633622389600\cos(\upsilon) \\
&\quad +320849521200\upsilon^2\cos(\upsilon)\big), \\
\hat{d}_2 &= -\frac{11}{67980387195968400\upsilon^2}\big(475862710371778800\cos(\upsilon) - 475862710371778800 - 10954502218055\upsilon^8 \\
&\quad - 19827612932157450\upsilon^4 + 591617761016679\upsilon^6 + 221008661886302325\upsilon^2\big).
\end{aligned} \tag{13}$$

Using the Taylor series, we obtain the following expressions in powers of υ

$$\begin{aligned}
\hat{b}_1 &= -\frac{95}{39} - \frac{11}{135}\upsilon^2 + \frac{3207538391}{791840164500}\upsilon^4 - \frac{2107925159}{99771860727000}\upsilon^6 - \frac{467}{279417600}\upsilon^8 + \frac{2503}{239740300800}\upsilon^{10} - \frac{697}{14384418048000}\upsilon^{12} \\
&\quad + \frac{1189}{6846982990848000}\upsilon^{14} + \cdots, \\
\hat{b}_2 &= \frac{89332243}{33052692} + \frac{11}{135}\upsilon^2 - \frac{182712277}{41134554000}\upsilon^4 + \frac{14305181}{592337577600}\upsilon^6 + \frac{1}{518400}\upsilon^8 - \frac{1}{80870400}\upsilon^{10} + \frac{1}{16982784000}\upsilon^{12} \\
&\quad - \frac{1}{4619317248000}\upsilon^{14} + \cdots, \\
\hat{d}_1 &= -\frac{95}{39} - \frac{2161814503}{211157377200}\upsilon^4 + \frac{2231638589}{15203331158400}\upsilon^6 - \frac{1301}{69854400}\upsilon^8 + \frac{461}{3353011200}\upsilon^{10} - \frac{353}{479480601600}\upsilon^{12} \\
&\quad + \frac{4817}{1611054821376000}\upsilon^{14} + \cdots, \\
\hat{d}_2 &= \frac{362030669}{132210768} + \frac{7688021}{685575900}\upsilon^4 - \frac{5416301}{39489171840}\upsilon^6 + \frac{11}{518400}\upsilon^8 - \frac{1}{6220800}\upsilon^{10} + \frac{1}{1132185600}\upsilon^{12} \\
&\quad - \frac{1}{271724544000}\upsilon^{14} + \cdots.
\end{aligned} \tag{14}$$

When $\upsilon \to 0$, the obtained coefficients of the adapted fourth-order method become the constant coefficients of the counterpart method in the RKN6(4)6 ER approach. Similarly, for the sixth order method, if we consider as unknowns b_1, b_3, d_1, d_2 in Equations (8)–(11), we obtain the following solution:

$$\begin{aligned}
b_1 &= -\frac{1}{42762720\upsilon^3\left(-19971 + 370\upsilon^2\right)}\big(-246844018344\upsilon^3 - 1708028562240\upsilon - 854014281120\cos(\upsilon)\upsilon - 133805178\upsilon^7 \\
&\quad + 2562042843360\sin(\upsilon) + 15822206400\cos(\upsilon)\upsilon^3 + 102741600\upsilon^4\sin(\upsilon) - 142335713520\upsilon^2\sin(\upsilon) + 5236458112\upsilon^5 \\
&\quad +5513365\upsilon^9\big), \\
b_3 &= \frac{1}{9483840\upsilon^3\left(-19971 + 370\upsilon^2\right)}\big(568205305920\sin(\upsilon) + 49062096657\upsilon^3 - 568205305920\upsilon + 112736484\upsilon^7 \\
&\quad -3889500606\upsilon^5 - 1359380\upsilon^9\big), \\
d_1 &= -\frac{1}{16470275421600\upsilon^2}\big(-16470275421600\sin(\upsilon)\upsilon - 179561239\upsilon^{10} - 610541811587880\upsilon^2 + 50150562660900\upsilon^4 \\
&\quad +1268211207463200(1 - \cos(\upsilon)) + 28384598151\upsilon^8 - 1628608402980\upsilon^6 + 106949840400\upsilon^2\cos(\upsilon)\big) \\
d_2 &= -\frac{11}{1722169808964532800\upsilon^2}\big(12055188662751729600(\cos(\upsilon) - 1) - 299466115430688\upsilon^8 \\
&\quad +16743317587155180\upsilon^6 - 502299527614655400\upsilon^4 + 5934809032148813985\upsilon^2 + 1922730181460\upsilon^{10}\big).
\end{aligned} \tag{15}$$

Again, using the Taylor series, we obtain that

$$\begin{aligned}
b_1 &= -\frac{341}{780} - \frac{1}{71096760}\upsilon^4 - \frac{72439753}{41649619556160}\upsilon^6 + \frac{267152203513}{1848410115902380800}\upsilon^8 + \frac{37844826222673}{7998162992015396840640 0}\upsilon^{10} \\
&\quad + \frac{113190494349090388999}{421690666619744254403672601600 0}\upsilon^{12} + \frac{449194205218849539466465 3}{1145335465216556100638621391611289600 0}\upsilon^{14} + \cdots, \\
b_3 &= \frac{2853}{11840} + \frac{1}{71096760}\upsilon^4 - \frac{8251967}{7572658101120}\upsilon^6 + \frac{914422348337}{16635691043121427200}\upsilon^8 + \frac{9270389295341171}{172760320627532571757824 00}\upsilon^{10} \\
&\quad + \frac{42215705341778568959}{345019636325245299057550310400 0}\upsilon^{12} + \frac{40905128292997919257052 71}{1874185306718000891954107731727564800 0}\upsilon^{14} + \cdots, \\
d_1 &= -\frac{341}{780} - \frac{7447}{2437603200}\upsilon^6 - \frac{5478895}{709488787392}\upsilon^8 + \frac{461}{3353011200}\upsilon^{10} - \frac{353}{479480601600}\upsilon^{12} + \frac{4817}{1611054821376000}\upsilon^{14} + \cdots, \\
d_2 &= \frac{29774625727}{50240091840} + \frac{7447}{2437603200}\upsilon^6 + \frac{1176527}{131630572800}\upsilon^8 - \frac{1}{6220800}\upsilon^{10} + \frac{1}{1132185600}\upsilon^{12} - \frac{1}{271724544000}\upsilon^{14} + \cdots.
\end{aligned} \tag{16}$$

Again, when $\upsilon \to 0$, the above values of b_1, b_3, d_1, and d_2 in case of the higher order fitted method become the constant coefficients of the counterpart method in the RKN6(4)6 ER pair.

The new adapted RKN embedded method with the above coefficients depending on υ and the remaining coefficients of the RKN6(4)6 ER method will be named as EETFRKN6(4)6 ER.

4 | ORDER OF ACCURACY AND ERROR ANALYSIS

This section presents the local truncation error (LTE) of the new methods and the corresponding orders of convergence for scalar problems. They are obtained considering the common Taylor series expansions. The LTE at x_{n+1} of the exact solution y and its first derivative y' are

$$\begin{aligned}
LTE &= y_{n+1} - y(x_n + h), \\
LTE_{der} &= y'_{n+1} - y'(x_n + h).
\end{aligned} \tag{17}$$

Proposition 1. *The principal terms of the Local truncation error for y and y', that is, $PLTE$ and $PLTE_{der}$ of the adapted method of lower order are*

$$\begin{aligned}
PLTE &= -\frac{h^5}{748440}(792w^2 f_x + 792w^2 f_y y' + 2379y' f_{yy} y'' + 793 f_{xxx} + 2379(y')^2 f_{xyy} + 792 f_y f_x + 2379y'' f_{xy} \\
&\quad + 792(f_y)^2 y' + 793(y')^3 f_{yyy} + 2379y' f_{yxx}) + O(h^6), \\
PLTE_{der} &= \frac{h^5}{120}(4y' f_{xxxy} + (f_y)^2 y'' + 6y'' f_{yxx} + 3(y'')^2 f_{yy} + 4 f_{xy} f_x + 6(y')^2 f_{xxyy} + 4(y')^3 f_{xyyy} + f_y f_{xx} \\
&\quad + 6(y')^2 f_{yyy} y'' + 5(y')^2 f_{yy} f_y + f_{xxxx} + 12y' f_{xyy} y'' + 6 f_y y' f_{xy} + 4y' f_{yy} f_x + (y')^4 f_{yyyy}) + O(h^6),
\end{aligned}$$

where all the functions that appear on the right hand sides are applied at x_n, from which we can infer that it has algebraic order four.

Proposition 2. *Similarly, concerning the higher order method in the new RKN pair, we have the following PLTE and $PLTE_{der}$:*

$$PLTE = -\frac{h^7}{213290280}(w^4 f_y y' - (f_y)^2 f_x - (f_y)^3 y' + w^4 f_x) + O(h^8),$$

$$\begin{aligned}PLTE_{der} =& \frac{h^7}{5040}(81(y')^2 f_{yyy} f_y y'' + 60y' f_{yyy} f_x y'' + 102y' f_y f_{xy} y'' + 66y' f_{yy} f_{xy} y'' + 30y' f_{yy} f_y f_x + f_{xxxxxx}\\
&+ 15(y'')^3 f_{yyy} + 45(y'')^2 f_{xxyy} + 18y''(f_{xy})^2 + (f_y)^3 y'' + 6(y')^5 f_{xyyyyy} + 6f_{xxx} f_{xy} + f_y f_{xxxx} + (f_y)^2 f_{xx}\\
&+ 20 f_x f_{xxxy} + 15 f_{yxx} f_{xx} + 15(y')^2 f_{xxxxyy} + 20(y')^3 f_{xxxyyy} + 15(y')^4 f_{xxyyyy} + 6y' f_{xxxxxy} + 10 f_{yy}(f_x)^2\\
&+ 15y'' f_{xxxxy} + (y')^6 f_{yyyyyy} + 18(y'')^2 f_{yy} f_y + 90y' f_{xyyy}(y'')^2 + 45(y')^2 f_{yyyy}(y'')^2 + 21(y')^2 f_{yy}(f_y)^2\\
&+ 21(y')^4 f_y f_{yyyy} + 21(y')^4 f_{yyy} f_{yy} + 33(y')^2 (f_{yy})^2 y'' + 60(y')^3 f_{xyyyy} y'' + 60y' f_{xxyyy} y'' + 90(y')^2 f_{xxyyy} y''\\
&+ 60y'' f_{xyy} f_x + 21 f_y y'' f_{yxx} + 15y'' f_{yy} f_{xx} + 15(y')^4 f_{yyyyy} y'' + 10 f_y f_x f_{xy} + 48y' f_{xy} f_{yxx} + 12(f_y)^2 y' f_{xy}\\
&+ 64(y')^3 f_y f_{xyyy} + 36(y')^3 f_{yyy} f_{xy} + 60y' f_{xxyy} f_x + 60(y')^2 f_{xyyy} f_x + 20(y')^3 f_{yyyy} f_x + 15(y')^2 f_{yyy} f_{xx}\\
&+ 78(y')^2 f_{xyy} f_{xy} + 66(y')^2 f_y f_{xxyy} + 33(y')^2 f_{yy} f_{yxx} + 30y' f_{xyy} f_{xx} + 24 f_y y' f_{xxxy} + 6y' f_{yy} f_{xxx}\\
&+ 48(y')^3 f_{yy} f_{xyy}) + O(h^8),\end{aligned}$$

where the functions appearing in the right hand sides are applied at x_n, from which we can infer that it has algebraic order six. Therefore, we have that the adapted embedded RKN pair maintains the orders of the counterpart original RKN pair.

4.1 | Stability analysis

Applying the newly developed RKN method to the test equation $y'' = -w^2 y$, the linear stability analysis is derived. Setting $\tilde{h} = v^2 = w^2 h^2$, we get that the approximate solution satisfies the following recurrence equation:

$$L_{n+1} = E(\tilde{h}) L_n,$$

with

$$L_{n+1} = \begin{bmatrix} y_{n+1} \\ h y'_{n+1} \end{bmatrix}, L_n = \begin{bmatrix} y_n \\ h y'_n \end{bmatrix}, E(\tilde{h}) = \begin{bmatrix} 1 - \tilde{h} b^T N^{-1} e & 1 - \tilde{h} b^T N^{-1} c \\ -\tilde{h} d^T N^{-1} e & 1 - \tilde{h} d^T N^{-1} c \end{bmatrix}, N = I + \tilde{h} A,$$

where $A = \left(a_{ij}\right)_{6\times 6}$ is the lower triangular matrix containing the coefficients, I is the identity matrix of dimension six, and

$$b = [b_1, b_2, b_3, b_4, b_5, b_6]^T,\ d = [d_1, d_2, d_3, d_4, d_5, d_6]^T,\ e = [1, 1, 1, 1, 1, 1]^T,\ c = [c_1, c_2, c_3, c_4, c_5, c_6]^T.$$

It is assumed that for sufficiently small values of v, the stability matrix $E(\tilde{h})$ has complex conjugates eigenvalues.[16] In view of the test equation used, an oscillatory numerical solution should be obtained. The eigenvalues of $E(\tilde{h})$ determine the oscillatory character of the approximate solutions. We note that the characteristic equation of $E(\tilde{h})$ is given by

$$\lambda^2 - tr(E(\tilde{h}))\lambda + det(E(\tilde{h})) = 0. \tag{18}$$

Theorem 3 (Anastassi and Kosti[10]). *If we apply to the common test equation $y'' = -w^2 y$, the RKN scheme in (2)–(4), we get the formula for calculating directly the phase-lag (or dispersion error) $\Psi(v)$ given by*

$$\Psi(v) = v - \arccos\left(\frac{tr(E(\tilde{h}))}{2\sqrt{det(E(\tilde{h}))}}\right). \tag{18}$$

If $\Psi(v) = O(v^{l+1})$, then the method is said to have phase-lag order l. For an explicit RKN method, $tr(E(\tilde{h}))$ and $det(E(\tilde{h}))$ are polynomials in v (in case of an implicit RKN method these would be rational functions).

Definition 3 (Anastassi and Kosti[10]). For the RKN method given in (2)–(4), the value $\beta(v) = 1 - \sqrt{det(E(\tilde{h}))}$ is called the amplification error (or dissipative error). If $\beta(v) = O(v^{s+1})$, then the method is said to have amplification error of order s.

Furthermore, we study the stability property of the developed methods when applied to the test equation, $y'' = -w^2 y$.

Definition 4 (Anastassi and Kosti[10]). The interval $I = (0, \tilde{h}_a), \tilde{h}_a \in \mathbb{R}^+ \cup \{+\infty\}$, so that $v \in (0, \tilde{h}_a)$ is called

1. the interval of stability of the RKN method, if $\tilde{h}_a$ is the highest value for which $|\lambda| < 1$.
2. the interval of periodicity of the RKN method, if $\tilde{h}_a$ is the highest value for which $|\lambda| = 1$, and $[tr(E(\tilde{h}))]^2 - 4det(E(\tilde{h})) < 0$ (the eigenvalues are complex conjugate).

- If $(0, \tilde{h}_{stab})$ lies in the stability interval, then $\tilde{h}_{stab}$ is called the stability boundary.
- If $(0, \tilde{h}_{per})$ lies in the periodicity interval, then $\tilde{h}_{per}$ is called the periodicity boundary.
- If $\tilde{h}_{stab} = \infty$, then the RKN method is A-stable.[10]
- If $\tilde{h}_{per} = \infty$, then the RKN method is P-stable.[10]

The resulting adapted methods of order 6 and 4 are both "almost" P-stable, but they have no intervals of stability. Using the Mathematica software, we have found that the eigenvalues for the two methods are $\lambda_1 = e^{-i\sqrt{\tilde{h}}}$, $\lambda_2 = e^{i\sqrt{\tilde{h}}}$, and also the traces and the determinants of the matrix $E(\tilde{h})$ for the two methods are $tr(E(\tilde{h})) = 2\cos\left(\sqrt{\tilde{h}}\right)$ and $det(E(\tilde{h})) = 1$. We have obtained that $\Psi(v) = v - \arccos\left(\frac{2\cos(v)}{2\sqrt{1}}\right) = 0$, $\beta(v) = 1 - \sqrt{det(E(\tilde{h}))} = 0$ which implies that the methods are dispersive of order infinity and dissipative of order infinity.

Theorem 4. *The trigonometrically adapted methods developed in this paper are "almost" P-stable, for the set* $\Upsilon := \{v^2 \in \mathbb{R} - v^2 \neq (n\pi)^2, n \in \mathbb{N}\}$.

The theorem can be proved by considering the fact that the trigonometrically adapted methods developed here have respectively the following, $tr(E(\tilde{h})) = 2\cos(v)$ and $det(E(\tilde{h})) = 1$, and from Franco et al,[17] the trigonometrically adapted methods are "almost" P-stable, if and only if for every $v > 0$, $det(E(\tilde{h})) = 1$ and $|tr(E(\tilde{h}))| < 2$. Implying $|tr(E(\tilde{h}))| = |2\cos(v)| < 2$, for $v \neq n\pi, n \in \mathbb{N}$.

5 | NUMERICAL EXAMPLES

To assess the performance of the proposed method, we will consider some well-known pairs of RKN methods for numerical comparisons:

- EETFRKN6(4)6ER: The new adapted embedded RKN pair developed in Section 3,
- RKN6(4)6 ER: The optimized non-FSAL embedded RKN algorithm of orders 6(4) with six stages derived by El-Mikkawy and Rahmo,[3]
- RKN6(4)6ER-PFAF: The phase- and amplification-fitted RKN pair of orders 6(4) developed in Anastassi and Kosti,[10]
- RKN6(4)6FM: The well-known 6(4) family of RKN formulae derived by Dormand et al.[2]

We will used them to solve some well-known oscillatory IVPs:

Example 1. The Model Problem in Medvedev et al.[18]:

The first choice is the test equation problem

$$y'' = -25y,\ y(0) = 0,\ y'(0) = 5, x \in [0, 10],$$

whose exact solution is given by

$$y(x) = \sin(5x).$$

To apply the method developed here and the method in Anastassi and Kosti,[10] we consider $w = 5$. We note that the proposed method solves this problem exactly provided that exact arithmetic is used. The errors obtained are due to round-off effects.

Example 2. Inhomogeneous Problem in Monovasilis et al.[19]:

$$y'' = -v^2 y + (v^2 - 1)\sin x,\ y(0) = 1,\ y'(0) = v + 1, x \in [0, 10].$$

We consider the case of $v = 10$, for which the exact solution is given by

$$y(x) = \sin(10x) + \cos(10x) + \sin(x).$$

To apply the method developed here and the method in Anastassi and Kosti,[10] we consider $w = 10$.

Example 3. Inhomogeneous Problem in Senu:[20]

$$y'' = -y + \frac{1}{1000}\cos(x),\ y(0) = 1,\ y'(0) = 0, x \in [0, 10].$$

The exact solution is

$$y(x) = \cos(x) + \frac{1}{2000}x\sin(x).$$

Now, in the adapted EETFRKN6(4)6ER method and in the RKN6(4)6ER-PFAF method, we consider $w = 1$.

Example 4. The Orbital Problem in Kalogiratou et al.[21]:

$$\begin{aligned} y_1'' &= -y_1 + \frac{1}{1000}\cos(x),\ y_1(0) = 1,\ y_1'(0) = 0, \\ y_2'' &= -y_2 + \frac{1}{1000}\sin(x),\ y_2(0) = 0,\ y_2'(0) = \frac{9995}{10000}, x \in [0, 10]. \end{aligned}$$

The exact solution is

$$y_1(x) = \cos(x) + \frac{1}{2000}x\sin(x),$$

$$y_2(x) = \sin(x) - \frac{1}{2000}x\cos(x).$$

Now we consider $w = 1.0$ to use the proposed method and the one in Anastassi and Kosti.[10]

Example 5. Inhomogeneous System in Senu et al.[22]:

$$y'' + \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix} y = \epsilon \begin{pmatrix} \frac{93}{2}\cos 2x & -\frac{99}{2}\sin 2x \\ \frac{93}{2}\sin 2x & -\frac{99}{2}\cos 2x \end{pmatrix},\ y(0) = \begin{pmatrix} -1+\epsilon \\ 1 \end{pmatrix},\ y'(0) = \begin{pmatrix} -10 \\ 10 + 2\epsilon \end{pmatrix},\ x \in [0, 10].$$

The exact solution is given by

$$y(x) = \begin{pmatrix} \epsilon\cos(2x) - \cos(10x) - \sin(10x) \\ \epsilon\sin(2x) + \cos(10x) + \sin(10x) \end{pmatrix},$$

where $\epsilon = 10^{-1}$, to apply the method developed here and the method in Anastassi and Kosti,[10] we consider $w = 10$.

Example 6. A Non-linear System in Vyver:[23]

$$\begin{aligned} y_1'' + w^2 y_1 &= \frac{2y_1y_2 - \sin(2wx)}{({y_1}^2 + {y_2}^2)^{\frac{3}{2}}}, y_1(0) = 1, y_1'(0) = 0, \\ y_2'' + w^2 y_2 &= \frac{{y_1}^2 - {y_2}^2 - \cos(2wx)}{({y_1}^2 + {y_2}^2)^{\frac{3}{2}}}, y_2(0) = 0, y_2'(0) = w,\ x \in [0, 10], \end{aligned}$$

with a known solution given by

$$y_1(x) = \cos(wx),$$

$$y_2(x) = \sin(wx).$$

In order to use the adapted schemes for solving this problem, we have considered that $w = 5$.

Example 7. Almost Periodic Problem in Vyver:[23]

$$y_1'' = -y_1 + \epsilon\cos(\Psi x),\ y_1(0) = 1,\ y_1'(0) = 0,$$
$$y_2'' = -y_2 + \epsilon\sin(\Psi x),\ y_2(0) = 0,\ y_2'(0) = 1,\ x \in [0, 5].$$

The exact solution is

$$y_1(x) = \frac{(1-\epsilon-\Psi^2)}{(1-\Psi^2)}\cos(x) + \frac{\epsilon}{(1-\Psi^2)}\cos(\Psi x),$$

TABLE 2 Data for Example 1

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
10^{-4}	EETFRKN6(4)6ER	22	162	6	1.116688(−7)	0.076
	RKN6(4)6 ER	164	994	2	1.700453(−8)	0.093
	RKN6(4)6ER-PFAF	169	1044	6	3.344658(−5)	0.138
	RKN6(4)6FM	83	503	1	3.888508(−6)	0.084
10^{-6}	EETFRKN6(4)6ER	36	366	30	3.531841(−11)	0.099
	RKN6(4)6 ER	318	1918	2	1.619526(−10)	0.127
	RKN6(4)6ER-PFAF	319	1929	3	2.377939(−7)	0.147
	RKN6(4)6FM	317	1912	2	8.861495(−10)	0.100
10^{-10}	EETFRKN6(4)6ER	61	581	43	9.203748(−14)	0.111
	RKN6(4)6 ER	2370	14235	3	8.418821(−13)	0.178
	RKN6(4)6ER-PFAF	2370	14235	3	8.414380(−13)	0.309
	RKN6(4)6FM	1975	11940	18	6.077361(−13)	0.119
10^{-12}	EETFRKN6(4)6ER	74	564	24	3.475345(−14)	0.192
	RKN6(4)6 ER	4575	27465	3	3.408412(−12)	0.417
	RKN6(4)6ER-PFAF	4575	27465	3	3.411896(−12)	0.779
	RKN6(4)6FM	4566	27411	3	3.410508(−12)	0.402

TABLE 3 Data for Example 2

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
10^{-2}	EETFRKN6(4)6ER	54	389	13	8.289042(−4)	0.040
	RKN6(4)6 ER	172	1047	3	4.680593(−6)	0.040
	RKN6(4)6ER-PFAF	184	1179	15	6.482959(−3)	0.043
	RKN6(4)6FM	87	532	2	6.183292(−4)	0.041
10^{-4}	EETFRKN6(4)6ER	184	1304	40	3.322960(−7)	0.043
	RKN6(4)6 ER	330	1995	3	4.641362(−8)	0.044
	RKN6(4)6ER-PFAF	339	2094	12	1.682397(−5)	0.062
	RKN6(4)6FM	318	2063	31	1.641347(−6)	0.045
10^{-6}	EETFRKN6(4)6ER	355	2370	48	3.600293(−10)	0.042
	RKN6(4)6 ER	1061	6541	35	9.420464(−11)	0.043
	RKN6(4)6ER-PFAF	1066	6591	39	9.513662(−7)	0.046
	RKN6(4)6FM	636	3831	3	3.361068(−9)	0.043
10^{-8}	EETFRKN6(4)6ER	967	5992	38	2.061684(−12)	0.045
	RKN6(4)6 ER	2455	14750	4	2.965073(−12)	0.046
	RKN6(4)6ER-PFAF	2456	14761	5	6.665539(−9)	0.048
	RKN6(4)6FM	1823	11113	35	4.879741(−11)	0.046

TABLE 4 Data for Example 3

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	5	30	0	1.521173(−3)	0.057
10^{-2}	RKN6(4)6 ER	12	72	0	7.079259(−6)	0.066
	RKN6(4)6ER-PFAF	2	4 204	12	2.209235(−2)	0.069
	RKN6(4)6FM	8	48	0	5.092014(−4)	0.066
	EETFRKN6(4)6ER	8	48	0	2.140179(−6)	0.051
10^{-4}	RKN6(4)6 ER	22	132	0	8.057457(−8)	0.058
	RKN6(4)6ER-PFAF	22	132	0	1.285246(−5)	0.104
	RKN6(4)6FM	22	132	0	1.443515(−7)	0.055
	EETFRKN6(4)6ER	22	132	0	1.129209(−10)	0.058
10^{-6}	RKN6(4)6 ER	80	480	0	6.799228(−12)	0.067
	RKN6(4)6ER-PFAF	80	480	0	1.940115(−14)	0.064
	RKN6(4)6FM	42	252	0	2.932989(−9)	0.068
	EETFRKN6(4)6ER	40	240	0	1.266653(−12)	0.062
10^{-8}	RKN6(4)6 ER	154	924	0	8.015810(−14)	0.079
	RKN6(4)6ER-PFAF	154	924	0	1.934564(−14)	0.077
	RKN6(4)6FM	154	924	0	9.603152(−13)	0.070

TABLE 5 Data for Example 4

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	5	30	0	1.521173(−3)	0.057
10^{-2}	RKN6(4)6 ER	12	72	0	7.079259(−6)	0.067
	RKN6(4)6ER-PFAF	29	259	17	1.480990(−2)	0.082
	RKN6(4)6FM	12	72	0	1.229031(−5)	0.061
	EETFRKN6(4)6ER	8	48	0	2.140179(−6)	0.059
10^{-4}	RKN6(4)6 ER	22	132	0	8.057457(−8)	0.061
	RKN6(4)6ER-PFAF	22	132	0	1,390225(−5)	0.064
	RKN6(4)6FM	22	132	0	2.020373(−7)	0.063
	EETFRKN6(4)6ER	22	132	0	1.129209(−10)	0.063
10^{-6}	RKN6(4)6 ER	80	480	0	6.799228(−12)	0.071
	RKN6(4)6ER-PFAF	80	480	0	8.182344(−14)	0.075
	RKN6(4)6FM	42	252	0	3.859971(−9)	0.067
	EETFRKN6(4)6ER	40	240	0	1.266653(−12)	0.070
10^{-8}	RKN6(4)6 ER	154	924	0	8.015810(−14)	0.098
	RKN6(4)6ER-PFAF	154	924	0	2.448042(−14)	0.089
	RKN6(4)6FM	154	924	0	1.164041(−12)	0.080

TABLE 6 Data for Example 5

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	55	395	13	3.966215(−4)	0.098
10^{-2}	RKN6(4)6 ER	172	1047	3	4.680587(−6)	0.130
	RKN6(4)6ER-PFAF	184	1179	15	6.482023(−3)	0.154
	RKN6(4)6FM	87	532	2	6.183238(−4)	0.102
	EETFRKN6(4)6ER	177	1132	14	2.039512(−7)	0.048
10^{-4}	RKN6(4)6 ER	330	1995	3	4.641357(−8)	0.066
	RKN6(4)6ER-PFAF	339	2094	12	1.682027(−5)	0.082
	RKN6(4)6FM	318	2063	31	1.641367(−6)	0.055
	EETFRKN6(4)6ER	331	2061	15	3.259013(−10)	0.138
10^{-6}	RKN6(4)6 ER	1061	6541	35	9.423040(−11)	0.295
	RKN6(4)6ER-PFAF	1066	6591	39	9.511759(−7)	0.371
	RKN6(4)6FM	636	3831	3	3.361052(−9)	0.193
	EETFRKN6(4)6ER	654	4009	17	1.883126(−12)	0.342
10^{-8}	RKN6(4)6 ER	2455	14750	4	3.011633(−12)	0.787
	RKN6(4)6ER-PFAF	2456	14761	5	6.664011(−9)	0.940
	RKN6(4)6FM	1823	11113	35	4.880385(−11)	0.406

TABLE 7 Data for Example 6

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	48	318	6	4.082247(−5)	0.069
10^{-2}	RKN6(4)6 ER	86	526	2	1.587808(−6)	0.078
	RKN6(4)6ER-PFAF	93	603	9	5.819300(−3)	0.078
	RKN6(4)6FM	43	263	1	2.159180(−4)	0.069
	EETFRKN6(4)6ER	90	580	8	1.030174(−7)	0.081
10^{-4}	RKN6(4)6 ER	165	1000	2	1.560154(−8)	0.122
	RKN6(4)6ER-PFAF	169	1044	6	3.347975(−5)	0.135
	RKN6(4)6FM	83	503	1	3.978592(−6)	0.083
	EETFRKN6(4)6ER	325	1995	9	3.708764(−11)	0.062
10^{-6}	RKN6(4)6 ER	318	1918	2	1.574824(−10)	0.084
	RKN6(4)6ER-PFAF	414	2499	3	2.389390(−7)	0.085
	RKN6(4)6FM	318	1918	2	1.187567(−9)	0.130
	EETFRKN6(4)6ER	623	3793	11	6.373790(−13)	0.200
10^{-8}	RKN6(4)6 ER	1228	7383	3	9.065856(−13)	0.238
	RKN6(4)6ER-PFAF	1228	7383	3	6.167400(−12)	0.301
	RKN6(4)6FM	614	3694	2	2.231952(−11)	0.116

TABLE 8 Data for Example 7

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	4	24	0	4.920604(−7)	0.064
10^{-2}	RKN6(4)6 ER	6	36	0	3.655632(−6)	0.066
	RKN6(4)6ER-PFAF	14	119	7	4.984947(−3)	0.098
	RKN6(4)6FM	6	36	0	4.286415(−6)	0.082
	EETFRKN6(4)6ER	5	30	0	2.139670(−7)	0.078
10^{-4}	RKN6(4)6 ER	11	66	0	3.631281(−8)	0.081
	RKN6(4)6ER-PFAF	11	66	0	6.040571(−6)	0.085
	RKN6(4)6FM	11	66	0	9.022879(−8)	0.085
	EETFRKN6(4)6ER	8	48	0	1.620388(−9)	0.080
10^{-6}	RKN6(4)6 ER	40	240	0	3.297806(−12)	0.100
	RKN6(4)6ER-PFAF	40	240	0	7.827072(−14)	0.091
	RKN6(4)6FM	21	126	0	1.924265(−9)	0.084
	EETFRKN6(4)6ER	12	72	0	2.884804(−12)	0.072
10^{-8}	RKN6(4)6 ER	77	462	0	3.563816(−14)	0.096
	RKN6(4)6ER-PFAF	77	462	0	1.035283(−14)	0.102
	RKN6(4)6FM	77	462	0	5.923040(−13)	0.102

TABLE 9 Data for Example 8

TOL	Method	NSTEP	NFE	RSTEP	MAXER	CPU(s)
	EETFRKN6(4)6ER	39	299	13	8.462269(−4)	0.100
10^{-2}	RKN6(4)6 ER	108	663	3	2.050108(−6)	0.182
	RKN6(4)6ER-PFAF	120	795	15	4.061431(−3)	0.200
	RKN6(4)6FM	54	334	2	2.672139(−4)	0.123
	EETFRKN6(4)6ER	65	460	14	4.217526(−6)	0.131
10^{-4}	RKN6(4)6 ER	208	1263	3	2.047826(−8)	0.264
	RKN6(4)6ER-PFAF	217	1362	12	1.677241(−5)	0.167
	RKN6(4)6FM	208	1263	3	7.776034(−8)	0.254
	EETFRKN6(4)6ER	213	1353	15	4.358650(−10)	0.085
10^{-6}	RKN6(4)6 ER	401	2421	3	2.069079(−10)	0.160
	RKN6(4)6ER-PFAF	804	4864	8	9.517935(−7)	0.332
	RKN6(4)6FM	401	2421	3	1.489622(−9)	0.146
	EETFRKN6(4)6ER	401	2491	17	8.791813(−13)	0.185
10^{-8}	RKN6(4)6 ER	1543	9278	4	1.222695(−12)	0.501
	RKN6(4)6ER-PFAF	1544	9289	5	6.771624(−9)	0.542
	RKN6(4)6FM	772	4647	3	2.768529(−11)	0.269

$$y_2(x) = \frac{(1 - \epsilon\Psi - \Psi^2)}{(1 - \Psi^2)} \sin(x) + \frac{\epsilon}{(1 - \Psi^2)} \sin(\Psi x),$$

where $\epsilon = 0.001$ and $\Psi = 0.1$.

For the adapted methods, we have taken $w = 1$.

Example 8. System of Non-linear Oscillators in Medvedev et al.[24]:

$$y_1'' = -w^2 y_1 - \alpha y_1 ({y_1}^2 + {y_2}^2)^2,\ y_1(0) = 1,\ y_1'(0) = 0,$$
$$y_2'' = -w^2 y_2 - \alpha y_2 ({y_1}^2 + {y_2}^2)^2,\ y_2(0) = 0,\ y_2'(0) = w + \epsilon,\ x \in \left[0, \frac{20\pi}{w + \epsilon}\right].$$

The exact solution is

$$y_1(x) = \cos(w + \epsilon)x,$$

$$y_2(x) = \sin(w + \epsilon)x,$$

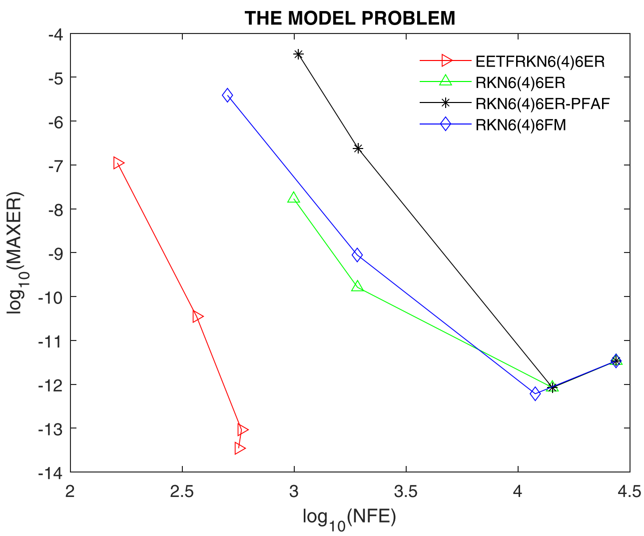

FIGURE 1 Efficiency curves for Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]

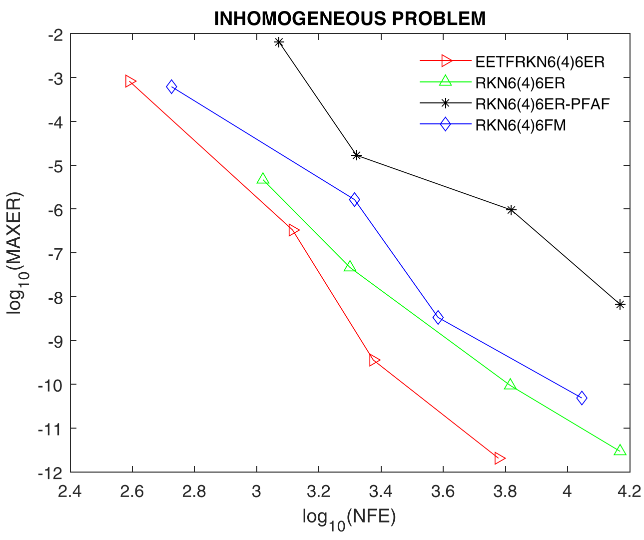

FIGURE 2 Efficiency curves for Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]

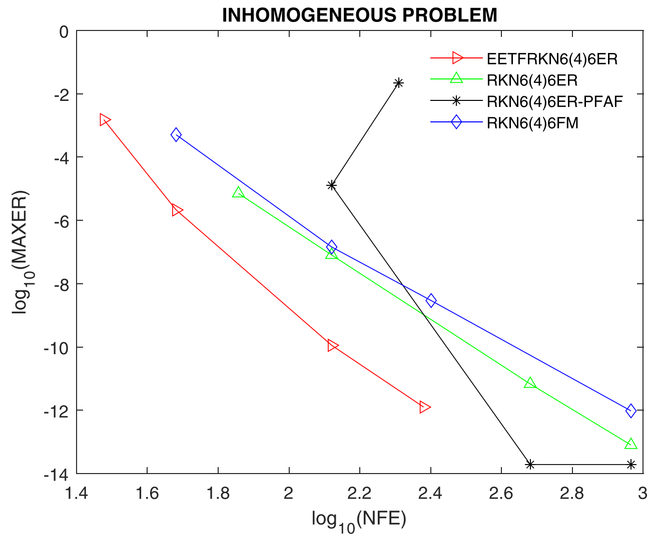

FIGURE 3 Efficiency curves for Example 3 [Colour figure can be viewed at wileyonlinelibrary.com]

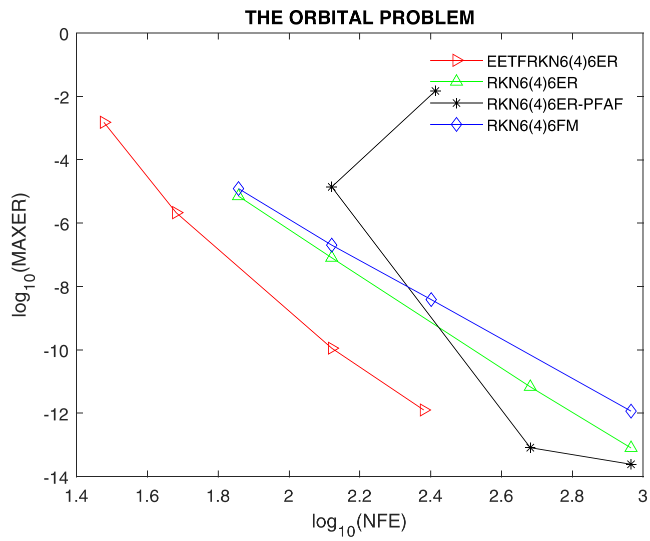

FIGURE 4 Efficiency curves for Example 4 [Colour figure can be viewed at wileyonlinelibrary.com]

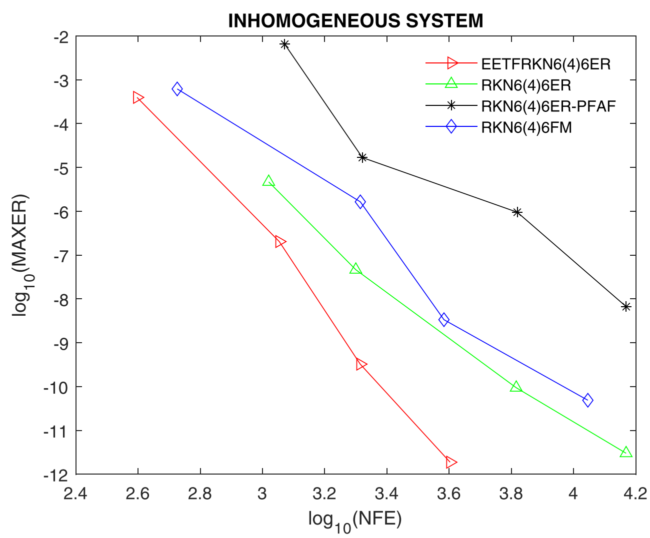

FIGURE 5 Efficiency curves for Example 5 [Colour figure can be viewed at wileyonlinelibrary.com]

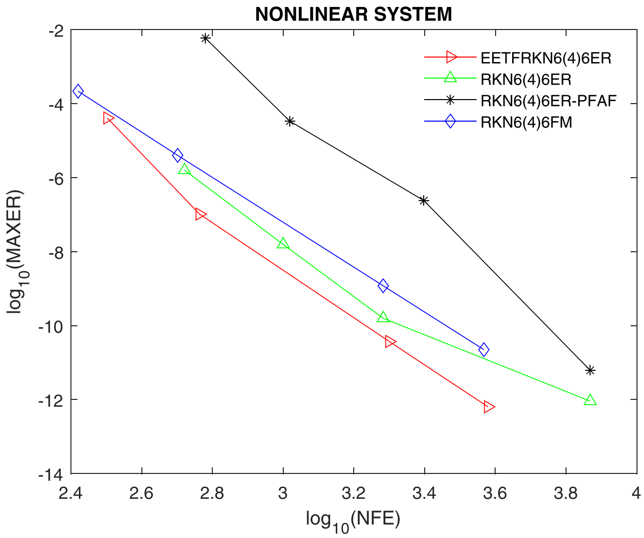

FIGURE 6 Efficiency curves for Example 6 [Colour figure can be viewed at wileyonlinelibrary.com]

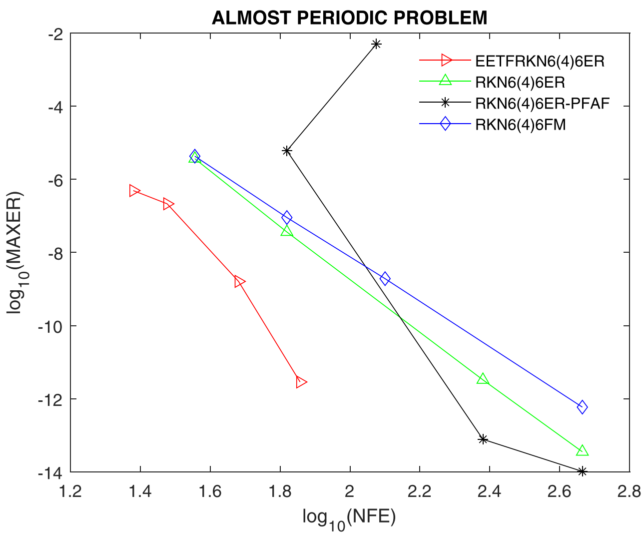

FIGURE 7 Efficiency curves for Example 7 [Colour figure can be viewed at wileyonlinelibrary.com]

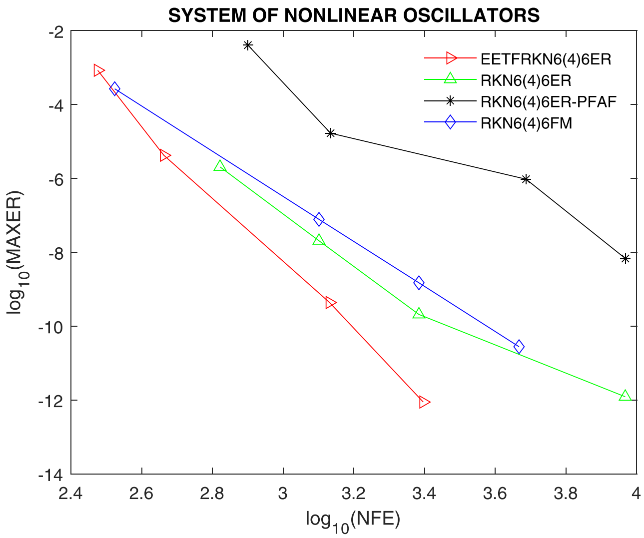

FIGURE 8 Efficiency curves for Example 8 [Colour figure can be viewed at wileyonlinelibrary.com]

where $\epsilon = 0.001$, $w = 10$, and $\alpha = \epsilon(2w + \epsilon)$. Now, to use the adapted schemes we have considered that $w = 10$.

The obtained data are collected in Tables 2–9, where we have considered different tolerances, TOL. The tables present the usual values as

- NFE: number of function evaluations,
- NSTEP: number of steps,
- RSTEP: number of rejected steps,
- MAXER: maximum absolute errors,
- CPU: computational time in seconds

We can see that the proposed method presents very good results concerning the number of steps and computational time.

To further show the robustness and performance of the proposed method EETFRKN6(4)6 ER, we present the efficiency curves of EETFRKN6(4)6 ER compared to other existing RKN methods of orders 6(4) with six stages. Figures 1–8 show the efficiency curves for the examples considered, where one can observe the good performance of the proposed method. We utilized the following tolerances: Tol $= 10^{-2k}$, $k = 2, 3, 5, 6$ for problem 1, and $k = 1, 2, 3, 4$, for problems 2, 3, 4, 5,6,7, and 8.

6 | DISCUSSION OF RESULTS

The newly developed method (EETFRKN6(4)6ER) has the fewest error norm, the fewest number of function evaluations per steps, and the fewest CPU time; meaning that it has high efficiency and accuracy when solving all the given modeled problems as shown in Tables 2–9 and in Figures 1–8. Therefore, the EETFRKN6(4)6ER is suitable for the numerical solution of the problem in (1) showing a better performance than other existing 6(4) RKN pairs with six stages appeared in the literature.

7 | FINAL COMMENTS

We have used the technique for developing a trigonometrically fitted method based on the 6(4) embedded pair of El-Mikkawy and Rahmo[3] and obtained a new adapted RKN embedded pair. The developed method has eight variable coefficients that depend on the product of the frequency of the method w, and the step-size h.[25,26] Also, we obtained the local truncation errors of the underlying methods in EETFRKN6(4)6ER, confirming the preservation of the orders of accuracy of the counterpart pair. We have also shown the stability properties of both RKN methods. The numerical examples show that EETFRKN6(4)6ER is more efficient than various RKN pairs used for comparisons.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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