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More Effective Results for Testing Oscillation of Non-Canonical Neutral Delay Differential Equations

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Abstract: In this work, we address an interesting problem in studying the oscillatory behavior of solutions of fourth-order neutral delay differential equations with a non-canonical operator. We obtained new criteria that improve upon previous results in the literature, concerning more than one aspect. Some examples are presented to illustrate the importance of the new results.

Keywords: neutral differential equations; fourth order; oscillatory behavior; non-canonical case

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1. Introduction

We direct our attention during this work to studying the oscillatory behavior of the solutions of the neutral delay differential equation (NDDE):

$$\left(a\cdot\left((u+p\cdot(u\circ\tau))'''\right)^{\beta}\right)'(t)+\left(q\cdot(u\circ\sigma)^{\beta}\right)(t)=0,\ t\geq t_0 \quad (1)$$

in the non-canonical case, that is, when:

$$A_0(t_0):=\int_{t_0}^{\infty}a^{-1/\beta}(\kappa)\mathrm{d}\kappa<\infty.$$

Furthermore, we assume that β is a ratio of odd positive integers, a, τ, σ, p and q are in $C[t_0,\infty)$, a is positive, a', p and q are non-negative, $p<1$, $q\neq 0$ on any half line $[t_*,\infty)$ for all $t_*\geq t_0$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$, $(f\circ g)(t)=f(g(t))$ and:

$$A_k(t):=\int_t^{\infty}A_{k-1}(\varrho)\mathrm{d}\varrho,\ \text{for } k=1,2,$$

A solution u of the Equation (1) means a function in $C([t_*,\infty),\mathbb{R})$, which satisfies:

$$u+p\cdot(u\circ\tau)\in C^3[t_*,\infty),\ \ a\cdot\left((u+p\cdot(u\circ\tau))'''\right)^{\beta}\in C^1[t_*,\infty),$$

and also satisfies (1) on $[t_*,\infty)$. We will only consider solutions that are not identically zero eventually. A solution u of (1) is called *oscillatory* if it is neither positive nor negative, ultimately; otherwise, it is called *non-oscillatory*.

Differential equations with a neutral argument have interesting applications in problems of real-world life. In the networks containing lossless transmission lines, the neutral differential equations appear in the modeling of these phenomena as is the case of high-speed computers. In addition, second order neutral equations appear in the theory of automatic control and in aeromechanical systems in which inertia plays an important role. Moreover, second order delay equations play an important role in the study of vibrating masses attached to an elastic bar, as the Euler equation, see: [1–3].

To the best of our knowledge, the number of works dealing with the study of higher-order neutral differential equations in the non-canonical case is much smaller than those that deal with equations in the canonical case (see [4–16]). On the other hand, it is easy to find many works that have dealt with non-canonical higher-order equations with delay but not neutral (see for example [17–20]).

When studying the oscillation of the NDDEs in (1) in the non-canonical case, one of the most interesting goals is to find criteria that ensure the non-existence of Kneser solutions (solutions which satisfy $(-1)^k(u+p\cdot(u\circ\tau))^{(k)}(t)>0$ for $k=0,1,2,3$, $t\in[t_0,\infty)$). This is because most of the relationships commonly used are not valid in this case.

For second-order equations, in an interesting work, Bohner et al. [21] addressed this problem, obtaining the following restriction for the solution and a related function:

$$u>\left(1-p\cdot\frac{A_0\circ\tau}{A_0}\right),$$

where u is a Kneser-type solution. This relationship allowed the authors to find many new criteria that simplified and improved their previous results in the literature. The first interesting problem was how to extend Bohner's results in [21] to the even-order equations.

Recently, by using comparison techniques, Li and Rogovchenko [22] studied the oscillatory behavior of the even-order neutral delay differential equation:

$$\left(a\cdot\left((u+p\cdot(u\circ\tau))^{(n-1)}\right)^{\alpha}\right)'(t)+\left(q\cdot(u\circ\sigma)^{\beta}\right)(t)=0, \tag{2}$$

where $n\geq 4$ is an even number. However, the results in [22] depend on the existence of three unknown functions that satisfy certain conditions, and there is no general rule on how to choose these functions. So another interesting problem is how to find criteria that do not include unknown functions.

Theorem 1 ([22] Theorem 6)**.** *Let $n\geq 4$ be even and $0<\alpha=\beta\leq 1$. Assume that $0\leq p(t)\leq p_0<\infty$ for some constant p_0:*

$$\tau'\geq\tau_*>0 \text{ and } \tau\circ\sigma=\sigma\circ\tau \tag{3}$$

and there exist three functions $\eta_1,\eta_2,\eta_3\in C([t_0,\infty),\mathbb{R})$ such that:

$$\eta_1(t)\leq\sigma(t)\leq\eta_2(t),\ \eta_1(t)\leq\tau(t)\leq t<\eta_2(t),\ \eta_3(t)\geq\sigma(t),\ \eta_3(t)>t$$

and:

$$\lim_{t\to\infty}\eta_1(t)=\infty.$$

Suppose also that:

$$\frac{\tau_*\left(\tau_*+p_0^{\beta}\right)^{-1}}{((n-1)!)^{\beta}}\liminf_{t\to\infty}\int_{\tau^{-1}(\eta_1(t))}^{t}Q(s)\left(\frac{(\eta_1(s))^{n-1}}{a^{1/\beta}(\eta_1(s))}\right)^{\beta}\mathrm{d}s>\frac{1}{\mathrm{e}},$$

$$\frac{\tau_*\left(\tau_*+p_0^{\beta}\right)^{-1}}{((n-2)!)^{\beta}}\liminf_{t\to\infty}\int_{t}^{\eta_2(t)}\left(Q(s)\left(\sigma^{n-2}(s)\right)^{\beta}(A_0(\eta_2(s)))^{\beta}\right)\mathrm{d}s>\frac{1}{\mathrm{e}}$$

and:

$$\frac{\tau_*\left(\tau_*+p_0^{\beta}\right)^{-1}}{((n-3)!)^{\beta}}\liminf_{t\to\infty}\int_t^{\eta_3(t)}\left(Q(s)\left(\int_{\eta_3(s)}^{\infty}\left((\eta-\eta_3(s))^{n-3}A_0(\eta)\right)\mathrm{d}\eta\right)^{\beta}\right)\mathrm{d}s>\frac{1}{\mathrm{e}},$$

where $Q(t)=\min\{q(t),q(\tau(t))\}$*. Then, every solution of (2) is oscillatory.*

In this work, we will address all the interesting problems above by obtaining a new relationship between the solution and a related function (as an extension of Bohner's results in [21]). Furthermore, the new criteria ensure the oscillation of all the solutions of (1), and are distinguished by the following:

- They do not require unknown functions;
- They do not need condition (3).

In order to prove our main results, we will use the following lemmas.

Lemma 1 ([23] Lemma 2.2.1)**.** *Let* $\phi\in C^n([t_0,\infty),(0,\infty))$ *and* $\phi^{(n)}(t)$ *be of constant sign on* $[t_1,\infty)$ *with* $t_1\geq t_0$*. Then, there exists an integer* $\kappa\in[0,n]$*, with* $n+\kappa$ *even if* $\phi^{(n)}(t)\geq 0$*, or* $n+\kappa$ *odd if* $\phi^{(n)}(t)\leq 0$*, such that:*

$$\kappa>0 \text{ yields } \phi^{(j)}(t)>0 \text{ for } j=0,1,...,\kappa-1,$$

and

$$\kappa\leq n-1 \text{ yields } (-1)^{\kappa+j}\phi^{(j)}(t)>0 \text{ for } j=\kappa,\kappa+1,...,n-1.$$

Lemma 2 ([17])**.** *Assume that* $\phi\in C^m([t_0,\infty),\mathbb{R}^+)$*,* $\phi^{(m)}$ *is not identically zero on a subray of* $[t_0,\infty)$ *and* $\phi^{(m)}$ *is of fixed sign. Suppose that* $\phi^{(m-1)}\phi^{(m)}\leq 0$ *for* $t\in[t_1,\infty)$*, where* $t_1\geq t_0$ *is large enough. If* $\lim_{t\to\infty}\phi(t)\neq 0$*, then there exists a* $t_\lambda\in[t_1,\infty)$ *such that:*

$$\phi\geq\frac{\lambda}{(m-1)!}t^{m-1}\left|\phi^{(m-1)}\right|,$$

for every $\lambda\in(0,1)$ *and* $t\in[t_\lambda,\infty)$*.*

Lemma 3 ([21] Lemma 2.6)**.** *Assume that* K_i *is a real number for* $i=1,2,3$*,* $K_2>0$*, and* β *is a ratio of odd positive integers. Then, for all* $w\in\mathbb{R}$ *:*

$$K_1w-K_2(w-K_3)^{(\beta+1)/\beta}\leq K_1K_3+\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{K_1^{\beta+1}}{K_2^{\beta}}.$$

2. Main Results

First, we will proceed to classify the set of positive solutions of (1) according to the behavior of its derivatives. To facilitate the calculations, we adopt the following notations: $z:=u+p\cdot(u\circ\tau)$, and:

$$Q(t):=q(t)\left(1-p(\sigma(t))\frac{A_2(\tau(\sigma(t)))}{A_2(\sigma(t))}\right)^{\beta}.$$

We assume that u is a positive solution of (1). Note that from the definition of z, we have that $z(t)>0$; moreover, from (1) it is $\left(a(t)(z'''(t))^{\beta}\right)'\leq 0$. This implies that $a(t)(z'''(t))^{\beta}$ is non-increasing and of constant sign, and thus, since $a(t)>0$, we have that $(z'''(t))^{\beta}$ is of constant sign, and so is $z'''(t)$.

According to Lemma 1 with $n = 3$, there exists an integer κ with:

$$\kappa = \begin{cases} 1 \text{ or } 3 & \text{if } z'''(t) > 0; \\ 0 \text{ or } 2 & \text{if } z'''(t) < 0. \end{cases}$$

Thus, we get that:

$$\begin{array}{ll} z''' > 0 & \begin{cases} (1)\ \kappa = 1, & z > 0, \quad z' > 0, \quad z'' < 0 \\ (2)\ \kappa = 3, & z > 0, \quad z' > 0, \quad z'' > 0 \end{cases} \\ z''' < 0 & \begin{cases} (3)\ \kappa = 0, & z > 0, \quad z' < 0, \quad z'' > 0 \\ (4)\ \kappa = 2, & z > 0, \quad z' > 0, \quad z'' > 0 \end{cases} \end{array}$$

Moreover, if $z'''(t) > 0$, $a'(t) > 0$ and $\left(a(t)(z'''(t))^{\beta}\right)' \leq 0$, then $z^{(4)}(t) < 0$. Then, we eventually obtain the following three exclusive cases:

D1 : $z^{(i)}(t) > 0$ for $i = 0, 1, 3$, and $z^{(4)}(t) < 0$;
D2 : $z^{(i)}(t) > 0$ for $i = 0, 1, 2$, and $z^{(3)}(t) < 0$;
D3 : $z^{(i)}(t) > 0$ for $i = 0, 2$, and $z^{(j)}(t) < 0$ for $j = 1, 3$ (note that in this case u is a Kneser solution).

Lemma 4. *If $u(t)$ is a Kneser solution of (1), then the function z/A_2 is increasing, eventually.*

Proof. Based on the positivity of the solution u, it follows from (1) that $a(t)(z'''(t))^{\beta}$ is non-increasing. Then, taking into account that we are in case **D3**, we have that:

$$-z''(t) \leq \int_t^{\infty} \frac{1}{a^{1/\beta}(\varrho)} a^{1/\beta}(\varrho) z'''(\varrho) \mathrm{d}\varrho \leq a^{1/\beta}(t) z'''(t) A_0(t), \tag{4}$$

which leads to:

$$\left(\frac{z''(t)}{A_0(t)}\right)' = \frac{A_0(t)z'''(t) + a^{-1/\beta}(t)z''(t)}{A_0^2(t)} \geq 0.$$

Therefore, we have that z''/A_0 is an increasing function, and thus:

$$-z'(t) \geq \int_t^{\infty} \frac{z''(\varrho)}{A_0(\varrho)} A_0(\varrho) \mathrm{d}\varrho \geq \frac{z''(t)}{A_0(t)} A_1(t),$$

which implies that:

$$\left(\frac{z'(t)}{A_1(t)}\right)' = \frac{A_1(t)z''(t) + A_0(t)z'(t)}{A_1^2(t)} \leq 0.$$

By using a similar approach, it is easy to conclude that $-A_1(t)z(t) \leq z'(t)A_2(t)$, and so $z(t)/A_2(t)$ is an increasing function. □

Theorem 2. *Assume that there exist some $t_1 \geq t_0$ such that $A_2(t) > p(t)A_2(\tau(t))$ for $t \geq t_1$. If there exists a function $\theta \in C([t_0, \infty), (0, \infty))$ such that:*

$$\limsup_{t \to \infty} \frac{A_2^{\beta}(t)}{\theta(t)} \int_{t_1}^{t} \left(\theta(h) Q(h) - \frac{1}{(\beta+1)^{\beta+1}} \frac{(\theta'(h))^{\beta+1}}{\theta^{\beta}(h) A_1^{\beta}(h)} \right) \mathrm{d}h > 1, \tag{5}$$

then, the Equation (1) has no Kneser solutions.

Proof. We proceed by contradiction. Assuming that u is a Kneser solution of (1) on $[t_1, \infty)$, where $t_1 \geq t_0$. As in the proof of Lemma 4, we arrive at (4). Integrating (4) from t to ∞ and taking into account the behavior of the derivatives of z, we obtain:

$$z'(t) \leq a^{1/\beta}(t) z'''(t) A_1(t), \tag{6}$$

and integrating again, we obtain:

$$z(t) \geq -a^{1/\beta}(t) z'''(t) A_2(t). \tag{7}$$

By Lemma 4, we have that $z(t)/A_2(t)$ is an increasing function, and hence $z(\tau(t)) \leq (A_2(\tau(t))/A_2(t)) z(t)$. Thus, it follows from the definition of z that:

$$u(t) \geq z(t)\left(1 - p(t)\frac{A_2(\tau(t))}{A_2(t)}\right),$$

which together with (1) gives:

$$\left(a(t)(z'''(t))^{\beta}\right)' \leq -Q(t) z^{\beta}(\sigma(t)). \tag{8}$$

Now, we define the function:

$$\mathbf{T}(t) := \theta(t)\left(\frac{a(t)(z'''(t))^{\beta}}{z^{\beta}(t)} + \frac{1}{A_2^{\beta}(t)}\right).$$

It follows readily from (7) that $T(t) \geq 0$ for $t \geq t_1$. Moreover, we have that:

$$\mathbf{T}'(t) = \frac{\theta'(t)}{\theta(t)}\mathbf{T}(t) + \theta(t)\left(\frac{\left(a(t)(z'''(t))^{\beta}\right)'}{z^{\beta}(t)} - \frac{a(t)(z'''(t))^{\beta}}{z^{\beta+1}(t)}\beta z'(t) + \frac{\beta A_1(t)}{A_2^{\beta+1}(t)}\right).$$

Now, using the inequalities in (6) and (8), we obtain that:

$$\begin{aligned}
\mathbf{T}'(t) &\leq \frac{\theta'(t)}{\theta(t)}\mathbf{T}(t) + \theta(t)\left(-Q(t)\frac{z^{\beta}(\sigma(t))}{z^{\beta}(t)} - \beta a^{1+1/\beta}(t) A_1(t)\left(\frac{z'''(t)}{z(t)}\right)^{\beta+1} + \frac{\beta A_1(t)}{A_2^{\beta+1}(t)}\right) \\
&\leq \frac{\theta'(t)}{\theta(t)}\mathbf{T}(t) - \theta(t)Q(t) - \beta\frac{A_1(t)}{\theta^{1/\beta}(t)}\left(\mathbf{T}(t) - \frac{\theta(t)}{A_2^{\beta}(t)}\right)^{1+1/\beta} + \theta(t)\frac{\beta A_1(t)}{A_2^{\beta+1}(t)}.
\end{aligned} \tag{9}$$

Using Lemma 3 with $K_1 := \theta'/\theta$, $K_2 := \beta A_1 \theta^{-1/\beta}$, $K_3 := \theta A_2^{-\beta}$ and $w := T$, we obtain:

$$\begin{aligned}
\mathbf{T}'(t) &\leq -\theta(t)Q(t) + \frac{1}{(\beta+1)^{\beta+1}}\frac{(\theta'(t))^{\beta+1}}{\theta^{\beta}(t)A_1^{\beta}(t)} + \frac{\theta'(t)}{A_2^{\beta}(t)} + \theta(t)\frac{\beta A_1(t)}{A_2^{\beta+1}(t)} \\
&= -\theta(t)Q(t) + \frac{1}{(\beta+1)^{\beta+1}}\frac{(\theta'(t))^{\beta+1}}{\theta^{\beta}(t)A_1^{\beta}(t)} + \left(\frac{\theta(t)}{A_2^{\beta}(t)}\right)'.
\end{aligned}$$

Integrating the above inequality from t_1 to t, we have:

$$\begin{aligned}\int_{t_1}^{t}\left(\theta(h)Q(h)-\frac{1}{(\beta+1)^{\beta+1}}\frac{(\theta'(h))^{\beta+1}}{\theta^{\beta}(h)A_1^{\beta}(h)}\right)\mathrm{d}h &\leq \left.\left(\frac{\theta(h)}{A_2^{\beta}(h)}-\mathbf{T}(h)\right)\right|_{t_1}^{t} \\ &= \left.-\theta(h)\frac{a(h)(z'''(h))^{\beta}}{z^{\beta}(h)}\right|_{t_1}^{t} \\ &\leq -\theta(t)\frac{a(t)(z'''(t))^{\beta}}{z^{\beta}(t)}. \end{aligned} \tag{10}$$

From (7), we see that $-a(z''')^{\beta}z^{-\beta}\leq 1/A_2^{\beta}$ and so (10) becomes:

$$\frac{A_2^{\beta}(t)}{\theta(t)}\int_{t_1}^{t}\left(\theta(h)Q(h)-\frac{1}{(\beta+1)^{\beta+1}}\frac{(\theta'(h))^{\beta+1}}{\theta^{\beta}(h)A_1^{\beta}(h)}\right)\mathrm{d}h\leq 1. \tag{11}$$

The obtained inequality (11) conflicts with the condition (5), and this contradiction ends the proof. □

3. Discussion and Examples

In the following theorem, we present sufficient conditions for the oscillation of all solutions of (1).

Theorem 3. *Assume that there exist some $t_1\geq t_0$ such that $A_2(t)>p(t)A_2(\tau(t))$, and that for some constant $\lambda_0\in(0,1)$, the first-order delay differential equation:*

$$\psi'(t)+\left(\frac{\lambda_0}{6}\sigma^3(t)\right)^{\beta}\frac{G(t)}{a(\sigma(t))}\psi(\sigma(t))=0 \tag{12}$$

is oscillatory, and that for some constant $\lambda_1\in(0,1)$, it is:

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\frac{\lambda_1^{\beta}}{(2!)^{\beta}}\sigma^{2\beta}(h)G(h)A_0^{\beta}(h)-\frac{\beta^{\beta+1}a^{-1/\beta}(h)}{(\beta+1)^{\beta+1}A_0(h)}\right)\mathrm{d}h=\infty, \tag{13}$$

where $G:=q(1-p(\sigma))^{\beta}$, for $t\geq t_1$. If (5) holds, then every solution of (1) is oscillatory.

Proof. Assume that (1) has a positive solution u. From (1), we have:

$$\left(a(t)\left(z'''(t)\right)^{\beta}\right)'=-q(t)u^{\beta}(\sigma(t))\leq 0. \tag{14}$$

According to Lemma 1 and taking into account the order of the equation in (1), we eventually obtain the following three exclusive cases **D1–D3**.

First, suppose that case **D1** holds. From the definition of z, we have:

$$u(t)=z(t)-p(t)u(\tau(t))\geq(1-p(t))z(t). \tag{15}$$

Using (14) in (15) gives:

$$\left(a(t)\left(z'''(t)\right)^{\beta}\right)'\leq -q(t)(1-p(\sigma(t)))^{\beta}z^{\beta}(\sigma(t)). \tag{16}$$

Using Lemma 2 with $m=4$, we have:

$$z(t)\geq\frac{\lambda t^3}{3!}z'''(t), \tag{17}$$

for every $\lambda \in (0,1)$. From (16) and (17), we obtain:

$$\left(a(t)\left(z'''(t)\right)^{\beta}\right)' + G(t)\left(\frac{\lambda\sigma^{3}(t)}{6}\right)^{\beta}\left(z'''(\sigma(t))\right)^{\beta} \leq 0.$$

Letting $\psi(t) = a(t)(z'''(t))^{\beta}$. Clearly, ψ is a positive solution of the first-order delay differential inequality:

$$\psi'(t) + G(t)\left(\frac{\lambda\sigma^{3}(t)}{6a^{1/\beta}(\sigma(t))}\right)^{\beta}\psi(\sigma(t)) \leq 0. \tag{18}$$

It follows from [24] [Theorem 1] that the corresponding differential Equation (12) also has a positive solution for all $\lambda \in (0,1)$, which is a contradiction.
We then assume that case **D2** holds. We define the function Φ by

$$\Phi(t) = \frac{a(t)(z'''(t))^{\beta}}{(z''(t))^{\beta}}. \tag{19}$$

Then, $\Phi(t) < 0$ for $t \geq t_1$. Noting that $a(t)\left(z^{(n-1)}(t)\right)^{\beta}$ is decreasing, we have:

$$a^{1/\beta}(s)z'''(s) \leq a^{1/\beta}(t)z'''(t),\ s \geq t \geq t_1. \tag{20}$$

Multiplying (20) by $a^{-1/\beta}(s)$ and integrating it on $[t,\infty)$, we obtain:

$$0 \leq z''(t) + a^{1/\beta}(t)z'''(t)A_0(t),$$

that is:

$$-\frac{a^{1/\beta}(t)z'''(t)A_0(t)}{z''(t)} \leq 1.$$

From (19), we see that:

$$-\Phi(t)A_0^{\beta}(t) \leq 1. \tag{21}$$

Differentiating (19), we have:

$$\Phi'(t) = \frac{\left(a(t)(z'''(t))^{\beta}\right)'}{(z''(t))^{\beta}} - \frac{\beta a(t)(z'''(t))^{\beta+1}}{(z''(t))^{\beta+1}},$$

which, in view of (1) and (19), becomes:

$$\Phi'(t) = -\frac{q(t)u^{\beta}(\sigma(t))}{(z''(t))^{\beta}} - \frac{\beta\Phi^{(\beta+1)/\beta}(t)}{a^{1/\beta}(t)}. \tag{22}$$

Taking into account the fact that $z'(t) > 0$ and the definition of $z(t)$, we deduce that (15) holds. Hence, (22) becomes:

$$\Phi'(t) \leq -\frac{q(t)(1-p(\sigma(t)))^{\beta}z^{\beta}(\sigma(t))}{(z''(t))^{\beta}} - \frac{\beta\Phi^{(\beta+1)/\beta}(t)}{a^{1/\beta}(t)}. \tag{23}$$

Using Lemma 2 with $m = 2$, we find:

$$z(t) \geq \frac{\lambda t^{2}}{2!}z''(t),$$

for all sufficiently large t and for every $\lambda \in (0,1)$. Then, (23) becomes:

$$\Phi'(t) \le -q(t)(1-p(\sigma(t)))^{\beta}\left(\frac{\lambda\sigma^2(t)}{2!}\right)^{\beta}\frac{(z''(\sigma(t)))^{\beta}}{(z''(t))^{\beta}} - \frac{\beta\Phi^{(\beta+1)/\beta}(t)}{a^{1/\beta}(t)}.$$

Since $t \ge \sigma(t)$ and $z''(t)$ is decreasing, we have:

$$\Phi'(t) \le -q(t)(1-p(\sigma(t)))^{\beta}\left(\frac{\lambda\sigma^2(t)}{2!}\right)^{\beta} - \frac{\beta\Phi^{(\beta+1)/\beta}(t)}{a^{1/\beta}(t)}. \tag{24}$$

Multiplying (24) by $A_0^{\beta}(t)$ and integrating it into $[t_1, t]$, we obtain:

$$\begin{aligned} 0 \ \ \ge \ \ & A_0^{\beta}(t)\Phi(t) - A_0^{\beta}(t_1)\Phi(t_1) + \int_{t_1}^{t}\frac{\beta A_0^{\beta-1}(s)}{a^{1/\beta}(s)}\Phi(s)\mathrm{d}s + \int_{t_1}^{t}\frac{\beta A_0^{\beta}(s)}{a^{1/\beta}(s)}\Phi^{(\beta+1)/\beta}(s)\mathrm{d}s \\ & + \int_{t_1}^{t} q(s)(1-p(\sigma(s)))^{\beta}\left(\frac{\lambda\sigma^2(s)}{2!}\right)^{\beta}A_0^{\beta}(s)\mathrm{d}s. \end{aligned}$$

Setting $A = A_0^{\beta}(s)/a^{1/\beta}(s)$, $B = A_0^{\beta-1}(s)/a^{1/\beta}(s)$ and $w = -\Phi(s)$, and using the inequality:

$$Bw - Aw^{(\alpha+1)/\alpha} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}$$

we obtain:

$$\int_{t_1}^{t}\left(\frac{\lambda_1^{\beta}}{(2!)^{\beta}}\sigma^{2\beta}(h)G(h)A_0^{\beta}(h) - \frac{\beta^{\beta+1}a^{-1/\beta}(h)}{(\beta+1)^{\beta+1}A_0(h)}\right)\mathrm{d}h \le \frac{\Phi(t_1)}{A_0^{-\beta}(t_1)} + 1,$$

which contradicts (13).

Finally, we suppose that case **D3** holds. From Theorem 2, we obtain a contradiction. The proof of the theorem is complete. □

Corollary 1. *Assume that there exist some $t_1 \ge t_0$ such that $A_2(t) > p(t)A_2(\tau(t))$, and (5), (13) hold for some constant $\lambda_1 \in (0,1)$ and for $t \ge t_1$. If:*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\left(\frac{\lambda_0}{6}\sigma^3(h)\right)^{\beta}\frac{G(h)}{a(\sigma(h))}\mathrm{d}h > \frac{1}{\mathrm{e}}, \tag{25}$$

then every solution of (1) is oscillatory.

Proof. Using Theorem 2.1.1 in [25], we obtain that Equation (12) is oscillatory under the condition (25). Therefore, the proof is the same as that of Theorem 3. □

Example 1. *Consider the fourth-order equation:*

$$(t^4(u(t) + p_0u(\lambda t))''')' + q_0u(\mu t) = 0,\ t \ge 1, \tag{26}$$

where $\lambda, \mu \in (0,1)$, $p_0 \in (0,\lambda)$ and $q_0 > 0$. It is easy to verify that $A_0(t) = \frac{1}{3t^3}$, $A_1(t) = \frac{1}{6t^2}$ and $A_2(t) = \frac{1}{6t}$. Using Theorem 2 and choosing $\theta(t) = A_2(t)$, we have that Equation (26) has no Kneser solutions if:

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(q_0\left(1-\frac{p_0}{\lambda}\right)\frac{1}{6} - \frac{1}{4}\right)\frac{1}{h}\mathrm{d}h > 1,$$

and this is satisfied when:

$$q_0 > \frac{6\lambda}{4(\lambda - p_0)}. \tag{27}$$

Remark 1. *It is easy to see that the results in [22] are difficult to apply, because there are no clear rules or guidelines for selecting the unknown functions η_i which must meet a set of conditions. However, by choosing $\eta_3 = 1 + \lambda$ in Theorem 8 in [22], we deduce that Equation (26) has no Kneser solutions if:*

$$q_0 > \frac{6(\lambda + p_0)(\lambda + 1)}{\lambda \mathrm{e} \ln(1 + 1/\lambda)}. \tag{28}$$

In the special case where $\lambda = 1/2$ and $p_0 = 1/4$, the conditions (27) and (28) become $q_0 > 3.0$ and $q_0 > 4.5206$, respectively. Therefore, our new results provide more precise criteria for the non-existence of Kneser solutions.

Example 2. *Consider the fourth-order equation:*

$$\left(\mathrm{e}^{\beta t}\left((u(t) + p_0 u(t - \tau_0))'''\right)^{\beta}\right)' + q_0 \mathrm{e}^{\beta t} u^{\beta}(t - \sigma_0) = 0, \tag{29}$$

where $\tau_0, \sigma_0, q_0 > 0$ and $p_0 \in [0, \mathrm{e}^{-\tau_0})$. It is easy to verify that $A_k(t) = \mathrm{e}^{-t}$ for $k = 0, 1, 2$, and:

$$Q(t) := q_0 \mathrm{e}^{\beta t}(1 - p_0 \mathrm{e}^{\tau_0})^{\beta}.$$

Note that (13) and (25) are directly satisfied. Finally, taking $\theta(t) = \mathrm{e}^{-\beta t}$, it is a simple task to check that condition (5) is true whenever:

$$q_0(1 - p_0 \mathrm{e}^{\tau_0})^{\beta} > \frac{\beta^{\beta+1}}{(\beta + 1)^{\beta+1}}. \tag{30}$$

Thus, from Corollary (1), every solution of (29) is oscillatory if (30) holds.

Remark 2. *In Example 2, in the non-neutral case, that is, $p_0 = 0$, the oscillation condition of the Equation (29) becomes:*

$$q_0 > \frac{\beta^{\beta+1}}{(\beta + 1)^{\beta+1}},$$

which is the same condition obtained in [18,19].

4. Conclusions

In this work, a new criterion was established to determine the non-existence of the so-called Kneser solutions of a class of even-order NDDEs. Using this criterion, some conditions to ensure the oscillation of all solutions of the studied equation were established. The conditions obtained do not use unknown functions and provide more precise results than those presented in [22]. Moreover, by studying the non-canonical case, our results complement the results in [4–7,14].

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