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A second-derivative functionally fitted method of maximal order for oscillatory initial value problems

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Abstract
This paper deals with the construction of a functionally fitted method for solving first-order differential systems whose solutions present an oscillatory behaviour. The method incorporates the second derivative to obtain better accuracies and is developed on the basis that it provides no errors when the true solution is a linear combination of some trigonometric and exponential functions containing a parameter. The main properties of the method are presented, showing a fourth-order convergence. Some numerical experiments are included to show the good performance of the proposed method.

Keywords Collocation · Exponential function · Functionally fitted · Maximal order · Second derivative · Trigonometric function

Mathematics Subject Classification 65L05 · 65L06

1 Introduction

In this article, a second derivative functionally fitted method (SDFFM) is constructed and applied for solving a first-order initial-value problem (IVP) of the form

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$$v'(x) = f\left(x, v(x)\right), v\left(x_0\right) = v_0, \quad x \in [a, b], \tag{1}$$

whose solutions exhibit oscillatory behaviour, where $v \in \mathbb{R}^d$, $f:\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function that satisfies a Lipschitz condition to guarantee the existence of a unique solution (see Lambert 1991), and d is the dimension of the system.

These kinds of IVPs are often encountered in science and engineering, a number of which are listed by Ehigie et al. (2017). There are many numerical methods available in the mathematical literature to solve the problem in (1) when the solution is oscillatory, which according to Yakubu et al. (2018) can be classified into two, viz: methods with constant coefficients and methods whose coefficients depend on the frequency of the problem. Methods with constant coefficients are basically based on the use of polynomial basis functions (see Lambert 1973; Brugnano and Triginate 1988; Jator and Oladejo 2017; Sunday et al. 2013; Enright 1974; Lambert and Watson 1976; Jator 2010). Most of these methods do not perform well in case of oscillatory solutions due to the nature of the solutions (Ehigie et al. 2017). It is against this drawback that many numerical methods, otherwise known as functionally fitted methods, have tried to take advantage of this special property of the solution that may be known in advance.

Functionally fitted methods, according to Nguyen et al. (2006), are generalizations of collocation techniques to integrate an IVP exactly if its solution is a linear combination of a chosen set of basis functions. When these basis functions are chosen as the power functions, we recover classical algebraic collocation methods. Functionally fitted numerical algorithms whose coefficients depend on the frequency started with the elegant work by Gautschi (1961) who proposed the Adams and Störmer adapted-type methods considering trigonometric polynomials. In this light, Neta and Ford (1984), Neta (1986) and Sanugi and Evans (1989) proposed adapted methods of Nyström and Milne–Simpson type, backward differentiation formula, and leap-frog and Runge–Kutta, respectively. All of these methods are implemented in a step by step approach. Many extensions of such methods have been proposed, which include Mixed interpolation methods (Duxbury 1999; Coleman and Duxbury 2000), exponential fitting methods (Franco 2002, 2003, 2006; Ixaru et al. 2002; Vanden Berhe and Van Daele 2007; Vanden Berghe et al. 2001; Martin-Vaquero and Vigo-Aguiar 2008; Vigo-Aguiar and Simos 2001; You and Chen 2013; Konguetsof and Simos 2003; Fang and Wu 2008; Fang et al. 2009; Conte et al. 2020), trigonometrically fitted methods (Jator and collaborators, Jator et al. 2013; Ngwane and Jator 2013, 2014, 2015; Ndukum et al. 2016; Ramos and Vigo-Aguiar 2010, 2014; Vigo-Aguiar and Ramos 2015; Monovasilis et al. 2017; Abdulganiy and collaborators, Abdulganiy et al. 2017a, b, 2018). In all these extensions, the basis functions considered is either the set $\left\{1, x, x^2, \ldots, x^n, \cos\left(\omega x\right), \sin\left(\omega x\right)\right\}$ or $\left\{1, x, x^2, \ldots, x^n, \mathrm{e}^{\omega x}, \mathrm{e}^{-\omega x}\right\}$. Other basis functions are possible, as listed in Nguyen et al. (2007).

In this paper, a second derivative functionally fitted method is introduced using the multistep collocation technique for which the approximate interpolating function is a linear combination of monomials, trigonometric terms and exponential terms. Specifically, we propose a method which integrates exactly the kind of IVPs in (1) when the solutions are expressed as linear combinations of the set $\left\{1, \sin\left(\omega x\right), \cos\left(\omega x\right), \mathrm{e}^{\omega x}, \mathrm{e}^{-\omega x}\right\}$. This basis function is considered for its ease to be analysed and the provision of an improved extension for solving initial value problems with oscillatory solutions.

This paper is arranged as follows: the construction of SDFFM is discussed in Sect. 2. The basic properties of the method are presented in Sect. 3, while the implementation and some numerical experiments are presented in Sect. 4. Finally, Sect. 5 concludes the paper.

2 Construction of SDFFM

To solve the IVP in Eq. (1), we proceed by considering that we have a scalar equation and assuming that the true solution $v(x)$ can be approximated by a fitted function $I(x, u)$ which incorporates a parameter u. It will be seen that the coefficients of this function may be expressed in terms of v (the dependent variable), f (the first derivative) and g (the second derivative) evaluated at different grid points. It is worth noting that the coefficients of SDFFM are selected so that it integrates the IVP (1) exactly when the solutions are members of the linear space spanned by $P = \left\{1, \sin(\omega x), \cos(\omega x), e^{\omega x}, e^{-\omega x}\right\}$. Customarily, the notations v_j, $f_j = f\left(x_j, v_j\right)$ and $g_j = g\left(x_j, v_j\right)$ are approximations of the exact values $v\left(x_j\right)$, $v'\left(x_j\right) = f\left(x_j, v\left(x_j\right)\right)$ and $v''\left(x_j\right) = g\left(x_j, v\left(x_j\right)\right)$, respectively, where $g(x, v(x)) = f'(x, v(x))$, $x_j = x_0 + jh$, $j = 1, 2, \ldots, N$, with $x_0 = a$, $x_N = b$, and $h = (x_N - x_0)/N$ the step length.
We consider that the exact solution $v(x)$ can be approximated by a fitted function $I(x, u)$ defined by

$$v(x) \cong I(x, u) = a_0 + a_1 \sin(\omega x) + a_2 \cos(\omega x) + a_3 e^{\omega x} + a_4 e^{-\omega x}, \tag{2}$$

where the parameter u is given by $u = \omega h$. Considering this approximation, we explicitly demand that the following system of five equations must be satisfied

$$\begin{cases} I(x_n, u) = v_n, \\ I'(x, u)\big|_{x=x_{n+j}} = f_{n+j}, \quad j = 0, 1, \\ I''(x, u)\big|_{x=x_{n+j}} = g_{n+j}, \quad j = 0, 1. \end{cases} \tag{3}$$

We now state the proposition that aids the construction of the continuous method as follows:

Proposition 1 *Let assume that* $I(x, u)$ *satisfies the system of equations given in* (3)*. Define the following vectors,* $K = (v_n, f_n, f_{n+1}, g_n, g_{n+1})^T$, $P = \left\{1, \sin(\omega x), \cos(\omega x), e^{\omega x}, e^{-\omega x}\right\}$, *and the matrix*

$$W = \begin{bmatrix} 1 & \sin(\omega x_n) & \cos(\omega x_n) & e^{\omega x_n} & e^{-\omega x_n} \\ 0 & \omega \cos(\omega x_n) & -\omega \sin(\omega x_n) & \omega e^{\omega x_n} & -\omega e^{-\omega x_n} \\ 0 & \omega \cos(\omega x_{n+1}) & -\omega \sin(\omega x_{n+1}) & \omega e^{\omega x_{n+1}} & -\omega e^{-\omega x_{n+1}} \\ 0 & -\omega^2 \sin(\omega x_n) & -\omega^2 \cos(\omega x_n) & \omega^2 e^{\omega x_n} & \omega^2 e^{-\omega x_n} \\ 0 & -\omega^2 \sin(\omega x_{n+1}) & -\omega^2 \cos(\omega x_{n+1}) & \omega^2 e^{\omega x_{n+1}} & \omega^2 e^{-\omega x_{n+1}} \end{bmatrix}.$$

Then, the continuous approximation that will be used to generate the SDFFM can be expressed as

$$I(x, u) = \sum_{j=0}^{4} \frac{det\left(W_j\right)}{det(W)} P_j(x) \tag{4}$$

where W_j *is obtained after replacing the* $j-th$ *column of* W *by* K.

Proof The proof can be readily obtained, similarly to the one given in Abdulganiy et al. (2018). □

Remark 1 We emphasize that the equation in (4) provides a continuous approximation of the true solution, and has the form

$$I(x,u) = v_n + h\left(\beta_0(x,u) f_n + \beta_1(x,u) f_{n+1}\right) + h^2\left(\delta_0(x,u) g_n + \delta_1(x,u) g_{n+1}\right). \quad (5)$$

Our method is obtained from the continuous form (5) by evaluating it at $x = x_{n+1}$, which results in

$$v_{n+1} = v_n + h\left(\beta_0 f_n + \beta_1 f_{n+1}\right) + h^2\left(\delta_0 g_n + \delta_1 g_{n+1}\right), \quad (6)$$

where

$$\begin{aligned}
\beta_0 &= \frac{(\cos(u) + \sin(u) - 1)\,e^{2u} - \cos(u) + 1 + (-2\,e^u + 1)\sin(u)}{u\left(e^{2u}\cos(u) + \cos(u) - 2\,e^u\right)}\\
\beta_1 &= \frac{(\cos(u) + \sin(u) - 1)\,e^{2u} - \cos(u) + 1 + (-2\,e^u + 1)\sin(u)}{u\left(e^{2u}\cos(u) + \cos(u) - 2\,e^u\right)}\\
\delta_0 &= \frac{(\sin(u) - 1)\,e^{2u} + 2\,e^u\cos(u) - \sin(u) - 1}{u^2\left(e^{2u}\cos(u) + \cos(u) - 2\,e^u\right)}\\
\delta_1 &= \frac{(-\sin(u) + 1)\,e^{2u} - 2\,e^u\cos(u) + \sin(u) + 1}{u^2\left(e^{2u}\cos(u) + \cos(u) - 2\,e^u\right)}.
\end{aligned} \quad (7)$$

The formula in (6) can be written in the form

$$v_{n+1} - v_n = h\phi_{f,g}(v_n, v_{n+1}; u, h),$$

where the subscript indicates that the dependence of ϕ on v_n, v_{n+1} is through the functions f and g. Thus, the numerical solution of the problem in (1) is the one given by

$$\begin{cases} v_{n+1} - v_n = h\phi_{f,g}(v_n, v_{n+1}; u, h), \\ v_0 = v(x_0), \qquad\qquad n = 1, 2, \ldots, N-1. \end{cases} \quad (8)$$

(Note that the two coefficients β_0 and β_1 are the same, and $\delta_0 = -\delta_1$. This symmetry of the coefficients greatly simplifies the formulas of the method and its implementation).
For small values of u, the coefficients of the method may be subject to heavy cancellations and in that case the Taylor series expansion of the coefficients must be used (Lambert 1973). The series expansion of each of the coefficients up to $O\left(u^{16}\right)$ are as follows

$$\begin{aligned}
\beta_0 = \beta_1 &\simeq \frac{1}{2} + \frac{1}{1440}u^4 + \frac{1}{725760}u^8 + \frac{2879}{1046139494400}u^{12} + \frac{3911}{711374856192000}u^{16},\\
\delta_0 = -\delta_1 &\simeq \frac{1}{12} + \frac{1}{6720}u^4 + \frac{71}{239500800}u^8 + \frac{59}{99632332800}u^{12} + \frac{863449}{729870602452992000}u^{16}.
\end{aligned} \quad (9)$$

It is interesting to note that as $u \to 0$ in the power series expansion of the coefficients, methods based on polynomial basis are recovered (Lambert 1973). In this case, we recover the fourth-order symmetric Obrechkoff method given by

$$v_{n+1} = v_n + \frac{h}{2}\left(f_n + f_{n+1}\right) + \frac{h^2}{12}\left(g_n - g_{n+1}\right).$$

3 Characteristics of SDFFM

The basic properties of SDFFM are analysed in this section. We study the local truncation error, order, error constant, zero stability, convergence and linear stability.

3.1 Local truncation error and order

Proposition 2 *The SDFFM has a local truncation error of the form* $LTE = C_5 h^5 \left(\omega^4 v'(x_n) - v^{(5)}(x_n)\right) + O\left(h^6\right)$.

Proof Assuming that $v(x)$ is a sufficiently differentiable function, we consider the Taylor series expansions at $x = x_n$ of $v(x_n + jh)$, $v'(x_n + jh)$ and $v''(x_n + jh)$, $j = 0, 1$. We replace into the formula (6) the approximate values for the exact ones, that is, $v_{n+1} \to v(x_{n+1})$, $f_{n+j} \to v'\left(x_{n+j}\right)$, $g_{n+j} \to v''\left(x_{n+j}\right)$, together with the coefficients given in (7). After simplifying, we obtain

$$
\begin{aligned}
\text{LTE} &= v(x_n + h) - \left(v(x_n) + h \sum_{j=0}^{1} \beta_j(u) v'(x_n + jh) + h^2 \sum_{j=0}^{1} \delta_j(u) v''(x_n + jh) \right) \\
&= C_5 h^5 \left(\omega^4 v'(x_n) - v^{(5)}(x_n) \right) + O\left(h^6\right).
\end{aligned}
$$

Consequently, the principal term of the local truncation Error (LTE) of SDFFM is obtained as

$$
\text{LTE}(ppal.term) = -\frac{1}{720} h^5 \left(\omega^4 v'(x_n) - v^{(5)}(x_n) \right),
$$

which indicates that the order (p) of SDFFM is at least $p = 4$. □

Remark 2 According to Butcher (2008), a linear $k-$step method of order p is said to be of maximal order if $p = 2k + 2$. Since the SDFFM is of order $p = 4$ with error constant $C_5 = -\frac{1}{720}$, we, therefore, remark that SDFFM is a maximal order method.

Proposition 3 *The local truncation error of SDFFM preserves its basis functions.*

Proof Solving the differential equation $v^{(5)}(x) - \omega^4 v'(x) = 0$ results in the fundamental set of solutions $\left\{1, \sin(\omega x), \cos(\omega x), e^{\omega x}, e^{-\omega x}\right\}$ which are the basis functions of the SDFFM. □

3.2 Zero stability

The concept of stability is very important in numerical analysis. Generally speaking, a method is stable if small perturbations in the data cause small variations in the solution provided by it. Zero-stability is a type of stability that is concerned with the behaviour of a numerical scheme when $h \to 0$. Any numerical method for solving (1) will produce errors that can be interpreted as if we were solving a perturbed problem of the form

$$
\begin{cases}
z_{n+1} - z_n = h\left(\phi_{f,g}(z_n, z_{n+1}; u, h) + \delta_n\right), \\
z_0 = v(x_0) + \delta_0, \quad n = 0, 1, 2, \ldots, N-1.
\end{cases}
$$

According to Lambert (1991) the zero-stability may be defined as follows:

Definition 1 Let $\delta = \{\delta_i\}_{i=0}^{N-1}$, $\delta^* = \left\{\delta_i^*\right\}_{i=0}^{N-1}$ be any two perturbations of (8) and let $\{z_i\}_{i=0}^{N}$, $\left\{z_i^*\right\}_{i=0}^{N}$ be the corresponding solutions, respectively. Then if there exist constants K and h_0 such that for all $0 < h < h_0$ it is

$$\|z_i - z_i^*\| \leq K\epsilon, \quad i = 0, 1, \ldots, N,$$

whenever

$$\|\delta_i - \delta_i^*\| \leq \epsilon, \quad i = 0, 1, \ldots, N,$$

we say that the method (8) is zero-stable.

From a practical point of view, zero-stability is concerned with the roots of the characteristic polynomial of the difference equation in (8) when $h \to 0$ (see Lambert 1991), which is addressed in the following result.

Proposition 4 *The SDFFM is zero-stable.*

Proof As $h \to 0$ in (6), we found out that

$$v_{n+1} - v_n = 0,$$

which is normalized to obtain the first characteristic equation given by

$$\rho\,(z) = z - 1 = 0.$$

According to Lambert (1973), a numerical method is zero-stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple. Since the root of $\rho\,(z) = 0$ satisfies $|z| \leq 1$, then, the SDFFM is zero stable. □

3.3 Consistency and convergence

According to Lambert (1973), a sufficient condition for a numerical method to be consistent is that $p \geq 1$. We, therefore, remark that the SDFFM is consistent. The convergence of the SDFFM is established since zero-stability + consistence ⇒ convergence (Fatunla 1991).

3.4 Linear stability

Proposition 5 *Applying the SDFFM in (6) to the test equation* $v' = \lambda v$ *yields* $v_{n+1}=M\,(\tau, u)\, v_n$, *where*

$$M\,(\tau, u) = \left(1-\tau\beta_1-\tau^2\delta_1\right)^{-1}\left(1+\tau\beta_0+\tau^2\delta_0\right).$$

Proof We substitute $v' = \lambda v$ and $v'' = \lambda^2 v$ in (6) to obtain

$$\left(1-\tau\beta_1-\tau^2\delta_1\right) v_{n+1} - \left(1+\tau\beta_0+\tau^2\delta_0\right) v_n = 0 \quad (10)$$

where the stability parameter is denoted as $\tau = h\lambda$. It follows from (10) that

$$v_{n+1} = M\,(\tau, u)\, v_n.$$

□

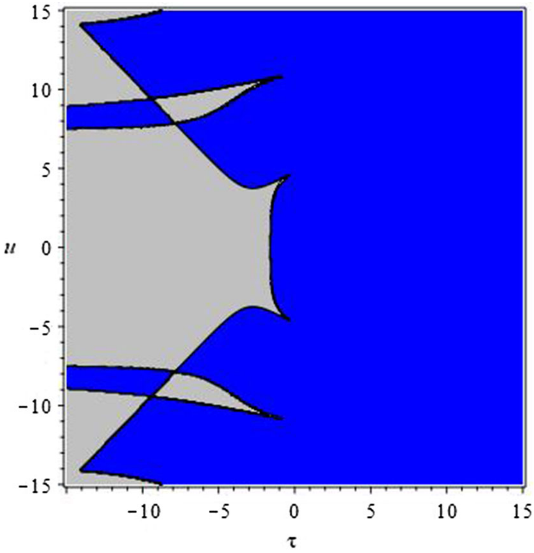

Fig. 1 $\tau - u$ Stability region for SDFFM

Corollary 1 *The rational function* $M(\tau, u)$ *is specified by*

$$M(\tau, u) = \frac{P(\tau, u)}{Q(\tau, u)}, \tag{11}$$

where

$$\begin{aligned} P(\tau, u) = &-(u\cos(u) + \tau\ (\sin(u) - 1))(\tau + u)\,\mathrm{e}^{2u} - \left(2\tau^2\mathrm{e}^u + u(u - \tau)\right)\cos(u) \\ &+ \tau\ \left(2u\mathrm{e}^u - u + \tau\right)\sin(u) + 2u^2\mathrm{e}^u - \tau\ (u - \tau), \end{aligned}$$

$$\begin{aligned} Q(\tau, u) = &((u\cos(u) + \tau\ (1 - \sin(u)))(\tau - u))\,\mathrm{e}^{2u} - \left(2\tau^2\mathrm{e}^u - u(\tau + u)\right)\cos(u) \\ &- \tau\ \left(2u\mathrm{e}^u - u - \tau\right)\sin(u) + 2u^2\mathrm{e}^u + \tau\ (\tau + u). \end{aligned}$$

This stability function determines the stability region of the SDFFM according to the following definition.

Definition 2 (Coleman and Ixaru 1996) The region of stability of a numerical method for solving (1) is the region in the $(\tau, u)-$plane throughout which $|M(\tau, u)| \leq 1$.

Since the stability function depends on two parameters τ and u, we plot the stability region of the SDFFM in the (τ, u) −plane as shown in Fig. 1, the blue coloured region being the stability region.

4 Implementation

The SDFFM with angular frequency ω is implemented for solving the IVP in (1) on the interval $[x_0, x_N]$. This interval is partitioned with $\mathrm{N} \in \mathbb{Z}$, $\mathrm{N} > 0$ for a fixed step length such that $h = \frac{b-a}{N}$. The first numerical result v_1 for $n = 0$ in (6), is produced over the subinterval $[x_0, x_1] = [x_0,\ x_0 + h]$, since v_0 is known from the IVP under consideration. For the second solution, for $n = 1$ in (6), the value of v_2 is obtained over the subinterval $[x_1, x_2] =$

Table 1 Comparison of results for Example 1

h	SDFFM		TBNM		TIRKN3		RADAU5	
	NFE	Err	NFE	Err	NFE	Err	NFE	Err
$\frac{1}{30}$	602	4.0×10^{-5}	602	2.1×10^{-4}	907	2.5×10^{-4}	853	2.2×10^{-4}
$\frac{1}{60}$	1202	2.5×10^{-6}	1202	1.3×10^{-5}	1288	6.6×10^{-6}	1208	4.4×10^{-4}
$\frac{1}{80}$	1602	7.7×10^{-7}	1602	4.1×10^{-6}	1682	7.0×10^{-6}	1639	6.0×10^{-6}

$[x_0 + h,\ x_0 + 2h]$ as v_1 is known from the previous step. This procedure is continued for $n = 2, \ldots, N-1$ iteratively to obtain the numerical solution to equation (1) on the entire interval of integration over non-overlapping subintervals $[x_0, x_1], [x_1, x_2], \ldots, \left[x_{N-1}, x_N\right]$.

Note that on each step the equation in (6) must be solved, for which we used the Newton's method, taking as starting value the approximate value v_n. If we write the equation in (6) as $F(v_{n+1}) = 0$, the stopping criteria used for the Newton's method are $|v_{n+1}^{i+1} - v_{n+1}^{i}| < 10^{-10}$ and $|F(v_{n+1}^{i+1})| < 10^{-10}$, taking the maximum number of iterations equal to 50.

4.1 Numerical examples

To study the numerical efficiency of the SDFFM, we integrate a number of problems with oscillatory solutions. The maximum global error of the approximate solution is calculated on the grid points $\{x_n\}_{n=0}^{N}$ as $Err = \|v\,(x_n) - v_n\|_\infty$, and the computational efficiency is shown by plotting the logarithm of the maximum global error ($\log(Err)$) against the logarithm of the number of function evaluations ($\log(NFE)$). We note that the fitting frequencies used in the numerical experiments have been obtained from the problems referenced from the literature. However, the strategies for the frequency choice considered in Ramos and Vigo-Aguiar (2010), Vigo-Aguiar and Ramos (2014) and Vanden Berghe et al. (2001) can be explored.

4.2 Example 1: non linear Strehmel–Weiner problem

Consider the non-linear second-order IVP in Nguyen et al. (2007) and Jator et al. (2013) in the interval $0 \leq x \leq 10$ given by

$$v_1''\,(x) = (v_1\,(x) - v_2(x))^3 + 6368 v_1\,(x) - 6384 v_2\,(x) + 42\cos\,(10x)\,,\ v_1\,(0) = 0.5,\ \ v_1'\,(0) = 0,$$
$$v_2''\,(x) = -(v_1\,(x) - v_2(x))^3 + 12768 v_1\,(x) - 12784 v_2\,(x) + 42\cos(10x)\,,\ v_2\,(0) = 0.5,\ \ v_2'\,(0) = 0,$$

with solution in closed form given by $v_1\,(x) = v_2\,(x) = \cos(4x)\ -\ \frac{\cos(10x)}{2}$. This problem is solved to establish the performance of SDFFM on a non linear problem. Numerical results of the maximum global errors of SDFFM with $\omega = 4$ are compared with the fourth-order trigonometrically fitted Numerov method (TBNM) of Jator et al. (2013) and the TIRKN3 and RADAU5 listed in Nguyen et al. (2007) which are presented in Table 1. Figure 2 displays the efficiency curves of the methods considered for comparisons and that of the SDFFM.

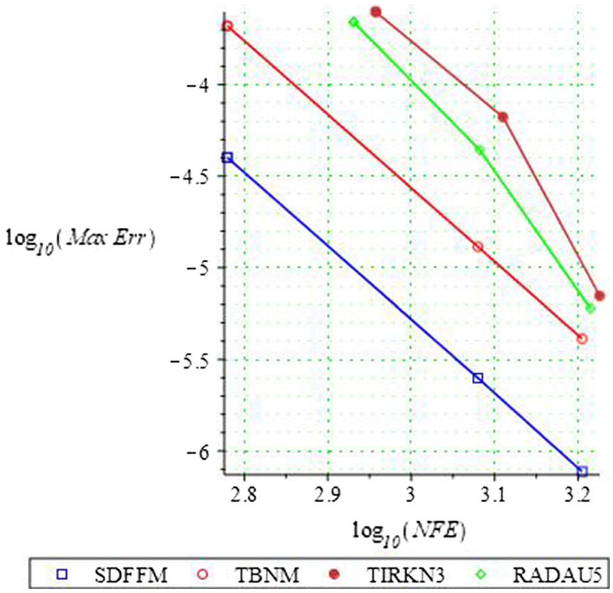

Fig. 2 Efficiency curves for Example 1

4.3 Example 2: orbital problem

We consider the famous almost periodic problem introduced by Stiefel and Bettis in 1969, given by

$$z''(x) + z(x) = 0.001e^{ix},\ i = \sqrt{-1},$$
$$z\,(0) = 1,\ z'\,(0) = 0.995i, \quad z \in \mathbf{C}.$$

Setting $z(x) = v_1(x) + iv_2(x)$, the above differential problem can be written equivalently as

$$v_1''(x) + v_1(x) = 0.001\cos(x)\ , \quad v_1\,(0) = 1, \quad v_1'\,(0) = 0$$
$$v_2''(x) + v_2(x) = 0.001\sin(x)\ , \quad v_2\,(0) = 0, \quad v_2'\,(0) = 0.995$$

with exact solution given by

$$v_1\,(x) = \cos(x)\ + 0.0005x\sin(x)$$
$$v_2\,(x) = \sin(x)\ - 0.0005x\cos(x)\ .$$

According to Lambert and Watson (1976), the analytical solution represents the perturbed motion of a circular orbit in the complex plane. The point $z(x)$ spirals slowly outwards such that its distance from the origin at any time x is given by

$$\gamma(x) = \sqrt{(v_1(x))^2 + (v_2(x))^2} = \sqrt{1 + (0.0005x)^2}.$$

This problem has been considered in the literature within different intervals of integration and different forms of errors have been computed for different values of the step size. Lambert and Watson (1976) and Jator (2010) solved the problem numerically to obtain the values of $v_1\,(x)$ and $v_2\,(x)$ using a sixth-order symmetric multistep method (SMM) and a seventh-order

Table 2 Computed values of γ, $x = 40\pi$, $\gamma(x) = 1.001972$

Methods	h				
	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$\frac{\pi}{6}$	$\frac{\pi}{9}$	$\frac{\pi}{12}$
SDFFM	1.001964	1.001969	1.001954	1.001972	1.001972
HLMM	1.001983	1.001974	1.001972	1.001983	1.001972
SMM	1.003067	1.002217	1.002047	1.001987	1.001973

Table 3 Comparison of results for Example 2, $\gamma(x) = 1.001972$, $x = 40\pi$

Methods	$h = \frac{\pi}{4}$	$h = \frac{\pi}{5}$	$h = \frac{\pi}{6}$	$h = \frac{\pi}{9}$	$h = \frac{\pi}{12}$
	Err(γ) Err(z)	Err(γ) Err(z)	Err(γ) Err(z)	Err(γ) Err(z)	Err(γ) Err(z)
SDFFM	8.02×10^{-6}	3.33×10^{-6}	1.80×10^{-5}	3.23×10^{-8}	1.02×10^{-9}
	1.30×10^{-4}	5.30×10^{-5}	2.60×10^{-5}	5.1×10^{-7}	1.60×10^{-8}
HLMM	1.10×10^{-5}	1.58×10^{-6}	2.97×10^{-7}	1.21×10^{-8}	1.24×10^{-9}
	2.17×10^{-4}	3.95×10^{-5}	9.51×10^{-6}	3.89×10^{-7}	3.96×10^{-8}
SMM	1.10×10^{-3}	2.45×10^{-4}	7.50×10^{-5}	6.00×10^{-6}	1.00×10^{-6}
	3.13×10^{-2}	7.30×10^{-3}	2.30×10^{-3}	1.88×10^{-4}	3.30×10^{-5}

hybrid linear multistep method (HLMM), respectively. The results for γ at $x = 40\pi$, which corresponds to 20 orbits of the point $z(x)$, are compared for the SDFFM and the related results of Lambert and Watson (1976) and Jator (2010) taking $\omega = 1$ as presented in Table 2. The errors in the computed values of z and γ are defined as follows

$$\text{Err}\,(x_i) = |z\,(x_i) - z_i| = \sqrt{(v_1\,(x_i) - v_{1\text{i}})^2 + (v_2\,(x_i) - v_{2\text{i}})^2}$$

$$\text{Err}\,(\gamma) = |\gamma\,(x) - \gamma| = \sqrt{(v_1(x))^2 + (v_2(x))^2} - \sqrt{v_1{}^2 + v_2{}^2}$$

and are displayed in Table 3.

Obviously, all the solutions obtained for SDFFM spiral outward for all step lengths. This is in agreement with the theoretical solution as well as with the results provided by the schemes by Lambert and Watson (1976) and Jator (2010).

From the results in Table 3, it is clear that SDFFM is more accurate than SMM of sixth order, and competes favourably with HLMM of order seven. Similarly, for the same problem, we confirm in Table 4 and Fig. 3 that SDFFM is more accurate and efficient compared to the sixth order methods in Cash (1984), Thomas (1988) or Xiang and Thomas (2002), and competes favourably with the seventh order method in Jator (2010).

Table 4 Comparison of absolute errors, $\mathrm{Err}(\gamma)$, $x = 40\pi$

h	SDFFM		Cash		Thomas		Xiang-Thomas		HLMM	
	NFE	Err	NFE	Err	NFE	Err	NFE	Err	NFE	Err
$\frac{\pi}{4}$	322	8.02×10^{-6}	640	1.25×10^{-5}	480	1.21×10^{-5}	480	1.22×10^{-4}	325	1.10×10^{-5}
$\frac{\pi}{5}$	402	3.33×10^{-6}	800	3.39×10^{-6}	600	2.71×10^{-5}	600	1.68×10^{-6}	403	1.58×10^{-6}
$\frac{\pi}{6}$	482	1.60×10^{-7}	960	1.19×10^{-6}	720	1.10×10^{-5}	720	7.29×10^{-7}	481	2.97×10^{-7}
$\frac{\pi}{9}$	722	3.23×10^{-8}	1440	9.79×10^{-8}	1080	8.91×10^{-7}	1080	6.28×10^{-8}	721	1.21×10^{-8}
$\frac{\pi}{12}$	962	1.02×10^{-9}	1920	1.71×10^{-8}	1440	6.28×10^{-8}	1440	4.25×10^{-9}	961	1.24×10^{-9}

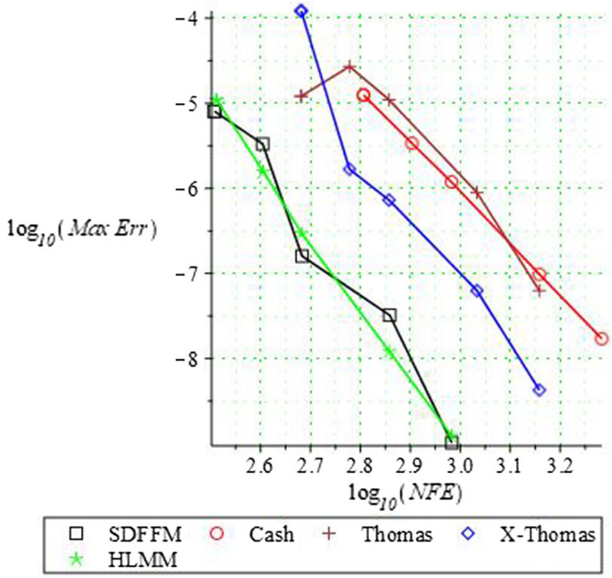

Fig. 3 Efficiency curves for Example 2

4.4 Example 3: Kepler problem

As our third example, we consider the classical Kepler problem which was solved in Jator et al. (2013) and Ndukum et al. (2016)

$$v_1^{''}(x) = -\frac{v_1(x)}{r^3}, \qquad v_1(0) = 1-\epsilon,\, v_1^{'}(0) = 0$$
$$v_2^{''}(x) = -\frac{v_2(x)}{r^3}, \qquad v_2(0) = 0,\, v_2^{'}(0) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}$$

where $r = \sqrt{v_1(x)^2 + v_2(x)^2}$ and ϵ $(0 \le \epsilon \le 1)$ is the eccentricity of the orbit. The solution of this problem in closed form is

$$v_1(x) = \cos(\lambda) - \epsilon,\;\; v_2(x) = \sqrt{1-\epsilon^2}\sin(\lambda),$$

where λ is the solution of the Kepler's equation $\lambda = x + \epsilon\sin(\lambda)$. We integrate the Kepler problem on the interval $[0,5\pi]$ with the fitting parameter $\omega = 1$ with eccentricity $\epsilon = 0.05$. The results presented in Table 5 and Fig. 4 show the competitiveness of the SDFFM with the fourth-order methods TBNM and BTFEBDM3 in Jator et al. (2013) and Ndukum et al. (2016), respectively.

4.5 Example 4: pertubed system

As our fourth experiment, we consider the following non-linear perturbed system on the range [0, 10] with $\epsilon = 10^{-3}$

$$v_1^{''}(x) = \epsilon\varphi_1(x) - 25v_1(x) - \epsilon\left(v_1(x)^2 + v_2(x)^2\right) \qquad v_1(0) = 1 \;\;,\; v_1^{'}(0) = 0,$$
$$v_2^{''}(x) = \epsilon\varphi_2(x) - 25v_2(x) - \epsilon\left(v_1(x)^2 + v_2(x)^2\right) \qquad v_2(0) = \epsilon \;\;,\; v_2^{'}(0) = 5,$$

Table 5 Comparison of results for Example 3

SDFFM			TBNM			BTFEBDM3		
N	Err	NFE	N	Err	NFE	N	Err	NFE
200	2.69×10^{-6}	402	200	6.63×10^{-3}	402	200	1.83×10^{-2}	402
400	3.43×10^{-8}	802	400	9.87×10^{-4}	802	450	5.84×10^{-4}	902
1200	1.22×10^{-9}	2402	1200	1.97×10^{-7}	2402	1200	1.17×10^{-5}	2402
2000	3.17×10^{-10}	4002	2000	3.62×10^{-9}	4002	2100	1.23×10^{-6}	4202

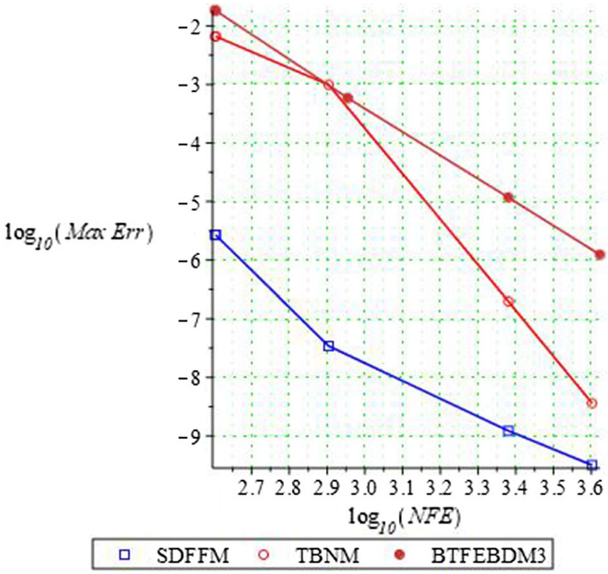

Fig. 4 Efficiency curves for Example 3

where

$$\varphi_1(x) = 1 + \epsilon^2 + 2\epsilon \sin\left(5x + x^2\right) + 2\cos\left(x^2\right) + \left(25 - 4x^2\right)\sin\left(x^2\right),$$
$$\varphi_2(x) = 1 + \epsilon^2 + 2\epsilon \sin\left(5x + x^2\right) - 2\sin\left(x^2\right) + \left(25 - 4x^2\right)\cos\left(x^2\right).$$

The solution in closed form is given by $v_1(x) = \cos(5x) + \epsilon \sin\left(x^2\right),\ v_2(x) = \sin(5x) + \epsilon \cos\left(x^2\right)$. The performance of SDFFM with the fitting parameter selected as $\omega = 5$ in comparison with a fifth-order TFARKN method by Fang et al. (2009), a fifth-order TRI5 method by Fang and Wu (2008), sixth-order methods DIS6 and ZER6 both in Franco (2006), are presented in Table 6 and Fig. 5, respectively.

Table 6 Comparison of results for Example 4

SDFFM			TRI5			DIS6			ZER6			TFARKN		
N	Err	NFE	N	Err	NFE	N	Err	NFE	N	Err	NFE	N(rej)	Err	NFE
40	5.37×10^{-9}	82	40	2.0×10^{-5}	230	40	6.31×10^{-4}	300	80	7.94×10^{-5}	490	29(6)	1.58×10^{-3}	250
80	8.32×10^{-11}	162	80	1.26×10^{-7}	300	80	5.01×10^{-6}	450	160	1.00×10^{-6}	700	88(9)	5.01×10^{-6}	450
160	1.29×10^{-12}	322	160	7.94×10^{-9}	500	160	2.51×10^{-8}	700	320	3.16×10^{-8}	1400	262(8)	1.26×10^{-8}	1000
320	2.00×10^{-14}	642	320	1.00×10^{-10}	950	320	3.16×10^{-10}	1250	640	6.31×10^{-10}	2600	811(4)	3.98×10^{-11}	3250

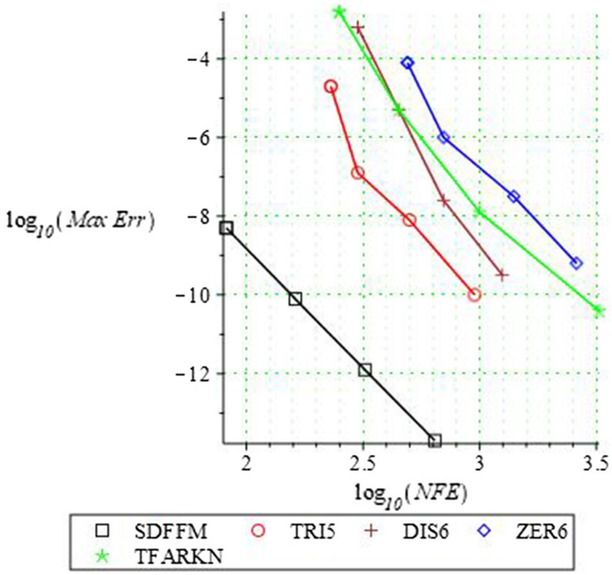

Fig. 5 Efficiency curves for Example 4

Table 7 Comparison of results for Example 5

h	SDFFM	EF IMP3	EFRK3	EFLMM3
0.1	1.00×10^{-23}	1.24×10^{-09}	6.03×10^{-06}	2.41×10^{-04}
0.05	6.90×10^{-26}	6.95×10^{-11}	6.66×10^{-07}	2.36×10^{-05}
0.025	2.82×10^{-28}	9.62×10^{-12}	8.00×10^{-08}	2.52×10^{-06}

4.6 Example 5: nearly sinusoidal problem

As our last experiment, we consider the system of two equations commonly referred to as Lambert system (Lambert 1991), on the range [0, 10] with $\omega = 1$

$$v_1'(x) + 2v_1(x) - v_2(x) = \sin(\omega x),$$
$$v_2'(x) + (\beta + 2)v_1(x) - (\beta + 1)v_2(x) = (\beta + 1)(\sin(\omega x) - \cos(\omega x)),$$

with initial conditions $v_1(0) = 2$ and $v_2(0) = 3$. The exact solutions of this system $v_1(x) = 2\exp(-x) + \sin(\omega x)$ and $v_2(x) = 2\exp(-x) + \cos(\omega x)$, which according to Lambert (1991) is independent of β. This system is used in Lambert (1991) to illustrate the concept of stiffness and its numerical consequences with $\beta = -3$ and $\beta = -1000$ for non stiff and stiff cases, respectively.

We present the comparison between our obtained results and EF IMP3, EFRK3, and EFLMM3 stated in Conte et al. (2020), Vanden Berghe et al. (2001) and Ixaru et al. (2002), respectively, for $\beta = -1000$ in Table 7 and Fig. 6. We observe from Table 7 that SDFFM produces smaller errors with respect to other methods it is compared to. Consequently, SDFFM is a competitive numerical integrator.

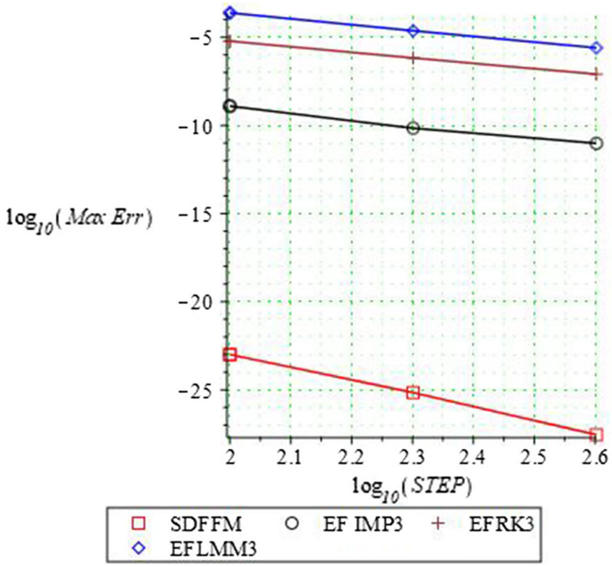

Fig. 6 Efficiency curves for Example 5

5 Conclusions

The presented functionally fitted method of order four has been designed to deal with first-order differential systems whose solutions are oscillatory. Some numerical experiments have been included, showing a better performance of the proposed method compared with other methods in the literature, even of higher order. In our future work we will try to extend this kind of approach for solving problems whose solutions contain two frequencies.

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Conflict of interest The authors declare no conflict of interest.

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